# VARIATIONAL FORMULATION OF SOFTENING PHENOMENA IN FRACTURE MECHANICS: THE ONE-DIMENSIONAL CASE

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**Abstract**

We show that discrete models of atoms subject to nearest-neighbour non-linear interactions approximate continua allowing for softening and fracture. A detailed study is carried out of local minima and stationary points. Scale effects are discussed.

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## 1. Introduction

Diagrams from tension tests on bars of macroscopically homogeneous materials typically exhibit a behaviour that can be described as follows. Let $l$ denote the length of the specimen at rest, and $d$ its elongation. If we plot the stress $\sigma$ as a function of $d$ we notice that it is increasing up to a certain critical value of the elongation, and then it decreases, up to a final threshold where fracture occurs (see, e.g., [7], Chapter 2).

This behaviour is usually explained by assuming that, when the stress reaches a critical value (maximum tensile stress), a confined fracture zone appears. As a consequence, the body is gradually unloaded as the damage increases. This phenomenon can be theoretically analyzed by means of a model, where the softening of the material due to the damage within the fracture zone is taken into account (see, e.g., [15]). It can be assumed that the width of the fracture zone is initially zero (see [16]), so that it is represented by a point $x_0$ in the reference configuration $(0,l)$. If $w$ denotes the additional deformation within this zone, it is assumed that the stress transmitted can be expressed as $g(w)$, for some decreasing function $g$. If $\sigma = f(\varepsilon)$ is the stress-strain relation in the undamaged region, the equilibrium condition is given by $f(\varepsilon) = g(w)$. This leads to a variational principle, with an energy given by

$$\int_0^l F(\dot{u}(x))\, dx + G([u](x_0))\,. \tag{1.1}$$

where $F$ and $G$ are the primitives of $f$ and $g$, respectively, $u$ represents the displacement, which is thought to be smooth except at $x_0$, and $[u](x_0) = u(x_0+) - u(x_0-)$ is the jump of $u$ at $x_0$. The equilibrium configuration corresponding to the elongation $d$ has a displacement $u$ which minimizes (1.1) among all functions which satisfy the boundary conditions $u(0) = 0$ and $u(l) = d$.

This model, and in particular the fact that the damaged zone has width zero in the reference configuration, can be justified starting from a simple discrete model with concentrated masses. We will study the properties of the equilibria for the discrete problem, and then we will obtain the variational model described by (1.1) as the number of concentrated masses tends to infinity.

In the discrete model the bar is identified with a system of $n+1$ equally spaced material points interacting through an array of $n$ non-linear springs connecting neighbouring points. We suppose that the force due to each spring depends on its relative elongation $\varepsilon$ following a law $\sigma = \psi_n(\varepsilon)$.

In order to obtain the qualitative behaviour described at the beginning of the introduction, we assume that there exist two constants $\varepsilon_n^{\rm ult}$ and $\varepsilon_n^{\rm frac}$, with $0 < \varepsilon_n^{\rm ult} < \varepsilon_n^{\rm frac}$, such that $\psi_n$ is increasing on $(-\infty, \varepsilon_n^{\rm ult}]$, $\psi_n$ is decreasing on $[\varepsilon_n^{\rm ult}, \varepsilon_n^{\rm frac}]$, and vanishes on $[\varepsilon_n^{\rm frac}, +\infty)$. This means that each spring has a (non-linear) elastic behaviour up to the critical value $\varepsilon_n^{\rm ult}$ of the relative elongation (ultimate strain), and that a softening phenomenon occurs between this value and the fracture threshold $\varepsilon_n^{\rm frac}$. In addition we assume as usual that $\psi_n(0) = 0$, and $\psi_n(\varepsilon) \to -\infty$ as $\varepsilon \to -\infty$.

Let $x_n^i$, $i = 0, \ldots, n$, denote the locations of the material points in the reference configuration, and let $u_n^i$ denote the corresponding (longitudinal) displacements. If $\lambda_n$ is the distance of two neighbouring points and $\dot u_n^i = (u_n^i - u_n^{i-1})/\lambda_n$ is the relative elongation of the $i^{\rm th}$ spring, the energy of the system is

$$E_n(\{u_n^i\}) = \sum_{i=1}^n \lambda_n \Psi_n(\dot u_n^i)\,, \tag{1.2}$$

where $\Psi_n$ is the primitive of $\psi_n$ vanishing at $0$. The displacement $\{u_n^i\}$ corresponding to an equilibrium configuration in this discrete model is a stationary point for $E_n$ with appropriate boundary conditions.

We will prove (Theorems 2.2 and 2.3) that, if $\{u_n^i\}$ is an absolute minimum or a strict local minimum (i.e., a stable equilibrium) of the energy $E_n$, possibly perturbed by an external force, then there exists at most one index $j$ such that $\dot u_n^j > \varepsilon_n^{\rm ult}$. Therefore the softening phenomenon occurs at most in one spring. This "strain localization" (see [4] and [14]) is in accordance with the experimental data for concrete, which show that the width of the fracture zone is of the same order of the aggregate size.

In order to justify the continuous model (1.1) we consider the variational limit of the functionals (1.2) as $n$ tends to $+\infty$. Since we are interested in a model allowing for softening and fracture, we assume that the ultimate strains $\varepsilon_n^{\rm ult}$ and the ultimate stresses $\sigma_n^{\rm ult} := \psi_n(\varepsilon_n^{\rm ult}) = \max \psi_n$ are equibounded.

To describe the energies intervening in the limit model, we have to use the space $BV(0,l)$ of functions of bounded variation in $(0,l)$. Given $u \in BV(0,l)$, its distributional derivative $u'$ can be written as

$$u' = \dot u dx + \sum_{x \in S_u} [u](x)\delta_x + u'_c\,,$$

where $\dot{u}$ is the ordinary derivative of $u$, $S_u$ is the set of essential discontinuity points of $u$, $\delta_x$ is the Dirac mass concentrated at $x$, while the measure $u'_c$, called the Cantor part of $u'$, is a non-atomic measure on $(0,l)$, which is singular with respect to the Lebesgue measure. The jump term $[u](x)$ represents a singular strain which, in the reference configuration, is concentrated at the point $x$, while the measure $u'_c$ can be considered as a diffuse singular strain.

We will show (Theorem 3.1) that (up to a subsequence) the functionals $E_n$ $\Gamma$-converge to a functional of the form

$$\mathcal{F}(u) = \int_0^l \Phi(\dot{u})\,dx + \sigma^{\mathrm{ult}} u'_c(0,l) + \sum_{S_u} G([u])\,,$$

defined on functions $u \in BV_{\mathrm{loc}}(0,l)$ with $[u] \geq 0$ in $S_u$ and $u'_c \geq 0$ in $(0,l)$. This convergence result implies that the minimum values of $E_n$ for a given elongation $d$ converge to the minimum value of $\mathcal{F}$ with the same elongation, and (a suitable interpolation of) the corresponding displacements converge (up to a subsequence) to a displacement which minimizes $\mathcal{F}$ (Theorem 5.1). A similar result holds if external loads are added (Theorem 5.4).

