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Supercritical Conformal Metrics on Surfaces with Conical Singularities

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We study the problem of prescribing the Gaussian curvature on surfaces with conical singularities in supercritical regimes. Using a Morse-theoretical approach, we prove a general existence theorem on surfaces with positive genus, with a generic multiplicity result.

1 Introduction

The study of conformal metrics on surfaces with conical singularities dates back at least to Picard [30] and has been widely discussed in the last decades (see, e.g. [14, 16–19, 26, 32, 34, 39, 43, 44] and the references cited therein). In this paper, we are concerned with the construction of conformal metrics with prescribed Gaussian curvature on surfaces with conical singularities. We refer the reader, in particular, to [43] where a systematic analysis of this problem was initiated.

The above-mentioned results are the *singular* analog of the prescribed Gaussian curvature and Nirenberg problems (see [1, 3, 7–9, 28] and the references therein for further details).

Here, and in the rest of this paper, we denote by S a closed two-dimensional smooth surface without boundary. A conformal metric g_s on S is said to have a conical

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singularity of order $\alpha \in (-1, +\infty)$ (or of angle $\vartheta_\alpha = 2\pi(1+\alpha)$) at a given point $P_0 \in S$, if there exist local coordinates $z(P) \in \Omega \subset \mathbb{C}$ and $u \in C^0(\Omega) \cap C^2(\Omega \setminus \{P_0\})$ such that $z(P_0) = 0$ and

$$\tilde{g}_s(z) = |z|^{2\alpha} \, \mathrm{e}^u |\mathrm{d}z|^2, \quad z \in \Omega,$$

where $\tilde{g}_s$ is the local expression of g_s. The information concerning finitely many conical singularities is encoded in a divisor, which is the formal sum

$$\underline{\alpha}_m = \sum_{j=1}^{m} \alpha_j P_j, \quad m \in \mathbb{N}, \tag{1.1}$$

of the orders of the singularities $\{\alpha_1, \dots, \alpha_m\}$ times the singular points $\{P_1, \dots, P_m\}$. In particular, a metric g_s on S is said to represent the divisor $\underline{\alpha}_m$ if it has conical singularities of order α_j at point P_j for any $j \in \{1, \dots, m\}$. We will denote by $(S, \underline{\alpha}_m)$ the singular surface.

Let K be any Lipschitz function on S. We seek a conformal metric g on $(S, \underline{\alpha}_m)$ the Gaussian curvature of which is K.

The Euler characteristic of the singular surface $(S, \underline{\alpha}_m)$ (see [43]) is defined by

$$\chi(S, \underline{\alpha}_m) = \chi(S) + \sum_{j=1}^{m} \alpha_j,$$

where $\chi(S)$ is the Euler characteristic of S.

The Trudinger constant of the singular surface $(S, \underline{\alpha}_m)$ (see [15, 43]) is instead given by

$$\tau(S, \underline{\alpha}_m) = 2 + 2 \min_{j \in \{1, \dots, m\}} \min\{\alpha_j, 0\}.$$

According to the definitions in [43], the singular surface $(S, \underline{\alpha}_m)$ is said to be

$$\begin{cases} \text{subcritical} & \text{if } \chi(S, \underline{\alpha}_m) < \tau(S, \underline{\alpha}_m), \\ \text{critical} & \text{if } \chi(S, \underline{\alpha}_m) = \tau(S, \underline{\alpha}_m), \\ \text{supercritical} & \text{if } \chi(S, \underline{\alpha}_m) > \tau(S, \underline{\alpha}_m). \end{cases}$$

As far as one is interested in proving the existence of at least one conformal metric on $(S, \underline{\alpha}_m)$ with prescribed Gaussian curvature, the subcritical case is well understood. This is mainly due to the fact that on subcritical singular surfaces, the problem corresponds to minimizing a coercive functional (see [43]). On the contrary, much less is known concerning critical and supercritical singular surfaces.

We refer the reader to [16, 18, 19, 26, 31, 39, 44], for some positive results in this direction. In the same spirit of [22], Bartolucci and Tarantello obtained a result [2, Corollary 6] which, combined with Proposition 1.2, implies that: if $(S, \underline{\alpha}_m)$ is a supercritical singular surface with $\alpha_j > 0$, $j \in \{1, \dots, m\}$, $\chi(S) \leq 0$ and $4\pi\chi(S, \underline{\alpha}_m) \in (8\pi, 16\pi) \setminus \{8\pi(1+\alpha_j),\ j = 1, \dots, m\}$, then any positive Lipschitz continuous function K on S is the Gaussian curvature of at least one conformal metric on $(S, \underline{\alpha}_m)$. See also [13] for related issues.

In this paper, we will obtain a generalization of this result via a Morse-theoretical approach.

Let

$$\Gamma(\underline{\alpha}_m) = \left\{ \mu \in \mathbb{R}^+ \mid \mu = 8\pi k + 8\pi \sum_{j=1}^{m} (1+\alpha_j) n_j, \quad k \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N} \cup \{0\},\ n_j \in \{0, 1\} \right\}.$$

Our main result is the following.

