An Approximation Scheme for Evolutive Hamilton-Jacobi Equations

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ABSTRACT: An approximation scheme for the evolutive Hamilton-Jacobi equation associated with the finite horizon deterministic problem is presented. The scheme allows a variable time step and can be applied to structured and unstructured grids. Depending on the regularity of the value function different orders of convergence are proved. In particular, under Lipschitz assumptions the convergence behavior of conservative monotone hyperbolic schemes is retrieved. Namely, if $\frac{k}{h} \leq$ *constant* then we have convergence of order $\frac{1}{2}$, where k and h are the space and the time step, respectively.

KEYWORDS: Numerical schemes, viscosity solutions, evolutive Hamilton–Jacobi–Bellman equations.

1 Introduction

In this paper, we propose an algorithm for the approximation of the value function associated with the non-autonomous finite horizon deterministic problem.
Consider a system of controlled ordinary differential equations

$$\begin{aligned} &\dot{y}(t) = b(y(t), t, \alpha(t)) \qquad t \in (s, T), \\ &y(s) = x \end{aligned} \tag{1.1}$$

where $\alpha(\cdot) \in \mathcal{A} \equiv L^\infty([0,T]; A)$ is the *control*, A is a compact subset of $\mathbb{R}^m$, and the *state* $y \in \mathbb{R}^n$. The *cost functional* related to the finite horizon problem is

$$J(x, s, \alpha(\cdot)) \equiv \int_s^T f(y(t), t, \alpha(t)) e^{-\lambda(t-s)} \mathrm{d}t + \psi(y(T)) e^{-\lambda(T-s)} \tag{1.2}$$

where $\lambda > 0$ is a parameter, f is the *running cost* and ψ is the *final cost*.

We are interested in the numerical computation of the *value function* associated to this problem, namely

$$u(x,s) \equiv \inf_{\alpha(\cdot)\in\mathcal{A}} J(x,s,\alpha(\cdot)). \tag{1.3}$$

Capuzzo Dolcetta (1983) proposed a convergent time approximation of the value function for the autonomous infinite horizon case. Subsequentially, Capuzzo Dolcetta and Ishii (1984) proved convergence of order $\frac{\gamma}{2}$, where γ happens to be the Hölder exponent of the value function. Falcone in (1987) considered a time-space approximation based on the one proposed by Dolcetta and, in the case $\gamma = 1$, he proved convergence of the kind $\frac{k}{h}$, where k and h represent respectively the (fixed) space and the time steps. Other approximation schemes for first order Hamilton–Jacobi equations have been proposed by several authors using finite differences as in Crandall and Lions (1984) or finite elements as in Gonzales and Rofman (1985) (see also the book Kushner and Dupuis (1992) and the recent survey paper Falcone (1997) for a more extensive and up-dated list of references).
In this work, following the idea of Capuzzo Dolcetta (1983), we use the fact that u can be characterized (see Lions (1982)) as the unique viscosity solution of the Hamilton-Jacobi equation

$$\begin{aligned} &-\frac{\partial u}{\partial s} + \lambda u + \max_{a\in A}\{-b(x,s,a)\cdot\nabla_x u - f(x,s,a)\} = 0 \ \text{ in } \mathbb{R}^n \times [0,T) \\ &u(x,T) = \psi(x) \text{ in } \mathbb{R}^n, \end{aligned} \tag{1.4}$$

and as a first step we consider an approximation in time of (1.4), recursively defined. Our contribution here is to extend the scheme allowing a *variable* time step and deriving a-priori estimates on the rate of convergence in the L^∞ norm. A variable time step is mainly motivated by the necessity to have an accurate numerical resolution around the singularities of the solution (to be understood in this framework as the curves where the ∇u has jumps) and by the use of domain decomposition techniques without overlapping where handling the boundary conditions on the internal interfaces is crucial (see Camilli, Falcone, Lanucara and Seghini (1994) for more informations). We refer the reader interested in the analysis of the set of singularities of viscosity solutions to Hamilton-Jacobi-Bellman equations to the papers Fleming (1969) and Cannarsa and Soner (1987).
It should be noted that, in the Lipschitz case and after a suitable change of variable, our scheme belongs to a class of schemes introduced by Souganidis (1985) (see Remark 2.2). As a consequence the approximate solution u_h converges to u as $h^{1/2}$. If u is only Hölder continuous of order γ a combination of the arguments of Capuzzo Dolcetta and Ishii (1984), and Souganidis (1985) gives convergence of order $\gamma/2$. As second step, in order to get a complete discretized approximation of u, we consider a linear approximation in space of u_h denoting by u_h^k the solution of the fully discrete

scheme. We then prove that u_h^k converges to u as $\frac{k^\gamma}{h^{\gamma/2}} + h^{\gamma/2}$. Moreover, our proof can be easly modified to the infinite horizon case to obtain that the scheme proposed by Falcone in (1987) converges as $\frac{k^\gamma}{h^{\gamma/2}}$.

We assume in the sequel that for any (x,s), $(y,t) \in \mathbb{R}^n \times [0,T]$ and $a \in A$, there exists $\gamma \in (0,1]$ and positive constants such that

$$\begin{aligned} &|b(x,s,a) - b(y,t,a)| \le L_b(|x-y| + |s-t|), \\ &|b(x,s,a)| \le M_b \quad \text{and} \quad b(x,s,\cdot) \in C(A), \end{aligned} \tag{1.5}$$

$$\begin{aligned} &|f(x,s,a) - f(y,t,a)| \le L_f(|x-y|^\gamma + |s-t|^\gamma), \\ &|f(x,s,a)| \le M_f \quad \text{and} \quad f(x,s,\cdot) \in C(A), \end{aligned} \tag{1.6}$$

$$\begin{aligned} &|\psi(x) - \psi(y)| \le L_\psi |x-y|^\gamma, \\ &|\psi(x)| \le M_\psi. \end{aligned} \tag{1.7}$$

Note that under our assumptions (1.1) has a unique solution, see Coddington (1955).
Let us now recall the definition of viscosity solution and some properties of the value function that we will need afterwards.

