Lecture 2

In this lecture I will review some basic notions of differential geometry for surfaces in the space, and more generally for hyper-surfaces in $\mathbb{R}^{n+1}$.

2.1 First fundamental form

For every point $p$ in the surface $S$, we denote by $\text{Tau}(S,p)$ the tangent plane of $S$ at $p$. The first fundamental form $(I_p(S,p))$ is the quadratic form on $\text{Tau}(S,p)$ associated with the scalar product on $\text{Tau}(S,p)$.

In our setting, the scalar product is the one induced by the immersion of the Euclidean space and therefore

$$I_p(v) = |v|^2 \quad \forall v \in \text{Tau}(S,p)$$

Recall that from this quadratic form you can recover the scalar product:

$$\langle v, w \rangle_p = \frac{1}{2} \left[ I_p(v + w) - I_p(v) - I_p(w) \right]$$

Representation of $I_p$ using coordinates

Let $n = 2$ and $X : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a param. of the surface $S$. Since $\partial_1 X, \partial_2 X$ are a basis of $\text{Tau}(S,p)$ we can write every vector $v \in \text{Tau}(S,p)$ as

$$v = v_1 \partial_1 X + v_2 \partial_2 X$$

Hence

$$I_p(v) = E v_1^2 + 2F v_1 v_2 + G v_2^2$$

where

$$E = \partial_1 X \cdot \partial_1 X, \quad F = \partial_1 X \cdot \partial_2 X, \quad G = \partial_2 X \cdot \partial_2 X$$

in other words

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is the matrix associated to the quadratic form $I_p$ by choosing $\partial_1 X, \partial_2 X$ as a basis of $\text{Tau}(S,p)$.

The use of the letters $E, F, G$ for the coefficients of this matrix is classical and goes back to Gauss. In modern notation one denotes these coefficients by $g_{ij}$.

Incidentally

$$|\partial_1 X \wedge \partial_2 X| = \sqrt{EG-F^2}.$$

Indeed

$$|\partial_1 X \wedge \partial_2 X| = |\partial_1 X| |\partial_2 X| \sin \theta$$

$$= \sqrt{ |\partial_1 X|^2 |\partial_2 X|^2 (1-\cos \theta) }$$

$$= \sqrt{ |\partial_1 X|^2 |\partial_2 X|^2 (\partial_1 X \cdot \partial_2 X)^2 }$$

$$= \sqrt{EG-F^2}.$$

Remark

The second fundamental form is intrinsic (does not depend on the ambient space).
2.2 Second Fundamental Form

Let $S$ be an oriented surface, that is, for every point $p$ it is given a unit normal vector $N$ (depending continuously on $p$...)

Fix $p \in S$ and decompose $\mathbb{R}^{n+1}$ as $\text{Tau}(S,p) \oplus \text{Nov}(S,p) \oplus \text{Tau}(S,p) \oplus \mathbb{R}$ and write a point $p \in \mathbb{R}^{n+1}$ as $(x,y)$ accordingly.

Close to $p$, $S$ is described as the graph of some function $f : \text{Tau}(S,p) \to \mathbb{R}$ that is by the equation

$$N = f(x) = \langle Au, v \rangle + O(|v|^2)$$

The second fundamental form of $S$ at $p$ is precisely

$$\Pi_p(v) := \langle A_p v, v \rangle$$

(The map $p \mapsto A_p$ is known as the Weingarten map.)

2.3 Differential of the Gauss map

Let $S$ be an oriented surface, and for every $p \in S$ let $N = N(p)$ the orientation normal (unit) vector. The Gauss map of $S$ is precisely

$$N : S \to S^n \subset \mathbb{R}^{n+1}$$

Hence the differential of $N$ at $p \in S$ is a linear map

$$dN(p) : \text{Tau}(S,p) \to \text{Tau}(S^n, N(p)) \subset \mathbb{R}^{n+1}$$

Note that the tangent plane of $S^n$ at $N$ is exactly $N^\perp = \text{Tau}(S,p)$

Recall that the differential $dN(p)$ is related to the first order Taylor expansion of $N$ at $p$ : if we write a point $p'$ "close" to $p$ as $p + v + o(v)$ with $v \in \text{Tau}(S,p)$, then

$$N(p') = N(p + v + o(v)) = N(p) + dN(p) v + o(v)$$
Moreover, given any curve $\gamma$ on $S$ starting from $p$ we have
\[
\frac{d}{dt} N(\gamma)_{t=0} = dN(p) \cdot \dot{\gamma}(0)
\]
(Of course similar identities hold for the differential of any map, there is nothing specific of the Gauss map here.)

