SHARP BOUNDARY CONCENTRATION FOR A TWO-DIMENSIONAL NONLINEAR NEUMANN PROBLEM

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Abstract. We consider the elliptic equation $-\Delta u + u = 0$ in a bounded, smooth domain $\Omega \subset \mathbb{R}^2$ subject to the nonlinear Neumann boundary condition $\partial u/\partial\nu = |u|^{p-1}u$ on $\partial\Omega$ and study the asymptotic behavior as the exponent $p \to +\infty$ of families of positive solutions u_p satisfying uniform energy bounds. We prove energy quantization and characterize the boundary concentration. In particular we describe the local asymptotic profile of the solutions around each concentration point and get sharp convergence results for the L^∞-norm.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. This paper deals with the analysis of solutions of the boundary value problem

$$\begin{cases} \Delta u = u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where ν denotes the outer unit normal vector to $\partial\Omega$ and $p > 1$. Two dimensional elliptic equations with nonlinear Neumann boundary conditions arise in many fields (conformal geometry, corrosion modelling, etc...) see for instance [3, 7, 10, 11, 12, 13, 24, 25, 26, 28, 29, 30, 38] and in particular, [8, 9, 21, 36] where problem (1.1) is considered.

Observe that solutions to (1.1) correspond to critical points in $H^1(\Omega)$ of the free energy functional

$$E_p(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)\, dx - \frac{1}{p+1}\int_{\partial\Omega} u^{p+1}\, d\sigma,$$

and by the compact trace and Sobolev embeddings $H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$, one can derive the existence of at least a solution for any fixed exponent $p > 1$ by standard variational methods (see for instance [36]). For multiplicity results for p large enough see Castro ([8]) and for sign-changing solutions see for instance [28].

This paper is devoted to the study of the asymptotic behavior, as $p \to +\infty$, of general families of non-trivial solutions u_p to (1.1) under a uniform bound of their energy, namely we assume

$$p\int_\Omega (|\nabla u_p|^2 + u_p^2)\ \ dx \to \beta \in \mathbb{R}, \quad \text{as } p \to +\infty. \tag{1.2}$$

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In [36], and later in [9], this analysis has been carried out for the family of least energy solutions. Note that these solutions satisfy the condition

$$p\int_\Omega (|\nabla u_p|^2 + u_p^2)\ dx \to 2\pi e, \quad \text{as } p \to +\infty,$$

which is a particular case of (1.2). In [36] it was proved that least energy solutions remain bounded uniformly in p, and develop one peak on the boundary, whose location is controlled by the Green's function G for the Neumann problem

$$\begin{cases} \Delta_x G(x,y) = G(x,y) & \text{in } \Omega, \\ \frac{\partial G}{\partial \nu_x}(x,y) = \delta_y(x) & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

$y \in \partial\Omega$. Indeed the concentration point turns out to satisfy $\nabla_{\tau(x_0)} R(x_0) = 0$, where $\tau(x_0)$ denotes a tangent vector at the point $x_0 \in \partial\Omega$ and the Robin function is defined as $R(x) := H(x,x)$, where H is the regular part of G:

$$H(x,y) := G(x,y) - \frac{1}{\pi}\log\frac{1}{|x-y|}. \tag{1.4}$$

Later Castro [9] identified a limit problem by showing that a suitable scaling of the least energy solutions converges in $C^1_{loc}(\overline{\mathbb{R}^2_+})$ to the regular solution

$$U(t_1,t_2) = \log\left(\frac{4}{t_1^2 + (t_2+2)^2}\right) \tag{1.5}$$

of the Liouville problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial U}{\partial \nu} = e^U & \text{on } \partial\mathbb{R}^2_+ \\ \int_{\partial\mathbb{R}^2_+} e^U = 2\pi \text{ and } \sup_{\overline{\mathbb{R}^2_+}} U < +\infty. \end{cases} \tag{1.6}$$

He also proved that for least energy solutions

$$\|u_p\|_\infty \to \sqrt{e} \text{ as } p \to \infty,$$

as it had been previously conjectured in [36].

Observe that problem (1.1) also admits families of solutions which develop m boundary peaks as $p \to \infty$, for any integer $m \geq 1$, as proved in [8] and indeed, recently in [21], it has been proved that the boundary concentration behavior characterizes any family of solutions to (1.1) which satisfy the uniform energy bound (1.2) (i.e. not only the least energy ones).

In order to state the results of [21] we define, for a sequence $p_n \to +\infty$, the blow-up set $\mathcal{S}$ of the sequence $p_n u_{p_n}$, where u_{p_n} solves (1.1), to be the subset

$$\mathcal{S} := \{\bar{x} \in \overline{\Omega}\ :\ \exists\,(x_n)_n \in \overline{\Omega},\ x_n \to \bar{x},\ \text{with}\ \ p_n u_{p_n}(x_n) \to +\infty\}. \tag{1.7}$$

We summarize the results in [21] as follows:

Theorem I. *Let $(u_p)_p$ be a family of solutions of (1.1) satisfying (1.2). Then there exist $C, c, \tilde{c}, \tilde{C} > 0$ such that*

$$c \leq \|u_p\|_{L^\infty(\overline{\Omega})} \leq C, \quad \textit{for } p > 1 \tag{1.8}$$

$$\tilde{c} \leq p\int_{\partial\Omega} u_p^p d\sigma \leq \tilde{C}, \ \textit{for } p \textit{ large.} \tag{1.9}$$

Furthermore for any sequence $p_n \to +\infty$, there exists a subsequence (still denoted by p_n) such that the following statements hold true:

(1) *There exists an integer* $m \geq 1$, *a finite collection of* m *distinct points* $\bar{x}_i \in \partial\Omega$, $i = 1, \dots, m$, *such that the blow-up set* $\mathcal{S}$ *of the sequence* $p_n u_{p_n}$ *is given by*

$$\mathcal{S} = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m\}. \tag{1.10}$$

(2) *There exist* m *positive constants* $c_i > 0$, $i = 1, \dots, m$, *such that*

$$p_n u_{p_n}^{p_n} \overset{*}{\rightharpoonup} \sum_{i=1}^{m} c_i \delta_{\bar{x}_i} \text{ in the sense of Radon measures on } \partial\Omega$$

and

$$\lim_{n\to\infty} p_n u_{p_n} = \sum_{i=1}^{m} c_i G(., \bar{x}_i) \quad \text{in } C^1_{loc}(\overline{\Omega} \setminus \mathcal{S}),\ L^t(\Omega) \text{ and } L^t(\partial\Omega),\ \forall\, t \in [1, +\infty), \tag{1.11}$$

where G *is the Green's function for the Neumann problem* (1.3).

(3) *The points* $\bar{x}_i$, $i = 1, \dots, m$, *satisfy*

$$c_i \nabla_{\tau(\bar{x}_i)} H(\bar{x}_i, \bar{x}_i) + \sum_{h \neq i} c_h \nabla_{\tau(\bar{x}_i)} G(\bar{x}_i, \bar{x}_h) = 0, \tag{1.12}$$

where $\tau(\bar{x}_i)$ *is a tangent vector to* $\partial\Omega$ *at* $\bar{x}_i$ *and* H *is the regular part of* G *as defined in* (1.4).

This result shows boundary concentration at a finite number of points in $\mathcal{S} \subset \partial\Omega$, moreover by (1.11) and (A.4) it follows that in any compact subset of $\overline{\Omega} \setminus \mathcal{S}$

$$p u_p \leq C, \tag{1.13}$$

and so

$$\lim_{n\to+\infty} u_{p_n} = 0 \quad \text{in } C^1_{loc}(\overline{\Omega} \setminus \mathcal{S}). \tag{1.14}$$

Many questions arise from Theorem I:

- How does u_{p_n} behave *close* to the points $\bar{x}_i$?
  In particular, what is the asymptotic behavior of $\|u_{p_n}\|_\infty$?
- Can one compute the constants c_i which appear at points (2) and (3) in Theorem I?
- What one can say about the total energy of u_{p_n}?

Looking at the asymptotic results for least energy solutions ([36, 9]) and at the existence results of solutions with multiple concentrations points ([8]), it was conjectured in [21] that for general solutions of (1.1) under the uniform energy assumption (1.2) the constants c_i's must be all equal and that an asymptotic quantization of the energy must occur, more precisely it was conjectured that:

$$c_i = 2\pi\sqrt{e}, \quad \text{for } 1 \leq i \leq m, \tag{C1}$$

$$p_n \int_\Omega (|\nabla u_{p_n}|^2 + u_{p_n}^2)\, dx \to m \cdot 2\pi e; \tag{C2}$$

as $n \to \infty$, and furthermore that

$$\|u_p\|_{L^\infty(\overline{\Omega})} \to \sqrt{e}, \tag{C3}$$

as $p \to +\infty$.

Here we answer these questions, proving in particular (C1), (C2) and (C3).

Theorem 1.1. *Let $(u_p)_p$ be a family of solutions to (1.1) satisfying (1.2) and let $p_n \to +\infty$, as $n \to +\infty$, be the subsequence such that the statements in Theorem I hold true. Then*

(i)

$$c_i = \lim_{\delta \to 0} \lim_{n \to +\infty} p_n \int_{B_\delta(\bar{x}_i) \cap \partial\Omega} u_{p_n}^{p_n} dx = 2\pi\sqrt{e}, \quad \text{for } 1 \le i \le m;$$

(ii)

$$\lim_{\delta \to 0} \lim_{n \to \infty} \|u_{p_n}\|_{L^\infty(B_\delta(\bar{x}_i) \cap \overline{\Omega})} = \sqrt{e} \qquad \forall\, i = 1, \dots, m,$$

where $B_\delta(\bar{x}_i)$ is a ball of center at $\bar{x}_i$ and radius $\delta > 0$;

(iii)

$$\lim_{n \to \infty} p_n \int_\Omega (|\nabla u_{p_n}|^2 + u_{p_n}^2)\, dx = m \cdot 2\pi e;$$

(iv) let $\delta > 0$ be such that $B_{2\delta}(\bar{x}_i) \cap B_{2\delta}(\bar{x}_j) = \emptyset$ for $i \neq j$ and let $(y_{i,n})_n \subset \overline{B_\delta(\bar{x}_i) \cap \Omega}$, $i = 1, \dots, m$, be the sequences of local maxima of u_{p_n} around x_i, namely

$$u_{p_n}(y_{i,n}) := \|u_{p_n}\|_{L^\infty(B_\delta(\bar{x}_i) \cap \overline{\Omega})},$$

then $(y_{i,n})_n \subset \partial\Omega$, $\lim_{n \to +\infty} |y_{i,n} - \bar{x}_i| = 0$ and, setting $\mu_{i,n} := \left(p_n u_{p_n}(y_{i,n})^{p_n - 1}\right)^{-1} (\to 0)$, then

$$w_{i,n}(t) := \frac{p_n}{u_{p_n}(y_{i,n})} \Big(u_{p_n}(\Psi_i^{-1}(b_{i,n} + \mu_{i,n} t)) - u_{p_n}(y_{i,n}) \Big),$$

where $b_{i,n} = \Psi_i(y_{i,n})$, $t \in T_n := \{t \in \mathbb{R}^2 \,:\, b_{i,n} + \mu_{i,n} t \in \Psi_i(\overline{\Omega} \cap B_{R_i}(\bar{x}_i))\}$ and Ψ_i is a change of coordinates which flattens $\partial\Omega$ near $\bar{x}_i$ and $R_i > 0$ is a suitable radius (see Subsection 2.1).
Then

$$\lim_{n \to \infty} w_{i,n} = U \qquad \text{in } C^1_{loc}(\overline{\mathbb{R}^2_+}),$$

where U is the solution (1.5) of the Liouville problem (1.6).

Theorem 1.1 shows that the conjectures (C1) and (C2) are true, furthermore points (ii) and (iv) provide information on the solutions close to the concentration points $\bar{x}_i$ for p large, in particular we identify the same limit profile U around each concentration point. We stress that in [21] only the existence of a *first bubble* U was proved, scaling the solution around the sequence of global maxima, while the behavior around the other concentration points was unknown. Observe that U is the same profile describing the least energy solutions (for which $m = 1$, see [9]), and indeed our theorem, combined with the results in [21] (Theorem I), extends to general families of solutions the asymptotic results proved in [36, 9] for least energy solutions, thus giving a complete characterization of the asymptotic behavior for problem (1.1). We remark that the number m of concentration points coincides with the maximal number k of bubbles U which may appear as limit profiles (for details see Proposition 3.3 and (4.33) in Proposition 4.6).
We stress that from (1.12) and point *(i)* in Theorem 1.1 we also deduce that the concentration m-tuple $(\bar{x}_1, \dots, \bar{x}_m) \in (\partial\Omega)^m$ is a critical point of the function $\varphi_m : (\partial\Omega)^m \to \mathbb{R}$

$$\varphi_m(x_1, \dots, x_m) := \sum_{i=1}^m H(x_i, x_i) + \sum_{i \neq h}^m G(x_i, x_h). \tag{1.15}$$

We point out that (1.14) and (ii)-Theorem 1.1 clearly imply that also conjecture (C3) holds true:

Corollary 1.2. *Let $(u_p)_p$ be a family of solutions to (1.1) satisfying (1.2). Then*

$$\lim_{p\to+\infty}\|u_p\|_\infty=\sqrt{e}. \tag{1.16}$$

It is worth to remark the interesting analogy between the results here obtained for the Neumann problem (1.1) and those known for the Lane-Emden equation under Dirichlet boundary condition

$$\begin{cases}\Delta u=|u|^{p-1}u & \text{in } \Omega\\ u>0 & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{1.17}$$

The asymptotic behavior as $p\to+\infty$ of families $(u_p)_p$ of solutions of (1.17), under the assumption that condition (1.2) holds, is well understood after the works [31, 32, 1, 14, 15, 17, 37], and the results established therein can be tought as the analogs of Theorem I and Theorem 1.1. In particular it is known that u_p stays uniformly bounded and that, up to subsequences, peaks-up as N points in the domain Ω ([14]). Furthermore, it is proved ([15, 17, 37]) that (1.16) holds and that the concentration appears at a critical point of the functional (1.15), now defined on Ω^m, where G and H are respectively Green's and Robin's functions of $-\Delta$ in Ω under Dirichlet boundary conditions. Moreover there is quantization of the energy, since

$$\lim_{n\to\infty}p_n\int_\Omega(|\nabla u_{p_n}|^2+u_{p_n}^2)\,dx=N\cdot 8\pi e,$$

and limit profiles are identified.

In this work we perform a blow-up analysis for the solutions of problem (1.1) following the approach developed in [15, 17] in the framework of the Lane-Emden Dirichlet problem (1.17). Of course one has to be very careful since now we have a boundary concentration phenomenon due to the Neumann boundary condition, while the concentration for problem (1.17) is in Ω.

We prove Theorem 1.1 by first performing an exhaustion method which provides a construction of concentration points. This approach relies on the energy bound assumption (1.2) and comes with pointwise estimates of the solutions and with the description of the local asymptotic profile U. Similar methods have been exploited for more general 2D Dirichlet problems (see [18, 14]), also in higher dimension (see for instance [34, 19]). We have adapted the construction to deal with the Neumann problem, taking advantage also of the results in [21], this part can be found in Section 3.

Afterwards, in Section 4, we refine the asymptotic analysis, showing that one can actually scale the solutions around local maxima and deriving the sharp constants and the energy quantization. These proofs rely on a detailed local blow-up analysis, in particular we use a local Pohozaev identity (see the proof of Lemma 4.5), pointwise estimates of the rescaled functions (see Lemma 4.7) and exploit the Green representation formula for the solutions to (1.1) (see the proof of Proposition 4.8). Finally, at the end of Section 4, we complete the proof of Theorem 1.1.

We have postponed to Appendix A some technical estimates used throughout the paper.

2. Notations

We list here some notations used throughout the paper. First the coordinates of a point will be denoted as follows: $x = (x^1, x^2) \in \mathbb{R}^2$.
Next we denote the open ball centered at a point $q = (q^1, q^2) \in \mathbb{R}^2$ and radius $r > 0$ as $B_r(q) := \{x \in \mathbb{R}^2 \; : \; |x - q| < r\}$. We also define the open half ball as

$$B_r^+(q) := B_r(q) \cap \{x \in \mathbb{R}^2 \; : \; x^2 > q^2\}, \tag{2.1}$$

its flat boundary as

$$D_r(q) := B_r(q) \cap \{x \in \mathbb{R}^2 \; : \; x^2 = q^2\} \tag{2.2}$$

and its curved boundary as

$$S_r(q) := \{x \in \mathbb{R}^2 \; : \; |x - q| = r, \quad x^2 > q^2\}. \tag{2.3}$$

Moreover $\operatorname{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$. We stress that C will be a positive constant which can change from line to line.