Definition 1.1 *A function $u \in C(\mathbb{R}^n \times [0,T))$ is a viscosity solution* of (1.4.1) if for every $\phi \in C^1(\mathbb{R}^n \times [0,T))$ the following holds:
(i) if $u - \phi$ attains a local maximum at $(x_0, s_0) \in \mathbb{R}^n \times [0,T)$ then

$$\begin{aligned} &-\frac{\partial \phi(x_0,s_0)}{\partial s} + \lambda u(x_0,s_0) + \\ &\quad + \max_{a \in A}\{-b(x_0,s_0,a) \cdot \nabla_x \phi(x_0,s_0) - f(x_0,s_0,a)\} \le 0; \end{aligned}$$

(ii) if $u - \phi$ attains a local minimum at $(x_0, s_0) \in \mathbb{R}^n \times [0,T)$ then

$$\begin{aligned} &-\frac{\partial \phi(x_0,s_0)}{\partial s} + \lambda u(x_0,s_0) + \\ &\quad + \max_{a \in A}\{-b(x_0,s_0,a) \cdot \nabla_x \phi(x_0,s_0) - f(x_0,s_0,a)\} \ge 0. \end{aligned}$$

Moreover, if $u \in C(\mathbb{R}^n \times [0,T])$ and $u(x,T) = \psi(x)$ in $\mathbb{R}^n$ we say that u is a viscosity solution of (1.4).

Theorem 1.1 *Under the assumptions (1.5)-(1.7) the value function u is the viscosity solution in $BUC(\mathbb{R}^n \times [0,T])$ of (1.4), and for any (x,s), $(y,t) \in \mathbb{R}^n \times [0,T]$ satisfies*

$$|u(x,s) - u(y,t)| \le L_u(|x-y|^\gamma + |s-t|^\gamma).$$

For a proof of this theorem we refer to Crandall, Ishii and Lions (1992).

The next theorem tells us that the value function u verifies a so called dynamic programming principle.

Theorem 1.2 *Suppose (1.5)-(1.7), let $(x,s) \in \mathbb{R}^n \times [0,T]$, for each control α choose $\tau \in [s,T]$ (τ may depend on α), then we have:*

$$u(x,s) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_s^\tau f(y(t),t,\alpha(t)) e^{-\lambda(t-s)} dt + u(y(\tau),\tau) e^{-\lambda(\tau - s)} \right\}.$$

2 Time Approximation

As in the case of the autonomous infinite horizon problem, see Capuzzo Dolcetta (1983), the time approximation we consider is based on the fact that a direct discretization of the state and of the cost functional gives a discrete version of the dynamic programming principle in Theorem 1.2, which is nothing else but an approximation in time of the equation (1.4). For $(x,s) \in \mathbb{R}^n \times [0,T]$ fixed, given any $N \in \mathbb{N}$, set $h^N = \frac{T-s}{N}$ and for any $\alpha(\cdot) \in \mathcal{A}$ define

$$\begin{aligned} &y_{j+1} = y_j + h^N b(y_j, s_j, \alpha(s_j)) \qquad j = 0..(N-1), \\ &y_0 = x, \end{aligned} \tag{1.8}$$

where $s_j = s + jh^N$ with $j = 0..N$.
Moreover, consider

$$J_N(x,s,\alpha(\cdot)) \equiv \sum_{j=0}^{N-1} h^N f(y_j, s_j, \alpha(s_j)) e^{-\lambda j h^N} + \psi(y_N) e^{-\lambda(T-s)}, \tag{1.9}$$

$$u_N(x,s) \equiv \inf_{\alpha(\cdot) \in \mathcal{A}} J_N(x,s,\alpha(\cdot)). \tag{1.10}$$

Lemma 2.1 *Suppose (1.5)-(1.7), then for $n = 1..(N-1)$ we have:*

$$\begin{aligned} u_N(x,s) = \inf_{\alpha(\cdot) \in \mathcal{A}} \Bigg\{ &\sum_{j=0}^{n-1} h^N f(y_j, s_j, \alpha(s_j)) e^{-\lambda j h^N} + \\ &+ u_{N-n}(y_n, s_n) e^{-\lambda n h^N} \Bigg\}. \end{aligned} \tag{1.11}$$

Proof. Let $\bar{u}_N(x,s)$ be the right hand side of (1.11). By definition, for any $\epsilon > 0$ there exist $\alpha_0(\cdot)$, and $\alpha_1(\cdot) \in \mathcal{A}$ such that

$$\begin{aligned} \bar{u}_N(x,s) \quad &\le \sum_{j=0}^{n-1} h^N f(y_j^0, s_j, \alpha_0(s_j)) e^{-\lambda j h^N} + \\ &+ u_{N-n}(y_n^0, s_n) e^{-\lambda n h^N} \le \bar{u}_N(x,s) + \epsilon, \end{aligned} \tag{1.12}$$

and, since $\frac{T-s_n}{N-n} = h^N$,

$$u_{N-n}(y_n^0, s_n)e^{-\lambda n h^N} \leq$$
$$\leq \left\{ \sum_{j=0}^{(N-n)-1} h^N f(y_j^1, s_{n+j}, \alpha_1(s_{n+j}))e^{-\lambda j h^N} + \right. \tag{1.13}$$
$$\left. +\psi(y_{N-n}^1)e^{-\lambda(T-s_n)} \right\} e^{-\lambda n h^N} \leq u_{N-n}(y_n^0, s_n)e^{-\lambda n h^N} + \epsilon,$$

