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Morse index and sign-changing bubble towers for Lane–Emden problems

Francesca De Marchis · Isabella Ianni · Filomena Pacella

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Abstract We consider the semilinear Lane–Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{E}_p)$$

where $p > 1$ and Ω is a smooth bounded symmetric domain of $\mathbb{R}^2$. We show that for families (u_p) of sign-changing symmetric solutions of $(\mathcal{E}_p)$ an upper bound on their Morse index implies concentration of the positive and negative part, $u_p^{\pm}$, at the same point, as $p \to +\infty$. Then, an asymptotic analysis of u_p^+ and u_p^- shows that the asymptotic profile of (u_p), as $p \to +\infty$, is that of a tower of two different bubbles.

Keywords Superlinear elliptic boundary value problem · Sign-changing solution · Asymptotic analysis · Bubble towers · Morse index

Mathematics Subject Classification 35B05 · 35B06 · 35J91

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F. De Marchis
University of Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Rome, Italy

I. Ianni
Second University of Napoli, V.le Lincoln 5, 81100 Caserta, Italy

F. Pacella (✉)
University of Roma Sapienza, P.le Aldo Moro 8, 00185 Rome, Italy
e-mail: pacella@mat.uniroma1.it

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1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^2$ and consider the Lane–Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where $p > 1$.

The aim of the paper is to show, under some symmetry assumption on Ω, a relation between the Morse index of sign-changing symmetric solutions of (1.1) and their asymptotic profile, as $p \to +\infty$.

In order to state precisely our result, we need to introduce some notations. For a given family (u_p) of sign-changing solutions of (1.1), we denote by

- $u_p^+ = \max(0, u_p)$, $u_p^- = -\min(0, u_p)$
- $\mathcal{N}_p^{\pm} \subset \Omega$ the positive/negative nodal domain of u_p, i.e., $\mathcal{N}_p^{\pm} = \{x \in \Omega : u_p(x) \gtrless 0\}$
- NL_p the nodal line of u_p, i.e., $NL_p = \{x \in \Omega : u_p(x) = 0\}$
- $x_p^{\pm}$ the maximum/minimum point in Ω of u_p, i.e., $u_p(x_p^{\pm}) = \pm\|u_p^{\pm}\|_\infty$
- $\mu_p^{\pm} := \frac{1}{\sqrt{p|u_p(x_p^{\pm})|^{p-1}}}$
- $\widetilde{\Omega}_p^{\pm} := \frac{\Omega - x_p^{\pm}}{\mu_p^{\pm}} = \{x \in \mathbb{R}^2 : x_p^{\pm} + \mu_p^{\pm}x \in \Omega\}$.

Recalling that the Morse index $m(v)$ of a solution v of a problem of type (1.1) is the number of the negative eigenvalues of the linearized operator at v, we state our main result:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded smooth domain containing the origin O and invariant under the action of a cyclic group G of rotations about the origin with order $|G| \geq 2$. Let (u_p) be a family of sign-changing G-symmetric solutions of (1.1) with two nodal regions such that*

$$p \int_\Omega |\nabla u_p|^2 dx \overset{p\to+\infty}{\longrightarrow} \beta, \quad \textit{for some} \quad \beta \in \mathbb{R}, \tag{1.2}$$

and

$$m(u_p) < |G| + 1. \tag{1.3}$$

Then, assuming w.l.o.g. that $\|u_p\|_\infty = \|u_p^+\|_\infty$, we have

(i) $|x_p^{\pm}| \to O$ *as* $p \to +\infty$,
(ii) NL_p *shrinks to the origin as* $p \to +\infty$,
(iii) *the rescaled function* $v_p^+(x) := p\frac{u_p(x_p^+ + \mu_p^+ x) - u_p(x_p^+)}{u_p(x_p^+)}$ *defined in* $\widetilde{\Omega}_p^+$ *converges (up to a subsequence) to the regular solution U of*

$$\begin{cases} -\Delta U = e^U & \textit{in} \quad \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U dx = 8\pi \end{cases} \tag{1.4}$$

with $U(0) = 0$, in $C^1_{loc}(\mathbb{R}^2)$,
(iv) *the rescaled function* $v_p^-(x) := p\frac{u_p(x_p^- + \mu_p^- x) - u_p(x_p^-)}{u_p(x_p^-)}$ *defined in* $\widetilde{\Omega}_p^-$ *converges in* $C^1_{loc}(\mathbb{R}^2 \setminus \{x_\infty\})$ *(up to a subsequence) to a singular solution V of*

$$\begin{cases} -\Delta V = e^V + H\delta_{x_\infty} & \textit{in} \quad \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^V dx < \infty \end{cases} \tag{1.5}$$

where H is a negative suitable constant and δ_{x_∞} is the Dirac measure centered at $x_\infty = -\lim_{p\to+\infty} \frac{x_p^-}{\mu_p^-} \neq 0$,

(v) $\sqrt{p}u_p \to 0$ *in* $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$ *as* $p \to +\infty$.

The assertions of the above theorem show that both u_p^+ and u_p^- concentrate at the same point which is the origin and, after suitable rescalings, they have the limit profile of a regular and a singular solution of the Liouville equation in the plane. So the limit profile of u_p, as $p \to +\infty$, is that of a *tower of two different bubbles*.

Remark 1.2 According to the classification in [11], if $H \notin -4\pi\mathbb{N}$, the solutions of (1.5) are radial with respect to x_∞, while, if $H \in -4\pi\mathbb{N}$, they can be either radial with respect to x_∞ or invariant under the action of a cyclic group of rotations of order $\frac{H}{4\pi} + 1$ (which in our case should be at least $|G|$) about x_∞. We refer to Proposition 3.5 for further details.

