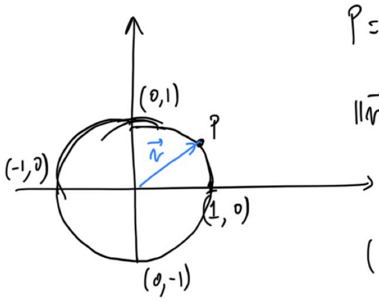
Trigonometria



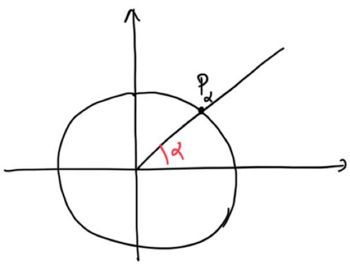
$$P = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\|\vec{v}\| = 1$$

$$\|\vec{v}\|^2 = 1$$

$$(x, y)\begin{pmatrix} x \\ y \end{pmatrix} = 1$$

$$\boxed{x^2 + y^2 = 1}$$

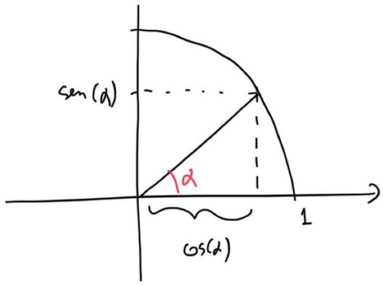


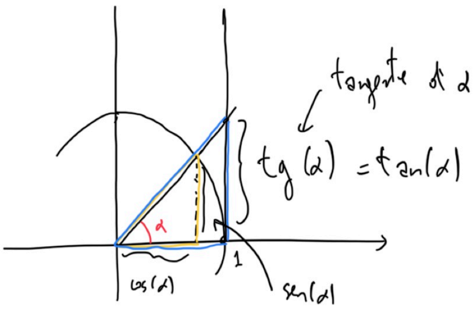
Per definizione, le coordinate di P_α si indicano con

$$P_\alpha = \begin{pmatrix} \cos(\alpha) \\ \mathrm{sen}(\alpha) \end{pmatrix}$$

coseno di α

seno di α





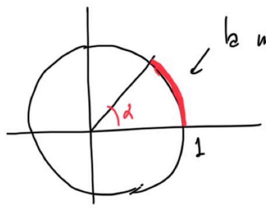
$$\cos(\alpha) : 1 = \text{sen}(\alpha) : \text{tg}(\alpha)$$

$$\Rightarrow \frac{\cos(\alpha)}{1} = \frac{\text{sen}(\alpha)}{\text{tg}(\alpha)} \rightarrow \boxed{\text{tg}(\alpha) = \frac{\text{sen}(\alpha)}{\cos(\alpha)}}$$

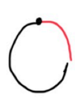
$P_\alpha = \begin{pmatrix} \text{sen}(\alpha) \\ \cos(\alpha) \end{pmatrix}$ appartiene alla circonferenza di raggio 1 centrata in O, per costruzione. Quindi le sue coordinate soddisfano l'equazione della circonferenza.

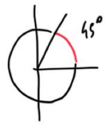
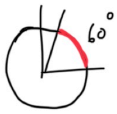
$$\rightarrow \boxed{\cos^2(\alpha) + \text{sen}^2(\alpha) = 1}$$

La misura degli angoli in radianti



la misura di questo arco è, per definizione, la misura di α in radianti

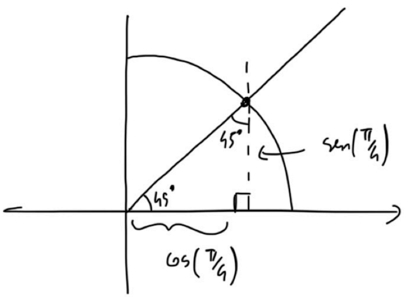
Misura in gradi	Arco	Misura in radianti
360°		2π
180°		π
90°		$\pi/2$

45°		$\pi/4$
60° $(3 \cdot 60° = 180°)$		$\pi/3$
30°		$\pi/6$

Angolo in gradi	Angolo in radianti	cos	sen
360°	2π	1	0
180°	π	-1	0
90°	$\pi/2$	0	1
270°	$\frac{3}{2}\pi$	0	-1