here y_j^0 and y_j^1 are the solutions of (1.8) for $\alpha_0(\cdot)$, x and $\alpha_1(\cdot)$, y_n^0, respectively. Set

$$\bar{\alpha}(t) = \begin{cases} \alpha_0(t) & t \in [s, s_n] \\ \alpha_1(t) & t \in (s_n, T], \end{cases}$$

then $\bar{\alpha}(\cdot) \in \mathcal{A}$, and

$$\bar{y}_j = \begin{cases} y_j^0 & j = 0..n \\ y_{j-n}^1 & j = (n+1)..N \end{cases}$$

is the solution of (1.8) for $\alpha(\cdot) = \bar{\alpha}(\cdot)$.
As a consequence, we have

$$u_N(x,s) \leq J_N(x, s, \bar{\alpha}(\cdot)) = \sum_{j=0}^{n-1} h^N f(y_j^0, s_j, \alpha_0(s_j))e^{-\lambda j h^N} +$$
$$+ \textstyle\sum_{j=n}^{N-1} h^N f(y_{j-n}^1, s_j, \alpha_1(s_j))e^{-\lambda j h^N} + \psi(y_{N-n}^1)e^{-\lambda(T-s)},$$

therefore, using (1.13) we get

$$u_N(x,s) \leq \sum_{j=0}^{n-1} h^N f(y_j^0, s_j, \alpha_0(s_j))e^{-\lambda j h^N} + u_{N-n}(y_n^0, s_n)e^{-\lambda n h^N} + \epsilon,$$

and, due to (1.12), $u_N(x,s) \leq \bar{u}_N(x,s) + 2\epsilon$ for any $\epsilon > 0$. We obtain the other inequality in a similar way, thus the theorem follows. □

Remark 2.1 *Equation (1.11) for n=1 yields a discrete version of the dynamic programming principle, that is*

$$u_N(x,s) = \inf_{\alpha(\cdot)\in\mathcal{A}} \{h^N f(x,s,\alpha(s)) +$$
$$+ e^{-\lambda h^N} u_{N-1}(x + h^N b(x,s,\alpha(s)), s + h^N)\} \tag{1.14}$$

Moreover, $s = T$ implies $h^N = 0$ for any $N > 0$, hence (1.10) leads $u_N(x,T) = \psi(T)$.

The previous remark suggests an approximation in time of the value function $u(x,s)$. Equation (1.14) will be modified in order to allow a variable time step. This makes the method more flexible and efficient.

Consider a partition $P \equiv \{0 = s_0 < s_1 < .. < s_N = T\}$ of the interval $[0,T]$, and set $h = \max_j h_j$, where $h_{j+1} = s_{j+1} - s_j$ $(j = 0..(N-1))$, then for any $(x,s) \in \mathbb{R}^n \times [0,T]$ define

$$\begin{aligned} u_h(x,s) = \quad & \inf_{a\in A}\{(s_{j+1}-s)f(x,s,a) + e^{-\lambda(s_{j+1}-s)} \\ & u_h(x+(s_{j+1}-s)\,b(x,s,a), s_{j+1})\} \quad \text{if } s \in [s_j, s_{j+1}) \\ u_h(x,T) \quad & = \psi(x), \end{aligned} \tag{1.15}$$

with $j = 0..(N-1)$.

Remark 2.2 *If we perform the change of variable $t = T - s$, it is straightforward to see that, when $\gamma = 1$, (1.15) is a particular case of the schemes consider by Souganidis (1985).*

Lemma 2.2 *Assume (1.5)-(1.7), for any $x, y \in \mathbb{R}^n$ and $j = 0..(N-1)$ we have*

1. $|u_h(x,s_j) - u_h(y,s_j)| \le L_1 e^{L_b^\gamma (N-j)h^\gamma} |x-y|^\gamma$
2. $|u_h(x,s_j) - u_h(x,T)| \le L_2((T-s_j) + (N-j)h^\gamma)$,
3. *let $j \neq 0$, $s = s_j$ and $t \in [s_{j-1}, s_j)$ then*
$$|u_h(x,s_j) - u_h(x,t)| \le L_3(|s_j - t| + e^{L_b^\gamma (N-j)h^\gamma} |s_j - t|^\gamma),$$
4. *if $s, t \in [s_j, s_{j+1})$*
$$|u_h(x,s) - u_h(x,t)| \le L_4(|s-t| + e^{L_b^\gamma (N-j)h^\gamma} |s-t|^\gamma),$$
5. *$u_h(x,s)$ is bounded and uniformly continuous in $\mathbb{R}^n \times [0,T]$,*

where L_1, L_2, L_3 and L_4 are positive constants that do not depend on P.

Proof. We start by noticing that since $\psi(x)$ is continuous, then the inf in (1.15) is actually a min. In fact, if $u_h(x,s_j)$ is continuous in x, then the inf in the definition of $u_h(x,s_{j-1})$ is a min, so *1.* can be derived for $j-1$ which gives $u_h(x,s_{j-1})$ continuous in x and so on down to $j = 0$.
For $s = T$, we defined u_h to be ψ, thus by (1.7) we get

$$\begin{aligned} & |u_h(x,s_{N-1})| \le M_f h_N + e^{-\lambda h_N} M_\psi, \\ & |u_h(x,s_{N-1}) - u_h(y,s_{N-1})| \le \\ & \qquad \le (h_N L_f + L_\psi e^{-\lambda h_N}(1 + L_b^\gamma h_N^\gamma))\,|x-y|^\gamma, \\ & |u_h(x,s_{N-1}) - u_h(x,s_N)| \le \\ & \qquad \le h_N(M_f + \lambda M_\psi) + L_\psi e^{-\lambda h_N} M_b^\gamma h_N^\gamma. \end{aligned} \tag{1.16}$$