In order to illustrate the relationship between the sequence $\{\psi_n\}$ and the functions $\Phi$ and $G$ which appear in the functional $\mathcal{F}$, we introduce two sequences of functions which describe the different behaviour of $\psi_n$ in its increasing and decreasing branches, respectively. Let $f_n$ be any increasing function coinciding with $\psi_n$ on $(-\infty, \varepsilon_n^{\mathrm{ult}}]$, and let $g_n$ be the rescaled function

$$g_n(w) = \psi_n\Big(\frac{w}{\lambda_n} + \varepsilon_n^{\mathrm{ult}}\Big) \qquad w \geq 0\,.$$

The need of a different scale in the fracture zone was discussed in [18]. As the functions $f_n$ and $g_n$ are monotonic, it is not restrictive to assume that, up to a subsequence, they converge pointwise to two functions $f$ and $g$, respectively. Then the constant $\sigma^{\mathrm{ult}}$ and the functions $G$ and $\Phi$ are defined as follows: $\sigma^{\mathrm{ult}} = g(0+)$, while $G$ and $\Phi$ are the primitives, vanishing at $0$, of $g$ and $\min\{f, \sigma^{\mathrm{ult}}\}$, respectively.

In Sections 6 and 7, we will consider the simplified functional $\mathcal{E}$ defined on all piecewise smooth functions $u$ as

$$\mathcal{E}(u) = \int_0^l F(\dot{u})\,dx + \sum_{S_u} G([u]), \tag{1.3}$$

where $F$ is the primitive of $f$ vanishing at $0$. We will show that $\mathcal{F}$ and $\mathcal{E}$ have the same local minimizers with prescribed Dirichlet boundary conditions, and that this remains true if we add a dead load (Theorem 7.2). The $\Gamma$-convergence result mentioned above implies therefore that the minimum values of $E_n$ for given elongation and external load converge to the minimum value of $\mathcal{E}$ with the same elongation and load, and (a suitable interpolation of) the corresponding displacements converge (up to a subsequence) to a displacement which minimizes $\mathcal{E}$. This approximation property provides a justification of the choice of an energy $\mathcal{E}$ of the form (1.3) for a continuous model allowing for damage and fracture, and shows that such functionals are the only ones that can be obtained by a limiting procedure starting from a discrete model with non-linear springs with the qualitative properties described above.

We perform a detailed analysis of the stationary points of $\mathcal{E}(u) - \int_0^l hu\,dx$, where $h$ is the density of an external force. We show in particular that a minimum energy configuration exists for every elongation $d$ (Theorem 7.6). Moreover, we prove that for every stationary point there exists $\sigma$ such that

$$f(\dot{u}) + H = \sigma \quad \text{in } (0,l)\setminus S_u\,, \qquad g([u]) + H = \sigma \quad \text{in } S_u\,,$$

where $H$ is a primitive of $h$ (Theorem 6.2). In this case all strict local minimizers (which correspond to stable equilibria) have at most one jump point (Theorem 7.5). As a consequence we recover that the damaged zone for stable equilibria in the continuous limit is concentrated in a single point, and this justifies the use of the functional (1.1). Similar results in the case $h = 0$, but with a more general $g$, can be found in [11].

In the last section, we analyze the scale effect in the model, under monotonicity assumptions on $l\,f^{-1} + g^{-1}$. For simplicity, we describe here the special case when $f$ is linear on $\mathbf{R}$ and $g$ is affine on the interval $[0, w^{\rm frac}]$, and vanishes for values exceeding the fracture threshold $w^{\rm frac}$. If $\varepsilon^{\rm ult}$ is the relative elongation defined by the condition $f(\varepsilon^{\rm ult}) = \sigma^{\rm ult}$, then the behaviour of stable stationary points is different in the case when $l\,\varepsilon^{\rm ult} < w^{\rm frac}$ or $w^{\rm frac} < l\,\varepsilon^{\rm ult}$. Notice that the alternative depends only on the length $l$ of the bar. In the case $l\,\varepsilon^{\rm ult} < w^{\rm frac}$, three distinct regimes are possible, depending on $d$. For $d < l\,\varepsilon^{\rm ult}$ the only solution has no jumps (elastic regime); for $l\,\varepsilon^{\rm ult} < d < w^{\rm frac}$ we have a solution with a jump and non-zero strain (damaged regime); if $d > w^{\rm frac}$ the only solution has one jump point $x_0$ with $[u](x_0) = d$, and $\dot{u}(x) = 0$ for $x \neq x_0$ (fractured regime). If $w^{\rm frac} < l\,\varepsilon^{\rm ult}$ only two stable regimes are possible: the elastic regime for $d < l\,\varepsilon^{\rm ult}$ and the fractured one for $d > w^{\rm frac}$. Thus, for $w^{\rm frac} < d < l\,\varepsilon^{\rm ult}$ two stable solutions are possible, and in addition there is at least one unstable stationary solution. In particular, if we let $d$ increase continuously from 0 to a value larger than $l\,\varepsilon^{\rm ult}$, a discontinuous transition from the elastic to the fractured regime takes place at some value of $d$ between $w^{\rm frac}$ and $l\,\varepsilon^{\rm ult}$, and no softening phenomenon occurs in the process. This analysis explains some scale effects observed in fracture mechanics. In particular it agrees with the observation that the fracture tends to be brittle if the length of the specimen is large enough.

## 2. The Discrete Model

In the discrete model we consider a bar of length $l>0$ as a system of $n+1$ masses located at the points $x_n^i = i\,\lambda_n$, $i=0,...,n$, equally spaced in the interval $[0,l]$, with mutual distance $\lambda_n = l/n$. In this section we keep $n$ fixed and we study the properties of the equilibrium configurations. In Sections 3, 4 and 5 we shall study the limits of the minimum energy configurations as $n$ tends to infinity.

We consider only the problem of longitudinal displacements. We model the behaviour of this system of $n+1$ material points as depending on an array of $n$ non-linear springs connecting neighbouring points. We suppose that the tension $\sigma$ due to each spring depends on its relative elongation $\varepsilon$ following a constitutive relation $\sigma = \psi_n(\varepsilon)$, where $\psi_n\colon \mathbf{R} \to [-\infty,+\infty)$ is continuous and satisfies

$$\lim_{\varepsilon\to-\infty} \psi_n(\varepsilon) = -\infty\,. \tag{2.1}$$

We assume there exist three constants $\varepsilon_n^{\rm min}$, $\varepsilon_n^{\rm ult}$, and $\varepsilon_n^{\rm frac}$, with $-\infty \le \varepsilon_n^{\rm min} < 0 < \varepsilon^{\rm ult} < \varepsilon_n^{\rm frac} \le +\infty$, such that $\psi_n(\varepsilon) = -\infty$ for $\varepsilon \le \varepsilon_n^{\rm min}$ (incompenetrability), $\psi_n$ is increasing on $(\varepsilon_n^{\rm min}, \varepsilon_n^{\rm ult}]$ (elastic behaviour), $\psi_n(0)=0$ (stress-free reference configuration), $\psi_n$ is positive and decreasing on $[\varepsilon_n^{\rm ult}, \varepsilon_n^{\rm frac})$ (softening regime), and $\psi_n(\varepsilon)=0$ for $\varepsilon \ge \varepsilon_n^{\rm frac}$ (fracture). The constant $\varepsilon_n^{\rm ult}$ is the ultimate strain (i.e., the maximum strain in the elastic regime), while $\varepsilon_n^{\rm frac}$ is the fracture strain. The constant $\sigma_n^{\rm ult} := \psi_n(\varepsilon_n^{\rm ult}) = \max\psi_n$ is ultimate tensile stress (i.e., the maximum possible tension of the springs).

