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Classical and non-classical tools for minimal surfaces*

* These notes contain some short parts that were not included in the lectures for lack of time.
Lecture 1

1.0 Outline of the lectures

- recalling some basic notions related to the Plateau problem & minimal surfaces, i.e., area, area formula, first variation of the area...
- recalling some basic facts from the theory of holomorphic functions and differential geometry of surfaces in the space which will be used extensively in the lectures by prof. Hildebrandt.
- giving a brief and self-contained introduction to the theory of finite perimeter sets as a tool to prove existence results for variational problems related to area-minimization
- outlining a few applications of the theory of finite perimeter sets (to give an idea of where this theory can be fruitfully applied...).

1.1 Hausdorff measure

**Purpose:** Intrinsic (and widely accepted) notion of length and area for subsets of higher-dimensional spaces (the point is to avoid parametrizations).

**Definition of Hausdorff measure**

Fix a real number \( d \geq 0 \) (the dimension). Given a set \( E \subset \mathbb{R}^n \) define, for every \( s > 0 \)

\[
\mathcal{H}^d_s(E) := \inf \left\{ \sum \text{diam}(E_i)^d \mid \{E_i\} \text{ countable cover of } E \text{ s.t. diam}(E_i) \leq s \right\}
\]

and

\[
\mathcal{H}^d_s(E) := \lim_{s \to 0} \mathcal{H}^d_s(E) = \sup_{s > 0} \mathcal{H}^d_s(E)
\]

**Remarks**

- \( \mathcal{H}^d \) is a \( s \)-additive measure on Borel sets
- \( \mathcal{H}^d \) is invariant under isometries (rigid motions) and scales as one would expect from a \( d \)-dimensional measure:
  \[ \mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E) \]
- \( \mathcal{H}^n = \mathcal{L}^n \) = Lebesgue measure on \( \mathbb{R}^n \). In fact
  \( \mathcal{H}^d \) = usual \( d \)-dimensional volume of subsets of \( d \)-dimensional plane (or smooth \( d \)-dim. submanifolds). For this we need the Riemann constant \( \mathcal{A}_d \) with \( \mathcal{A}_d = \text{vol. unit ball in } \mathbb{R}^d \).
1.3

- Why do we need to take the limit $S \rightarrow 0$ in the definition? Indeed this is not needed for $d = n$: $H^n = H^S \equiv \mathbb{R}$ for every $S \geq 0$. But it is needed for $d < n$ (for instance, if $E$ is a bounded curve of infinite length, you have $H^1(E) = \infty$ but $H^d_n(E) < \infty$ for every $S > 0$).

- Finally, $H^d$ can be defined of $E \subset \mathbb{R}^n$ with $X$ metric space, and it is intrinsic in the sense that it depends only on the restriction of the distance to the set $E$. In fact holds more: if $d$ and $d'$ are distances on $X$ such that $d'(x,y) = d(x,y) + O(d'(x,y))$ as $d(x,y) \rightarrow 0$, then $d$ and $d'$ induce the same Hausdorff measures; that is, $H^d$ depends only on the asymptotic behaviour of the distance on close points (only the metric matters).

Hausdorff dimension

Note that given $E$ and $d < d'$ then $H^d(E) = 0 \Rightarrow H^{d'}(E) = \infty$ ("a surface has infinite length") $H^d(E) < \infty \Rightarrow H^d(E) = 0$ ("a curve has zero area")

This motivates the following definition of Hausdorff dimension:

$$\dim_H(E) := \inf \{d | H^d(E) = 0 = \sup \{d | H^d(E) < \infty\}$$

1.2 Area formula

Aim: compute effectively the area (Haussdorff meas) using a parametrization of the set.

Given a map $X : D \rightarrow \mathbb{R}^n$, with $D$ open set in $\mathbb{R}^d$, which parametrizes the $d$-dimensional surface $S = X(D)$, then $H^d(S)$ is given by a suitable integral formula.

Standard assumptions:
- $X$ of class $C^1$, injective (more or less), maximal rank (in most points).

First case:
- $d = 2 \land m = 3$
- (surfaces in space)

Let $X = x(u) = X(u,v,z)$, $\partial u$, $\partial z$ partial der of $X$.

