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Il Meccanismo di Higgs-Braut-Englert

Tesi di Laurea

Relatore:
Prof.
Domenico Fiorenza

Laureando:
Giuseppe Malavolta
Matricola: 1136452

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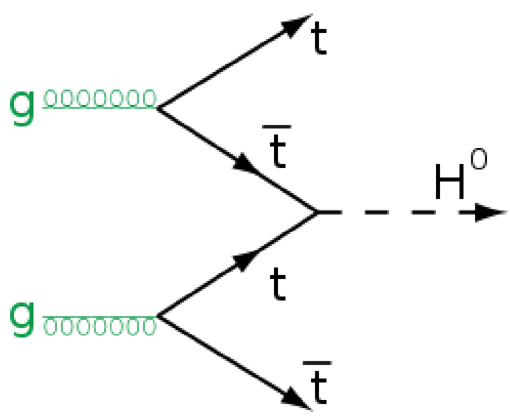
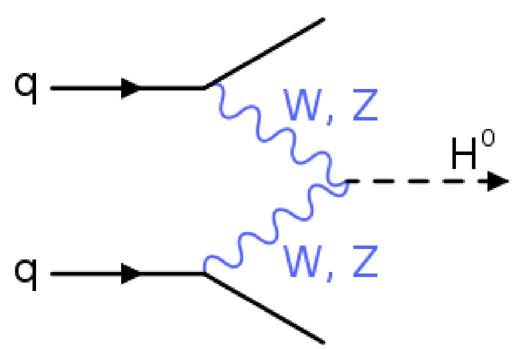
Giuseppe Malavolta

The Higgs-Braut-Englert Mechanism

Master Thesis in Mathematics





To R. and E.

Contents

In the title page: two Feynman diagrams schematizing the interaction processes which will hopefully occur at CERN L.H.C. resulting in the creation of a Higgs boson H. The first diagram represents the decay of two gluons g in a top/antitop couple $t/\bar{t}$, which combines in a Higgs boson. In the second one each of the quarks q emits a W or a Z boson, which combine to make a Higgs boson H.

Introduction

In this master thesis I present the algebraic and geometric tools necessary for understanding a famous subnuclear model of the elementary particles: the *Higgs-Braut-Engler mechanism.*

The idea, inspiring this thesis, was born in september 2009. During that period, the *Large Hadron Collider* was finally completed and started the warmup leading to the first collision of march 2010 at a record energy of 3.5 TeV. As a mathematical student with physical interests, I simply could not ignore this event and I decided to attend a theoretical physics class for learning about the *quantum field theory* and the *standard model* unifying theory. Due to the physical purposes of the class, the experimental aspects of the standard model have been deeply discussed and pretty much exhausted, while the mathematical aspects were introduced only to make all the necessary calculations (such as calculating the scattering sections or the conserved currents). Attending this class has been quite a unique experience in my study course and inspired me to further investigate the mathematical aspects of the standard model. So I looked for an advisor to support me in making this investigation a master thesis, and I met Domenico Fiorenza. It was immediately clear that the work was huge: the standard model has countless mathematical aspects and exhausting everything appeared an unreasonable duty for a master thesis. Accordingly, we decided to limit ourselves to the geometric and algebraic aspects and focus on a model that is central in the theory: the H.B.E. mechanism.

The standard model of the particle is a unifying theory of three out four of the fundamental interactions of the nature between the elementary particles: *the electromagnetism, the weak and the strong interactions.* The path leading to the standard model was initiated by Maxwell in 1873 with the *"Treatise on Electricity and Magnetism"*, where a theory unifying the electric and magnetic phenomena was outlined. Subsequently, in the 20th century, once the Rutherford atom model was accepted and confirmed by the experiments, a research of the subnuclear interactions could be started.

During 1930s Enrico Fermi studied the decay of the nuclei and arranged a theory for the *weak interaction.* The term *weak* refers to the significant lower energy at which the processes of decay occur respect to the eletromagnetic and strong interaction phenomena. It is important to remark that in the Fermi's theory the interactions are supposed pointwise, while the electromagnetism has a particle responsible for carrying the interaction: the photon.

For unifying the electromagnetism and the weak interaction, it was necessary to introduce the analogue of the photon in the weak theory. This was accomplished by Sheldon Glashow and Luciano Maiani in 1961: they formulated the electroweak unification by introducing the intermediate bosons W and Z for the weak interactions. The Glashow-Maiani theory is a general case of the Fermi's theory, once the field of the W and the Z are assumed constant, and thus the interaction is considered pointwise.

A further improvement to the electroweak theory was proposed six years later by Steven Weinberg and Abdus Salam. They introduced a new boson, responsible for generating the masses: the Higgs boson. The mechanism of mass generation through the Higgs boson is the *Higgs-Braut-Engler mechanism*. The improvement, Weinberg and Salam made to the Glashow-Maiani theory, was providing a model with *exact gauge symmetries*.

The final step towards the standard model was the formulation of the strong interaction in terms of *quarks* interacting via *gluons*. Murray Gell-Mann and George Zweig, in 1964, conjecture, for the first time, a theory of the interaction in the nuclei where protons and neutrons were composites of quarks. The first experimental evidence of the existence of the quarks arrived in 1969 at the *Standford Linear Accelerator*. Subsequently, having in mind the electroweak theory, in 1974, Harald Fritzsch and Murray Gell-Mann, postulated the existence of the gluons as intermediate bosons of the strong interaction. Once again the Higgs boson was responsible for generating the masses. The experimental evidence of the existence of the gluons was found in 1979 at PETRA in Hamburg. In the same year, the electroweak missing bosons, W and Z, are found by a Carlo Rubbia and Simon Van der Meer experiment at *CERN*.

So far, the only particle to be observed for completing the picture of the Standard model is the Higgs boson. This is one of the goals of the CERN L.H.C.

It is worth to furnish a brief guide for the reader.

In the first section we provide a mathematical characterization of the fundamental physical parameters of a elementary particle: *the mass, the spin and the helicity*. These are the parameters classifying the irreducible representations of the Poincar group, i.e. the group that conserves the inner product of the Minkowski space. At this stage, the fundamental Klein-Gordon equation will be presented as the differential equation associated to the Casimir element $\|P\|^2$.

In the second section we introduce two famous equations of the elementary particles: the Dirac and the massive vectorial field equations. In particular we will be looking at the equations as *equivariant maps* and we will explain in which sense the Dirac and massive vectorial field operators can be considered as *factorizations* of Klein-Gordons. Furthermore we will recognize in the solutions of each equation a representation of the Poincar group.

In the third and last section, the Higgs-Braut-Englert mechanism is finally introduced. Due to the work done in Sections 1 and 2, once the lagrangian of the model is written, it is sufficient to look at the action to recognize the type of the particles occurring in the model. The scalar particle that will appear in the H.B.E. model is the *Higgs boson*. Furthermore we will present the use of the mechanism in the standard model: in general and in the case of the weak interaction.

In appendix we leave some classic topic of Lie algebras theory and differential geometry, for providing a quick reference that supports the three main sections. In particular the Appendix A supports Section 1 and the Appendixes B and C support Section 3

The thesis material was, in first instance, developed on the n-category laboratory, *a wiki-lab for collaborative work on Mathematics, Physics and Philosophy.* Further informations, together with a digital version of the thesis, can be found at *ncatlab.org*.

Roma,
winter 2011.
Giuseppe Malavolta

1 Mass, helicity and spin

1.1 Symmetries

In Bourbaki's parlance, given any mathematical struture $\mathcal{S}$ its *symmetries* are the structure-preserving invertible self-transformations of $\mathcal{S}$. In the more modern categorical language this is expressed by saying that the symmetries of an object X are the elements of its automorphism group.

Concretely, the situation one is customary faced with is the following: one has a basic strcuture $\mathcal{S}_{\text{basic}}$ which is enriched with an additional structure, becoming $\mathcal{S}_{\text{rich}}$. Symmetries of $\mathcal{S}_{\text{rich}}$ are then symmetries of $\mathcal{S}_{\text{basic}}$ preserving the additional structure.

Example. 1. In linear algebra one can consider a real vector space V and endow it with a metric g; the symmetries of (V, g) are then the isometries, i.e., the linear automorphisms $\varphi : V \to V$ such that $\varphi^* g = g$. Note that $\mathrm{Aut}(V, g)$ is the stabilizer of g for the action of $\mathrm{Aut}(V)$ on the space of symmetric bilinear forms on V. Similarly, if V is even dimensional, one can endow it with a symplectic form ω, and consider the group of symplectic transformations of (V, ω). This example immediately generalizes from vector spaces to differential manifolds: one considers isometries of (pseudo-)Riemannian manifolds and symplectomorphisms of symplectic manifolds.

Example. 2. In classical mechanics one considers a differential manifold M endowed with a Lagrangian, i.e., a smooth function $L : TM \to \mathbb{R}$. Symmetries of such a Lagrangian system are diffeomorphisms $\varphi : M \to M$ such that $(d\varphi)^* L = L$, where $d\varphi : TM \to TM$ is the differential of φ.

Example. 3. The Lagrangian example above has a natural generalization in Hamiltonian mechanics. An Hamiltonian system is a triple (M, ω, H) where (M, ω) is a symplectic manifold, and $H : M \to \mathbb{R}$ is a smooth function, called the *Hamiltonian*. A symmetry of an Hamiltonian system is a symplectomorphism φ of (M, ω) such that $\varphi^* H = H$.

In all of the above examples, symmetries are not just a group: they are a Lie group (eventually an infinite-dimensional one). So it is meaningful to talk of *infinitesimal symmetries*. Mathematically speaking, these are elements of the Lie algebra of the Lie group of symmetries. It is interesting to remark that in infinite-dimensional situations, the Lie algebra structure of the vector space of infinitesimal symmetries is a perfectly well define object, even when a rigorous infinite-dimensional Lie group structure on the group od symmetries is not defined.

Example. 4. In example 3, infinitesimal symmetries are vector fields X on M such that $\mathcal{L}_X \omega = 0$ and $\mathcal{L}_X H = 0$, where $\mathcal{L}_X$ denotes the Lie derivative along X. Among these symmetries, of particular interest are the Hamiltonian ones, i.e., vector fields of the form $X_f = \{f, -\}$, where f is a smooth function on M and $\{-, -\}$ is the Poisson bracket induced by the symplectic struture of

M. Of the two conditions an Hamiltonian vector field has to satisfy in order to eb an infinitesimal symmetry of the system, the first one, $\mathcal{L}_{X_f}\omega = 0$ is always satisfied (this is Liouville's theorem), whereas the second one, i.e., $\mathcal{L}_{X_f}H = 0$ is equivalent to $\{f, H\} = 0$. In other terms, the Lie algebra of infinitesimal Hamiltonian symmetries is identified with the Lie algebra of smooth functions on M Poisson-commuting with H (modulo the constants, i.e., the kernel of $f \mapsto X_f$). In terms of classical mechanics, each infinitesimal Hamiltonian symmetry f is a *constant of motion*, i.e., if $x : \mathbb{R} \to M$ is the time evolution of the Hamiltonian system with initial datum $x(0) = x_0$, then $f(x(t)) = f(x_0)$ for every $t \in \mathbb{R}$.

Example. 5. In quantum mechanics one considers a noncommutative version of example 4, where smooth functions are replaced by self-adjoint operators and Poisson brackets are replaced by commutators. More precisely, quantum states are norm 1 vectors in an Hilbert space $\mathcal{H}$ and the probability of a transition of the system from a state ϕ to a state ψ is

$$P(\phi \to \psi) = |\langle\phi|\psi\rangle|^2.$$

Symmetries of the system will have to preserve all these transition probabilities, so they will be unitary operators $U : \mathcal{H} \to \mathcal{H}$. Moreover, the system is endowed with a distinguished self-adjoint operator, the Hamiltonian $H : \mathcal{H} \to \mathcal{H}$, and the Poisson-commutation relation $\{f, H\} = 0$ of Hamiltonain mechanics is translated in the commutation relation $[U, H] = 0$. This in particular implies that any symmetry U of $(\mathcal{H}, H)$ preserves the H-eigenspaces decomposition of $\mathcal{H}$. In the quantum mechanics parlance, the scalars in the spectrum of the operator H are called the *energy levels* of the Hamiltonian.

Remark. 6. In the above example we restricted our attention to unitary operators by the principle of "symmetries of the rich structure are a subgroup of symmetries of the basic structure", and assuming that the basic structure was that of an Hilbert space, so that its symmetries were the automorphisms of $\mathcal{H}$ as a Banach space. One could however consider more general symmetries, by looking at $\mathcal{H}$ just as a set endowed with the function

$$\begin{aligned} P : \mathcal{H} \times \mathcal{H} &\to \mathbb{R} \\ (\phi, \psi) &\mapsto |\langle\phi|\psi\rangle|^2. \end{aligned}$$

and then consider the set of all invertible self-transformations of the set $\mathcal{H}$ preserving P. But this actually brings in nothing new: a P-preserving self-transformation of $\mathcal{H}$ is either a unitary transformation of $\mathcal{H}$ or a unitary transformation of $\overline{\mathcal{H}}$ (Wigner's theorem, [Wig59]).

1.2 Groups of symmetries

For a given object X in some category $\mathcal{C}$, the complete description of the whole group of symmetries $\mathrm{Aut}(X)$ may be an extremely difficult problem. In concrete situations one is often more interested in distinguished (in some sense) and well-behaved subgroups of $\mathrm{Aut}(X)$, or more in general in group representation with values in $\mathrm{Aut}(X)$. For instance, if G is a Lie group, a realization of G as a symmetry group for a quantum mechanical system is a group homomorphism

$$\rho : G \to U(\mathcal{H})$$

which is smooth in the sense that for any two states ϕ and ψ in $\mathcal{H}$, the "matrix coefficient" $\langle \phi | \rho(g) \cdot \psi \rangle$ is a smooth complex-valued function on G. Here we are denoting by $U(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. We will also occasionally meet linear representations

$$\rho : G \to \mathrm{Aut}(\mathcal{H}),$$

where $\mathrm{Aut}(\mathcal{H})$ is the group of automorphisms of $\mathcal{H}$ in the category of Hilbert spaces. Since the category of Hilbert spaces is a full subcategory of Banach spaces, $\mathrm{Aut}(\mathcal{H})$ is the group of automorphisms of $\mathcal{H}$ as a Banach space, i.e., the group of continuous invertible linear endomorphisms of $\mathcal{H}$ with a continuous inverse.

Let us now focus on unitary representations. If $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ which is stable under the unitary action of the symmetry group G, then also $\mathcal{H}_0^\perp$ is G-stable. This immediately implies the following result.

Proposition. 7. *All unitary representations of a group G are completely reducible, i.e., are direct sums of irreducible representations.*

When G is compact, each linear representation $G \to \mathrm{Aut}(\mathcal{H})$ is actually an unitary representation (up to conjugation). Indeed, let μ_G be the Haar measure of G, i.e., the unique normalized biinvariant measure on G and set

$$\langle\langle \phi | \psi \rangle\rangle = \int_G \langle \rho(g) \cdot \phi | \rho(g) \cdot \psi \rangle d\mu_G(g).$$

Then $\langle\langle - | - \rangle\rangle$ is an inner product on $\mathcal{H}$ inducing an Hilbert space structure equivalent to the original one, and the representation ρ is manifestly unitary with respect to this new inner product. So, by the above proposition, we obtain that all continuous linear representations of a compact group G on an Hilbert space are completely reducible.

A less trivial result is the following.

Proposition. 8. *Let G be a compact Lie group (or, more in general, a compact topological group). Then, irreducible representations of G are finite dimensional.*

This is proved by showing that if G is compact then each unitary representation of G is the direct sum of its finite-dimensional subrepresentations. A complete proof of this statement can be found, e.g., in [BtD85].

Combining the two propositions above, one obtains the following principle: *to understand compact Lie groups of symmetries in quantum mechanics, one has to study their finite dimensional irreducible representations.*

1.3 From Lie groups to Lie algebras

Throughout this section we will assume the reader is familiar with the basics of the theory of Lie algebra and of the Lie groups/Lie algebra correspondence. So we will directly focus on the examples we will need in the sequel. Details on the general theory can be found, e.g., in [FuH91].

If the Lie group G is connected, then its group structure is entirely determined by an arbitrary small neighborhood of the identity element e and so, ultimately, by the Lie algebra structure of its tangent space at the identity. The Lie algebra $\mathfrak{g} = T_eG$ is called the Lie algebra of the Lie group G; it is canonically identified with the Lie algebra of left-invariant vector fields on G by the Lie algebra homomorphism

$$\begin{aligned}\{\text{left invariant vector fields on } G\} &\xrightarrow{\sim} \mathfrak{g}\\ X &\mapsto X_e.\end{aligned}$$

If $\{\mathbf{e}_i\}_{i\in I}$ is a linear basis of $\mathfrak{g}$, then the Lie algebra structure of $\mathfrak{g}$ is completely encoded in the *structure constants* of the Lie bracket:

$$[\mathbf{e}_i, \mathbf{e}_j] = f_{ij}^k \mathbf{e}_k.$$

Example. 9. The Lie algebra $\mathfrak{so}_3$ of the Lie group $SO(3)$ of rotations in $\mathbb{R}^3$ is the Lie algebra of 3×3 real antisymmetric matrices. Its canonical basis is given by the generators of rotations around the x-, y-, and z-axis, respectively:

$$X_1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}; \quad X_3 = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

These are colloquially called the *infinitesimal rotations* around the coordinate axes. It is starightforward to check the commutation relations

$$[X_1, X_2] = X_3; \qquad [X_2, X_3] = X_1; \qquad [X_3, X_1] = X_2,$$

i.e.,

$$[X_i, X_j] = \epsilon_{ijk}\delta^{kl} X_l.$$

In the physics literature it is customary to consider the complexified Lie algebra $\mathfrak{so}_{3;\mathbb{C}} = \mathfrak{so}_3 \otimes \mathbb{C}$ with basis $\{J_1, J_2, J_3\}$, where $J_i = \mathbf{i}X_i$. The commutation relations of the J_i are

$$[J_i, J_j] = \mathbf{i}\epsilon_{ijk}\delta^{kl} J_l.$$

Example. 10. The Lie algebra $\mathfrak{su}_2$ of the Lie group $SU(2)$ is the Lie algebra of 2×2 complex anti-Hermitean matrices. The inner product $(A, B) \mapsto \mathrm{tr}(A^*B)$ makes $\mathfrak{su}_2$ a 3-dimensional Euclidean space. The action of $SU(2)$ by conjugation on $\mathfrak{su}_2$ is isometric with respect to this inner product, thus giving a Lie group homomorphism $SU(2) \to O(3)$. Since $SU(2)$ is connected, the image of this homomorfism is necessarily contained in the connected component $SO(3)$. Moreover, one checks that $SU(2) \to SO(3)$ is actually surjective and that its kernel is $\{\mathrm{Id}, -\mathrm{Id}\}$. Thus the homomorphism $SU(2) \to SO(3)$ is a degree 2 covering map, and since $SU(2)$ is simply connected this exhibits $SU(2)$ as the universal cover of $SO(3)$:

$$SU(2) \to SU(2)/\{\pm\mathrm{Id}\} \cong SO(3).$$

A covering map of Lie groups induces an isomorphism of the corresponding Lie algebras. So we have a natural isomorphism $\mathfrak{su}_2 \cong \mathfrak{so}_3$. Explicitly, a basis of $\mathfrak{su}_2$ is given by

$$Y_1 = \begin{pmatrix} \mathbf{i}/2 & 0 \\ 0 & -\mathbf{i}/2 \end{pmatrix}; \quad Y_2 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}; \quad Y_3 = \begin{pmatrix} 0 & \mathbf{i}/2 \\ \mathbf{i}/2 & 0 \end{pmatrix}.$$

One has

$$[Y_1, Y_2] = Y_3; \qquad [Y_2, Y_3] = Y_1; \qquad [Y_3, Y_1] = Y_2,$$

and the isomorphism $\mathfrak{su}_2 \xrightarrow{\sim} \mathfrak{so}_3$ is manifest. In physics literature, it is customary to consider the complexified Lie algebra $\mathfrak{su}_{2;\mathbb{C}} = \mathfrak{su}_2 \otimes \mathbb{C}$ with basis $\{J_1, J_2, J_3\}$ given by $J_i = \mathbf{i}Y_i$. The commutation relations among the J_i are, clearly,

$$[J_i, J_j] = \mathbf{i}\epsilon_{ijk}\delta^{kl}J_l,$$

as in the $\mathfrak{so}_{3;\mathbb{C}}$ case. Note that

$$J_1 = -\frac{1}{2}\sigma_3; \qquad J_2 = -\frac{1}{2}\sigma_2; \qquad J_3 = -\frac{1}{2}\sigma_1$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ are the *Pauli matrices*.

Example. 11. The Lie group $SO(3)$ acts as a group of diffeomorphisms on $\mathbb{R}^3$. Hence we have an injective group homomorphisms $SO(3) \to \mathrm{Diff}(\mathbb{R}^3)$ inducing an injective Lie algebra homomorphism

$$\mathfrak{so}_3 \hookrightarrow \{\text{left } SO(3)\text{-invariant vector fields on } \mathbb{R}^3\}.$$

In particular, we can see X_i as a $SO(3)$-invariant vector field on $\mathbb{R}^3$. Explicitly,

$$X_1 = x^2\partial_3 - x^3\partial_2; \qquad X_2 = x^3\partial_1 - x^1\partial_3; \qquad X_3 = x^1\partial_2 - x^2\partial_1,$$

which is best written in the compact form

$$X_i = \epsilon_{ijk}x^j\partial^k$$

where we have used the metric in $\mathbb{R}^3$ to raise the index in the derivation, i.e., $\partial^i = \delta^{ij}\partial_j$. Complexifying this construction, we can identify the basis elements J_i of $\mathfrak{so}_{3;}$ with complex vector field on $\mathbb{R}^3$. It is customary to write

$$J^{ij} = \epsilon^{ijk}J_k = \mathbf{i}(x^i\partial^j - x^j\partial^i).$$

With this notation the commutation relations read

$$[J^{ij}, J^{kl}] = \mathbf{i}(\delta^{jk}J^{il} + \delta^{il}J^{jk} - \delta^{ik}J^{jl} - \delta^{jl}J^{ik}).$$

1.4 Universal enveloping algebras

Any associative algebra A defines a Lie algebra A_{Lie} simply by taking the commutator as the Lie bracket:

$$[a, b] = ab - ba.$$

This construction is a functor

$$\text{Associative algebras} \to \text{Lie algebras}.$$

Remarkably, this functor has a left adjoint

$$U : \text{Lie algebras} \to \text{Associative algebras}.$$

That is, given a Lie algebra $\mathfrak{g}$ there exists an associative algebra $U(\mathfrak{g})$ sucht that for any associative algebra A,

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, A_{\text{Lie}}) = \text{Hom}_{\text{Assoc}}(U(\mathfrak{g}), A).$$

In other words, any Lie algebra morphism $\mathfrak{g} \to A_{\text{Lie}}$ can be uniquely extended to an associative algebra morphism $U(\mathfrak{g}) \to A$. Since $U(\mathfrak{g})$ is characterized by an universal property, it is unique up to natural isomorphism. The algebra $U(\mathfrak{g})$ is called the *universal envelping algebra* of $\mathfrak{g}$. Note that, taking $A = U(\mathfrak{g})$ one obtatins a canonical Lie algebra morphism $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})_{\text{Lie}}$, induced by the identity of $U(\mathfrak{g})$. This morphism is actually injective and $U(\mathfrak{g})$ is generated, as an associative algebra, by the image of $\iota_{\mathfrak{g}}$ (Poincaré-Birkhoff-Witt theorem).

Example. 12. A typical situation in which $U(\mathfrak{g})$ occurs is the following. Consider a linear representation of a Lie group G, i.e., a Lie group homomorphism $G \to GL(V)$, where V is some vector space. This induces a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V) = \text{End}(V)_{\text{Lie}}$, and so an associative algebra homomorphism

$$U(\mathfrak{g}) \to \text{End}(V).$$

Example. 13. Let M be a differential manifold, and let $\mathcal{X}(M)$ be the Lie algebra of vector fields on M. Finally, let $\mathcal{D}(M)$ be the associative algebra of differential operators on M. Then $\mathcal{X}(M)$ is a sub-Lie algebra of $\mathcal{D}(M)_{\text{Lie}}$. If a smooth action of a Lie group G on M is given, the above inclusion is refined to an inclusion $\mathcal{H}(M)^G \hookrightarrow \mathcal{D}(M)^G_{\text{Lie}}$ of left G-invariant vector fields on M into the Lie algebra of left G-invariant differential operators on M. In particular, if $M = G$ acting on itself by left multilication, we get a Lie algebra inclusion

$$\mathfrak{g} \hookrightarrow \mathcal{D}(G)^G_{\text{Lie}},$$

where we have used the identification of $\mathfrak{g}$ with left-invariant vector fields on G. By the universal property of $U(\mathfrak{g})$, this gives an associative algebra morphism from $U(\mathfrak{g})$ to $\mathcal{D}(G)^G$, which turns out to be an isomorphism:

$$U(\mathfrak{g}) \xrightarrow{\sim} \{\text{left-invariant differential operators on } G\}.$$

More in general, if M is a differential manifold with a smooth G-action, we have a natural morphism of asociative algebras

$$U(\mathfrak{g}) \to \{\text{left } G\text{-invariant differential operators on } M\}.$$

1.5 Casimir elements and invariant differential operators

By definition, a Casimir element for a Lie algebra $\mathfrak{g}$ is an element in the center of its universal enveloping algebra, i.e. is an element C in $U(\mathfrak{g})$ such that $[C, x] = 0$ for any x in $U(\mathfrak{g})$. Since the algebra $U(\mathfrak{g})$ is generated by the linear subspace $\mathfrak{g}$, this is equivalent to $[C, x] = 0$ for any x in $\mathfrak{g}$. As remarked in the previous section, if $\mathfrak{g}$ is the Lie algebra of a Lie group G, then $U(\mathfrak{g})$ is identified with the algebra of left-invariant differential operators on G. Therefore Casimir elements are identified with biinvariant differential operators on G. More in general, if M is a differential manifold endowed with a smooth G-action, Casimir elements induce G-biinvariant differential operators on M. In what follows we will say G-invariant differential operator to mean that a differential operator is biinvariant, whereas we will always say left G-invariant or right G-invariant when the operator is invariant only with respect to left- or right- translations.

Example. 14. For $\mathfrak{su}_2 \cong \mathfrak{so}_3$ a Casimir operator is

$$\|J\|^2 = \delta^{ij} J_i J_j.$$

Indeed,

$$\begin{aligned}
[J_i, \|J\|^2] &= \delta^{jk}[J_i, J_j J_k] \\
&= \mathbf{i}\,\delta^{jk}([J_i, J_j]J_k + J_j[J_i, J_k]) \\
&= \mathbf{i}\,\delta^{jk}(\epsilon_{ijl}\delta^{lm} J_m J_k + \epsilon_{ikl}\delta^{lm} J_j J_m) \\
&= \mathbf{i}\sum_{j,l}(\epsilon_{ijl} J_l J_j + \epsilon_{ijl} J_j J_l) \\
&= \mathbf{i}\sum_{j,l}(\epsilon_{ijl} + \epsilon_{ilj}) J_l J_j = 0.
\end{aligned}$$

The operator $\|J\|^2$ is called *angular momentum operator* in physics. If $\rho : SU(2) \to U(\mathcal{H})$ is a unitary representation of $SU(2)$, then the image of $\mathfrak{su}_2$ in $\mathrm{End}(\mathcal{H})$ consists of anti-Hermitean operators. Since multiplication by $\mathbf{i}$ turns an anti-Hermitean operator into an Hermitean one, the operators (corresponding to the) J_i are Hermitean, and so is also

$$\|J\|^2 : \mathcal{H} \to \mathcal{H}$$

since

$$(\|J\|^2)^* = (\delta^{ij} J_i J_j)^* = \delta_{ij} J_j^* J_i^* = \delta_{ij} J_j J_i = \|J\|^2$$

since δ is real and symmetric. In particular $\|J\|^2$ is diagonalizable, its spectrum is real, and its eigenspaces are subrepresentations of ρ. This means that J^2 acts as a scalar on irreducible unitary representations of $SU(2)$. This scalar is a numerical invariant attached to irreducible complex representations of $\mathfrak{su}_2 \cong \mathfrak{so}_3$. More precisely, we will see in Section 1.13 that for any nonnegative half-integer ℓ there exist exactly one unitary irreducible representation ρ_ℓ of dimension $2\ell+1$, and that these are all possible unitary irreducible representations of $SU(2)$. The operator $\|J\|^2$ acts as the multiplication by $\ell(\ell+1)$ in the representation ρ_ℓ.