2.1. Change of coordinates which straightens out $\partial\Omega$ near a point on $\partial\Omega$.

We assume that $\partial\Omega \in C^2$. We fix a point on $\partial\Omega$ that we denote by $Q \in \partial\Omega$, in the following the change of coordinates defined below will be applied around the points in $S = \{\bar{x}_1, \ldots, \bar{x}_m\}$ (see Theorem 1.1) and around limit points of suitable special sequences (see Section 3).
It can be proved that there exist $R > 0$ and a C^2 function $\rho : \mathbb{R} \to \mathbb{R}$ such that, up to reordering the coordinates and reorienting the axis

$$\Omega \cap B_R(Q) = \{x = (x^1, x^2) \in B_R(Q) \; : \; x^2 > \rho(x^1)\}$$
$$\partial\Omega \cap B_R(Q) = \{x = (x^1, x^2) \in B_R(Q) \; : \; x^2 = \rho(x^1)\}.$$

Furthermore, up to a suitable translation of the axis we can assume that

$$Q = (0, 0)$$

so that

$$\rho(0) = 0$$

and, up to a suitable rotation of the axis, we can also assume that

$$\rho'(0) = 0.$$

We define the map

$$y = \Psi(x) \tag{2.4}$$

defined by

$$\begin{cases} y^1 = x^1 \\ y^2 = x^2 - \rho(x^1) \end{cases}$$

then Ψ is one-to-one and $det(J_\Psi) = 1$. Note that Ψ is a C^2 function which maps $\overline{\Omega} \cap B_R(Q)$ into a subset of the the half-plane, more precisely

$$\Omega_Q := \Psi(\Omega \cap B_R(Q)) \subset \{y = (y^1, y^2) \; : y^2 > 0\}$$
$$\partial^F \Omega_Q := \Psi(\partial\Omega \cap B_R(Q)) \subset \{y = (y^1, y^2) \; : y^2 = 0\}$$

and the point $Q = 0$ is mapped to the origin.

We define

$$\widetilde{u}_p(y) := u_p(\Psi^{-1}(y)), \quad \text{for all } y \in \Omega_Q \cup \partial^F \Omega_Q \tag{2.5}$$

then (see [21])

$$\begin{cases} \Delta \widetilde{u}_p - \widetilde{u}_p - 2\rho'(y^1)\frac{\partial^2 \widetilde{u}_p}{\partial y^1 \partial y^2} - \rho''(y^1)\frac{\partial \widetilde{u}_p}{\partial y^2} + (\rho'(y^1))^2\frac{\partial^2 \widetilde{u}_p}{\partial (y^2)^2} = 0 & \text{in } \Omega_Q \\ \frac{\partial \widetilde{u}_p}{\partial y^2}\left[-1 + \rho'(y^1) - (\rho'(y^1))^2\right] = \widetilde{u}_p^p & \text{in } \partial^F\Omega_Q \end{cases}$$

Let $x_p \in \overline{\Omega}$ be a family of points such that $Q = \lim_p x_p$ and $\mu_p := \left(pu_p(x_p)^{p-1}\right)^{-1} \to 0$. Hence for p large $x_p \in B_R(Q)$ and so the point

$$q_p := \Psi(x_p) \in \Omega_Q \cup \partial^F\Omega_Q$$

is well defined.
We scale $\widetilde{u}_p$ around q_p, setting

$$z_p(t) := \frac{p}{\widetilde{u}_p(q_p)}\left(\widetilde{u}_p(q_p + \mu_p t) - \widetilde{u}_p(q_p)\right), \qquad \text{for } t \in T_{Q,p} \cup \partial^F T_{Q,p}$$

where

$$T_{Q,p} := \{t = (t^1, t^2) \in \mathbb{R}^2 \ : \ q_p + \mu_p t \ \in \Omega_Q\} \tag{2.6}$$

$$\partial^F T_{Q,p} := \{t = (t^1, t^2) \in \mathbb{R}^2 \ : \ q_p + \mu_p t \ \in \partial^F\Omega_Q\} \tag{2.7}$$

Let us observe that we can choose $\bar{R} > 0$ such that

$$B_{\frac{\bar{R}}{\mu_p}}(0) \cap \{t \ : \ t^2 > -\frac{q_p^2}{\mu_p}\} \subset T_{Q,p} \tag{2.8}$$

and

$$B_{\frac{\bar{R}}{\mu_p}}(0) \cap \{t \ : \ t^2 = -\frac{q_p^2}{\mu_p}\} \subset \partial^F T_{Q,p} \tag{2.9}$$

Indeed, let us fix $\bar{R} > 0$ so that

$$B^+_{2\bar{R}}(0) \subset \Omega_Q \qquad \left(\text{and } D_{2\bar{R}}(0) := B_{2\bar{R}}(0) \cap \{y^2 = 0\} \subset \partial^F\Omega_Q\right).$$

Since $q_p \to 0$ (because $x_p \to Q$) then, for p large, $|q_p| \leq \bar{R}/2$ hence

$$B_{\bar{R}}(q_p) \cap \{y^2 \geq 0\} \subset B_{2\bar{R}}(0)^+ \cup D_{2\bar{R}}(0).$$

(2.8) and (2.9) follow observing that

$$q_p + \mu_p t \in B_{\bar{R}}(q_p) \cap \{y^2 \geq 0\} \ \Leftrightarrow \ \begin{cases} |t| \leq \frac{\bar{R}}{\mu_p} \\ t^2 \geq -\frac{q_p^2}{\mu_p} \end{cases}.$$

The function z_p satisfies

$$\begin{cases} L_p z_p - (\mu_p)^2 z_p = p(\mu_p)^2 & \text{in } B_{\frac{\bar{R}}{\mu_p}}(0) \cap \{t \ : \ t^2 > -\frac{q_p^2}{\mu_p}\} \\ N_p z_p = \left(1 + \frac{z_p}{p}\right)^p & \text{in } B_{\frac{\bar{R}}{\mu_p}}(0) \cap \{t \ : \ t^2 = -\frac{q_p^2}{\mu_p}\} \end{cases}$$

where, since $\rho'(0) = 0$ and ρ'' is continuous:

$$L_p := \Delta - 2\rho'(q_p^1 + \mu_p t^1)\frac{\partial^2}{\partial t^1 \partial t^2} - \mu_p \rho''(q_p^1 + \mu_p t^1)\frac{\partial}{\partial t^2} + [\rho'(q_p^1 + \mu_p t^1)]^2 \frac{\partial^2}{\partial (t^2)^2} \underset{p \to +\infty}{\longrightarrow} \Delta$$

and

$$N_p := \left[-1 + \rho'(q_p^1 + \mu_p t^1) - [\rho'(q_p^1 + \mu_p t^1)]^2\right]\frac{\partial}{\partial t^2} \underset{p \to +\infty}{\longrightarrow} \frac{\partial}{\partial \nu}.$$

Thanks to these convergences one can restrict to consider the case when $\partial\Omega$ is flat near Q, since the same arguments adapt to the non-flat case (see for instance [21, 9]).

3. Exhaustion of concentration points

Given a family (u_p) of solutions of (1.1), for a sequence $p_n \to +\infty$ we define the *concentration set* $\widetilde{\mathcal{S}}$ of u_{p_n} as

$$\widetilde{\mathcal{S}} := \left\{x \in \overline{\Omega} \ : \ \exists\, (x_n)_n \in \overline{\Omega},\ x_n \to x,\ \text{with}\ \ p_n u_{p_n}^{p_n-1}(x_n) \to +\infty\right\} \subset \overline{\Omega}. \tag{3.1}$$

Clearly

$$\widetilde{\mathcal{S}} \subseteq \mathcal{S} = \{\bar{x}_1, \ldots, \bar{x}_m\}\ (\subset \partial\Omega), \tag{3.2}$$

where $\mathcal{S}$ is the blow-up set of the sequence $p_n u_{p_n}$ (see (1.7) for the definition) characterized in [21] (see Theorem I in the Introduction). Indeed by the definition of $\widetilde{\mathcal{S}}$, for any $x \in \widetilde{\mathcal{S}}$ there exists a sequence $x_n \in \overline{\Omega}$, $x_n \to x$ such that

$$p_n u_{p_n}^{p_n-1}(x_n) \to +\infty,$$

then clearly

$$p_n u_{p_n}(x_n) \to +\infty,$$

hence, by the definition of $\mathcal{S}$, $x \in \mathcal{S}$ and (3.2) is proved.
As a consequence, up to reordering the points in $\mathcal{S}$, there exists $N \le m$ such that

$$\widetilde{\mathcal{S}} = \{\bar{x}_1, \ldots, \bar{x}_N\}. \tag{3.3}$$

In this section we will prove the existence of a maximal number k of *concentrating sequences* x_n for the set $\widetilde{\mathcal{S}}$, satisfying specific properties, in particular we get pointwise estimates and a description of u_p close to the points of $\widetilde{\mathcal{S}}$. The main result is contained in Proposition 3.3 below.

We introduce some notation. For $l \in \mathbb{N} \setminus \{0\}$ families of points $(x_{i,p})_p \subset \overline{\Omega}$, $i = 1, \ldots, l$, such that

$$p u_p^{p-1}(x_{i,p}) \to +\infty \quad \text{as} \quad p \to +\infty, \tag{3.4}$$

we define the parameters

$$\mu_{i,p} := \left(p u_p^{p-1}(x_{i,p})\right)^{-1} \quad (\to 0, \quad \text{as } p \to +\infty), \tag{3.5}$$

and introduce the following properties:

$(\mathcal{P}_1^l)$ For any $i, j \in \{1, \ldots, l\}$, $i \neq j$,

$$\lim_{p \to +\infty} \frac{|x_{i,p} - x_{j,p}|}{\mu_{i,p}} = +\infty.$$

$(\mathcal{P}_2^l)$ For any $i \in \{1, \ldots, l\}$,

$$\lim_{p \to +\infty} \frac{dist(x_{i,p}, \partial\Omega)}{\mu_{i,p}} = 0.$$

$(\mathcal{P}_3^l)$ For any $i = 1, \ldots, l$, let $Q_i \in \partial\Omega$ be such that

$$Q_i := \lim_p x_{i,p},$$

let Ψ_i be the change of coordinates which straightens $\partial\Omega$ in a neighborhood of Q_i of radius $R_i > 0$, let $q_{i,p} = \Psi_i(x_{i,p})$ and let

$$z_{i,p}(t) := \frac{p}{u_p(x_{i,p})}\Big(u_p\big(\Psi_i^{-1}(q_{i,p} + \mu_{i,p} t)\big) - u_p(x_{i,p})\Big) \quad \text{for } t \in T_{i,p} \cup \partial^F T_{i,p}, \tag{3.6}$$

where $T_{i,p} := T_{Q_{i,p}}$, see (2.6)-(2.7) in Section 2 for the notations.
Then

$$T_{i,p} \cup \partial^F T_{i,p} \to \overline{\mathbb{R}^2_+} \qquad \text{and} \quad z_{i,p}(t) \longrightarrow U(t) \qquad \text{in } C^1_{loc}(\overline{\mathbb{R}^2_+}) \text{ as } p \to +\infty, \tag{3.7}$$

where U is the function in (1.5).

$(\mathcal{P}_4^l)$ There exists $C>0$ such that

$$pR_{l,p}(x)u_p^{p-1}(x)\leq C$$

for all $p>1$ and all $x\in\overline{\Omega}$. where $R_{l,p}$ is the function

$$R_{l,p}(x):=\min_{i=1,\dots,l}|x-x_{i,p}|,\ \forall x\in\overline{\Omega}. \tag{3.8}$$

Remark 3.1. *If we assume that there exist $l\in\mathbb{N}\setminus\{0\}$ families of points $(x_{i,p})_p\subset\overline{\Omega}$, $i=1,\dots,l$ which satisfies (3.4) and such that property $(\mathcal{P}_4^l)$ holds true, then it is clear that the concentration set defined in (3.1) reduces to*

$$\widetilde{\mathcal{S}}=\left\{\lim_{p\to+\infty}x_{i,p},\ i=1,\dots,l\right\}.$$

Lemma 3.2. *If there exists $l\in\mathbb{N}\setminus\{0\}$ such that the properties $(\mathcal{P}_1^l)$, $(\mathcal{P}_2^l)$ and $(\mathcal{P}_3^l)$ hold for families $(x_{i,p})_{i=1,\dots,l}$ of points satisfying (3.4), then*

$$p\int_\Omega(|\nabla u_p|^2+u_p^2)\,dx\geq 2\pi\sum_{i=1}^{l}\alpha_i^2+o_p(1)\quad as\ p\to+\infty,$$

where $\alpha_i:=\liminf_{p\to+\infty}u_p(x_{i,p})$ $(\geq 1,$ by (3.4)).

Proof. Let us fix $i\in\{1,\dots,l\}$. Since $\lim_{p\to+\infty}x_{i,p}=Q_i\in\partial\Omega$ and $\lim_{p\to+\infty}\mu_{i,p}=0$, then for any $R>0$ and p sufficiently large $B_{R\mu_{i,p}}(x_{i,p})\cap\partial\Omega$ is contained in a small ball centered at Q_i where we can straighten the boundary as in Subsection 2.1.
Let us assume w.l.o.g. that $Q_i=0$. We claim that

$$(B_{\frac{R\mu_{i,p}}{3}}(q_{i,p})\cap\{y^2=0\})\subset\Psi(B_{R\mu_{i,p}}(x_{i,p})\cap\partial\Omega), \tag{3.9}$$

where $q_{i,p}:=\Psi(x_{i,p})$. Given Ψ as in (2.4), it follows that Ψ^{-1} is a C^2 function in a neighbourhood of $(0,0)=\Psi(Q_i)$ furthermore $D\Psi^{-1}(0,0)=I$. Thus

$$\exists\,\delta>0\quad\text{such that}\quad\|D\Psi^{-1}(y)\|\leq 3\quad\forall\,y\in\overline{B_\delta^+(0,0)}. \tag{3.10}$$

Since $\lim_{p\to+\infty}q_{i,p}=\Psi(Q_i)=(0,0)$, then for p sufficiently large we have

$$(B_{\frac{R\mu_{i,p}}{3}}(q_{i,p})\cap\{y^2=0\})\subset\overline{B_\delta^+(0,0)}.$$

Let $y=(y^1,0)\in(B_{\frac{R\mu_{i,p}}{3}}(q_{i,p})\cap\{y^2=0\})$ then $y=\Psi(x)$ where

$$\begin{aligned}|x-x_{i,p}| &= |\Psi^{-1}(y)-\Psi^{-1}(q_{i,p})|\\ &\leq \sup_{\overline{B_\delta^+(0,0)}}\|D\Psi^{-1}\||y-q_{i,p}|\\ &\overset{(3.10)}{\leq} R\mu_{i,p}.\end{aligned}$$

This proves (3.9).