Then it is intuitively clear that

$$\text{Area}(S) = H^d(S) = \int_D |\partial u \wedge \partial z| \, du \, dz$$
In particular \( \partial_1 X \) and \( \partial_2 X \) span the tangent plane to \( S \) at \( X \) and therefore

\[
N = \frac{\partial_1 X \wedge \partial_2 X}{|\partial_1 X \wedge \partial_2 X|}
\]

is a normal (unit) vector to \( S \) at \( X \).

Notice that therefore a parametrization \( X \) defines implicitly an orientation of the parametrized surface \( S \) (either in terms of choice of a basis of \( \text{T}_x(S) \) or, equivalently, in terms of choice of a normal vector).

**Second case:**

c \( = \) n arbitrary.

Let \( S \) be a d-dimensional surface in \( \mathbb{R}^n \) parametrized by \( X: \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^n \). Then

\[
\text{Vol}_d(S) = \mathcal{H}^d(S) = \int_{\Omega} J \, du_1 \ldots du_d
\]

where \( J = J(u) \) is the d-dimensional volume of the parallelepiped spanned by the vectors \( \partial_1 X, \ldots, \partial_d X \) (the columns of the matrix \( VX \)).

How do we compute the Jacobian determinant \( J \)?

Taking any isometry \( R \) from \( \text{T}_x(S) = \text{Span}\{X_1, \ldots, X_d\} \) to \( \mathbb{R}^d \) we have

\[
J = \left| \det (R \cdot VX) \right|
\]

recall that \( \det(R) = \det A \)

\[
R = R^T \iff \sqrt{\det((R \cdot VX)^T (R \cdot VX))} = \sqrt{\det((VX)^T VX)}
\]

Binet's formula

\[
\Rightarrow \sqrt{\sum_{H = \text{deg}} \det H}^2
\]

Hence

\[
\text{Vol}_d(S) = \mathcal{H}^d(S) = \int_{\Omega} J \, du_1 \ldots du_d = \int_{D} \sqrt{\det(VX^T VX)}
\]

\[
= \int_{D} \sqrt{\sum_{H = \text{deg}} \det H}^2
\]

**Remarks**

- Binet's formula states that for a \( n \times d \) matrix \( A \) there holds \( \det(A^T A) = \sum_{H = \text{deg}} \det H^2 \).

For \( n = 3, d = 2 \) it has the following interpretation:

If \( R \) is a subset of a plane \( \mathbb{R}^2 \) and \( R_1, R_2, R_3 \) are the projection on the coordinate planes, then

\[
\text{Area}(R) = \sqrt{\sum (\text{Area}(R_i))^2}
\]
- Area of the graph of a function.
Let \( X(u) = (u, f(u)) \) be the standard parameterization of the graph \( \Gamma \) of the function \( f : \mathbb{R}^d \to \mathbb{R}^m \).

Then, if \( m = 1 \) we recover the usual formula
\[
\text{Vol}_d (\Gamma) = \int_D \sqrt{1 + |\nabla p|^2} \, d^d x.
\]

However this formula is no longer correct for \( m > 1 \), indeed in this case,
\[
\text{Vol}_d (\Gamma) = \int_D \sqrt{1 + \sum_{H \text{ square neighborhood of } \nabla f} (\det(H))^2} \, d^d x.
\]

- Another important variant of the area formula in case \( X \) is not injective involves the degree instead of the multiplicity:
\[
\int_{p \in S} \deg (X, p) \, d\mathbb{H}^d (p) = \int_{D \uparrow p} \pm J(w) \, d\mathbb{H}^d (w)
\]

number of \( u \in D \) s.t. \( X(u) = p \), counted \( \pm 1 \) if \( \nabla X(u) \) is orient. preserving, counted -1 if \( \nabla X(u) \) is orient. reversing.

the sign \( \pm \) is +1 if \( \nabla X(u) \) is orient. preserving, and -1 if it is orient. reversing.

This is known as "oriented" area formula.

- If \( X \) is not injective the area formula must be corrected to take into account
multiplicity
\[
\int_{p \in S} \#(X^{-1}(p)) \, d\mathbb{H}^d (p) = \int_D \sqrt{\det(\nabla X \cdot \nabla X)} \, d^d x.
\]

number of \( u \in D \) s.t. \( X(u) = p \), i.e., multiplicity of \( X \) at \( p \).
1.3 Plateau's Problem (simplest formulation)

Given a closed (regular) curve $\Gamma$ in $\mathbb{R}^3$, find the surface $S$ with boundary $\Gamma$ with minimal area.