Example. 15. For the defining representation of $SO(3)$ on $\mathbb{R}^3$, the elements J_i in $\mathrm{End}(\mathbb{R}^3)$ are described in Example 9. One therefore sees that

$$\|J\|^2 = 2\mathrm{Id}.$$

Complexifying this representation one obtains the $\ell = 1$ representation of $SU(2)$.

Example. 16. The defining representation of $SU(2)$ on $\mathbb{C}^2$ is the $\ell = 1/2$ representation of $SU(2)$. For this representation, the elements J_i in $\mathrm{End}(\mathbb{C}^2)$ are described in Example 10. One therefore sees that

$$\|J\|^2 = \frac{3}{4}\mathrm{Id} = \frac{1}{2}\left(\frac{1}{2}+1\right)\mathrm{Id}$$

in this case.

1.6 The Lorentz group

Let $\mathbb{R}^{1,3}$ be the standard Minkowski space with metric η of signature $(+,-,-,-)$. The Lorentz group is the group $O(1,3)$ of isometries of η. In matrix form, an element Λ of the Lorenz group is a 4×4 real matrix Λ^i_j such that

$$\Lambda^i_k \eta_{ij} \Lambda^j_l = \eta_{kl}$$

The determinant map

$$\det : O(1,3) \to O(1) = \{\pm 1\}$$

is surjective since

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is an element of $O(1,3)$. The subgroup of Lorentz transformations of determinant 1 is called the subgroup of *proper* Lorentz transformations, and is denoted $SO(1,3)$. In contrast with what happens with $O(4)$, where $SO(4)$ is the connected component of the identity, the group of proper Lorentz transformations is not connected. Indeed, since the columns of Λ^i_j are an η-orthonormal basis of $\mathbb{R}^4$, the first column Λ^i_0 satisfies $\eta(\Lambda_0, \Lambda_0) = 1$, i.e.,

$$(\Lambda^0_0)^2 - (\Lambda^1_0)^2 - (\Lambda^2_0)^2 - (\Lambda^3_0)^2 = 1.$$

This implies $(\Lambda^0_0)^2 \geq 1$ and so there are two disjoint possibilities: $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. In the first case the Lorenz transformation is called *orthocronous*, in the seconda case *anorthocronous*. An example of anorthochronous Lorenz transformation is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The map $O(1,3) \to \{\pm 1\}$ given by $\Lambda \mapsto \mathrm{sgn}(\Lambda_0^0)$ is actually a group homomorphism, and so orthocronous Lorentz transformations are a subgroup of $O(1,3)$. One can then show that the connected component of the identity in $O(1,3)$ is precisely the subgroup $SO^+(1,3)$ of proper orthocronous Lorentz transformations. In particular, $O(1,3)$ has exactly four connected components.

The group $SO^+(1,3)$ is a 6-dimensional connected Lie group. It is not simply connected, and its universal cover is the Lie group $SL(2;\mathbb{C})$. This is conveniently seen as follows. Let $\mathfrak{her}_2$ be 4-dimensional real vector space of 2×2 Hermitean matrices. Since the determinant of an Hermitean matrix is a real number, we have a quadratic form

$$\det : \mathfrak{her}_2 \to \mathbb{R}$$

whose signature turns out to be $(1,3)$. This is seen by the linear isomorphism $\mathbb{R}^4 \to \mathfrak{her}_2$ given by

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} x^0 + x^3 & x^1 - \mathbf{i}x^2 \\ x^1 + \mathbf{i}x^2 & x^0 - x^3 \end{pmatrix}.$$

Indeed,

$$\det \begin{pmatrix} x^0 + x^3 & x^1 - \mathbf{i}x^2 \\ x^1 + \mathbf{i}x^2 & x^0 - x^3 \end{pmatrix} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

The group $SL(2;\mathbb{C})$ acts on $\mathfrak{her}_2$ by

$$A \mapsto PAP^*,$$

and this action clearly preserves the quadratic form det, so that we get a group homomorphism $SL(2;\mathbb{C}) \to O(1,3)$. Since $SL(2;\mathbb{C})$ is connected, the image of this homomorphism is contained in $SO^+(1,3)$; moreover the morphism induced at the Lie algebra level is an isomorphism and so $SL(2;\mathbb{C}) \to SO^+(1,3)$ is a covering. The kernel of this map is $\{\pm \mathrm{Id}\}$ so $SL(2;\mathbb{C}) \to SO^+(1,3)$ is a two-fold covering; moreover, since $SL(2;\mathbb{C})$ is simply connected, this is the universal covering of $SO^+(1,3)$:

$$SL(2,\mathbb{C}) \to SL(2,\mathbb{C})/\{\pm \mathrm{Id}\} \cong SO^+(1,3).$$

The Lorentz Lie algebra The Lie algebra $\mathfrak{so}_{1,3}$ of the Lorentz group is a 6-dimensional real Lie algebra. As a matrix algebra, it is the algebra of 4×4 real matrices A_j^i such that

$$A_k^i \eta_{il} + \eta_{ki} A_l^i = 0, \quad \text{for every } k, l.$$

As above, we will be interested in the complexification $\mathfrak{so}_{1,3;\mathbb{C}} = \mathfrak{so}_{1,3} \otimes \mathbb{C}$. A linear basis of $\mathfrak{so}_{1,3;\mathbb{C}}$ is given by the six matrices

$$J^{ab} = \mathbf{i}\,(\eta^{ac} E_c^b - E_c^a \eta^{cb}), \qquad a < b,$$

where E^i_j is the elementary matrix with 1 in position (i,j) and 0 elsewhere. Indeed,

$$(J^{ab})^i_k\eta_{il} + \eta_{ki}(J^{ab})^i_l = \mathbf{i}\,(\eta^{ac}\delta^{bi}\delta_{ck} - \delta^{ai}\delta_{ck}\eta^{cb})\eta_{il} + \mathbf{i}\,\eta_{ki}(\eta^{ac}\delta^{bi}\delta_{cl} - \delta^{ai}\delta_{cl}\eta^{cb})$$
$$= \mathbf{i}\,(\eta^{ak}\eta_{bl} - \eta^{kb}\eta_{al} + \eta_{kb}\eta^{al} - \eta_{ka}\eta^{lb}) = 0,$$

where we used the identities $\eta_{ij} = \eta_{ji}$ and $\eta^{ij} + \eta_{ij}$ for ay i,j. The commutation relations of the basis elements J^{ab} are

$$[J^{ab}, J^{cd}] = \mathbf{i}\,(\eta^{bc}J^{ad} + \eta^{ad}J^{bc} - \eta^{ac}J^{bd} - \eta^{bd}J^{ac}),$$

where we have set $J^{aa} = 0$ and $J^{ba} = -J^{ab}$ if $a < b$. It is convenient to represent the basis elements J^{ab} as $SO^+(1,3)$ left-invariant vector fields on $\mathbb{R}^4$, as we did for the Lie algebra of $SO(3)$:

$$J^{ab} = x^a\partial^b - x^b\partial^a,$$

where $\partial^i = \eta^{ij}\partial_j$. The group $SO(3)$ embeds in $SO^+(1,3)$ via

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

and so $\mathfrak{so}_3$ is a Lie subalgebra of $\mathfrak{so}_{1,3}$. The generators of the copy of $\mathfrak{so}_3$ inside $\mathfrak{so}_{1,3}$ are clearly J^{12}, J^{23} and J^{13}. For $i,j,k \in \{1,2,3\}$ one writes

$$L_i = \epsilon^{ijk}J^{jk}$$

so that

$$L_1 = J^{23}; \qquad J_2 = -J^{13}; \qquad L_3 = J^{12}.$$

In the physics lieterature, the generators L_1, L_2, L_3 are called the *infinitesimal rotations* aroud the spatial axes of $\mathbb{R}^{1,3}$. Note that these precisely corresponds to the elements od $\mathfrak{so}_3$ we denoted by J_1, J_2, J_3 in 9.

The reamining three generators of $\mathfrak{so}_{1,3}$ are the elements

$$K_i = J^{0i} = \mathbf{i}(x^0\partial^i - x^i\partial^0) = -\mathbf{i}(x^0\partial_i + x^i\partial_0).$$

These are called the *boosts* in the physics literature. The commutation relations of the generators J^{ab} are conveniently written in terms of the L_i and the K_j:

$$[L_i, L_j] = \mathbf{i}\epsilon_{ijk}L_k; \qquad [K_i, K_j] = -\mathbf{i}\epsilon_{ijk}L_k; \qquad [L_i, K_j] = \mathbf{i}\epsilon_{ijk}K_k.$$

Therefore, if we consider the complex basis

$$J_i^+ = \frac{1}{2}(L_i + \mathbf{i}K_i); \qquad J_i^- = \frac{1}{2}(L_i - \mathbf{i}K_i), \qquad i = 1,2,3,$$

the commutation relations become

$$[J_i^+, J_j^+] = \mathbf{i}\epsilon_{ijk}J_k^+; \qquad [J_i^-, J_j^-] = \mathbf{i}\epsilon_{ijk}J_k^-; \qquad [J_i^+, J_j^-] = 0.$$

So we have an isomorphism of complex Lie algebras

$$\mathfrak{so}_{1,3;\mathbb{C}} \cong \mathfrak{su}_{2;\mathbb{C}} \oplus \mathfrak{su}_{2;\mathbb{C}}.$$

It follows that irreducible complex representations of the Lorentz Lie algebra are tensor products of an irreducible representation of the "left" $\mathfrak{su}_2$ subalgebra and of an irreducible representation of the "right" $\mathfrak{su}_2$ subalgebra. In particular, as we will show in Section 1.13, complex irreducible representations of $\mathfrak{so}_{1,3}$ are indexed by a pair of nonnegative half integers (j^+, j^-), and we will write

$$(j^+, j^-) = (j^+, 0) \otimes (0, j^-)$$

to mean that the $\mathfrak{so}_{1,3}$ representation indexed by (j^+, j^-) is the tensor product of the representation of the left copy of $\mathfrak{su}_2$ indexed by j^+ and of the representation of the right copy of $\mathfrak{su}_2$ indexed by j^-.

1.7 The Poincaré group and its Lie algebra

We now extend the Lorentz group by adding to it translations of $\mathbb{R}^4$. More precisely, we are considering the semidirect product

$$\mathcal{P} = \mathbb{R}^4 \rtimes O(1,3),$$

where the Lorentz group acts on $\mathbb{R}^4$ by its defining representation. In particular we have a shoert exact sequence

$$0 \to \mathbb{R}^4 \to \mathcal{P} \to O(1,3) \to 1.$$

This semidirect product is the group of isometries of $\mathbf{R}^{1,3}$ as a Minkowskian manifold, and is called the Poincaré group. It is a (noncompact) 10-dimensional Lie group. Its Lie algebra $\mathfrak{p}$ is obtained by adding to the Lorentz Lie algebra $\mathfrak{so}_{1,3}$ four new generators, corresponding to a basis of the abelian Lie algebra $\mathbb{R}^4$ of *infinitesimal translations* of $\mathbb{R}^{1,3}$. It is customary to take as basis of $\mathbb{R}^4$ the infinitesimal translations along the coodinate axis. In the identification of elements of $\mathfrak{p}$ with vector fields on $\mathbb{R}^{1,3}$ these are just the coordinate vector fields ∂_i. In the complexified Poincaré Lie algebra we set

$$P^i = \mathbf{i}\partial^i = \mathbf{i}\eta^{ij}\partial_j.$$

The commutation relations involving the new generators P^i are then easily seen to be

$$[P^a, P^b] = 0; \qquad [P^a, J^{bc}] = \mathbf{i}(\eta^{ab}P^c - \eta^{ac}P^b).$$

A Casimir element for $\mathfrak{p}$ is

$$\|P\|^2 = P_a P^a = \eta_{ab} P^a P^b.$$

This is an immediate consequence of the fact that the Poincaré group acts isometrically on $\mathbb{R}^{1,4}$. Indeed, one trivially has $[\|P\|^2, P^a] = 0$ and, if $\Lambda = (\Lambda^a_b)$ is an element in the Lorentz subgroup, then

$$\Lambda^{-1}(P^a)\Lambda = \Lambda^a_b P^b,$$

where we have used that, since Λ is an element of the Lorentz group,

$$(\Lambda^{-1})^a_b = \eta_{bc}\Lambda^c_d\eta^{da}.$$

Hence

$$\Lambda^{-1}(\|P\|^2)\Lambda = \eta_{ab}\Lambda^a_c\eta_{ab}\Lambda^b_d P^c P^d = \eta_{cd}P^c P^d = \|P\|^2.$$

And the invariance of $\|P\|$ under the conjugacy action of the Lorentz group immediately gives the invariance of $\|P\|$ under the adjoint action of the Lorentz Lie algebra.

If one prefers a direct Lie algebra computation, then we trivially have $[\|P\|^2, P^a] = 0$, and

$$\begin{aligned}
[\|P\|^2, J^{bc}] &= \eta_{da}P^d[P^a, J^{bc}] + \eta_{da}[P^d, J^{bc}]P^a \\
&= \mathbf{i}(\eta_{da}P^d(\eta^{ab}P^c - \eta^{ac}P^b) + \eta_{da}(\eta^{db}P^c - \eta^{dc}P^b)P^a) \\
&= \mathbf{i}(\delta^b_d P^d P^c - \delta^c_d P^d P^b + \delta^b_a P^c P^a - \delta^c_a P^b P^a) \\
&= 0.
\end{aligned}$$

Therefore, if

$$\mathcal{P} \to U(\mathcal{H})$$

is a unitary representation, the Casimir element $\|P\|^2$ will act as an Hermitean operator

$$\|P\|^2 : \mathcal{H} \to \mathcal{H}$$

and so $\|P\|^2$-eigenspaces will be subrepresentations of $\mathcal{H}$. In particular, $\|P\|^2$ will act as a real scalar on every irreducible unitary representation of $\mathcal{P}$. This scalar is called the *mass* of the irreducible unitary representation. This is the first ingredient for Wigner's classification of irreducible unitary representations of the Poincaré group, we will describe in Section 1.9.

As remarked in Section 1.5, the Casimir operator $\|P\|^2$ corresponds to a (second-order) Poincaré biinvariant differential operator on $\mathbb{R}^{1,3}$. Explicitly, this second order operator is

$$-\Box = -\partial_a\partial^a = -\eta_{ab}\partial^a\partial^b.$$

So $\|P\|^2$ corresponds to the opposite of the *D'Alembert operator* on $\mathbb{R}^{1,3}$ and the eigenstates equation $\|P\|^2\phi = m^2\phi$ becomes the *Klein-Gordon equation*

$$(\Box + m^2)\phi = 0.$$

We will come back to this in Section 1.14

1.8 Induced representations and equivariant bundles

In this section we present the induced representation method. Essentially it consists in building a representation of a Lie group G from a representation of a closed Lie subgroup $H \subseteq G$.

Definition. 17. Given a representation V of H, the induced representation of G, usually written $\mathrm{Ind}_H^G(V)$ is by definition the vector space

$$\mathrm{Ind}_H^G(V) = \{\text{smooth functions } f : G \to V \text{ such that } f(hg) = hf(g) \quad \forall h \in H\}$$

with the obvious G-action on it.

This construction has a nice geometric interpretation: $\mathrm{Ind}_H^G(V)$ can be naturally realized as the space of sections of a G-equivariant vector bundle on a G-homogeneous space. Recall that a G-equivariant vector bundle $\mathcal{V}$ on a G-manifold M is the datum of a lifting of the G-action on M to a G-action on the total space of $\mathcal{V}$ which is linear on the fibers. Given G, H and V as before, define the vector bundle $G \times_H V$ over G/H by

$$G \times_H V = G \times V / \sim$$

where $\sim$ is the equivalence relation

$$(gh, v) \sim (g, hv) \quad \forall g \in G, \forall h \in H, \forall v \in V.$$

The projection map $\pi : G \times_H V \to G/H$ is given by $\pi(g, v) = gH$ and the G-action is given by $g'(g, v) = (g'g, v)$, for $g' \in G$. Denote by $\Gamma(G/H, G \times_H V)$ the vector space of sections of $G \times_H V$. This vector space carries a natural action of gra$G$: if σ is a section of $G \times_H V \to G/H$ and g is an element of G, then

$$(g\sigma)_x := g(\sigma_{g^{-1}\ x}).$$

There is a natural isomorphism of representations of G between $\Gamma(G/H, G \times_H V)$ and $\mathrm{Ind}_H^G(V)$. Indeed, given a section $s \in \Gamma(G/H, G \times_H V)$, let $f_s \in \mathrm{Ind}_H^G(V)$ be the function $f_s : G \to V$ defined by

$$f_s(g) = g^{-1}(s(g)).$$

Conversely, given $f \in \mathrm{Ind}_H^G(V)$, let $s_f \in \Gamma(G/H, G \times_H V)$ be section

$$s_f(g) = (g, f(g)).$$

It is straightforward to check that the given construction define morphisms of G-representations between $\Gamma(G/H, G \times_H V)$ and $\mathrm{Ind}_H^G(V)$ which are inverse each other.

The induced representation construction also has an important functroial interpretation: it is the adjoint of the restriction functor

$$|_H : G\text{-representation} \to H\text{-representations},$$

i.e. if V is a representation of H and W is a representation of G, then there is a natural isomorphism

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) = \mathrm{Hom}_H(W|_H, V).$$

This is known as *Frobenius reciprocity formula.*

Let us remark that the bigger the subgroup H is taken, the smaller the induced representation results. For instance if $H = \{e\}$ and $V = \mathbb{C}$, then $\mathrm{Ind}_H^G(V) = C^\infty(G;\mathbb{C})$, which is an enormous space.

It is also interesting to remark that in general, even if the representation V of H is irreducible, we can't state anything on the irreducibility of $\mathrm{Ind}_H^G(V)$. On the other hand, if $\mathrm{Ind}_H^G(V)$ is irreducible, then clearly the H-representation V is irreducible, since a splitting of V induces a splitting of $\mathrm{Ind}_H^G(V)$.

1.9 Wigner's theorem

Every unitary irreducible representation of the Poincaré group is induced by a representation of the stabiler subgroup $O(1,3)_{\vec{p}}$ of some point $\vec{p}$ for the standard action of the Lorentz group on $\mathbb{R}^{1,3}$, a result originally due to Wigner [Wig39]. Furthermore, in the next section we will classify all the possibilities for $\mathcal{P}_{\vec{p}}$.

Let

$$\mathcal{P} \to U(\mathcal{H})$$

be a unitary representation of the Poicaré group on an Hilbert space $\mathcal{H}$, and let

$$\mathfrak{p}_{\mathbb{C}} \to \mathrm{End}(\mathcal{H})$$

be the induced Lie algebra represention of the complexified Poincaré Lie algebra. As in the previous sections, let P^μ the generators of the translations or $\mathbb{R}^4$, multiplied by $\mathbf{i}$. Since $\mathcal{P}$ acts on $\mathcal{H}$ by unitary operators, the real Lie algebra $\mathfrak{p}$ acts on $\mathcal{H}$ by anti-Hermitean operators, and so the P^μ act as Hermitean operators. In particular, they are diagonalizable, with a real spectrum. Since they commute, they are simultaneously diagonalizable, and the Hilbert space $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = \bigoplus_p \mathcal{H}_{\vec{p}},$$

with $\vec{p} = (p^0, p^1, p^2, p^3)$ ranging in $\mathbb{R}^4$. In the above orthogonal decomposition, $\mathcal{H}_{\vec{p}}$ denotes the $\vec{p}$-eigenspace for the P^μ: elements of $\mathcal{H}_{\vec{p}}$ are vectors $|\vec{p}\rangle$ of $\mathcal{H}$ such that

$$P^\mu|\vec{p}\rangle = p^\mu|\vec{p}\rangle, \qquad \text{for } \mu = 0, \dots, 3.$$

The action of the translations subgroup of $\mathcal{P}$ is then recovered simply by exponentiation:

$$(\mathrm{Id}, \vec{a})|\vec{p}\rangle = e^{-\mathbf{i}a_\mu P^\mu}|\vec{p}\rangle = e^{-\mathbf{i}a_\mu p^\mu}|\vec{p}\rangle = e^{-\mathbf{i}(\vec{a}|\vec{p})}|\vec{p}\rangle,$$

where $(\vec{a}|\vec{p}) = \eta(\vec{a}, \vec{p})$ is the Minkowski inner product on $\mathbb{R}^{1,3}$. So in particular each subspace $\mathcal{H}_{\vec{p}}$ is stable for the action of the translation subgroup of $\mathcal{P}$, which acts by scalar multiplication on each $\mathcal{H}_{\vec{p}}$.

We are thus left with the problem of describing the action of the Lorentz group $O(3,1)$: given an eigenvector $|\vec{p}\rangle$, we want to see where it is mapped by an element $\Lambda \in O(3,1)$. The key to answer this question is to notice that the

Lie subalgebra of infinitesimal translation is preserved by the conjugacy action of P on $\mathfrak{p}_{\mathbb{C}}$. More precisely, recall that if $\Lambda = (\Lambda^\mu_\nu)$, we have

$$\Lambda^{-1}(P^\mu)\Lambda = \Lambda^\mu_\nu P^\nu,$$

and let us act on $\Lambda|\vec{p}\rangle$ with the operator P^μ. We have

$$P^\mu\Lambda|\vec{p}\rangle = \Lambda(\Lambda^{-1}P^\mu\Lambda)|\vec{p}\rangle = \Lambda(\Lambda^\mu_\nu P^\nu|\vec{p}\rangle) = \Lambda(\Lambda^\mu_\nu p^\nu|\vec{p}\rangle) = (\Lambda^\mu_\nu p^\nu)\,\Lambda|\vec{p}\rangle.$$

Therefore $\Lambda|\vec{p}\rangle$ is a P^μ-eigenvector, with eigenvalue $(\Lambda\vec{p})^\mu$. We express this in compact form as

$$\Lambda|\vec{p}\rangle = |\Lambda\vec{p}\rangle.$$

In other words, an element Λ in the Lorentz group induces an isomorphism

$$\Lambda : \mathcal{H}_{\vec{p}} \xrightarrow{\sim} \mathcal{H}_{\Lambda\vec{p}}.$$

Moreover, if $\pi_{\vec{p}} : \mathcal{H} \to \mathcal{H}_{\vec{p}}$ denotes the projection on the $\vec{p}$-eigenspace, then

$$\pi_{\vec{p}} = \Lambda^{-1}\pi_{\Lambda\vec{p}}\Lambda,$$

for any Λ in the Lorentz group. Indeed, both sides act as the zero operator on an eigenvector $|\vec{q}\rangle$ with $\vec{q} \neq \vec{p}$, whereas, acting on $|\vec{p}\rangle$ we have

$$\Lambda^{-1}\pi_{\Lambda\vec{p}}\Lambda|\vec{p}\rangle = \Lambda^{-1}\Lambda|\vec{p}\rangle = \pi_{\vec{p}}|\vec{p}\rangle,$$

since $\Lambda|\vec{p}\rangle$ is an element in the eigenspace $\mathcal{H}_{\Lambda\vec{p}}$. Summing up, for a fixed $\vec{p}_0$, the direct sum

$$\bigoplus_{\vec{p}\in O(1,3)\vec{p}_0} \mathcal{H}_{\vec{p}},$$

where $\vec{p}$ ranges in the Lorentz orbit of $\vec{p}_0$, is a subrepresentation of the representation of $\mathcal{P}$ we started with. So if the original representatin was irreducible and $\vec{p}$ is in the spectrum of the P^μ, then

$$\mathcal{H} = \bigoplus_{\vec{q}\in O(1,3)\vec{p}_0} \mathcal{H}_{\vec{p}}.$$

Let $m^2 = \|\vec{p}_0\|^2 = p_{0\mu}{p_0}^\mu$. The Lorentz group preserves the Minkowski norm, and, if $m^2 \neq 0$, acts transitively on the set on norm m^2 vectors, so that $\vec{p}$ is in the Lorentz orbit of $\vec{p}_0$ if and only if $\|p\|^2 = m^2$. Therefore, if $m^2 \neq 0$ we have found

$$\mathcal{H} = \bigoplus_{\|\vec{p}\|^2=m^2} \mathcal{H}_{\vec{p}}.$$

Note that from this, in particular we recover that the Casimir element $\|P\|^2$ acts as the scalar m^2 on $\mathcal{H}$, so that m^2 is the *mass* of the irreducible representation $\mathcal{H}$, in the notations of Section 1.7. Indeed, by the above decomposition, for any eigenvector $|\vec{p}\rangle$ in $\mathcal{H}$ we have

$$\|P\|^2|\vec{p}\rangle = \eta_{\mu\nu}P^\mu P^\nu|\vec{p}\rangle = \eta_{\mu\nu}p^\mu p^\nu|\vec{p}\rangle = m^2|\vec{p}\rangle.$$

Denote by X_{m^2} the *mass m^2 hyperboloid*, i.e., the set

$$X_{m^2} = \{\vec{p} \in \mathbb{R}^{1,3} \text{ such that } \|\vec{p}\|^2 = m^2\}$$

Then the collection of eigenspaces $\mathcal{H}_{\vec{p}}$ is a vector bundle $\mathcal{E}_{m^2}$ over X_{m^2}, and the Hilbert space $\mathcal{H}$ is naturally identified with the Hilbert space of L^2-sections of this bundle (with respect to the spectral measure on X_{m^2}). Indeed, each vector ψ in $\mathcal{H}$ is identified with the section σ_ψ defined by $\sigma_\psi : \vec{p} \mapsto \pi_{\vec{p}}(\psi)$, where $\pi_{\vec{p}} : \mathcal{H} \to \mathcal{H}_{\vec{p}}$ is the projection on the $\vec{p}$-eigenspace. Moreover the vector bundle $\mathcal{E}$ is clearly $O(1,3)$-equivariant, and the map

$$\sigma : \mathcal{H} \xrightarrow{\sim} \{\text{sections of } \mathcal{E}_{m^2} \text{ over } X_{m^2}\}$$

is an isomorphism of representations of $O(1,3)$. This last statement is nothing but the identity $\pi_{\vec{p}} = \Lambda^{-1}\pi_{\Lambda\vec{p}}\Lambda$ derived above. Therefore we see that we are precisely in the situation described in Section 1.8: let $\mathcal{P}$ act on $\mathbb{R}^{1,3}$ via if we denote by $O(1,3)_{\vec{p}_0}$ the stabilizer of $\vec{p}_0$ under the Lorentz action, then the Lorentz orbit of $\vec{p}$ is

$$X_{m^2} = O(1,3)/O(1,3)_{\vec{p}_0}$$

and the representation of the Lorentz subgroup of $\mathcal{P}$ on $\mathcal{H}$ is the representation induced on the space of sections of an equivariant bundle on a Lorentz-homogeneous space by a representation of the stabilizer subgroup $O(1,3)_{\vec{p}_0}$ on the fiber $\mathcal{H}_{\vec{p}}$. The description of the Poincaré action is completed by recalling that the subgroup of translations acts by scalar mutiplication by $e^{\mathbf{i}(\vec{a}|\vec{p})}$ on the fibers.