The first results for problem (1.1) about the existence of sign-changing solutions whose positive and negative part concentrate at the same point have been obtained in [10] for nodal radial solutions in the ball and in [8] for nodal symmetric solutions similar to those considered in Theorem 1.1. As compared to [8], the main difference is that there a relation between the asymptotic energy β [see (1.2)] of the solutions and the order of the group G was exploited, while here we use the bound (1.3) on the Morse index.

We believe that this connection between the Morse index and the limit profile of the solutions is the real novelty of our result. It shows once again a deep relation between the information obtained by the linearization and the qualitative properties of the solutions.

Our assumption (1.3) also allows to weaken the hypothesis on the order of the symmetry group G which, in [8], was assumed to be: $|G| \geq 4e$. On the other side, it should be said that, generally, energy conditions are easier to be checked than Morse index bounds. Indeed, in [7] solutions satisfying the energy bound stated in [8] have been proved to exist. Another difference with the result in [8] is that here for the asymptotic analysis of u_p^- we are not able to exclude the nonradiality of v_p^-.

Let us observe that the assumptions of Theorem 1.1 are reasonable since the G-symmetric solutions found recently in [7] in the case $|G| \geq 4$, have two nodal regions, satisfy (1.2) and we conjecture, supported by numerical evidence and asymptotic computations, that their Morse index should be 4. Let us recall that for some symmetric sign-changing solutions a lower bound on their Morse index can be obtained, as proved in [1]. This shows in particular that the Morse index of sign-changing radial solutions in a ball is at least 4 and, we expect that in the case of least energy radial sign-changing solutions in a ball, their Morse index is exactly 4, as we are going to prove in a paper in preparation.

The Theorem 1.1 will follow from a slightly more general result where the assumption (1.3) is substituted by the condition

$$\max\{m(u_p^+), m(u_p^-)\} < |G|. \tag{1.6}$$

Indeed, since the Morse index $m(u_p)$ of a solution u_p of (1.1) is always larger or equal to $m(u_p^\pm) + 1$, it is obvious that (1.3) implies (1.6).

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected smooth domain containing the origin O and invariant under the action of a cyclic group G of rotations about the origin with $|G| \geq 2$.*

If (u_p) is a family of sign-changing solutions of (1.1) with two nodal regions satisfying (1.2) and (1.6) then the assertions (i)–(v) hold.

The proof of Theorem 1.3 (and hence of Theorem 1.1) is based on several results proved in [8]. Let us point out that a crucial initial step is to show that the solutions considered have the property that their nodal line neither touches the boundary of Ω, nor passes through the origin, i.e., for u_p holds:

$$NL_p \cap \partial\Omega = \emptyset \quad \text{and} \quad O \notin NL_p. \tag{1.7}$$

Since the solution u_p considered in the above theorem has two nodal regions, (1.7) is a consequence of the following general result whose proof is exactly the same as that of [7, Lemma 4.1] and [7, Lemma 4.3] (written there for $|G| \geq 4$).

Proposition 1.4 *If G is a cyclic group of rotations about the origin with $|G| \geq 2$ then any G-symmetric nodal solution u_p of (1.1) such that $\sharp(u_p) \leq |G|$ satisfies (1.7), where $\sharp(u_p)$ is the number of nodal domains of u_p.*

We believe that (1.7) is the crucial qualitative property of the solutions which yields the concentration of u_p^+ and u_p^- at the same point.

Moreover, let us observe that for sign-changing solutions with any number of nodal regions in any G-symmetric domain Ω the condition (1.3) implies the properties in (1.7). Indeed, we know (cfr. [2]) that

$$\sharp(u_p) \leq m(u_p).$$

hence (1.3) yields

$$\sharp(u_p) \leq m(u_p) \leq |G|, \tag{1.8}$$

so that again by Proposition 1.4 we get (1.7).

The outline of the paper is as follows. In Sect. 2, we recall or prove some results in general bounded, not necessarily symmetric domains. In Sect. 3, we give the proof of Theorem 1.3 as consequence of other results concerning the asymptotic analysis of the negative parts (u_p^-) in G-symmetric domains.

2 Preliminary results in general bounded domains

In order to prove Theorem 1.3, we follow the scheme of the proof of [8, Theorem 1.2], showing that all the steps can be obtained under the new assumptions of this paper.

We start introducing some notations and recalling some results obtained in [8] on the asymptotic behavior of a family (u_p) of solutions of (1.1), in a general smooth bounded domain Ω, satisfying the energy condition (1.2).

Given a family (u_p) of solutions of (1.1) and assuming that there exist $n \in \mathbb{N} \setminus \{0\}$ families of points $(x_{i,p})$, $i = 1, \ldots, n$ in Ω such that

$$p|u_p(x_{i,p})|^{p-1} \to +\infty \quad \text{as} \quad p \to +\infty, \tag{2.1}$$

we define the parameters $\mu_{i,p}$ by

$$\mu_{i,p}^{-2} = p|u_p(x_{i,p})|^{p-1}, \quad \text{for all} \quad i = 1, \ldots, n. \tag{2.2}$$

By (2.1) it is clear that $\mu_{i,p} \to 0$ as $p \to +\infty$ and that

$$\forall \epsilon > 0 \ \exists \, p_{i,\epsilon} \ \text{such that} \ |u_p(x_{i,p})| \geq 1 - \epsilon, \quad \forall p \geq p_{i,\epsilon}. \tag{2.3}$$