(Here "find" means "prove the existence of ".)

Parametric approach

Among all $X: \mathcal{D} \to \mathbb{R}^3$ such that $X$ restricted to $\partial \mathcal{D}$ parametrizes $\Gamma$, find the one such that

$$F(X) = \text{Area}(X(\mathcal{D})) = \int_{\mathcal{D}} |\partial X| \, \text{d}x$$

is minimal.

(To be discussed: a) which domain $\mathcal{D}$ we consider? just a disc? b) what is the regularity of $X$? c) is $X$ injective?)

Naive attempt

One can imagine finding a minimizer of $F$ by a standard semicontinuity-and-compactness strategy, that is, by the following two steps:

1. Proving that $F$ is weakly lower semicontinuous on a suitable class $\mathcal{Y}$ of Sobolev maps $X: \mathcal{D} \to \mathbb{R}^3$

2. Proving that the set $\mathcal{Y}_m := \{X | F(X) \leq m\}$ is bounded (in the Sobolev norm under consideration) at least for some $m > \inf F$.

Indeed $\mathcal{Y}_m$ is weakly closed because $F$ is weakly lower semicontinuous, and therefore if it is bounded it is weakly compact (at least if $F$ is a reflexive space). Thus a standard argument shows that $F$ attains a minimum on $\mathcal{Y}_m$ and therefore on $\mathcal{Y}$.

It is worth to see what happens in the case we are interested in.

It turns out that semicontinuity is (essentially) OK, but the problem is compactness.
1.4 Semicontinuity of \( F(X) = \int_D |\partial X \Lambda \partial X| \) 

We recall two basic facts:

**Fact 1** If \( v_n \rightharpoonup v \) in \( L^p(D, \mathbb{R}^n) \) and \( f: \mathbb{R}^n \rightarrow [0, +\infty] \) is a convex lower s.c. function then

\[
\int_D f(v) \leq \liminf_{n \to \infty} \int_D f(v_n)
\]

This is essentially due to the fact that \( f \) is the upper envelope of affine functions, which allows to write the integral \( \int f(v) \) as a (localized) upper envelope of weakly continuous functionals.

**Fact 2** If \( u_n \rightharpoonup u \) in \( W^{1,p}(D, \mathbb{R}^n) \) with \( p \geq 2 \), then \( \nabla u_n \rightharpoonup \nabla u \) in \( L^p \) and \( \det \nabla u_n \rightharpoonup \det \nabla u \) in \( L^{p/2} \) (some extra care should be taken for the case \( p = 2 \)).

The key point for (\( \ast \)) is the identity

\[
\det \nabla u = \partial_1 \partial_3 u_2 - \partial_2 \partial_3 u_1 = \partial_1 (u_1 \partial_3 u_2) - \partial_2 (u_1 \partial_3 u_2)
\]

Indeed \( u^{(1)} \rightharpoonup u \) in \( W^{1,p} \) implies \( u^{(1)} \rightharpoonup u \) in \( L^p \) and \( \nabla u^{(1)} \rightharpoonup \nabla u \) in \( L^p \). Hence \( u^{(1)} \partial_3 u_2^{(1)} \rightharpoonup u_1 \partial_3 u_2 \) in \( L^{p/2} \) and \( \partial_1 (u^{(1)} \partial_3 u_2^{(1)}) \rightharpoonup \partial_1 (u_1 \partial_3 u_2) \) in the sense of distributions, hence \( \det \nabla u^{(1)} \rightharpoonup \det \nabla u \) in the sense of distributions, and the rest follows by the fact that \( \det \nabla u \) are uniformly bounded a.e. in \( L^{p/2} \).

Putting together these two facts we obtain that

\[
F(X) = \int_D |\partial X \Lambda \partial X| = \int \sqrt{\sum_{1 \leq i < j \leq 2} (\det \nabla(X(x), x))}^2
\]

is weakly lower s.c. in \( W^{1,p} \) for \( p \geq 2 \).

The key point is that \( |\partial X \Lambda \partial X| \) is a convex functions of the determinants of the \( 2 \times 2 \) minors of \( \nabla X \). At the same way one shows that

\[
F(u) = \int_D f(\nabla u)
\]

is weakly lower s.c. if \( f \) is any convex functions of \( \nabla u \) and its square minors (what is called a "polyconvex" function).
1.5 Lack of compactness (coercivity) 

for $F(x)$.