The situation for $m^2 = 0$ is completely similar, but we have to distinguish two cases. Indeed the Lorentz action on the set of zero-norm vectors is not transitive but there are two orbits: one consisting of the 0 vector alone (the *vacuum state* of physics parlance), and the other consisting of all nonzero zero-norm vectors of $\mathbb{R}^{1,3}$ (the *light cone* in the physics jargon). Also, almost nothing changes if instead of the Poincaré group $\mathcal{P}$ we consider the universal cover of the connected component of the identity:

$$\tilde{\mathcal{P}} = \mathbb{R}^4 \rtimes \widetilde{SO^+}(1,3).$$

Indeed, nothing changes at the Lie algebra level, and at the group level the only difference is that the orbit space for the action of the universal cover $SL(2;\mathbb{C})$ of the proper orthochronous Lorentz group $SO^+(1,3)$ on $\mathbb{R}^{1,3}$ is more refined than the orbit space for the full Lorentz group.

Therefore we have finally proven the following.

Theorem. 18 (Wigner)**.** *Irreducible unitary representation of the Poincaré group $\mathcal{P}$ are classified by pairs $(\mathcal{O}, s)$, where $\mathcal{O}$ is a Lorentz orbit in $\mathbb{R}^{1,3}$ and s is (the isomorphism class of) an irreducible representation of the stabilizer $O(1,3)_{\vec{p}}$ of a point $\vec{p}$ in $\mathcal{O}$ under the Lorentz action. Moreover, the $(\mathcal{O}, s)$-representation is induced by the s-representation of $O(1,3)_{\vec{p}}$. The analogous statement holds for the universal cover $\mathbb{R}^4 \rtimes SL(2;\mathbb{C})$ of the identity element in the Poincaré group.*

The stabilizer subgroups of $SL(2;\mathbb{C})$ in Wigner's theorem are called *little groups* in the physics terminology.

1.10 $SL(2;\mathbb{C})$-orbits on $\mathbb{R}^{1,3}$ and elemetary particles

We now exhibit the classification $SL(2;\mathbb{C})$-orbits on $\mathbb{R}^{1,3}$ and the corresponding little groups. In view of Wigner's theorem this gives a classification of the irreducible unitary representations of $\mathcal{P}$. When the unitary irreducible representation of the little group involved is finite dimensional, the corresponding representation of $\mathbb{R}^4 \rtimes SL(2;\mathbb{C})$ is called an *elementary particle* in physics.

1. The origin $\{0\} \in \mathbb{R}^4$ is a singleton orbit, stabilized by the whole $SL(2;\mathbb{C})$. As we are going to show below, the only finite dimensional unitary irreducible representation of $SL(2;\mathbb{C})$ is the trivial one. The corresponding representation of the Poincaré group is called the *vacuum state* in physics.

2. $\{\vec{p} \in \mathbb{R}^{1,3} | p_\mu p^\mu = 0, p_0 > 0\}$. Up to rescaling, a representative for this orbit is $\vec{p}_0 = (1,0,0,1)$. Recalling the definition of the $SL(2;\mathbb{C})$-action on $\mathbb{R}^{1,3}$, the corresponding little group is

$$H = \{A \in SL_2(\mathbb{C}) | AM_{\vec{p}_0}A^* = M_{\vec{p}_0}\}, \quad M_{\vec{p}_0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the equation $AM_{\vec{p}_0}A^* = M_{\vec{p}_0}$ becomes

$$\begin{pmatrix} |a|^2 & a\bar{c} \\ c\bar{a} & |c|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and so we find $|a| = 1$, $b \in \mathbb{C}$ and $c = 0$, i.e.,

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix}, |a| = 1 \right\}.$$

Consider now the bijection $\mathbb{C} \times U(1) \leftrightarrow H$ given by

$$(z,\zeta) \leftrightarrow \begin{pmatrix} z & \bar{z}\zeta \\ 0 & \bar{z} \end{pmatrix}$$

With this notation, the multiplication in H reads

$$(z,\zeta)(z'\zeta') \leftrightarrow \begin{pmatrix} z & \bar{z}\zeta \\ 0 & \bar{z} \end{pmatrix} \begin{pmatrix} z' & \bar{z}'\zeta' \\ 0 & \bar{z}' \end{pmatrix} = \begin{pmatrix} zz' & z\bar{z}'\zeta' + \bar{z}\bar{z}'\zeta \\ 0 & \bar{z}\bar{z}' \end{pmatrix} \leftrightarrow (zz', z^2\zeta' + \zeta).$$

Hence, the group H is identified the semidirect product $\mathbb{C} \rtimes U(1)$ with $U(1)$ acting on $\mathbb{C}$ by $(z,\zeta) \mapsto z^2\zeta$. This is the double cover of the semidirect product of $\mathbb{C}$ with $U(1)$ given by the standard $U(1)$ action $(z,\zeta) \mapsto z\zeta$.

Since the standard $U(1)$-action on $\mathbb{C}$ is naturally identified with the $SO(2)$-action on $\mathbb{R}^2$, the little group H is the double cover of the group $SE(2)$ of orientation preserving (affine) isometries of the Euclidean plane. Its finite dimensional unitary irreducible representations are classified by an half-integer ε, called the *helicity* of the representation. The corresponding elementary particle is called a *massless helicity ε particle*.

3. $\{p \in \mathbb{R}^4 | p_\mu p^\mu = 0, p_0 < 0\}$. The little group is the analogous of the precedent case. The corresponding elementary particle is called a *massless helicity ε antiparticle*.

4. $\{p \in \mathbb{R}^4 | p_\mu p^\mu > 0, p_0 > 0\}$. Up to rescaling, a representative for this orbit is $\vec{p}_0 = (1, 0, 0, 0)$, which corresponds to the Hermitean matrix

$$M_{\vec{p}_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the little group is

$$H = \{A \in SL(2; \mathbb{C}) | A\mathrm{Id}A^* = \mathrm{Id}\} = SU(2).$$

We will show next that irreducible unitary representations of $SU(2)$ are classified by a nonnegative half-integer ℓ called the *spin*. The corresponding elementary particle is called a *mass m spin ℓ particle*.

5. $\{p \in \mathbb{R}^4 | p_\mu p^\mu > 0, p_0 < 0\}$. The little group is the analogous of the previous case. The corresponding elementary particle is called a *mass m spin ℓ antiparticle*.

6. $\{p \in \mathbb{R}^4 | p_\mu p^\mu < 0\}$ Up to rescaling, a representative for this orbit is $\vec{p}_0 = (0, 1, 0, 0)$. The little group for this orbit is then:

$$H = \{A \in SL(2; \mathbb{C}) | AM_{\vec{p}_0}A^* = M_{\vec{p}_0}\}, \qquad M_{\vec{p}_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It is convenient to rewrite the defining relation for elements of H as $A = M_{\vec{p}_0}(A^*)^{-1}M_{\vec{p}_0}^{-1}$. The right-hand side of this expression is

$$M_{\vec{p}_0}(A^*)^{-1}M_{\vec{p}_0}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \bar{A},$$

so that $H = SL(2; \mathbb{R})$. Corresponding elementary particles are *nonphysical* since they would have negative mass-square.

1.11 Finite dimensional unitary representations of $SL(2, \mathbb{C})$

Here we prove that the only finite dimensional unitary representation of $SL(2; \mathbb{C})$ is the trivial one. We borrow the following argument from [KnT00], where the case of $SL(2; \mathbb{R})$-representations is treated. Let

$$\rho : SL(2; \mathbb{C}) \to U(n)$$

be a continuous reperesentation, and consider the subset of $SL(2;\mathbb{C})$ consisting of the matrices of the form

$$A(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \qquad z \neq 0.$$

All these matrices have the same Jordan normal form, so they are in the same $SL(2;\mathbb{C})$-conjugacy class. Therefore their images $\rho(A(z))$ are in the same $U(n)$-conjugacy class. Let us call this conjugacy class C. Since $U(n)$ is compact, conjugacy classes in $U(n)$ are closed, so

$$\mathrm{Id}_{U(n)} = \lim_{z\to 0} \rho(A(z))$$

is an element of C (here we have used the continuity of ρ). But this means that C is the conjugacy class of the identity in $U(n)$ and so consists of the identity alone. Since by construction $\rho(A(z))$ lies in C for every $z \neq 0$, we have thus shown that $\rho(A(z)) = \mathrm{Id}_{U(n)}$ for every z in $\mathbb{C}$. The identical argument applies to the matrices

$$B(w) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \qquad w \neq 0.$$

The matrices $A(z)$ and $B(w)$ generate $SL(2;\mathbb{C})$, as is easily seen by noticing that the two matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generate the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Therefore, ρ is the trivial representation.

It is worth remarking that $SL(2;\mathbb{C})$ admits nontrivial infinite dimensional unitary representations, see, e.g. [DaN67].

1.12 Finite dimensional unitary representations of $\mathbb{C}\rtimes U(1)$

In this section we show that irreducible unitary finite-dimensional representations of the double cover $\widetilde{SE}(2)$ of the group $SE(2)$ of affine isometries of $\mathbb{R}^2$ are classified by an half-integer ε; moreover ε is an integer precisely when the given reresentation $\widetilde{SE}(2) \to U(n)$ factors through $\widetilde{SE}(2) \to SE(2)$. We thank Andrea Maffei for having shown us this proof.

We begin by recalling that $\widetilde{SE}(2)$ is the semidirect product $\mathbb{C} \rtimes U(1)$, with $U(1)$ acting on $\mathbb{C}$ by $(z, \zeta) \mapsto z^2\zeta$. Let now

$$\rho : \mathbb{C} \rtimes U(1) \to U(n)$$

be a finite dimensional representation. The restriction of ρ to the normal subgroup $\mathbb{C}$ gives a representation

$$\rho|_{\mathbb{C}} : \mathbb{C} \to U(n)$$

of the real Lie group $\mathbb{C}$ on the complex vector space $\mathbb{C}^n$. This gives a splitting

$$\mathbb{C}^n = \bigoplus_{\varphi \in I} V_\varphi$$

where I is a finite subset of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$ and V_φ is the subspace of $\mathbb{C}^n$ where $\mathbb{C}$ acts via ρ as

$$\rho(1, \zeta)(\vec{v}) = e^{\mathbf{i}\varphi(\zeta)}\vec{v}.$$

We now look at the $U(1)$-action. Since $(1, z^2\zeta) \cdot (z, 0) = (z, 0) \cdot (1, \zeta)$ we have

$$\rho(z, 0) : V_\varphi \to V_{z^{-1*}\varphi},$$

for any $\varphi \in I$, where $(z^{-1^*}\varphi)(\zeta) = \varphi(z^{-2}\zeta)$. Indeed, if $\vec{v}$ is an element of V_φ, then

$$\rho(1, \zeta)\left(\rho(z, 0)\vec{v}\right) = \rho(z, 0)\left(\rho(1, z^{-2}\zeta)\vec{v}\right) = e^{\mathbf{i}\varphi(z^{-2}\zeta)}\rho(z, 0)\vec{v}.$$

Therefore $U(1)$ acts as a permutation group on the elemnets of the finite set I. Since $U(1)$ is connected, this permutation action is trivial, so $z^{-1^*}\varphi = \varphi$ for every $\varphi \in I$. But the only $U(1)$-invariant element in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$ is the zero morphism, hence the normal subgroup $\mathbb{C}$ of $\mathbb{C} \rtimes U(1)$ acts trivially on $\mathbb{C}^n$ and the representation ρ factors through $\mathbb{C} \rtimes U(1) \to U(1)$. We are therefore reduced to classify irreducible unitary representations of $U(1)$. Since $U(1)$ is an abelian Lie group, these are all 1-dimensional, and so everything boils down to the problem of describing Lie group homomorphisms from $U(1)$ to itself. And it is well known (and easy to show) that these are all of the form $z \mapsto z^k$ for some integer k. Writing $k = 2\varepsilon$ for an half-integer ε we finally find that irreducible unitary representations of the double cover $\widetilde{SE}(2)$ of the group $SE(2)$ of affine isometries of $\mathbb{R}^2$ are classified by an half-integer ε, and that ε is an integer precisely when the given representation factors through $\widetilde{SE}(2) \to SE(2)$. In the language of particle physics, the half-integer ε is called the *helicity* of the massless particle corresponding to the given irreducible representation of $\widetilde{SE}(2)$.

1.13 Irreducible representations of $SU(2)$

Since the Lie group $SU(2)$ is compact, all its irreducible representations are finite dimensional. Moreover, since $SU(2)$ is simply connected, its representation theory is equivalent to the representation theory of its Lie algebra. A set of generators for $\mathfrak{su}(2)$ is given by Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By complexifying the Lie algebra $\mathfrak{su}_2$ we obtain

$$\mathfrak{su}_{2;\mathbb{C}} = \mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C}).$$

The latter is the Lie algebra of the complex Lie group $SL(2;\mathbb{C})$. Therefore, we are reduced to studying complex finite dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. We ar going to show that there is exactly one such representation (up to isomorphism), for any dimension, which can be explicitly described. This is conveniently done by fixing the following set of generators for $\mathfrak{sl}_2(\mathbb{C})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutation rules

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Having fixed this notation, we have

Theorem. 19. *For each nonnegative half-integer l there exists a unique (up to isomorphism) irreducible complex linear representation of $\mathfrak{sl}_2(\mathbb{C})$ on a complex vector space of dimension $2\ell + 1$. Moreover, there is a basis*

$$\{v_{-2\ell}, v_{-2\ell+2}, \dots, v_{2\ell-2}, v_{2\ell}\}$$

of V such that:

1. $hv_{2(\ell-j)} = 2(\ell - j)v_i$,
2. $ev_{2(\ell-j)} = j(2\ell - j + 1)v_{2(\ell-j+1)}$, *with* $ev_{2\ell} = 0$,
3. $fv_{2(\ell-j)} = v_{2(\ell-j-1)}$, *with* $fv_{-2\ell} = 0$.

Proof. Let V be a complex linear irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $2\ell + 1$, and denote by V_λ the λ-eigenspace of V with respect to the operator h. We begin by showing that

$$e : V_\lambda \to V_{\lambda+2}; \qquad f : V_\lambda \to V_{\lambda-2}.$$

Indeed,

$$h(ev) = ehv + [h, e]v = e(\lambda v) + 2ev = (\lambda + 2)ev,$$

and similarly for f. Assume $V_\lambda \neq \{0\}$. By finite dimensionality of V, only finitely many eigenspaces $V_{\lambda+2k}$ can be nonzero, so there exists $k_0 \in \mathbb{N}$ with $V_{\lambda+2k_0} \neq 0$ and $V_{\lambda+2k_0+2} = 0$. Pick a nonzero vector $v_{2\ell}$ in $V_{\lambda+2k_0}$ and for any $j \geq 0$, set $v_{2(\ell-j)} = f^j v_{2\ell}$. Since $v_{2(\ell-j)}$ is an element of $V_{\lambda+2(k_0-j)}$ there exist a minimum k such that $v_{2(\ell-(k+1))} = 0$. Then the vectors $v_{2\ell}, v_{2(\ell-1)}, \dots, v_{2(\ell-k)}$ are linearly independent (since they are nonzero and belong to h-eigenspaces for distinct eigenvalues). Now we show that $V = \mathrm{span}\{v_{2(\ell-k)}, \dots, v_{2\ell}\}$. Since by hypothesis V is irreducible, it is sufficient to show that $\mathrm{span}\{v_{2(\ell-k)}, \dots, v_{2\ell}\}$ is stable under e, f, h. The f- and h-stability is obvious, so we needonly to check the e-stability. This results from

$$ev_{2(\ell-j)} = j(\lambda + 2k_0 - j + 1)v_{2(\ell-j+1)},$$

with $v_{2(\ell-j)} = 0$ for $j < 0$ and $j > k$, which we prove inductively. Assume we have proved the statement for j. Then for $j+1$ we have:

$$\begin{aligned} ev_{2(\ell-(j+1))} &= ef \cdot v_{2(\ell-j)} \\ &= hv_{2(\ell-j)} + fev_{2(\ell-j)} \\ &= (\lambda + 2k_0 - 2j)v_{2(\ell-j)} + j(\lambda + 2k_0 - j + 1)fv_{2(\ell-j+1)} \\ &= (j+1)(\lambda + 2k_0 - j)v_{2(\ell-j)} \end{aligned}$$

So $V = \mathrm{span}\{v_{2(\ell-k)}, \ldots, v_{2\ell}\}$. In particular $\dim V = k+1$ and so $k = 2\ell$. This means that our basis for V is actually

$$\{v_{-2\ell}, v_{-2\ell+2}, \ldots, v_{2\ell-2}, v_{2\ell}\}.$$

It is now easy to show that $\lambda + 2k_0 = 2\ell$. Indeed, since $h = [e, f]$ is a commutator, h is traceless in any representation. Computing the trace of h on the vector space $\mathrm{span}\{v_{2(\ell-j)}\}_{j=0,\ldots,2\ell}$ we therefore find

$$0 = \sum_{j=0}^{2\ell} (\lambda + 2k_0 - 2j) = (\lambda + 2k_0 - 2\ell)(2\ell + 1).$$

Therefore, our basis vectors satisfy:

- $hv_{2(\ell-j)} = 2(\ell - j)v_i$,
- $ev_{2(\ell-j)} = j(2\ell - j + 1)v_{2(\ell-j+1)}$, with $ev_{2\ell} = 0$,
- $fv_{2(\ell-j)} = v_{2(\ell-j-1)}$, with $fv_{-2\ell} = 0$.

This shows that up to isomorphisms, a $2\ell + 1$-dimensional irreducible representation od $\mathfrak{sl}_2(\mathbb{C})$ is completely determined by its dimension. To prove the existence of an irreducible $2\ell+1$-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$, consider the free vector space over the set $\{v_{-2\ell}, v_{-2\ell+2}, \ldots, v_{2\ell-2}, v_{2\ell}\}$ and define an $\mathfrak{sl}_2(\mathbb{C})$-action by defining the action of the generators e, f, h on the basis elements $v_{2(\ell-j)}$ by the above formulas. To see that the representation defined this way is irreducible, let U be a nonzero invariant subspace. Since U is invariant under $h\cdot$, U is spanned by a nonempty subset of the basis $\{v_{2(\ell-j)}\}$. By repeatedly applying e and f to any vector in the basis $\{v_{2(\ell-j)}\}$ one obtains all the others (up to nonzero scalar multiples), and so $U = V$. Δ

Remark. 20. In Section 1.5 we anticipated that the momentum operator $\|J\|^2$ acts as the scalar $\ell(\ell+1)$ on the $2\ell+1$ irreducible representation V of $SU(2)$. We can now prove this statement. Let $\{v_{2(\ell-j)}\}_{j=0,\ldots,2\ell}$ be a distinguished basis of V as described above. Since $\|J\|^2$ acts as a scalar on V, we can compute this scalar simply by looking at the action of $\|J\|^2$ on the vector $v_{2\ell}$. We have

$$J_1 = -\frac{1}{2}h; \qquad J_2 = \frac{\mathbf{i}}{2}(e - f); \qquad J_3 = -\frac{1}{2}(e + f)$$

and so

$$\|J\|^2 v_{2\ell} = \frac{1}{4}(h^2 + 2ef + 2fe)v_{2\ell} = \frac{1}{4}(4\ell^2 + 4\ell)v_{2\ell} = \ell(\ell+1)v_{2\ell}.$$

1.14 The Klein-Gordon equation

As an illustrative example of the above construction, we investigate the solutions of the Klein-Gordon equation on $\mathbb{R}^{1,3}$. We begin by considering the defining action of the Poincaré group on $\mathbb{R}^{1,3}$, lifted to an action of $\mathbb{R}^4 \rtimes SL(2;\mathbb{C})$. Since the standard Lesbegue measure on $\mathbb{R}^{1,3}$ is translation- and Lorentz-invariant, we have an induced unitary representation on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{1,3})$ of square-integrabel functions on $\mathbb{R}^{1,3}$. Passing to Lie algebras, the (universal cover of the) Poincaré group acts by vector fields, i.e., by derivations. A major technical point to be stressed is that the first-order differential operator one gets this way, i.e., corresponding to elements in $\mathfrak{p}_{\mathbb{C}}$ are only *densely defined* on $\mathcal{H}$. For instance, the element P^μ corresponds to the derivation $\mathbf{i}\partial^\mu$. This means that $\mathfrak{p}_{\mathbb{C}}$ can not be handled as a Lie algebra of operators on $\mathcal{H}$ within the framework of classical Hilbert spaces. A rigorous treatment can be given within the framework of *rigged Hilbert spaces/Gelfan'd triples*, see [GeV64]. Yet, such a rigorous treatment is not the aim of this note, so we will kindly ask the reader to pretend that $\vec{x} \mapsto e^{-\mathbf{i}(\vec{p}|\vec{x})}$ is a square-integrable function on $\mathbb{R}^{1,3}$.

Let now $m^2 > 0$ and consider the m^2-eigenspace equation

$$\|P\|^2\phi = m^2\phi$$

for the Casimir element $\|P\|^2$. Since P^μ acts on $\mathcal{H}$ as $\mathbf{i}\partial^\mu$, this equation is the Klein-Gordon equation on $\mathbb{R}^{1,3}$:

$$(\Box + m^2)\phi = 0,$$

where $\Box = \partial_\mu\partial^\mu$ is the D'Alembert operator on $\mathbb{R}^{1,3}$. Therefore, we see that the space of solutions of the Klein-Gordon equation is a representation of the Poincaré group. Obviously, this could be directly seen by the manifest Poincaré-invariance of the D'Alembert operator. What we have gained by deriving this from the abstract nonsense of Casimir operators is that we now see the Klein-Gordon equation as a stand-alone equation, but as a piece of the larger picture of Wigner's investigation of unitary representations of the Poincaré group. And in the larger picture we know that the representation of the Poincaré group given by solutions of the Klein-Gordon equation is induced by a representation of the little group at $\vec{p}$ on $\mathcal{H}_{\vec{p}}$, where $\vec{p}$ is a point in $\mathbb{R}^{1,3}$ with $\|\vec{p}\|^2 = m^2$. By definition $\mathcal{H}_{\vec{p}}$ is the $\vec{p}$-eigenspace for the action of the infinitesimal translations P^μ, hence it is defined as the space of the joint solutions of the first-order differential equations

$$\mathbf{i}\partial^\mu\phi = p^\mu\phi.$$

Therefore we find that $\mathcal{H}_{\vec{p}}$ consists of the functions

$$\phi(\vec{x}) = e^{-\mathbf{i}(\vec{p}|\vec{x})}\phi_0, \qquad \phi_0 \in \mathbb{C}.$$

In particular it is a 1-dimensional space, and so for $m^2 > 0$ the Klein-Gordon equation describes a spin 0 mass m elementary particle. We will come back to this when discussing the *massive scalar field*.

Since $\mathcal{H}_{\vec{p}}$ is 1-dimensional, the general discussion in Section 1.9 tells us that the space of solutions of the Klein-Gordon equation is naturally identified with the space of sections of a line bundle over the mass m^2 hyperboloid. This can be nicely interpreted as the Fourier transform of the Klein-Gordon equation,

$$(\|p\|^2 - m^2)\hat{\phi}(\vec{p}) = 0$$

telling us that $\hat{\phi}$ is a distribution on $\mathbb{R}^{1,3}$ which is supported on the mass m^2 hyperboloid $\|p\|^2 - m^2$. If $\vec{p}_0$ is a vector in $\mathbb{R}^{1,3}$, the $\vec{p}_0$-eigenspace equations become, via the Fourier transform,

$$(p^\mu - p_0{}^\mu)\hat{\phi}(\vec{p}) = 0,$$

which show that $\vec{p}_0$-eigenstates are distributions supported at the point $\vec{p}$. Since a distribution supported on a point is a finite linear combination of derivatives of δ-functions at that point, we have

$$\hat{\phi} = \sum_{|I| \leq n} a_I \partial_I \delta_{\vec{p}_0}$$

for suitable coefficients a_I in $\mathbb{C}$, see, e.g., [Vl79]. The condition that $\hat{\phi}$ is annihilated by the ideal $(p^\mu - p_0{}^\mu)_{\mu=0,\ldots,3}$ then implies that all the coefficients a_I with $|I| > 0$ vanish, i.e. $\hat{\phi}$ is a scalar multiple of the Dirac's δ at $\vec{p}_0$:

$$\hat{\phi}(\vec{p}) = \sigma(\vec{p})\delta_{\vec{p}_0}(\vec{p}),$$

and σ is naturally interpreted as a section of the trivial line bundle over the mass m^2 hyperboloid.

2 Dirac operators and the massive vector field

2.1 Equivariant maps and G-algebras

Let now M be a differential manifold, and A be a $\mathbb{K}$-algebra, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then we can consider the algebra

$$C^\infty(M; A)$$

of smooth maps from M to A. If A is finite-dimensional, then $C^\infty(M; A) \cong C^\infty(M; \mathbb{K}) \otimes A$. If a Lie group G is given, acting on the manifold M as a group of diffeomorphisms, and on the algebra A as a group of algebra automorphisms, then we can consider the subalgebra

$$C^\infty(M; A)^G$$

of G-equivariant maps from M to A, i.e., the algebra of maps $\Phi : M \to A$ making the diagram

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d, "g"'] & A \arrow[d, "g"] \\ M \arrow[r, "\Phi"] & A \end{tikzcd}$$

commute for every g in G. Note that, if A is finite-dimensional, this is nothing but the algebra of G-invariant elements in the tensor product $C^\infty(M; \mathbb{C}) \otimes A$ for the standard action of G on $C^\infty(M; \mathbb{K})$, namely $(g, \Phi) \mapsto (g^{-1})^*\Phi$. As a particular case, when the G-manifold M is a vector space V on which G acts linearly, we can consider the algebra of *polynomial* G-equivariant A-valued maps on V,

$$\mathrm{Sym}^\bullet(V; A)^G.$$

A remarkable case we will consider is the following: G is a Lie group acting on a vector space V, and we take $A = \mathrm{End}(V)$ with its natural G-algebra structure given by conjugation action of G. In this case, if $\Phi : M \to \mathrm{End}(V)$ is a G-equivariant map, the for every point x in X, the G-action on V induces an isomorphism between $\ker(\Phi_x)$ and $\ker(\Phi_{gx})$, for any x in M. Indeed, if v is an element of $\ker(\Phi_x)$, then

$$\Phi_{gx}(gv) = g(\Phi_x(g^{-1}gv)) = g(\Phi_x(v)) = 0.$$

This means that the family of subspaces $\ker(\Phi_x)$ of V defines a G-equivariant vector bundle on the orbit of a point x in M. We therefore have a representation of G on the sections of this bundle; this representation is the representation induced by the action of the stabilizer G_x of x on the space $\ker(\Phi_x)$. We will coem back to this example when studying the representations of the Poincaré group associated with the Dirac operator and with the massive vector field operator.