Let $\Psi_n\colon \mathbf{R} \to [0,+\infty]$ be defined by

$$\Psi_n(\varepsilon) := \int_0^\varepsilon \psi_n(s)\,ds, \tag{2.2}$$

so that we have $\Psi_n'(\varepsilon) = \psi_n(\varepsilon)$ in $(\varepsilon_n^{\min}, +\infty) = \{\psi_n \neq -\infty\}$. If $u_n^i$ denotes the longitudinal displacement of the point $x_n^i$, then the internal energy of the system is given by

$$E_n(\{u_n^i\}) := \sum_{i=1}^{n} \lambda_n \Psi_n(\dot{u}_n^i)\,, \tag{2.3}$$

where $\dot{u}_n^i = (u_n^i - u_n^{i-1})/\lambda_n$ is the relative elongation of the $i^{\text{th}}$ spring, connecting $x_n^{i-1}$ to $x_n^i$. Note that $\dot{u}_n^i$ coincides with the constant derivative of the affine interpolation between $u_n^{i-1}$ and $u_n^i$ on the interval $[x_n^{i-1}, x_n^i]$.

Suppose now that an external force $\lambda_n h_n^i$ is acting on the material point $x_n^i$ for $i = 1, \dots, n-1$. The total force acting on $x_n^i$ is then

$$\psi_n(\dot{u}_n^{i+1}) - \psi_n(\dot{u}_n^i) + \lambda_n h_n^i\,, \tag{2.4}$$

and the total energy of the system is given by

$$E_n^h(\{u_n^i\}) := \sum_{i=1}^{n} \lambda_n \Psi_n(\dot{u}_n^i) - \sum_{i=1}^{n-1} \lambda_n h_n^i u_n^i\,. \tag{2.5}$$

An equilibrium configuration of the system with prescribed displacements $u_n^0 = 0$ and $u_n^n = d$ is then a stationary point of the constrained problem

$$\min\{E_n^h(\{u_n^i\}) : u_n^0 = 0,\ u_n^n = d\}\,. \tag{2.6}$$

The necessary and sufficient condition for stationarity is precisely that the force given by (2.4) is zero for $i = 1, \dots, n-1$.

For $i = 2, \dots, n$ let $H_n^i := \lambda_n h_n^1 + \cdots + \lambda_n h_n^{i-1}$ be the resultant of the external forces acting on the first $i-1$ points. If we set $H_n^1 := 0$, the stationarity can be written in the following form: there exists a constant $\sigma_n$ such that

$$\psi_n(\dot{u}_n^i) + H_n^i = \sigma_n \qquad \text{for } i = 1, \dots, n\,. \tag{2.7}$$

Since $H_n^1 = 0$, the previous formula shows that $\sigma_n \leq \sigma_n^{\text{ult}}$. For future use we note that the energy (2.5) can be written as

$$E_n^h(\{u_n^i\}) = \sum_{i=1}^{n} \lambda_n \Psi_n(\dot{u}_n^i) + \sum_{i=1}^{n} \lambda_n H_n^i \dot{u}_n^i - H_n^n d\,. \tag{2.8}$$

We want now to study the properties of the local minima of the constrained problem (2.6), which correspond to stable equilibria of the system. We begin with a lemma.

**Lemma 2.1.** *Let $\{u_n^i\}$ be a local minimum of the constrained problem (2.6). Suppose that there are two distinct indices $j$ and $k$ such that $\dot u_n^j > \varepsilon_n^{\rm ult}$ and $\dot u_n^k > \varepsilon_n^{\rm ult}$. Then $\dot u_n^j > \varepsilon_n^{\rm frac}$ and $\dot u_n^k > \varepsilon_n^{\rm frac}$. Moreover $H_n^j = H_n^k$.*

*Proof.* Let us prove that $\dot u_n^k > \varepsilon_n^{\rm frac}$. Suppose, by contradiction, that $\varepsilon_n^{\rm ult} < \dot u_n^k \le \varepsilon_n^{\rm frac}$. Since $\psi_n$ is non-increasing on $[\varepsilon_n^{\rm ult}, +\infty)$ and decreasing on $[\varepsilon_n^{\rm ult}, \varepsilon_n^{\rm frac}]$, we have

$$\Psi_n(\dot u_n^j + \varepsilon) - \Psi_n(\dot u_n^j) + \Psi_n(\dot u_n^k - \varepsilon) - \Psi_n(\dot u_n^k) < \big(\psi_n(\dot u_n^j) - \psi_n(\dot u_n^k)\big)\varepsilon \tag{2.9}$$

for $\varepsilon > 0$ small enough. Let $\{v_n^i\}$ be defined by the equalities $v_n^0 = 0$, $\dot v_n^j = \dot u_n^j + \varepsilon$, $\dot v_n^k = \dot u_n^k - \varepsilon$, and $\dot v_n^i = \dot u_n^i$ for all other indices $i$. As a consequence of these equalities we have $v_n^n = u_n^n = d$. Since $\{u_n^i\}$ is a local minimum, if $\varepsilon > 0$ is small enough we have $E_n^h(\{u_n^i\}) \le E_n^h(\{v_n^i\})$. Using (2.8) we obtain

$$\begin{aligned} &\Psi_n(\dot u_n^j + \varepsilon) + \Psi_n(\dot u_n^k - \varepsilon) + H_n^j(\dot u_n^j + \varepsilon) + H_n^k(\dot u_n^k - \varepsilon) \\ &\qquad \ge \Psi_n(\dot u_n^j) + \Psi_n(\dot u_n^k) + H_n^j \dot u_n^j + H_n^k \dot u_n^k . \end{aligned} \tag{2.10}$$

The stationarity condition (2.7) gives $H_n^j - H_n^k = \psi_n(\dot u_n^k) - \psi_n(\dot u_n^j)$, so that (2.10) contradicts (2.9). This proves that $\dot u_n^k > \varepsilon_n^{\rm frac}$.

The proof for $\dot u_n^j$ is analogous. The equality $H_n^j = H_n^k$ follows from the stationarity condition (2.7) and from the fact that $\psi_n(\dot u_n^j) = \psi_n(\dot u_n^k) = 0$. □

**Theorem 2.2.** *Let $\{u_n^i\}$ be a strict local minimum of the constrained problem (2.6). Then there is at most one index $j$ such that $\dot u_n^j > \varepsilon_n^{\rm ult}$.*

*Proof.* If there are two distinct indices $j$ and $k$ such that $\dot u_n^j > \varepsilon_n^{\rm ult}$ and $\dot u_n^k > \varepsilon_n^{\rm ult}$, then by Lemma 2.1 we have $\dot u_n^j > \varepsilon_n^{\rm frac}$, $\dot u_n^k > \varepsilon_n^{\rm frac}$, and $H_n^j = H_n^k$. Since $\Psi_n$ is constant in $(\varepsilon_n^{\rm frac}, +\infty)$, we can prove that $\{u_n^i\}$ is not a strict local minimum by using the configuration $\{v_n^i\}$ introduced in the proof of Lemma 2.1, which, for $\varepsilon > 0$ small enough, does not change the energy $E_n^h$. □

**Theorem 2.3.** *Assume that $d > l\,\varepsilon_n^{\min}$ (or $d = l\,\varepsilon_n^{\min}$ and $\Psi_n(\varepsilon_n^{\min}) < +\infty$). Then the constrained problem (2.6) attains its absolute minimum, and for every minimum point $\{u_n^i\}$ there is at most one index $j$ such that $\dot u_n^j > \varepsilon_n^{\rm ult}$. For this index $j$ we have $H_n^j = \min_i H_n^i$. Conversely, for every $j$ such that $H_n^j = \min_i H_n^i$ there exists a minimum* *point of (2.6) such that* $\dot u_n^i \le \varepsilon_n^{\rm ult}$ *for every* $i \ne j$.