Unfortunately the class $S^m = \{x | F(x) \in M\}$ is not bounded in any reasonable Sobolev norm.

The reason is that $F$ is invariant under re-parametrization, that is,

$$F(x) = F(x \circ \phi)$$

for every diffeomorphism $\phi : D \rightarrow D$.

But taking suitable $\phi$ we can make any sobolev norm of $x \circ \phi$ as large as we want.

On the other hand one can use the invariance of $F$ to restrict the search of a minimizer to a much smaller and better behaved class!

This will indeed be the starting point of prof. Hildebrandt's lectures.

Note that indeed a similar approach is routinely used when looking for curves of minimal length on a surface $S$ (geodesics). Indeed using the fact that every curve $\gamma : [0,1] \rightarrow S$ admits a re-parametrization with constant speed one finds out that

$$\text{minimization of } \int_0^1 |\dot{\gamma}| \text{ over all } \gamma : [0,1] \rightarrow S$$

$$\text{II}$$

$$\text{minimization of } \int_0^1 |\gamma| \text{ over all } \gamma \text{ such that } |\dot{\gamma}| \text{ is constant}$$

$$\text{II}$$

$$\text{minimization of } \int_0^1 |\gamma|^2 \text{ over all } \gamma : [0,1] \rightarrow S.$$
1.6 Conformal parametrizations

Note that for every $3 \times 2$ matrix $M = (H_1, H_2)$ there holds

$$|M| = |M_1| |M_2| \sin \theta$$

$$\leq |M_1| |M_2|$$

$$\leq \frac{1}{2} (|M_1|^2 + |M_2|^2)$$

$$= \frac{1}{2} |M|^2 \quad \text{Euclidean norm of } M = \left(\frac{\partial}{\partial x} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \right)^T$$

Thus $|M_1 \wedge M_2| \leq \frac{1}{2} |M|^2$ and moreover holds if and only if $M$ satisfies

a) $M_1 \cdot M_2 = 0$ (i.e. $\theta = \frac{\pi}{2}$) & $|M_1| = |M_2|$  
\[\uparrow\]

b) $M$ is a composition of an isometry and an homothety  
\[\uparrow\]

c) $M$ preserves angles between vectors, that is $\frac{M_1 \cdot M_2}{|M_1| |M_2|} = \frac{V_1 \cdot V_2}{|V_1| |V_2|}$  
\[\uparrow\]

d) $M$ preserves orthogonality, that is $V_1 \cdot V_2 = 0 \Rightarrow M_1 \cdot M_2 = 0$. \(\star\)

[a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) immediate.  
d) $\Rightarrow$ a) : apply $\star$ with $V_1 = e_1, V_2 = e_2$ and then with $V_1 = e_1 + e_3, V_2 = e_1 - e_3$.  

The matrices $M$ satisfying a) or b) or c) or d) are called conformal.

A consequence of previous computations is that

for a generic parametrization $X:D \rightarrow \mathbb{R}^3$ there holds

$$\int_D \|
abla X \times \delta x\| \leq \frac{1}{2} \int_D \|
abla X\|^2$$

$$\text{area functional} \uparrow$$

$$\text{Dirichlet funct.} \uparrow$$

and holds if $X$ is conformal, that is, $\nabla X$ is a conformal matrix at (almost) every point of $D$.

Of course the existence of conformal parametrization for all surfaces cannot be taken for granted, but this will not be discussed in this lecture. I will instead spend some time to discuss conformal changes of variables in the plane.
1.7 Conformal maps in the plane

A map \( f : D \subset \mathbb{R}^2 \to \mathbb{R}^2 \) is conformal if \( \partial_x f \cdot \partial_y f = 0 \) and \( |\partial_x f| = |\partial_y f| \) at every point, that is

\[
\nabla f \in M \cup M'
\]

\[
\left\{ \begin{align*}
(\begin{array}{cc}
a & b \\
b & a
\end{array}) \mid a, b \in \mathbb{R} \end{align*} \right\} \cup \left\{ \begin{align*}
(\begin{array}{cc}
a & b \\
b & -a
\end{array}) \mid a, b \in \mathbb{R} \end{align*} \right\}
\]

Now, \( \nabla f \in M \) is equivalent to say that \( f \) satisfies the Cauchy-Riemann equations, that is, \( f \) is holomorphic (and the identification \( \alpha \cdot \beta = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \) gives the identification of the complex derivative \( f' \) with the Jacobian matrix \( \nabla f \).