Then, we define the concentration set

$$\mathcal{S} = \left\{ \lim_{p\to+\infty} x_{i,p}, \quad i = 1, \dots, n \right\} \subset \bar{\Omega} \tag{2.4}$$

and the function

$$R_{n,p}(x) = \min_{i=1,\dots,n} |x - x_{i,p}|, \quad \forall x \in \Omega. \tag{2.5}$$

Finally, we introduce the following properties:

($\mathcal{P}_1^n$) For any $i, j \in \{1, \dots, n\}$, $i \neq j$,

$$\lim_{p\to+\infty} \frac{|x_{i,p} - x_{j,p}|}{\mu_{i,p}} = +\infty.$$

($\mathcal{P}_2^n$) For any $i = 1, \dots, n$,

$$v_{i,p}(x) := \frac{p}{u_p(x_{i,p})}(u_p(x_{i,p} + \mu_{i,p}x) - u_p(x_{i,p})) \longrightarrow U(x)$$

in $C^1_{loc}(\mathbb{R}^2)$ as $p \to +\infty$, where

$$U(x) = \log\left(\frac{1}{1 + \frac{1}{8}|x|^2}\right)^2 \tag{2.6}$$

is the solution of $-\Delta U = e^U$ in $\mathbb{R}^2$, $U \leq 0$, $U(0) = 0$ and $\int_{\mathbb{R}^2} e^U = 8\pi$.

($\mathcal{P}_3^n$) There exists $C > 0$ such that

$$pR_{n,p}(x)^2 |u_p(x)|^{p-1} \leq C$$

for all p sufficiently large and all $x \in \Omega$.

The following results have been obtained in [8].

Lemma 2.1 *Let (u_p) be a family of solutions to (1.1) satisfying (1.2). Then,*

(*i*) *If u_p changes sign, then $\|u_p^\pm\|^{p-1}_{L^\infty(\Omega)} \geq \lambda_1$ where $\lambda_1 := \lambda_1(\Omega)$ is the first eigenvalue of the operator $-\Delta$ in $H^1_0(\Omega)$. In particular for the points $x_p^\pm$, where the maximum and the minimum are achieved, the analogous of (2.1) and (2.3) hold.*

(*ii*) *If, for $n \in \mathbb{N}\setminus\{0\}$, the properties ($\mathcal{P}_1^n$) and ($\mathcal{P}_2^n$) hold for families $(x_{i,p})_{i=1,\dots,n}$ of points satisfying (2.1), then*

$$p\int_\Omega |\nabla u_p|^2\, dx \geq 8\pi \sum_{i=1}^n \alpha_i^2 + o_p(1) \text{ as } p \to +\infty,$$

where $\alpha_i := \liminf_{p\to+\infty} |u_p(x_{i,p})|$.

Proof See [8, Lemma 2.1]. □

Proposition 2.2 *Let (u_p) be a family of solutions to (1.1) and assume that (1.2) holds. Then, there exist $k \in \mathbb{N}\setminus\{0\}$ and k families of points $(x_{i,p})$ in Ω, $i = 1, \dots, k$ such that, after passing to a sequence, (2.1), ($\mathcal{P}_1^k$), ($\mathcal{P}_2^k$), and ($\mathcal{P}_3^k$) hold. Moreover, given any family of points $x_{k+1,p}$, it is impossible to extract a new sequence from the previous one such that ($\mathcal{P}_1^{k+1}$), ($\mathcal{P}_2^{k+1}$), and ($\mathcal{P}_3^{k+1}$) hold with the sequences $(x_{i,p})$, $i = 1, \dots, k+1$. At last, we have*

$$\sqrt{p}u_p \to 0 \quad in \quad C^1_{loc}(\bar{\Omega} \setminus \mathcal{S})\, as\, p \to +\infty. \tag{2.7}$$

Proof See [8, Proposition 2.2]. □

Proposition 2.2 was inspired by the paper [9] where positive solutions of semilinear elliptic problems with critical exponential nonlinearities in 2-dimension were studied. Its proof is based on an induction argument, namely one first proves that $(\mathcal{P}_1^1)$, $(\mathcal{P}_2^1)$ hold for points $x_{1,p}$ where u_p achieves $\|u_p\|_\infty$ (actually $(\mathcal{P}_1^1)$ is trivially verified), and then one shows that if $(\mathcal{P}_1^n)$, $(\mathcal{P}_2^n)$ are satisfied for some $n \in \mathbb{N}\setminus\{0\}$ then either $(\mathcal{P}_3^n)$ holds true or there exists a point $x_{n+1,p}$, such that the $(n+1)$-tuple $x_{1,p}, \ldots, x_{n+1,p}$ fulfills $(\mathcal{P}_1^{n+1})$, $(\mathcal{P}_2^{n+1})$. The procedure necessarily stops by virtue of Lemma 2.1 and assumption (1.2).

Moreover, one can easily derive the following corollary.

Corollary 2.3 *Under the assumptions of Proposition 2.2 if the solutions u_p are sign-changing it follows that*

$$\frac{dist(x_{i,p}, \partial\Omega)}{\mu_{i,p}} \overset{p\to+\infty}{\to} +\infty \quad and \quad \frac{dist(x_{i,p}, NL_p)}{\mu_{i,p}} \overset{p\to+\infty}{\to} +\infty \quad for\ all \quad i \in \{1,\ldots,k\}$$

where, as in Sect. 1, NL_p denotes the nodal line of u_p.