Theorem 1.1. Let $(S, \underline{\alpha}_m)$ be a supercritical singular surface with $\alpha_j > 0$, $j \in \{1, \dots, m\}$, $\chi(S) \leq 0$ and $4\pi\chi(S, \underline{\alpha}_m) \notin \Gamma(\underline{\alpha}_m)$. Then, any positive Lipschitz continuous function K on S is the Gaussian curvature of at least one conformal metric on $(S, \underline{\alpha}_m)$. □

We attack this problem by a variational approach as first proposed in [3] and then pursued by many authors (see, e.g. [1, 16, 28, 43] and the references cited therein). Proposition 1.2 allows us to reduce the problem to a scalar differential equation on S. To state it, we need to introduce some notation. Let g_0 be any smooth conformal metric on S, $Q \in S$ be a given point and $G(P, Q)$ be the solution of (see [1])

$$-\Delta_0 G(P, Q) = \delta_Q - \frac{1}{|S|} \quad \text{in } S, \qquad \int_S G(P, Q)\, \mathrm{d}V_{g_0}(P) = 0,$$

where δ_Q denotes the Dirac delta with pole Q, Δ_0 the Laplace–Beltrami operator associated with g_0, and $|S|$ the area of S with respect to the volume form $dV_{\{g_0\}}$ induced by g_0. For a given divisor $\underline{\alpha}_m$, we define

$$h_m(P) = 4\pi \sum_{j=1}^{m} \alpha_j G(P, P_j).$$

Let us also denote by K_0 the (smooth) Gaussian curvature induced by g_0. Then we have the following proposition.

Proposition 1.2. Let $\alpha_j > 0$ for $j = 1, \dots, m$, K a Hölder continuous function on S, and suppose that $\chi(S, \underline{\alpha}_m) > 0$. The metric

$$g = \lambda \frac{\mathrm{e}^{-h_m}\, \mathrm{e}^{u}}{\int_S 2K\, \mathrm{e}^{-h_m}\, \mathrm{e}^{u}} g_0, \quad \text{with } \lambda = 4\pi\chi(S, \underline{\alpha}_m),$$

is a conformal metric on $(S, \underline{\alpha}_m)$ with Gaussian curvature K if and only if u is a classical solution to

$$-\Delta_0 u = \lambda \frac{2K\, \mathrm{e}^{-h_m}\, \mathrm{e}^{u}}{\int_S 2K\, \mathrm{e}^{-h_m}\, \mathrm{e}^{u}\, \mathrm{d}V_{g_0}} - 2K_0 - \frac{4\pi}{|S|}\sum_{j=1}^{m} \alpha_j \quad \text{in } S. \tag{1.2}$$

□

The proof of Proposition 1.2 is rather standard and is postponed to the Appendix. By using it, we are reduced to finding a classical solution of (1.2), that is, by standard elliptic regularity theory, a critical point $u \in H(S)$ of

$$J_\lambda(u) = \int_S |\nabla u|^2\, \mathrm{d}V_{g_0} - \lambda \log\left(\int_S 2K\, \mathrm{e}^{-h_m}\, \mathrm{e}^{u}\, \mathrm{d}V_{g_0}\right), \tag{1.3}$$

where $H(S) = \{u \in H^1(S) \mid \int_S u = 0\}$ and λ satisfies the Gauss–Bonnet constraint

$$\lambda = \int_S 2K\, \mathrm{e}^{-h_m}\, \mathrm{e}^{u}\, \mathrm{d}V_{g_0} = 4\pi\chi(S) + 4\pi\sum_{j=1}^{m}\alpha_j = 4\pi\chi(S, \underline{\alpha}_m). \tag{1.4}$$

By means of Proposition 1.2, Theorem 1.1 will follow immediately from the next result.

Theorem 1.3. Let S be a closed surface of positive genus, $K_0 \in L^s(S)$ for some $s > 1$ and K any positive Lipschitz function on S. Suppose, moreover, that $\alpha_j \geq 0$ for $j \in \{1, \dots, m\}$. Then, for any $\lambda \in (8\pi, +\infty) \setminus \Gamma(\underline{\alpha}_m)$, there exists at least one critical point $u \in H(S)$ for J_λ. □

Remark 1.4. As a consequence of the results in [30] (see also [29]) and in [2], it is straightforward to verify that our proof of Theorem 1.3 works whenever K is positive and Hölder continuous in S and Lipschitz continuous in a neighborhood of $\{P_1, \ldots, P_m\}$. We conclude that the result of Theorem 1.1 holds also under these assumptions on K. □

We notice that in case $\alpha_j = 0$, $j \in \{1, \ldots, m\}$, since $\Gamma(\underline{\alpha}_m) = 8\pi\mathbb{N}$, we come up with another proof of the existence of solutions for the mean field equation (1.2) (see [5]) for $\lambda \in (8\pi, +\infty) \setminus 8\pi\mathbb{N}$, previously obtained in [10] and more recently in [23, 35] (see also [20, 42]). In the same spirit of [24, 35], other positive results concerning the existence of solutions for (1.2) have been derived in [37]. Other results, in the same direction of [10], have been recently announced in [11] (see [12]).

Let us observe, in particular, that if $\chi(S, \underline{\alpha}_m) \leq 0$, then $(S, \underline{\alpha}_m)$ is subcritical. Therefore, as far as we are concerned with supercriticality, there is no loss of generality in assuming $\chi(S, \underline{\alpha}_m) > 0$. We also remark that if $\chi(S, \underline{\alpha}_m) \leq 0$, a set of much more detailed results concerning the prescribed Gaussian curvature problem are at hand (see [43]).

We are also able to prove the following generic multiplicity result, where $\mathcal{M}$ stands for the space of all C^2 Riemannian metrics on S equipped with the C^2 norm.