*Proof.* Since by (2.8) the energy $E_n^h$ can be written in the form

$$\sum_{i=1}^n \lambda_n \Psi_n(\dot u_n^i) + \sum_{i=1}^n \lambda_n (H_n^i - \min_k H_n^k + 1)\dot u_n^i + (\min_k H_n^k - 1 - H_n^n) d\,,$$

it follows easily from (2.1) and (2.2) that every minimizing sequence for problem (2.6) is bounded, and this implies that the absolute minimum is achieved.

If there are two distinct indices $j$ and $k$ such that $\dot u_n^j > \varepsilon_n^{\rm ult}$ and $\dot u_n^k > \varepsilon_n^{\rm ult}$, then by Lemma 2.1 we have $\dot u_n^j > \varepsilon_n^{\rm frac}$, $\dot u_n^k > \varepsilon_n^{\rm frac}$, and $H_n^j = H_n^k$. Let $\{v_n^i\}$ be the configuration defined by the equalities $v_n^0 = 0$, $\dot v_n^j = \dot u_n^j + \dot u_n^k - \varepsilon_n^{\rm ult}$, $\dot v_n^k = \varepsilon_n^{\rm ult}$, and $\dot v_n^i = \dot u_n^i$ for all other indices $i$. Clearly we have $v_n^n = d$. As $H_n^j = H_n^k$ and $\Psi_n(\varepsilon) = \Psi_n(\varepsilon_n^{\rm frac}) > \Psi_n(\varepsilon_n^{\rm ult})$ for $\varepsilon \ge \varepsilon_n^{\rm frac}$, we conclude that $E_n^h(\{v_n^i\}) < E_n^h(\{u_n^i\})$, which contradicts the minimality of $\{u_n^i\}$. Therefore there is at most one index $j$ such that $\dot u_n^j > \varepsilon_n^{\rm ult}$.

If there exists another index $k$ such that $H_n^j > H_n^k$, then we consider the new configuration $\{z_n^i\}$ defined by $z_n^0 = 0$, $\dot z_n^j = \dot u_n^k$, $\dot z_n^k = \dot u_n^j$, and $\dot z_n^i = \dot u_n^i$ for all other indices. This configuration satisfies $z_n^n = d$ and has smaller energy than $\{u_n^i\}$. This contradicts the minimality of $\{u_n^i\}$ and shows that $H_n^j = \min_i H_n^i$. Conversely, if $k$ is another index such that $H_n^k = \min_i H_n^i$, then the configuration $\{z_n^i\}$ defined above has the same energy as $\{u_n^i\}$. Therefore it is a minimum point of (2.6) such that $\dot z_n^i \le \varepsilon_n^{\rm ult}$ for every $i \ne k$. □

When $h_n^i = 0$ for $i = 1, \dots, n-1$ we obtain the following well known result.

**Proposition 2.4.** *Let* $\{u_n^i\}$ *be an absolute minimum, or a strict local minimum, of the functional* $E_n$ *with the constraints* $u_n^0 = 0$ *and* $u_n^n = d$. *Then one of the following conditions is satisfied:*

(a) $\dot u_n^i = d/l \le \varepsilon_n^{\rm ult}$ *for* $i = 1, \dots, n$;

(b) *there exist a constant* $\varepsilon < \varepsilon_n^{\rm ult}$ *and an index* $j$ *such that* $\dot u_n^j > \varepsilon_n^{\rm ult}$, $\psi_n(\dot u_n^j) = \psi_n(\varepsilon)$, *and* $\dot u_n^i = \varepsilon$ *for every* $i \ne j$.

*Proof.* The result follows easily from the stationarity condition (2.3) and from Theorems 2.2 and 2.3. □

## 3. Limits of Discrete Models

To study the limits of the discrete models introduced in Section 2 as the number of particles tends to infinity, we will use the space $BV(0,l)$ of *functions of bounded variation*, which is defined as the set of functions $u \in L^1(0,l)$ such that the distributional derivative $u'$ is a bounded Radon measure on $(0,l)$. It is easy to see that for such functions the left and right essential limits $u(x-)$ and $u(x+)$ exist at every point. If we define $[u](x) := u(x+) - u(x-)$ and $S_u := \{x \in (0,l) : \ u(x-) \neq u(x+)\}$, then we can write

$$u' = \dot{u} dx + \sum_{x \in S_u} [u](x) \delta_x + u'_c \,,$$

where $\dot{u}$ is the ordinary derivative of $u$, which is defined almost everywhere on $(0,l)$, $\delta_x$ is the Dirac mass concentrated at $x$, while the measure $u'_c$, called the Cantor part of $u'$, is a non-atomic measure on $(0,l)$, which is singular with respect to the Lebesgue measure. The measure

$$u'_s := \sum_{x \in S_u} [u](x) \delta_x + u'_c$$

is the singular part of the measure $u'$ with respect to the Lebesgue measure.

The space $SBV(0,l)$ of *special functions of bounded variation* is defined as the space of those functions $u \in BV(0,l)$ such that $u'_c$ is identically zero. The space considered here is the one-dimensional case of a more general class of spaces introduced by De Giorgi and Ambrosio in [10].

We adopt the standard notation of measure theory. In particular, if $\mu$ is a signed measure, then $\mu^+$, $\mu^-$, and $|\mu|$ are the positive part, the negative part, and the total variation of $\mu$, respectively. If $E$ is a set, its characteristic function $\chi_E$ is defined by $\chi_E(x) = 1$ for $x \in E$ and $\chi_E(x) = 0$ for $x \notin E$.

Since we shall frequently use truncations, for $a$, $b \in [-\infty, +\infty]$ we set $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$, $a^+ := a \vee 0$, and $a^- := -(a \wedge 0)$.

To study the asymptotic behaviour of the minimum points of a sequence of functionals we shall use the notion of $\Gamma$-convergence, for which we refer to [9]. We recall that a sequence $\{\mathcal{F}_n\}$ of functionals $\mathcal{F}_n : L^1(0,l) \to [0,+\infty]$ is said to $\Gamma$*-converge* in $L^1(0,l)$ to a functional $\mathcal{F} : L^1(0,l) \to [0,+\infty]$ if for every $u \in L^1(0,l)$ the following conditions hold:

(i) (lower semicontinuity inequality) for every sequence $\{u_n\}$ converging to $u$ in $L^1(0,l)$ we have $\mathcal{F}(u) \le \liminf_n \mathcal{F}_n(u_n)$;

(ii) (existence of a recovery sequence) there exists a sequence $\{u_n\}$ converging to $u$ in $L^1(0,l)$ such that $\mathcal{F}(u) \geq \limsup_n \mathcal{F}_n(u_n)$.