Example. 21. If $A = \mathbb{K}$ with the trivial G-action, then $C^\infty(M;\mathbb{K})^G$ is the subalgebra of $C^\infty(M;\mathbb{K})$ consisting of G-invariant functions. If V is a $\mathbb{K}$-vector space, then the inclusion $\mathbb{K} \hookrightarrow \mathrm{End}(V)$ of $\mathbb{K}$ as the subalgebra of scalar endomorphisms of V is G-equivariant, and so induces a natural morphism of algebras of G-equivariant maps

$$C^\infty(M;\mathbb{K})^G \hookrightarrow C^\infty(M;\mathrm{End}(V))^G.$$

Example. 22. Let V be a real vector space endowed with a nondegenerate symmetric pairing, and let $O(V)$ be its isometry group. The pairing on V induces an isomorphism between V and its dual, so we have a quadratic map $\Phi : V \to \mathrm{End}(V)$ sending a vector v to the endomorphisms $|v\rangle\langle v|$. The map Φ is clearly $O(V)$-equivariant. Indeed, for any v in V and for every g in $O(V)$, we have

$$(g \cdot \Phi_v)(w) = |gv\rangle\langle v|g^{-1}w\rangle = |gv\rangle\langle gv|w\rangle,$$

i.e., $g \cdot \Phi_v = \Phi_{gv}$. We will come back to this example when discussing the massive scalar field.

Example. 23. Let V be a real vector space endowed with a nondegenerate symmetric pairing, and let $Cl(V)$ be its *Clifford algebra*, i.e., the quotient of the free associative algebra with unit on V by the ideal I_V generated by the elements $v \cdot w + w \cdot v - 2\langle v|w\rangle$. Since the isometry group $O(V)$ preserves the the ideal I_V, the canonical linear morphism $V \hookrightarrow Cl(V)$ is $O(V)$-equivariant. We will come back to this example when describing the Dirac operator.

2.2 The Skolem-Noether theorem

We now study more in detail the case $A = \mathrm{End}(V)$. In the examples in the previous section we have seen that $\mathrm{End}(V)$ can be given a G-algebra structure via a representation of G on V. The Skolem-Noether theorem says this is essentially the only possibility. To state it precisely, we need recalling the following.

Definition. 24. A $\mathbb{K}$-algebra S is *simple* if it has no nontrivial two-sided ideal; it is *central simple* if it is simple and $Z(S) = \mathbb{K}$.

Example. 25. Let V be a finite dimensional vector space over $\mathbb{K}$; then the algebra $\mathrm{End}(V)$ is a central simple $\mathbb{K}$-algebra.

Theorem. 26 (Skolem-Noether)**.** *Let S be a finite dimensional central simple $\mathbb{K}$-algebra, and let R be a simple $\mathbb{K}$-algebra. If $f, g : R \to S$ are nonzero homomorphisms, then there is an inner automorphism $\alpha : S \to S$ such that $\alpha f = g$. In particular, if $R = S$, taking $f = \mathrm{Id}_S$ one sees that every automorphism of S is inner.*

As recalled above, the algebra of endomorphisms of a finite dimensional vector space is central simple, therefore every automorphism of $\mathrm{End}(V)$ is inner. Since this is the only case of the Skolem-Noether theorem we will be concerned with, let us give a direct and elementary proof of this particular case, addressing the reader to [FaD93] for the general case. To prove that every automorphism of $\mathrm{End}(V)$ is inner, fix a basis $\{e_1, \ldots, e_n\}$ of V and denote by $E_i^j : V \to V$ the endomorphism of V defined by mapping e_i to e_j and all the other basis vectors to zero. The endomorphisms E_i^j are a linear basis for $\mathrm{End}(V)$. Let now $\varphi : \mathrm{End}(V) \to \mathrm{End}(V)$ be an automorphism, and let $F_i^j = \varphi(E_i^j)$. Then the F_i^j are a linear basis of endomorphisms of V and satisfy the same relations as the E_i^j. In particular, they are rank 1 operators. Fix a nonzero vector f in the image of F_1^1 and set $f_i = F_1^j f$. Then the vectors f_i are a basis of V and so there exists an element g in $GL(V)$ such that $g(e_i) = f_i$ for any i in $1, 2 \ldots, n$. It is then immediate to see that $F_i^j = gE_i^j g^{-1}$, i.e., $\varphi(E_i^j) = gE_i^j g^{-1}$ for any i, j. Since the E_i^j are a linear basis of $\mathrm{End}(V)$ this proves that φ is the inner automorphism induced by g.

The result just proved can be stated as follows: there is a surjective group homomorphism

$$GL(V) \to \mathrm{Aut}(\mathrm{End}(V)).$$

It is immediate to determine the kernel of this homomorphism: an automorphism of V acts trivially by conjugation on $\mathrm{End}(V)$ if and only if it is a scalar. Therefore we have:

Proposition. 27. *If V is a finite dimensional vector space over $\mathbb{K}$, then there is a short exact sequence*

$$1 \to \mathbb{K}^* \to GL(V) \to \mathrm{Aut}(\mathrm{End}(V)) \to 1,$$

i.e., the group of automophisms of the algebra $\mathrm{End}(V)$ *is naturally identified with the group $PGL(V)$ of projective linear transformations of V.*

Assume now a Lie group G is acting on $\mathrm{End}(V)$ as a group of algebra automorphisms. By the above proposition this is equivalent to the datum of a *projective representation* of G:

$$G \to PGL(V).$$

If $\mathbb{K} = \mathbb{C}$, then the inclusion $SL(V) \hookrightarrow GL(V)$ induces an isomorphism

$$SL(V)/\mu_n \xrightarrow{\sim} PGL(V)$$

where μ_n is the group of n-th roots of unity. Since μ_n is a discrete subgroup of the center of $SL(V)$, we find that $SL(V)$ is the universal cover of $PGL(V)$. In conclusion, an action of a Lie group G as algebra automorphism group of $\mathrm{End}(V)$, with V a finite dimensional complex vector space induces a group homomorphism

$$\tilde{G} \to SL(V),$$

where $\tilde{G}$ is the universal cover of the connected component of the identity of G, lifting the projective representation $G \to PGL(V)$.

We are going to show in the following section how his construction, applied to the natural acion of $O(1,3)$ on the complexified Clifford algebra $Cl(1,3)_{\mathbb{C}}$ induces a four dimensional representation of $\mathrm{Spin}(1,3) \cong SL(2;\mathbb{C})$, which splits into the direct sum of two spin 1/2 representations of $SU(2)$.

2.3 The Dirac isomorphism

The Dirac isomorphism is an algebra isomorphism between the complexified Clifford algebra $Cl(1,3)_{\mathbb{C}}$ and the algebra $M_4(\mathbb{C})$ of 4×4 complex matrices. We will describe this isomorphism in an intrisic way in Appendix B. For what matters here we can give an explicit construction working with a fixed orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{R}^{1,3}$, e.g., the standard one. In terms of this basis, the complexified Clifford algebra $Cl(1,3)_{\mathbb{C}}$ is the free associative algebra ober $\mathbb{C}$ generatd by the elements e_i with the relations

$$e_i \cdot e_j + e_j \cdot e_i = 2\eta_{ij}$$

Therefore, giving an algebra homomorphism

$$Cl(1,3)_{\mathbb{C}} \to M_4(\mathbb{C})$$

is equivalent to giving four matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $M_4(\mathbb{C})$ such that

$$\{\gamma_i, \gamma_j\} = 2\eta_{ij}\mathrm{Id},$$

where we have written $\{\gamma_i, \gamma_j\}$ for $\gamma_i\gamma_j + \gamma_j\gamma_i$. Writing $\gamma^i = \eta^{ij}\gamma_j$, this is in turn equivalent to giving four matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ in $M_4(\mathbb{C})$ such that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathrm{Id}.$$

Such a set of matrices has been fisrt described by Paul Dirac, and is nowadays known as the set of *Dirac γ-matrices*. They are

$$\gamma^0 = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3,$$

where the σ_i are the Pauli matrices. In completely explicit form,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}; \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is then easy to see that the algebra homomorphims $Cl(1,3)_{\mathbb{C}} \to M_4(\mathbb{C})$ induced by the Dirac γ-matrices is actually an isomorphism. Indeed, both algebras are 16-dimensional over $\mathbb{C}$ and the Dirac homomorphism is easily seen to be injective.

2.4 Spinors and half-spinors

The group $O(1,3)$ of linear isometries of $\mathbb{R}^{1,3}$ acts as a group of algebra automorphisms on the Clifford algebra $Cl(1,3)$ and so also on its complexification. Therefore, via the Dirac isomorphism we get an action of $O(1,3)$ on the algebra $\mathrm{End}(\mathbb{C}^4)$ of endomorphisms of $\mathbb{C}^4$. As we have shown in the previous sections, this induces a 4-dimensional representation of the universal cover $SL(2;\mathbb{C})$ of $SO(1,3)^+$. The 4-dimensional complex vector space $\mathbb{C}^4$ endowed with this representation is called the *space of spinors*. In more concrete terms, we have the following situation: consider the $SL(2;\mathbb{C})$-action on $\mathbb{R}^{1,3}$ induced by the projection $SL(2;\mathbb{C}) \to SO(1,3)^+$, and denote by

$$S : SL(2;\mathbb{C}) \to SL(4;\mathbb{C})$$

the 4-dimensional representation of $SL(2;\mathbb{C})$ induced by the $O(1,3)$-action on $Cl(1,3)$; finally, denote by $\rho : Cl(1,3)_{\mathbb{C}} \to \mathrm{End}(\mathbb{C}^4)$ the Dirac isomorphism. Then for any v in $\mathbb{R}^{1,3}$ and any Λ in $SL(2;\mathbb{C})$ we have

$$\rho(\Lambda \cdot v) = S(\Lambda)\rho(v)S(\Lambda^{-1}).$$

In particular, if Λ fixes v, then $S(\Lambda)$ commutes with $\rho(v)$. This means that each eigenspace for the action of $\rho(v)$ on $\mathbb{C}^4$ carries a representation of the stabilizer subgroup of v. When $v = e^0 = (1,0,0,0)$, we have seen in Section 1 that the stabilizer subgroup is $SU(2)$. Since $\rho(e^0) = \gamma^0$, we therefore see that the space $\mathbb{C}$ of spinors splits into the direct sum of two representations of $SU(2)$, corresponding to the $+1$ and to the -1 γ^0-eigenspace. Both thses representations of $SU(2)$ are two-dimensional and irreducible, i.e., they are two copies of the spin $1/2$ representation. Let us denote this decomposition of $\mathbb{C}^4$ as

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad \phi, \chi \in \mathbb{C}^2.$$

The two vectors ϕ and χ are called the *half-spinor* components of the spinor ψ. The subspaces of half-spinors of type ϕ and χ are preserved by γ^0 and interchanged by γ^i for $i = 1,2,3$. More precisely, one has

$$\gamma^0 \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi \\ -\chi \end{pmatrix}; \qquad \gamma^i \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \sigma_i \chi \\ -\sigma_i \phi \end{pmatrix}, \qquad i = 1,2,3.$$

It is sometomes convenient to make the change of variables

$$\psi^+ = \phi + \mathrm{i}\chi; \qquad \psi^- = \phi - \mathrm{i}\chi.$$

Now, the subspaces of half-spinors of type $\psi^\pm$ are preserved by γ^i for $i = 1,2,3$, and are interchanged by γ^0. More precisely,

$$\gamma^0 \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix}; \qquad \gamma^i \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = -\mathbf{i} \begin{pmatrix} \sigma_i \psi^+ \\ \sigma_i \psi^- \end{pmatrix}, \qquad i = 1,2,3,$$

or, in compact notation,

$$\gamma^0 \psi^\pm = \psi^\mp; \qquad \gamma^i \psi^\pm = -\mathbf{i}\sigma_i \psi^\pm, \qquad i = 1,2,3.$$

2.5 Twisted Casimir elements

Let G be a Lie group acting on an associative algebra A as a group of algebra automorphisms. Since algebar morphisms are in particular linear morphisms, we have a Lie group homomorphism

$$G \to \mathrm{Aut}_{\mathrm{Alg}}(A) \subseteq \mathrm{Aut}_{\mathrm{Vect}}(A)$$

and hence a Lie algebra morphism

$$\mathfrak{g} \to \mathrm{Der}(A) \subseteq \mathrm{End}_{\mathrm{Vect}}(A)$$

Here $\mathrm{Der}(A)$ is the Lie algebar of *derivations* of A, i.e., the Lie algebra of linear operators $X : A \to A$ such that

$$X(ab) = X(a)b + aX(b).$$

For an element x in $\mathfrak{g}$, let us denote by $X_x : A \to A$ the corresponding derivation, and let us write $[x, a]$ for $X_x(a)$, so that

$$[x, ab] = [x, a]b + a[x, b].$$

This notation is compatible with the Lie algebra structure on A_{Lie}. Indeed we have

$$[x, [a, b]] = [x, ab - ba] = [x, a]b + a[x, b] - [x, b]a - b[x, a] = [[x, a], b] + [a, [x, b]]$$

and, since $X : \mathfrak{g} \to \mathrm{Der}(A)$ is a Lie algebra homomorphism

$$\begin{aligned}[][x, y], a] &= X_{[x,y]}(a) = [X_x, X_y](a) = X_x(X_y(a)) - X_y(X_x(a)) \\ &= [x, [y, a]] - [y, [x, a]].\end{aligned}$$

In other words, we are dealing with a natural Lie algebra structure on the direct sum of the vector spaces A and $\mathfrak{g}$: the *semidirect product* Lie algebra $A_{\mathrm{Lie}} \rtimes \mathfrak{g}$. Note that A_{Lie} is an ideal of $A_{\mathrm{Lie}} \rtimes \mathfrak{g}$ and we have the Lie algebra extension

$$0 \to A_{\mathrm{Lie}} \to A_{\mathrm{Lie}} \rtimes \mathfrak{g} \to \mathfrak{g} \to 0.$$

In a similar fashion, one builds the *A-twisted universal enveloping algebra* $U(\mathfrak{g})_A$ as the free associative algebra generated by A and $U(\mathfrak{g})$ modulo the relations coming from A and from $U_{\mathfrak{g}}$, and the relations $xa - ax = [x, a]$ and $1_A x = x$, where 1_A is the unit element of A, for any a in A and any x in $\mathfrak{g}$. Note that these relations imply that every element in $U(\mathfrak{g})_A$ can be written as a finite sum

$$\sum a_I x_I, \qquad a_I \in A, x_I \in U(\mathfrak{g}).$$

Recalling that $U(\mathfrak{g})$ is the algebra of left G-invariant differential operators on G, this gives an interpretation of $U(\mathfrak{g})_A$ as an algebra of differential operators on G with coefficients in A.

Definition. 28. A *twisted Casimir element* of $\mathfrak{g}$ is an element in the centralizer of $U(\mathfrak{g})$ in $U(\mathfrak{g})_A$.

In particular, each Casimir element is also a twisted Casimir element. Also note that, since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ as an associative algebra, twisted Casimir elements are precisely those elements Ω in $U(\mathfrak{g})_A$ such that

$$[x, \Omega] = 0, \qquad \text{for all } x \in \mathfrak{g}.$$

Example. 29. The trivial example is clearly the trivial twist, given by $A = \mathbb{K}$ with the trivial $\mathfrak{g}$-action. In this case one has $U(\mathfrak{g})_A = U(\mathfrak{g})$, and twisted Casimir elements are precisely the Casimir elements of $\mathfrak{g}$.

A less trivial example is obtained starting with a linear representation of a Lie group H acting isometrically on a real vector space V endowed with a nondegenerate symmetric bilinear pairing. Namely, let G be the semidirect product $V \rtimes H$ and make G act on the algebra $\mathrm{End}(V)$ by conjugation via the projection $G \to H$. Then, by the general argument above we can consider the twisted enveloping algebra $U(\mathfrak{g})_{\mathrm{End}(V)}$.

The vector space V is an abelian Lie subalgebra of $\mathfrak{g}$ and acts trivially on $\mathrm{End}(V)$. Hence $U(\mathfrak{g})_{\mathrm{End}(V)}$ contains a copy of the associative algebra

$$\mathrm{Sym}^\bullet(V) \otimes \mathrm{End}(V).$$

Since the pairing on V identifies elements in the symmetric algebra $\mathrm{Sym}^\bullet(V)$ with polynomial functions on V, we see that $U(\mathfrak{g})_{\mathrm{End}(V)}$ contains a copy of the algebra

$$\mathrm{Sym}^\bullet(V, \mathrm{End}(V))$$

of $\mathrm{End}(V)$-valued polynomial functions on V. Since the action of H on V is isometric, the morphism $V \to V^*$ induced by the pairing, i.e., $|v\rangle \mapsto \langle v|$, is H-equivariant. Indeed

$$\langle v | h^{-1} w\rangle = \langle hv | w\rangle,$$

i.e., $h \cdot \langle v| = |hv\rangle$, for any h in H. This means that the pairing on V induces an isomorphism of H-modules

$$\mathrm{Sym}^\bullet(V) \otimes \mathrm{End}(V) \cong \mathrm{Sym}^\bullet(V, \mathrm{End}(V)).$$

In particular, the algebra of H-invariant elements of $\mathrm{Sym}^\bullet(V, \mathrm{End}(V))$ is naturally identified with the algebra

$$\mathrm{Sym}^\bullet(V, \mathrm{End}(V))^H$$

of H-equivariant polynomial fucntions from V to $\mathrm{End}(V)$. Passing to Lie algberas, we therefore see that $[x, \Omega] = 0$ for any Ω in $\mathrm{Sym}^\bullet(V, \mathrm{End}(V))^H \subseteq U(\mathfrak{g})_{\mathrm{End}(V)}$, and any $x \in \mathfrak{h}$. On the other hand, elements in the abelian Lie subalgebra V of $\mathfrak{g}$ act trivially both on $\mathrm{Sym}^\bullet(V)$ and on $\mathrm{End}(V)$. Hence the whole Lie algebra $\mathfrak{g}$ acts trivially on the copy of $\mathrm{Sym}^\bullet(V, \mathrm{End}(V))^H$ in $U(\mathfrak{g})_{\mathrm{End}(V)}$. In conclusion, we have proved the following.

Proposition. 30. *Let H be a group of isometries for a vector space V endowed with a nondegenerate symmetric bilinear form, and let $\mathfrak{g}$ be the Lie algebra of the semidirect product $V \rtimes H$. Then each H-equivariant polynomial map from V to* $\mathrm{End}(V)$ *defines a twisted Casimir element in* $U(\mathfrak{g})_{\mathrm{End}(V)}$.

Note that via the inclusion

$$\mathrm{Sym}^{\bullet}(V, \mathrm{End}(V))^H \hookrightarrow \mathrm{Sym}^{\bullet}(V, \mathrm{End}(V))^H$$

the above proposition tells that H-invariant polynomials on V give rise to Casimir elements for $\mathfrak{g}$. Moreover, by complexification, each H-equivariant polynomial map from V to $\mathrm{End}(V_{\mathbb{C}})$ induces an $\mathrm{End}(V_{\mathbb{C}})$-twisted Casimir element for the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

2.6 The mass-square operator revisited

As a first application of the formalism just described we recover the Casimir element $\|P\|^2$ for the Poincaré Lie algebra. Recall that the Poincaré group is obtained as the semidirect product of the Lorentz group $O(1,3)$ with the Minkowski space $\mathbb{R}^{1,3}$. Hence any Lorentz-invariant polynomial function on $\mathbb{R}^{1,3}$ will produce a Casimir element for the Poincaré Lie algebra.The square norm of a vector is a quadratic function on $\mathbb{R}^{1,3}$ and it is trivially Lorentz-invarinat, o it gives us a Casimir element. If we denote by $p^0, \ldots, p^3$ the standard coordinates on $\mathbb{R}^{1,3}$ then the square norm of a vector reads $\vec{p} \mapsto \|\vec{p}\|^2 = \eta_{\mu\nu} p^\mu p^\nu$. If we denote by P^μ the vector in $\mathbb{R}^{1,3}$ corresponding to the linear operator p^μ via the Minkowski metric, then the Casimir operator associated with the square norm of a vector is seen to be

$$\|P\|^2 = \eta_{\mu\nu} P^\mu P^\nu.$$

Thus, we recover the fact that, as anticipated, the deep reason why $\|P\|^2$ is a Casimir operator for the Poincaré Lie algebra is that the norm square is a Lorentz-invariant function.

2.7 The Dirac operator

In a completely similar fashion we can obtain the *Dirac operator* as a twisted Casimir operator for the complexified Poincaré Lie algebra. Recall from example 23 that we have a Lorentz invariant linear map

$$\mathbb{R}^{1,3} \to Cl(1,3)$$

given by the linear embedding of the Minkowski space in its Clifford algebra. By complexification and Dirac isomorphism we therefore obtain a Lorentz-equivariant linear map

$$\mathbb{R}^{1,3} \to Cl(1,3)_{\mathbb{C}} \cong \mathrm{End}(\mathbb{C}^4)$$

Then, by the general construction described in Section 2.5, this gives an $\mathrm{End}(\mathbb{C}^4)$-twisted Casimir element for the complexified Poincaré Lie algebra. This element is the Dirac operator. To obtain an explicit expression, let $\{e_0, \ldots, e_3\}$ be the standard basis of $\mathbb{R}^{1,3}$ and let $\{p^0, \ldots, p^3\}$ be the corresponding coordinates. Then the Lorentz-invariant linear map $\mathbb{R}^{1,3} \to \mathrm{End}(\mathbb{C}^4)$ we are considering is nothing but the Dirac map $e_i \mapsto \gamma_i$, where $\gamma_i = \eta_{ij}\gamma^i$, and $\{\gamma^0, \ldots, \gamma^3\}$ are Dirac's γ-matrices. Hence, the Dirac map written in coordinates is

$$\vec{p} \mapsto \not{p},$$

where we have adopted Feynman slash-notation:

$$\not{p} = \gamma_\mu p^\mu.$$

The corresponding twisted Casimir element is

$$\not{P} = \gamma_\mu P^\mu.$$

2.8 The massive vector field operator

As our last example, we describe the massive vector field operator and show how its construction pefectly parallels the construction of the Dirac operator. Recall from example 2.1 that we have a natural Lorentz-equivariant quadratic map

$$\mathbb{R}^{1,3} \to \mathrm{End}(\mathbb{R}^4)$$

given by $|v\rangle \mapsto |v\rangle\langle v|$. On the other hand, via the Lorentz-equivariant embedding $\mathbb{R} \hookrightarrow \mathrm{End}(\mathbb{R}^4)$, each Lorentz-invariant polynomial function on $\mathbb{R}^{1,3}$ gives a Lorentz-equivariant polynomial function on $\mathbb{R}^{1,3}$ with values in $\mathrm{End}(\mathbb{R}^4)$. So, in particular, for any scalar m, the function mapping a vector v in the Minkowski space to $(\|v\|^2 - m^2)\mathrm{Id}_{\mathbb{R}^4}$ is a Lorentz-equivarinat polynomial. Summing up, we can consider the Lorentz-equivariant polynomial mapping a vector v in $\mathbb{R}^{1,3}$ in the endomorphism

$$\Phi(v) = (\|v\|^2 - m^2)\mathrm{Id}_{\mathbb{R}^4} - |v\rangle\langle v|$$

or $\mathbb{R}^4$. The corresponding twisted Casimir operator is called the *mass m vector field operator*. As above, to get an explicit expression we can work in coordinates:

$$\Phi(\vec{p})^\mu_\nu = (\|p\|^2 - m^2)\delta^\mu_\nu - p^\mu p_\nu.$$

Therefore, denoting by E^ν_μ the 4×4 matrix with 1 in position (μ, ν) and 0 elsewhere, the corresponding twisted Casimir operator is

$$\left((\|P\|^2 - m^2)\delta^\mu_\nu - P^\mu P_\nu\right) E^\nu_\mu.$$

2.9 Factorizing Klein-Gordon operator: the Dirac case

Both the Dirac operator and the massive vector field operator appears as factors of the $\mathrm{End}(\mathbb{C}^4)$-twisted Klein-Gordon operator

$$(\|P\|^2 - m^2)\mathrm{Id}_{\mathbb{C}^4}.$$

Indeed, the Lorentz-invariant polynomial $\|p\|^2 - m^2$ is irreducible as an element of $\mathrm{Sym}^\bullet(\mathrm{R}^{1,3}, \mathbb{C})^{SL(2;\mathbb{C})}$, but it admits nontrivial factorizations in the algebra of Lorentz-invariant matrix valued polynomials

$$\mathrm{Sym}^\bullet(\mathrm{R}^{1,3}, \mathrm{End}(\mathbb{C}^4))^{SL(2;\mathbb{C})}.$$

The first of this factorization is the one that led Paul Dirac to the introduction of the γ-matrices. Indeed, it is immediate to check that the Dirac map $\not{p}$ is a square root of $\|p\|^2$:

$$\not{p}^2 = (\gamma_\mu p^\mu)(\gamma_\nu p^\nu) = (\gamma_\mu \gamma_\nu) p^\mu p^\nu = \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} p^\mu p^n u = \eta_{\mu\nu} p^\mu p^\nu = \|p\|^2,$$

where we have used the commutativity of the p^μ. In particular, we see that if ψ is an m-eigenvalue for $\not{P}$, then ψ is an m^2-eigenvalue for $\|P\|^2$. This can be rephrased by saying that a solution ψ of the Dirac equation

$$(\not{P} - m)\psi = 0$$

is automatically a solution of the Klein-Gordon equation $(\|P\|^2 - m^2)\psi = 0$. For instance, if we realize the $\mathrm{End}(\mathbb{C}^4)$-twisted operators P^μ as the derivations $\mathbf{i}\partial^\mu$ on the space of $\mathbb{C}^4$-valued functions on $\mathbb{R}^{1,3}$ (the spinor fields of physics parlance), then we recover that the solutions of the Dirac equation

$$(\not{\partial} + \mathbf{i}m)\psi = 0$$

are automatically solutions of the Klein-Gordon equation

$$(\Box + m^2)\psi = 0.$$

This phenomenon can nicely be read in terms of the Lorentz-equivariant factorization of the Klein-Gordon polynomial given by

$$\pi_+(\vec{p})\pi_-(\vec{p}) = \|p\|^2 - m^2$$

given by

$$\pi_-(\vec{p}) = \not{p} - m\mathrm{Id}_{\mathbb{C}^4},$$
$$\pi_+(\vec{p}) = \not{p} + m\mathrm{Id}_{\mathbb{C}^4}.$$

Indeed by this factorization we see that the kernel of the operator corresponding to π_- is contained in the kernel of the operator corresponding to $\|p\|^2 - m^2$. Also note that the factors π_+ and π_- commute. In terms of differential operators, this is the Dirac factorization

$$(\not{\partial} - \mathbf{i}m)(\not{\partial} + \mathbf{i}m) = \Box + m^2.$$

2.10 Factorizing Klein-Gordon: the massive vectorial field case

The factorization of the Klein-Gordon polynomial in the algebra of Lorentz-invariant $\mathrm{End}(\mathbb{C}^4)$-valued polynomials on $\mathbb{R}^{1,3}$ is not unique. We now show how also the Lorentz-invariant polynomial

$$K(\vec{p})^\mu_\nu = (\|p\|^2 - m^2)\delta^\mu_\nu - p^\mu p_\nu$$

giving the mass m vector field operator appears as a factor of $\|p\|^2 - m^2$. This is conveniently seen by recalling the coordinate-free expression for K:

$$K(v) = (\|v\|^2 \mathrm{Id}_{\mathbb{C}^4} - |v\rangle\langle v| - m^2 \mathrm{Id}_{\mathbb{C}^4}).$$

One has

$$\big(|v\rangle\langle v|\big)^2 = \|v\|^2\, |v\rangle\langle v|$$

and so

$$|v\rangle\langle v| \left(\|v\|^2 \mathrm{Id}_{\mathbb{C}^4} - |v\rangle\langle v|\right) = 0.$$

Hence

$$\left(\mathrm{Id}_{\mathbb{C}^4} - \frac{1}{m^2}|v\rangle\langle v|\right) K(v) = K(v) + |v\rangle\langle v| = \|v\|^2 - m^2 \mathrm{Id}_{\mathbb{C}^4}.$$

Note that also

$$L(v) = \mathrm{Id}_{\mathbb{C}^4} - \frac{1}{m^2}|v\rangle\langle v|$$

is Lorentz-equivarinat and so $L(v)K(v) = \|v\|^2 - m^2$ is a Lorentz-equivariant factorization of the Klein-Gordon polynomial. Also note that, as in the Dirac case, the factors K and L commute. Passing to the associated operators, we therefore see that any solution of the mass m vector field equation

$$\left(\partial^\mu \partial_\nu - (\Box + m^2)\delta^\mu_\nu\right) A_\mu = 0$$

is automatically a solution of the Klein-Gordon equation $(\Box + m^2)A = 0$, which justifies the interpretation of the parameter m in $K(\vec{p})$ as the mass.