**Fundamental identity**

The differential of the Gauss map is related to the second fundamental form by the following identity: for $v \in T_{p}(S,p)$
\[
\Pi_{p}(v) = \langle -dN(p) \circ v, v \rangle
\]

**Proof**

As before we identify $\mathbb{R}^{m \times 1}$ with $T_{p}(S,p) \oplus \mathbb{R}$

\[
N(p') = N(t + v + o(v)) = \frac{(-NpV + \alpha(v), 1)}{\sqrt{1 + |\alpha(v)|^2}}
\]

\[
= \frac{(-Apv + \alpha(v), 1)}{\sqrt{1 + |Apv + \alpha(v)|^2}} = \left[\frac{-Apv}{1 + |Apv + \alpha(v)|^2}\right] + o(v)
\]

\[
\Pi_{p}(v) = \alpha(0) + o(v)
\]

\[
N(p) \quad dN(p)v
\]

Hence $-dN(p) = Ap$, and in particular
\[
\Pi_{p}(v) = \langle Apv; v \rangle = \langle -dN(p)v; v \rangle
\]

**2.4 Curvatures**

The second fundamental form $\Pi_{p}$ is a quadratic form on $T_{p}(S,p)$ associated to the self-adjoint linear map $Ap : T_{p}(S,p) \rightarrow T_{p}(S,p)$

\[
-\Pi_{p}(v) = \langle Av; v \rangle
\]

Being self-adjoint, $Ap$ admits $n$ real eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ with corresponding eigenvectors $e_{1}, \ldots, e_{n}$.

(Note that $\lambda_{i}$ and $e_{i}$ do not depend on the choice of a basis on $T_{p}(S,p)$.)

$\lambda_{1}, \ldots, \lambda_{n}$ principal curvatures (of $S$ at $p$)

$e_{1}, \ldots, e_{n}$ principal directions

In fact one is often more interested in the coefficients of the characteristic polynomial of $Ap$, that is, the elementary symmetric functions of the eigenvalues $\lambda_{i}$:

\[
P_{A}(\lambda) = \det(\lambda I - A)
\]

\[
= \Pi(\lambda - \lambda_{i})
\]

\[
= \lambda^{m} - [(\lambda_{1} + \cdots + \lambda_{n}) \lambda^{m-1} + (\lambda_{1} \lambda_{2} + \cdots + \lambda_{n} \lambda_{n}) \lambda^{m-2} + \cdots + (-1)^{m} \lambda_{1} \lambda_{2} \cdots \lambda_{n}]
\]
In particular
\[ \lambda_1 + \ldots + \lambda_n = \text{trace of } A \]
is called mean curvature (of \( S \) at \( p \)) and usually denoted by \( H \).

**Remarks**
- The mean curvature is sometimes defined as \( \frac{1}{n} (\lambda_1 + \ldots + \lambda_n) \).
- It is sometimes convenient to define the mean curvature vector \( \vec{H} = H \cdot N \).

While \( H \) depends on the choice of the orientation, \( \vec{H} \) does not.

### 2.5 Mean curvature and Gauss curvature

Let \( r \) be a 2-dim. surface \((n=2)\).

Given \( p \in S \), let \( \lambda_1, \lambda_2 \) be the principal curvatures of \( S \) at \( p \). Then we set

\[
\begin{align*}
    H & = \text{mean curvature} \\
    & = \lambda_1 + \lambda_2 = \text{trace of } A_p = -\text{trace } dN(p) \\
    K & = \text{Gauss curvature} \\
    & = \lambda_1 \lambda_2 = \det A_p = \det (dN(p))
\end{align*}
\]

And using coordinates...