As a consequence, for any $i \in \{1,\ldots,k\}$, letting $\mathcal{N}_{i,p} \subset \Omega$ be the nodal domain of u_p containing $x_{i,p}$ and setting $u_p^i := u_p\chi_{\mathcal{N}_{i,p}}$ (χ_A is the characteristic function of the set A), then the scaling of u_p^i around $x_{i,p}$:

$$z_{i,p}(x) := \frac{p}{u_p(x_{i,p})}\left(u_p^i(x_{i,p} + \mu_{i,p}x) - u_p(x_{i,p})\right), \tag{2.8}$$

defined on $\widetilde{\mathcal{N}}_{i,p} := \frac{\mathcal{N}_{i,p} - x_{i,p}}{\mu_{i,p}}$, converges to U in $C^1_{loc}(\mathbb{R}^2)$, where U is the function defined in (2.6).

Proof See [8, Corollary2.4]. □

We point out that, since we are assuming without loss of generality that $\|u_p\|_\infty = \|u_p^+\|_\infty$, we can take x_p^+ as the point $x_{1,p}$ so that directly from the proof of Proposition 2.2 we get the following result for the rescaling about x_p^+.

Proposition 2.4 *Let (u_p) be a family of solutions to* (1.1) *satisfying* (1.2)*. Then, the rescaled functions*

$$v_p^+(x) := \frac{p}{u_p(x_p^+)}\left(u_p(x_p^+ + \mu_p^+ x) - u_p(x_p^+)\right) \tag{2.9}$$

defined on $\widetilde{\Omega}_p^+$ (see Sect. 1 for the definition) converge to U in $C^1_{loc}(\mathbb{R}^2)$, where U is the function introduced in (2.6).

Now, we prove a general proposition on the sign of the first eigenvalue of the linearized operators at $u_p^\pm$:

$$L_p^\pm := -\Delta - p|u_p^\pm|^{p-1},$$

in the space $H_0^1(\mathcal{N}_p^\pm)$, respectively. Let us denote by $\lambda_j^\pm$, $j = 1, 2, \ldots$, respectively, their eigenvalues with homogeneous Dirichlet boundary conditions, and let $m(u_p^\pm)$ be the Morse index of $u_p^\pm$ in $\mathcal{N}_p^\pm$, namely $\lambda_j^\pm < 0$, for $j = 1, \ldots, m(u_p^\pm)$ and $\lambda^\pm_{m(u_p^\pm)+1} \geq 0$. Moreover, for a domain $B \subseteq \mathcal{N}_p^\pm$, we denote by $\lambda_j^\pm(B)$, $j = 1, 2, \ldots$ the Dirichlet eigenvalues of $L_p^\pm$ in B.

Proposition 2.5 *Let (u_p) be a family of solutions to (1.1) satisfying (1.2), and let $(x_{i,p}) \subset \Omega$, $i = 1, \dots, k$ be families of points as in Proposition 2.2. Then, there exists $\bar{r} > 0$ such that*

$$\lambda_1^+ \left(B_{\bar{r}\mu_{i,p}}(x_{i,p})\right) < 0 \quad \textit{for large } p, \quad \textit{if } (x_{i,p}) \subset \mathcal{N}_p^+$$
$$\lambda_1^- \left(B_{\bar{r}\mu_{i,p}}(x_{i,p})\right) < 0 \quad \textit{for large } p, \quad \textit{if } (x_{i,p}) \subset \mathcal{N}_p^-$$

where $B_{\bar{r}\mu_{i,p}}(x_{i,p})$ are the balls centered in $x_{i,p}$ of radius $\bar{r}\mu_{i,p}$.

Proof Without loss of generality, by (2.1), we may assume that either $(x_{i,p}) \subset \mathcal{N}_p^+$ or $(x_{i,p}) \subset \mathcal{N}_p^-$, for p large. We give the proof in the case $(x_{i,p}) \subset \mathcal{N}_p^+$, the other case being similar.

Let us consider the linear operators

$$\widetilde{L_{i,p}^+} := -\Delta - \frac{|u_p^+(\mu_{i,p}x + x_{i,p})|^{p-1}}{|u_p(x_{i,p})|^{p-1}}$$

in the space $H_0^1(\widetilde{\mathcal{N}}_{i,p}^+)$ where $\widetilde{\mathcal{N}}_{i,p}^+ := \{x \in \mathbb{R}^2 : x_{i,p} + \mu_{i,p}x \in \mathcal{N}_p^+\}$.

Since for any function $v \in H_0^1(\widetilde{\mathcal{N}}_p^+)$ we have that the rescaled function $w(x) = v(\mu_{i,p}x + x_{i,p})$ belongs to $H_0^1(\widetilde{\mathcal{N}}_{i,p}^+)$, we get that the Dirichlet eigenvalues $\widetilde{\lambda}_j^{i,+}$, $j = 1, 2, \dots$ of $\widetilde{L_{i,p}^+}$ satisfy

$$\widetilde{\lambda}_j^{i,+} = \lambda_j^+ \frac{1}{p|u_p(x_{i,p})|^{p-1}}, \quad j = 1, 2, \dots.$$

Moreover, for any subset $B \subseteq \mathcal{N}_p^+$, letting $\widetilde{B}_{i,p} := \{x \in \mathbb{R}^2 \ : \ \mu_{i,p}x + x_{i,p} \in B\} \subseteq \widetilde{\mathcal{N}}_{i,p}^+$, then the Dirichlet eigenvalues of $\widetilde{L_{i,p}^+}$ in $\widetilde{B}_{i,p}$ are

$$\widetilde{\lambda_j}^{i,+}(\widetilde{B}_{i,p}) := \lambda_j^+(B) \frac{1}{p|u_p(x_{i,p})|^{p-1}}, \quad j = 1, 2, \dots.$$