Let us write, for any $R>0$, recalling the definition of $\widetilde{u}_p$ in (2.5)

$$\begin{aligned} p\int_{B_{R\mu_{i,p}}(x_{i,p})\cap\partial\Omega} u_p^{p+1}\,d\sigma(x) &\geq p\int_{\Psi_i(B_{R\mu_{i,p}}(x_{i,p})\cap\partial\Omega)} \widetilde{u}_p^{p+1}(y)\,d\sigma(y)\\ &\overset{(3.9)}{\geq} p\int_{B_{\frac{R\mu_{i,p}}{3}}(q_{i,p})\cap\{y^2=0\}} \widetilde{u}_p^{p+1}(y)\,d\sigma(y)\\ &\geq p\mu_{i,p}\int_{B_{\frac{R}{3}}(0)\cap\{t^2=-\frac{q_{i,p}^2}{\mu_{i,p}}\}} \widetilde{u}_p^{p+1}(q_{i,p}+\mu_{i,p}t)\,dt^1\\ &\overset{(3.5)}{\geq} u_p(x_{i,p})^2\int_{B_{\frac{R}{3}}(0)\cap\{t^2=-\frac{q_{i,p}^2}{\mu_{i,p}}\}} \frac{\widetilde{u}_p^{p+1}(q_{i,p}+\mu_{i,p}t)}{u_p^{p+1}(x_{i,p})}\,dt^1\\ &\geq u_p(x_{i,p})^2\int_{B_{\frac{R}{3}}(0)\cap\{t^2=-\frac{q_{i,p}^2}{\mu_{i,p}}\}} \left(1+\frac{z_{i,p}(t)}{p}\right)^{p+1}dt^1. \quad (3.11)\end{aligned}$$

Thanks to $(\mathcal{P}_3^l)$, we have

$$\|z_{i,p}-U\|_{L^\infty(\overline{B_{R/3}^+})} = o_p(1) \quad \text{as } p\to+\infty. \tag{3.12}$$

Thus by (3.11), (3.12) and Fatou's lemma

$$\liminf_{p\to+\infty}\left(p\int_{B_{R\mu_{i,p}}(x_{i,p})\cap\partial\Omega} u_p^{p+1}\,d\sigma(x)\right) \geq \alpha_i^2\int_{B_{R/3}(0)\cap\{t^2=0\}} e^{U(t)}\,dt^1. \tag{3.13}$$

Moreover by virtue of $(\mathcal{P}_1^l)$ it is not hard to see that $B_{R\mu_{i,p}}(x_{i,p})\cap B_{R\mu_{j,p}}(x_{j,p})=\emptyset$ for all $i\neq j$. Hence, in particular, thanks to (3.13)

$$\liminf_{p\to+\infty}\left(p\int_{\partial\Omega} u_p^{p+1}\,d\sigma(x)\right) \geq \sum_{i=1}^{l}\left(\alpha_i^2\int_{B_{R/3}(0)\cap\{t^2=0\}} e^{U(t)}\,dt^1\right).$$

At last, since this holds for any $R>0$, we get

$$p\int_\Omega(|\nabla u_p|^2+u_p^2)\,dx = p\int_{\partial\Omega} u_p^{p+1}\,d\sigma(x) \geq \sum_{i=1}^{l}\alpha_i^2\int_{\partial\mathbb{R}^2_+} e^{U(t)}\,dt^1 + o(1) = 2\pi\sum_{i=1}^{l}\alpha_i^2 + o(1),$$

as $p\to+\infty$. $\square$

Using an exhaustion method, we establish the existence of a maximal number k of "bubbles" U appearing about the points of the boundary subset $\widetilde{\mathcal{S}}$.

Proposition 3.3. *Let (u_p) be a family of solutions to (1.1) and assume that (1.2) holds. Then after passing to a subsequence $p_n\to+\infty$ as $n\to+\infty$, there exist an integer $k\geq 1$ and k families of points (x_{i,p_n}) in $\overline{\Omega}$ $i=1,\dots,k$ such that $(\mathcal{P}_1^k)$, $(\mathcal{P}_2^k)$, $(\mathcal{P}_3^k)$ and $(\mathcal{P}_4^k)$ hold. Moreover given any family points x_{k+1,p_n}, it is impossible to extract a new sequence from the previous one such that $(\mathcal{P}_1^{k+1})$, $(\mathcal{P}_2^{k+1})$, $(\mathcal{P}_3^{k+1})$ and $(\mathcal{P}_4^{k+1})$ hold with the sequences (x_{i,p_n}), $i=1,\dots,k+1$. Furthermore, there exists $N\leq\min\{m,k\}$ (where m is the number of points of the set $\mathcal{S}$) such* *that, up to reordering the points $\bar{x}_i \in \mathcal{S}$, it holds*

$$\widetilde{\mathcal{S}} = \left\{ \lim_{n\to+\infty} x_{i,p_n},\ i = 1,\dots,k \right\} = \{\bar{x}_1,\dots,\bar{x}_N\}, \tag{3.14}$$

where $\widetilde{\mathcal{S}}$ is the concentration set defined in (3.1).

Remark 3.4. *The point x_{1,p_n} can be taken to be a maximum point of u_{p_n} in $\overline{\Omega}$, hence it belongs to $\partial\Omega$, see STEP 1 below. The other sequences x_{i,p_n}, $i = 2,\dots,k$ are instead in $\overline{\Omega}$.*
Observe also that the number N of distinct points in $\widetilde{\mathcal{S}}$ satisfies $N \le k$.

Proof. For simplicity throughtout the proof we will denote any sequence $p_n \to +\infty$ as $n \to +\infty$ simply by p.

STEP 1. We show that there exists a family $(x_{1,p})$ of points in Ω such that, after passing to a subsequence $(\mathcal{P}_2^1)$ and $(\mathcal{P}_3^1)$ hold.

Let us choose $x_{1,p}$ be a point in $\overline{\Omega}$ where u_p achieves its maximum. In [21] it has been proved that $x_{1,p} \in \partial\Omega$ and that it satisfies $(\mathcal{P}_3^1)$.

STEP 2. We assume that $(\mathcal{P}_1^n)$, $(\mathcal{P}_2^n)$ and $(\mathcal{P}_3^n)$ hold for some $n \in \mathbb{N}\setminus\{0\}$. Then we show that either $(\mathcal{P}_1^{n+1})$, $(\mathcal{P}_2^{n+1})$ and $(\mathcal{P}_3^{n+1})$ hold or $(\mathcal{P}_4^n)$ holds, namely there exists $C > 0$ such that

$$pR_{n,p}(x)u_p^{p-1}(x) \le C$$

for all $x \in \Omega$, with $R_{n,p}$ defined as in (3.8).

Let $n \in \mathbb{N}\setminus\{0\}$ and assume that $(\mathcal{P}_1^n)$, $(\mathcal{P}_2^n)$ and $(\mathcal{P}_3^n)$ hold while

$$\sup_{x\in\overline{\Omega}} \left(pR_{n,p}(x)u_p^{p-1}(x) \right) \to +\infty \quad \text{as } p \to +\infty. \tag{3.15}$$

We let $x_{n+1,p} \in \overline{\Omega}$ be such that

$$pR_{n,p}(x_{n+1,p})u_p^{p-1}(x_{n+1,p}) = \sup_{x\in\overline{\Omega}} \left(pR_{n,p}(x)u_p^{p-1}(x) \right). \tag{3.16}$$

By (3.15), (3.16) and since Ω is bounded it is clear that

$$pu_p^{p-1}(x_{n+1,p}) \to +\infty \quad \text{as } p \to +\infty$$

and

$$\liminf_{p\to+\infty} u_p(x_{n+1,p}) \ge 1. \tag{3.17}$$

We will prove that $(\mathcal{P}_1^{n+1})$, $(\mathcal{P}_2^{n+1})$ and $(\mathcal{P}_3^{n+1})$ hold with the added sequence $(x_{n+1,p})$.

Proof of $(\mathcal{P}_1^{n+1})$.

We first claim that

$$\frac{|x_{i,p} - x_{n+1,p}|}{\mu_{i,p}} \to +\infty \quad \text{as } p \to +\infty \tag{3.18}$$

for all $i = 1,\dots,n$ and $\mu_{i,p}$ as in (3.5).
Let us assume by contradiction that there exists $i \in \{1,\dots,n\}$ such that $|x_{i,p} - x_{n+1,p}|/\mu_{i,p} \to R$ as $p \to +\infty$ for some $R \ge 0$. Then the points $x_{i,p}$ and $x_{n+1,p}$ are close to each other and by virtue of $(\mathcal{P}_2^n)$, they are very close to the boundary of Ω. Let us denote $q_{i,p} := \Psi_i(x_{i,p})$ and

$q_{n+1,p} := \Psi_i(x_{n+1,p})$ where Ψ_i is the function defined by (2.4) around the boundary point $Q_i := \lim_{p\to+\infty} x_{i,p}$. Since Ψ_i is C^2, $(|q_{i,p} - q_{n+1,p}|/\mu_{i,p})_p$ is bounded. Up to subsequence, $|q_{i,p} - q_{n+1,p}|/\mu_{i,p} \to R'$ as $p \to +\infty$ for some $R' \geq 0$. Thanks to $(\mathcal{P}_3^n)$, we get

$$\begin{aligned}
\lim_{p\to+\infty} p|x_{i,p} - x_{n+1,p}|u_p^{p-1}(x_{n+1,p}) &= \lim_{p\to+\infty} \frac{|x_{i,p} - x_{n+1,p}|}{\mu_{i,p}} \left(\frac{u_p(x_{n+1,p})}{u_p(x_{i,p})}\right)^{p-1} \\
&= \lim_{p\to+\infty} \frac{|x_{i,p} - x_{n+1,p}|}{\mu_{i,p}} \left(\frac{u_p(\Psi_i^{-1}(q_{n+1,p}))}{u_p(x_{i,p})}\right)^{p-1} \\
&= \lim_{p\to+\infty} \frac{|x_{i,p} - x_{n+1,p}|}{\mu_{i,p}} \left(1 + \frac{z_{i,p}(\mu_{i,p}^{-1}(q_{n+1,p} - q_{i,p}))}{p}\right)^{p-1} \\
&= \frac{4R}{(t^1)^2 + (t^2+2)^2} < +\infty \text{ (where } (t^1)^2 + (t^2)^2 = R'^2),
\end{aligned}$$

against (3.15) and (3.16), thus (3.18) holds.
Setting

$$\mu_{n+1,p} := \left[p u_p^{p-1}(x_{n+1,p})\right]^{-1} \to 0 \text{ as } p \to +\infty, \tag{3.19}$$

by (3.15) and (3.16) we deduce that

$$\frac{R_{n,p}(x_{n+1,p})}{\mu_{n+1,p}} \to +\infty \quad \text{as } p \to +\infty. \tag{3.20}$$

Then (3.18), (3.20) and $(\mathcal{P}_1^n)$ imply that $(\mathcal{P}_1^{n+1})$ holds with the added sequence $(x_{n+1,p})$.

Proof of $(\mathcal{P}_2^{n+1})$.
Let us prove that for any $S > 0$

$$\sup_{B_{S\mu_{n+1,p}}(x_{n+1,p})\cap\overline{\Omega}} \frac{u_p(x)}{u_p(x_{n+1,p})} \leq 1 + O\left(\frac{S\mu_{n+1,p}}{(p-1)R_{n,p}(x_{n+1,p})}\right). \tag{3.21}$$

Let $x \in B_{S\mu_{n+1,p}}(x_{n+1,p}) \cap \overline{\Omega}$, since $x_{n+1,p}$ satisfies (3.16),

$$R_{n,p}(x)u_p^{p-1}(x) \leq R_{n,p}(x_{n+1,p})u_p^{p-1}(x_{n+1,p}).$$

Furthermore $|x - x_{n+1,p}| \leq S\mu_{n+1,p}$, thus

$$\begin{aligned}
R_{n,p}(x) &\geq \min_{i=1,\dots,n} |x_{n+1,p} - x_{i,p}| - |x - x_{n+1,p}| \\
&\geq R_{n,p}(x_{n+1,p}) - S\mu_{n+1,p}.
\end{aligned}$$

Then, since for p large by (3.20), $R_{n,p}(x_{n+1,p}) - S\mu_{n+1,p} > 0$

$$\begin{aligned}
u_p^{p-1}(x) &\leq \frac{R_{n,p}(x_{n+1,p})}{R_{n,p}(x_{n+1,p}) - S\mu_{n+1,p}} u_p^{p-1}(x_{n+1,p}) \\
&\leq \frac{1}{1 - \frac{S}{R_{n,p}(x_{n+1,p})}\mu_{n+1,p}} u_p^{p-1}(x_{n+1,p}) \\
&\leq \left(1 + O\left(\frac{S\mu_{n+1,p}}{R_{n,p}(x_{n+1,p})}\right)\right) u_p^{p-1}(x_{n+1,p}),
\end{aligned}$$

thus (3.21) is proved.

Let us now introduce the rescaled function

$$v_{n+1,p}(t) := \frac{p}{u_p(x_{n+1,p})}[u_p(x_{n+1,p} + \mu_{n+1,p}t) - u_p(x_{n+1,p})], \forall t \in \widetilde{\Omega}_{n+1,p} := \mu_{n+1,p}^{-1}(\Omega - x_{n+1,p}). \tag{3.22}$$

Observe that by definition for $t \in \widetilde{\Omega}_{n+1,p} \cap B_S(0)$

$$v_{n+1,p}(t) = p\left(\frac{u_p(x)}{u_p(x_{n+1,p})} - 1\right), \tag{3.23}$$

where $x := x_{n+1,p} + \mu_{n+1,p}t \in \Omega \cap B_{S\mu_{n+1,p}}(x_{n+1,p})$, hence by (3.21) and (3.20) it follows that for any $S > 0$ one has

$$\limsup_{p\to+\infty} \sup_{\widetilde{\Omega}_{n+1,p}\cap B_S(0)} v_{n+1,p} \le 0. \tag{3.24}$$

Next to show $(\mathcal{P}_2^{n+1})$ we argue by contradiction assuming that $\lim_{p\to+\infty} dist(x_{n+1,p}, \partial\Omega)\mu_{n+1,p}^{-1} \neq 0$. Up to a subsequence two cases may occur:

(1) $dist(x_{n+1,p}, \partial\Omega)\mu_{n+1,p}^{-1} \longrightarrow L > 0$,
(2) $dist(x_{n+1,p}, \partial\Omega)\mu_{n+1,p}^{-1} \longrightarrow +\infty$.

Case (1). Let us start by the first case. We have $x_{n+1,p} \longrightarrow Q_{n+1} \in \partial\Omega$. We may assume without loss of generality that the unit outward normal to $\partial\Omega$ at Q_{n+1} is $-e^2$. For simplicity we will also assume that $\partial\Omega$ is flat near Q_{n+1}, we point out that all our arguments can be adapted to the non-flat case considering the change of coordinates which straightens out $\partial\Omega$ near Q_{n+1}, introduced in Section 2.1 (see for instance [9, Theorem 3]). The flatness assumption means that the function Ψ_{n+1} in (2.4) is the identity, namely that there exists $R := R_{n+1} > 0$ such that $\Omega \cap B_R^+(Q_{n+1}) = B_R^+(Q_{n+1})$ and $\partial\Omega \cap \partial B_R^+(Q_{n+1}) = \overline{D_R(Q_{n+1})}$.
In particular, for p large one has that $x_{n+1,p}^2 = dist(x_{n+1,p}, \partial\Omega)$, so that by assumption

$$\frac{x_{n+1,p}^2}{\mu_{n+1,p}} \longrightarrow L \tag{3.25}$$

as $p \to +\infty$.
Let us project $x_{n+1,p}$ on the boundary defining the point $\hat{x}_{n+1,p} := (x_{n+1,p}^1, 0) (\in \partial\Omega)$, and let us set

$$\begin{aligned} s_{n+1,p}(t) \quad &:= \quad v_{n+1,p}\Big(t^1, t^2 - \frac{x_{n+1,p}^2}{\mu_{n+1,p}}\Big), \qquad \forall t \in \widehat{\Omega}_{n+1,p} := \mu_{n+1,p}^{-1}(\Omega - \hat{x}_{n+1,p}) \\ &\overset{(3.22)}{=} \quad \frac{p}{u_p(x_{n+1,p})}[u_p(\hat{x}_{n+1,p} + \mu_{n+1,p}t) - u_p(x_{n+1,p})]. \end{aligned} \tag{3.26}$$