If \( X: D \to \mathbb{R}^2 \) is a parametrization of \( S \), we can write every \( v \in \text{Tan}(S, p) \) as

\[ V = V_1 \partial_1 X + V_2 \partial_2 X. \]

Then the second fund. form is given by

\[ II_p(V) = e V_1^2 + 2f V_1 V_2 + g V_2^2. \]

where
\[
\begin{align*}
    e & = II_p(\partial_1 X) = \langle -dN \partial_1 X, \partial_1 X \rangle \\
    & = \langle N, \partial_1^2 X \rangle \leftarrow \text{deviate the identity} \\
    f & = -dN \partial_1 X, \partial_2 X = \langle N, \partial_2 \partial_1 X \rangle \leftarrow \langle N, \partial_1 \partial_2 X \rangle = 0 \\
    g & = II_p(\partial_2 X) = -dN \partial_2 X, \partial_2 X = \langle N, \partial_2^2 X \rangle \leftarrow \langle N, \partial_2 \partial_2 X \rangle = 0
\end{align*}
\]

In other words \( (e, f, g) \) is the matrix assoc. to the quadratic form \( II_p \) by choosing \( \partial_1 X, \partial_2 X \) as a basis of \( \text{Tan}(S, p) \), that is

\[
(e, f, g) = dX^* A \cdot dX
\]

\[ \text{Hence, } \]

\[
e g - f^2 = \det(e, f, g)
= (\det dX)^2 \det A
= (E G - F^2) \cdot K
\]

that is

\[
K = -\frac{eg-f^2}{EG-F^2}
\]

Note that \( e, f, g, E, F, G \) can be computed using \( X \) and its first and second order derivatives.
If in addition $X$ is conformal...

that is, $E = G = |\partial x|^2 = |\partial_x X| = \frac{1}{2} |\partial X|^2$ and $F = \partial_x \partial_x X = 0$, then we have a simple formula also for the mean curvature $H$.

Indeed in this case $dX$ is of the form $\mathbf{c} \cdot \mathbf{R}$ with $\mathbf{R} : \mathbb{R} \rightarrow \text{Tau}(S, p)$ an isometry and $\mathbf{c} = \frac{1}{2} |\nabla X|$. Hence

$$e + g = \text{tr} \left( \begin{pmatrix} \mathbf{c} \cdot \mathbf{R}^* \mathbf{c} \cdot \mathbf{R} \\ 0 \end{pmatrix} \right) = \text{tr} \left( dX^* A dX \right) = \text{tr} \left( \mathbf{c}^2 \cdot \mathbf{R}^* \mathbf{R} \right) = \mathbf{c}^2 \cdot \text{tr}(A) = \mathbf{c}^2 \cdot H$$

that is

$$H = \frac{e + g}{\mathbf{c}^2} = \frac{\left< \nabla X, \partial_x X + \partial_x^2 X \right> - \left< \nabla X, \Delta X \right>}{\frac{1}{4} |\nabla X|^2 - \frac{1}{4} |\partial X|^2}$$

Next we use curvatures to write two useful formulas: one for the volume of the tubular neighborhood of a surface, and one for the first variation of the area of a surface.

2.6 Volume of the tubular neighbourhood

Let $\Omega$ be a bounded open set in $\mathbb{R}^3$ with a regular boundary $S = \partial \Omega$ oriented by the outer normal $\mathbf{N}$ (at least of class $C^2$).

For every $r > 0$ let $\Omega_r$ be the $r$-neighbourhood of $\Omega$

$$\Omega_r := \{ x \in \Omega : \text{dist}(x, \partial \Omega) < r \}$$

We want to compute the volume of $\Omega_r$.

If $r$ is sufficiently small, then $\Omega_r \setminus \Omega$ is parametrized by

$$\Psi : \Omega \times [0, r] \rightarrow \mathbb{R}^3$$

$$\Psi : (p, t) \rightarrow p + t \mathbf{N}(p)$$

To compute $\text{vol}(\Omega_r)$ we need to compute the differential of $\Psi$ and its determinant.

$$d\Psi : (p, t) : \text{Tau}(S, p) \times \mathbb{R} \rightarrow \mathbb{R}^3$$

$$d\Psi = dp + t d\mathbf{N} + N dt$$

Now we identify $\mathbb{R}^3$ with $\text{Tau}(S, p) \times \text{Nov}(S, p)$, choose $e_1, e_2$ orthonormal basis of $\text{Tau}(S, p)$
and use \(e_1, e_2, N\) as an orthonormal basis of \(\mathbb{R}^3\). With respect to this basis, the matrix associated to \(dp\) is

\[
\mathbf{M} = \begin{pmatrix}
1 - tA & 0 \\
0 & 1
\end{pmatrix}^{2+1}
\]

where \(A\) is the matrix associated to \(-dN\) and therefore represents the second fundamental form of \(S\) at \(p\).