As a consequence, to prove the thesis is equivalent to show that there exists $\bar{r} > 0$ such that

$$\widetilde{\lambda_1}^{i,+}(B_{\bar{r}}(0)) < 0 \ \text{ for large } p, \tag{2.10}$$

where $B_{\bar{r}}(0)$ is the ball centered in 0 and radius $\bar{r}$. To prove (2.10), we consider the functions

$$w_{i,p} := x \cdot \nabla z_{i,p} + \frac{2}{p-1} z_{i,p} + \frac{2p}{p-1},$$

where $z_{i,p}$ is the function defined in (2.8). We have that $w_{i,p}$ satisfies $\widetilde{L_{i,p}^+}(w_{i,p}) = 0$ and $w_{i,p}(0) \to 2$. Moreover, as $z_{i,p}(x) \to U(x) = \log\left(\frac{1}{(1+\frac{1}{8}|x|^2)^2}\right)$, we also get that $w_{i,p}(x) \to -\frac{4r^2}{8+r^2} + 2$, for $|x| = r$, and so, for large r, $w_{i,p}(x) \to \alpha < 0$ for $x \in \partial B_r(0)$. For such r's, let us define $A_{i,p} := \{x \in B_r(0) : w_{i,p} > 0\}$, and let us define $\bar{w}_{i,p} = w_{i,p}$ in $A_{i,p}$ and $\bar{w}_{i,p} \equiv 0$ in $B_r(0) \setminus A_{i,p}$.

Then, $\bar{w}_{i,p} \in H_0^1(B_r(0))$ and for $\bar{r} > r$

$$\widetilde{\lambda_1}^{i,+}(B_{\bar{r}}(0)) < \widetilde{\lambda_1}^{i,+}(B_r(0)) \leq \int_{B_r(0)} |\nabla \bar{w}_{i,p}|^2 - \int_{B_r(0)} \frac{|u_p^+(\mu_{i,p}x + x_{i,p})|^{p-1}}{|u_p(x_{i,p})|^{p-1}} \bar{w}_{i,p}^2 = 0,$$

which proves the assertion. □

3 Results for symmetric domains and proof of Theorem 1.3

All we have proved in the previous section holds regardless the symmetry of Ω. In the sequel using the symmetry and the assumption on the Morse index (1.6), we will derive more specific and precise results.

Thus, let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded smooth domain containing the origin and invariant under the action of a cyclic group G of rotations about the origin with $|G| \geq 2$. Let us consider a family (u_p) of sign-changing G-symmetric solutions as in the statement of Theorem 1.3. We apply Proposition 2.2 which gives a maximal number k of families of points $(x_{i,p})$, $i = 1, \ldots, k$, in Ω such that, up to a sequence, (P_1^k), (P_2^k) and (P_3^k) hold for our solutions. We start with the following result.

Proposition 3.1 *Let Ω be as in Theorem 1.3, and let (u_p) be a family of sign-changing G-symmetric solutions of (1.1) satisfying (1.2). Let $(x_{i,p}) \subset \Omega$, $i = 1, \ldots, k$ be a family of points as in Proposition 2.2. If $(x_{i,p}) \subset \mathcal{N}_p^+$, for p large, then assume that $m(u_p^+) < |G|$ otherwise, if $(x_{i,p}) \subset \mathcal{N}_p^-$, for p large, then assume that $m(u_p^-) < |G|$. Then,*

$$\frac{|x_{i,p}|}{\mu_{i,p}} \text{ is bounded.}$$

In particular $|x_{i,p}| \to 0$.

Proof We prove the assertion in the case $(x_{i,p}) \subset \mathcal{N}_p^+$, the other case being similar. Moreover, in order to simplify the notation, we drop the dependence on i namely we set $x_p := x_{i,p}$ and $\mu_p := \mu_{i,p}$. Let $h := |G|$ and assume by contradiction that there exists a sequence $p_n \to +\infty$ such that $\frac{|x_{p_n}|}{\mu_{p_n}} \to +\infty$. Then, since the h distinct points $g^j x_{p_n}$ (where the (g^j)'s are the element of G), $j = 0, \ldots, h-1$, are the vertex of a regular polygon centered in O, we have that $d_n := |g^j x_{p_n} - g^{j+1} x_{p_n}| = 2\widetilde{d}_n \sin\frac{\pi}{h}$, where $\widetilde{d}_n := |g^j x_{p_n}|$, $j = 0, \ldots, h-1$. Hence, we also have that $\frac{d_n}{\mu_{p_n}} \to +\infty$.

Let

$$R_n := \min\left\{\frac{d_n}{3}, \frac{d(x_{p_n}, \partial\Omega)}{2}, \frac{d(x_{p_n}, NL_{p_n})}{2}\right\}, \tag{3.1}$$

then by construction

$$\begin{aligned} &B_{R_n}(g^j x_{p_n}) \subseteq \mathcal{N}_{p_n}^+ \quad \text{for} \quad j = 0, \ldots, h-1, \\ &B_{R_n}(g^j x_{p_n}) \cap B_{R_n}(g^l x_{p_n}) = \emptyset, \quad \text{for} \quad j \neq l \end{aligned} \tag{3.2}$$