Theorem 1.5. Under the hypotheses of Theorem 1.3, with $\lambda \in (8N\pi, 8(N+1)\pi) \setminus \Gamma(\underline{\alpha}_m)$, and (g_0, K) in an open and dense subset of $\mathcal{M} \times C^{0,1}(S)$, J_λ admits at least $\binom{N+\mathfrak{g}-1}{\mathfrak{g}-1} = \frac{(N+\mathfrak{g}-1)!}{N!(\mathfrak{g}-1)!}$ critical points, where $\mathfrak{g}$ is the genus of S. □

We prove Theorems 1.3 and 1.5 using a variational and Morse-theoretical approach, looking at topological changes in the structure of sublevels of J_λ. For the regular case (with $\alpha_j = 0$, $j = 1, \ldots, m$), it was shown in [36] that for $\rho \in (8N\pi, 8(N+1)\pi)$, $N \in \mathbb{N}$, high sublevels have trivial topology, while low sublevels are homotopically equivalent to formal barycenters of S of order $\mathbb{N}$. By this, we mean the family of unit measures which are supported in at most N points of S.

Here we use a related argument: even if we do not completely characterize the topology of low sublevels, we are still able to retrieve some partial information. In particular, we embed a *bouquet* of circles, $B^{\mathfrak{g}}$, in S which does not intersect the singular points, and we construct a global projection of S onto $B^{\mathfrak{g}}$. The latter map induces a projection from the barycenters of S onto those of $B^{\mathfrak{g}}$ and we show that the latter set embeds nontrivially into arbitrarily low sublevels of J_λ. More precisely, we prove that low sublevels are noncontractible, yielding Theorem 1.3 and that their Betti numbers are comparable to those of the barycenters of the *bouquet*, which gives Theorem 1.5.

The paper is organized as follows. In Section 2, we recall some preliminary facts regarding some analytical issues (improved Moser–Trudinger inequalities, compactness results) and some topological ones (notions in algebraic topology and Morse theory). Finally in Section 3, we prove our main theorems analyzing the topology of sublevels of J_λ in terms of the barycenters of $B^{\mathfrak{g}}$, the Betti numbers of which are computed explicitly.

1.1 Notation

For $P \in S$ and $Q \in S$, let us denote by $d_0(P, Q)$ the geodesic distance induced by g_0 and for any couple of sets $\omega_1 \in S$ and $\omega_2 \in S$,

$$\mathrm{dist}(\omega_1, \omega_2) = \inf_{P \in \omega_1,\, Q \in \omega_2} d_0(P, Q).$$

For a metric space X and for $N \in \mathbb{N}$, we define the following family of probability measures, known in literature as formal barycenters of X of order N:

$$X_N = \left\{ \sum_{i=1}^{N} t_i \delta_{x_i} : t_i \in [0, 1], \sum_{i=1}^{N} t_i = 1, x_i \in X \right\}. \tag{1.5}$$

In the rest of this paper, we will denote by $\int_S \cdot$ the Lebesgue integral with respect to the volume form induced by g_0.

2 Preliminaries

We divide this section into an analytical part and a topological one.

2.1 Analytical preliminaries

We will need the following lemmas whose proofs can be found in [23, Lemma 3.2]. This kind of "distribution of mass" analysis was introduced in [16].

Lemma 2.1. For any integer $\ell \geq 1$, let $\omega_1, \omega_2, \dots, \omega_{\ell+1}$ be open sets in S satisfying

$$\mathrm{dist}(\omega_i, \omega_j) \geq \sigma_0 > 0 \quad \forall\, i \neq j,$$

for some $\sigma_0 > 0$. For any $\gamma_0 \in \left(0, \frac{1}{\ell+1}\right)$, and for any $\tilde{\varepsilon}_0 > 0$, there exist $C = C(S, \ell, \sigma_0, \tilde{\varepsilon}_0, \gamma_0)$ such that

$$\log \int_S e^u \leq C + \frac{\int_S |\nabla u|^2}{16\pi(\ell+1) - \tilde{\varepsilon}_0}, \tag{2.1}$$

for any $u \in H(S)$ satisfying

$$\int_{\omega_j} \frac{e^u}{\int_S e^u} \geq \gamma_0 \quad \forall\, j \in \{1, \ldots, \ell+1\}. \qquad \square$$

Using this result and a covering lemma, one can then characterize the concentration properties of the functions in $H(S)$ with low energy (see [23, Lemma 3.4]).

Lemma 2.2. Assuming $N \geq 1$ and $\lambda \in (8\pi N, 8\pi(N+1))$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$, there exists a large positive constant $L = L(\varepsilon, r)$ such that for every $u \in H(s)$ with $J_\lambda(u) \leq -L$, there exist N points $\{p_{1,u}, p_{2,u}, \ldots, p_{N,u}\} \subset S$ such that

$$\int_{S \setminus \bigcup_{i=1}^N B_r(p_{i,u})} \frac{e^u}{\int_S e^u} < \varepsilon. \tag{2.2}$$

$\square$

Lemma 2.2 implies that the unit measure $\frac{e^u}{\int_S e^u}$ resembles a finite linear combination of Dirac deltas with at most N elements: one is then induced to consider the family of formal barycenters of S of order N (see the Notation). These considerations can be made rigorous in the sense specified by the following result.