$\operatorname{sen}(\pi/4) = ?$ $\quad \cos(\pi/3) = ?$ $\quad \operatorname{sen}(\pi/6) = ?$

$\pi/4$

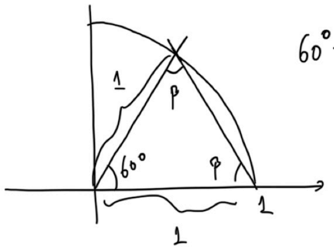


$$\left.\begin{array}{l} \operatorname{sen}(\pi/4) = \cos(\pi/4) \\ \operatorname{sen}^2(\pi/4) + \cos^2(\pi/4) = 1 \end{array}\right\} \rightarrow 2\operatorname{sen}^2(\pi/4) = 1$$

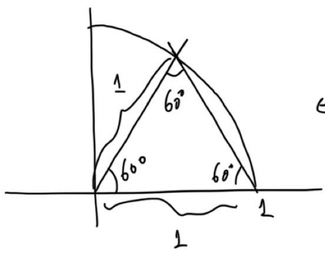
$$\rightarrow \operatorname{sen}^2\left(\pi/4\right) = \frac{1}{2} \Rightarrow \operatorname{sen}\left(\pi/4\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \cos\left(\pi/4\right) = \frac{\sqrt{2}}{2}$$

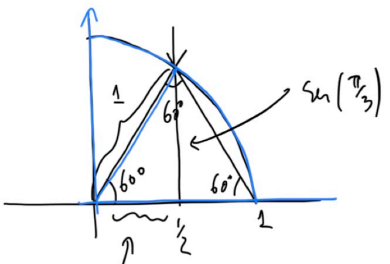
$\pi/3$



$$60° + 2\beta = 180°$$
$$\Rightarrow 2\beta = 120°$$
$$\Rightarrow \beta = 60°$$



← è un triangolo equilatero



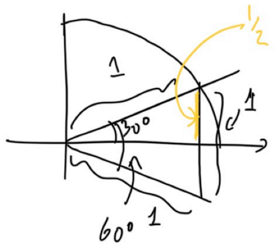
$\cos(60°) = \cos(\pi/3) = \frac{1}{2}$

$$\operatorname{sen}^2(\pi/3) + \cos^2(\pi/3) = 1$$

$$\rightarrow \operatorname{sen}^2(\pi/3) + \left(\frac{1}{2}\right)^2 = 1 \rightarrow \operatorname{sen}^2\left(\frac{\pi}{3}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\operatorname{sen}(\pi/3) = \frac{\sqrt{3}}{2}$$

$\pi/6$

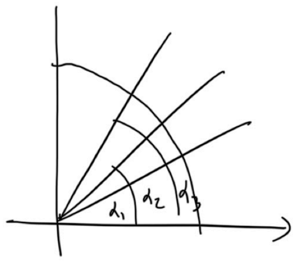


$\mathrm{sen}(\pi/6) = 1/2$

Ragionando come prima

$\cos(\pi/6) = \sqrt{3}/2$

Trucco mnemonico



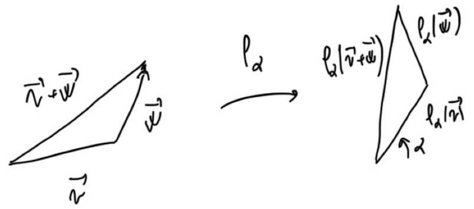
	sen	cos
$\alpha_1 = \pi/6$	$\sqrt{1}/2 = 1/2$	$\sqrt{3}/2$
$\alpha_2 = \pi/4$	$\sqrt{2}/2 = 1/\sqrt{2}$	$\sqrt{2}/2 = 1/\sqrt{2}$
$\alpha_3 = \pi/3$	$\sqrt{3}/2$	$\sqrt{1}/2 = 1/2$

$\mathrm{sen}(\alpha+\beta) = ?$ $\qquad \cos(\alpha+\beta) = ?$

Chiamiamo ρ_α la rotazione attorno all'origine di angolo α

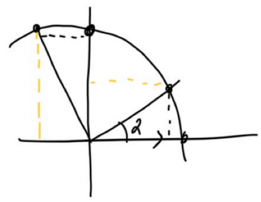
$\rho_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ ed è un'applicazione lineare:

ruotare $\vec{v}+\vec{w}$ è la stessa cosa che prima ruotare $\vec{v}$, poi ruotare $\vec{w}$ e infine sommarli.