and by virtue of Corollary 2.3

$$\frac{R_n}{\mu_{p_n}} \to +\infty. \tag{3.3}$$

By Proposition 2.5 it follows that $\lambda_1^+\left(B_{\bar r \mu_{p_n}}(x_{p_n})\right) < 0$ for large n. So by the G-symmetry of $u_{p_n}^+$ and the invariance of the laplacian by orthogonal transformations, it is easy to see that $\lambda_1^+(B_{\bar r \mu_{p_n}}(g^j x_{p_n})) < 0$, for each $j = 0, \ldots, h-1$. Hence, by the variational characterization of the first eigenvalue, there exists $\varphi_j \in H_0^1(B_{\bar r \mu_{p_n}}(g^j x_{p_n}))$, such that

$$R(v) := \frac{\int_{\mathcal{N}_{p_n}^+}\left[|\nabla v|^2 - p|u_p^+|^{p-1}v^2\right]}{\|v\|_2^2} \geq R(\varphi_j) = \lambda_1^+(B_{\bar r \mu_{p_n}}(g^j x_{p_n})) < 0,$$

for any $v \in H_0^1(B_{\bar r \mu_{p_n}}(g^j x_{p_n}))$, $v \neq 0$, $j = 0, \ldots, h-1$.

Let $W := span\{\varphi_0, \dots, \varphi_{h-1}\}$, then by (3.3) it follows that for p large $B_{\bar{r}\mu_{p_n}}(g^j x_{p_n}) \subseteq B_{R_n}(g^j x_{p_n})$, hence $W \subset H_0^1(\mathcal{N}_{p_n}^+)$ and also, by (3.2), $\dim W = h$ and $R(v) \leq \sum_{j=0}^{h-1} R(\varphi_j) < 0$ for any $v \in W$.

Hence, using the variational characterization of the h-th eigenvalue, it follows that $\lambda_h^+ < 0$, namely $m(u_{p_n}^+) \geq h$, a contradiction. $\square$

Now, we state several results which can be obtained exactly in the same way as for analogous results in [8]. They will be important steps for the proof of Theorem 1.3.

Proposition 3.2 *Under the same assumptions as in Theorem 1.3, we have:*

(i) $NL_p \cap \partial\Omega = \emptyset$ *and* $O \notin NL_p$.
(ii) $O \in \mathcal{N}_p^+$ *for p large.*
(iii) $x_{i,p} \in \mathcal{N}_p^+$ *for p large and* $i = 1, \dots, k$.
(iv) The maximal number k of families of points $(x_{i,p})$, $i = 1, \dots, k$, *for which* (P_1^k), (P_2^k) *and* (P_3^k) *hold is* 1.
(v) There exists $C > 0$ *such that for any family* $(x_p) \subset \Omega$, *one has*

$$\frac{|x_p|}{\mu(x_p)} \leq C \tag{3.4}$$

for p large, where $\mu(x_p)$ *is defined by* $(\mu(x_p))^{-2} = p|u_p(x_p)|^{p-1}$.

Proof As already observed in the Introduction, (i) is a consequence of Proposition 1.4 which applies to any G-symmetric solution having two nodal domains. Once property (i) is proved the (ii)–(v) follow as in [8, Corollary 3.5, Proposition 3.6 and Corollary 3.7]. $\square$

By Lemma 2.1 and Proposition 3.2 for the minimum points x_p^-, we then have

$$\frac{|x_p^-|}{\mu_p^-} \leq C, \tag{3.5}$$

so there are two possibilities: either $\frac{|x_p^-|}{\mu_p^-} \to \ell > 0$ or $\frac{|x_p^-|}{\mu_p^-} \to 0$ as $p \to +\infty$, up to subsequences. A crucial point of the proof is to exclude the latter case.

Proposition 3.3 *There exists* $\ell > 0$ *such that, up to a subsequence,*

$$\frac{|x_p^-|}{\mu_p^-} \to \ell \quad as \quad p \to +\infty.$$

Let us define

$$x_\infty := -\lim_{p\to+\infty} \frac{x_p^-}{\mu_p^-}, \quad |x_\infty| = \ell > 0. \tag{3.6}$$

Proof See [8, Proposition 4.2]. $\square$

Next, even if we have no information on the geometry of the nodal line we are able to show that the nodal line shrinks to the origin faster than μ_p^- as $p \to +\infty$.

Proposition 3.4 *We have*

$$\frac{\max_{y_p \in NL_p} |y_p|}{\mu_p^-} \to 0 \quad as \quad p \to +\infty.$$

Proof See [8, Proposition 4.3] □

These two last propositions allow to characterize the behavior of the rescaled solutions about x_p^-.

Proposition 3.5 *The scaling of u_p around x_p^-*

$$v_p^-(x) := \frac{p}{u_p(x_p^-)}\Big(u_p(\mu_p^- x + x_p^-) - u_p(x_p^-)\Big) \tag{3.7}$$

defined on $\widetilde{\Omega}_p^-$ converges (passing to a subsequence) in $C^1_{loc}(\mathbb{R}^2 \setminus \{x_\infty\})$ to the function $V(x - x_\infty)$, where V is a singular solution of

$$\begin{cases} -\Delta V = e^V + H\delta_0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^V dx < \infty. \end{cases} \tag{3.8}$$

for some negative H, and x_∞ is the point defined in (3.6). More precisely, letting ℓ be as in (3.6), then:

- *either V is the radial singular solution of (3.8), for some negative $H = H(\ell)$,*
$$V = V_{rad,\ell}(x) := \log\left(\frac{2\alpha^2\beta^\alpha|x|^{\alpha-2}}{(\beta^\alpha + |x|^\alpha)^2}\right) \quad x \in \mathbb{R}^2 \setminus \{0\},$$
where $\alpha = \sqrt{2\ell^2 + 4}$ and $\beta = \ell\left(\frac{\alpha+2}{\alpha-2}\right)^{1/\alpha}$,
- *or V is the $(\eta + 1)$-symmetric solution of (3.8), for $H = -4\pi\eta$, which in complex notations can be expressed as follows*
$$V = V_{\eta,\ell}(z) := \log\left(\frac{8(\eta+1)^2\lambda|z|^{2\eta}}{(1 + \lambda|z^{\eta+1} - c|^2)^2}\right) \quad z \in \mathbb{C} \setminus \{0\},$$
where $(\eta+1)$ is an integer multiple of $|G|$, $\lambda = \frac{(\ell^2+2\eta^2)^2}{8(\eta+1)^2\ell^{2\eta+4}}$, $c = (-x_\infty)^{\eta+1}(1-\frac{4\eta(\eta+1)}{\ell^2+2\eta^2})$.