Lemma 2.3. If $\lambda \in (8N\pi, 8(N+1)\pi)$ with $N \geq 1$, then there exists a continuous projection $\Psi : \{J_\lambda \leq -L\} \to S_N$. $\square$

This is exactly the map Ψ defined in Lemma 4.9 of [23]. On the other hand, for what concerns the embedding of the space of formal barycenters S_N, into arbitrarily low sublevels, the statement of Proposition 5.1 in [23], cannot be applied as it stands to the singular case. To state the adapted version, we need to introduce the following family of test functions.

For $\delta > 0$ small, consider a smooth non-decreasing cut-off function $\chi_\delta : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties

$$\begin{cases} \chi_\delta(t) = t, & \text{for } t \in [0, \delta] \\ \chi_\delta(t) = 2\delta & \text{for } t \geq 2\delta \\ \chi_\delta(t) \in [\delta, 2\delta], & \text{for } t \in [\delta, 2\delta]. \end{cases}$$

Then given $\sigma \in S_N$, $\sigma = \sum_{i=1}^N t_i \delta_{x_i}$ ($\sum_{i=1}^N t_i = 1$) and $\mu > 0$, we define $\varphi_{\mu,\sigma} : S \to \mathbb{R}$ by

$$\varphi_{\mu,\sigma}(y) = \log \sum_{i=1}^N t_i \left(\frac{\mu}{1 + \mu^2 \chi_\delta^2(\mathrm{d}_i(y))} \right)^2 - \log(\pi), \tag{2.3}$$

where we have set

$$\mathrm{d}_i(y) = \mathrm{d}_0(y, x_i), \qquad x_i, y \in S. \tag{2.4}$$

We point out that, since the distance is a Lipschitz function, $\varphi_{\mu,\sigma}(y)$ is also Lipschitz in y, and hence it belongs to $H^1(S)$. Let us denote by $\tilde{\varphi}_{\mu,\sigma}$ the normalized functions $\varphi_{\mu,\sigma} - \bar{\varphi}_{\mu,\sigma} \in H(S)$.

By using Lemma 2.2 and by arguing as in [24] we obtain the following result.

Proposition 2.4. Suppose $\lambda \in (8N\pi, 8(N+1)\pi)$ with $N \geq 1$. Let $\tilde{\varphi}_{\mu,\sigma}$ be the functions defined above and let K be a compact subset of $S \setminus \{P_1, \ldots, P_m\}$. Then,

$$\frac{e^{\tilde{\varphi}_{\mu,\sigma}}}{\int_S e^{\tilde{\varphi}_{\mu,\sigma}}} \rightharpoonup \sigma \quad \text{and} \quad J_\lambda(\tilde{\varphi}_{\mu,\sigma}) \to -\infty \quad \text{uniformly for } \sigma \in K_N \quad \text{as } \mu \to \infty, \tag{2.5}$$

where K_n denotes the set of formal barycenters of order N supported in K. □

We will need some compactness properties for (1.2), relying on the following result (see [2]).

Theorem 2.5 ([2]). Let K be a positive Lipschitz function on S and let $\tilde{h} = K\,\mathrm{e}^{-h_m}$. Let u_i solve (1.2) with $\alpha_j > 0$, $p_j \in S$, and $\lambda = \lambda_i$, $\lambda_i \to \bar{\lambda}$. Suppose that $\int_S \tilde{h}\,\mathrm{e}^{u_i}\,\mathrm{d}V_g \leq C_1$ for some fixed $C_1 > 0$. Then along a subsequence u_{i_k}, one of the following alternatives hold:

(i) u_{i_k} is uniformly bounded from above on S;

(ii) $\max_S(u_{i_k} - \log \int_S \tilde{h} e^{u_{i_k}}) \to +\infty$ and there exists a finite blow-up set $\Sigma = \{q_1, \ldots, q_l\} \subset S$ such that

  (a) for any $s \in \{1, \ldots, l\}$, there exist $x_n^s \to q_s$ such that $u_{i_k}(x_n^s) \to +\infty$ and $u_{i_k} \to -\infty$ uniformly on the compact sets of $S \setminus \Sigma$,

  (b) $\lambda_{i_k} \frac{\tilde{h}\,\mathrm{e}^{u_{i_k}}}{\int_S \tilde{h}\,\mathrm{e}^{u_{i_k}}\,\mathrm{d}V_g} \rightharpoonup \sum_{s=1}^l \beta_s \delta_{q_s}$ in the sense of measures, with $\beta_s = 8\pi$ for $q_s \neq \{p_1, \ldots, p_m\}$, or $\beta_s = 8\pi(1 + \alpha_j)$ if $q_s = p_j$ for some $j = \{1, \ldots, m\}$. In particular, one has that

$$\bar{\lambda} \in \Gamma(\underline{\alpha}_m).$$

□

From the above result, we obtain immediately the following corollary.

Corollary 2.6. Suppose we are in the above situation and that $\lambda \notin \Gamma(\underline{\alpha}_m)$. Then the solutions of (1.2) stay uniformly bounded in $C^2(S)$. □

Corollary 2.6 is a compactness criterion useful to bypass the Palais–Smale condition, which is not known for the functional J_λ. This corollary, combined with the arguments in [33] (proved for the regular case, but adapting in a straightforward way to the singular one) allows to prove the next alternative.