We can choose $\delta > 0$ such that $B_\delta^+(\hat{x}_{n+1,p}) \subset B_R^+(Q_{n+1})$, hence

$$B^+_{\frac{\delta}{\mu_{n+1,p}}}(0) \subset \widehat{\Omega}_{n+1,p} \quad \text{ and } \quad D_{\frac{\delta}{\mu_{n+1,p}}}(0) \subset \partial\widehat{\Omega}_{n+1,p}$$

and, by (1.1), the rescaled function $s_{n+1,p}$ solves the system

$$\begin{cases} -\Delta s_{n+1,p} + \mu_{n+1,p}^2 s_{n+1,p} = -\mu_{n+1,p}^2 p & \text{in } B^+_{\frac{\delta}{\mu_{n+1,p}}}(0), \\ \dfrac{\partial s_{n+1,p}}{\partial\nu} = \left(1 + \dfrac{s_{n+1,p}}{p}\right)^p & \text{on } D_{\frac{\delta}{\mu_{n+1,p}}}(0), \end{cases} \tag{3.27}$$

furthermore for any $\sigma > 0$, by (3.25), there exists $S > 0$ such that $B^+_\sigma(0) \subset B_S(0, \frac{x^2_{n+1,p}}{\mu_{n+1,p}}) \cap \widehat{\Omega}_{n+1,p}$, then

$$\begin{aligned}\limsup_{p\to+\infty} \sup_{B^+_\sigma(0)} s_{n+1,p}(t) &\leq \limsup_{p\to+\infty} \sup_{B_S(0,\frac{x^2_{n+1,p}}{\mu_{n+1,p}})\cap\widehat{\Omega}_{n+1,p}} v_{n+1,p}(t^1, t^2 - \frac{x^2_{n+1,p}}{\mu_{n+1,p}}) \\ &\leq \limsup_{p\to+\infty} \sup_{B_S(0)\cap\widetilde{\Omega}_{n+1,p}} v_{n+1,p}(t) \overset{(3.24)}{\leq} 0. \end{aligned} \tag{3.28}$$

Arguing similarly as in the proof of [9, Lemma 2], we will prove that for any $r > \frac{L}{3}$ there exist $C > 0$, $p_r > 1$ and $\alpha \in (0,1)$ such that

$$\|s_{n+1,p}\|_{C^{1,\alpha}(B^+_r(0))} \leq C, \qquad \text{for any } p > p_r. \tag{3.29}$$

We first observe that for any fixed $q \geq 2$ and for p sufficiently large

$$\int_{B^+_{4r}(0)} |p\mu^2_{n+1,p}|^q \, dx = o_p(1) \tag{3.30}$$

and

$$\begin{aligned}\int_{D_{4r}(0)} \left|1 + \frac{s_{n+1,p}(t)}{p}\right|^{pq} d\sigma &= \frac{1}{\mu_{n+1,p}} \int_{D_{4r\mu_{n+1,p}}(\hat{x}_{n+1,p})} \left(\frac{u_p(x)}{u_p(x_{n+1,p})}\right)^{pq} d\sigma \\ &\leq \frac{p}{u_p^2(x_{n+1,p})} \sup_{x\in D_{4r\mu_{n+1,p}}(\hat{x}_{n+1,p})} \left(\frac{u_p(x)}{u_p(x_{n+1,p})}\right)^{p(q-1)-1} \int_{\partial\Omega} u_p^{p+1}(x)\, d\sigma \\ &\leq C, \end{aligned} \tag{3.31}$$

where in the last inequality we used (3.17), $D_{4r\mu_{n+1,p}}(\hat{x}_{n+1,p}) \subset B_{cr\mu_{n+1,p}}(x_{n+1,p}) \cap \bar{\Omega}$ for some constant $c > 0$ (being $r > \frac{L}{3}$), (3.21), (3.20) and the energy bound (1.2), since for a solution u_p one has $\int_\Omega (|\nabla u_p|^2 + u_p^2)\, dx = \int_{\partial\Omega} u_p^{p+1} d\sigma$.
Let us now consider the solution w_p to

$$\begin{cases} -\Delta w_p + \mu^2_{n+1,p} w_p = -p\mu^2_{n+1,p} & \text{in } B^+_{4r}(0), \\ \dfrac{\partial w_p}{\partial \nu} = \left(1 + \dfrac{s_{n+1,p}}{p}\right)^p & \text{on } D_{4r}(0), \\ w_p = 0 & \text{on } S_{4r}(0). \end{cases} \tag{3.32}$$

By (3.30) and (3.31), with $q = 2$, the existence of such $w_p \in H^1(B^+_{4r}(0))$ is guaranteed by Lax-Milgram. Furthermore arguing as in [35, Theorem 5.3], we have by (3.30) and (3.31), that $w_p \in W^{\frac{1}{2}+t,q}(B^+_{4r}(0))$ with the uniform bound

$$\|w_p\|_{W^{\frac{1}{2}+t,q}(B^+_{4r}(0))} \leq C\left(\|\mu^2_{n+1,p} p\|_{L^q(B^+_{4r}(0))} + \left\| \left(1 + \frac{s_{n+1,p}}{p}\right)^p \right\|_{L^q(D_{4r}(0))} \right) \leq C, \tag{3.33}$$

for $q > 4$ and $0 < t < 2/q$.
In particular, by Sobolev embeddings, $\|w_p\|_{L^\infty(B^+_{4r}(0))} \leq C$, so we can define the function

$$\varphi_p := w_p - s_{n+1,p} + \|w_p\|_{L^\infty(B^+_{4r}(0))} + 1,$$

which solves

$$\begin{cases} -\Delta \varphi_p + \mu^2_{n+1,p} \varphi_p = \mu^2_{n+1,p}(\|w_p\|_{L^\infty(B^+_{4r})} + 1) & \text{in } B^+_{4r}(0), \\ \frac{\partial \varphi_p}{\partial \nu} = 0 & \text{on } D_{4r}(0), \end{cases}$$

furthermore, since $s_{n+1,p} \leq 1$ in $B^+_{4r}(0)$ for p sufficiently large by (3.28), then

$$\varphi_p \geq 0 \qquad\qquad \text{in } B^+_{4r}(0).$$

We define, for $t = (t^1, t^2) \in B_{4r}(0)$, the function

$$\hat{\varphi}_p(t) = \begin{cases} \varphi_p(t) & \text{if } t^2 \geq 0 \\ \varphi_p(t^1, -t^2) & \text{if } t^2 < 0, \end{cases}$$

which turns out to be a non-negative weak solution to

$$-\Delta\varphi + \mu^2_{n+1,p}\varphi = \mu^2_{n+1,p}(\|w_p\|_{L^\infty(B^+_{4r}(0))} + 1) \qquad \text{in } B_{4r}(0).$$

Therefore by the Harnack inequality we get for every $a \geq 1$

$$\begin{aligned} \left(\fint_{B_{3r}(0)} \hat{\varphi}^a_p \right)^{\frac{1}{a}} &\leq C \left(\inf_{B_{3r}(0)} \hat{\varphi}_p + \|\mu^2_{n+1,p}(\|w_p\|_{L^\infty(B^+_{4r}(0))} + 1)\|_{L^2(B_{4r}(0))} \right) \\ &\overset{3r>L\,+\,(3.25)}{\leq} C \left(\varphi_p(0, \frac{x^2_{n+1,p}}{\mu_{n+1,p}}) + \mu^2_{n+1,p}C \right) \\ &\leq C \left(2\|w_p\|_{L^\infty(B^+_{4r}(0))} + 1 + \mu^2_{n+1,p}C \right) \\ &\leq C, \end{aligned}$$

where we have used that $s_{n+1,p}(0, \frac{x^2_{n+1,p}}{\mu_{n+1,p}}) = 0$ and that $\|w_p\|_{L^\infty(B^+_{4r}(0))} \leq C$. Then

$$\|\varphi_p\|_{L^a(B_{3r}(0))} \leq C|B_{3r}(0)|^{\frac{1}{a}} \leq C \quad \text{for any } p > p_r \text{ and for any } a > 1.$$

Finally by interior elliptic regularity

$$\|\hat{\varphi}_p\|_{W^{2,q}(B_{2r}(0))} \leq C \left(\|\mu^2_{n+1,p}(\|w_p\|_{L^\infty(B^+_{4r}(0))} + 1)\|_{L^q(B_{3r}(0))} + \|\hat{\varphi}_p\|_{L^q(B_{3r}(0))} \right) \leq C. \tag{3.34}$$

Being $s_{n+1,p} = w_p + \|w_p\|_{L^\infty(B^+_{4r}(0))} + 1 - \varphi_p$, combining (3.33) and (3.34) we obtain

$$\|s_{n+1,p}\|_{W^{\frac{1}{2}+t,q}(B^+_{2r}(0))} \leq C \quad \text{for } q > 4,\ 0 < t < \frac{2}{q} \text{ and } p > p_r.$$

At last by the Morrey embedding theorem we get that

$$\|s_{n+1,p}\|_{C^{0,\alpha}(B^+_{2r}(0))} \leq C \quad \text{for some } \alpha > 0$$

and in turn, by Schauder estimates for the Neumann problem, we get

$$\|s_{n+1,p}\|_{C^{1,\alpha}(B^+_r(0))} \leq C \left(\| - \mu^2_{n+1,p}p\|_{L^\infty(B^+_{2r}(0))} + \|(1 + \frac{s_{n+1,p}}{p})^p\|_{C^{0,\alpha}(D_{2r}(0))} + \|s_{n+1,p}\|_{C^{0,\alpha}(B^+_{2r}(0))} \right) \leq C$$

for any $p > p_r$, so (3.29) holds true.

By (3.29) and the regularity theory of elliptic equations, we derive that, up to a subsequence,

$$s_{n+1,p} \to \tilde{U} \text{ in } C^1_{loc}(\overline{\mathbb{R}^2_+}) \text{ as } p \to \infty, \tag{3.35}$$

where, by (3.27) and (3.28), $\tilde{U}$ satisfies the following problem

$$\begin{cases} \Delta\tilde{U} = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial\tilde{U}}{\partial\nu} = e^{\tilde{U}} & \text{on } \partial\mathbb{R}^2_+ \\ \tilde{U} \leq 0 & \text{in } \mathbb{R}^2_+. \end{cases} \tag{3.36}$$

Furthermore $s_{n+1,p}(0, \frac{x^2_{n+1,p}}{\mu_{n+1,p}}) = 0$, for any p, then by (3.25) it follows that $\tilde{U}(0, L) = 0$.
Let us now prove that

$$\int_{\partial \mathbb{R}^2_+} e^{\tilde{U}} < \infty. \tag{3.37}$$

Let $R > 0$, then for any $|t^1| < R$ we have

$$(p+1)\left[\log\left(1 + \frac{s_{n+1,p}(t^1, 0)}{p}\right) - \frac{s_{n+1,p}(t^1,0)}{p+1}\right] \underset{p\to+\infty}{\longrightarrow} 0,$$

so we can use Fatou's Lemma and (3.35) to write

$$\begin{aligned}
\int_{-R}^{R} e^{\tilde{U}(t^1,0)} dt^1 \quad &\leq \quad \int_{-R}^{R} e^{s_{n+1,p}(t^1,0)+(p+1)\left[\log\left(1+\frac{s_{n+1,p}(t^1,0)}{p}\right)-\frac{s_{n+1,p}(t^1,0)}{p+1}\right]} dt^1 + o_p(1) \\
&\leq \quad \int_{B_R(0)\cap\{t^2=0\}} \left(1 + \frac{s_{n+1,p}(t)}{p}\right)^{p+1} dt^1 + o_p(1) \\
&\leq \quad \int_{B_R(0)\cap\{t^2=0\}} \frac{u_p^{p+1}(\hat{x}_{n+1,p} + \mu_{n+1,p} t)}{u_p^{p+1}(x_{n+1,p})} dt^1 + o_p(1) \\
&\leq \quad \mu_{n+1,p}^{-1} \int_{B_{R\mu_{n+1,p}}(\hat{x}_{n+1,p})\cap\{x^2=0\}} \frac{u_p^{p+1}(x)}{u_p^{p+1}(x_{n+1,p})} dx^1 + o_p(1) \\
&\leq \quad \frac{p}{u_p(x_{n+1,p})^2} \int_{\partial\Omega} u_p^{p+1}(x) d\sigma(x) + o_p(1) \\
&\overset{(3.17)}{\leq} \quad \frac{p}{(1-\varepsilon)^2} \int_{\partial\Omega} u_p^{p+1}(x) d\sigma(x) + o_p(1) \overset{(1.2)}{\leq} C < +\infty,
\end{aligned}$$

so that $e^{\tilde{U}} \in L^1(\partial\mathbb{R}^2)$.
Using now (3.36), (3.37) and the classification due to P. Liu [27] (see also [39]), we obtain

$$\tilde{U}(t^1, t^2) = \log \frac{2\eta_2}{(t^1 - \eta_1)^2 + (t^2 + \eta_2)^2} \quad \text{where } \eta_1 \in \mathbb{R} \text{ and } \eta_2 > 0. \tag{3.38}$$

Remark 3.5. *Notice that what we have proven up to now in Case* (1) *holds true also if*

$$dist(x_{n+1,p}, \partial\Omega)\mu_{n+1,p}^{-1} \longrightarrow L = 0,$$

in particular we get that the rescaled function s_{n+1} defined in (3.26) *converges to $\tilde{U}$ (see* (3.35)*), which in this case satisfies the conditions $\tilde{U}(0,0) = 0$ and $\tilde{U} \leq 0$, that imply that $\eta_1 = 0$ and $\eta_2 = 2$, namely that $\tilde{U} \equiv U$, where U is the function defined in* (1.5)*.*

We remark that $\tilde{U}$ is a radial and decreasing function with respect to the point $(\eta_1, -\eta_2)$. We have

$$\tilde{U}(\eta_1, 0) > \tilde{U}(0, L) = 0$$

which contradicts the fact that $\tilde{U} \leq 0$.
This rules out Case (1), namely the possibility that $dist(x_{n+1,p}, \partial\Omega)\mu_{n+1,p}^{-1} \longrightarrow L > 0$.

Case (2). In the sequel we will prove that the second case i.e. $dist(x_{n+1,p}, \partial\Omega)\mu_{n+1,p}^{-1} \longrightarrow +\infty$ can not occur too. Let $v_{n+1,p}$ be the rescaled function defined in (3.22). By (1.1), $v_{n+1,p}$ solves

$$-\Delta v_{n+1,p} + \mu_{n+1,p}^2 v_{n+1,p} = -\mu_{n+1,p}^2 p \quad \text{in } \widetilde{\Omega}_{n+1,p}.$$

Since for any $R>0$, $B_R(0)\subset\widetilde{\Omega}_{n+1,p}$ for p large enough, it follows that $\widetilde{\Omega}_{n+1,p}$ converges to the whole plane $\mathbb{R}^2$.
Furthermore, from (3.17), (3.19) and the uniform bound (1.8) we get

$$|\Delta v_{n+1,p}|\le|\mu^2_{n+1,p}v_{n+1,p}|+|\mu^2_{n+1,p}p|\le C \text{ in } \widetilde{\Omega}_{n+1,p}, \tag{3.39}$$

namely $v_{n+1,p}$ are functions with uniformly bounded laplacian in $B_R(0)$, moreover $v_{n+1,p}(0)=0$.
Let us decompose

$$v_{n+1,p}=\varphi_p+\psi_p \quad \text{ in } \widetilde{\Omega}_{n+1,p}\cap B_R(0),$$

with $-\Delta\varphi_p=-\Delta v_{n+1,p}$ in $\widetilde{\Omega}_{n+1,p}\cap B_R(0)$ and $\psi_p=v_{n+1,p}$ in $\partial(\widetilde{\Omega}_{n+1,p}\cap B_R(0))$. Using (3.39) by standard elliptic theory, we see that φ_p is uniformly bounded in $\widetilde{\Omega}_{n+1,p}\cap B_R(0)$. The function ψ_p is harmonic in $\widetilde{\Omega}_{n+1,p}\cap B_R(0)$ and bounded from above by (3.24). By the Harnack inequality, either ψ_p is uniformly bounded in $B_R(0)$, or it tends to $-\infty$ on each compact set of $B_R(0)$. The second alternative cannot happen because, by definition, $\psi_p(0)=v_{n+1,p}(0)-\varphi_p(0)=-\varphi_p(0)\ge -C$. Hence $v_{n+1,p}$ is uniformly bounded in $B_R(0)$ for all $R>0$. After passing to a subsequence, standard elliptic theory implies that $v_{n+1,p}$ is bounded in $C^2_{loc}(\mathbb{R}^2)$. Thus

$$v_{n+1,p}\to V \text{ in } C^1_{loc}(\mathbb{R}^2) \text{ as } p\to\infty, \tag{3.40}$$

where $V\in C^1(\mathbb{R}^2)$ is a harmonic function satisfying $V(0)=0$ and $V\le 0$ by (3.24).
So by a Liouville type theorem

$$V\equiv 0. \tag{3.41}$$

Let now $Q_{n+1}=\lim_{p\to\infty}x_{n+1,p}$. By (3.2) and (3.19) it follows that $Q_{n+1}\in\mathcal{S}\subset\partial\Omega$, where $\mathcal{S}$ is the concentration set in (1.7). In order to simplify the exposition, we will assume in the sequel that $\partial\Omega$ is flat near the point Q_{n+1}. This flatness assumption means that there exists $R_0>0$ such that

$$\Omega\cap B^+_{R_0}(Q_{n+1})=B^+_{R_0}(Q_{n+1}) \text{ and } D_{R_0}(Q_{n+1})\subset\partial\Omega.$$