Proof Let us consider the translations of (3.7):

$$s_p^-(x) := v_p^-\left(x - \frac{x_p^-}{\mu_p^-}\right) = \frac{p}{u_p(x_p^-)}(u_p(\mu_p^- x) - u_p(x_p^-)), \qquad x \in \frac{\Omega}{\mu_p^-}$$

which solve

$$\begin{cases} -\Delta s_p^-(x) = \left|1 + \frac{s_p^-(x)}{p}\right|^{p-1}\left(1 + \frac{s_p^-(x)}{p}\right) & x \in \frac{\Omega}{\mu_p^-} \\ s_p^-(\frac{x_p^-}{\mu_p^-}) = 0 & \\ s_p^-(x) \le 0 & x \in \frac{\Omega}{\mu_p^-}. \end{cases}$$

Observe that $\frac{\Omega}{\mu_p^-} \to \mathbb{R}^2$ as $p \to +\infty$.

We claim that for any fixed $r > 0$, $|-\Delta s_p^-|$ is bounded in $\frac{\Omega}{\mu_p^-} \setminus B_r(0)$.

Indeed Proposition 3.4 implies that if $x \in \frac{\mathcal{N}_p^+}{\mu_p^-}$, then $|x| \le \frac{\max_{z_p \in NL_p} |z_p|}{\mu_p^-} < r$, for p large, hence

$$\left(\frac{\Omega}{\mu_p^-} \setminus B_r(0)\right) \subset \frac{\mathcal{N}_p^-}{\mu_p^-} \quad \text{for} \quad p \text{ large}$$

and so the claim follows observing that for $x \in \frac{\mathcal{N}_p^-}{\mu_p^-}$, then $|-\Delta s_p^-(x)| \leq 1$.

Hence, by the arbitrariness of $r > 0$, we have that $s_p^- \to V$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$ where V is a solution of

$$-\Delta V = e^V \quad \text{in } \mathbb{R}^2 \setminus \{0\}$$

which satisfies $V \leq 0$ and $V(-x_\infty) = 0$ where x_∞ is defined in (3.6).

Moreover $e^V \in L^1(\mathbb{R}^2)$, indeed for any $r > 0$ and for any $\varepsilon \in (0, 1)$

$$\begin{aligned}\int_{B_{\frac{1}{r}}(0)\setminus B_r(0)} e^V \,\mathrm{d}x &\leq \int_{B_{\frac{1}{r}}(0)\setminus B_r(0)} \frac{|u_p(\mu_p^- x)|^{p+1}}{|u_p(x_p^-)|^{p+1}} \mathrm{d}x + o_p(1)\\ &= \frac{p}{|u_p(x_p^-)|^2} \int_{B_{\frac{\mu_p^-}{r}}(0)\setminus B_{r\mu_p^-}(0)} |u_p(y)|^{p+1} dy + o_p(1)\\ &\overset{Lemma\ 2.1\ (i)}{\leq} \frac{p}{(1-\varepsilon)^2} \int_\Omega |u_p(y)|^{p+1} dy + o_p(1) \overset{(1.2)}{<} +\infty.\end{aligned}$$

Observe that if V was a classical solution of $-\Delta V = e^V$ in the whole $\mathbb{R}^2$ then necessarily $V(x) = U(x + x_\infty)$. As a consequence, $v_p^-(x) = s_p^-(x + \frac{x_p^-}{\mu_p^-}) \to V(x - x_\infty) = U(x)$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{x_\infty\})$. Observe that since $x_\infty = -\lim_p \frac{x_p^-}{\mu_p^-}$, then [8, Proposition 3.8] applies, implying that $\frac{|x_p^-|}{\mu_p^-} \to 0$ as $p \to +\infty$, and this is in contradiction with Proposition 3.3.

Thus, by [4–6] and the classification given in [3], we have that V solves, for some $\eta > 0$, the following entire equation

$$\begin{cases} -\Delta V = e^V - 4\pi\eta\delta_0 & \text{in } \mathbb{R}^2\\ \int_{\mathbb{R}^2} e^V \mathrm{d}x = 8\pi(1+\eta), \end{cases} \tag{3.9}$$

where δ_0 denotes the Dirac measure centered at the origin.

Since s_p^- is G-symmetric, by the classification of [11] either V is radial or $\frac{\eta+1}{|G|} \in \mathbb{N}$ and V is $(\eta+1)$-symmetric.

If V is radial, then $V(r)$ satisfies

$$\begin{cases} -V'' - \frac{1}{r}V' = e^V & \text{in } (0, +\infty)\\ V \leq 0\\ V(\ell) = V'(\ell) = 0 \end{cases}.$$

The solutions of this problem are

$$V(r) = \log\left(\frac{4}{\delta^2} \frac{e^{\frac{\sqrt{2}}{\delta}(\log r - y))}}{\left(1 + e^{\frac{\sqrt{2}}{\delta}(\log r - y))}\right)^2}\right) - 2\log r \tag{3.10}$$

for $\delta > 0$, $y \in \mathbb{R}$.