It is then easy to prove by induction the inequalities below

$$|u_h(x,s_j)| \le M_f\left(h_{j+1}+\sum_{n=j+2}^{N} h_n\, e^{-\lambda\sum_{m=j+1}^{n-1} h_m}\right)+M_\psi\, e^{-\lambda\sum_{m=j+1}^{N} h_m}, \quad (1.17)$$

$$\begin{aligned}|u_h(x,s_j)-u_h(y,s_j)| \le \\ \le \Bigg[L_f\left(h_{j+1}+\sum_{n=j+2}^{N} h_n\, e^{-\lambda\sum_{m=j+1}^{n-1} h_m}\prod_{m=j+1}^{n-1}(1+L_b^\gamma h_m^\gamma)\right)+ \\ + e^{-\lambda\sum_{m=j+1}^{N} h_m} L_\psi \prod_{m=j+1}^{N}(1+L_b^\gamma h_m^\gamma)\Bigg]|x-y|^\gamma, \end{aligned} \quad (1.18)$$

$$\begin{aligned}|u_h(x,s_j)-u_h(x,T)| \quad &\le (M_f+\lambda M_\psi)(h_{j+1}+\sum_{n=j+2}^{N} h_n\, e^{-\lambda\sum_{m=j+1}^{n-1} h_m})+ \\ &+L_\psi M_b^\gamma\Big(\sum_{n=j+1}^{N} h_n^\gamma\, e^{-\lambda\sum_{m=j+1}^{n} h_m}\Big), \end{aligned} \quad (1.19)$$

where $j = 0..(N-2)$.
The inequalities *1.* and *2.* directly follow from (1.18), and (1.19) with

$$L_1 = (L_f T + L_\psi) \quad \text{and} \quad L_2 = M_f + \lambda M_\psi + L_\psi M_b^\gamma.$$

Now, (1.17) implies $|u_h(x,s)| \le 2TM_f + M_\psi$, which together with *1.* yield *3.*, and *4.* if we notice that for $s \in [s_{N-1}, s_N)$

$$|u_h(x,s)-u_h(x,T)| \le |T-s|M_f + e^{-\lambda(T-s)}L_\psi M_b^\gamma |T-s|^\gamma + \lambda M_\psi |T-s|,$$

$$|u_h(x,s)-u_h(x,t)| \le (M_f+L_f)|s-t| + L_\psi(L_b^\gamma + M_b^\gamma)|s-t|^\gamma.$$

Moreover,

$$L_3 = M_f+\lambda(2TM_f+M_\psi)+L_1M_b^\gamma \quad \text{and} \quad L_4 = M_f+L_f+L_1(L_b^\gamma+M_b^\gamma).$$

Finally, the uniform continuity is a consequence of *1.* ,*3.* , and *4.* □

Remark 2.3 *In the case $\gamma = 1$, starting from the inequality (1.8), it is possible to prove the Lipschitz continuity of $u_h(x,s)$.*

A combination of the arguments used by Capuzzo Dolcetta and Ishii (1984) and by Souganidis (1985) gives convergence of order $\frac{\gamma}{2}$ of u_h to u as $h \to 0$.

Theorem 2.1 *Let u_h be defined as in (1.15), and u the value function defined in (1.3), then if the assumptions (1.5)-(1.7) are verified we have for any $(x,s) \in \mathbb{R}^n \times [0,T]$*

$$|u_h(x,s) - u(x,s)| \le C h^{\frac{\gamma}{2}}, \tag{1.20}$$

where C depends on T but not on the partion P.

Proof. Given $\epsilon, \alpha > 0$ we define

$$\varphi(y,t,x,s) = u_h(y,t) - u(x,s) + \beta_\epsilon(y-x) + \eta_\alpha(t-s),$$

where $\beta_\epsilon(x) = -\frac{|x|^2}{\epsilon^2}$, and $\eta_\alpha(s) = -\frac{|s|^2}{\alpha^2}$. By Theorem 1.1 and Lemma 2.1, for any $\delta > 0$ there exist $(y_1,t_1),(x_1,s_1) \in \mathbb{R}^n \times [0,T]$ such that

$$\varphi(y_1,t_1,x_1,s_1) > \sup \varphi - \delta.$$

We choose $\theta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\theta(y_1,x_1) = 1$ and $0 \le \theta(y,x) \le 1$, $|D\theta| < 1$, so that for any $\delta \in (0,1)$

$$\zeta(y,t,x,s) = \varphi(y,t,x,s) + \delta\theta(y,x)$$

has a max at a point (y_0,t_0,x_0,s_0), with $y_0, x_0 \in \mathrm{supp}\theta$ and $t_0, s_0 \in [0,T]$. Therefore, if we set

$$\phi(x,s) = u_h(y_0,t_0) + \beta_\epsilon(y_0-x) + \eta_\alpha(t_0-s) + \delta\theta(y_0,x),$$

$u(x_0,s_0) - \phi(x_0,s_0) = -\zeta(y_0,t_0,x_0,s_0)$ is a local min for $u(x,s) - \phi(x,s)$. Recalling the definition of ζ, for any $(y,t),(x,s) \in \mathbb{R}^n \times [0,T]$ we have

$$\begin{aligned} &u_h(y_0,t_0) - u(x_0,s_0) + \beta_\epsilon(y_0-x_0) + \eta_\alpha(t_0-s_0) + \delta\theta(y_0,x_0) \ge \\ &\ge u_h(y,t) - u(x,s) + \beta_\epsilon(y-x) + \eta_\alpha(t-s) + \delta\theta(y,x). \end{aligned} \tag{1.21}$$