$\rho_\alpha \longleftrightarrow$ matrice 2×2 le cui colonne sono

$$\rho_\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e \quad \rho_\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\rho_\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) \\ \operatorname{sen}(\alpha) \end{pmatrix}$$

$$\rho_\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\operatorname{sen}(\alpha) \\ \cos(\alpha) \end{pmatrix}$$

$$\rho_\alpha \longleftrightarrow \begin{pmatrix} \cos(\alpha) & -\operatorname{sen}(\alpha) \\ \operatorname{sen}(\alpha) & \cos(\alpha) \end{pmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\rho_\beta"] \arrow[rr, bend right, "\rho_{\alpha+\beta}"'] & \mathbb{R}^2 \arrow[r, "\rho_\alpha"] & \mathbb{R}^2 \end{tikzcd}$$

$$\begin{pmatrix} \cos(\alpha+\beta) & -\operatorname{sen}(\alpha+\beta) \\ \operatorname{sen}(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\operatorname{sen}(\alpha) \\ \operatorname{sen}(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\operatorname{sen}(\beta) \\ \operatorname{sen}(\beta) & \cos(\beta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \operatorname{sen}(\alpha)\operatorname{sen}(\beta) & \cdots \\ \operatorname{sen}(\alpha)\cos(\beta) + \cos(\alpha)\operatorname{sen}(\beta) & \cdots \end{pmatrix}$$

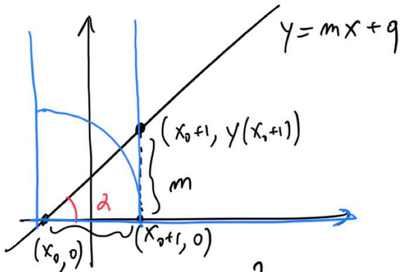
$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \operatorname{sen}(\alpha)\operatorname{sen}(\beta) \\ \operatorname{sen}(\alpha+\beta) &= \operatorname{sen}(\alpha)\cos(\beta) + \cos(\alpha)\operatorname{sen}(\beta) \end{aligned}}$$

Casi particolari $\alpha = \beta$

$$\cos(2\alpha) = \cos^2(\alpha) - \operatorname{sen}^2(\alpha) = 1 - 2\operatorname{sen}^2(\alpha)$$
$$= 2\cos^2(\alpha) - 1$$

$$\operatorname{sen}(2\alpha) = 2\operatorname{sen}(\alpha)\cos(\alpha)$$

Perché è interessante $\operatorname{tg}\alpha$?



$\alpha = ?$

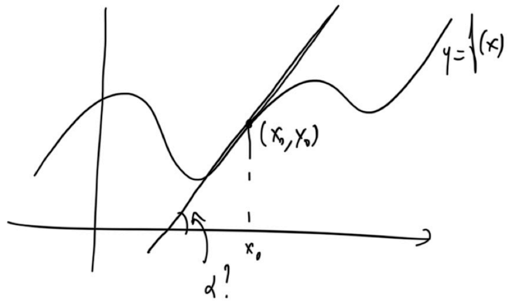
$$y(x) = mx + q$$
$$y(x_0) = 0 \rightarrow mx_0 + q = 0$$
$$y(x_0+1) = m(x_0+1) + q = mx_0 + m + q =$$
$$= \underbrace{mx_0 + q}_{0} + m = m$$

$$\boxed{\operatorname{tg}\alpha = m}$$

Il coefficiente angolare di $y = mx + q$ è la tangente dell'angolo α che la retta forma con l'asse delle x.