On the other hand, \( \nabla f \in M' \) means that \( f \) is antiholomorphic.

Let us review a few useful facts.

**FACT 1**

A conformal change of variable preserves Dirichlet energy and conformality. If \( \phi : D \subset \mathbb{R}^2 \to D \subset \mathbb{R}^2 \) is conformal and bijective, for every \( u : D \to \mathbb{R} \) (or \( \mathbb{R}^m \)) there holds

\[
\int_D |\nabla u|^2 = \int_{D'} |\nabla (u \circ \phi)|^2.
\]

Moreover, for every \( X : D \to \mathbb{R}^3 \) which is conformal, \( X \circ \phi \) is conformal too.

**FACT 2**

If \( D \) is connected and \( f : D \to \mathbb{R}^2 \) is conformal, then \( f \) is either holomorphic or antiholomorphic.

For \( f \) of class \( C^2 \), define \( A = \{ z \in D : \nabla f(z) \in M \setminus \{0\} \} \), \( A' = \{ z \in D : \nabla f(z) \in M' \setminus \{0\} \} \) and \( A_0 = \{ z \in D : \nabla f(z) = 0 \} \). Set

\[
g(z) = \begin{cases} \frac{\partial f}{\partial \bar{z}}(z) & \text{if } z \in A \\ 0 & \text{if } z \in A_0 \\ \frac{\partial f}{\partial z}(z) & \text{if } z \in A'
\end{cases}
\]

Then \( g \) is holomorphic, and therefore is either constantly 0 or \( g(0) \) is a discrete set, that is \( A_0 \) is discrete, which implies that \( A \) or \( A' \) is empty (because they cannot be separated by a discrete set).

**FACT 3**

\( \text{holo} \circ \text{holo} = \text{holo} \) \((\equiv \, M \times M \subset M)\),
\( \text{holo} \circ \text{antiholo} = \text{antiholo} \) \((\equiv \, M \times M' \subset M')\),
\( \text{antiholo} \circ \text{holo} = \text{antiholo} \) \((\equiv \, M' \times M' \subset M')\),
\( \text{antiholo} \circ \text{antiholo} = \text{holo} \) \((\equiv \, M' \times M \subset M)\).

In particular, every antiholomorphic function can be written as \( \bar{f(z)} \) or \( f(z) \) with \( f \) holomorphic.
As a corollary of facts 2 and 3, conformal maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ reduce essentially to holomorphic maps.

**FACT 4**

If $f$ is holomorphic and injective then $f' \neq 0$ (i.e., $\nabla f \neq 0$) at every point. Hence $f'$ is holomorphic too.

Assume $f'(z_0) = 0$. Then $f$ can be written as $f(z) = f(z_0) + (z - z_0) g(z)$ for some $n \geq 2$, $g$ holomorphic with $g(z_0) \neq 0$ (just write the Taylor series of $f$ at $z_0$...). Since $g(z_0) \neq 0$, it can be locally written as $h^m(z)$ and therefore $f(z) = f(z_0) + ((z - z_0) h(z))^n$, which is clearly not injective in any neighbourhood of $z_0$....

**FACT 5**

If $D$ is the disc $(D = \{z : |z| < 1\})$ an holomorphic function $f : D \to \mathbb{C}$ is determined (up to a purely imaginary constant) by the restriction of its Real Part $\text{Re} f$ to the boundary of $D$.

Indeed $f(z) = \sum_{n \geq 0} a_n z^n \Rightarrow \text{Re}(e^{i\theta}) = \sum_{n \geq 0} \frac{1}{2} a_n e^{i\theta} + \frac{1}{2} a_n e^{-i\theta} \Rightarrow$ the Fourier coefficients of $\text{Re}(e^{i\theta})$ give $a_0$ and $a_n$ for every $n \geq 1$.

**QUESTION**

How many are the holomorphic bijections from a domain $D$ into itself? and from $D$ into another domain $D'$?

The answer is: very few! This is already hinted by fact 5, and indeed we have:

**FACT 6**

If $D$ is the disc, the holomorphic homeomorphisms of $D$ into itself are of the form

$$\frac{f(z)}{g(z)} = \frac{b z + a}{1 + a z} \quad (\star)$$

with $a, b \in \mathbb{C}$, $|b| = 1$ and $|a| < 1$.