Now, taking in (1.21) $x = y = y_0$ and $s = s_0$, $t = t_0$ we obtain, if $\delta < \frac{1}{\epsilon^2}$

$$|y_0 - x_0| \le C_{\epsilon,\delta}\, \epsilon^{\frac{2}{2-\gamma}}, \tag{1.22}$$

where $C_{\epsilon,\delta} = \max\left\{ \left(\frac{L_u}{(1-\delta\epsilon^2)^2} \right)^{\frac{1}{2-\gamma}}, (L_u+\delta)^{\frac{1}{2-\gamma}} \right\}$. Similarly, we find

$$|s_0 - t_0| \le \alpha^{\frac{2}{2-\gamma}} L_u^{\frac{1}{2-\gamma}}. \tag{1.23}$$

Always from (1.21), using (1.22) and (1.23), for $x = y$ and $s = t$ we derive

$$\begin{aligned} &u_h(x,s) - u(x,s) \le \\ &\quad \le u_h(y_0,t_0) - u(x_0,s_0) + C_{\epsilon,\delta}^2\, \epsilon^{\frac{2\gamma}{2-\gamma}} + \alpha^{\frac{2\gamma}{2-\gamma}} L_u{}^{\frac{2}{2-\gamma}} + 2\delta. \end{aligned} \tag{1.24}$$

We need to consider three different cases:
(case1) $t_0 = T$. In this case $u_h(y_0, T) = \psi(y_0) = u(y_0, T)$, so Theorem 1.1 with (1.22)-(1.24) yields

$$u_h(x,s) - u(x,s) \leq (C^{\gamma}_{\epsilon,\delta} + C^2_{\epsilon,\delta})\epsilon^{\frac{2\gamma}{2-\gamma}} + L_u^{\frac{2}{2-\gamma}}\alpha^{\frac{2\gamma}{2-\gamma}} + 2\delta,$$

and since if $\delta \to 0$ then $C_{\epsilon,\delta} \to$ constant, taking $\epsilon = \alpha = h^{\frac{2-\gamma}{4}}$ we have (1.20).
(case2) $t_0 \neq T$, $s_0 = T$. We have $u(x_0, T) = \psi(x_0) = u_h(x_0, T)$, moreover we know $t_0 \in [s_{j-1}, s_j)$ for some j, so that *2.* and *3.* of Lemma 2.2 lead to

$$u_h(y_0, t_0) - u(y_0, T) \leq C((T - s_j) + (N - j)h^{\gamma} + h + e^{L_b^{\gamma}(N-j)h^{\gamma}}|s_j - t_0|^{\gamma})$$

On the other hand, (1.23) yields

$$|s_j - T| \leq \alpha^{\frac{2}{2-\gamma}} L_u^{\frac{1}{2-\gamma}}$$
$$(N - j) \leq \tfrac{\alpha^{\frac{2}{2-\gamma}}}{h_{min}} L_u^{\frac{1}{2-\gamma}},$$

where $h_{min} = \min_{1 \leq j \leq N} h_j$. Combining these with (1.24), we derive

$$u_h(x,s) - u(x,s) \leq C\Bigg[h + \alpha^{\frac{2\gamma}{2-\gamma}} + \alpha^{\frac{2}{2-\gamma}} + \\ + \left(\frac{\alpha^{\frac{2}{2-\gamma}}}{h_{min}} + e^{L_b^{\gamma} L_u^{\frac{1}{2-\gamma}} h^{\frac{\gamma}{2}}}\right)h^{\gamma} + (C^{\gamma}_{\epsilon,\delta} + C^2_{\epsilon,\delta})\,\epsilon^{\frac{2\gamma}{2-\gamma}}\Bigg] + 2\delta.$$

Hence, the choices $\epsilon = h^{\frac{2-\gamma}{4}}$ and $\alpha = (h_{min}h^{-\frac{\gamma}{2}})^{\frac{2-\gamma}{2}}$ if $h_{min} < 1$, and $\alpha = h^{-\frac{\gamma(2-\gamma)}{4}}$ otherwise, give us (1.20). If $h > 1$ we use $h \leq T^{1-\frac{\gamma}{2}}h^{\frac{\gamma}{2}}$, $h^{-\frac{\gamma}{2}} \leq 1 \leq h^{\frac{\gamma}{2}}$ and so on.
(case3) $s_0, t_0 \neq T$. Since u is a viscosity solution of (1.4) there exists $a^* \in A$ such that

$$-\frac{\partial\phi(x_0,s_0)}{\partial s} + \lambda u(x_0,s_0) + \\ -b(x_0,s_0,a^*)\cdot\nabla_x\phi(x_0,s_0) - f(x_0,s_0,a^*) \geq 0, \qquad (1.25)$$

and by definition of u_h if $t_0 \in [s_{j-1}, s_j)$, then

$$u_h(y_0,t_0) - (s_j - t_0)f(y_0,t_0,a^*) + \\ -e^{-\lambda(s_j - t_0)}u_h(y_0 + (s_j - t_0)b(y_0,t_0,a^*), s_j) \leq 0. \qquad (1.26)$$