$$\left(\alpha = \operatorname{arctg}(m)\right)$$

Esempio

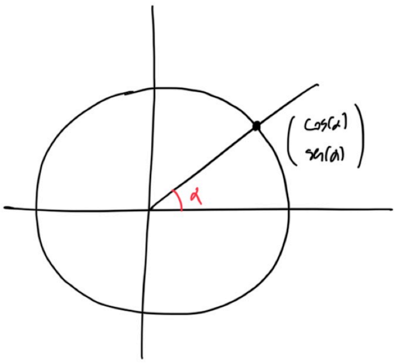


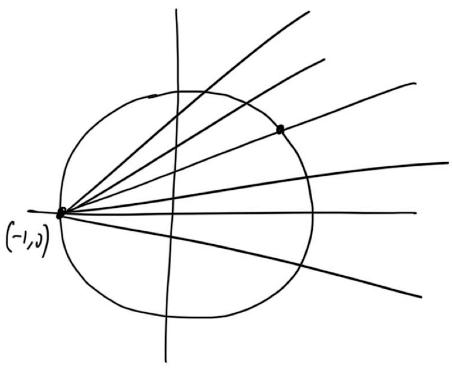
la retta tg in $(x_0, f(x_0))$ ha coefficiente angolare $f'(x_0)$ (per definizione di derivata)

$$\Rightarrow \quad \mathrm{tg}\, \alpha = f'(x_0)$$

$$\Rightarrow \quad \alpha = \mathrm{arctg}\, f'(x_0)$$

Esprimere $\mathrm{sen}(\alpha)$ e $\cos(\alpha)$ per mezzo di $\mathrm{tg}\left(\frac{\alpha}{2}\right)$.





Devo descrivere il fascio di rette per $(-1,0)$

Se $\vec{v}$ è un vettore che dà la direzione alla retta $\Rightarrow$ equazioni parametriche per r sono

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} + t\vec{v}$$

$$\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} + t\begin{pmatrix}a\\b\end{pmatrix}$$

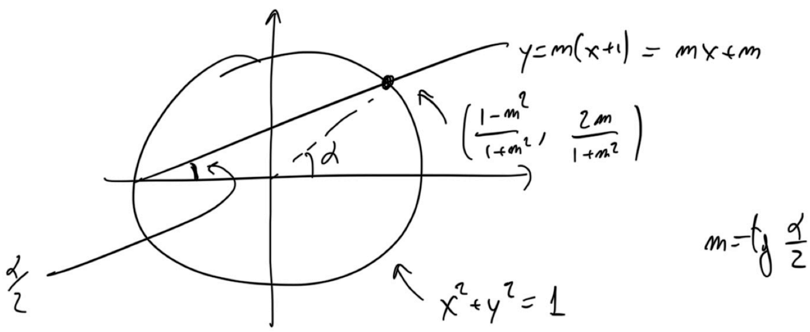
$$\begin{pmatrix}x+1\\y\end{pmatrix} = t\begin{pmatrix}a\\b\end{pmatrix} \iff \det\begin{pmatrix}x+1 & a\\ y & b\end{pmatrix} = 0$$

$$b(x+1) - ay = 0$$

Se $a \neq 0 \rightarrow y = \frac{b}{a}(x+1)$

$y = m(x+1)$ ← fascio di rette per $(-1,0)$

(in più c'è la retta verticale $x=-1$)



$$x^2 + m^2(x+1)^2 = 1$$

$$(m^2+1)x^2 + 2m^2 x + (m^2-1) = 0$$

$$x_{1/2} = \frac{-2m^2 \pm \sqrt{4m^4 - 4(m^2+1)(m^2-1)}}{2(m^2+1)}$$

$$= \frac{-2m^2 \pm \sqrt{4m^4 - 4(m^4-1)}}{2(m^2+1)}$$

$$= \frac{-2m^2 \pm \sqrt{\cancel{4m^4} - \cancel{4m^4} + 4}}{2(m^2+1)} = \frac{-2m^2 \pm 2}{2(m^2+1)}$$

$$\frac{-2\cancel{(m^2+1)}}{2\cancel{(m^2+1)}} = -1$$

$$\frac{\cancel{2}(1-m^2)}{\cancel{2}(1+m^2)}$$

$$y = m(x+1) = m\left(\frac{1-m^2}{1+m^2}+1\right) =$$

$$= m\left(\frac{1-\cancel{m^2}+1+\cancel{m^2}}{1+m^2}\right) = \frac{2m}{1+m^2}$$

$$\begin{cases} \cos(\alpha) = \dfrac{1-m^2}{1+m^2} = \dfrac{1-tg^2(\alpha/2)}{1+tg^2(\alpha/2)} \\ \\ sen(\alpha) = \dfrac{2m}{1+m^2} = \dfrac{2\,tg(\alpha/2)}{1+tg^2(\alpha/2)} \end{cases}$$