We also assume that near Q_{n+1}, $\partial\Omega$ is contained in the hyperplane $x^2=0$ and the unit outward normal to $\partial\Omega$ at Q_{n+1} is $(-e^2)$ where e^2 is the second element of the canonical basis of $\mathbb{R}^2$.
W.l.o.g. we can also assume that

$$B_{R_0}(Q_{n+1})\cap\mathcal{S}=\{Q_{n+1}\}. \tag{3.42}$$

Now, inspired by Guo-Liu [24] (see page 750), we define the function

$$W_p(x)=\frac{p}{u_p^2(x_{n+1,p})}\int_{-s_0}^{s_0}u_p^{p+1}(x+(s,0))ds, \quad \forall x\in\overline{B^+_{2s_0}(Q_{n+1})},$$

where $0<s_0<R_0/4$. We have

$$\begin{aligned}\Delta_x\left(u^{p+1}(x+(s,0))\right) &= (p+1)u_p^p(x+(s,0))\,\Delta_x u_p(x+(s,0))\\ &\quad +(p+1)\,p\,u_p^{p-1}(x+(s,0))\,|\nabla_x u_p(x+(s,0))|^2\\ &\ge (p+1)\,u_p^p(x+(s,0))\,\Delta_x u_p(x+(s,0))\\ &= (p+1)\,u_p^{p+1}(x+(s,0))\\ &\ge 0 \quad \forall x\in B^+_{2s_0}(Q_{n+1})\cup S_{2s_0}(Q_{n+1}) \text{ and } \forall s\in[-s_0,s_0].\end{aligned}$$

Hence W_p is a subharmonic function in $B^+_{2s_0}(Q_{n+1})$.
By (1.13) and (3.42), for p large,

$$|u_p(y)|\le C_1\frac{1}{p}, \quad \forall\, y\in\overline{S_{2s_0}(Q_{n+1})},$$

thus

$$W_p \to 0 \quad \text{uniformly in } \overline{S_{2s_0}(Q_{n+1})}. \tag{3.43}$$

Furthermore for each $y \in D_{2s_0}$ we have

$$W_p(y) \leq \frac{p}{u_p^2(x_{n+1,p})} \int_{\partial\Omega} u_p^{p+1}(x)\, d\sigma(x) \overset{(1.2),(3.17)}{\leq} C_2. \tag{3.44}$$

Combining (3.43) with (3.44) and using the maximum principle we get

$$W_p(x) \leq C \text{ for all } x \in \overline{B^+_{2s_0}(Q_{n+1})}. \tag{3.45}$$

On the other hand, we have for $k > C$ and p sufficiently large

$$\begin{aligned}
W_p(x_{n+1,p}) &= \frac{p}{u_p^2(x_{n+1,p})} \int_{-s_0}^{s_0} u_p^{p+1}(x_{n+1,p} + (s,0))ds \\
&\overset{(3.19)}{\geq} \frac{p}{u_p^2(x_{n+1,p})} \int_{-k\mu_{n+1,p}}^{k\mu_{n+1,p}} u_p^{p+1}(x_{n+1,p} + (s,0))ds \\
&\geq \frac{p\ \mu_{n+1,p}}{u_p^2(x_{n+1,p})} \int_{-k}^{k} u_p^{p+1}(x_{n+1,p} + \mu_{n+1,p}(t,0))dt \\
&\geq \int_{-k}^{k} \left(\frac{u_p(x_{n+1,p} + \mu_{n+1,p}(t,0))}{u_p(x_{n+1,p})} \right)^{p+1} dt \\
&\geq \int_{-k}^{k} \left(1 + \frac{v_{n+1,p}(t,0)}{p} \right)^{p+1} dt \\
&\overset{(3.40)}{=} \int_{-k}^{k} e^{V(t,0)} dt + o(1) \overset{(3.41)}{=} 2k + o(1) > \tfrac{3}{2}C,
\end{aligned}$$

which is in contradiction with (3.45).
Hence, we have proved that Case (2) cannot occur.
So, up to a subsequence,

$$\lim_{p\to\infty} dist(x_{n+1,p}, \partial\Omega)\mu_{n+1,p}^{-1} = 0$$

and $(\mathcal{P}_2^{n+1})$ holds with the added points $(x_{n+1,p})$.

Proof of $(\mathcal{P}_3^{n+1})$.
Since $(\mathcal{P}_2^{n+1})$ holds, by Remark 3.5, assuming the flatness of $\partial\Omega$ near Q_{n+1}, we have that

$$s_{n+1,p} \to U \quad \text{in } C^1_{loc}(\overline{\mathbb{R}^2_+})$$

where U is the limit function in (1.5).
By the definition of $s_{n+1,p}$ (see (3.26)), since $x^2_{n+1,p}/\mu_{n+1,p} \to 0$, we conclude that also

$$v_{n+1,p} \to U \quad \text{in } C^1_{loc}(\overline{\mathbb{R}^2_+}),$$

where $v_{n+1,p}$ are the rescaled functions introduced in (3.22).
In the flat case this proves that $(\mathcal{P}_3^{n+1})$ holds with the added points $(x_{n+1,p})$, since the rescaled function $z_{n+1,p}$ in (3.6) coincide with the rescaled functions $v_{n+1,p}$. The non-flat case can be obtained in the same way (see [9, Theorem 3]), thus *STEP 2.* is proved.

STEP 3. We complete the proof of Proposition 3.3.

From *STEP 1.* we have that $(\mathcal{P}_1^1)$, $(\mathcal{P}_2^1)$ and $(\mathcal{P}_3^1)$ hold. Then, by *STEP 2.*, either $(\mathcal{P}_4^1)$ holds or $(\mathcal{P}_1^2)$, $(\mathcal{P}_2^2)$ and $(\mathcal{P}_3^2)$ hold. In the first case the assertion is proved with $k = 1$. In the second case we go on and proceed with the same alternative until we reach a number $k \in \mathbb{N} \setminus \{0\}$ for which $(\mathcal{P}_1^k)$, $(\mathcal{P}_2^k)$, $(\mathcal{P}_3^k)$ and $(\mathcal{P}_4^k)$ hold up to a sequence. Note that this is possible because the solutions u_p satisfy (1.2) and Lemma 3.2 holds and hence the maximal number k of families of points for which $(\mathcal{P}_1^k)$, $(\mathcal{P}_2^k)$, $(\mathcal{P}_3^k)$ hold must be finite.

Moreover, given any other family of points $x_{k+1,p}$, it is impossible to extract a new sequence from it such that $(\mathcal{P}_1^{k+1})$, $(\mathcal{P}_2^{k+1})$, $(\mathcal{P}_3^{k+1})$ and $(\mathcal{P}_4^{k+1})$ hold together with the points $(x_{i,p})_{i=1,..,k+1}$. Indeed if $(\mathcal{P}_1^{k+1})$ was verified then

$$\frac{|x_{k+1,p} - x_{i,p}|}{\mu_{k+1,p}} \to +\infty \quad \text{as } p \to +\infty, \quad \text{for any } i \in \{1, \dots, k\},$$

but this would contradict $(\mathcal{P}_4^k)$.

Finally let us recall that by Remark 3.1

$$\widetilde{\mathcal{S}} = \left\{ \lim_{p \to +\infty} x_{i,p},\ i = 1, \dots, k \right\}$$

hence (3.14) follows from (3.3).

□

4. Refined analysis

We know that the solutions of (1.1) satisfy Theorem I in the Introduction. In particular, for a sequence of positive solutions u_{p_n}, the blow-up set $\mathcal{S}$ of $p_n u_{p_n}$ is a discrete subset of $\partial\Omega$

$$\mathcal{S} = \{\bar{x}_1, \dots, \bar{x}_m\} \subset \partial\Omega$$

(see (1.7) for the definition of $\mathcal{S}$ and (1)-Theorem I for its characterization).
Moreover we have seen in Proposition 3.3 that, up to reordering the points $\bar{x}_i$, the concentration set $\widetilde{\mathcal{S}}$ of u_{p_n}, defined in (3.1), satisfies

$$\widetilde{\mathcal{S}} = \{\bar{x}_1, \dots, \bar{x}_N\}, \qquad \text{with } N \le \min\{m, k\},$$

where k is the maximal number of bubbles U in Proposition 3.3.

Thanks to the local analysis developed in the previous section, we can actually deduce the following.

Proposition 4.1.

$$\mathcal{S} = \widetilde{\mathcal{S}} \qquad \textit{and so in particular} \quad m = N \le k.$$

Proof. It is enough to show that $\mathcal{S} \subseteq \widetilde{\mathcal{S}}$, namely that $m \le N$. Let us assume by contradiction that $\bar{x}_m \in \mathcal{S} \setminus \widetilde{\mathcal{S}}$. Since $\bar{x}_m \in \mathcal{S}$, by definition there exists $x_n \in \overline{\Omega}$, $x_n \to \bar{x}_m$ such that $p_n u_{p_n}(x_n) \to +\infty$. Since $\bar{x}_m \notin \widetilde{\mathcal{S}}$ then there exist $r > 0$ such that $\overline{B_r(\bar{x}_m)} \cap \widetilde{\mathcal{S}} = \emptyset$ and that definitively $x_n \in \overline{B_r(\bar{x}_m)} \cap \overline{\Omega} =: K$. Next we show that $(\mathcal{P}_4^k)$, which holds by Proposition 3.3, implies that

$$\max_K p_n |u_{p_n}| \le C, \tag{4.1}$$

thus reaching a contradiction.
Let $\delta := dist(K, \widetilde{\mathcal{S}})/2 > 0$. For $y \in K$ we have

$$u_{p_n}(y) = \int_{\partial\Omega} G(x,y)\frac{\partial u_{p_n}}{\partial \nu}\ d\sigma(x) = \int_{\partial\Omega} G(x,y)u_{p_n}^{p_n}(x)\ d\sigma(x), \tag{4.2}$$

where $G(.,.)$ is the Green function satisfying (1.3). We split the integral over $\partial\Omega$ into two parts, the integral over $B_\delta(y) \cap \partial\Omega$ and the integral over $\partial\Omega \setminus B_\delta(y)$.
On the one hand, if $x \in B_\delta(y)$ we get $d(x, \widetilde{\mathcal{S}}) \geq \delta > 0$ and so $R_{k,p_n}(x) \geq c > 0$ for n large. From $(\mathcal{P}_4^k)$ we derive

$$u_{p_n}^{p_n-1}(x) \leq \frac{C}{p_n}, \quad \forall x \in B_\delta(y).$$

Hence

$$\int_{B_\delta(y)\cap\partial\Omega} G(x,y)u_{p_n}^{p_n}(x)\ d\sigma(x) \leq C\left(\frac{1}{p_n}\right)^{\frac{p_n}{p_n-1}} \int_{\partial\Omega} G(x,y)\ d\sigma(x) \leq \frac{C}{p_n} \tag{4.3}$$

where we have used the fact that $G(.,y)$ is integrable on $\partial\Omega$ which follows from Lemma A.1 in the Appendix.
On the other hand if $x \in \partial\Omega \setminus B_\delta(y)$ then $G(x,y) \leq C$ by using (A.4) since $|x-y| > \delta > 0$. Thus we get

$$\int_{\partial\Omega\setminus B_\delta(y)} G(x,y)u_{p_n}^{p_n}(x)dx \leq c_K \int_{\partial\Omega} u_{p_n}^{p_n}(x)dx \overset{(1.9)}{\leq} \frac{c_K}{p_n}. \tag{4.4}$$

Combining (4.2) with (4.3) and (4.4) we deduce (4.1).

□

We conclude the subsection with a result which will be of help in computing the constants c_i which appear in (1.13) and (1.12).

Lemma 4.2. *Let $p_n \to +\infty$, as $n \to +\infty$ and $c_i > 0$ be as in Theorem I, then*

$$c_i = \lim_{\delta\to 0}\lim_{n\to\infty} p_n \int_{B_\delta(\bar{x}_i)\cap\partial\Omega} u_{p_n}^{p_n}\, dx\ , \quad i = 1, \dots, m.$$

Proof. We retrace the proof of (1.13) in order to characterize the constants c_i. Let us observe that, since the points $\bar{x}_i$'s are isolated, there exists $\delta > 0$ such that $B_{2\delta}(\bar{x}_i) \cap B_{2\delta}(\bar{x}_j) = \emptyset$. Then by the Green representation formula for each $x \in \overline{\Omega} \setminus \mathcal{S}$ we have

$$\begin{aligned} p_n u_{p_n}(x) &= p\int_{\partial\Omega} G(x,y)u_{p_n}^{p_n}(y)\, d\sigma(y) \\ &= p_n \sum_{i=1}^m \int_{B_\delta(\bar{x}_i)\cap\partial\Omega} G(x,y)u_{p_n}^{p_n}(y)\, d\sigma(y) + p_n\int_{\partial\Omega\setminus\cup_i B_\delta(\bar{x}_i)} G(x,y)u_{p_n}^{p_n}(y)\, d\sigma(y) \\ &= p_n \sum_{i=1}^m \int_{B_\delta(\bar{x}_i)\cap\partial\Omega} G(x,y)u_{p_n}^{p_n}(y)\, d\sigma(y) + o_n(1), \end{aligned}$$

where in the last equality we have used that $p_n u_{p_n}$ is bounded on compact sets of $\overline{\Omega}\setminus\mathcal{S}$ and the fact that $\int_{\partial\Omega} G(x,y)\ d\sigma(y) \leq C$ (from Lemma A.1 in the Appendix).
Furthermore by the continuity of $G(x,\cdot)$ in $\overline{\Omega}\setminus\{x\}$ and by (1.9) we obtain, up to a subsequence, that

$$\lim_{n\to+\infty} p_n u_{p_n}(x) = \sum_{i=1}^m c_i G(x,\bar{x}_i), \qquad \text{where} \quad c_i := \lim_{\delta\to 0}\lim_{n\to\infty} p_n \int_{B_\delta(\bar{x}_i)\cap\partial\Omega} u_{p_n}^{p_n}\, dx.$$

$\square$

4.1. **Scaling around local maxima.**

Let $x_{i,p_n} \in \overline{\Omega}$, $i = 1, \dots, k$, be the maximal number of concentrating sequences in Proposition 3.3. W.l.o.g. we can relabel them and assume for the first m sequences that

$$x_{j,p_n} \to \bar{x}_j, \ \forall j = 1, \dots, m \quad \text{and} \quad \mathcal{S} = \{\bar{x}_1, \ \bar{x}_2, \ \dots, \bar{x}_m\} \tag{4.5}$$

In order to simplify the exposition, we will assume in the sequel that $\partial\Omega$ is flat near $\bar{x}_j$ for all $j = 1, \dots, m$. This flatness assumption means that there exists $R_j > 0$ such that

$$\Omega \cap B^+_{R_j}(\bar{x}_j) = B^+_{R_j}(\bar{x}_j) \text{ and } D_{R_j}(\bar{x}_j) \subset \partial\Omega, \text{ for all } j = 1, \dots, m, \tag{4.6}$$

and moreover $\Psi_j \equiv \mathrm{Id}$. Since the $\bar{x}_j$'s are distinct, it follows that there exists $r \in (0, \min_{j=1,\dots,m} R_j/4)$ such that

$$B^+_{4r}(\bar{x}_\ell) \cap B^+_{4r}(\bar{x}_j) = \emptyset, \ B^+_{4r}(\bar{x}_j) \subset \Omega, \text{ for all } \ell, j = 1, \dots, m, \ell \neq j. \tag{4.7}$$

Lemma 4.3. *Let $m \in \mathbb{N} \setminus \{0\}$ be as in (4.5) and let $r > 0$ be as in (4.7). Let us define $y_{j,n} \in \overline{B^+_{2r}(\bar{x}_j)}, j = 1, \dots, m$, such that*

$$u_{p_n}(y_{j,n}) := \max_{\overline{B^+_{2r}(\bar{x}_j)}} u_{p_n}(x). \tag{4.8}$$

Then, for any $j = 1, \dots, m$ and as $n \to \infty$:

(i)
$$\varepsilon_{j,n} := \left[p_n u_{p_n}^{p_n - 1}(y_{j,n})\right]^{-1} \longrightarrow 0. \tag{4.9}$$

(ii)
$$y_{j,n} \longrightarrow \bar{x}_j \quad \text{and} \quad y_{j,n} \in \partial\Omega \text{ for } n \text{ large}. \tag{4.10}$$

(iii)
$$\frac{|y_{i,n} - y_{j,n}|}{\varepsilon_{j,n}} \longrightarrow +\infty \text{ for any } i = 1, \dots, m, i \neq j.$$

(iv) Defining:
$$w_{j,n}(y) := \frac{p_n}{u_{p_n}(y_{j,n})}(u_{p_n}(y_{j,n} + \varepsilon_{j,n} y) - u_{p_n}(y_{j,n})), \quad y \in \Omega_{j,n} := \frac{\Omega - y_{j,n}}{\varepsilon_{j,n}}, \tag{4.11}$$

then
$$\lim_{n\to\infty} w_{j,n} = U \quad \text{in } C^1_{loc}(\mathbb{R}^2_+) \tag{4.12}$$

with U as in (1.5).