Hence

\[
\det \mathbf{M} = \det \begin{pmatrix}
1 - tA & 0 \\
0 & 1
\end{pmatrix}
\]

\[
= \det (1 - tA)
\]

\[
= 1 - \text{tr}(A) \cdot t + \det(A) \cdot t^2
\]

\[
= 1 - H \cdot t + K \cdot t^2
\]

and then

\[
\text{Vol}(S_r) = \text{Vol}(S_0) + \int_0^r \int_{S_0} \det \mathbf{M} \, dt \, dp
\]

\[
= \text{Vol}(S_0) + \int_0^r \int_{S_0} (1 - \text{tr}(A)) \cdot t + K \cdot t^2 \, dt \, dp
\]

\[
= \text{Vol}(S_0) + \int_0^r \int_{S_0} r - \frac{1}{2} H(p) r^2 + K(p) r^3 \, dp
\]

and finally

\[
\text{Vol}(S_r) = \text{Vol}(S_0) + \text{Area}(S) \cdot r
\]

\[
- \frac{1}{2} \int_S H \cdot r^2 + \frac{1}{3} \int_S K \cdot r^3
\]

Thus the volume of \(S_r\) is a polynomial of degree three in \(r\), at least for \(r\) sufficiently small, that is, within the validity of the tubular neighborhood theorem.

(For every \(x \in S_r\) there exists a unique point \(\pi(x)\) on \(S = S_r\) which minimizes the distance from \(x\)).

In particular is \(S_2\) is convex the previous formula holds for all \(r > 0\).

Then by approximation one can show that for every convex body \(S_2\), not necessarily with a regular boundary, \(\text{Vol}(S_r)\) is a polynomial with degree \(r\).

This way one can give a meaning to \(\int_S H\) and \(\int_S K\) for every convex surface \(S\).

In fact, there is a local version of the previous formula:

\[
\text{Vol}(R_r) = \text{Area}(R) \cdot r
\]

\[
- \frac{1}{2} \int_R H \cdot r^2 + \frac{1}{3} \int_R K \cdot r^3
\]
Using this formula one can give a meaning to $S_R^H$ and $S_R^K$ for every region $R$ of every convex surface $S$.

This remark is the starting point for the definition of curvature measures of convex surfaces (Alexandrov, Federer).

2.7 First variation of the area

Now we want to compute the first variation of the area. This means that given an hypersurface $S$ in $\mathbb{R}^{n+1}$ we want to find a formula for

$$\frac{d}{dh} \text{Vol}_n(S_{h})\bigg|_{h=0}$$

where $S_h$ is a one-parameter family of surfaces such that $S_0 = S$, or, if you like, a curve in the space of surfaces passing through $S$ at time $h=0$ (thus the previous derivative can be seen as the partial derivative of the function $\text{Vol}_n$ at $S$ in the direction defined by the curve $h \to S_h$).

To this end, we must first a) define the class of "admissible variations" that is, which maps $h \to S_h$ to consider, and then b) compute explicitly $\text{Vol}_n(S_{h})$.

a) We choose a vector field $\eta$ normal to $S$ (not necessarily with norm $=1$). $\eta$ will be as regular as needed in the following computation.

Then if $N$ is the orienting (unitary) normal field, we can write $\eta$ as $\eta = \varphi \cdot N$

where $\varphi$ is a given real-valued function on $S$.

For every $h \in R$ we set

$$S_h := \{ p + h \eta(p) \mid p \in S \}$$

where $S_h$ is a regular surface only for $h$ sufficiently small....
Thus $S_p$ can be parametrized by
$$\psi_p: S \rightarrow S_p$$
$$p \mapsto p + \eta(p)$$
Let's compute the determinant of $d\psi_p$:
$$d\psi_p(p) = dp + \eta \cdot dp$$
recall that $\eta = \varphi N$,
$$\eta = \varphi dN + \nu N \nu$$
tangential component normal component

Thus $d\psi_p(p)$ is a linear map from $T_p(S, p)$ to $\mathbb{R}^{n+1}$, choosing any orthonormal basis $e_1, \ldots, e_n$ for $T_p(S, p)$ and using $e_1, \ldots, e_n, N$ as an orthonormal basis of $\mathbb{R}^{n+1}$, we represent $d\psi_p(p)$ by the $(n+1) \times n$ matrix
$$M = \left[ \begin{array}{cccc}
Id & \varphi A
\end{array} \right]$$
where $A = -dN$ represent the second fundamental form of $S$ at $p$.