In order to apply $\Gamma$-convergence to our asymptotic analysis, we have to consider the discrete energies $E_n$ as defined on $L^1(0,l)$. For this purpose, for every $n$ we consider the space $\mathcal{A}_n$ of all continuous functions $u$ on $[0,l]$ which are affine on $[x_n^{i-1}, x_n^i]$ for all $i$. For every function $u_n \in \mathcal{A}_n$ we set $u_n^i = u(x_n^i)$ and $\dot{u}_n^i = (u_n^i - u_n^{i-1})/\lambda_n$, so that $\dot{u}_n^i$ is the constant value of the derivative of $u_n$ in $(x_n^{i-1}, x_n^i)$. We consider the energy functional $\mathcal{E}_n$ defined on $L^1(0,l)$ by

$$\mathcal{E}_n(u_n) = \int_0^l \Psi_n(\dot{u}_n)\,dx = \sum_{i=1}^n \lambda_n \Psi_n(\dot{u}_n^i) = E_n(\{u_n^i\}) \tag{3.1}$$

for $u_n \in \mathcal{A}_n$, and by $\mathcal{E}_n(u_n) = +\infty$ for $u_n \notin \mathcal{A}_n$. It is clear that all minimum problems for $\mathcal{E}_n$ with prescribed boundary conditions are equivalent to the corresponding minimum problems for $E_n$, in the sense that they have the same minimum values and the minimum points of $\mathcal{E}_n$ are the affine interpolations of the minimum points of $E_n$.

Our asymptotic analysis is very general, and will be done under weaker assumptions than those considered in the previous section. Therefore we are not allowed to use the simple structure of the minimizers of problem (2.6), which is valid only under the assumptions considered in Section 2.

We do not require $\psi_n$ to be continuous; we assume only that $\psi_n(0) = 0$ and that $\psi_n$ is non-negative on $[0,+\infty)$, non-decreasing on $(-\infty, \varepsilon_n^{\mathrm{ult}}]$, and non-increasing on $[\varepsilon_n^{\mathrm{ult}}, +\infty)$, with $0 < \varepsilon_n^{\mathrm{ult}} < +\infty$. These weaker assumptions include also the case of plastic behaviour, which corresponds to intervals where $\psi_n$ is constant. In order to pass to the limit in the minimum problems, we assume that (2.1) holds uniformly with respect to $n$. Moreover we assume that there exist $\varepsilon_* < +\infty$ and $\sigma_* < +\infty$ such that $\varepsilon_n^{\mathrm{ult}} \leq \varepsilon_*$ and $\sigma_n^{\mathrm{ult}} = \psi_n(\varepsilon_n^{\mathrm{ult}}) = \max \psi_n \leq \sigma_*$ for every $n$. These hypotheses imply that there exists a non-decreasing continuous function $\psi_*\colon \mathbf{R} \to \mathbf{R}$ such that $\psi_n(\varepsilon) \leq \psi_*(\varepsilon)$ and $\psi_*(\varepsilon) \to -\infty$ as $\varepsilon \to -\infty$. Therefore the functions $\Psi_n$ defined by (2.2) satisfy the inequality

$$\Psi_n(\varepsilon) \geq \Psi_*(\varepsilon) \qquad \text{for every } \varepsilon \leq 0\,, \tag{3.2}$$

where $\Psi_*$ is the primitive of $\psi_*$ vanishing at $0$ and, consequently,

$$\lim_{\varepsilon \to -\infty} \frac{\Psi_*(\varepsilon)}{|\varepsilon|} = +\infty\,. \tag{3.3}$$

We are now in a position to state the main theorem of this section.

**Theorem 3.1.** *There exists a subsequence, still denoted by* $\{\mathcal{E}_n\}$*, which* $\Gamma$*-converges to a functional* $\mathcal{F}: L^1(0,l) \to [0,+\infty]$ *such that*

$$\mathcal{F}(u) = \int_0^l \Phi(\dot u)\,dx + \sigma^{\mathrm{ult}} u_c'(0,l) + \sum_{S_u} G([u])\,, \tag{3.4}$$

*if* $u \in BV_{\mathrm{loc}}(0,l)$ *and* $u_s' \geq 0$ *in* $(0,l)$*, while* $\mathcal{F}(u) = +\infty$ *for all other functions* $u \in L^1(0,l)$*. In (3.4) the function* $\Phi: \mathbf{R} \to [0,+\infty]$ *is convex and lower semicontinuous, the function* $G: [0,+\infty) \to [0,+\infty)$ *is concave and continuous,* $\Phi(0) = G(0) = 0$*,* $\sigma^{\mathrm{ult}} = G'(0+) \leq \sigma_*$*,* $\Phi(\varepsilon)/|\varepsilon| \to +\infty$ *as* $\varepsilon \to -\infty$*,* $\Phi(\varepsilon) < +\infty$ *for* $\varepsilon \geq 0$*, and* $\Phi'(\varepsilon) = \sigma^{\mathrm{ult}}$ *for* $\varepsilon \geq \varepsilon_*$*. Moreover, every functional of the form (3.4) can be obtained as* $\Gamma$*-limit of* $\{\mathcal{E}_n\}$ *for a suitable choice of* $\{\psi_n\}$*.*

In Sections 4 and 5 we shall use this result to obtain the convergence of the minimum points of the constrained problems (2.6).

We shall see in Remark 6.6 that $\sigma^{\mathrm{ult}}$ is the maximum stress allowed in the continuous model (ultimate tensile stress).

**Remark 3.2.** Note that $\Phi(\varepsilon)/\varepsilon \to \sigma^{\mathrm{ult}}$ as $\varepsilon \to +\infty$. Therefore, if $\sigma^{\mathrm{ult}} > 0$, there exist two constants $A > 0$ and $B \geq 0$ such that $\Phi(\varepsilon) \geq A|\varepsilon| - B$ for every $\varepsilon \in \mathbf{R}$ and, consequently, $\mathcal{F}(u) < +\infty$ implies $u \in BV(0,l)$. Indeed, if $\mathcal{F}(u) < +\infty$, then (3.4) implies that $\dot u \in L^1(0,l)$, $|u_c'|(0,l) < +\infty$, and $\sum_{S_u} G([u]) < +\infty$. As $G$ is non-decreasing and $G'(0+) = \sigma^{\mathrm{ult}} > 0$, we have $G(1) > 0$, so that there is only a finite number of points $x$ with $[u](x) > 1$. As $G$ is concave and $G(0) = 0$, we have $G([u](x)) \geq G(1)[u](x)$ for every $x$ such that $0 \leq [u](x) \leq 1$. Consequently $\sum_{S_u} [u] < +\infty$.

If $\sigma^{\mathrm{ult}} = 0$, then $G(w) = 0$ for every $w \geq 0$ and $\Phi(\varepsilon) = \Phi(\varepsilon \wedge 0)$. Therefore the functional $\mathcal{F}$ becomes

$$\mathcal{F}(u) = \int_0^l \Phi(\dot u \wedge 0)\,dx\,,$$

if $u \in BV_{\mathrm{loc}}(0,l)$ and $u_s' \geq 0$, while $\mathcal{F}(u) = +\infty$ otherwise. This is the one-dimensional case of the energy functional for masonry-like structures studied in [17], [12], [2], and [3].