Consider (1.21) for $y = y_0 + (s_j - t_0)b(y_0, t_0, a^*)$, $x = x_0$ and $t = s_j$, $s = s_0$, then (1.26) becomes

$$(1 - e^{-\lambda(s_j - t_0)})u_h(y_0,t_0) - (s_j - t_0)f(y_0,t_0,a^*) +$$

$$+e^{-\lambda(s_j-t_0)}\Big[(s_j-t_0)\nabla_y\beta_\epsilon(y_0-x_0)\cdot b(y_0,t_0,a^*)-\frac{1}{\epsilon^2}(s_j-t_0)^2$$
$$|b(y_0,t_0,a^*)|^2-\frac{2}{\alpha^2}(s_j-t_0)(t_0-s_0)-\frac{1}{\alpha^2}(s_j-t_0)^2\Big]+$$
$$+e^{-\lambda(s_j-t_0)}(s_j-t_0)[\delta\, b(y_0,t_0,a^*)\cdot\nabla_y\theta(y',x_0)]\le 0, \qquad (1.27)$$

for some y'.
Dividing (1.27) by $(s_j-t_0)>0$, and subtracting (1.25) from it, we have

$$\frac{(1-e^{-\lambda(s_j-t_0)})}{(s_j-t_0)}u_h(y_0,t_0)-\lambda u(x_0,s_0)+f(x_0,s_0,a^*)-f(y_0,t_0,a^*)+$$
$$+\nabla_y\beta_\epsilon(y_0-x_0)\cdot\Big(e^{-\lambda(s_j-t_0)}b(y_0,t_0,a^*)-b(x_0,s_0,a^*)\Big)-\delta M_b-$$
$$-\frac{1}{\alpha^2}(s_j-t_0)+\delta\nabla_x\theta(y_0,x_0)\cdot b(x_0,s_0,a^*)-\frac{1}{\epsilon^2}(s_j-t_0)M_b^2+$$
$$+\Big(1-e^{-\lambda(s_j-t_0)}\Big)\frac{2}{\alpha^2}(t_0-s_0)\le 0,$$

which implies

$$\frac{(1-e^{-\lambda(s_j-t_0)})}{(s_j-t_0)}u_h(y_0,t_0)-\lambda u(x_0,s_0)\le L_f(|y_0-x_0|^\gamma+|t_0-s_0|^\gamma)+$$
$$+2\delta M_b+(s_j-t_0)(\frac{1}{\alpha^2}+\frac{M_b^2}{\epsilon^2})+\frac{2}{\alpha^2}\lambda(t_0-s_0)(s_j-t_0)+$$
$$+\frac{2}{\epsilon^2}|y_0-x_0|\Big[\lambda(s_j-t_0)M_b+L_b(|y_0-x_0|+|t_0-s_0|)\Big].$$

Once again, setting $\epsilon=\alpha=h^{\frac{2-\gamma}{4}}$, we obtain (1.20). Note that if $h>1$ we use $h\le T^{1-\frac{\gamma}{2}}h^{\frac{\gamma}{2}}$ and $h^{\frac{\gamma}{2}-\frac{1}{2}}\le 1\le T^{\frac{1}{2}}h^{\frac{\gamma}{2}}$ and so on.
Using the fact that there exists a^* for which the min in (1.15) is obtained, we can prove in a similar way the other inequality and the theorem follows.
□

3 Space approximation

We will consider a space approximation of $u_h(x,s)$. As we said the algorithm can work on structured or unstructured grids, we will give the convergence proof for a general unstructured grid.
For simplicity, we assume $\Omega\subset\mathbb{R}^n$ to be a polyhedron and consider a finite family T_i of simplices which give a regular triangulation of $\Omega=\cup_i T_i$.
Let M be the number of vertices, x_i, of the triangulation. Set

$$k\equiv\max_i\{\text{diam } T_i\},\ G\equiv\{x_i, i=1..M\} \text{ and } S\equiv\{s_j, j=0..N\}.$$

We suppose, for any $(x_i, s_j) \in G \times S$, and $a \in A$

$$x_i + h_j b(x_i, s_j, a) \in \Omega, \tag{1.28}$$

then for $x \in \Omega$, and $t \in [0, T]$ we define

$$\begin{aligned} u_h^k(x,s) &= \sum_{i=1}^{M} \lambda_i(x) u_h^k(x_i, s_{j+1}) \qquad \text{if } s \in [s_j, s_{j+1}), \\ u_h^k(x,T) &= \psi(x) \qquad \text{if } s = T, \end{aligned} \tag{1.29}$$

with $x = \sum_{i=1}^{M} \lambda_i(x) x_i$, where $\sum_{i=1}^{M} \lambda_i(x) = 1$, $\lambda_i(x) > 0$, and

$$\begin{aligned} u_h^k(x_i, s_j) &= \min_{a \in A} \{ h_{j+1} f(x_i, s_j, a) + e^{-\lambda h_{j+1}} \sum_{l=1}^{M} \lambda_l^i(a, s_j) u_h^k(x_l, s_{j+1}) \}, \\ u_h^k(x,T) &= \psi(x), \end{aligned} \tag{1.30}$$

$j = 0..(N-1)$, here $x_i + h_{j+1} b(x_i, s_j, a) = \sum_{l=1}^{M} \lambda_l^i(a, s_j) x_l$.

Lemma 3.1 *Suppose (1.5)-(1.7) and let $u_h^k(x,s)$ be defined as in (1.28)-(1.30), for any $x_i \in I$ and $j = 0..(N-2)$ we have*

$$|u_h^k(x_i, s_j) - u_h^k(x_i, T)| \le (M_f + \lambda M_\psi)(h_{j+1} + \sum_{n=j+2}^{N} h_n \, e^{-\lambda \sum_{m=j+1}^{n-1} h_m})$$

$$+ L_\psi M_b^\gamma (\sum_{n=j+1}^{N} h_n^\gamma \, e^{-\lambda \sum_{m=j+1}^{n} h_m}) + L_\psi k^\gamma (\sum_{n=j+1}^{N} e^{-\lambda \sum_{m=j+1}^{n} h_m}). \tag{1.31}$$

Moreover,

$$|u_h^k(x_i, s_{N-1}) - u_h^k(x_i, T)| \le (M_f + \lambda M_\psi) h_N + L_\psi e^{-\lambda h_N} (k^\gamma + h_N^\gamma M_b^\gamma)$$

Proof. The result follows by induction using the definition of $u_h^k(x,s)$.