2.11 Twisted Casimir elements and representations of the Poincaré group

We now use the formalism of twisted Casimir operators induced by equivariant maps to investigate the representations of the (universal cover of the) Poincaré group associated with the Dirac operator and the massive vector field operator. The idea is that given a Lorentz-equivariant polynomial map

$$\Phi : \mathbb{R}^{1,3} \to \mathrm{End}(\mathbb{C}^4)$$

and a point $\vec{p}_0$ in $\mathbb{R}^{1,4}$ we have two natural ways to build a representation of $\mathbb{R}^4 \rtimes SL(2;\mathbb{C})$ associated with these data.

The first way consists in making $SL(2;\mathbb{C})$ act on $\mathbb{R}^{1,3}$ via the projection $SL(2;\mathbb{C}) \to SO(1,3)^+$. Since the map Φ is Lorentz-equivariant, the family of subspaces $\ker(\Phi_{\vec{p}})$ of $\mathbb{C}^4$ gives an $SL(2;\mathbb{C})$-equivariant bundle over the $SL(2;\mathbb{C})$-orbit of $\vec{p}_0$, see Section 2.1. This representation is induced by the representation of the stabilizer subgroup $SL(2;\mathbb{C})_{\vec{p}_0}$ on $\ker(\Phi_{\vec{p}_0})$. Finally, an element $\vec{a}$ in the subgroup $\mathbb{R}^4$ of $\mathbb{R}^4 \rtimes SL(2;\mathbb{C})$ acts on sections by fiberwise multiplication by the character $e^{-\mathbf{i}(\vec{a}|\vec{p})}$.

The second way consists in considering the trivial $\mathbb{C}^4$-bundle over $\mathbb{R}^{1,3}$, and to consider its space of sections $\mathcal{H}$, which, following the physicists' language, we will call *spinor fields*. The space $\mathcal{H}$ carries a natural unitary action of the Poincaré group by pull-back. The Lorentz-equivariant polynomial Φ gives a twisted Casimir element Ω, which acts on the space of spinors as a differential operator with coefficients in $\mathrm{End}(\mathbb{C}^4$. Since $[x,\Omega]=0$ for any x in the Poincaré Lie algebra, its eigenspaces, and so in particular iits kernel, are Poincaré-invariant subspaces of the space of spinors. On the other hand, recalling the construction in Section 1, the subspace

$$\bigoplus_{\vec{p}\in SL(2;\mathbb{C})\cdot\vec{p}_0} \mathcal{H}_{\vec{p}},$$

where $\mathcal{H}_{\vec{p}}$ is the $\vec{p}$-eigenspace for the action of the infinitesimal translations P^μ on spinors, is a subrepresentation of $\mathcal{H}$. Hence the intersection

$$\ker(\Omega) \cap \left(\bigoplus_{\vec{p}\in SL(2;\mathbb{C})\cdot\vec{p}_0} \mathcal{H}_{\vec{p}} \right)$$

is a representation of $\mathbb{R}^4 \rtimes SL(2;\mathbb{C})$, and by Wigner's theorem it is induced by a representation of the stabilizer of $\vec{p}_0$ in $SL(2;\mathbb{C})$ on the vector space

$$\ker(\Omega) \cap \mathcal{H}_{\vec{p}_0}.$$

Therefore, to show that the two constructions are equivalent we only need to show that there is a natural isomorphism of representations of $SL(2;\mathbb{C})_{\vec{p}_0}$

$$\ker(\Omega) \cap \mathcal{H}_{\vec{p}_0} \xrightarrow{\sim} \ker(\Phi_{\vec{p}_0}).$$

This is easy: by definition $\mathcal{H}_{\vec{p}_0}$ is the $\vec{p}_0$-eigenspace for the action of the infinitesimal translations P^μ, hence it is defined as the space of the joint solutions of the first-order differential equations

$$\mathbf{i}\partial^\mu \psi = p_0^\mu \psi,$$

with ϕ a $\mathbb{C}^4$-valued function on $\mathbb{R}^{1,3}$. Therefore we find that $\mathcal{H}_{\vec{p}}$ consists of the functions

$$\phi(\vec{x}) = e^{-\mathbf{i}(\vec{p}_0|\vec{x})}\psi_0, \qquad \psi_0 \in \mathbb{C}^4.$$

In particular it is a 4-dimensional space, and $\psi \mapsto \psi(0)$ is an isomorphism between $\mathcal{H}_{\vec{p}_0}$ and $\mathbb{C}^4$. By the way the twisted Casimir element Ω is constructed from Φ, it is immediate to see that Ω acts on an element ψ of $\mathcal{H}_{\vec{p}_0}$ as $\Phi_{\vec{p}_0}$ acts on the corresponding element $\psi(0)$ in $\mathbb{C}^4$, i.e. we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}_{\vec{p}_0} \arrow[r, "\Omega"] \arrow[d, "\mathrm{ev}_0"', "\wr"] & \mathcal{H}_{\vec{p}_0} \arrow[d, "\wr"', "\mathrm{ev}_0"] \\ \mathbb{C}^4 \arrow[r, "\Phi_{\vec{p}_0}"] & \mathbb{C}^4, \end{tikzcd}$$

where ev_0 is the evaluation at 0 and where we have used the fact that, since $[P^\mu, \Omega] = 0$, the operator Ω preserves the eigenspaces of the operators P^μ. And this concludes the proof.

Example. 31. The Dirac operator $\not{P}$ acts on the space of spinors as the operator

$$\mathbf{i}\not{\partial} = \mathbf{i}\gamma^\mu \partial_\mu.$$

For any real scalar $m > 0$ the *Dirac equation*

$$(\not{\partial} + \mathbf{i}m)\phi = 0$$

determines a representation of the (universal cover of the) Poincaré group. Recall that the Lorentz-equivariant polynomial inducing the Dirac operator is $p^\mu \gamma_\mu$, so that the the Lorentz-equivariant polynomial Φ corresponding to the Dirac equation is

$$\Phi_{\vec{p}} = p^\mu \gamma_\mu - m\mathrm{Id}_{\mathbb{C}^4}.$$

Picking $\vec{p}_0 = (m, 0, 0, 0)$ we see that the stabilizer subgroup of $\vec{p}_0$ is $SU(2)$ and that the irreducible representation corresponding to the Dirac equation is induced by a representation of $SU(2)$ on the kernel of $\Phi_{\vec{p}_0}$. Since

$$\Phi_{\vec{p}_0} = m(\gamma_0 - \mathrm{Id}_{\mathbb{C}^4}) = m \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

the kernel of $\Phi_{\vec{p}_0}$ is a 2-dimensional complex vector space: it is precisely the space of half-spinors of type ϕ we met in Section 1. Moreover, as we saw in Section 1, an m-eigenvector for the Dirac operator $\not{P}$ is a m^2-eigenvector for the mass-square operator $\|P\|^2$. So we see that the Dirac equation describes a mass m spin 1/2 particle. Analogously, also the conjugate Dirac equation

$$(\not{\partial} - \mathbf{i}m)\psi = 0$$

describes a mass m spin 1/2 particle, and the $SU(2)$-representation involved in this case is the one on the space of half-spinors of type χ.

2.12 A spin 1 particle

We conclude by discussing the mass m vector field equation

$$\left(\partial^\mu\partial_\nu - (\Box + m^2)\delta^\mu_\nu\right) A_\mu = 0.$$

The treatment is completely analogous to the Dirac equation. The mass m vector field operator is induced by the Lorentz-equivariant polynomial

$$\Phi_{\vec{p}} = (\|p\|^2 - m^2)\mathrm{Id}_{\mathbb{C}^4} - |\vec{p}\rangle\langle\vec{p}|.$$

Fixing $m > 0$ and choosing $\vec{p}_0 = (m, 0, 0, 0)$, we see that the representation of $SU(2)$ corresponding to the massive vector field equation is a linear representation on the vector space

$$\ker(\Phi_{\vec{p}_0}) = \ker \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

i.e., it is a 3-dimensional representation on the subspace of $\mathbb{C}^4$ spanned by the last three vectors of the standard basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{C}^4$. This is actually an irreducible representation. To see this recall that the $SL(2;\mathbb{C})$-action on $\mathbb{R}^{1,3}$ is defined by identifying the Minkowski space with the space of 2×2 Hermitean matrices, and then making an element A in $SL(2;\mathbb{C})$ act on a Hermitean matrix M by

$$M \mapsto AMA^*.$$

This means that the restriction of this $SL(2;\mathbb{C}$-representation to the subgroup $SU(2)$ is the representation of $SU(2)$ by conjugation on the space of 2×2 Hermitean matrices. This is not irreducible, but splits into the direct sum of two irreducible representations: the trivial 1-dimensional representation and the 3-dimensional representation given by the action of $SU(2)$ on the space of traceless 2×2 Hermitean matrices. Multiplication by $\mathbf{i}$ identifies this action with the adjoint action of $SU(2)$ on $\mathfrak{su}_2$, and so its complexification is an irreducible complex 3-dimensional representation of $SU(2)$. To identify this representation with the one arising from the massive vector field operator, just recall from Section 1 that the linear isomorphism $\mathbb{R}^{1,3} \to \mathfrak{her}_2$ maps the subspace spanned by the first vector e_1 of the standard basis of $\mathbb{R}^{1,3}$ to the subspace of scalar Hermitean matrices, and the subspace spanned by the three basis vectors $\{e_2, e_3, e_4\}$ to the subspace of traceless Hermitean matrices. Passing to complexification we therefore see that the 3-dimensional complex $SU(2)$-representation associated with the mass m vector field equation is the representation given by $SU(2)$ acting by conjugation on the complex vector space of 2×2 traceless matrices, and so it corresponds to a spin 1 particle. Moreover, we have seen in section 1 that a zero vector for the mass m vector field operator is an m^2-eigenvector for the mass-square operator $\|P\|^2$, so the mass m vector field equation

$$\left(\partial^\mu\partial_\nu - (\Box + m^2)\delta^\mu_\nu\right) A_\mu = 0$$

describes a mass m spin 1 particle.

3 The Higgs-Braut-Englert mechanism

3.1 Interactions

We sketch the transition between a free theory to an interaction theory. Even if we are not giving the details, we want to clarify why the product of fields is referred as an interaction between particles. The big step in the transition is changing the definition of field. Recall that in the free theory a field ϕ can be looked as a section of a vector bundle over the Minkowski space, i.e. a field is a function in some L^2-like hilbert space. In the theory with interaction, instead, a field ϕ is function from the Minkowski space taking values in the operators of an Hilbert space $\mathcal{H}$ into itself. In particular denote $|p\rangle \in \mathcal{H}$ the state of a particle of momentum p. Then we will write

$$\phi|p\rangle$$

intending the action of ϕ over the state $|p\rangle$. Furthermore we will be writing

$$\langle q|\phi|p|\rangle$$

intending the action of ϕ over $|p\rangle$ projected over $|q\rangle$. This is also called the probability amplitude for the state $|p\rangle$ to collapse via ϕ to a state $|q\rangle$. Remark that $\langle p|q\rangle$ is the inner product of $\mathcal{H}$. *An interaction is a non zero probability amplitude* between two states in $\mathcal{H}$.

We present as an example the case of two K-G fields interacting.

Example. 32. Recall the Klein-Gordon equation for a complex scalar field ϕ

$$(\Box - m^2)\phi = 0$$

We find solution that are periodic of period L in each coordinate. A planar wave $\phi_{\vec{n}}(x)$ solves K-G

$$\phi_{\vec{n}}(x) = N(\omega)[a(p)e^{-i\omega x^0 - k\cdot x} + c(p)e^{i(\omega x^0 + k\cdot x})]$$

for $p^2 = m^2$, $p = (p^0, \frac{2\pi}{L}n^1, \frac{2\pi}{L}n^2, \frac{2\pi}{L}n^3)$, $n^i \in \mathbb{N}$, $p^0 = \omega = \sqrt{m^2 + \mathbf{k}^2}$. A formal solution for K-G is obtained by overlapping the waves ϕ_n

$$\phi(x) = \sum_{\vec{n}} \phi_{\vec{n}}(x).$$

Each planar wave $\phi_{\vec{n}}(x)$ can be considered as an operator over an Hilbert space $\mathcal{H}$ of the state of the scalar and charged particles of mass m. Define

$$a(p)|0\rangle = |p,+\rangle,$$

and

$$a^*(p)|p,+\rangle = |0\rangle.$$

Then a, a^* are the operators of annihilation/creation for a positive charged particle. The analogous relation are given for $c(p)$ over $|p, -\rangle$. A solution for K-G is now an operator over an Hilbert space. An example of interaction between two Klein-Gordon fields ϕ, ψ can be provided as follows. Let ϕ be the field acting over the Hilbert space $\mathcal{H}_1$ of the particles of mass m_1 and ψ the field acting over the Hilbert space $\mathcal{H}_2$ the particles of mass m_2. An interaction between these two types of particles is given by the product $\phi\psi$, since the probability amplitude

$$\langle p|\phi\psi|q\rangle, \quad |q\rangle \in \mathcal{H}_1, \quad |p\rangle \in \mathcal{H}_2$$

is not zero.

3.2 The Higgs-Braut-Englert action

Let $L \to M$ be an Hermitean line bundle over a Lorentian manifold M. The *Higgs-Braut-Englert Lagrangian density* is

$$\mathcal{L}(\nabla, \phi) = \|\nabla\phi\|^2 - V(\phi) + \frac{1}{4e^2}\|F_\nabla\|^2,$$

where ∇ is a $U(1)$-connection on L, ϕ is a section of L, F_∇ is the curvature of ∇, and V is the quartic potential

$$V(\phi) = \mu^2\|\phi\|^2 + \lambda\|\phi\|^4.$$

The coupling constant e is called the *electric charge* of the theory. In the above formula

$$\|\nabla\phi\|^2 = \langle\nabla\phi \wedge \star\nabla\phi\rangle_L$$

is the scalar field given by the norm of $\nabla\phi$ in the Hermitean metric of L, and

$$\|F_\nabla\|^2 = \langle F_\nabla \wedge \star F_\nabla\rangle_{\mathfrak{u}_1}$$

is the norm of the curvature form of ∇ in the standard metric on $\mathfrak{u}_1$. The gauge group of the theory is the group $U(1)$; an element α of $C^\infty(M, U(1))$ acts on the fields as

$$\begin{aligned} \nabla &\mapsto \nabla^\alpha = \alpha\nabla\alpha^{-1} \\ \phi &\mapsto \alpha\phi. \end{aligned}$$

The Lagrangian $\mathcal{L}$ is clearly invariant under the action of these gauge transformations:

$$\mathcal{L}(\nabla^\alpha, \alpha \cdot \phi) = \mathcal{L}(\nabla, \phi).$$

The total energy density corresponding to the given Lagrangian is

$$\|\nabla\phi\|^2 + \frac{1}{4e^2}\|F_\nabla\|^2 + V(\phi)$$

and so the minimum energy field configurations are (∇_0, ϕ_0) with ∇_0 a flat connection on L, and ϕ_0 is a ∇_0-flat section such that $r = \|\phi_0\|$ minimizes $V(r) = \mu^2 r^2 + \lambda r^4$.

In order to have a stable theory, the potential $V(r)$ has to be coercive, and so $\lambda > 0$. The coefficient μ^2 can instead assume both signs. For $\mu^2 \geq 0$, the unique minimum of $V(r)$ is $r = 0$. A much more interesting situation is obtained with $\mu^2 < 0$. In this case $r = 0$ is still a critical point, but it a local maximum so it corresponds to an unstable equilibrium; the minimum is instead attained at

$$r = \sqrt{-\frac{\mu^2}{2\lambda}}.$$

Moreover, up to replacing $V(\phi)$ with $V(\phi) + \frac{\mu^4}{4\lambda}$, we may assume that the minimum energy is zero. The condition $\|\phi_0\| = \sqrt{-\mu^2/2\lambda} > 0$ then implies that ϕ_0 is an everywhere nonzero section of the line bundle L. Hence ϕ_0 gives a trivialization of L. Choosing the norrmalized section $\phi_0/\|\phi_0\|$ as a trivializing section, we get an Hermitean trivialization:

$$L \cong M \times \mathbb{C}$$

with the standard Hermitean metric. In this trivialization, the section ϕ_0 becomes the constant function $\sqrt{-\mu^2/2\lambda}$ on M, and the connection ∇_0 is the de Rham differential.

3.3 Vacuum state oscillations

Having fixed a minimum configuration (a *vacuum state*) we can investigate its infinitesimal perturbations (*oscillations*). Mathematically, this corresponds to making the change of variables

$$\phi = \phi_0 + \frac{1}{\sqrt{2}}\sigma; \qquad \nabla = \nabla_0 + ie\omega.$$

Here σ is the difference of two sections of L and so it is a section of L, whereas $ie\omega$ is the difference of two $\mathfrak{u}_1$ connections and so is a $\mathfrak{u}_1$-valued 1-form on M. In particular, ω is a real valued 1-form. As remarked in Section 3.2, the section ϕ_0 induces a trivialization of L and so σ is the datum of a complex valued function

$$\sigma = \sigma_1 + i\sigma_2 : M \to \mathbb{C}$$

with σ_1 and σ_2 two real scalar fields. Also, the connection ∇_0 is identified with the de Rham differential d. Since ϕ_0 is a positive real number, for small enough values of the perturbation σ, the field $\phi : M \to \mathbb{C}$ will take its values in a small disk certered away from zero, and so ϕ can be written as

$$\phi = e^{ia}\rho$$

for suitable real valued funcions $a, \rho : M \to \mathbb{R}$. This means that by the gauge action we can make the perturbed field $\phi = \phi_0 + \sigma/\sqrt{2}$ be a real field. More precisely, up to gauge equivalence each field configuration (ϕ, ∇) close enough to the vaccum configuration (ϕ_0, ∇_0) is equivalent (by a *unique* gauge equivalence) to a field configuration of the form $(\rho, d+ie\omega)$, with ρ a *real* scalar field. This choice of gauge fixing is called the *unitary gauge*. Note that, since ϕ_0 is a real scalar, in the unitary gauge also the perturbation field σ is a real scalar. Hence, the fields in the *gauge fixed Lagrangian* are a real scalar field σ and a real valued 1-form ω. The field σ, that vanishes in the unitary gauge, is the field of the Goldstone boson. We will show that such a boson is massless and that is carried by a certain class of lagrangians in Appendix C

3.4 The gauge fixed Lagrangian

As remarked in the previous section, the fields involved in the gauge fixed Lagrangian are a real scalar field σ and a real valued 1-form ω. The gauge fixed lagrangian density, expressed in terms of these fields reads

$$\mathcal{L}_{\mathrm{gf}}(\omega,\sigma) = \frac{1}{2}\|d\sigma\|^2 + e^2(\phi_0 + \frac{\sigma}{\sqrt{2}})^2\|\omega\|^2 - V(\phi_0 + \frac{\sigma}{\sqrt{2}}) + \frac{1}{4}\|d\omega\|^2,$$

where, as above, $\phi_0 = \sqrt{-\mu^2/\lambda}$. Therefore, expanding up to second order in σ and ω, and shifting the potential $V(\phi)$ so that its minimum is zero, we find

$$\begin{aligned}\mathcal{L}_{\mathrm{gf}}(\omega,\sigma) &= \frac{1}{2}\|d\sigma\|^2 - \frac{\mu^2}{2\lambda}e^2\|\omega\|^2 + \mu^2\|\sigma\|^2 + \frac{1}{4}\|d\omega\|^2 + \cdots \\ &= \left(\frac{1}{2}\|d\sigma\|^2 - \frac{1}{2}{M_\sigma}^2\|\sigma\|^2\right) + \left(\frac{1}{4}\|d\omega\|^2 + \frac{1}{2}{M_\omega}^2\|\omega\|^2\right) + \cdots,\end{aligned}$$

where we have set

$${M_\sigma}^2 = -2\mu^2 = 4\lambda\|\phi_0\|^2; \qquad {M_\omega}^2 = -\frac{\mu^2}{\lambda}e^2 = 2e^2\|\phi_0\|^2.$$

In the density $\mathcal{L}_{\mathrm{gf}}$ the terms in σ and ω are not coupled: this is due to the fact that by considering the terms in the lagrangians up to second order in the fields, we exclude the interaction terms (in the sense of Section 3.1). Hence we can study two lagrangians independently. The terms in σ are

$$\mathcal{L}_\sigma = \frac{1}{2}\|d\sigma\|^2 + M_\sigma^2\|\sigma^2\|.$$

We recognize the density of Klein-Gordon action. Thus σ is a scalar field that represents a mass M_σ and spin 0 particle. The physicist call this particle *the Higgs boson*. The terms in ω result

$$\mathcal{L}_\omega = -M_\omega^2\|\omega\|^2 + \frac{1}{4}\|d\omega\|^2.$$

We recognize the density of the massive vectorial field action. Thus ω is the field of a spin 1 and mass M_ω particle. The physicists call the particle of ω an *intermediate boson*.

Let us focus on the field ϕ of the Higgs boson. Recall that we wrote $\phi = \phi_0 + \frac{1}{\sqrt{2}}\sigma$. In particular remark that the constant ϕ_0 in the density $\mathcal{L}_{gf}$

appeared coupled with the terms of order 2 of ω. These terms are exactly the difference between the lagrangian of the Maxwell theory (see Appendix B) and the lagrangian of the massive vectorial field. Hence these terms make the difference between a massive boson and a massless one. In this sense, the Higgs boson is referred as the particle responsible of generating the masses. Remark that by looking at its potential, the higgs boson auto-interacts with itself, generating its very mass.

3.5 The standard model

As an application of the theory developed so far we present the *standard model* of the subnuclear particles. In nature there are four fundamental interactions

- The *weak* interaction, responsible of the radiative phenomena.
- The *strong* interaction, that occurs inside the nuclei of the atoms.
- The *electromagnetism*, responsible of the interaractions between charged particles.
- The *gravitation*, responsible of the attractive phenomena between massive bodies.

The standard model unifies in a unique theory the weak, strong and electromagnetic interactions. In this final section we sketch a procedure that, through the H.B.E. mechanism, leads to the lagrangians of the standard model.

Recall that so far we have introduced

1. The fundamental equations for a generic particle of mass m and spin s in the free case, profusely studied in Section 1.
2. The tools for writing lagrangians for the fields invariant under gauge transformations (see Appendix B).
3. A particle capable of "generating" the masses: the Higgs boson.

This is all the technology we need to approach the standard model. In general for each interaction one can proceed as follows.

1. Consider the fields of the particles subjected to the interaction.
2. Write a Yang-Mills lagrangian for the particles, invariant under a certain group of gauge transformations and ignore all the mass terms.
3. Add the higgs boson lagrangian (the H.B.E. lagrangian of Section 3.2), and let the interactions with the boson generate all the needed masses.

Example. 33. As an application of the procedure, enlisted above, we present the lagrangian describing the weak interactions between leptons. We consider a couple of leptons: the electron e and its associated neutrino ν_e. We introduce the field ϕ as the couple

$$\psi = \begin{pmatrix} e \\ \nu_e \end{pmatrix}$$

Thus the field ψ is the section of a vector bundle E, with base space M, the Minkowski space, and fiber $\mathbb{C}^4 \otimes \mathbb{C}^4$ (e and ν_e take values in $\mathbb{C}^4$). In the free theory each field e, ν_e satisfies the Dirac equation for spin $1/2$ and mass $m = m_e, m_\mu$. We want to ignore the masses both for the electron and the neutrino. First, we want to produce a lagrangian that is invariant under the gauge transformation with group $SU(2)$, that is

$$\psi' = e^{i\alpha(x)\frac{\tau_i}{2}}\psi, \quad i = 1, 2, 3;$$

where τ_i is the i-th Pauli's matrix. For obtaining a gauge invariant lagrangian density it is sufficient to pick the lagrangian with global symmetries

$$\mathcal{L}_{\text{global}} = e^\dagger \not{\partial} e + \nu_e^\dagger \not{\partial} \nu_e$$

and then apply the pure Yang-Mills theory (see Appendix B). The resulting invariant lagrangian is

$$\mathcal{L}_{ew} = -e^\dagger \not{D} e + \nu_e^\dagger \not{D} \nu_e + \frac{1}{4}\|W\|^2$$

The covariant derivative D results

$$D_\mu = \partial_\mu + igW_\mu \cdot \frac{\tau}{2}$$

where W is a connection over the vector bundle E. Remark that as $W_\mu \cdot \tau$ we intend the saturation of the three components of W_μ with the three Pauli's matrix. This is a case where the connection is effectively a *matrix* of 1 forms, and thus the resulting Yang-Mills theory is non-commutative. The particle associated to the field W is the*intermediate boson* of the weak theory. At this stage, if one repeats the calculations, which are the analogous of those of the previous section, it results that all the particles W, e, ν_e are massless. Following the procedure, it is now time to introduce the Higgs boson. Define a couple

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}.$$

We add the following lagrangian density to $\mathcal{L}_{ew}$

$$\mathcal{L}_\phi = (D_\mu \phi^\dagger)(D^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda)\phi^\dagger \phi)^2,$$

where $\mu^2 < 0$. With this lagrangian ϕ takes a non-zero expectation value, say $\bar{\phi}$ in the vacuum state. It results

$$\bar{\phi} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \quad \eta = \sqrt{\frac{-\mu^2}{2\lambda}}$$

(see Appendix C). In the unitary gauge ϕ results

$$\phi = \begin{pmatrix} 0 \\ \eta + \frac{\sigma(x)}{\sqrt{2}} \end{pmatrix}. \tag{1}$$

Now the usual calculations yield that the boson W is a mass M_W and spin 1 particle. The only thing left is adding the mass term of the electron, since the mass of the neutrino is several orders of magnitude lower than m_e. We exploit again the Higgs boson field ϕ. Define

$$\mathcal{L}_{e\phi} = g_e(e^\dagger \phi e_R + e^\dagger \phi^\dagger e)$$

where

$$e_{L,R} = \frac{1 \pm \gamma^5}{2} e.$$

The new piece of lagrangian density results gauge invariant. Furthermore by expanding the lagrangian according to the expression of ϕ in the 1 we have

$$\mathcal{L}_{e\phi} = g_e \eta e^\dagger e + \ldots \text{ interaction terms } \ldots$$

that is the missing mass term in the Dirac lagrangian density. Finally we can write the complete lagrangian density describing the *weak* interactions between the electron and its neutrino

$$\begin{aligned} \mathcal{L} &= -e^\dagger \not{D} e + \nu_e^\dagger \not{D} \nu_e + \frac{1}{4}\|W\|^2 + \\ &+ g_e(e^\dagger \phi e_R + e^\dagger \phi^\dagger e) + (D_\mu \phi^\dagger)(D^\mu \phi) + \\ &- \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \end{aligned}$$

We resume in a table the needed gauge symmetries and field choices for obtaining the actual theory

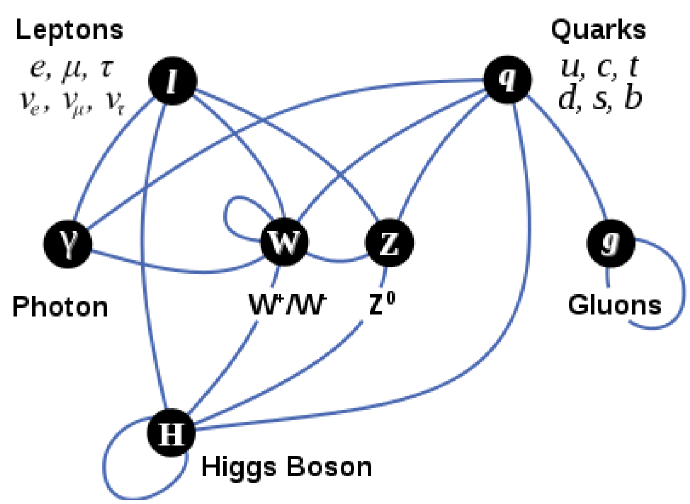
Interaction	Group	Particles
E-M	$U(1)$	any charged particle
Weak	$SU(2)$	leptons
Strong	$SU(3)$	quarks

In the case of a particle that is subjected to more than a single interaction, it is sufficient to consider as the gauge group, the product of the gauge groups associated to each interaction. For example the electron is subjected to both the electro-magnetic and the weak interactions, hence one writes a Yang-Mills lagrangian for the gauge group $G = U(1) \times SU(2)$. In this sense the standard model is a *unifying theory* of three out four fundamental interactions in nature.