We first check that the maps given by $(\star)$ are homeomorphisms of $D$ into itself (they are obviously holomorphic). Consider indeed all projective transformations of the projective line $P\mathbb{C} = \mathbb{C} \cup \{\infty\}$, namely the maps $g(z) = \frac{Az + B}{Cz + D}$ with $AB - CD \neq 0$.

Imposing that four points of the circle $S'$ (e.g., $\pm 1, \pm i$) are mapped into the circle it is enough to guarantee that all $S'$ is mapped
onto $S^1$ because projective transformations maps conics in conics, and one conic is determined by four points. These conditions give all the maps of the form (*) with $|b|=1$. The condition $|a|<1$ is obtained by requiring that $0$ is mapped inside $D$ and not outside.

To prove that there are no other holomorphic homeomorphisms of $D$ besides those in ($\ast$), it is enough to show that every such homeomorphism $f$ which satisfies the additional constraint $f(0)=0$ is a rotation, that is $f(z)=\lambda z$ for some $\lambda$ with $|\lambda|=1$ (if $f$ does not satisfy $f(0)=0$, by composing with a suitable map $w$ ($\ast$) we get $\tilde{f}$ s.t. $\tilde{f}(0)=0$; if $\tilde{f}$ is a rotation then $f$ is of the form ($\ast$)). Indeed we have $|f(z)|=1$ for every $z \in \partial D=S^1$ (because $f$ maps $\partial D$ to $\partial D$) and since $f(z)$ is holomorphic also $w_0$, the maximum principle yields $|f(z)|=1$ in $D$ and therefore $f(z)=\text{constant of modulus 1}$ in $D$ (an holomorphic function with constant modulus is constant).

Remark. By fact 6, an holomorphic homeomorphism $f$ of the disc $D$ is uniquely determined by assigning the values $f(z_1), f(z_2), f(z_3) \in S^1$ for given $z_1, z_2, z_3 \in S^1$.

FACT 7

Given the rigidity of holomorphic automorphisms of the disc (fact 6) it is rather surprising that the following holds:

**Riemann Mapping Theorem**

If $A$ is a simply connected open set in $\mathbb{C}$ with $A \neq \mathbb{C}$, then there exists an holomorphic homeomorphism $f : D \to A$, where $D$ is the disc. Moreover if the boundary of $A$ is sufficiently regular, then $f$ extends to an homeomorphism of $D \cup \overline{A}$.

FACT 8

From the Riemann mapping theorem one might guess that if two open set $A_1$ and $A_2$ are homeomorphic then they are also isomorphic (holomorphically homeomorphic). It is not so: let $D_{R,R'}$ denote the annulus $\{z \in \mathbb{C} | R_1 < |z| < R_2 \}$; then $D_{R,R'}$ is isomorphic to $D_{R',R}$, if and only if $R/R' = R'/R$.

Assume for simplicity $R=R'=1$, and let $f : D_1 \to D_2$ be an isomorphism. Up to an inversion we can also assume that $f$ maps $S^1$ to $S^1$ and $rS^1$ to $rS^1$. Then $f$ can be extended by reflection in an isomorphism $\tilde{f} : D_{1,1} \to D_{1,1}$.
Take indeed
\[ \tilde{f}(z) = \begin{cases} f(z) & \text{if } r \leq |z| \leq s, \\ \frac{|z^2|}{|f(r/2)|} & \text{if } |z| < r. \end{cases} \]

By iterating this reflection procedure, one gets an \textit{holomorphic homeomorphism} \( \tilde{f} : D \rightarrow D \) such that
\[ |f(z)| = r^n = |z|^n \quad \text{for every } z \text{ s.t. } |z| = r. \]

But then \( \frac{|z^n|}{|z|r^n} \) must be equal to 1, that is \( r = r' \).

\textbf{Final Remark}

We have seen that holomorphic homeomorphisms in the plane are quite rigid, and of course the same holds for conformal homeomorphisms (cfr. fact 2 above).

In higher dimension the situation is even worse: let \( \tilde{f} : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a conformal map, that is,
\[ \tilde{f} \cdot \tilde{f}' = 0 \quad \forall i,j; \quad |\tilde{f}| = |\tilde{f}'| \quad \forall i,j. \]

Then \( \tilde{f} \) is (locally) a composition of similarities, isometries (rotations and reflections) and inversions \( (x \mapsto \frac{x}{|x|^p}) \).

This class is unsuitable for any variational application....