Lemma 2.7. If $\lambda \notin \Gamma(\underline{\alpha}_m)$ and if J_λ has no critical levels inside some interval $[a, b]$, then $\{J_\lambda \leq a\}$ is a deformation retract of $\{J_\lambda \leq b\}$. □

Remark 2.8. As far as we are concerned with the approach presented in this paper, it seems not easy to remove the hypothesis on the positivity of K. The difficulties are inherited by the lack of concentration-compactness-quantization results (in the same spirit of [2, 4, 29]) for solutions of (1.2) with K possibly changing sign or even just nonnegative. Actually, our analysis relies heavily on Theorem 2.5 (see also results in [4, 29]) where this hypothesis is required (see [38] for related issues in the regular case).

However, the necessary condition imposed by the Gauss–Bonnet constraint (1.4) just reads

$$\int_S 2K\,\mathrm{e}^{-h_m}\,\mathrm{e}^{u} = 4\pi\,\chi(S, \underline{\alpha}_m),$$

so that, in principle, there should be no obstructions (as in the regular and subcritical cases [28, 43]) in finding conformal metrics on supercritical singular surfaces of positive genus with Gaussian curvature just assumed to be positive somewhere. □

This remark motivates the following question: Is it true that any Lipschitz continuous function on S can be realized as the Gaussian curvature of a conformal metric on a supercritical surface satisfying the hypotheses of Theorem 1.1?

2.2 Topological and Morse-theoretical preliminaries

This section is devoted to collect some classical and more recent results concerning the topological structure of the sublevels of J_λ and of Morse functionals. We will also give a short review of basic notions of algebraic topology needed to get the multiplicity estimate.

Throughout, the sign $\simeq$ will refer to homotopy equivalences, while $\cong$ will refer to homeomorphisms between topological spaces or isomorphisms between groups. Given a pair of spaces (X, A), we will denote by $H_q(X, A)$ the relative qth homology group with coefficient in $\mathbb{Z}$ and by $\tilde{H}_q(X) := H_q(X, x_0)$ the reduced homology with coefficient in $\mathbb{Z}$, where $x_0 \in X$. Finally, if X, Y, are two topological spaces and $f : X \to Y$ is a continuous function, we will denote by $f_* : H_q(X) \to H_q(Y)$, for $q \in \mathbb{N}$, the pushforward induced by f.

Since the functional J_λ stays uniformly bounded on the solutions of (1.2) (by Corollary 2.6), the deformation Lemma 2.7 can be used to prove that it is possible to retract the whole Hilbert space $H(S)$ onto a high sublevel $\{J_\lambda \le b\}$, $b \gg 0$ (see [36, Corollary 2.8], for the regular case: also for this issue, only minor changes are required). More precisely one has the following.

Proposition 2.9. If $\lambda \notin \Gamma(\underline{\alpha}_m)$ and if b is sufficiently large positive, the sublevel $\{J_\lambda \le b\}$ is a deformation retract of $H(S)$ and hence is contractible. □

We recall next a classical result in Morse theory: Morse inequalities.

Theorem 2.10 (see, e.g. [6, Theorem 4.3])**.** Let M be a Hilbert manifold and $f \in C^2(M; \mathbb{R})$ be a Morse function (i.e., all critical points are nondegenerate) satisfying the (PS)-condition. Let a, b $(a < b)$ be regular values for f and

$$C_q(a, b) := \#\{\text{critical points of } f \text{ in } \{a \le f \le b\} \text{ with index } q\},$$

$$\beta_q(a, b) := \operatorname{rank}(H_q(\{f \le b\}, \{f \le a\})).$$

Then

$$\sum_{q=0}^{n} (-1)^{n-q} C_q(a, b) \ge \sum_{q=0}^{n} (-1)^{n-q} \beta_q(a, b), \quad n = 0, 1, 2, \dots \text{ (strong inequalities)},$$

$$C_q(a, b) \ge \beta_q(a, b), \quad q = 0, 1, 2, \dots \text{ (weak inequalities)}.$$ □

As already remarked in [21], the (PS)-condition can be replaced by the request that appropriate deformation lemmas hold true for f. In particular, a flow defined by Malchiodi [36] allows us to adapt to J_λ the classical deformations lemmas [6, Lemma 3.2 and Theorem 3.2] needed so that Theorem 2.10 can be applied for $M = H(S)$ and $f = J_\lambda$, under the further assumption that all the critical points of J_λ are nondegenerate.

To sum up, if J_λ is a Morse functional and a and b are regular values for J_λ, then the weak and the strong inequalities are verified. For the regular case, De Marchis showed [21] that it is possible to apply a transversality result due to Saut and Temam [41] which guarantees that generically all the critical points of the Euler functional are nondegenerate. In fact, exactly the same procedure allows us to obtain the following statement (see the proof of [21, Theorem 1.5] for details).

Proposition 2.11. For $\lambda \notin \Gamma(\underline{\alpha}_m)$ and for (g_0, K) in an open and dense subset of $\mathcal{M} \times C^{0,1}(S)$, J_λ is a Morse functional. □

Let now recall some well-known definitions in algebraic topology.

Join. The join of two spaces X and Y is the space of all segments "joining points" in X to points in Y. It is denoted by $X * Y$ and is the identification space

$$X * Y := X \times [0,1] \times Y/(x,0,y) \sim (x',0,y),\ (x,1,y) \sim (x,1,y') \quad \forall\, x, x' \in X,\ \forall\, y,\ y' \in Y.$$

Wedge sum. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \amalg Y$ obtained by identifying x_0 and y_0 to a single point. If $\{x_0\}$ (resp. $\{y_0\}$) is a closed subspace of X (resp. Y), that is, a deformation retract of some neighborhood in X (resp. Y), then $\tilde{H}_q(X \vee Y) \cong \tilde{H}_q(X) \bigoplus \tilde{H}_q(Y)$, provided that the wedge sum is formed at basepoints x_0 and y_0.