Finally, for completeness, we present a picture of the subnuclear particles with the occurring interactions. All the lagrangians of the standard model will result as a special case of the general procedure we described. Note that the charges are given as multiples of the elementary charge e.c. and the masses in multiplies of the electron volt.

	Particle	Spin	Charge	Mass
Leptons	electron e	1/2	-1 e.c.	0.5 MeV
	muon μ	1/2	-1 e.c.	105 MeV
	tauon τ	1/2	-1 e.c.	1776 MeV
	ν_e	1/2	neutral	< 1.5 eV
	ν_μ	1/2	neutral	< 1.5 eV
	ν_τ	1/2	neutral	< 1.5 eV
Quarks	up	1/2	+2/3 e.c.	1.7-3.3 MeV
	down	1/2	-1/3 e.c.	4.1-5.8 MeV
	charm	1/2	+2/3 e.c.	1270 MeV
	strange	1/2	-1/3 e.c.	101 MeV
	top	1/2	+2/3 e.c.	172000 MeV
	bottom	1/2	-1/3 e.c.	4190 MeV
Intermediate bosons	W	1	+1 e.c.	80 GeV
	Z	0	neutral	91 GeV
	gluons	1	neutral	massless
	photon	1	neutral	massless
?	Higgs	0	neutral	?

The occurring interactions are resumed in the graph.



A Spin groups and Clifford algebras

A.1 Semisimple Lie algebra representations

In this section we provide an algorithmic procedure for describing the representations of an arbitrary complex semisimple Lie algebra $\mathfrak{g}$. The presentation follows closely [FuH91] and is tuned for understanding the representation theory in the thesis. Recall that a semisimple Lie algebra $\mathfrak{g}$ is an algebra endowed with a Lie bracket $[.,.]$ with no nonzero solvable ideals I, i.e. the series $[I,I], [[I,I],[I,I]], \ldots$ is not zero for every ideal $I \subset \mathfrak{g}$. We resume the procedure in seven steps.

Step 0. *Verify that your Lie algebra is semisimple.*

Step 1. *Find an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acting diagonally.* Such an $\mathfrak{h}$ is an abelian subalgebra that acts diagonally on one faithful representation of $\mathfrak{g}$. Moreover, in order that the restriction of a representation V of $\mathfrak{g}$ to $\mathfrak{h}$ carry the greatest possible information about V, $\mathfrak{h}$ should clearly be maximal among abelian, diagonalizable, subalgebras. Such an subalgebra is called a *Cartan subalgebra.*

Step 2. *Let $\mathfrak{h}$ act on $\mathfrak{g}$ by the adjoint representation, and decompose $\mathfrak{g}$ accordingly.* By the choice of $\mathfrak{h}$, its action on any representation of $\mathfrak{g}$ will be diagonalizable; applying this to the adjoint representation we arrive at a direct sum decomposition, called the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha), \tag{2}$$

where the action of $\mathfrak{h}$ preserves each $\mathfrak{g}_\alpha$ and acts on it by scalar multiplication by the linear functional $\alpha \in \mathfrak{h}^*$; that is, for any $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ we will have

$$ad(H)(X) = \alpha(H) \cdot X.$$

The second direct sum in the expression 2 is over a finite set of eigenvalues $\alpha \in \mathfrak{h}^*$; these eigenvalues are called the *roots* of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha$ are called the *root spaces.* Of course, $\mathfrak{h}$ itself is just the eingenspace for the action of $\mathfrak{h}$ corresponding to the eigenvalue 0; so that we will be referring $\mathfrak{g}_0$ as $\mathfrak{h}$. The set of all roots is usually denoted $R \subset \mathfrak{h}^*$.

We can picture the structure of the Lie algebra in terms of the diagram of its roots. Let V be a representation of $\mathfrak{g}$, thus for $v \in V$ and $H \in \mathfrak{h}$ it results

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H,X](v) \\ &= X(\alpha \cdot v) + 2X(v) \\ &= (\alpha + 2) \cdot X(v), \end{aligned}$$

by this *fundamental calculation* it results that in the case of the adjoint representation $\mathfrak{g}_\alpha$ carries $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$. To complete the picture of the roots we need some other fact that we report without proof

1. each root space $\mathfrak{g}_\alpha$ is one dimensional.
2. R will generate a lattice $\Lambda_R \subset \mathfrak{h}^*$ of rank equal to the dimension of $\mathfrak{h}$.
3. R is symmetric about the origin, i.e., if $\alpha \in R$ is a root, then $-\alpha \in R$ is a root as well.

Step 3. *Find the distinguished subalgebras* $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}$. Let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be a root space, one dimensional by 1 of the previous step. Then by 3, there is another root space $\mathfrak{g}_{-\alpha}$ and their commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a subspace of $\mathfrak{h}$, of dimension at most one. The adjoint action of the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ thus carries each of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ into itself; so that the direct sum

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}],$$

is a subalgebra of $\mathfrak{g}$. It can be showed that $\mathfrak{s}_\alpha$ is isomorphic to $\mathfrak{sl}_2$.

Step 4. *Use the integrality of the eigenvalues of the* H_α. The distinguished elements $H_\alpha \in \mathfrak{h}$ found above are important first of all because, by the analysis of the representations of $\mathfrak{sl}_2$ carried out in section 1, in any representation of $\mathfrak{s}_\alpha$, consequently in every representation of $\mathfrak{g}$, all the eigenvalues of the action H_α must be integers. Thus every eigenvalue $\beta \in \mathfrak{h}^*$ of every representation of $\mathfrak{g}$ must assume integer values on all the H_α. We correspondingly let Λ_W be the set of linear functionals $\beta \in \mathfrak{h}^*$ that are integer valued on all the H_α; Λ_W will be a lattice, said the *wight lattice* of $\mathfrak{g}$. *All weights of all representations of* $\mathfrak{g}$ *will lie in* Λ_W. In particular $R \subset \Lambda_W$, since the roots are the weights of the adjoint representation. Hence $\Lambda_R \subset \Lambda_W$.

Step 5. *Use the symmetry of the eigenvalues of the* H_α. The integrality of the eigenvalues of the H_α under any representation is only half of the story; it is also true that they are symmetric about the origin in $\mathbb{Z}$. To express this for any α we introduce an product on $\mathfrak{g}$, *the Killing form.* For each pair of elements $X, Y \in \mathfrak{g}$, define

$$B(X,Y) = Tr\left(\mathrm{ad}(X)\mathrm{ad}(Y) : \mathfrak{g} \to \mathfrak{g}\right).$$

One can show that B is a positive definite on the real subspace of $\mathfrak{h}$ spanned by the vectors $\{H_\alpha : \alpha \in R\}$. We will be writing $\langle .,. \rangle = B(.,.)$. We can now define the involution W_α on the vector space $\mathfrak{h}^*$ with $+1$-eigenspace the hyperplane

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* : \langle H_\alpha, \beta \rangle = 0\},$$

and -1-eigenspace the line spanned by α itself. In english, W_α is the reflection in the plane Ω_α with axis the line spanned by α

$$W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha.$$

Let $\mathcal{M}$ the group generated by these involutions; $\mathcal{M}$ is called the *Weyl group* of the Lie algebra $\mathfrak{g}$.

Now suppose that V is any representation of $\mathfrak{g}$, with eigenspace decomposition $V = \oplus V_\beta$. The weights β appearing in this decomposition can then be broken up into equivalence classes mod α, and the direct sum

$$V_{[\beta]} = \oplus_{n \in \mathbb{Z}} V_{\beta + n\alpha},$$

of the eigenspaces in a given equivalence class will be a subrepresentation of V for $\mathfrak{s}_\alpha$. It follows then that the set of weights of V congruent to any given β mod α will be invariant under the involution W_α; in particular it results that *the set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group.*

Step 6. *Choose a direction in $\mathfrak{h}^*$.* By this we mean a real linear functional l in the lattice Λ_R irrational with respect to this lattice. This gives us a decomposition of the set

$$R = R^+ \cup R^-$$

where $R^+ = \{\alpha : l(\alpha) > 0\}$. The set R^+ is called the set of *positive roots*, while R^- is the set of *negative roots*. Once an ordering of the roots is given we can introduce the *highest weight vector* $v \in V$ for a representation V of $\mathfrak{g}$ as the vector in the kernel of the action of $\mathfrak{g}_\alpha$, $\alpha \in R^+$. The results for the highest weight vector are resumed in the following proposition, that we report without proof.

Proposition. 34. *For any semisimple complex Lie algebra $\mathfrak{g}$*

1. *every finite dimensional representation V of $\mathfrak{g}$ possesses a highest weight vector;*
2. *the subspaces W of V generated by the images of a highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$ is an irreducible representation in V;*
3. *an irreducible representation possesses a unique highest weight vector up to scalars.*

Before approaching the final step, we want to introduce a bit more of terminology. By what we have seen, the highest weight of any representation of V will be a weight α satisfying $\alpha(H_\gamma) \geq 0$ for every $\gamma \in R^+$. The locus $\mathcal{W}$, in the real span of the roots, oh points satisfying these inequalities, is called closed *Weyl chamber* associated to the ordering of the roots. A Weyl chamber could also be described as the closure of a connected component of the complement of the union of the hyperplane Ω_α.

Step 7. *Classify the irreducible, finite-dimensional representations of $\mathfrak{g}$.* All the step are leading to the fundamental existence and uniqueness theorem, that we report without proof

Theorem. 35. *For any α in the intersection of the Weyl chamber $\mathcal{W}$ associated to the ordering of the roots with the weight lattice Λ_W, there exists a unique irreducible, finite-dimensional representation Γ_α of $\mathfrak{g}$ with highest weight α; this gives a bijection between $\mathcal{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$. The weights of Γ_α will consist of those elements of the weight lattice congruent to α modulo the root lattice Λ_R and lying in the convex hull of the set of points in $\mathfrak{h}^*$ conjugate to α under the Weyl group.*

A.2 The representations of $\mathfrak{so}(Q)$

In this section we will consider a vector space V equipped with a bilinear form Q. We define the group $SO(Q)$ to be the group in $End(V)$ preserving the bilinear form

$$SO(Q) = \{g \in End(V) | Q(gv, gv) = Q(v, v), v \in V\}.$$

A distinction must be done between the odd and the even dimension case. We want to write Q in terms of a basis of V, and here is where the cases of even and odd dimension first separate. In the even case, where $Dim(V) = 2n$, we will choose a basis for V such that:

$$Q(e_j, e_j + n) = Q(e_j + n, e_j) = 1$$

and

$$Q(e_i, e_j) = 0, \text{if } j \neq i \pm n.$$

The bilinear form Q may be expressed as:

$$Q(x, y) = x^t M y$$

where M is a $2n \times 2n$-matrix given in block form as

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The group $SO_{2n}(\mathbb{C})$ is the group of $2n \times 2n$-matrices A satisfying:

$$M = A^t M A$$

and the Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$ correspondingly the space of matrices X satisfying the relation

$$X^t M + MX = 0.$$

Writing a $2n \times 2n$-matrix X in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

we have

$$X^t M = \begin{pmatrix} C^t & A^t \\ D^t & B^t \end{pmatrix}.$$

and

$$MX = \begin{pmatrix} C & D \\ A & B \end{pmatrix}.$$

Then we have that the off-diagonal blocks B and C of X are skew-symmetric, and the diagonal blocks A and D are negative transposes of each other In odd dimension the canonical form for Q is slightly different. We can write

$$Q(x, y) = x^t M y$$

with the $2n + 1 \times 2n + 1$-matrix M

$$M = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The bottom right block is 1×1. Analogously the Lie algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ is the algebra of the matrices X satisfying the relation $X^t M + MX = 0$. Given X

$$X = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix},$$

then it is equivalent to saying that B,C are skew-symmetric and A and D negative transposes of each other. Moreover $E = -H^t$, $F = -G^t$, and $J = 0$. Referring to the general semisimple Lie algebra theory, we want to find a Cartan sublagebra. With these choices in the odd and even case, we may take the subalgebra $\mathfrak{h} \subset \mathfrak{so}_m\mathbb{C}$ of diagonal matrices. As usual we find which endomorphisms are allowed eigenvectors for the action of $\mathfrak{h}$. It results that

$$Y_{i,j} = E_{i,n+j} - E_{j,n+i}$$

$$Z_{i,j} = E_{n+i,j} - E_{n+j,i}$$

are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues $L_i + L_j$ and $-L_i - L_j$, respectevely. In sum the set of the roots of the Lie algebra $\mathfrak{so}_m\mathbb{C}$ are the vectors $\{\pm L_i \pm L_j\}_{i \neq j} \subset \mathfrak{h}^*$. Next we should proceed by depicting a graphics of the weights and build the allowed weighs with the several copies of $\mathfrak{sl}_2\mathbb{C}$ contained in $\mathfrak{so}_m\mathbb{C}$. The final result is slightly different in even and odd dimension again.

In the even case, recall that the weight lattice is generated by $L_1, \ldots, L_n$ together with the further vector $(L_1 + \cdots + L_n)/2$. The weyl chamber, on the other hand, is the cone:

$$\mathcal{W} = \{\sum a_i L_i | a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

Note that the Weyl chamber is a simplicial cone, with faces corresponding to the n planes $a_1 = a_2, \ldots a_{n-1} = a_n$, the edges of the Weyl chamber are thus the rays generated by the vectors $L_1, L_1 + L_2, \ldots L_1 + \ldots L_n$ and $L_1 + \cdots + L_{n-1} - L_n$. We see from this that the intersection of the weight lattice with the closed Weyl cone is a free semigroup generated by fundamental weights, in this case the vectors $L_1, L_1+L_2, \dots L_1+\dots L_{n-2}$ and the vectors

$$\alpha = (L_1 + \dots L_n)/2, \qquad \beta = (L_1 + \dots L_{n-1} - L_n)/2.$$

As before, the obvious place to start to look for irreducible representations is among the exterior power of the standard representation. This almost works: we have

Theorem. 36. *The exterior powers $\Lambda^k V$ of the standard representation V of $\mathfrak{so}_m\mathbb{C}$ are irreducible for $k = 1, 2, \dots, n-1$. The exterior power $\Lambda^k V$ has exactly two irreducible factors.*

Proof. Consider the matrix of the metric in the standard form and $e_1, \dots, e_2n$ the associated orthonormal basis. The group $SO_{2n}(\mathbb{C})$ contains a subgroup G of automorphisms of the space $V = \mathbb{C}^2 m$ preserving the decomposition

$$V = \mathbb{C}\{e_1, \dots e_n\} \oplus \mathbb{C}\{e_{n+1}, \dots e_{2n}\}$$

. In matrices

$$G = \left\{ \begin{pmatrix} X & 0 \\ 0 & X^{t^{-1}} \end{pmatrix}, X \in GL_n(\mathbb{C}) \right\}.$$

The corresponding subalgebra is

$$\mathfrak{s} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}, X \in \mathfrak{sl}_n\mathbb{C} \right\} \subset \mathfrak{so}_{2n}\mathbb{C}.$$

Denote by W the standard representation of $\mathfrak{sl}_n\mathbb{C}$. The restriction of the standard representation V of $\mathfrak{so}_{2n}\mathbb{C}$ to the subalgebra $\mathfrak{s}$ splits

$$V = W \oplus W^*$$

into a direct sum of W and its dual. We have correspondingly

$$\Lambda^k V = \bigoplus_{a+b=k} (\Lambda^a W \otimes \Lambda^b W^*)$$

We also can say how each factor on the right-hand side of this expression decomposes as a representation of $\mathfrak{sl}_n\mathbb{C}$: we have the contraction maps

$$\Psi_{a,b} : \Lambda^a W \otimes \Lambda^b W^* \to \Lambda^{a-1} W \otimes \Lambda^{b-1} W^*$$

and the kernel of $\Psi_{a,b}$ is the irreducible representation $W^{a,b}$ with highest weight $2L_1 + \dots + 2L_a + L_{a+1} + \dots + L_{n-b}$. The restriction of $\Lambda^k V$ to $\mathfrak{s}$ is thus given by

$$\Lambda^k V = \bigoplus_{a+b \leq k a+b=k(2)} W^{a,b},$$

where the actual highest weight factor in the summand $W^{a,b} \subset \Lambda^k V$ is the vector

$$w^{a,b} = e_1 \wedge \dots \wedge e_a \wedge e_{2n-b+1} \wedge \dots \wedge e_{2n} \wedge Q^{(k-a-b)/2} =$$

$$= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge (\sum (e_i \wedge e_{n+i})^{(k-a-b)/2}.$$

Now, all the vectors $w^{a,b}$ have distinct weights and it follows that any highest weight vector for the action of $\mathfrak{so}_{2n}\mathbb{C}$ on $\Lambda^k V$ will be a scalar multiple of one of the $w^{a,b}$. It will thus suffice in order to show that $\Lambda^k V$ is irreducible for $k < n$, to exhibit for each (a, b) with $a + b \leq k$ other than $(k, 0)$ a positive root α such that the image $\mathfrak{g}_\alpha(w^{a.b}) \neq 0$. This is simplest in the case $a + b = k$. We have

$$\begin{aligned} Y_{a+1,n-b+1}(w^{a,b}) &= \\ (E_{a+1,2n-b+1} - E_{n-b+1,n+a+1})(e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n}) &= \\ w^{a+1,b-1} &\neq 0, \end{aligned}$$

and $Y_{i,j}$ is the generator of the positive root space $\mathfrak{g}_{L_i+L_j}$. For $a + b < k$ the calculation is just slightly different. This concludes the proof for $k < n$. The proof for $a + b = k = n$ requires only one further step: we have to check the vectors $w^{a,b}$ with $a + b = k = n$ to see if any of them might be highest weight vectors for $\mathfrak{so}_{2n}\mathbb{C}$. In fact (as tthe statement of the theorem implies), two of them are: one can check that $w^{n,0}$ and $w^{n-1,1}$ are killed by every positive root space $\mathfrak{g}_{L_i+L_j}$. To see that no other vector $w^{n-a,a}$ is, look at the action of $Y_{a+1,a+2} \in \mathfrak{g}_{L_{a+1}+L_{a+2}}$, we have

$$\begin{aligned} Y_{a+1,a+2}(w^{a,n-a}) &= \\ E_{a+1,n+a+2} - E_{a+2,n+a+1}(e_1 \wedge \cdots \wedge e_a \wedge e_{n+a+1} \wedge \cdots \wedge e_{2n}) &= \\ = e_1 \wedge \cdots \wedge e_a \wedge e_{a+1} \wedge e_{n+a+1} \wedge e_{n+a+3} \wedge \cdots \wedge e_{2n} & \\ -e_1 \wedge \cdots \wedge e_a \wedge e_{a+2} \wedge e_{n+a+2} \wedge \cdots \wedge e_{2n} &\neq 0. \end{aligned}$$

Δ

Remark that we have not all the irreducible representation of $\mathfrak{so}_{2n}\mathbb{C}$. By the theorem the exterior powers $\Lambda^0 V, \dots \Lambda^{n-2} V$ provide us the irreducible representatios with highest weight the fundamental weight along the first $n-2$ edges of the Weyl chamber (of course the eterior power $\Lambda^{n-2} V$ is irreducible but its highest weight does not lie in the Weyl chamber). Even if we have decomposed the representation $\Lambda^n V$ in two irreducible subrepresentation, we have that such two representations have no primitive highest weight: they are divisible by 2. Thus, given the theorem, we have constructed exactly one-half of the irreducible representation for $\mathfrak{so}_{2n}\mathbb{C}$, those whose highest weights lies in the sublattice $\mathbb{Z}\{L_1, \dots, L_n\}$. Explicitly any weight in the Weyl chamber can be written as

$$\begin{aligned} \gamma &= a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-2}(L_1 + \cdots + L_{n-2}) + \\ &\quad + a_{n-1}(L_1 + \cdots + L_{n-1} - L_n)/2 + a_n(L_1 + \cdots + L_n)/2 \end{aligned}$$

with $a_i \in \mathbb{N}$. If $a_{n-1} + a_n$ is even, with $a_{n-1} \geq a_n$ we see that the representation

$$Sym^{a_1} V \otimes \cdots \otimes Sym^{a_{n-2}} V \otimes Sym^{a_{n-1}}(\Lambda^{n-1} V) \otimes Sym^{(a_{n-1}-a_n)/2}(\Gamma_{2\beta})$$

will contain an irreducible representation Γ_γ with highest weight γ. If $a_{n-1} \leq a_n$ we will find Γ_γ inside

$$Sym^{a_1} V \otimes \cdots \otimes Sym^{a_{n-2}} V \otimes Sym^{a_{n-1}}(\Lambda^{n-1} V) \otimes Sym^{(a_n - a_{n-1})/2}(\Gamma_{2\alpha})$$

There remains a problem of constructing the irreducible representations Γ_γ whose highest weight γ involves an odd number of α's and β's. To do this, we clearly have to exhibit irreducible representations Γ_α and Γ_β. These are called the spin representations of $\mathfrak{so}_{2n}\mathbb{C}$. When such a representations will be constructed, the representation Γ_γ will be found in the tensor product:

$$Sym^{a_1} V \otimes \cdots \otimes Sym^{a_{n-2}} V \otimes Sym^{a_{n-1}}(\Gamma_\beta) \otimes Sym^{(a_n}(\Gamma_\alpha)$$

In the odd case, recall that the weight lattice of $\mathfrak{so}_{2n+1}\mathbb{C}$ is generated by $L_1, \ldots, L_n$ together with the further vector $(L_1 + \cdots + L_n)/2$. The Weyl chamber is the cone:

$$\mathcal{W} = \left\{ \sum a_i L_i | a_1 \geq a_2 \geq \cdots \geq a_n \right\}.$$

Again the intersection of the Weyl chamber with the weight lattice is a free semigroup, in this case generated by the fundamental weights $\omega_1 = L_1$, $\omega_2 = L_1 + L_2$, $\omega_{n-1} = L_1 + \cdots + L_{n-1}$ and $\alpha = /L_1 + \cdots + L_n)/2$. The standard representation do serve to generate all the irreducible representations whose highest weights are in the sublattice $\mathbb{Z}\{L_1, \ldots, L_n\}$. The following theorem guarantees us that the exterior power of the standard representation are irreducible.

Theorem. 37. *For $k = 0, 1, \ldots, n$ the exterior power $\Lambda^k V$ of the standard representation V of $\mathfrak{so}_{2n+1}\mathbb{C}$ is the irreducible representation with highest weight $L_1 + \ldots L_n$.*

Proof. We don't report the proof, since it is the analogous of the even case. Δ

We have thus constructed one-half of the irreducible representations of $\mathfrak{so}_{2n+1}\mathbb{C}$: any weight γ in the closed Weyl chamber can be written

$$\begin{aligned} \gamma &= a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1}) + \\ &\quad + a_n(L_1 + \cdots + L_n)/2, \end{aligned}$$

with $a_i \in \mathbb{N}$. If a_n is even, the representation

$$Sym^{a_1} V \otimes \cdots \otimes Sym^{a_n}(\Lambda^n V)$$

will contain an irreducible rep with highest weight γ. Yet again we are still missing any representation whose weights involve odd mutiples of α (remark that theorem 37 does not consider odd multiple of α). For constructing these, we have to exhibit an irreducible representation Γ_α with highest weight α. This is the spin representation of $\mathfrak{so}_{2n+1}\mathbb{C}$. Once this is done, we will have constructed all the representations of $\mathfrak{so}_{2n+1}\mathbb{C}$. For any γ as above the tensor

$$Sym^{a_1} V \otimes \cdots \otimes Sym^{a_{n-1}}(\Lambda^{n-1} V) \otimes Sym^{a_n}(\Gamma_\alpha)$$

will contain a copy of Γ_γ. Remark that the result is very similar to the even case, the difference is born by the irreducibility of the n-th exterior power of the standard in the odd case. We will recover the missing representation for $\mathfrak{so}_m\mathbb{C}$ in the following section, as the representations of the universal covering of $SO(m)$, the group $Spin(Q)$.

A.3 Clifford algebras

For introducing the Spin group, it is necessary to define first a Clifford algebra. It is given a vector space V, equipped with a quadratic form Q. A Clifford algebra $Cl(V,Q)$ is an associative algebra with unit 1, which contains and is generated by V, with the relations:

$$v \cdot w + w \cdot v = 2Q(v,w), v, w \in V. \tag{3}$$

Otherwise, a Clifford algebra can be defined as the algebra satisfying the following universal property: given E an associative algebra with unit, and given a linear mapping $j : V \to E$ such that $j(v)^2 = Q(v,v)\dot{1}$ for $\forall v \in V$, then there should be a unique homomorphism of algebras from $Cl(V,Q)$ to E extending j. The Clifford algebra can be realized in the tensor algebra

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus \dots$$

and then taking the quotient $Cl(V,Q) = T(V)/I(Q)$, where $I(Q)$ is the two sided ideal generated by all elements of the form $v \otimes v - Q(v,v) \cdot 1$. It is automatic that this $Cl(V,Q)$ satisfies the universal property. With the following lemma we realize a basis for $Cl(V,Q)$.

Lemma. 38. Let $v_1, \dots, v_n$ be a basis for V, then the products $e_I = e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_k}$ for $I = \{i_1 < \dots < i_k\}$, and with $e_\emptyset = 1$, form a basis for $Cl(V,Q)$.

Proof. From the equations

$$e_i \cdot e_j + e_j \cdot e_i = 2Q(e_i, e_j),$$

it follows that the elements e_I generate $Cl(V,Q)$. Their independence follows by remarking that e_I and e_J are independent if length$(I) \neq$ length(J) and e_I and e_J with same length are independent if and only if I and J are not two permutations of the same set. Thus $\{1, e_I\}_{I \text{ multi-index}}$ is a basis $Cl(V,Q$. Δ

Remark. 39. When $Q = 0$ the Clifford algebra is just the exterior algebra ΛV.

Remark. 40. Due to the lemma the dimension of $Cl(V,Q)$ is $2^{dim(V)}$ and the canonical map from V to $Cl(V,Q)$ is an embedding.

Since the ideal $I(Q) \subset T(V)$ is generated by elements of even degree, the Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$ grading:

$$Cl(Q) = C^{\text{even}} \oplus C^{\text{odd}} = C^{+} \oplus C^{-}$$

with

$$C^{+}C^{-} \subset C^{-}, C^{+}C^{+} \subset C^{+}, C^{-}C^{-} \subset C^{+}.$$

The set C^{+} is spanned by the products of an even number of elements in V, similarly the set C^{-}. is spanned by the product of an odd number of elements in V. In particular C^{+} is a subalgebra of dimension 2^{m-1}. Remark that as an associative algebra, $Cl(V,Q)$ can be realized as a Lie algebra with bracket $[a,b] = ab - ba$. From now on we will assume that Q is non-degenerate. The goal of this section will be realizing the non-integer spin representation for $SO(Q)$ (due to general theory for compact Lie group, the finite dimensional irreducible representation for the group and the algebra are the same). This will be achieved in two steps:

- embedding $\mathfrak{so}(Q)$ inside C^{+}.
- identifying the Clifford algebra with one or two copies of matrix algebras.