(v)
$$\liminf_{n\to\infty} \frac{p_n}{u_{p_n}(y_{j,n})} \int_{D_r(y_{j,n})} u_{p_n}^{p_n}(x)\, d\sigma(x) \geq 2\pi.$$

Remark 4.4. *(i) is the analogue of (3.4)-(3.5) for the families of points $y_{j,n}$, $j = 1, \dots, m$. (iii) and (iv) are respectively properties $(\mathcal{P}^m_1)$ and $(\mathcal{P}^m_3)$ introduced in Section 3. Moreover by (i) we get*

$$\liminf_{n\to\infty} u_{p_n}(y_{j,n}) \geq 1 \tag{4.13}$$

and by (ii) we also deduce property $(\mathcal{P}^m_2)$ and that for any $\delta \in (0, 2r)$ there exists $n_\delta \in \mathbb{N}$ such that

$$y_{j,n} \in D_\delta(\bar{x}_j), \quad \text{for } n \geq n_\delta. \tag{4.14}$$

Proof. (i): let $\bar{x}_j \in \mathcal{S} = \widetilde{\mathcal{S}}$ by Proposition 4.1 then there exists a sequence $x_n \to \bar{x}_j$ such that $p_n u_{p_n}^{p_n-1}(x_n) \to +\infty$ as $n \to \infty$. Hence $x_n \in B_r(\bar{x}_j)$ for n large and the assertion follows observing that by definition $u_{p_n}(y_{j,n}) \geq u_{p_n}(x_n)$.

(ii): Assume by contradiction that $y_{j,n}$ does not converge to $\bar{x}_j$, then up to a subsequence (that we still denote by $y_{j,n}$) $y_{j,n} \to \widetilde{x}$ such that $(2r \geq)\ |\bar{x}_j - \widetilde{x}| \geq \delta > 0$. But then by (1.13) in Theorem I

$$p_n u_{p_n}(y_{j,n}) = \sum_{j=1}^{m} c_j G(\widetilde{x}, \bar{x}_j) + o_n(1) = O(1). \tag{4.15}$$

Moreover, since $\bar{x}_j \in \mathcal{S}$, there exists a sequence $x_n \in \overline{\Omega}$ such that $x_n \to \bar{x}_j$ and $p_n u_{p_n}(x_n) \to +\infty$. Hence $x_n \in B_r(\bar{x}_j)$ for n large and by definition $u_{p_n}(y_{j,n}) \geq u_{p_n}(x_n)$, which is in contradiction with (4.15), as a consequence $y_{j,n} \to \bar{x}_j$.

Recall that $y_{j,n}$ satisfies (4.8) and $\Delta u_{p_n} = u_{p_n} > 0$ in $B_{2r}^+(\bar{x}_j)$. If by contradiction $y_{j,n} \in B_{2r}^+(\bar{x}_j)$, then $\Delta u_{p_n}(y_{j,n}) \leq 0$, which is impossible. Hence $y_{j,n} \in \partial B_{2r}^+(\bar{x}_j) = D_{2r}(\bar{x}_j) \cup S_{2r}(\bar{x}_j)$. Since $y_{j,n} \longrightarrow \bar{x}_j \in \partial\Omega$, we obtain $y_{j,n} \in D_{2r}(\bar{x}_j) \subset \partial\Omega$ for n large.

(iii): Just observing that by construction $|y_{i,p} - y_{j,p}| \geq 4r$ if $i \neq j$.

(iv): Observe that (ii) and (i) imply that for any $R > 0$ there exists $n_R \in \mathbb{N}$ such that

$$B_R^+(0) \subset B^+_{\frac{2r}{\varepsilon_{j,n}}}\left(\frac{\bar{x}_j - y_{j,p}}{\varepsilon_{j,n}}\right) \subset \Omega_{j,n} \quad \text{for} \quad n \geq n_R. \tag{4.16}$$

Indeed for n large $y_{j,n} \in D_r(\bar{x}_j)$ by (ii) and and $R\varepsilon_{j,n} < r$ by (i). As a consequence $B^+_{R\varepsilon_{j,n}}(y_{j,n}) \subset B_{2r}^+(\bar{x}_j) \subset \Omega$ for n large, which gives (4.16) by scaling back.

From (4.16) and the arbitrariness of R we deduce that the set $\Omega_{j,n} \to \mathbb{R}^2_+$ as $n \to \infty$. (4.12) is then obtained similarly as in the proof of Proposition 3.3-$(\mathcal{P}_3^{n+1})$.

(v): using (4.14) we have that $y_{j,n} \in D_r(\bar{x}_j)$ for large n and so $B_r^+(y_{j,n}) \subset B_{2r}^+(\bar{x}_j) \subset \Omega$ for n large, namely, by scaling

$$B^+_{\frac{r}{\varepsilon_{j,n}}}(0) \subset \Omega_{j,n}, \qquad \text{for } n \text{ large} \tag{4.17}$$

By scaling and passing to the limit as $n \to +\infty$, by (i), (iv) and Fatou's Lemma one has

$$\begin{aligned} \liminf_{n\to\infty} \frac{p_n}{u_{p_n}(y_{j,n})} \int_{D_r(y_{j,n})} u_{p_n}^{p_n}(x)\, d\sigma(x) &= \liminf_{n\to\infty} \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} \left(1 + \frac{w_{j,n}(y)}{p_n}\right)^{p_n} d\sigma(y) \\ &\geq \int_{\partial\mathbb{R}^2_+} e^{U(y)}\, d\sigma(y) = 2\pi \end{aligned}$$

which gives (v). $\square$

Lemma 4.5. *Let $r>0$ be as in (4.7) and $y_{j,n}$ for $j=1,\dots,m$ be the local maxima of u_{p_n} as in (4.8). Let us define*

$$\beta_{j,n} := \frac{p_n}{u_{p_n}(y_{j,n})} \int_{D_r(y_{j,n})} u_{p_n}^{p_n}(x)\, d\sigma(x), \qquad \textit{for } j=1,\dots,m. \tag{4.18}$$

Then

$$\lim_{n\longrightarrow+\infty} \beta_{j,n} = 2\pi. \tag{4.19}$$

Proof. Fix $j\in\{1,\dots,m\}$. By Lemma 4.3-(v) we already know that

$$\liminf_{n\to+\infty} \beta_{j,n} \geq 2\pi,$$

so we have to prove only the opposite inequality:

$$\limsup_{n\to+\infty} \beta_{j,n} \leq 2\pi. \tag{4.20}$$

For $\delta\in(0,r)$ by (4.7)

$$B_\delta^+(\bar{x}_j)\subset\Omega \tag{4.21}$$

and we define

$$\alpha_{j,n}(\delta) := \frac{p_n}{u_{p_n}(y_{j,n})} \int_{D_\delta(\bar{x}_j)} u_{p_n}^{p_n}(x)\, d\sigma(x). \tag{4.22}$$

In order to prove (4.20) it is sufficient to show that

$$\lim_{\delta\to0}\limsup_{n\to+\infty} \alpha_{j,n}(\delta) \leq 2\pi \tag{4.23}$$

since (4.20) will follow observing that

$$\beta_{j,n} = \alpha_{j,n}(\delta) + \frac{p_n}{u_{p_n}(y_{j,p_n})} \int_{D_r(y_{j,n})\setminus D_\delta(\bar{x}_j)} u_{p_n}^{p_n}(x)\, d\sigma(x) = \alpha_{j,n}(\delta) + o_n(1), \tag{4.24}$$

where the second term goes to zero as $n\to\infty$ because $y_{j,n}\in\overline{D_{2r}(\bar{x}_j)}$. Indeed $D_r(y_{j,n})\setminus D_\delta(\bar{x}_j)\subset D_{3r}(\bar{x}_j)\setminus D_\delta(\bar{x}_j)\subset\partial\Omega\setminus\mathcal{S}$ and we know that for any compact subset of $\partial\Omega\setminus\mathcal{S}$ the limit (1.13) holds and $\liminf_{n\to\infty} u_{p_n}(y_{j,n})\geq1$ by (4.13).

In the rest of the proof we show (4.23).
Let us consider the local Pohozaev identity for problem (1.1) in the set $B_\delta^+(\bar{x}_i)$:

$$\int_{B_\delta^+(\bar{x}_i)} u_{p_n}^2\, dx = \int_{\partial B_\delta^+(\bar{x}_i)} \frac12\langle x-\bar{x}_i,\nu\rangle(|\nabla u_{p_n}|^2+u_{p_n}^2) - \langle x-\bar{x}_i,\nabla u_{p_n}\rangle\frac{\partial u_{p_n}}{\partial\nu}\, d\sigma(x), \tag{4.25}$$

where ν is the outer unitary normal vector to $\partial B_\delta^+(\bar{x}_i)$ in x. Recalling that we have assumed that $\partial\Omega$ is flat near $\bar{x}_i$ (see (4.6)), then we have $\nu=-e^2$ on $D_\delta(\bar{x}_i)$, so that $\langle x-\bar{x}_i,\nu\rangle=0$ and $\langle x-\bar{x}_i,\nabla u\rangle=(x-\bar{x}_i)^1\frac{\partial u}{\partial x^1}$ for each $x\in D_\delta(\bar{x}_i)$. Furthermore on $S_\delta(\bar{x}_i)$ we have $\nu=\frac{x-\bar{x}_i}{\delta}$ and $\langle x-\bar{x}_i,\nabla u\rangle=\delta\frac{\partial u}{\partial\nu}$. Hence from (4.25) and by integrating by part we get

$$\begin{aligned}\frac{1}{p_n+1}\int_{D_\delta(\bar{x}_i)} u_{p_n}^{p_n+1}\, d\sigma &= \int_{B_\delta^+(\bar{x}_i)} u_{p_n}^2\, dx - \frac{\delta}{2}\int_{S_\delta(\bar{x}_i)}\left(|\nabla u_{p_n}|^2+u_{p_n}^2-2\left(\frac{\partial u_{p_n}}{\partial\nu}\right)^2\right)d\sigma\\ &\quad+\left[(x-\bar{x}_i)^1\frac{u_{p_n}^{p_n+1}}{p_n+1}\right]_{\bar{x}_i^1-\delta}^{\bar{x}_i^1+\delta}.\end{aligned}$$

Multiplying the last equation by p_n^2 we obtain

$$\begin{aligned}\frac{p_n^2}{p_n+1}\int_{D_\delta(\bar{x}_i)} u_{p_n}^{p_n+1}\,d\sigma \quad &\geq \quad -\frac{\delta}{2}\int_{S_\delta(\bar{x}_i)} |p_n\nabla u_{p_n}|^2\,d\sigma - \frac{\delta}{2}\int_{S_\delta(\bar{x}_i)} (p_n u_{p_n})^2\,d\sigma \\ &\quad + \delta\int_{S_\delta(\bar{x}_i)}\left(p_n\frac{\partial u_{p_n}}{\partial\nu}\right)^2 d\sigma + \ p_n^2\left[(x-\bar{x}_i)^1\frac{u_{p_n}^{p_n+1}}{p_n+1}\right]_{\bar{x}_i^1-\delta}^{\bar{x}_i^1+\delta} \\ &=: \quad T_1+T_2+T_3+T_4. \end{aligned} \tag{4.26}$$

Next we analyze the behavior of the four terms in the right hand side.

Recall that, by (1.11), $p_n u_{p_n} \to \sum_{j=1}^m c_j G(\cdot,\bar{x}_j)$ in $C^1_{loc}(\overline{B_r^+(\bar{x}_i)}\setminus\{\bar{x}_i\})$. Moreover, using Lemma A.3, for $\delta\in(0,r)$ we have

$$\sum_{j=1}^m c_j G(x,\bar{x}_j) = \frac{c_i}{\pi}\log\frac{1}{|x-\bar{x}_i|} + O(1)\ , \qquad \sum_{j=1}^m c_j\nabla G(x,\bar{x}_j) = \ -\frac{c_i}{\pi}\frac{x-\bar{x}_i}{|x-\bar{x}_i|^2} + O(1) \tag{4.27}$$

for each $x\in\overline{B_\delta^+(\bar{x}_i)}\setminus\{\bar{x}_i\}$. By the uniform convergence of the derivative of $p_n u_{p_n}$ on compact sets combined with (4.27), passing to the limit we have

$$T_1 = -\frac{\delta}{2}\int_{S_\delta(\bar{x}_i)}|p_n\nabla u_{p_n}|^2\,d\sigma \underset{n\to\infty}{\longrightarrow} -\frac{\delta}{2}\int_{S_\delta(\bar{x}_i)}\left(-\frac{c_i}{\pi}\frac{x-\bar{x}_i}{|x-\bar{x}_i|^2}+O(1)\right)^2 d\sigma(x) = -\frac{c_i^2}{2\pi}+O(\delta)$$

$$T_2 = -\frac{\delta}{2}\int_{S_\delta(\bar{x}_i)}(p_n u_{p_n})^2\,d\sigma \underset{n\to\infty}{\longrightarrow} -\frac{\delta}{2}\int_{S_\delta(\bar{x}_i)}\left(\frac{c_i}{\pi}\log\frac{1}{|x-\bar{x}_i|}+O(1)\right)^2 d\sigma(x) = O(\delta^2\log^2\delta)$$

$$T_3 = \delta\int_{S_\delta(\bar{x}_i)}\left(p_n\frac{\partial u_{p_n}}{\partial\nu}\right)^2 d\sigma \underset{n\to\infty}{\longrightarrow} \delta\int_{S_\delta(\bar{x}_i)}\left(-\frac{c_i}{\pi}\frac{\langle x-\bar{x}_i,\nu(x)\rangle}{|x-\bar{x}_i|^2}+O(1)\right)^2 d\sigma(x) = \frac{c_i^2}{\pi}+O(\delta)$$

and also

$$T_4 \leq \frac{2p_n}{p_n+1}\delta\|p_n u_{p_n}^{p_n+1}\|_{L^\infty(\partial D_\delta(\bar{x}_i))} \overset{(1.13)}{=} o_n(1)O(\delta).$$

So by (4.26) and recalling the definition of $\alpha_{i,n}$

$$\begin{aligned}\alpha_{i,n}(\delta)u_{p_n}(y_{i,n})^2 \quad &\overset{(4.22)}{=} \quad u_{p_n}(y_{i,n})\ p_n\int_{D_\delta(\bar{x}_i)} u_{p_n}^{p_n}(x)\,d\sigma(x) \geq p_n\int_{D_\delta(\bar{x}_i)} u_{p_n}^{p_n+1}(x)\,d\sigma(x) \\ &\overset{(4.26)}{\geq} \quad \frac{c_i^2}{2\pi}+O(\delta)+o_n(1), \end{aligned} \tag{4.28}$$

but by Lemma 4.2, (4.21) and (4.22)

$$c_i = \lim_{\delta\to0}\lim_{n\to\infty} p_n\int_{D_\delta(\bar{x}_i)} u_{p_n}^{p_n}\,dx = \lim_{\delta\to0}\lim_{n\to\infty}\alpha_{i,n}(\delta)u_{p_n}(y_{i,n}). \tag{4.29}$$

Combining (4.28) and (4.29) we get (4.23). □

Lemma 4.5 immediately implies the following result.