Smash product. Inside a product space $X \times Y$, there are copies of X and Y, namely $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $x_0 \in X$ and $y_0 \in Y$, respectively. These two copies of X and Y in $X \times Y$ intersect only at the point (x_0, y_0), so their union can be identified with the wedge sum $X \vee Y$. The smash product $X \wedge Y$ is then defined to be the quotient $X \times Y / X \vee Y$. For example, $S^n \wedge S^m \cong S^{n+m}$.

Suspension. The k-fold (*unreduced*) suspension of X is defined to be $S^{k-1} * X$, while the k-fold *reduced* suspension is the smash product $S^k \wedge X$. A useful property of the reduced suspension is that, for any q, $n \geq 0$, $\tilde{H}_q(X) \cong \tilde{H}_{q+n}(S^n \wedge X)$. It is crucial to note that reduced and unreduced constructions are homotopically equivalent constructions for the spaces we will deal with. In the following, we will often use the latter property for replacing, in some results of [27], the unreduced suspension by the reduced one.

Reduced symmetric product. We denote by $\overline{SP}^k(X)$ the kth reduced symmetric product which is the symmetric smash product $X^{(k)}/\mathfrak{S}_k$, where $X^{(k)}$ is the k-fold smash product of X with itself and $\mathfrak{S}_k$ is the permutation group. We set $\overline{SP}^0(X) = S^0$. A theorem by Dold [25, Theorem 7.2] implies that the homology of reduced symmetric products only depends on the homology of the underlying space. Moreover, it has been proved that $\overline{SP}^k(X \vee Y) = \bigvee_{r+s=k} \overline{SP}^r(X) \wedge \overline{SP}^s(Y)$; finally, in the case of the 2-sphere, $\overline{SP}^k(S^2) \cong S^{2k}$ (see [27, Theorem 1.3 and Corollary 4.3]).

3 Proof of the Theorems

We first make the following claim, the proof of which follows from Propositions 3.1 and 3.2.

Claim. For $\lambda \in (8\pi N, 8\pi(N+1)) \setminus \Gamma(\underline{\alpha}_m)$, choosing L sufficiently large positive, one has that

$$\beta_{2N-1}(L, -L) \geq \binom{N+\mathfrak{g}-1}{\mathfrak{g}-1} = \frac{(N+\mathfrak{g}-1)!}{N!(\mathfrak{g}-1)!}. \qquad \square$$

Once the claim is proved, the conclusion of Theorem 1.3 follows from Lemma 2.7. To prove Theorem 1.5, it is instead sufficient to apply Proposition 2.11 and Theorem 2.10 (using the observations after it) with $a = -L$ and $b = L$.

Proposition 3.1. There exists $L > 0$ sufficiently large such that, for any $q \in \mathbb{N}$, $\beta_q(L, -L) \geq \beta_q(B_N^{\mathfrak{g}})$, where $B_N^{\mathfrak{g}}$ is the space of formal barycenters on a bouquet of $\mathfrak{g}$ circles, with $\mathfrak{g}$ the genus of S. $\square$

We recall that a space $B^{\mathfrak{g}}$ is a bouquet of $\mathfrak{g}$ circles if $B^{\mathfrak{g}} = \bigcup_{j=1}^{\mathfrak{g}} A_j$, with A_j homeomorphic to S^1 and $A_i \cap A_j = \{P\}$, and P is called the center of the bouquet. In the above statement, $\beta_q(B_N^{\mathfrak{g}})$ stands for the qth Betti number of $B_N^{\mathfrak{g}}$, namely the rank of $H_q(B_N^{\mathfrak{g}})$.

Proof. Proposition 2.9 implies that $\{J_\lambda \leq L\}$ is contractible (for L sufficiently large). Thus, from the exactness of the homology sequence

$$\cdots \to \tilde{H}_q(\{J_\lambda \leq -L\}) \to \tilde{H}_q(\{J_\lambda \leq L\}) \to H_q(\{J_\lambda \leq L\}, \{J_\lambda \leq -L\}) \to \tilde{H}_{q-1}(\{J_\lambda \leq -L\}) \to \cdots$$

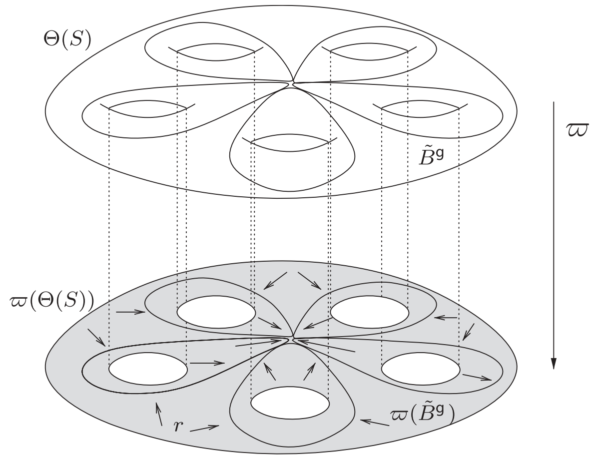

Fig. 1. $\tilde{B}^{\mathfrak{g}}$ embedded in $\Theta(S)$ and their projections.