To prove Theorem 3.1 and to obtain an explicit dependence of $\Phi$, $\sigma^{\mathrm{ult}}$, and $G$ on the sequence $\{\psi_n\}$, it is convenient to consider separately the behaviour of $\psi_n$ in the half-lines $(-\infty, \varepsilon_n^{\mathrm{ult}}]$ and $[\varepsilon_n^{\mathrm{ult}}, +\infty)$. To this aim for every $n$ we introduce a non-decreasing function $f_n\colon \mathbf{R} \to [-\infty, +\infty]$ satisfying

$$f_n(\varepsilon) = \psi_n(\varepsilon) \qquad \text{for all } \varepsilon \le \varepsilon_n^{\text{ult}}\,. \tag{3.5}$$

Note that $f_n \ge \psi_n$ on the whole $\mathbf{R}$, and $f_n \ge \sigma_n^{\text{ult}} = \psi_n(\varepsilon_n^{\text{ult}}) = \max \psi_n$ on $[\varepsilon_n^{\text{ult}}, +\infty)$. We also consider the rescaled function $g_n\colon [0, +\infty) \to [0, +\infty)$ defined by

$$g_n(w) := \psi_n(\frac{w}{\lambda_n} + \varepsilon_n^{\text{ult}}) \tag{3.6}$$

so that

$$\psi_n(\varepsilon) = \begin{cases} f_n(\varepsilon) & \text{if } \varepsilon \le \varepsilon_n^{\text{ult}}, \\ g_n(\lambda_n(\varepsilon - \varepsilon_n^{\text{ult}})) & \text{if } \varepsilon \ge \varepsilon_n^{\text{ult}}. \end{cases} \tag{3.7}$$

Finally, we define $F_n\colon \mathbf{R} \to [0, +\infty]$ and $G_n\colon [0, +\infty) \to [0, +\infty)$ by

$$F_n(\varepsilon) := \int_0^\varepsilon f_n(s)\, ds\,, \qquad G_n(w) := \int_0^w g_n(s)\, ds\,. \tag{3.8}$$

Thus it follows that

$$\Psi_n(\varepsilon) = \begin{cases} F_n(\varepsilon) & \text{if } \varepsilon \le \varepsilon_n^{\text{ult}}, \\ F_n(\varepsilon_n^{\text{ult}}) + \dfrac{1}{\lambda_n} G_n(\lambda_n(\varepsilon - \varepsilon_n^{\text{ult}})) & \text{if } \varepsilon \ge \varepsilon_n^{\text{ult}}. \end{cases} \tag{3.9}$$

Since the functions $f_n$ are non-decreasing and the functions $g_n$ are non-increasing, by Helly's theorem there exist two subsequences, still denoted by $\{f_n\}$ and $\{g_n\}$, such that $\{f_n\}$ converges pointwise to a non-decreasing function $f\colon \mathbf{R} \to [-\infty, +\infty]$ and $\{g_n\}$ converges pointwise to a non-increasing function $g\colon [0, +\infty) \to [0, +\infty)$.

As $f_n(\varepsilon_n^{\text{ult}}) = \psi_n(\varepsilon_n^{\text{ult}}) = g_n(0) \le \sigma_*$, the monotonicity of $f_n$ and $g_n$ yields $f_n(\varepsilon) \ge g_n(w)$ for $\varepsilon \ge \varepsilon_*$ and $w \ge 0$, while $\sigma_* \ge g_n(w)$ for $w \ge 0$. This implies $f(\varepsilon) \ge g(0+)$ for $\varepsilon \ge \varepsilon_*$ and $\sigma_* \ge g(0+)$. We define $\sigma^{\text{ult}} := g(0+)$ and

$$F(\varepsilon) := \int_0^\varepsilon f(s)\, ds\,, \qquad G(w) := \int_0^w g(s)\, ds\,, \tag{3.10}$$

for $\varepsilon \in \mathbf{R}$ and $w \ge 0$. Let $\varepsilon^{\min} = \inf\{\varepsilon \in \mathbf{R} : \ f(\varepsilon) > -\infty\}$ and $\varepsilon^{\max} = \sup\{\varepsilon \in \mathbf{R} : f(\varepsilon) < +\infty\}$. Since $f$ is non-increasing and $f(0) = 0$, we have $\varepsilon^{\min} \le 0 \le \varepsilon^{\max}$ and $F(\varepsilon) < +\infty$ for every $\varepsilon \in (\varepsilon^{\min}, \varepsilon^{\max})$. It is easy to see that $F_n \to F$ uniformly on compact sets of $(\varepsilon^{\min}, \varepsilon^{\max})$ and that $G_n \to G$ uniformly on compact sets of $[0, +\infty)$.

Finally we define

$$\overline{F}(\varepsilon) := \int_0^\varepsilon f(s) \wedge \sigma^{\rm ult} ds \,. \tag{3.11}$$

Note that $\overline{F}(\varepsilon) = F(\varepsilon)$ for $\varepsilon \le \varepsilon^{\rm ult}$ and $\overline{F}(\varepsilon) = F(\varepsilon^{\rm ult}) + \sigma^{\rm ult}(\varepsilon - \varepsilon^{\rm ult})$ for $\varepsilon \ge \varepsilon^{\rm ult}$, where $\varepsilon^{\rm ult} = \sup\{\varepsilon \in \mathbf{R} : f(\varepsilon) < \sigma^{\rm ult}\}$. Since $f(\varepsilon) \ge \sigma^{\rm ult}$ for $\varepsilon \ge \varepsilon_*$, we have $\varepsilon^{\rm ult} \le \varepsilon_* < +\infty$. Moreover, by (3.5) we have $f(\varepsilon) \le \psi_*(\varepsilon)$ for $\varepsilon \le 0$, so that

$$\overline{F}(\varepsilon) = F(\varepsilon) \ge \Psi_*(\varepsilon) \qquad \text{for } \varepsilon \le 0 \,. \tag{3.12}$$

Let $\mathcal{E}\colon L^1(0,l) \to [0,+\infty]$ be the functional defined by

$$\mathcal{E}(u) = \int_0^l F(\dot u)\, dx + \sum_{S_u} G([u]) \,, \tag{3.13}$$

if $u \in SBV_{\rm loc}(0,l)$ and $[u] \ge 0$, and equal to $+\infty$ otherwise. We shall consider also the functional $\overline{\mathcal{E}}\colon L^1(0,l) \to [0,+\infty]$ defined by

$$\overline{\mathcal{E}}(u) = \int_0^l \overline{F}(\dot u)\, dx + \sigma^{\rm ult} u_c'(0,l) + \sum_{S_u} G([u]) \,, \tag{3.14}$$

if $u \in BV_{\rm loc}(0,l)$ and $u_s' \ge 0$, and equal to $+\infty$ otherwise. As in Remark 3.2 we can prove that, if $\sigma^{\rm ult} > 0$, then $\mathcal{F}(u) < +\infty$ implies $u \in BV(0,l)$. Using the arguments of [5] it is possible to prove that $\overline{\mathcal{E}}$ is the lower semicontinuous envelope of $\mathcal{E}$ in $L^1(0,l)$. We shall prove in Sections 6 and 7 that for a wide class of boundary value problems the functionals $\mathcal{E}$ and $\overline{\mathcal{E}}$ have the same stationary points and the same minimizers.

Theorem 3.1 is an immediate consequence of the following result, which shows also that the functions $\Phi$ and $G$ which appear in (3.4) are the functions $\overline{F}$ and $G$ defined by (3.11) and (3.10) respectively.