Proposition 4.6. *Let $r>0$ be as in (4.7) and let $y_{j,n}$, for $j=1,\dots,m$, be the local maxima of u_{p_n} as in (4.8), where m is the number of points in the concentration set $\mathcal{S}$. Let us consider a subsequence of p_n (which we still denote by p_n) such that*

$$m_j := \lim_{n\to\infty} u_{p_n}(y_{j,n}) = \lim_{r\to0}\lim_{n\to\infty}\|u_{p_n}\|_{L^\infty(\overline{B_{2r}(\bar{x}_j)\cap\Omega})} \tag{4.30}$$

is well defined for $j = 1, \ldots, m$. Then one has

$$c_j = 2\pi \cdot m_j; \tag{4.31}$$

$$\lim_{n\to\infty} p_n \int_\Omega \left(|\nabla u_{p_n}|^2 + u_{p_n}^2\right) \, dx = 2\pi \sum_{j=1}^m m_j^2; \tag{4.32}$$

and

$$m = k, \tag{4.33}$$

where c_j's are the constant in Theorem I and $k \in \mathbb{N} \setminus \{0\}$ is the maximal number of bubbles given by Proposition 3.3.

Proof. Observe that, by (1.8), m_j is well defined for a suitable subsequence of p_n and, furthermore $1 \le m_j < \infty$ for any $j = 1, \ldots, m$, by (4.13).

(4.31) follows from some argument already used in the proof of Lemma 4.5 (see (4.22), (4.24)), indeed we have

$$c_j = \lim_{\delta\to 0}\lim_{n\to\infty} p_n \int_{D_\delta(\bar{x}_j)} u_{p_n}^{p_n}(x)\, d\sigma(x) \overset{(4.22)}{=} \lim_{\delta\to 0}\lim_{n\to\infty} \alpha_{j,n}(\delta) u_{p_n}(y_{j,n}) \overset{(4.24)}{=} \lim_{n\to\infty} \beta_{j,n} u_{p_n}(y_{j,n}) = 2\pi \cdot m_j,$$

where the last equality follows from Lemma 4.5.

Next we prove (4.32). Observe that

$$\begin{aligned} p_n \int_\Omega \left(|\nabla u_{p_n}|^2 + u_{p_n}^2\right) \, dx &= p_n \int_{\partial\Omega} u_{p_n}^{p_n+1} \, d\sigma \\ &= \sum_{j=1}^m p_n \int_{D_r(\bar{x}_j)} u_{p_n}^{p_n+1} \, d\sigma + p_n \int_{\partial\Omega \setminus \cup_{j=1}^m D_r(\bar{x}_j)} u_{p_n}^{p_n+1} \, d\sigma \\ &\overset{(1.13)}{=} \sum_{j=1}^m p_n \int_{D_r(\bar{x}_j)} u_{p_n}^{p_n+1} \, d\sigma + o_n(1). \end{aligned} \tag{4.34}$$

Moreover

$$p_n \int_{D_r(\bar{x}_j)} u_{p_n}^{p_n+1} \, d\sigma = p_n \int_{D_{\frac{r}{2}}(y_{j,n})} u_{p_n}^{p_n+1} \, d\sigma + o_n(1), \tag{4.35}$$

since for n large enough $D_{\frac{r}{3}}(\bar{x}_j) \subset D_{\frac{r}{2}}(y_{j,n}) \subset D_r(\bar{x}_j)$ so that

$$p_n \int_{D_r(\bar{x}_j) \setminus D_{\frac{r}{2}}(y_{j,n})} u_{p_n}^{p_n+1} \, d\sigma \le p_n \int_{\{x \in D_r(\bar{x}_j),\ \frac{r}{3} < |x - \bar{x}_j| < r\}} u_{p_n}^{p_n+1} \, d\sigma \overset{(1.13)}{=} o_p(1).$$

Let us consider the remaining term in the right hand side of (4.35) and prove that

$$\lim_{n\to\infty} p_n \int_{D_{\frac{r}{2}}(y_{j,n})} u_{p_n}^{p_n+1} \, d\sigma = 2\pi \cdot m_j^2. \tag{4.36}$$

On the one side, since the families of points $y_{j,n}$, $j = 1, \ldots, m$, satisfy properties $(\mathcal{P}_1^m)$, $(\mathcal{P}_2^m)$ and $(\mathcal{P}_3^m)$ introduced in Section 3 (see Remark 4.4), similarly as in the proof of Lemma 3.2, using (4.30), we obtain that

$$\liminf_{n\to\infty} p_n \int_{D_{\frac{r}{2}}(y_{j,n})} u_{p_n}^{p_n+1} \, d\sigma \ge 2\pi \cdot m_j^2. \tag{4.37}$$

On the other side, since $B^+_{\frac{r}{2}}(y_{j,n}) \subset B^+_{2r}(\bar{x}_j) \subset \Omega$ for n large,

$$p_n \int_{D_{\frac{r}{2}}(y_{j,n})} u_{p_n}^{p_n+1}\, d\sigma \le u_{p_n}(y_{j,n})\, p_n \int_{D_r(y_{j,n})} u_{p_n}^{p_n}\, d\sigma \overset{(4.18)}{=} u_{p_n}(y_{j,n})^2 \beta_{j,n},$$

so Lemma 4.5 implies that

$$\limsup_{n\to\infty} p_n \int_{D_{\frac{r}{2}}(y_{j,n})} u_{p_n}^{p_n+1}\, d\sigma \le 2\pi \cdot m_j^2. \tag{4.38}$$

(4.36) then follows by combining (4.37) and (4.38). Finally (4.34), (4.35) and (4.36) imply (4.32).

Next we show that the points $y_{j,n},\ j = 1, \dots, m$, also satisfy property $(\mathcal{P}_4^m)$, namely that there exists $C > 0$ such that

$$p_n R_{m,p_n}(x) u_{p_n}^{p_n-1}(x) \le C \quad \forall x \in \overline{\Omega} \tag{4.39}$$

where $R_{m,p_n}(x) := \min_{j=1,\dots,m} |x - y_{j,n}|$. Arguing by contradiction we suppose that

$$\sup_{x\in\overline{\Omega}} \left(p_n R_{m,p_n}(x) u_{p_n}^{p_n-1}(x) \right) \to +\infty \quad \text{as } n \to +\infty$$

and let $y_{m+1,n} \in \overline{\Omega}$ be such that

$$p_n R_{m,p_n}(y_{m+1,n}) u_{p_n}^{p_n-1}(y_{m+1,n}) = \sup_{x\in\overline{\Omega}} \left(p_n R_{m,p_n}(x) u_{p_n}^{p_n-1}(x) \right). \tag{4.40}$$

By (4.40) and since Ω is bounded it is clear that

$$p_n u_{p_n}^{p_n-1}(y_{m+1,n}) \to +\infty \quad \text{as } n \to +\infty.$$

Taking the sequences of local maxima $y_{j,n}$ for $j = 1, \dots, m$ and the added sequence $y_{m+1,n}$, similarly as in the proof of Proposition 3.3, we then get that $(\mathcal{P}_1^{m+1})$, $(\mathcal{P}_2^{m+1})$ and $(\mathcal{P}_3^{m+1})$ hold. Applying now Lemma 3.2 for the families of points $(y_{i,n})_{i=1,\dots,m+1}$ and using (4.30) we obtain

$$p_n \int_\Omega \left(|\nabla u_{p_n}|^2 + u_{p_n}^2 \right) dx \ge 2\pi \sum_{i=1}^m m_i^2 + 2\pi m_{m+1}^2 + o_n(1) \overset{(4.13)}{\ge} 2\pi \sum_{i=1}^m m_i^2 + 2\pi + o_n(1) \text{ as } n \to +\infty,$$

thus

$$\lim_{n\to\infty} p_n \int_\Omega |\nabla u_{p_n}|^2 + u_{p_n}^2\, dx > 2\pi \sum_{i=1}^m m_i^2$$

which contradicts (4.32) concluding the proof of $(\mathcal{P}_4^m)$.
At last in order to derive (4.33), let us consider k families of points $x_{1,p_n}, x_{2,p_n}, \dots x_{k,p_n} \in \overline{\Omega}$ as in the statement of Proposition 3.3. By virtue of Proposition 4.1

$$\mathcal{S} = \{\bar{x}_1, \dots, \bar{x}_m\} = \{\lim_{n\to+\infty} x_{i,p_n} : i \in \{1, \dots, k\}\}.$$

Given $i \in \{1, \dots, k\}$, let $j \in \{1, \dots, m\}$ be such that $\lim_{n\to+\infty} x_{i,p_n} = \bar{x}_j$. Next, recalling that $\{y_{1,n}, y_{2,n}, \dots, y_{m,n}\}$ satisfy $(\mathcal{P}_4^m)$ and applying (4.39) at x_{i,p_n} we get

$$p|x_{i,p_n} - y_{j,n}| u_{p_n}^{p_n-1}(x_{i,p_n}) \overset{(4.7)+(4.8)}{=} p R_{m,p_n}(x_{i,p_n}) u_{p_n}^{p_n-1}(x_{i,p_n}) \le C.$$

So in particular, up to a subsequence

$$\left| \frac{y_{j,n} - x_{i,p_n}}{\mu_{i,p_n}} \right| \le C.$$

As a consequence, up to a subsequence, since $y_{j,n} \in \partial\Omega$ and x_{i,p_n} satisfies $(\mathcal{P}_2^k)$, there exists $\hat{t}_{i,j} \in \partial\mathbb{R}^2_+$ such that

$$t_{i,j,n} := \frac{y_{j,n} - x_{i,p_n}}{\mu_{i,p_n}} \to \hat{t}_{i,j}.$$

By (4.8), (3.6) and (3.7)

$$0 \leq \frac{p_n}{u_{p_n}(x_{i,p_n})}(u_{p_n}(y_{j,n}) - u_{p_n}(x_{i,p_n})) = z_{i,p_n}(t_{i,j,n}) \to U(\hat{t}_{i,j}) \leq 0.$$

Thus, by (1.5) $\hat{t}_{i,j} = 0$, then

$$\frac{|y_{j,n} - x_{i,p_n}|}{\mu_{i,p_n}} = o_n(1). \tag{4.41}$$

In conclusion, let us suppose by contradiction that $k > m$, then there exists

$$i, \ell \in \{1, \ldots, k\}, \quad i \neq \ell, \quad \text{such that} \quad \lim_{n\to+\infty} x_{i,p_n} = \lim_{n\to+\infty} x_{\ell,p_n} = \bar{x}_j \quad \text{for some } j \in \{1, \ldots, m\}.$$

In addition w.l.o.g. let us assume that up to a subsequence $\mu_{i,p_n} \geq \mu_{\ell,p_n}$.
By (4.41)

$$\begin{aligned} \frac{|x_{i,p_n} - x_{\ell,p_n}|}{\mu_{i,p_n}} &\leq \frac{|x_{i,p_n} - y_{j,n}|}{\mu_{i,p_n}} + \frac{|x_{\ell,p_n} - y_{j,n}|}{\mu_{i,p_n}} \\ &\leq \frac{|x_{i,p_n} - y_{j,n}|}{\mu_{i,p_n}} + \frac{|x_{\ell,p_n} - y_{j,n}|}{\mu_{\ell,p_n}} = o_n(1), \end{aligned}$$

which is a contradiction against property $(\mathcal{P}_1^k)$ for $x_{1,p_n}, x_{2,p_n}, \ldots x_{k,p_n}$. ☐

Next we give a decay estimate for the rescaled functions $w_{j,n}$ which will be fundamental to compute the constants m_i's.

Lemma 4.7. *For any $\gamma \in (0,2)$ there exists $R_\gamma > 1$ and $n_\gamma \in \mathbb{N}$ such that*

$$w_{j,n}(z) \leq (2-\gamma)\log\frac{1}{|z|} + \widetilde{C}_\gamma, \quad \forall j = 1, \ldots, k \tag{4.42}$$

for some $\widetilde{C}_\gamma > 0$ provided $R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{j,n}}$, $z \in D_{\frac{r}{\varepsilon_{j,n}}}(0)$ and $n \geq n_\gamma$.
As a consequence

$$0 \leq \left(1 + \frac{w_{j,n}(z)}{p_n}\right)^{p_n} \leq \begin{cases} 1 & \text{for } |z| \leq R_\gamma \\ C_\gamma \frac{1}{|z|^{2-\gamma}} & \text{for } R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{j,n}}. \end{cases} \tag{4.43}$$

Proof. Arguing similarly as in the proof of [15, Lemma 4.4] one can deduce a crucial pointwise estimate for $w_{j,n}$, namely it can be proved that for any $\varepsilon > 0$, there exist $R_\varepsilon > 1$ and $n_\varepsilon \in \mathbb{N}$ such that

$$w_{j,n}(y) \leq \left(\frac{\beta_{j,n}}{\pi} - \varepsilon\right)\log\frac{1}{|y|} + C_\varepsilon, \quad \forall j = 1, \ldots, m$$

for some $C_\varepsilon > 0$, provided $2R_\varepsilon \leq |y| \leq \frac{r}{\varepsilon_{j,n}}$, $y \in D_{\frac{r}{\varepsilon_{j,n}}}(0)$ and $n \geq n_\varepsilon$.
(4.42) then follows by Lemma 4.5. Finally (4.43) is a direct consequence of (4.42) (see for instance the proof of [17, Lemma 2.1] which can be easily adapted to this case). ☐

Proposition 4.8.

$$m_i = \sqrt{e}, \quad \forall i = 1, \ldots, m.$$

Proof. From (1.9)

$$c \le p_n \int_{\partial\Omega} u_{p_n}^{p_n}(x) d\sigma(x) \le C$$

hence, by the properties of the Green function G,

$$\begin{aligned} \int_{\partial\Omega\setminus D_{2r}(\bar{x}_j)} G(y_{j,n},x) u_{p_n}^{p_n}(x)\, d\sigma(x) &\le C_r \int_{\partial\Omega\setminus D_{2r}(\bar{x}_j)} u_{p_n}^{p_n}(x)\, d\sigma(x) \\ &\le C_r \int_{\partial\Omega} u_{p_n}^{p_n}(x)\, d\sigma(x) = O(\frac{1}{p_n}) \end{aligned} \tag{4.44}$$

and similarly, observing that (4.10) implies that for n large enough the points $y_{j,n} \in D_{r/2}(\bar{x}_j)$ and that $D_{r/2}(\bar{x}_j) \subset D_r(y_{j,n}) \subset D_{2r}(\bar{x}_j)$, also

$$\begin{aligned} \int_{D_{2r}(\bar{x}_j)\setminus D_r(y_{j,n})} G(y_{j,n},x) u_{p_n}^{p_n}(x)\, d\sigma(x) &\le \int_{\{x\in D_{2r}(\bar{x}_j),\ \frac{r}{2}<|x-\bar{x}_j|<2r\}} G(y_{j,n},x) u_{p_n}^{p_n}(x)\, d\sigma(x) \\ &\le C_{\frac{r}{2}} \int_{\partial\Omega} u_{p_n}^{p_n}(x)\, d\sigma(x) = O(\frac{1}{p_n}). \end{aligned} \tag{4.45}$$

Using the previous estimates and the Green representation formula, we then get

$$\begin{aligned} u_{p_n}(y_{j,n}) &= \int_{\partial\Omega} G(y_{j,n},x) u_{p_n}^{p_n}(x)\, d\sigma(x) \\ &= \int_{D_{2r}(\bar{x}_j)} G(y_{j,n},x) u_{p_n}^{p_n}(x)\, d\sigma(x) + \int_{\partial\Omega\setminus D_{2r}(\bar{x}_j)} G(y_{j,n},x) u_{p_n}^{p_n}(x)\, d\sigma(x) \\ &\overset{(4.44)-(4.45)}{=} \int_{D_r(y_{j,n})} G(y_{j,n},x) u_{p_n}^{p_n}(x)\, d\sigma(x) + o_n(1) \\ &\overset{(4.11)}{=} \frac{u_{p_n}(y_{j,n})}{p_n} \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} G(y_{j,n}, y_{j,n}+\varepsilon_{j,n}z) \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) + o_n(1) \\ &\overset{(1.4)}{=} \frac{u_{p_n}(y_{j,n})}{p_n} \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} H(y_{j,n}, y_{j,n}+\varepsilon_{j,n}z) \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) \\ &\quad -\frac{u_{p_n}(y_{j,n})}{\pi p_n} \int_{D_{\frac{r}{\varepsilon_{j,p}}}(0)} \log|z| \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p} d\sigma(z) \\ &\quad -\frac{u_{p_n}(y_{j,n})\log(\varepsilon_{j,n})}{\pi p_n} \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) + o_n(1) \\ &= A_n + B_n + C_n + o_n(1). \end{aligned} \tag{4.46}$$