we derive that

$$H_{q+1}(\{J_\lambda \leq L\}, \{J_\lambda \leq -L\}) \cong \tilde{H}_q(\{J_\lambda \leq -L\}), q \geq 0,$$

$$H_0(\{J_\lambda \leq L\}, \{J_\lambda \leq -L\}) = 0.$$

Now to obtain the thesis, it suffices to construct $j : B_N^{\mathfrak{g}} \to \{J_\lambda \leq -L\}$ and $f : \{J_\lambda \leq -L\} \to B_N^{\mathfrak{g}}$ such that $f \circ j$ is homotopically equivalent to the $\mathrm{Id}_{|B_N^{\mathfrak{g}}}$. In fact, if this is true, we have that

$$f_* \circ j_* = Id_{|H_*(B_N^{\mathfrak{g}})},$$

which implies that $\mathrm{rank}(H_q(\{J_\lambda \leq -L\})) \geq \mathrm{rank}(H_q(B_N^{\mathfrak{g}})) = \beta_q(B_N^{\mathfrak{g}})$.

In order to build these maps, we will regard $B^{\mathfrak{g}}$ as an appropriate subset of S: let us understand how.

Since any two differentiable, compact, orientable surfaces with the same genus are homeomorphic, we can consider an embedding Θ from S to $\mathbb{R}^3$ (with coordinates z_1, z_2, and z_3) such that in any hole passes a line parallel to the z_3-axis and moreover such that the projection on the plane $\{z_3 = 0\}$ is a circle with $\mathfrak{g}$ rounds holes as in Figure 1. Let us denote by ϖ the map projecting $\mathbb{R}^3$ onto the plane $\{z_3 = 0\}$.

In $\Theta(S \setminus \{P_1, \ldots, P_m\})$, it is clearly possible to find a bouquet of circles, $\tilde{B}^{\mathfrak{g}}$, verifying:

- $\varpi_{|\tilde{B}^{\mathfrak{g}}}$ is an homeomorphism,
- $\varpi(\tilde{B}^{\mathfrak{g}})$ is a bouquet having a hole of $\varpi(\Theta(S))$ in each loop,

- $\varpi(\tilde{B}^{\mathfrak{g}}) \cap \varpi(\{P_1, \ldots, P_m\}) = \emptyset$.

Then there exists a retraction $r : \varpi(\Theta(S)) \to \varpi(\tilde{B}^{\mathfrak{g}})$.

Let us set $B^{\mathfrak{g}} := \Theta^{-1}(\tilde{B}^{\mathfrak{g}})$, which is again a bouquet with $\mathfrak{g}$ loops.

We are, at last, in a position to define the desired maps.

$$\begin{gathered} j : B_N^{\mathfrak{g}} \longrightarrow \{J_\lambda \le -L\}, \\ \sigma = \sum_{i=1}^N t_i \delta_{b_i} (b_i \in B^{\mathfrak{g}}) \longmapsto \varphi_{\mu,\sigma}, \end{gathered} \tag{3.1}$$

$$\begin{gathered} f : \{J_\lambda \le -L\} \xrightarrow{\Psi} S_N \xrightarrow{\Upsilon} B_N^{\mathfrak{g}}, \\ u \longmapsto \Psi(u) = \sum_{i=1}^N t_i \delta_{x_i} \longmapsto \sum_{i=1}^N t_i \delta_{\Theta^{-1} \circ \varpi^{-1} \circ r \circ \varpi \circ \Theta(x_i)}. \end{gathered} \tag{3.2}$$

The fact that $f \circ j$ is homotopically equivalent to the identity on $B_N^{\mathfrak{g}}$ follows easily from Proposition 2.3 and the uniform continuity of Υ on $B_N^{\mathfrak{g}}$. ■

Proposition 3.2. $\beta_{2N-1}(B_N^{\mathfrak{g}}) = \binom{N+\mathfrak{g}-1}{\mathfrak{g}-1} = \frac{(N+\mathfrak{g}-1)!}{N!(\mathfrak{g}-1)!}$. □

Proof. Theorems 1.1 and 1.3 in [27] imply that for any $q \ge 0$,

$$\tilde{H}_q(B_N^{\mathfrak{g}}) \cong H_{q+1}(\overline{SP}^N(S^1 \wedge B^{\mathfrak{g}})).$$

Now notice that $S^1 \wedge B^{\mathfrak{g}}$ has the same homology of $\bigvee_{j=1}^{\mathfrak{g}} S^2$; hence, since the reduced symmetric product of a space only depends on its homology, it follows that for any $q \ge 0$,

$$\begin{aligned} \tilde{H}_q((B^{\mathfrak{g}})_N) &\cong H_{q+1}(\overline{SP}^N(S^1 \wedge B^{\mathfrak{g}})) \\ &\cong H_{q+1}\left(\overline{SP}^N\left(\bigvee_{j=1}^{\mathfrak{g}} S^2\right)\right) \\ &\cong [\text{property of the reduced symmetric product}] \\ &\cong H_{q+1}\left(\bigvee_{n_1+\cdots+n_{\mathfrak{g}}=N}\left(\bigwedge_{j=1}^{\mathfrak{g}} \overline{SP}^{s_j} S^2\right)\right) \end{aligned}$$

$$\cong \text{[property of the homology of the wedge sum]}$$

$$\cong \bigoplus_{n_1+\cdots+n_{\mathfrak{g}}=N} H_{q+1}\left(\bigwedge_{j=1}^{\mathfrak{g}}(\overline{SP}^{s_j}S^2)\right) \cong [\overline{SP}^{n}(S^2)\cong S^{2n}]$$

$$\cong \bigoplus_{n_1+\cdots+n_{\mathfrak{g}}=N} H_{q+1}(S^{2N}) \cong \begin{cases} \mathbb{Z}^s, & q=(2N-1), \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Here $s=\binom{N+\mathfrak{g}-1}{\mathfrak{g}-1}$ counts the number of tuples $(n_1,\ldots,n_{\mathfrak{g}})$ such that $\sum_{j=1}^{\mathfrak{g}} n_j=N$. The proof is thereby complete. ■

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Appendix

In this section, we prove Proposition 1.2.