**Theorem 3.3.** *Suppose that $\{f_n\}$ converges to $f$ pointwise in $\mathbf{R}$ and $\{g_n\}$ converges to $g$ pointwise in $[0,+\infty)$. Then $\{\mathcal{E}_n\}$ $\Gamma$-converges to $\overline{\mathcal{E}}$ in $L^1(0,l)$.*

*Proof.* We begin by proving the lower semicontinuity inequality. Let $\{u_n\}$ be a sequence which converges to $u$ in $L^1(0,l)$. We want to show that $\liminf_n \mathcal{E}_n(u_n) \ge \overline{\mathcal{E}}(u)$. It is not restrictive to suppose that $\{u_n\}$ converges to $u$ a.e. in $[0,l]$ and that the sequence

$\{\mathcal{E}_n(u_n)\}$ has a finite limit, so that, in particular, $u_n \in \mathcal{A}_n$ for $n$ large enough. Let us prove that

$$\sup_n \int_0^l (\dot u_n)^- dx < +\infty\,, \qquad \sup_n \int_\delta^{l-\delta} (\dot u_n)^+ dx < +\infty\,, \tag{3.15}$$

for every $0 < \delta < l/2$. The former inequality in (3.15) follows from (3.1), (3.2), and (3.3). To prove the latter inequality, we fix two points $a$ and $b$, with $0 < a < \delta$ and $l-\delta < b < l$, such that $\{u_n(a)\}$ and $\{u_n(b)\}$ converge to a finite limit. Then we have

$$\int_\delta^{1-\delta} (\dot u_n)^+ dx \le \int_a^b (\dot u_n)^+ dx = u_n(b) - u_n(a) + \int_a^b (\dot u_n)^- dx\,. \tag{3.16}$$

Since the right hand side is bounded, the proof of (3.15) is complete. From (3.15) it follows that $u \in BV_{\mathrm{loc}}(0,l)$ (see, e.g., [13], Theorem 1.9).

For every $n$ let $I_n \subset \{1,\dots,n\}$ be the set of indices such that $\dot u_n^i \le \varepsilon_n^{\mathrm{ult}}$ and let $J_n = \{1,\dots,n\} \setminus I_n$, i.e., the set of indices such that $\dot u_n^i > \varepsilon_n^{\mathrm{ult}}$. We define a new function $v_n$ on $(0,l]$, which is still affine on each open interval $(x_n^{i-1}, x_n^i)$, but may be discontinuous at some of the points $x_n^i$. On the intervals $(x_n^{i-1}, x_n^i]$ with $i \in I_n$ we set $v_n = u_n$. On the intervals $(x_n^{i-1}, x_n^i]$ with $i \in J_n$ the affine function $v_n$ is defined by the conditions $\dot v_n = \varepsilon_n^{\mathrm{ult}}$ and $v_n(x_n^i) = u_n^i$. Since $u_n$ and $v_n$ are affine on the intervals $(x_n^{i-1}, x_n^i]$, by an elementary computation we obtain

$$\int_{x_n^i}^{x_n^j} |v_n - u_n|\, dx \le \frac{\lambda_n}{2} \int_{x_n^i}^{x_n^j} (\dot u_n)^+ dx\,,$$

for $0 \le i < j \le n$. As $\{u_n\}$ converges to $u$ in $L^1(0,l)$, by (3.15) the previous inequalities imply that $\{v_n\}$ converges to $u$ in $L^1_{\mathrm{loc}}(0,l)$. Passing to a subsequence, we may assume that $\{v_n\}$ converges to $u$ a.e. on $(0,l)$, and the same argument used for $\{u_n\}$ shows that $\{v_n\}$ is bounded in $BV_{\mathrm{loc}}(0,l)$.

For every $n$ we have

$$\begin{aligned} \mathcal{E}_n(u_n) &= \sum_{i\in I_n} \lambda_n F_n(\dot u_n^i) + \lambda_n \# J_n\, F_n(\varepsilon_n^{\mathrm{ult}}) + \sum_{i \in J_n} G_n(u_n^i - u_n^{i-1} - \lambda_n \varepsilon_n^{\mathrm{ult}}) \\ &= \int_0^l F_n(\dot v_n)\, dx + \sum_{S_{v_n}} G_n([v_n])\,. \end{aligned} \tag{3.17}$$

where $\#A$ denotes the number of the elements of $A$. For each $k > 0$ let $f_n^k(\varepsilon) := (-k) \vee (f_n(\varepsilon) \wedge k)$, let $F_n^k$ be its primitive vanishing at $0$, and let

$$G_n^k(w) := \begin{cases} G_n(w) + \dfrac{w}{k} & \text{for } w \ge 0, \\ -kw & \text{for } w \le 0. \end{cases}$$

Then by (3.17) for every $0 < \delta < l/2$ and for every $k > 0$

$$\mathcal{E}_n(u_n) + \frac{1}{k} \sum_{S_{v_n} \cap (\delta, l-\delta)} [v_n] \geq \int_\delta^{l-\delta} F_n^k(\dot{v}_n)\, dx + \sum_{S_{v_n} \cap (\delta, l-\delta)} G_n^k([v_n]) \,.$$

Note that

$$\sum_{S_{v_n} \cap (\delta, l-\delta)} [v_n] \leq \sup_n |v_n'|(\delta, l-\delta) = c(\delta) < +\infty \,.$$

By Theorem 3.4 and Corollary 3.5 of [1] we have

$$\begin{aligned} &\liminf_{n \to +\infty} \mathcal{E}_n(u_n) + \frac{c(\delta)}{k} \\ &\qquad \geq \int_\delta^{l-\delta} F^k(\dot{u})\, dx + \sum_{S_u \cap (\delta, l-\delta)} G^k([u]) + \sigma^{\mathrm{ult}} (u_c')^+(\delta, l-\delta) + k\, (u_c')^-(\delta, l-\delta) \,, \end{aligned}$$

where $F^k$ and $G^k$ are the primitives vanishing at $0$ of the functions

$$\begin{aligned} f^k(\varepsilon) &:= (-k) \vee \big( (f(\varepsilon) \wedge \sigma^{\mathrm{ult}}) + \frac{1}{k} \big) \,, \\ g^k(w) &:= \begin{cases} (g(w) + \frac{1}{k}) \wedge \sigma^{\mathrm{ult}} & \text{for } w \geq 0, \\ -k & \text{for } w \leq 0, \end{cases} \end{aligned}$$

respectively. Taking the limit as $k \to +\infty$ and then as $\delta \to 0$ we obtain that $u_s' \geq 0$ in $(0, l)$ and

$$\liminf_{n \to +\infty} \mathcal{E}_n(u_n) \geq \int_0^l F(\dot{u})\, dx + \sum_{S_u} G([u]) + \sigma^{\mathrm{ult}} u_c'(0, l) \,.$$

Let $\mathcal{E}''$ be the $\Gamma$-limsup in $L^1(0, l)$ of the sequence $\{\mathcal{E}_n\}$ (see [9], Definition 4.1). To conclude the proof of the $\Gamma$-convergence it remains to show that for every function $u \in BV(0, l)$ with $u_s' \geq 0$ we have $\mathcal{E}''(u) \leq \overline{\mathcal{E}}(u)$; i.e., there exists a sequence $\{u_n\}$ of functions in $\mathcal{A}_n$ which converges to $u$ in $L^1(0, l)$ such that $\limsup_n \mathcal{E}_n(u_n) \leq \overline{\mathcal{E}}(u)$.

Let us define

$$\tilde{F}_n(\varepsilon) := \int_0^\varepsilon f_n(s) \wedge \sigma_*\, ds, \qquad \tilde{F}(\varepsilon) := \int_0^\varepsilon f(s) \wedge \sigma_*\, ds.$$

Then $\tilde{F}_n \geq \Psi_n$, $\tilde{F} \geq \overline{F}$, and $\tilde{F}_n \to \tilde{F}$ uniformly on compact sets of $(\varepsilon^{\min}, +\infty)$.