Since H satisfies (A.3) in the Appendix, by (4.9) and (4.10) we get

$$\lim_{n\to+\infty} H(y_{j,n}, y_{j,n}+\varepsilon_{j,n}z) = H(\bar{x}_j,\bar{x}_j), \quad \text{for any } z \in \partial\mathbb{R}^2_+,$$

so by (4.30), the convergence (4.12) and the uniform bounds in (4.43) we can apply the dominated convergence theorem, and since the function $z \mapsto 1/|z|^{2-\gamma}$ is integrable in $\{z \in \partial\mathbb{R}^2_+,\ |z| > R_\gamma\}$

choosing $\gamma \in (0,1)$ we deduce

$$\begin{aligned}\lim_{p\to+\infty} \quad & u_{p_n}(y_{j,n}) \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} H(y_{j,n}, y_{j,n}+\varepsilon_{j,n}z) \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) \\ & \overset{(4.12)}{=} m_j H(\bar{x}_j,\bar{x}_j) \int_{\partial\mathbb{R}^2_+} e^{U(z)}\, d\sigma(z) \overset{(1.6)}{=} 2\pi m_j H(\bar{x}_j,\bar{x}_j),\end{aligned}$$

from which

$$A_n := \frac{u_{p_n}(y_{j,n})}{p_n} \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} H(y_{j,n}, y_{j,n}+\varepsilon_{j,n}z) \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) = o_n(1). \tag{4.47}$$

For the second term in (4.46) we apply again the dominated convergence theorem, using (4.43) and observing now that the function $z \mapsto \log|z|/|z|^{2-\gamma}$ is integrable in $\{z \in \partial\mathbb{R}^2_+,\ |z| > R_\gamma\}$ and that $z \mapsto \log|z|$ is integrable in $\{z \in \partial\mathbb{R}^2_+,\ |z| \le R_\gamma\}$. Hence we get

$$\lim_{n\to+\infty} u_{p_n}(y_{j,n}) \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} \log|z| \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) = m_j \int_{\partial\mathbb{R}^2_+} \log|z| e^{U(z)}\, d\sigma(z) < +\infty$$

and this implies that

$$B_n := -\frac{u_{p_n}(y_{j,n})}{\pi p_n} \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} \log|z| \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) = o_n(1). \tag{4.48}$$

Finally for the last term in (4.46) let us observe that by the definition of $\varepsilon_{j,n}$ in (4.9)

$$\log\varepsilon_{j,n} = -(p_n-1)\log u_{p_n}(y_{j,n}) - \log p_n, \tag{4.49}$$

again by the dominated convergence theorem it follows

$$\begin{aligned} C_n \quad := \quad & -\frac{u_{p_n}(y_{j,n})\log(\varepsilon_{p,n})}{\pi p_n} \int_{D_{\frac{r}{\varepsilon_{j,n}}}(0)} \left(1+\frac{w_{j,n}(z)}{p_n}\right)^{p_n} d\sigma(z) \\ = \quad & -\frac{u_{p_n}(y_{j,n})\log(\varepsilon_{j,n})}{\pi p_n} \left(\int_{\partial\mathbb{R}^2_+} e^{U(z)}\, d\sigma(z) + o_n(1)\right) \\ = \quad & -\frac{u_{p_n}(y_{j,n})\log(\varepsilon_{j,n})}{\pi p_n}(2\pi + o_n(1)) \\ \overset{(4.49)}{=} \quad & u_{p_n}(y_{j,n}) \left[\frac{p_n-1}{p_n}\log u_{p_n}(y_{j,n}) + \frac{\log p_n}{p_n}\right](2+o_n(1)). \end{aligned} \tag{4.50}$$

Substituting (4.47), (4.48) and (4.50) into (4.46) we get

$$u_{p_n}(y_{j,n}) = u_{p_n}(y_{j,n}) \left[\frac{p_n-1}{p_n}\log u_{p_n}(y_{j,n}) + \frac{\log p_n}{p_n}\right](2+o_n(1)) + o_n(1),$$

passing to the limit as $n \to +\infty$ and using (4.30) we conclude that

$$\log m_j = \frac{1}{2}.$$

□

4.2. **The proof of Theorem 1.1.**

The statements of Theorem 1.1 have been proved in the various propositions obtained so far. In particular *(i)* is a consequence of Lemma 4.2, (4.31) and Proposition 4.8. *(ii)* derives from (4.30) and Proposition 4.8. The energy limit *(iii)* follows from (4.32) in Proposition 4.6, combined with Proposition 4.8. The statement *(iv)* is contained in Lemma 4.3 in the flat case, and can be easily extended to the non-flat case, similarly as in [21, 9], see Subsection 2.1.

Appendix A. Some properties of the Green function

Let $y \in \partial\Omega$ and let $G(x, y)$ be the Green function satisfying the Neumann problem (1.3). First note that $G \geq 0$ and by the classical strong maximum principle, for each $y \in \partial\Omega$ $G(., y)$ cannot attain its minimum in Ω. Also, by the Hopf lemma if $G(x, y) = 0$ for some $x, y \in \partial\Omega$, $x \neq y$ then the normal derivative $\frac{\partial G}{\partial \nu_x}(x, y)$ is negative, which is impossible. Therefore, for each $y \in \partial\Omega$ we have

$$G(\cdot, y) > 0 \text{ in } \overline{\Omega}. \tag{A.1}$$

By a compactness argument we can find a constant $c > 0$ such that $G(x, y) > c$ for all $y \in \partial\Omega$ and all $x \in \overline{\Omega}$.

Lemma A.1. *There exists a positive constant C_1 such that*

$$0 < G(x, y) \leq C_1 \left(|\log|x - y|| + 1 \right) \quad \text{for each } x \in \overline{\Omega} \setminus \{y\} \text{ and } y \in \partial\Omega.$$

Proof. By (1.4), we have

$$G(x, y) = \frac{1}{\pi} \log \frac{1}{|x - y|} + H(x, y) \tag{A.2}$$

where $\frac{1}{\pi} \log \frac{1}{|x-y|}$ is the singular part of G and $H(x, y)$ is the regular part of G. The function $H(., y)$ satisfies

$$\begin{cases} -\Delta_x H(x, y) + H(x, y) = -\dfrac{1}{\pi} \log \dfrac{1}{|x - y|} & \text{in } \Omega \\ \dfrac{\partial H}{\partial \nu_x}(x, y) = \dfrac{1}{\pi} \dfrac{\langle x - y, \nu(x) \rangle}{|x - y|^2} & \text{on } \partial\Omega. \end{cases}$$

Arguing as in [38] (see pages 834 and 835), we have

$$x \mapsto H(x, y) \in C^{1,\gamma}(\overline{\Omega}), \;\; y \mapsto H(x, y) \in C^{1,\gamma}(\partial\Omega, C^{1,\gamma}(\overline{\Omega})) \text{ and } \nabla_x H \in C(\overline{\Omega} \times \partial\Omega) \tag{A.3}$$

for any $\gamma \in (0, 1)$. The desired result follows from (A.1), (A.2) and (A.3). $\square$

As consequence of Lemma A.1, we have the following result.

Lemma A.2. *There exist $C_2, C_\delta > 0$ such that:*

$$G(x, y) \leq C_\delta \quad \forall\, |x - y| > \delta > 0, \tag{A.4}$$

$$|\nabla_x G(x, y)| \leq \frac{C_2}{|x - y|} \quad \forall\, x \in \overline{\Omega} \setminus \{y\}. \tag{A.5}$$

Proof. It is easy to see that (A.4) is a consequence of Lemma A.1.
By (A.2) we have

$$\nabla_x G(x, y) = -\frac{1}{\pi} \frac{x - y}{|x - y|^2} + \nabla_x H(x, y) \tag{A.6}$$

for each $x \in \overline{\Omega} \setminus \{y\}$. Hence (A.5) follows from (A.6) and (A.3). $\square$

Let $x_1, \ldots, x_n$ be n distinct points in $\partial\Omega$ and let r be some positive small constant such that $B_r(x_i) \cap B_r(x_j) =$ for all $1 \le i \ne j \le n$.

Lemma A.3. *Let $1 \le i \le n$ and let $(c_j)_{1 \le j \le n}$ be n real numbers. For each $x \in \overline{B_r(x_i) \cap \Omega} \setminus \{x_i\}$, we have*

$$\sum_{j=1}^{n} c_j G(x, x_j) = \frac{c_i}{\pi} \log \frac{1}{|x - x_i|} + O(1) \quad \textit{and} \quad \sum_{j=1}^{n} c_j \nabla G(x, x_j) = -\frac{c_i}{\pi} \frac{x - x_i}{|x - x_i|^2} + O(1).$$

Proof. Using Lemma A.2, for each $x \in \overline{B_r(x_i) \cap \Omega} \setminus \{x_i\}$ we have

$$\sum_{j=1}^{m} c_j G(x, x_j) = c_i G(x, x_i) + O(1) \quad \text{and} \quad \sum_{j=1}^{m} c_j \nabla G(x, x_j) = c_i \nabla G(x, x_i) + O(1).$$

Furthermore $G(x, x_i)$ satisfies (A.2) and (A.6), so that, by the regularity of H in (A.3) we obtain the desired result. □

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References

[1] Adimurthi, M. Grossi, *Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity*, Proc. Amer. Math. Soc. 132(4) (2004) 1013-1019.
[2] Adimurthi, M. Struwe, *Global compactness properties of semilinear elliptic equations with critical exponential growth*, J. Funct. Anal. 175(1) (2000) 125-167.
[3] M. Ben Ayed, H. Fourti, A. Selmi, *Harmonic Functions with nonlinear Neumann boundary condition and their Morse indices*, Nonlinear Analysis: Real World Applications 38 (2017) 96-112.
[4] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, *Uniform estimates for regularization of free boundary problems*, Analysis and partial differential equations (1990) 567-619.
[5] H. Brezis, F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations 16 (1991) 1223-1253.
[6] H. Brezis, W. Strauss, *Semi-linear second-order elliptic equations in L^1*, J. Math. Soc. Japan 25 (1973) 565-590.
[7] K. Bryan, M. Vogelius, *Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling*, Quart. Appl. Math. 60(4) (2002) 675–694.
[8] H. Castro, *Solutions with spikes at the boundary for a 2D nonlinear Neumann problem with large exponent*, J. Differential Equations 246 (2009) 2991-3037.
[9] H. Castro, *Asymptotic estimates for the least energy solution of a planar semi-linear Neumann problem*, Journal of Mathematical Analysis and Applications 428(1) (2015) 258-281.
[10] K. C. Chang, J. Q. Liu, *A prescribing geodesic curvature problem* Math. Z. 223 (1996), no. 2, 343-365.
[11] P. Cherrier, *Problèmes de Neumann non linéaires sur les varietes riemanniennes*, Journal of Functional Analysis 57 (1984) 154-206.
[12] S. Cruz-Blázquez, D. Ruiz, *Prescribing Gaussian and geodesic curvature on disks*. Adv. Nonlinear Stud. 18 (2018), no. 3, 453-468.
[13] J. Davila, M. del Pino, M. Musso, *Concentrating solutions in a two-dimensional elliptic problem with exponential Neumann data*, J. Functional Anal. 227 (2005) 430-490.
[14] F. De Marchis, I. Ianni, F. Pacella, *Asymptotic analysis and sign changing bubble towers for Lane-Emden problems*, JEMS 17(8) (2015) 2037-2068.
[15] F. De Marchis, I. Ianni, F. Pacella, *Asymptotic profile of positive solutions of Lane-Emden problems in dimension two*, J. Fix. Point Theory A. 19(1) (2017) 889-916.

[16] F. De Marchis, I. Ianni and F. Pacella, *Asymptotic analysis of the Lane-Emden problem in dimension two, contribution in the volume Partial Differential Equations arising from Physics and Geometry*, London Mathematical Society Lecture Note Series (No. 450), Cambridge University Press (2018).

[17] F. De Marchis, M. Grossi, I. Ianni and F. Pacella, *L^∞-norm and energy quantization for the planar Lane-Emden problem with large exponent*, Archiv der Mathematik 111 (2018) 421-429.

[18] O. Druet, *Multibumps analysis in dimension 2: quantification of blow-up levels*, Duke Math. J. 132(2) (2006) 217-269.

[19] O. Druet, E. Hebey and F. Robert, *Blow-up theory for elliptic PDEs in Riemannian geometry*, Math. Notes 45, Princeton Univ. Press, Princeton, N. J. 2004.

[20] L. C. Evans, *Partial differential equations.* American Mathematical society (1998). Third printing, 2002.

[21] H. Fourti, *Convergence for a planar elliptic problem with large exponent Neumann data*, J. Math. Anal. Appl. 502 (2021).

[22] Gilbarg D, Trudinger N S. *Elliptic partial differential equation of second order*. Springer-Verlag, 1977

[23] M. Grossi, C. Grumiau, F. Pacella, *Lane Emden problems: asymptotic behavior of low energy nodal solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire 30(1) (2013) 121-140.

[24] Y. X. Guo, J. Q. Liu, *Blow-up analysis for solutions of the Laplacian equation with exponential Neumann boundary condition in dimension two*, Commun. Contemp. Math. 8 (2006) 737-761.

[25] A. Jevnikar, R. López-Soriano, M. Medina, D. Ruiz, *Blow-up analysis of conformal metrics of the disk with prescribed Gaussian and geodesic curvatures*, preprint arXiv:2004.14680.

[26] O. Kavian, M. Vogelius, *On the existence and "blow-up" of solutions to a two-dimensional nonlinear boundary-value problem arising in corrosion modelling*, Proc. Roy. Soc. Edinburgh Section A 133(1) (2003) 119–149.

[27] P. Liu, *A Moser-Trudinger type inequality and blow-up analysis on Riemann surfaces*, dissertation, Der Fakultat fur Mathematik and Informatik der Universitat Leipzig (2001).

[28] A. Le, Z-Q. Wang, J. Zhou, *Finding Multiple Solutions to Elliptic PDE with Nonlinear Boundary Conditions*, J. Sci. Comput. 56 (2013) 591-615.

[29] R. López-Soriano, A. Malchiodi, D. Ruiz, *Conformal metrics with prescribed Gaussian and geodesic curvatures*, preprint arXiv:1806.11533.

[30] K. Medville, M. Vogelius, *Blow up behaviour of planar harmonic functions satisfying a certain exponential Neumann boundary condition*, SIAM J. Math. Anal., 36(6) (2005), 1772–1806.

[31] X. Ren, Xiaofeng, J. Wei, *On a two-dimensional elliptic problem with large exponent in nonlinearity*, Trans. Amer. Math. Soc. 343(2) (1994) 749-763.

[32] X. Ren, Xiaofeng, J. Wei, *Single-point condensation and least-energy solutions*, Proc. Amer. Math. Soc. 124(1) (1996) 111-120.

[33] M. Renardy and R. Rogers, *Introduction to partial differential equations,* Springer, Berlin, (1993).

[34] S. Santra, J. Wei, *Asymptotic behavior of solutions of a biharmonic Dirichlet problem with large exponents*, J. Anal. Math. 115 (2011) 1-31.

[35] E. Shamir, *Regularization of mixed second-order elliptic problems*, Israel J. Math. 6 (1968) 150-168.

[36] F. Takahashi, *Asymptotic behavior of least energy solutions for a 2D nonlinear Neumann problem with large exponent*, J. Math. Anal. Appl. 411(1) (2014) 95-106.

[37] P.-D. Thizy, *Sharp quantization for Lane–Emden problems in dimension two*, Pacific J. Math. 300 (2019) 491-497.

[38] Y. Wang, L. Wei, *Multiple boundary bubbling phenomenon of solutions to a Neumann problem*, Adv. Differential Equations 13 (2008) 829-856.

[39] T. Zhang, C. Zhou, *Classification of solutions for harmonic functions with Neumann boundary value*, Canad. Math. Bull. 61(2) (2018) 438-448.

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