Hence
$$H^k M = \left( I - \varphi \nu A \right) \nu \nu \varphi A \left( I - \varphi \nu A \right)$$
$$= I - \varphi \nu (A^k + A) + O(\varepsilon^2)$$
then
$$\sqrt{\det(H^k M)} = \sqrt{1 - 2 \varphi \nu tr(A) \varepsilon + O(\varepsilon^2)}$$
we use that
$$\det(1 + B) = 1 + \varphi \nu tr(A) \varepsilon + O(\varepsilon^2)$$
$$\sqrt{1 + hA} = 1 + \frac{1}{2} h + O(h^2)$$
mean curvature

then
$$\nu \mathbb{R}_n(S_p) = \int_S 1 - \varphi \nu H \varepsilon + O(\varepsilon^2)$$
$$= \nu \mathbb{R}_n(S) - h \int_S \varphi H + O(\varepsilon^2)$$
and finally
$$\frac{d}{dt} \nu \mathbb{R}_n(S_0)|_{t=0} = -\int_S \varphi H .$$
Conclusions

If $S$ minimize the area (the minimal volume) among all surfaces with prescribed boundary $\Gamma$ then at every point of $S$ there holds

$$\nabla \eta = 0$$

Given indeed any $\varphi: S \to \mathbb{R}$ such that $\varphi = 0$ on $\partial S$, we have that $\eta = 0$ on $\partial S$ and therefore $\partial S_{\eta} = \partial S$ (for $h$ sufficiently small). That is, $E_\eta$ is an "admissible variation" for the problem at hand.

Hence the minimality of $S$ implies

$$0 = \frac{\partial}{\partial h} \text{Vol}_h(S_\eta)\big|_{h=0} = \int_S H \varphi$$

and since this identity holds for every choice of $\varphi$ with $\varphi = 0$ on $\partial S$ we deduce

$$H = 0.$$ 

Remark

Why did we consider only vector fields $\eta$ orthogonal to $S$? Because considering non-orthogonal ones would not really give a larger class of variations $S_\eta$ (at least if we keep the boundary fixed).

Indeed if $S_\eta$ is the family generated by a vector field $\eta$, then "essentially the same" family can be obtained by replacing $\eta$ by its normal component $\eta_0$.

Essentially the same means that for the family $S_\eta$ generated by $\eta$ one has

$$\text{Vol}_h(S_\eta) = \text{Vol}_h(S_\eta) + O(h^2)$$

and then

$$\frac{\partial}{\partial h} \text{Vol}_h(S_\eta)\big|_{h=0} = \frac{\partial}{\partial h} \text{Vol}_h(S_\eta)\big|_{h=0}.$$

2.8 Variation of the area with prescribed volume

Consider now the following variation of the Plateau's problem:

among all set with prescribed volume find the one which minimizes the area of the boundary (isoperimetric problem).

If $A$ is a solution of this problem and
S is the boundary of A, then at every point of S there holds

\[ H = \text{constant} \]

Consider as before the family of boundaries \( S_{\lambda} \) associated to a normal vector field \( \eta = \phi \cdot \mathbf{N} \), and denote by \( A_{\lambda} \) the corresponding interiors.

Since \( \frac{d}{d\lambda} \text{Vol}(A_{\lambda}) \big|_{\lambda=0} = \int_S \phi \)

(we omit this computation, which is quite similar to one of the previous ones), then the admissible variations must satisfy

\[ \text{Vol}(A_{\lambda}) = \text{constant}, \quad \text{that is} \]

\[ \int_S \phi = 0 \]

And conversely for any \( \phi \) s.t. (1) holds, even if \( \text{Vol}(A_{\lambda}) \) is not constant, we can modify \( A_{\lambda} \) slightly so that the volume is constant.

Hence the minimality of A implies that

\[ \frac{d}{d\lambda} \text{Area}(S_{\lambda}) \big|_{\lambda=0} = \int_S H \phi = 0 \]

for all \( \phi \) s.t. (1) holds, that is, \( H \) is constant.