Remark 3.1 *From Lemma 3.1 we infer, for j=0..(N-1),*

$$\begin{aligned} &|u_h^k(x_i, s_j) - u_h^k(x_i, T)| \le \\ &\le (M_f + \lambda M_\psi)(T - s_j) + L_\psi (M_b^\gamma h^\gamma + k^\gamma)(N - j). \end{aligned} \tag{1.32}$$

Theorem 3.1 *Let $u(x,s)$ be the value function defined in (1.3), and suppose the hypotheses of Lemma 3.1 be verified, then for any $x \in \Omega$ and $s \in [0,T]$*

$$|u_h^k(x,s) - u(x,s)| \le C\left(\frac{k^\gamma}{h^{\gamma/2}} + h^{\gamma/2}\right), \tag{1.33}$$

if there exist positive constants such that

$$\text{for any } j = 1..N \qquad h \le C_1 h_j, \qquad \text{and} \qquad \frac{k}{h} \le C_2.$$

Proof. Following Theorem 2.1, define for $(x,s) \in \mathbb{R}^n \times [0,T]$ and $(x_i, s_j) \in G \times S$

$$\varphi(i,j,x,s) = u_h^k(x_i,s_j) - u(x,s) + \beta_\epsilon(x_i - x) + \eta_\alpha(s_j - s).$$

It is easy to verify that

$$\begin{aligned}\varphi_M \equiv & \sup_{(x_i,s_j)\in G\times S,\,(x,s)\in \mathbb{R}^n\times[0,T]} \varphi(i,j,x,s) = \\ & = \max_{(x_i,s_j)\in G\times S}\left[\sup_{(x,s)\in \mathbb{R}^n\times[0,T]} \varphi(i,j,x,s)\right] < \infty.\end{aligned}$$

Fix $\delta \in (0,1)$, then there exists $(x_{i_1}, s_{j_1}, x_1, s_1) \in G \times S \times \mathbb{R}^n \times [0,T]$ such that $\varphi(i_1,j_1,x_1,s_1) > \varphi_M - \delta$. Consider $\theta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\theta(x_{i_1}, x_1) = 1$, $0 \le \theta(y,x) \le 1, |D\theta| < 1$ and define

$$\zeta(i,j,x,s) = \varphi(i,j,x,s) + \delta\theta(x_{i_1}, x).$$

Since $(\mathrm{supp}\theta)_i \equiv \{x \, s.t. \, (x_i,x) \in \mathrm{supp}\theta\} \subset \mathbb{R}^n$ is compact, we claim

$$\begin{aligned}\zeta_M \equiv & \sup_{(x_i,s_j)\in G\times S,\,(x,s)\in \mathbb{R}^n\times[0,T]} \zeta(i,j,x,s) = \\ & = \max_{(x_i,s_j)\in G\times S}\left[\max_{(x,s)\in (supp\theta)_i\times[0,T]} \zeta(i,j,x,s)\right].\end{aligned}$$

Thus, $\zeta_M = \zeta(i_0,j_0,x_0,s_0)$ for some $(x_{i_0}, s_{j_0}, x_0, s_0) \in G \times S \times (\mathrm{supp}\theta)_{i_0} \times [0,T]$. Set $(y_0,t_0) = (x_{i_0}, s_{j_0})$, and define

$$\phi(x,s) = u_h^k(y_0,t_0) + \beta_\epsilon(y_0 - x) + \eta_\alpha(t_0 - s) + \delta\theta(y_0,x),$$

then $u(x_0,s_0) - \phi(x_0,s_0) = -\zeta(i_0,j_0,x_0,s_0)$ is a local min for $u(x,s) - \phi(x,s)$. So, for any $(x_i,s_j,x,s) \in G \times S \times \mathbb{R}^n \times [0,T]$

$$\begin{aligned}& u_h^k(y_0,t_0) - u(x_0,s_0) + \beta_\epsilon(y_0 - x_0) + \eta_\alpha(t_0 - s_0) + \delta\theta(y_0,x_0) \ge \\ & \ge u_h^k(x_i,s_j) - u(x,s) + \beta_\epsilon(x_i - x) + \eta_\alpha(s_j - s) + \delta\theta(x_i,x). \end{aligned}\tag{1.34}$$

At this point, as in Theorem 2.1, we can derive formulas (1.22), (1.23) and

$$\begin{aligned}& u_h^k(x_i,s_j) - u(x_i,s_j) \le \\ & \quad \le u_h^k(y_0,t_0) - u(x_0,s_0) + C_{\epsilon,\delta}^2\,\epsilon^{\frac{2\gamma}{2-\gamma}} + \alpha^{\frac{2\gamma}{2-\gamma}} L_u^{\frac{2}{2-\gamma}} + 2\delta. \end{aligned}\tag{1.35}$$

(case1) $t_0 = T$. Since by (1.30) $u_h^k(y_0, T) = \psi(y_0) = u(y_0, T)$, as in Theorem 2.1 (case1), we have $u_h^k(x_i, s_j) - u(x_i, s_j) \leq Ch^{\frac{\gamma}{2}}$.
(case2) $t_0 \neq T$, $s_0 = T$. In this case $u(x_0, T) = \psi(x_0) = u_h^k(x_0, T)$, so (1.35) reads

$$u_h^k(x_i, s_j) - u(x_i, s_j) \leq |u_h^k(y_0, t_0) - u_h^k(y_0, T)| + \\ + L_\psi \, C_{\epsilon,\delta}^{\gamma} \, \epsilon^{\frac{2\gamma}{2-\gamma}} + C_{\epsilon,\delta}^{2} \, \epsilon^{\frac{2\gamma}{2-\gamma}} + \alpha^{\frac{2\gamma}{2-\gamma}} L_u{}^{\frac{2}{2-\gamma}} + 2\delta.$$