Observe that from $V'(r) = 0$ we get $\frac{1-\sqrt{2}\delta}{1+\sqrt{2}\delta} = e^{\frac{\sqrt{2}}{\delta}(\log r - y)}$ and moreover $V(r) = 0$ for $r = \frac{\sqrt{1-2\delta^2}}{\delta}$. Hence, by $V(\ell) = V'(\ell) = 0$ it follows that $\ell^2 = \frac{1-2\delta^2}{\delta^2}$ which implies that $\delta = \frac{1}{\sqrt{2+\ell^2}}$. Inserting this estimate into (3.10) we get

$$V(r) = \log\left(\frac{2\alpha^2\beta^\alpha r^{\alpha-2}}{(\beta^\alpha + r^\alpha)^2}\right),$$

where $\alpha = \sqrt{2\ell^2+4}$ and $\beta = \ell\left(\frac{\alpha+2}{\alpha-2}\right)^{1/\alpha}$.

On the other hand, if $\frac{\eta+1}{|G|} \in \mathbb{N}$ and V is $(\eta+1)$-symmetric then there exists $\lambda > 0$ and $c \in \mathbb{C} \setminus \{0\}$ such that in complex notation

$$V(z) = \log\left(\frac{8(\eta+1)^2\lambda|z|^{2\eta}}{(1+\lambda|z^{\eta+1}-c|^2)^2}\right),$$

moreover $V(-x_\infty) = 0$ and $V(z) \le 0$ for any $z \in \mathbb{C}$.

Let $\zeta \in \mathbb{C}$ such that $\zeta^{\eta+1} = c$ and $\zeta = \sqrt[\eta+1]{|c|}e^{i\theta}$, $\theta \in [\theta_\infty - \frac{\pi}{\eta+1}, \theta_\infty + \frac{\pi}{\eta+1})$, where $-x_\infty = \ell e^{i\theta_\infty}$.

We first claim that

$$\zeta = \sqrt[\eta+1]{|c|}e^{i\theta_\infty}. \tag{3.11}$$

Let us suppose by contradiction that $\zeta = \sqrt[\eta+1]{|c|}e^{i\theta}$, $\theta \ne \theta_\infty$. We set $d := \partial B_\ell(0) \cap \{t\zeta : t > 0\}$. We know that $0 = V(-x_\infty) \ge V(d)$ and since $|-x_\infty|^{2\eta} = |d|^{2\eta} = \ell^{2\eta}$, then $|(-x_\infty)^{\eta+1} - c| \le |d^{\eta+1} - c|$ but this is false because $|d^{\eta+1}| = |(-x_\infty)^{\eta+1}| = \ell^{\eta+1}$ and $d^{\eta+1} = (\frac{|d|}{|\zeta|})^{\eta+1}c$. This proves (3.11).

Next, in order to compute λ and c in terms of x_∞ and η we set:

$$w = ze^{-i\theta_\infty} \quad \text{and} \quad \tilde{V}(w) := V(z) = \log\left(\frac{8(\eta+1)^2\lambda|w|^{2\eta}}{(1+\lambda|w^{\eta+1}-\tilde{c}|^2)^2}\right),$$

where $\tilde{c} = e^{-i(\eta+1)\theta_\infty}c \in \mathbb{R}^+$.

Let us consider the restriction of the argument of the logarithm to the positive real line, namely $g(s) := \frac{8(\eta+1)^2\lambda s^{2\eta}}{(1+\lambda(s^{\eta+1}-\tilde{c})^2)^2}$, $s \in (0, +\infty)$. Being $\tilde{V}(\ell) = V(-x_\infty) = 0 = \max_{\mathbb{C}} \tilde{V}$ we have that $g(\ell) = 1$ and $g'(\ell) = 0$. Imposing these two conditions, we get

$$8(\eta+1)^2\lambda\ell^{2\eta} = (1+\lambda(\ell^{\eta+1}-\tilde{c})^2)^2, \tag{3.12}$$

$$2\eta(1+\lambda(\ell^{\eta+1}-\tilde{c})^2)^2 - 4(\eta+1)\lambda\ell^{\eta+1}(1+\lambda(\ell^{\eta+1}-\tilde{c})^2)(\ell^{\eta+1}-\tilde{c}) = 0, \tag{3.13}$$

and in turn combining (3.13) and (3.12) we derive

$$(\ell^{\eta+1}-\tilde{c})\sqrt{\lambda} = \frac{\sqrt{2}\eta}{\ell}. \tag{3.14}$$

Substituting (3.14) in (3.12) we get

$$\lambda = \frac{(\ell+2\eta^2)^2}{8(\eta+1)^2\ell^{2\eta+4}}, \tag{3.15}$$

in turn by (3.14) and (3.15) we derive $\tilde{c} = \ell(1 - \frac{4\eta(\eta+1)}{\ell^2+2\eta^2})$. Thus, finally we have $c = (-x_\infty)^{\eta+1}(1 - \frac{4\eta(\eta+1)}{\ell^2+2\eta^2})$. □

Proof of Theorem 1.3 It follows from all previous results. More precisely, (i) follows from (3.4) and Lemma 2.1. The statement (ii) derives from Proposition 3.4. The asymptotic behavior of the rescaled functions v_p^+ and v_p^- is shown in Proposition 2.4 and Proposition 3.5

Finally, (v) is a consequence of Proposition 2.2. □

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