We can obtain the equation \( H = \text{constant} \) also using Lagrange multipliers: indeed, minimizing \( \text{Area}(S) \) under the constraint \( \text{Vol}(A) = \text{const.} \) is equivalent to minimizing \( \text{Area}(S) - \lambda \text{Vol}(A) \) (well, it is a matter of critical point, not minimizing...) and by the previous computations

\[ \frac{d}{d\lambda} \left[ \text{Area}(S_{\lambda}) - \lambda \text{Vol}(A_{\lambda}) \right] = \int_S (H - \lambda) \phi \]

and imposing this to be zero for all admissible variations implies \( H - \lambda = 0 \) for some \( \lambda \), that is, \( H = \text{constant} \).

2.3 Normal and geodesic curvature of a curve

Let \( \gamma \) be a curve in \( \mathbb{R}^3 \) parametrized by arc-length, unitary.

The the orienting tangent vector is

\[ \mathbf{t} = \dot{\gamma} \]

and the total curvature

\[ k = \mathbf{t} \cdot \mathbf{n} \]

is orthogonal to \( \mathbf{t} \), denote the identity \( \langle \mathbf{t}, \mathbf{n} \rangle = 1 \).

Now, if \( \gamma \) lies on the surface \( S \), \( \mathbf{N}, \mathbf{t}, \mathbf{n}, \mathbf{N} \times \mathbf{t} \) form an orthonormal system (at every point \( p \) in \( \gamma \)).
In particular we can write $\hat{\mathbf{c}}$ as a combination of $N$ and $\mathbf{NAC}$:

$$\hat{\mathbf{c}} = \mathbf{C}_1 \mathbf{N} + \mathbf{C}_2 \mathbf{NAC}$$

Called Normal Curvature of $\gamma$

and denoted by $k_n$

and Geodesic Curvature of $\gamma$

and denoted by $k_g$

$$= k_n \mathbf{N} + k_g \mathbf{NAC}$$

Deriving the identity $\langle \hat{\mathbf{c}}, \mathbf{N} \rangle = 0$ we get:

$$\langle \hat{\mathbf{c}}, \mathbf{N} \rangle + \mathbf{C}_2 \mathbf{NAC} = 0$$

Thus

$$\hat{\mathbf{c}} = \Pi_p(c) \mathbf{N} + k_g \mathbf{NAC}$$

2.10 Geodesic curvature and first variation of the length

Let $\gamma$ be a curve in $\mathbb{R}^3$ and let $\eta$ be a normal vector field ($\hat{\gamma} \cdot \eta = 0$).

For every $t$ set

$$\gamma_t(t) = \gamma(t) + \eta(t)$$

Then

$$\text{Length}(\gamma_t) = \int |\dot{\gamma}_t| dt = \int |\dot{\gamma}_t| + |\dot{\gamma}_t \cdot \eta| dt$$

$$= \int \sqrt{|\dot{\gamma}_t|^2 + \eta(t)^2} dt + 2 \eta \cdot \dot{\gamma} \cdot \eta$$

$$= \int \sqrt{1 + 2 \eta \cdot \dot{\gamma} + \eta(t)^2} dt$$

and then

$$\frac{d}{dt} \text{Length}(\gamma_t) = \int \dot{\gamma}_t \cdot \ddot{\gamma}_t = -\dot{\gamma} \cdot \eta$$

Assume now that $\gamma$ lies on the surface $S$ and minimizes length among all curves on $S$ with same endpoints. Then the admissible variations for $\gamma$ are those for which $\eta$ is tangent to $S$.

Well, this statement requires some justification.....
In other words \( \eta = \varphi \cdot (N \times z) \) (2.23)

with \( \varphi \) an arbitrary real function.

Hence the minimality of \( \gamma \) implies

\[
0 = \frac{d}{dt} \text{Length}(\gamma_t)_{t=0} = -\int z \cdot \gamma = -\int K g \cdot \varphi
\]

and then

\[
K g = 0.
\]

2.11 Gauss-Bonnet theorem

First version: global, no boundary

For every surface \( S \subseteq \mathbb{R}^3 \) compact and without boundary there holds

\[
\int_S K = 2\pi \chi(S)
\]

(GB1)

where \( \chi(S) \) is the Euler characteristic of \( S \).

The Euler characteristic of a surface (with or without boundary) is computed — as for polyhedral surfaces — by taking a "reasonable" triangulation of the surface and then setting

\[
\chi(S) = \text{number of faces (triangles)} - \text{number of edges} + \text{number of vertices}.
\]

Remarks

- \( \chi(S) \) does not depend on the triangulation: one shows that refining a triangulation does not change \( \chi(S) \) and then uses that given two triangulations, there exists another one finer than both.