For achieving the first goal we make explicit an isomorphism between $\Lambda^2 V$ with $\mathfrak{so}(Q)$. Recall that

$$\mathfrak{so}(Q) = \{X \in End(V) | Q(X,v,w) = Q(v,Xw) = 0, \text{for all } v,w \in V\}.$$

The isomorphism is given by

$$\Lambda^2 V = \mathfrak{so}(Q) \in End(V), \quad a \wedge b \to \phi_{a\wedge b}$$

for $a, b \in V$, where $\phi_{a\wedge b}$ is defined as:

$$\phi_{a\wedge b} = 2\left(Q(b,v)\cdot a - Q(a,v)\cdot b\right).$$

Then one calculates what the bracket on $\Lambda^2 V$ must be to make ϕ an isomorphism of Lie algebras:

$$\begin{aligned}
[\phi_{a\wedge b}, \phi_{c\wedge d}](v) &= \phi_{a\wedge b} \circ \phi_{c\wedge d}(v) - \phi_{c\wedge d} \circ \phi_{a\wedge b}(v) \\
&= 2\phi_{a\wedge b}(Q(d,v)c - Q(c,v)d) - 2\phi_{c\wedge d}(Q(b,v)a - Q(a,v)b) \\
&= 4Q(d,v)(Q(b,d)a - Q(a,d)b) \\
&\quad -4Q(c,v)(Q(b,d)a - Q(a,d)b) \\
&\quad -4Q(b,v)(Q(d,a)c - Q(c,a)d) \\
&\quad +Q(a,v)(Q(d,b)c - Q(c,b)d \\
&= 2Q(b,c)\phi_{a\wedge d} - 2Q(b,d)\phi_{a\wedge c} \\
&\quad -2Q(a,d)\phi_{c\wedge b} + 2Q(a,c)\phi_{d\wedge b}.
\end{aligned}$$

On the other hand, the bracket in the Clifford algebra satisfies

$$\begin{aligned}
[ab, cd] &= a\cdot b\cdot c\cdot d - c\cdot d\cdot a\cdot b \\
&= 2(Q(b,c)a\cdot d - a\cdot c\cdot b\cdot d) - (2Q(a,d)c\cdot b - c\cdot a\cdot d\cdot b) \\
&= 2Q(b,c)a\cdot d - (2Q(b,d)a\cdot c - a\cdot c\cdot d\cdot b) \\
&\quad -2(Q(a,d)c\cdot b + (2(Q(a,c)\cdot d\cdot b - a\cdot c\cdot d\cdot b) \\
&= 2Q(b,c)ad - 2Q(b,d)ac - 2Q(a,d)cb + 2Q(a,c)db.
\end{aligned}$$

It follows that the map $\psi : \Lambda^2 V \to Cl(V,Q)$ defined by

$$\psi(a\wedge b) = \frac{1}{2}(ab - ba) = ab - Q(a,b)$$

is a map of Lie algebras, and by looking at basis elements again one sees that it is an embedding. This proves:

Lemma. 41. The mapping $\psi\phi^{-1} : \mathfrak{so}(Q) \to Cl(V,Q)^{\text{even}}$ embeds $\mathfrak{so}(Q)$ as a Lie algebra in $Cl(V,Q)^{\text{even}}$.

We consider the even case. Write $V = W \oplus W'$, where W and W' are two n-dimensional isotropic spaces for Q. Recall that an isotropic space is a vector space where the form Q is zero. We can choose W as the vector space spanned by the first n basis vectors and W' by the last n, by fixing the non-degenerate form as the canonical symplectic matrix of dimension $2n$.

Lemma. 42. The decomposition $V = W \oplus W'$ determines an isomorphism of algebras

$$Cl(V,Q) = End(\Lambda W),$$

where $\Lambda W = \Lambda^0 W \oplus \Lambda^1 W \oplus \cdots \oplus \Lambda^n W$.

Proof. Mapping $Cl(V,Q)$ to the algebra $E = End(\Lambda W)$ is the same as defining a linear mapping $\phi : V \to E$ satisfying $\phi(v)^2 = Q(v,v)$. For doing this we construct two maps $lW \to E$ and $l'W' \to E$ such that

$$l(w)^2 = 0, \quad l'(w')^2 = 0,$$

and

$$l(w)\cdot l'(w') + l'(w')\cdot l(w) = 2(Q(w,w')\cdot 1 \tag{4}$$

for any $w \in W$, $w' \in W'$. For each $w \in W$, let $L_w \in E$ be left multiplication by w on the exterior algebra ΛW

$$L_w(\zeta) = w\wedge\zeta, \quad \zeta \in \Lambda W.$$

For $\theta \in W^*$, let D_θ be the derivation of ΛW such that $D_\theta(1) = 0$, $D_\theta(w) = \theta(w) \in \Lambda^0 W = \mathbb{C}$ for $w \in W$, and

$$D_\theta(\zeta\wedge\zeta) = D_\theta(\zeta)\wedge\zeta + (-1)^{\deg(\zeta)}\zeta\wedge D_\theta(\zeta).$$

Explicitly

$$D_\theta(w_1, \ldots, w_r) = \sum(-1)^{i-1}\theta(w_i)(\omega_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge \omega_r.$$

Now

$$l(w) = L_w \quad l'(w') = D_\theta$$

where $\theta \in W^*$ is defined by the identity $\theta(w) = 2Q(w, w')$ for all $w \in W$. The required equations 4 are satisfied: one checks directly on elements in $W = \Lambda^1 W$, and then that if they hold on ζ and ξ, they hold on $\zeta \wedge \xi$. Finally one may see that the resulting map in an isomorphism by looking at what happens to a basis. Δ

As consequence of the lemma we have a decomposition $\Lambda W = \Lambda^{\text{even}} W \oplus \Lambda^{\text{odd}} W$ into the sum of even and odd exterior powers, and $Cl(W, Q)^{\text{even}}$ respects this splitting. We deduce from Lemma an isomorphism:

$$Cl(V, Q)^{\text{even}} = End(\Lambda^{\text{even}} W) \oplus End(\Lambda^{\text{odd}} W).$$

Thus the following embedding of Lie algebras holds:

$$\mathfrak{so}(Q) \subset Cl(V, Q)^{\text{even}} = \mathfrak{gl}(\Lambda^{\text{odd}} W) \oplus \mathfrak{gl}(\Lambda^{\text{even}} W),$$

and hence we have two represenations of $\mathfrak{so}(Q)$, which we denote by:

$$S^+ = \Lambda^{\text{even}} W, \quad S^- = \Lambda^{\text{odd}} W.$$

With the following proposition, we want to prove that these are the wanted representations.

Proposition. 43. *The representations $S^\pm$ are the irreducible representations of $\mathfrak{so}(Q)$ with highest weights $\alpha = \frac{1}{2}(L_1 + \cdots + L_n)$ and $\beta = \frac{1}{2}(L_1 + \cdots + L_{n-1}$.*

Proof. We show that the natural basis vectors e_I for ΛW are weight vectors. Tracing through the isomorphism established above, we see that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+1})$ in $\Lambda^2 V$, which corresponds to $\frac{1}{2}(e_i \cdot e_{n+1} - 1)$ in $Cl(V, Q)$, which maps to

$$\frac{1}{2}(L_{e_i} D_{2e_i^*} - I) = L_{e_i} D_{e_i^*} - \frac{1}{2} I \in End(\Lambda W).$$

A simple calculation shows that

$$L_{e_i} D_{e_i^*}(e_I) = \begin{cases} e_I & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$$

Therefore, e_I spans a weight space with weight $\frac{1}{2}\left(\sum_{i \in I} L_i - \sum_{j \notin I} L_j\right)$. All such weights with given $|I|$ mod 2 are congruent by the Weyl group, so each of $S^+ = \Lambda^{\text{even}} W^+$ and $S^- = \Lambda^{\text{odd}} W$ must be an irreducible representation. The highest weights are easy to read off. For example, the highest weight for $\Lambda^{\text{even}}W$ is $\frac{1}{2}\sum L_i = \alpha$ if n is even, while if n is odd, its highest weight is β.

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For the odd case, write $V = W \oplus W' \oplus U$, where W and W' are n-dimensional isotropic subspaces, and U is a one-dimensional space perpendicular to them. For our standard Q on $\mathbb{C}^{2n+1}$, these are spanned by the first n, the second n, and the last basis vector. We emulate the even case, thus we find an isomorphism of $Cl(V,Q)$ with an algebra of endomorphism and then we discuss of the semi-integer irreducible representations.

Lemma. 44. The decomposition $V = W \oplus W' \oplus U$ determines an isomorphism of algebras

$$Cl(Q,V) = End(\Lambda W) \oplus End(\Lambda W').$$

Proof. Proceeding as in the even case, to map V to $E = End(\Lambda W)$, map $w \in W$ to L_w, and $w' \in W'$ to D_θ, where $\theta(w) = 2Q(w,w')$ as before. Let u_0 be the element in U such that $Q(u_0,u_0) = 1$ and send u_0 to the endomorphism that is the identity on $\Lambda^{\text{even}}W$, and minus the identity on $\Lambda^{\text{even}}W$. Since this involution skew communtes with all L_w and D_θ, the resulting map from $V = W \oplus W' \oplus U$ to E determines an algebra homomorphism from $Cl(V,Q)$ to E. The map to $End(\Lambda W')$ is defined similarly, swapping the roles of W and W'. Again one checks that the map is an isomorphism by looking at bases.

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Proposition. 45. *The representation $S = \Lambda W$ is the irreducible representation of $\mathfrak{so}(Q)$ with highest weight $\alpha = \frac{1}{2}(L_1 + \cdots + L_n)$.*

Proof. Exactly as in the even case, each e_I is an eigenvector with weight

$$\frac{1}{2}(\sum_{i\in I} L_i - \sum_{j\notin I} L_j).$$

This time all such weights are congruent modulo the Weyl group, so this must be an irreducible representation, and the highest weight is clearly $\frac{1}{2}(L_1+\cdots+L_n)$.

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Finally we want to find, inside the Clifford algebra, the Spin group. First define the anti-involution $x \to x^*$ determined by:

$$(v_1 \ldots v_r)^* = (-1)^r v_r \ldots v_1$$

for any $v_1 v_r \in V$. This operation $*$ is also called the conjugation. The Spin group can be defined as following:

$$Spin(Q) = \{x \in C(Q)^{\text{even}} | x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}.$$

From this definition we see that $Spin(Q)$ is a closed subgroup of the group of units in the even Clifford algebra. Any $x \in Spin(Q)$ determines an endomorphism ρ of V by:

$$\rho(x) \cdot v = x \cdot v \cdot x^*, v \in V$$

We want to see that the spin group is a covering of the orthogonal group.

Proposition. 46. *The mapping*

$$\rho : Spin(Q) \to SO(Q)$$

is a homomorphism, making $Spin(Q)$ a connected two-sheeted covering of $SO(Q)$. The kernel of ρ is $\{-1, 1\}$.

Proof. It is possible to show a stronger result. Define a larger subgroup, this time of the multiplicative group of $Cl(V, Q)$, by

$$Pin(Q) = \{x \in Cl(V, Q) | x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\},$$

and define a homomorphism

$$\rho : Pin(Q) \to O(Q), \quad \gamma(x) = \alpha(x) \cdot v \cdot x^*,$$

where $\alpha : Cl(V, Q) \to Cl(V, Q$ is the main involution. To see that ρ preservers the quadratic form Q, we use the fact that for $w \in V$, $Q(w, w) = ww = -ww^*$, and calculate

$$\begin{aligned} Q(\rho(x) \cdot v, \rho(x) \cdot v) &= -\alpha(x) \cdot v \cdot x^*(\alpha(x) \cdot v \cdot x^*)^* \\ &= -\alpha(x) \cdot v \cdot x^* \cdot x \cdot v^* \alpha(x)^* \\ &= -\alpha(x) \cdot v \cdot v^* \cdot \alpha(x^*) \\ &= Q(v, v) \cdot \alpha(x) \cdot \alpha(x^*) \\ &= Q(v, v) \cdot \alpha(xx^*) = Q(v, v). \end{aligned}$$

We claim next that ρ is surjective. This follows from the standard fact that the orthogonal group $O(Q)$ is generated by reflections. Indeed if R_w is the reflection in the hyperplane perpendicular to a vector w, normalized so that $Q(w, w) = -1$, it is easy to see that w is in $Pin(Q)$ and $\rho(w) = R_w$. In fact

$$w \cdot w^* = w \cdot (-w) = -Q(w, w) \cdot 1 = 1,$$

and so

$$\rho(w) \cdot w = \alpha(w) \cdot w \cdot w^* = -w \cdot 1 = -w.$$

If $Q(w, v) = 0$,

$$\rho(w) \cdot v = \alpha(w) \cdot v \cdot w^* = -w \cdot v \cdot w^* = v \cdot w \cdot w^* = v.$$

The next claim is that the kernel of ρ on the larger group $Pin(Q)$ is ± 1. Suppose x is in the kernel, and write $x = x_0 + x_1$ with $x_0 \in C^{\text{even}}$ and $x_1 \in C^{\text{odd}}$. Then

$x_0 \cdot v = v \cdot x_0$ for all $v \in V$, so x_0 is in the center of C. While $x_1 \cdot v = -v \cdot x_1$ for all $v \in V$. Using the identification of a Clifford algebra with a matrix algebra, one can calculate Then x_0 is in $\mathbb{C} \cdot 1$ and $x_1 = 0$. So $x = x_0$ is in $\mathbb{C}$ and $x^2 = 1$. So $x = \pm 1$.

It follows that if $R \in O(Q)$ is written as product of reflections $R = R_{w_1} \ldots R_{w_r}$, then the two elements in $\rho^{-1}(R)$ are $\pm w_1 \ldots w_r$.

To complete the proof, we must check that $Spin(Q)$ is connected or equivalently, that the two elements in the kernel of ρ can be connected by a path, since $SO(Q)$ is connected. We exhibit explicitly the path connecting ± 1. Let ξ, η be two elements in $Spin(Q)$, with $\xi \perp \eta$. By the relations 3, it results

$$\xi\eta\xi\eta = -\xi\xi\eta\eta = -1.$$

The condition $|\xi| = 1$ defines a sphere. For $dim(V) \geq 2$, the sphere is connected and a path ξ_t, with $|\xi_t| = 1, \quad \xi_0 = \eta$ and $\xi_1 = \xi$, exists. Then $\xi_t\eta\xi_t\eta$ connects ± 1.

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For completeness, we present some low dimensional accidental isomorphism

$$\begin{aligned}
Spin(1,1) &= GL_1(\mathbb{R}) \\
Spin(2,1) &= SL_2(\mathbb{R}) \\
Spin(3,1) &= SL_2(\mathbb{C}) \\
Spin(2,2) &= SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \\
Spin(3,2) &= Sp_4(\mathbb{R})
\end{aligned}$$

B Pure Yang-Mills theory

B.1 Connections and Curvature

In this section we want to introduce the concept of connection for a vector bundle in general. For short, a connection is the generalization of the concept of derivative for the sections of a vector bundle. We will be working with complex vector bundles that are *smooth*. The same results will hold for real vector bundles.

Let E be a complex smooth vector bundle over a manifold M. Consider the dual of the tangent space of M, say TM^*. The complex tensor product $E \otimes TM^*$ is also a complex vector bundle. The vector space of the smooth sections of this bundle will be denoted by $\Gamma(TM^* \otimes E)$. A connection on E is a $\mathbb{C}$ linear mapping

$$\nabla : \Gamma(E) \to \Gamma(TM^* \otimes E)$$

which satisfies the Leibniz formula

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for $f \in C^\infty(M, \mathbb{C})$ and $s \in \Gamma(E)$. The image $\nabla(s)$ is called the covariant derivative of s.

The connection ∇ can be written locally in coordinates. Choose a trivializing neighborhood U_α, and choose a basis $s_1, \dots s_n$ for the sections of $E|_{U_\alpha}$, so that every section can be written as a sum $f_1 s_1 + \dots + f_n s_n$ for f_i complex smooth functions.

Proposition. 47. *A connection ∇ on the trivial bundle $E|_{U_\alpha}$ is uniquely determined by $\nabla(s_1) \dots \nabla(s_n)$, which can be completely arbitrary smooth sections of the bundle $TM^* \otimes E$. Each of the sections $\nabla(s_i)$ can be written uniquely as a sum*

$$\sum \omega_{ij} \otimes s_j$$

where ω is a matrix of smooth 1-forms on U_α.

Due to the proposition a ∇ is given locally when a 1-form $A \in TM^*|_{U_\alpha}$ is given.

Example. 48. In the case of the tangent bundle, $E = TM$, a base of the sections can be easily calculated. Once a coordinated open U_α, the derivatives $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x^n}$ are a base for the sections over U_α. A connection is a linear combination

$$\sum (\omega_{ij} \otimes \frac{\partial}{\partial x_j})$$

with coefficients 1-forms on U_α. By writing a 1-form explicitly, it results

$$\sum (\Gamma^k_{ij} dx^i \otimes \frac{\partial}{\partial x_j}). \tag{5}$$

The coefficients Γ^k_{ij} are called the Christoffel symbols.

Example. 49. The flat connection is the connection such that the covariant derivatives $\nabla(s_1)\dots\nabla(s_n)$ are all zero, i.e. the connection matrices ω are zero. It is given by

$$\nabla\left(\sum f_i s_i\right) = \sum f_i \otimes s_i$$

Remark that the flat connection depends on the choice of the basis $\{s_i\}$.

A consequence of Proposition 47 is that a local connection can always be defined. Then each time we have a set of trivializing neighborhood $\{U_\alpha\}$ covering the base space M, we can define several ∇_α locally and then glue them together with a partition of unity to write a global connection. By asking the paracompactness of the base space M, we are guaranteed that a partition of the unity exists. This proves the following.

Proposition. 50. *Every smooth complex vector bundle E with paracompact base space M possesses a connection.*

Next let us consider the case of an induced vector bundle. Given a smooth map $g : M' \to M$ we can build the induced vector bundle $E' = g^*E$. Remark that there is a canonical $C^\infty(M, \mathbb{C})$-linear mapping

$$g^* : \Gamma(E) \to \Gamma(E').$$

Also any 1-form on M pulls back to a 1-form on M', so there is a canonical $C^\infty(M, \mathbb{C})$- linear mapping

$$g^* : \Gamma(TM^* \otimes E) \to \Gamma(TM'^* \otimes E')$$

The following guarantees us that the pull back of a connection is uniquely defined

Proposition. 51. *To each ∇ connection on E there corresponds one and only one connection $\nabla' = g^*\nabla$ on the induced bundle E' so that the following diagram is commutative*

$$\begin{tikzcd}
\Gamma(E) \arrow[r, "\nabla"] \arrow[d] & \Gamma(TM^* \otimes E) \arrow[d] \\
\Gamma(E') \arrow[r, "\nabla'"] & \Gamma(TM'^* \otimes E')
\end{tikzcd}$$

Proof. Over a trivializing neighborhood $U \subset M$, consider a basis for the sections $\Gamma(E)$. Then

$$\nabla(s_i) = \sum \omega_{ij} \otimes s_j.$$

We can lift the 1-forms ω_{ij} and the sections s_i to ω'_{ij} and s'_i, respectevely. If a lifted connection exists, then it is

$$\nabla'(s'_i) = \sum \omega'_{ij} \otimes s'_j.$$

The only verification to do is that ∇' is effectively a connection.

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We want to use the proposition 51 to construct something like a connection on the bundle $TM^* \otimes E$, once a connection ∇ on E is given. We will make use of ∇ together with the exterior power differentiation operator $d : \Gamma(TM^*) \to \Gamma(\Lambda^2 TM^*)$.

Proposition. 52. *Given ∇ there is one and only one $\mathbb{C}$ linear mapping*

$$\hat{\nabla} : \Gamma(TM^* \otimes E) \to \Gamma(\Lambda^2 TM^* \otimes E)$$

satisfying the Leibniz formula

$$\hat{\nabla}(\theta \otimes s) = d\theta \otimes s - \theta \wedge \nabla(s)$$

for every 1-form θ and every section $s \in \Gamma(E)$. Futhermore $\hat{\nabla}$ satisfies the identity

$$\hat{\nabla}(f(\theta \otimes s) = df \wedge (\theta \otimes s) + f\hat{\nabla}(\theta \otimes s).$$

Once we defined $\hat{\nabla}$, let us consider the composition $K = \nabla\hat{\nabla}$ of the two $\mathbb{C}$-linear mappings

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(TM^* \otimes E) \xrightarrow{\hat{\nabla}} \Gamma(\Lambda^2 TM^* \otimes E).$$

First remark that K is actually a $C^\infty(M, \mathbb{C})$-linear operator, for $f \in C^\infty(M, \mathbb{C})$ results

$$K(fs) = \hat{\nabla}(df \otimes s + f\nabla(s)) = 0 - df \wedge \nabla(s) + df \wedge \nabla(s) + fK(s) = fK(s)$$

Next the value of the section $K(s)$ at x depends only on $s(x)$. This is showed by supposing $s' = s$, then, in terms of a local basis $s_1 \dots s_n$ for sections, we have

$$s - s' = f_1 s_1 + \dots f_n s_n$$

near x, where $f_1(x) = \dots = f_n(x) = 0$. Hence

$$K(s') - K(s) = \sum f_i K(s_i)$$

vanishes at x. Hence we can state that the correspondance

$$s(x) \to K(s)(x)$$

defines a smooth section of the complex vector bundle $Hom(E, \Lambda^2 TM^* \otimes E)$. The section $K = K(\nabla)$ is called the *curvature tensor* of the connection ∇. Once a basis $\{s_i\}$ of the sections of $E|_{U_\alpha}$ is chosen in a trivializing neighborhood U_α, the section K has the local form

$$K(s_i) = \hat{\nabla}(\sum \omega_{ij} \otimes s_j) = \sum(\Omega_{ij} \otimes s_j)$$

The Ω matrix results

$$\Omega = \Omega_{ij} = d\omega_{ij} + \sum(\omega_{i\alpha} \wedge \omega_{\alpha j}).$$

In compact form

$$\Omega = d\omega + \omega \wedge \omega. \tag{6}$$

Through the curvature $K(\nabla)$ one can give a criteria of flatness of the connection ∇.

Theorem. 53. *A connection ∇ is flat if and only if its curvature $K(\nabla)$ vanishes identically.*

Proof. When a connection is flat, due to example 49, all $\nabla(s_i)$ vanish identically for a choice of the coordinates, and so, all the matrices $\omega_{ij} = 0$. Therefore also K vanishes, as it can be expressed in terms of the ω's. We are not providing a complete proof for the converse thus it is lengthy and technical. The idea is quite simple: the converse is showed once a basis such that $\nabla(s_i) = 0, \forall i$ is found. When this condition is written explicitly, it is necessary to provide a condition of the existence of the solution. This is the content of the theorem of Frobenius (see [Jo08], section 3.1).

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Next we enrich the structure of the vector bundle E by endowing it with a metric. Recall that a metric over E is the given of a function

$$\langle , \rangle : E \otimes E \to \mathbb{R}.$$

that is smooth and when restricted to the fiber $E_x \otimes E_x$ it is bilinear and non-degenerate. Thus if s and s' are smooth sections of E, then the inner product $\langle s, s' \rangle$ is a smooth real valued function on M. A connection ∇ over E is said to be compatible with the metric if the identity

$$d\langle s, s' \rangle = \langle \nabla s, s' \rangle + \langle s, \nabla s' \rangle$$

holds for all sections s, s'. One can give the compatibility condition in coordinate and show the following.

Proposition. 54. *Let $s_1, \dots s_n$ be an orthonormal basis for the sections of $E|_{U_\alpha}$, with U_α trivializing neighborhood for E, so that $\langle s_i, s_j \rangle = \delta_{ij}$. Then a connection ∇ on $E|_{U_\alpha}$ is compatible with the metric if and only if the associated connection matrix ω is skew-symmetric.*

Proof. The connection ∇ is compatible if and only if it results for $i \neq j$

$$0 = d\langle s_i, s_j \rangle = \langle \nabla s_i, s_j \rangle + \langle s_i, \nabla s_j \rangle = \sum \langle \omega_{ik} \otimes s_k, s_j \rangle + \sum \langle s_i, \omega_{jk} \otimes s_k \rangle = \omega_{ij} + \omega_{ji}$$

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Now we specialize to the case of $E = TM$. A connection ∇ on TM is said symmetric if the composition

$$\Gamma(TM) \xrightarrow{\nabla} \Gamma(TM \otimes TM) \xrightarrow{\Lambda} \Gamma(\Lambda^2 TM)$$

is equal to the exterior derivative d. In terms of local coordinates, referring to equation 5, it results that the Christoffel symbols Γ_{ij}^k must be symmetric in i, j. More generally a connection on TM^* is symmetric if and only if the second covariant derivative

$$\nabla(df) \in \Gamma(TM^* \otimes TM^*)$$

of an arbitrary smooth function f is a symmetric tensor. Once a trivializing neighborhood U_α is chosen, one can write the condition in terms of a local basis $\theta_1, \ldots, \theta_n$

$$\nabla d(f) = \sum a_{ij} \theta_i \otimes \theta_j,$$

with the coefficients satisfying the property $a_{ij} = a_{ji}$.

Asking a connection ∇ on TM to be symmetric *and* compatible with a metric is very restricting due to the following.

Proposition. 55. *There is a unique connection ∇ on TM that is compatible with a given metric g and is symmetric. Such a connection is called the* Levi-Civita connection.

Proof. The metric makes TM isomorphic to TM^*. Hence we work in TM^* for convenience. Choose a basis $\theta_1, \ldots, \theta_n$ for the sections of $TM^*|_{U_\alpha}$. We will show that there is one and only one skew-symmetric matrix ω of 1-forms such that

$$d\theta_k = \sum \omega_{kj} \wedge \theta_j,$$

defining a connection ∇ over U_α by the requirement that

$$\nabla(\theta_k) = \sum \omega_{kj} \otimes \theta_j.$$

it evidently follows that ∇ is the unique symmetric connection for $TM^*|_{U_\alpha}$, which is compatible with the metric. Since these local connections are unique, they agree on intersections $U_\alpha \cap U_\beta$ and so piece together to yield the required global connection. The proposition will follow by the following remark. A $n \times n \times n$ tensor A_{ijk} can be written uniquely as the sum of a tensor B_{ijk}, symmetric in i, j and a tensor C_{ijk} skew-symmetric in j, k. Now choosing A_{ijk} so that

$$d\theta_k = \sum A_{ijk} \theta_i \wedge \theta_j,$$

and setting $A_{ijk} = B_{ijk} + C_{ijk}$ as above, it follows that

$$d\theta_k = \sum C_{ijk} \theta_i \wedge \theta_j.$$

Evidently the 1-forms

$$\omega_{ij} = \sum C_{ijk} \theta_i$$

constitute the unique skew-symmetric matrix with

$$d\theta_k = \sum \omega_{kj} \wedge \theta_j.$$

This concludes the proof

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B.2 The Hodge $*$

We concluded the previous paragraph by introducing a metric and calculating a special connection compatible with the metric. It is only natural to introduce now the Hodge star isomorphism, that is in strict relationship with a metric.