Proof of Proposition 1.2. It is well known [28, 43] that $g=e^{2\tilde{w}}g_0$ is a conformal metric on $(S,\underline{\alpha}_m)$ with Gaussian curvature K if and only if

$$\begin{aligned} &-\Delta_0\tilde{w}=K\,e^{2\tilde{w}}-K_0 \quad \text{in } S\setminus\{P_1,\ldots,P_m\}, \\ &\frac{1}{2\pi}\int_S K\,e^{2\tilde{w}}=\chi(S)+\sum_{j=1}^{m}\alpha_j, \\ &\tilde{w}(\pi_j(z))=\alpha_j\log|z-z_j|+O(1), \quad z\in B_r(z_j),\ j\in 1,\ldots,m, \end{aligned} \tag{A.1}$$

where π_j is a set of local (complex) isothermal coordinates around $z_j=\pi_j^{-1}(P_j)$ (as induced by the g_0 partition of unity construction) and $r>0$, a suitably chosen positive small enough number. Let us define

$$w(P)=\tilde{w}(P)+2\pi\sum_{j=1}^{m}\alpha_j G(P,P_j). \tag{A.2}$$

Then w is a distributional solution of the equation

$$-\Delta_0 w = K\,\mathrm{e}^{-h_m}\,\mathrm{e}^{2w} - K_0 - \frac{2\pi}{|S|}\sum_{j=1}^{m}\alpha_j \quad \text{in } S\setminus\{P_1,\dots,P_m\}, \tag{A.3}$$

which also satisfies

$$\frac{1}{2\pi}\int_S K\mathrm{e}^{-h_m}\,\mathrm{e}^{2w} = \chi(S) + \sum_{j=1}^{m}\alpha_j, \tag{A.4}$$

and

$$w(\pi_j(z)) = \alpha_j \log|z-z_j| + 2\pi\sum_{\ell=1}^{m}\alpha_\ell G(\pi_j(z),\pi_\ell(z_\ell)) + \mathrm{O}(1), \quad z\in B_r(z_j),\ \ j\in 1,\dots,m.$$

However, it is also well known [1] that

$$G(P,P_j) = \frac{1}{2\pi}\log\left(d_0(P,P_j)\right) + \mathrm{O}(1), \quad P\simeq P_j,$$

where $d_0(\cdot,\cdot)$ is the geodesic distance defined by g_0. In particular, it is not too difficult to verify that

$$G(\pi_j(z),\pi_j(z_j)) = -\frac{1}{2\pi}\log|z-z_j| + \mathrm{O}(1), \quad z\simeq z_j, \tag{A.5}$$

and we readily conclude that

$$w(\pi_j(z)) = \mathrm{O}(1), \quad z\in B_r(z_j),\ \ j\in 1,\dots,m.$$

By standard elliptic theory, this condition implies that w is a distributional solution for (A.3) on S. In particular, by using (A.5) and the explicit expression of h_m, we see that e^{-h_m} is Hölder continuous in S, and the standard elliptic regularity theory shows that w is a classical solution for (A.3).

At this point, we conclude that if $u=2w$, then u is a classical solution for

$$-\Delta_0 u = 2K\,\mathrm{e}^{-h_m}\,\mathrm{e}^{u} - 2K_0 - \frac{4\pi}{|S|}\sum_{j=1}^{m}\alpha_j \quad \text{in } S, \tag{A.6}$$

and then setting

$$\lambda = 4\pi\left(\chi(S) + \sum_{j=1}^{m}\alpha_j\right),$$

and by using (A.4) we conclude that u is a classical solution for (1.2). Therefore, if

$$g = e^{2\tilde{w}} g_0 = e^{-h_m} e^u g_0 \equiv \lambda \frac{e^{-h_m} e^u}{\int_S 2K e^{-h_m} e^u} g_0$$

is a conformal metric on $(S, \underline{\alpha}_m)$ with Gaussian curvature K, then u is a classical solution for (1.2).

On the other hand, if u is a classical solution for (1.2), then (1.4) holds. Thus, we can define w by

$$2w = u + \log\lambda - \log\left(\int_S 2K e^{-h_m} e^u\right),$$

and come up with a classical solution for (A.3) on all S. At this point, we can use (A.2) to define $\tilde{w}$ and conclude that

$$\lambda \frac{e^{-h_m} e^u}{\int_S 2K e^{-h_m} e^u} g_0 = e^{-h_m} e^{2w} g_0 = e^{2\tilde{w}} g_0$$

is a conformal metric on $(S, \underline{\alpha}_m)$ with Gaussian curvature K. ■

Remark A.1. We remark that if $\alpha_i \in (-1, 0)$ for some $i \in I \subseteq \{1, \ldots, m\}$, then the statement of Proposition 1.2 still holds but for the condition of u being a classical solution, which should be replaced by $u \in C^2(S \setminus \{\cup_{i\in I} P_i\}) \cap C^0(S)$. □

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