![Refinement](basic step in refinement: added 2 faces, 3 edges, 1 vertex; \( \chi \) is the same)

- One can use also other polygons than just triangles.

![Refinement](triangulation of a pentagon: added 4 faces, 5 edges, 1 vertex; \( \chi \) is the same)
- Euler characteristic of the sphere
\[ \chi(S^2) = 6 - 9 + 5 = 2 \]

- Euler characteristic of a disc
\[ \chi(D) = 1 - 3 + 3 = 1 \]

- Euler characteristic of the torus
\[ \chi(T^2) = 2 - 3 + 1 = 0 \]

- Recall that the Gauss curvature \( K \) is the Jacobian determinant (with sign!) of the Gauss map \( N : S \to S^2 \). Hence the oriented area formula yields
\[ \int_S K = \int_{S^2} \deg(N, p) \, dp. \]

If \( \partial S = \emptyset \) then the degree \( \deg(N, p) \) does not depend on the point \( p \).

And therefore
\[ \int_S K = \deg(N) \cdot \text{Area}(S^2) = \deg(N) \cdot 4\pi. \]

This argument is almost sufficient to show that \( \int_S K \) is a topological invariant: if \( S \) and \( S' \) are isotopic (that is, there exists a one-parameter family of embedded surfaces \( S_t \) such that \( S_t \approx S_0 \) and \( S_t \approx S_1 \)) then \( \deg(N) = \deg(N') \) and therefore
\[ \int_S K = \int_{S'} K. \]

Second version of Gauss-Bonnet: local with bdry
Let \( R \) be an embedded disc. Then
\[ (GB2) \quad \int_K + \int_{\partial R} = 2\pi \]

Gauss curvature of \( R \)
Geodesic curvature of \( \partial R \)

Remarks
- Using the last formula one can compute the Gauss curvature of a surface \( S \) at a point \( p \) by taking a small disc-like neighbourhood of \( p \).
\[ K(p) = \frac{1}{\text{Area}(R)} \int_R K = \frac{2\pi - \int_{\partial R} K_g}{\text{Area}(R)} \]

Now, the geodesic curvature is "intrinsic," can be computed using only the notion of distance on \( S \), that is, the metric; this formula shows that the Gauss curvature is intrinsic, too. This is Gauss's Theorema Egregium.

Formula (GB2) can be extended to the case \( \partial R \) is piecewise smooth (we admit corners):

\[(GB2') \quad \int_K + \int_{\partial R} K_g + \sum_{\partial R} \alpha_i = 2\pi\]

The term \( \sum_{\partial R} \alpha_i \) accounts for the geodesic curvature of \( \partial R \), "concentrated," at the corners.

Formula (GB2') can be obtained from formula (GB2) by "smoothing out" corners.

Alternatively, note that the angle \( \alpha_i \) correspond to the "jump" of the tangent vector \( T \) to \( \partial R \) at a corner point \( p_i \) — that is, the distance on the sphere \( S^2 \) between the tangent vectors \( T^+ \) and \( T^- \) at the two sides of \( p_i \). Formally, this "jump" corresponds to the derivative of \( T \) (and therefore the curvature of \( \partial R \) concentrated at \( p_i \)).

To prove (GB2), consider the sphere-like surface \( R' \) obtained by taking two copies of \( R \) and gluing them at the boundary as in the figure.

Then a standard computation yields

\[ 2 \left[ \int_K + \int_{\partial R} K_g \right] = \int_R K \]

\[ \rightarrow \begin{array}{c}
\text{proceed as before} \\
\int_{S^2} \text{deg}(N',p) \, dp \\
\int_{S^2} \text{deg}(N') \cdot q_{11} = q_{11} \\
\text{if } R \text{ is "sufficiently flat", then } \text{deg}(N') \text{ is obviously } 1.
\end{array} \]
Third version of G-B.: global with body

Let $S$ be any compact surface with boundary. Then

\[ (GB3) \quad \int_S K + \int_{\partial S} k_g = 2\pi \chi(S) \]

\[ \uparrow \quad \text{canonically oriented} \quad \uparrow \quad \text{Euler chart of } S \]

- It's obvious how to modify this formula to include piecewise smooth boundaries.

- Formula (GB3) can be proved by taking a triangulation of $S$ and applying (GB2') to each triangle.

- Formula (GB4) is a particular case of (GB3).