It is given a n dimensional vector space V, equipped with a metric g. Consider the k-th exterior power $\Lambda^k V$. Remark that

$$dim\Lambda^k V = dim\Lambda^{n-k} V = \binom{n}{k}$$

Once an orientation is chosen, we can define the canonical isomorphism $*$. Let e^i be an oriented orthonormal basis of $V^* = \Lambda^1 V$. The Hodge $*$ is the linear map

$$* : \Lambda^k V \to \Lambda^{n-k} V$$
$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \epsilon_{i_1 \ldots i_n} g^{i_1 i_1} \ldots g^{i_k i_k} e^{i_{k+1}} \wedge \cdots \wedge e^{i_n}. \tag{7}$$

This definition does not depend on the choice of the orthonormal basis. By the rules of multilinear algebra

$$* \left(Ae^1 \wedge \cdots \wedge Ae^{i_k} \right) = det(A) * \left(e^1 \wedge \cdots \wedge e^{i_k} \right),$$

Thus if A is a orthonormal basis change preserving the orientation, $det(A) = 1$ and the definition of Hodge $*$ results well posed. Let us remark that the $*$ applied to a form ω returns a form $*\omega$ such that $\omega \wedge (*\omega)$ is some multiply of the volume form. We conclude this brief section with a useful example.

Example. 56. Let V be the space of the p-forms of the cotangent space of a manifold M, $V = T_x^* M$. We can define through the Hodge star a pointwise scalar product on $\Lambda^p T_x^* M$. Consider

$$\langle \omega_1, \omega_2 \rangle = *(\omega_1 \wedge *\omega_2),$$

the only non-trivial verification is verifying that is symmetric, i.e.

$$*(\omega_2 \wedge *\omega_1) = *(\omega_1 \wedge *\omega_2).$$

It is sufficient to show the relation on a basis $e_1, \ldots, e_d$ for $T_x^* M$. It results

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \cdots \wedge e_{i_p})) = *(e_1 \wedge \cdots \wedge e_d) = 1,$$

Since it is symmetric on a basis, so it is on the forms in $\Lambda^p T_x^* M$.

B.3 The Yang-Mills theory

In the previous section we considered the fields to be the section of a vector bundle E that was trivial, i.e.

$$E = \mathbb{R}^4 \times V.$$

Thus a field was an element $\Psi \in \Gamma(E)$. The resultant lagrangians were polynomial in the fields and could be written in terms of the standard derivative of $\mathbb{R}^n$ and possessed certain symmetries that the physicists call *global*.

According to the physical experience, it is naive to consider a global field field transformation: in the Minkowski's space-time frame there are points causally disconnected, then there is no reason of transforming the whole $\mathbb{R}^4$ in the same way. In this section we will introduce the gauge local transformations and we will write a lagrangian invariant for these.

Let P be a principal bundle with fiber a Lie group G and consider a section $\gamma \in \Gamma(P)$. An action of the group G is given on the fiber of a vector bundle E. A gauge transformation for a section $\phi \in \Gamma(E)$ of a vector bundle E is

$$\phi'(x) = \gamma(x)\phi(x),$$

(the action of the representation is omitted). As already stated, we want to produce a lagrangian L that is invariant under a gauge transformation. While writing the part of the potential of a gauge invariant lagrangian brings no problem at all, the derivative part appears immediately much more troublesome. First we cannot use the same derivative operator for $\phi(x)$ and $\phi(y)$, since they belong to the fibers E_x and E_y respectevely, and x, y can be in two different trivializations of the bundle. This is fixed by utilizing a connection (in the sense of section B.1) instead of a derivative. Secondly the connection has to be made gauge invariant.

Consider a vector bundle E with a metric structure and define a connection ∇ compatible with the metric. Recall that, due to section B.1, a connection is given, once a matrix of 1-form $A \in TM^*$ is given. Furthermore the compatibility condition for ∇ implies that the matrix A is skew-symmetric, i.e. $A \in \mathfrak{o}(n)$. Now we will produce a lagrangian for the connection represented as the matrix A. In the algebra $\mathfrak{o}(n)$ we have the scalar product defined by the Killing form

$$A \centerdot B = -tr(AB)$$

that is positive definite. Let us introduce for convenience the space Ad $E \subset$ End(E) of the skew-symmetric endomorphisms of the bundle E. For example the matrix A of a connection compatible with the metric is an element of $\Omega^1(\text{Ad } E)$, while its curvature $K = K(\nabla)$ is an element of $\Omega^2(\text{Ad } E)$. Next recall that we have a scalar product on the p-forms defined through the Hodge star (see section B.2)

$$\langle s_1 \otimes \omega_1, s_2 \otimes \omega_2 \rangle = s_1 s_2 \langle \omega_1, \omega_2 \rangle.$$

For extending the inner product on $\Omega^p(\mathrm{Ad}\ E)$ we assume that the base space M is oriented, compact. Then the following is well defined

$$(s_1 \otimes \omega_1, s_2 \otimes \omega_2) = \int \langle s_1 \otimes \omega_1, s_2 \otimes \omega_2 \rangle dV,$$

where dV is a volume form and $\omega_i \in \Omega^p(\mathrm{Ad}\ E)$. Finally the *Yang-Mills functional* $YM(\nabla)$ for a connection ∇ compatible with the metric results by definition

$$YM(\nabla) = (K, K) = \int_M \langle K, K \rangle dV.$$

We want to show that this functional is invariant for a gauge transformation and, through the principle of minimum action, find the equations A must satisfy. The space of all metric connections of E is an affine space, thus the difference of two connections is an element of $\Omega^1(\mathrm{Ad}\ E)$. For determining the Euler-Lagrange equations for the Yang-Mills functional, we may thus use the variations of the form

$$\nabla + tB, \quad B \in \Omega^1(\mathrm{Ad}\ E).$$

Then taken a section $s \in \Gamma(E)$, the curvature tensor $K(\nabla + tB)$ results

$$\begin{aligned} K(\nabla + tB)s &= (\nabla + tB)(\nabla + tB)s && (8) \\ &= \nabla^2 s + t\nabla(Bs) + tB \wedge \nabla s + t^2(B \wedge B)s && (9) \\ &= (K + t(\nabla B) + t^2(B \wedge B))s, && (10) \end{aligned}$$

we used the relation $\nabla(Bs) = (\nabla B)s - B \wedge \nabla s$. Then the derivative of the YM functional is

$$\begin{aligned} \frac{d}{dt} YM(\nabla + tB)_{|t=0} &= \frac{d}{dt} \int \langle K(\nabla + tB), K(\nabla + tB) \rangle dV && (11) \\ &= 2 \int \langle \nabla B, K(\nabla) \rangle dV. && (12) \end{aligned}$$

By using the adjoint operator ∇^* we have

$$\frac{d}{dt} YM(\nabla + tB)_{|t=0} = 2(B, \nabla^* K(\nabla))$$

It follows that ∇ is a critical point of the Yang-Mills functional if and only if

$$\nabla^* K(\nabla) = 0.$$

Such a ∇ is called the *Yang-Mills connection*. In coordinates one has

$$K(\nabla) = K_{ij} dx^i dx^j,$$

with

$$K_{ij} = \frac{1}{2} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j] \right),$$

then

$$d^*(K_{ij}dx^i dx^j) = -\frac{\partial K_{ij}}{\partial x^i} dx^j,$$

Since $\nabla = d + A$, the following holds

$$\nabla^* K(\nabla) = \left(-\frac{\partial K_{ij}}{\partial x^i} - [A_i, K_{ij}]\right) dx^j.$$

Finally the equation in coordinates reads as

$$\frac{\partial K_{ij}}{\partial x^i} + [A_i, K_{ij}] = 0, \quad \text{for } j = 0, 1, \ldots, rank(E).$$

Next we show that the YM functional is gauge invariant. Since a Yang-Mills connection is an element of $\Omega^1(\text{Ad } E)$, in particular a section $\gamma \in \Gamma(P)$ will be acting on a connection as

$$\gamma \bullet \nabla = \gamma^{-1} \nabla \gamma \tag{13}$$

This is a natural definition since we are requesting a gauge covariance to ∇, i.e. $\gamma \nabla' = \nabla \gamma$. The condition 13 is written explicitly

$$\gamma \bullet \nabla = \gamma^{-1} d\gamma + \gamma^{-1} A \gamma.$$

Remark that we can always find a section γ over a trivializing open set U such that

$$\gamma \bullet (\nabla) = \gamma^{-1} d\gamma + \gamma^{-1} A \gamma = 0, \tag{14}$$

since $A \in \Omega^1(\text{Ad } E)$, the equation 14 has a unique solution once a value $\gamma(x_0)$ is chosen. We have showed the following

Lemma. 57. Let ∇ be a connection of the vector bundle E over M. For any $x_0 \in M$ there exists a gauge transformation γ defined on some neighborhood of x_0 such that the gauge transformed connection $\gamma \bullet (\nabla)$ satisfies

$$\gamma \bullet (\nabla) = d \text{ at } x_0.$$

Such a transformation can always be chosen to be compatible with any structure preserved by ∇, in particular the metric.

Finally, the section $\gamma \in \Gamma(P)$ transforms in a trivializing neighborhood $U_\alpha \cap U_\beta$ as

$$\gamma_\beta = g_{\beta\alpha} \gamma_\alpha g_{\beta\alpha}^{-1}, \tag{15}$$

where g is an element of the structure group of the bundle. The curvature $K(\nabla)$ transforms as

$$\gamma \bullet K = \gamma^{-1} K \gamma$$

An orthogonal self map of E is an isometry of $\langle .,. \rangle$ and hence we obtain the following

$$\langle \gamma \bullet K, \gamma \bullet K \rangle = \langle K, K \rangle,$$

due to equation 15, the relation above behaves well with a change of trivializing neighborhood. We have showed the following

Theorem. 58. *The Yang-Mills functional is invariant under a gauge transformation. Hence also the set of critical points of YM is invariant. Thus, if ∇ is a Yang-Mills connection, so it is $\gamma\nabla$ for $\gamma \in \Gamma(P)$.*

The functional YM results to be a lagrangian for the field A. Hence a typical lagrangian for a field $\phi \in \Gamma(E)$ will be written as

$$\mathcal{L} = \mathcal{L}_\nabla + \mathcal{L}_{\text{pot}} + YM(\nabla)$$

where $\mathcal{L}_\nabla$ is the part in the covariant derivate ∇, the part $\mathcal{L}_{\text{pot}}$ is some gauge invariant potential and $YM(\nabla)$ is the lagrangian for the field A.

B.4 The Maxwell electromagnetic theory

The Maxwell electromagnetic field is the simplest example of an application of Yang-Mills theory, since the gauge group we will be considering is abelian. Consider a linear vector bundle E over the Minkowski space $\mathbb{R}^4$. We want to define a connection $\nabla = d + A$ over E compatible with the Minkowski metric and that is invariant under a gauge transformation. First we want to see what kind of gauge transformations we have to consider. Since E is linear, one has that the 1-form A for a fixed $x \in \mathbb{R}^4$ is a scalar in some unidimensional algebra. Thus, a priori, we should consider a gauge transformation in a principal bundle P with fiber $GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. The group $\mathbb{C} \setminus \{0\}$ is reduced by endowing the sections $\Gamma(E)$ with a L^2-like inner product. For $f, g \in \Gamma(E)$ define

$$\langle f, g \rangle = \int f\bar{g}dV$$

where dV is a volume form for $\mathbb{R}^4$. The problem of integrating over an infinite volume is fixed by considering the sections $f \in \Gamma(E)$ with finite L^2 norm. For now we can leave behind this question, since we are focusing on a connection ∇ and its 1-form A. The group, conserving pointwise the L^2 inner product, is obviously $U(1)$. Then we shall consider the principal bundle P over $\mathbb{R}^4$ with fiber $U(1)$. The gauge transformations $\gamma(x) \in \Gamma(P)$ result

$$\gamma(x) = e^{i\alpha(x)}, \quad \alpha \text{ a real function.}$$

We will proceed calculating

- the field equations for A,
- the Yang-Mills functional $YM(\nabla)$

Remark that since $A \in \Omega^1(\text{Ad } E)$ in coordinates we have

$$A = A_0dx^0 + \cdots + A_3dx^3.$$

Since A_i is scalar, a commutator $[A_i, A_j]$ yields zero. The equations for the components of A result

$$\partial_i(\partial_i A_j + \partial_j A_i) = 0, \quad j = 0, 1, 2, 3.$$

As usual the repeated indexes are summed and $\partial_i = \frac{\partial}{\partial x^i}$. Once we set $\Box = \partial_i \partial_i$, we have

$$\Box A_j + \partial_i \partial_j A_i = 0$$

The derivative part $\partial_i \partial_j A_i = \partial_j \mathrm{div}(A_i)$ is usually forced to zero with a gauge transformations, i.e. one finds $\gamma \in \Gamma(P)$ such that $\mathrm{div}(\gamma A_i) = 0$. Remark that the order of derivation for the smooth function A_i can be reverted. This is called by the physicists the *Lorentz gauge choice*. Recall that such a gauge choice is allowed due to Lemma 57.

Next we proceed by calculating the Yang-Mills functional $YM(\nabla)$. Recall that the YM functional is written in terms of the curvature F. According to the equation 6, the curvature results

$$F = dA + A \wedge A$$

By an easy calculation $A \wedge A = 0$, thus $F = dA = F^{ij} dx^i \wedge dx^j$. The matrix F^{ij} results

$$F = F^{ij} = \partial^i A^j - \partial^j A^i$$

The matrix F^{ij} is called by the physicists the *electromagnetic tensor*. It is skew-symmetric and is usually written as

$$F^{ij} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

The electromagnetic tensor satisfies the obvious condition $dF = 0$, that is written in coordinates

$$\begin{aligned} &\mathrm{curl}\vec{E} + \partial_0 \vec{B} = 0, \ \mathrm{div}\vec{B} = 0 \\ &\mathrm{curl}\vec{B} - \partial_0 \vec{E} = 0, \ \mathrm{div}\vec{E} = 0 \end{aligned} \tag{16}$$

where $\vec{E} = (E_1, E_2, E_3)$, $\vec{B} = (B_1, B_2, B_3)$. The equations 16 are the famous *Maxwell's equations in absence of charge and current.* The Yang-Mills functional results now

$$YM(\nabla) = \int F(*F) dV.$$

The density can be explicitly calculated by remarking that the matrix $(*F)^{ij}$ is the transpose of F^{ij}. We have

$$F^{\mu\nu} F_{\mu\nu} = \vec{E}^2 - \vec{B}^2,$$

that is the classic electromagnetic lagrangian density.

C The Goldstone Theorem

The Goldstone theorem plays a crucial role in the Higgs-Braut-Englert mechanism. It states that a certain class of lagrangians necessarily carry a massless scalar particle: the Goldstone boson. For convenience we present an example.

C.1 The "mexican" hat example

Consider the lagrangian density for a field $\phi : M \to \mathbb{C}$ (M being the Minkowski space),

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger - \mu^2 \phi\phi^\dagger - \lambda(\phi\phi^\dagger)^2, \tag{17}$$

for which the Euler-Lagrangian equation results

$$(\Box + \mu^2)\phi + 2\lambda\phi(\phi\phi^\dagger) = 0.$$

Define the transformations

$$\phi'(x) = e^{i\alpha}\phi(x); \qquad (\phi^\dagger)' = e^{-i\alpha}\phi^\dagger(x), \tag{18}$$

where α is a real phase. The density is clearly invariant under the 18, hence, through the Noether theorem, one calculates the current J^μ

$$J^\mu = i\left((\partial^\mu\phi)\phi^\dagger - \phi(\partial^\mu\phi^\dagger)\right)$$

that is conserved since

$$\partial^\mu J_\mu = 0.$$

It is convenient to introduce the real fields ϕ_1, ϕ_2, such that

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}.$$

The lagrangian 17, in terms of the real fields, figures as

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi_1\partial^\mu\phi_1 + \partial^\mu\phi_2\partial^\mu\phi_2) - \frac{\mu^2}{2}(\phi_1^2\phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2.$$

The potential is expressed in function of two parameters: λ, μ^2. Since λ is multiplying the dominant term at infinite, for obtaining a potential with lower bound, we force $\lambda > 0$. Conversely, we have two possibilities for μ: $\mu^2 > 0$ and $\mu^2 < 0$.

1. $\mu^2 > 0$. The potential is a function with a unique minimum configuration in $(\phi_1, \phi_2) = (0, 0)$. Thus the expectation value in the vacuum state for ϕ results

   $$\langle 0|\phi(x)|0\rangle = 0$$

   The vacuum state is symmetrical under the transformations 18. Hence, even if we can't solve exactly the E-L equations, we can study the pertubartions near the vacuum state. In the language of the bundles this is said that between the sections of the linear bundle $M \times \mathbb{C}$ we choose the section that is identically zero, and work with the variations of the zero section in $\Gamma(M \times \mathbb{C})$.

2. $\mu^2 < 0$. The points minimizing the potentials become a circle centered in the origin. Thus each pair $(\phi_1, \phi_2) = (\eta_1, \eta_2)$, with $\sqrt{\eta_1^2 + \eta^2}$ equal to the radius of the circle of the minimum state, makes the potential minimum. Thus the state of minimum energy is not symmetrical respect to the 18. By forcing to zero the derivatives of the potential and asking that the second component is zero, we can explicitly calculate η

$$\eta = \sqrt{\frac{-\mu^2}{2\lambda}}$$

Remark that the condition $\mu^2 < 0$ makes the square root real. Furthermore the expectation value of ϕ in the vacuum state is not zero

$$\langle 0|\phi(0)|0\rangle = \eta$$

We have found that the group of symmetries of the Lagrangian $\mathcal{L}$ is greater than the group of symmetries of the solutions of the E-L equations. This situation is referred as a spontaneous symmetry breaking mechanism. In the language of the bundles, we are choosing in $M \times \mathbb{C}$ the section $(\eta, 0)$ and we will work with the variations in $\Gamma(M \times \mathbb{C}$. This is done by defining

$$\phi(x) = \eta + \frac{\sigma_1(x) + i\sigma_2(x)}{\sqrt{2}},$$

with σ_i a real variations in $\Gamma M \times \mathbb{R}$. Then we can write the lagrangian 17 in terms of the variations. Remark that the terms of order greater than 2 describe the interactions between the fields, while the terms of order till 2, describe a free theory. Thus for recognizing the masses and the spin of the particles of the variations σ_i we want to write the terms till the second order. It results

$$\mathcal{L} = (\partial^\mu \sigma_1 \partial_\mu \sigma_1 + \partial^\mu \sigma_2 \partial_\mu \sigma_2) + \frac{1}{2} \sum_{i,j} M^2_{i,j} \sigma_i \sigma_j$$

The matrix M^2_{ij} takes the following values

$$\begin{aligned} M^2_{11} &= 4\lambda\eta^2 = -2\mu^2 \\ M^2_{12} &= M^2_{21} = 0 \\ M^2_{22} &= \mu^2 + 2\lambda\eta^2 = 0 \end{aligned}$$

The lagrangian till the second order results in two independent lagrangians Klein-Gordon-like for the fields σ_i. The mass in the sense of Section 1 results zero for the field σ_2. The scalar particle represented by σ_2 is the *Goldstone boson.*

C.2 The proof

We present now a general proof for the Goldstone theorem.

Theorem. 59. *It is given a lagrangian $L = L(\phi_i, \nabla\phi, t)$ with lagrangian density $\mathcal{L} = \mathcal{L}(\phi_i, \nabla\phi, t)$, where phi_i is a field, $\phi_i : \mathbb{R}^4 \to \mathbb{C}^n$. The lagrangian density is such that*

1. *It possesses an exact and global symmetry. According to the Noether theorem we have a conserved current J*

$$\partial_\mu J^\mu(x) = 0$$

   hence

$$\frac{dQ}{dt} = 0, \quad Q(t) = \int dx^1 dx^2 dx^3 J^0(x,t)$$

2. *There are fields $\{\phi_j\}_{j\in J}$ that are not invariant under the symmetry transformations.*

3. *A field ϕ has a vacuum expectation value not zero*

$$\langle 0|\phi(0)|0\rangle = \sqrt{2}\eta \neq 0.$$

Then exists a massless scalar particle: the Goldstone boson. Furthermore we can calculate the transition from the vacuum state to the state of one Goldstone boson with momentum p as

$$\langle p|\phi(0)|0\rangle = \frac{\sqrt{Z}}{\sqrt{(2\pi)^3 2\omega(p)}}$$

$$\langle 0|J^\mu(0)|p\rangle = \frac{p^\mu F}{\sqrt{(2\pi)^3 2\omega(p)}}$$

with $Z, F \neq 0$, and $\omega(p) = \sqrt{\boldsymbol{p}^2 + m^2}$.

Proof. The idea of the proof is to calculate a propagator that depends from the mass m and show that the expectation value in the vacuum of the field ϕ forces the mass to be null. It is not restricting to work with a linear complex field $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$ with $\phi_i \in \mathbb{R}$, given with the transformations

$$\phi' = e^{-iQ\alpha}\phi_2 e^{-iQ\alpha} = \phi - i\alpha[Q,\phi] + \mathcal{O}(\alpha^2),$$

where Q is the conserved charge given by the Noether theorem. The variations for ϕ_2 results

$$i\frac{\delta\phi_2(x,t)}{\alpha} = [Q, \phi_2(x)] = \int dy^1 dy^2 dy^3 [J^0(y,t), \phi_2(x,t)] = i\phi_i(x,t).$$

The time t can be chosen arbitrarily, since Q is conserved. The propagator we want to calculate is the Fourier transformation $F^\mu(q)$ of $\langle 0|T[J^\mu(x)\phi_2(0)]|0\rangle$, where T is the product:

$$T[\phi(x)\psi(y)] = \begin{cases} \phi(x)\psi(x), \text{ if } x^0 > y^0 \\ \psi(x)\phi(x), \text{ if } x^0 < y^0 \end{cases}.$$

By definition of Fourier transformation it results

$$F^\mu(q) = \int dx^0 dx^1 dx^2 dx^3 e^{iqx} \langle 0|T[J^\mu(x)\phi_2(0)]|0\rangle. \tag{19}$$

For convenience we will be writing the volume form $dx^0dx^1dx^2dx^3 = d^4x$. When the integration is over the spatial components only we will be writing $dx^1dx^2dx^3 = d^3x$. For the calculation we exploit the Kallen-Lehman representation for T (see [Ma11]). First we separate the cases of one particle state and many particles states

$$F^\mu(q) = F^\mu(q)_1 + F^\mu(q)_{>1}.$$

Then it results for $F^\mu(q)_1$, $x_0 > 0$

$$\begin{aligned}\langle 0|T[J^\mu(x)\phi_2(0)]|0\rangle_1 &= \int d^3p e^{-ipx} \langle 0|J^\mu(x)|p\rangle\langle p|\phi_2(0)|0\rangle \\ &= F\sqrt{Z} \int \frac{d^3p}{(2\pi)^3 2\omega(p)} e^{-ipx} p^\mu\end{aligned}$$

For $x^0 < 0$ we obtain an analogous expression with opposite energy $\omega(p)$. Thus we have

$$\langle 0|T[J^\mu(x)\phi_2(0)]|0\rangle_1 = \tag{20}$$

$$= F\sqrt{Z} \int \frac{d^3p}{(2\pi)^3 2\omega(p)} e^{-i\mathbf{p}x} (e^{-i\omega p x_0} \ \theta(x^0) + e^{i\omega(p)x_0}\theta(-x^0))p^\mu \tag{21}$$

There is a more convenient representation for the 21. Let C be a closed path around $p^0 = +\omega(p)$ in the complex plane. We define the complex integral over the path C^+

$$i\Delta^{(+)}(x) = \frac{1}{(2\pi)^4} \int_C^+ d^4p \ e^{-ipx} \frac{i}{p^2 - m^2}.$$

With a calculation, that we do not report since it is very typical, it results

$$i\Delta^{(+)}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(p)} e^{-ipx}.$$

Analogously one can define $i\Delta^{(-)}$ over the path C^- around $p^0 = -\omega(p)$. Thus, once we set

$$iD(x,m) = \theta(x^0)i\Delta^{(+)}(x) - \theta(-x^0)i\Delta^{(-)}(x),$$

the equation 19 is written

$$F^\mu(q)_1 = F\sqrt{Z}q^\mu \hat{D}(q,m).$$

where $\hat{D}$ is the stands for the Fourier transformation of D. Remark that D is the Feynmann propagator in literature. Conversely for the many-particle state, it results

$$F^\mu(q)_{>1} = q^\mu \int_{M_0}^{+\infty} \rho(M^2) \frac{dM^2}{q^2 - M^2}$$

where ρ is some integrable scalar function defining the mass density.

Finally we can show the Goldstone theorem. We calculate $q^\mu F^\mu$

$$\begin{aligned}
q_\mu F^\mu(q) &= q_\mu \int d^4x e^{iqx} \langle 0|T[J^\mu(x)\phi_2(0)]|0\rangle \\
&= \int d^4x(-i\partial_\mu e^{iqx})\langle 0|T[J^\mu(x)\phi_2(0)]|0\rangle \\
&= i\int d^4x e^{iqx}\partial_\mu \langle 0|T[J^\mu(x)\phi_2(0)]|0\rangle \\
&= i\int d^4x e^{iqx} \langle 0|T[\partial_\mu J^\mu(x)\phi_2(0)]|0\rangle + \\
&\quad +i\int d^4x e^{iqx}\delta(x^0)\langle 0|[J^0(x,0)\phi_2(0)]|0\rangle \\
&= i\int d^x e^{-iqx}\langle 0|T[J^0(x,0)\phi_2(0)]|0\rangle
\end{aligned}$$

In the limit $q_\mu \to 0$ we have

$$\lim_{q\to 0} q_\mu F^\mu(q) = i\langle 0|[Q,\phi_2(0)]|0\rangle = -\langle 0|\phi_1(0)]|0\rangle = -\sqrt{2}\eta \neq 0 \tag{22}$$

Recall that $F^\mu(q) = F^\mu(q)_1 + F^\mu(q)_{>1}$. In the limit, the term $F^\mu(q)_{>1}$ is zero. In fact

$$q_\mu F^\mu(q)_{>1} = q^2 \int_{M_0}^{+\infty} \rho(M^2)\frac{dM^2}{q^2 - M^2}$$

For $q^2 \to 0$ only the singular part of the integral counts, and it exists if $M_0 = 0$. We separate the singularity

$$\int_0^{+\infty} \rho(M^2)\frac{dM^2}{q^2 - M^2} = \int_0^{\bar{M}} \rho(M^2)\frac{dM^2}{q^2 - M^2} + \int_{\bar{M}}^{+\infty} \rho(M^2)\frac{dM^2}{q^2 - M^2}$$

with $\bar{M} > 0$. The first term is explicitly calculated

$$\int_0^{\bar{M}} \rho(M^2)\frac{dM^2}{q^2 - M^2} = \rho(0)log\left(\frac{q^2}{q^2 - \bar{M}^2}\right).$$

Thus the singularity is at most logaritmic and

$$\lim_{q^2\to 0} q_\mu F^\mu(q)_{>1} = 0.$$

Now the equation 22 figures as

$$\lim_{q\to 0} q_\mu F^\mu(q) = \lim_{q\to 0} q_\mu F^\mu(q)_1 = \sqrt{Z}F \lim_{q\to 0} \hat{D}(q,m) = -2\sqrt{2}\eta.$$

When sitting in a neighborhood of $q^2 = 0$

$$\hat{D}(q,m) \equiv \frac{-\sqrt{2}\eta}{\sqrt{Z}F}\frac{1}{q^2}.$$

The relation holds if $m = 0$, then

$$\hat{D}(q, m) \equiv \frac{1}{q^2}$$

and

$$\sqrt{Z} F = -\sqrt{2} \eta$$

that implies $Z, F \neq 0$.

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