But, by Remark 3.1 since $t_0 = s_{j_0}$,

$$|u_h^k(y_0, t_0) - u_h^k(y_0, T)| \leq (M_f + \lambda M_\psi)(T - s_{j_0}) + L_\psi(M_b^\gamma h^\gamma + k^\gamma)(N - j_0).$$

Hence, proceeding as in Theorem 2.1 (case2), we can prove that (1.33) holds true.
(case3) $s_0, t_0 \neq T$. First note that there exists $a^* \in A$ for which (1.25) is still true, and the analogous of (1.26) is

$$u_h^k(y_0, t_0) - h_{j_0+1} f(y_0, t_0, a^*) - e^{-\lambda h_{j_0+1}} \sum_{l=1}^{M} \lambda_l(a^*, s_{j_0}) u_h^k(x_l, s_{j_0+1}) \leq 0, \tag{1.36}$$

where $y_0 + h_{j_0+1} b(y_0, t_0, a^*) = \sum_{l=1}^{M} \lambda_l(a^*, s_{j_0}) x_l$.
Similarly to what was done in Theorem 2.1, we consider (1.34) for $x = x_0$, $x_i = x_l$, $s = s_0$, $s_j = s_{j_0+1}$, and we substitute it in (1.36) to obtain

$$(1 - e^{-\lambda h_{j_0+1}}) u_h^k(y_0, t_0) - h_{j_0+1} f(y_0, t_0, a^*) + e^{-\lambda h_{j_0+1}} \sum_{l=1}^{M} \lambda_l(a^*, s_{j_0}) \\ \left[\nabla_y \beta_\epsilon(y_0 - x_0) \cdot (x_l - y_0) - \frac{2}{\alpha^2}(t_0 - s_0) h_{j_0+1} - \frac{|x_l - y_0|^2}{\epsilon^2} - \right. \\ \left. - \frac{h_{j_0+1}^2}{\alpha^2} + \delta \nabla_x \theta(y_l, x_0) \cdot (x_l - y_0) \right] \leq 0, \tag{1.37}$$

again (1.25), and (1.37) give

$$\frac{(1 - e^{-\lambda h_{j_0+1}})}{h_{j_0+1}} u_h^k(y_0, t_0) - \lambda u(x_0, s_0) \leq \\ \leq L_f(|y_0 - x_0|^\gamma + |t_0 - s_0|^\gamma) + \nabla_y \beta_\epsilon(y_0 - x_0) \cdot \left[b(x_0, s_0, a^*) - \right. \\ \left. - e^{-\lambda h_{j_0+1}} \left(\sum_{l=1}^{M} \lambda_l(a^*, s_{j_0}) \frac{(x_l - y_0)}{h_{j_0+1}} \right) \right] + \\ + \frac{1}{\epsilon^2} \sum_{l=1}^{M} \lambda_l(a^*, s_{j_0}) \frac{|x_l - y_0|^2}{h_{j_0+1}} + \delta \sum_{l=1}^{M} \lambda_l(a^*, s_{j_0}) \frac{|x_l - y_0|}{h_{j_0+1}} + \\ + \frac{h_{j_0+1}}{\alpha^2} + \delta M_b + \frac{2}{\alpha^2} \lambda (t_0 - s_0) h_{j_0+1}. \tag{1.38}$$

On the other hand, $\lambda_l(a^*, s_{j_0}) \neq 0$ implies $|y_0 + h_{j_0+1} b(y_0, t_0, a^*) - x_l| \leq k$ thus, for any l, we know either $|y_0 - x_l| \leq k + h_{j_0+1} M_b$ or $\lambda_l(a^*, s_{j_0}) = 0$. Moreover, $\sum_{l=1}^{M} \lambda_l(a^*, s_{j_0})(x_l - y_0) = h_{j_0+1} b(y_0, t_0, a^*)$ hence, if we use (1.22) and (1.23), and we let δ tend to 0, (1.38) becomes

$$u_h^k(y_0, t_0) \quad -u(x_0, s_0) \leq C\Bigg(h_{j_0+1} + \epsilon^{\frac{2\gamma}{2-\gamma}} + \alpha^{\frac{2\gamma}{2-\gamma}} + h_{j_0+1}\epsilon^{\frac{2\gamma-2}{2-\gamma}} + \\ + \epsilon^{\frac{2\gamma-2}{2-\gamma}} \alpha^{\frac{2}{2-\gamma}} + \frac{1}{\epsilon^2}\left(\frac{k}{h_{j_0+1}^{1/2}} + h_{j_0+1}^{1/2} \right)^2 + \frac{h_{j_0+1}}{\alpha^2} \Bigg). \qquad (1.39)$$

If we choose $\alpha = h_{j_0+1}^{\frac{2-\gamma}{4}}$, and $\epsilon^2 = \frac{1}{(1+C_2)^{2-\gamma}} (\frac{k}{h_{j_0+1}^{1/2}} + h_{j_0+1}^{1/2})^{2-\gamma}$ by (1.35), and (1.39) we may conclude

$$u_h^k(x_i, s_j) - u(x_i, s_j) \leq C\left(\frac{k}{h^{1/2}} + h^{1/2} \right)^{\gamma}$$

and (1.33) follows. If $h > 1$ we use $h_{j_0+1} \leq T^{1-\frac{\gamma}{2}} h^{\frac{\gamma}{2}}$, and $h_{j_0+1} \leq T^{\frac{1}{2}} h^{\frac{1}{2}}$. □

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