Let us consider first a function $u \in SBV(0,l)$ such that $\#S_u < +\infty$, $[u] \geq 0$ in $S_u$, and $\varepsilon_1 \leq \dot u \leq \varepsilon_2$ a.e. in $(0,l)$, with $\varepsilon^{\min} < \varepsilon_1 < \varepsilon_2 < +\infty$. If $S_u = \emptyset$, we choose $u_n$ to be the affine interpolation of $u$ on $\{x_n^0, \dots, x_n^n\}$ and we get

$$\limsup_{n\to+\infty} \mathcal{E}_n(u_n) \leq \lim_{n\to+\infty} \sum_{i=1}^n \tilde F_n\Big(\frac{u(x_n^i) - u(x_n^{i-1})}{\lambda_n}\Big) \leq \int_0^l \tilde F(\dot u)\,dx\,,$$

where the last inequality follows from the uniform convergence of $\{\tilde F_n\}$ in the interval $[\varepsilon_1, \varepsilon_2]$ and from Jensen's inequality. Since $u \in W^{1,\infty}(0,l)$ in this case, it is easy to see that $\{u_n\}$ converges to $u$ in $W^{1,1}(0,l)$.

If $S_u \neq \emptyset$, by the local character of our arguments it is not restrictive to assume that $S_u$ contains exactly one point $x_0 \in (0,l)$. Hence, we can write $u = v + w$, where $v$ is a Lipschitz function in $[0,l]$ and $w = [u](x_0)\chi_{(x_0,l)}$. Let $v_n$ and $w_n$ be the affine interpolations of the values of $v$ and $w$ on the points $\{x_n^i\}$. It is easy to see that $\{v_n\}$ converges to $v$ in $W^{1,1}(0,l)$ and $\{w_n\}$ converges to $w$ in $L^1(0,l)$. Note that we have $\varepsilon_1 \leq \dot v_n \leq \varepsilon_2$ a.e. in $(0,l)$. We define $u_n = v_n + w_n$, which turns out to be the affine interpolation of the values of $u$ on the points $\{x_n^i\}$.

Let $i_n$ be the integer such that $x_0 \in [x^n_{i_n-1}, x^n_{i_n})$ and let $I_n$ and $J_n$ be defined as in the first part of the proof. Then $i_n \in J_n$ for $n$ large enough and, being $\Psi_n \leq \tilde F_n$, from Jensen's inequality we obtain

$$\begin{aligned}\mathcal{E}_n(u_n) \;=\; &\sum_{i\neq i_n} \lambda_n \Psi_n\Big(\frac{v(x_n^i) - v(x_n^{i-1})}{\lambda_n}\Big) + \lambda_n\, F_n(\varepsilon_n^{\mathrm{ult}}) \\ &+ G_n(v(x_n^{i_n}) - v(x_n^{i_n-1}) + [u](x_0) - \lambda_n \varepsilon_n^{\mathrm{ult}}) \\ \leq &\int_0^l \tilde F_n(\dot v)\,dx + \lambda_n\, F_n(\varepsilon_n^{\mathrm{ult}}) + G_n(v(x_n^{i_n}) - v(x_n^{i_n-1}) + [u](x_0))\,.\end{aligned}$$

From the uniform convergence of $\tilde F_n$ and $G_n$ we obtain that

$$\limsup_{n\to+\infty} \mathcal{E}_n(u_n) \leq \int_0^l \tilde F(\dot u)\,dx + G([u](x_0))\,.$$

Since this argument can be adapted to every function $u \in SBV(0,l)$ such that $\#S_u < +\infty$, $[u] \geq 0$ in $S_u$, and $\varepsilon_1 \leq \dot u \leq \varepsilon_2$ a.e. in $(0,l)$, with $\varepsilon^{\min} < \varepsilon_1 < \varepsilon_2 < +\infty$, for these functions we obtain

$$\mathcal{E}''(u) \leq \int_0^l \tilde F(\dot u)\,dx + \sum_{S_u} G([u])\,. \tag{3.18}$$

Let us consider now the general case of a function $u \in SBV(0,l)$ with positive jumps. It is not restrictive to suppose that $\dot{u} \geq \varepsilon^{\min}$ a.e. in $(0,l)$, otherwise the right hand side of (3.18) is $+\infty$. Let $S_u = \{x_1, x_2, \ldots\}$ and let $\varepsilon_k \to \varepsilon^{\min}$ such that $F(\varepsilon_k) < +\infty$. Let $u_k$ be the unique function in $SBV(0,l)$ which satisfies $u_k(0+) = u(0+)$ and

$$u_k' = \big((\dot{u} \vee \varepsilon_k) \wedge k\big)\, dx + \sum_{j=1}^{k} [u](x_j)\delta_{x_j}\,.$$

Since $u_k$ satisfies the conditions required in the previous step, by the lower semicontinuity of the $\Gamma$-limsup (see [9], Proposition 6.8) we have

$$\mathcal{E}''(u) \leq \liminf_{k\to+\infty} \mathcal{E}''(u_k) \leq \int_0^l \tilde{F}(\dot{u})\, dx + \sum_{S_u} G([u])\,.$$

If $\sigma^{\mathrm{ult}} > 0$, then $0 < G'(0+) = \sigma^{\mathrm{ult}} \leq \sigma_*$ and hence also $\tilde{F}' \wedge \sigma^{\mathrm{ult}} = f \wedge \sigma^{\mathrm{ult}}$ a.e. on $[0,+\infty)$. Therefore we can apply the relaxation results of [6] to obtain that $\overline{\mathcal{E}}$ is the lower semicontinuous envelope in $L^1(0,l)$ of the functional $\tilde{\mathcal{E}}$ defined by

$$\tilde{\mathcal{E}}(u) = \int_0^l \tilde{F}(\dot{u})\, dx + \sum_{S_u} G([u])\,,$$

if $u \in SBV(0,l)$ and $[u] \geq 0$, and by $\tilde{\mathcal{E}}(u) = +\infty$ for all other functions $u \in L^1(0,l)$. By the lower semicontinuity of $\mathcal{E}''$ this implies again that $\mathcal{E}'' \leq \overline{\mathcal{E}}$, as required.

If $\sigma^{\mathrm{ult}} = 0$, we can argue by comparison. Let $\mathcal{E}_n^k$, $k > 0$, be the functionals with integrands $\psi_n + \frac{1}{k}\chi_{(0,+\infty)}$. If $u \in BV(0,l)$ then by the previous step we have

$$\mathcal{E}''(u) = \Gamma\text{-}\limsup_{n\to+\infty} \mathcal{E}_n(u) \leq \Gamma\text{-}\limsup_{n\to+\infty} \mathcal{E}_n^k(u) \leq \int_0^l F(\dot{u} \wedge 0)\, dx + \frac{1}{k}(u')^+(0,l)\,,$$

and by the arbitrariness of $k$ we get $\mathcal{E}''(u) \leq \overline{\mathcal{E}}(u)$. If $u \in BV_{\mathrm{loc}}(0,l)$ and $\overline{\mathcal{E}}(u) < +\infty$, then $\dot{u} \wedge 0 \in L^1(0,l)$ by (3.3) and (3.12); consequently the functions $u_j = (-j) \vee (u \wedge j)$ belong to $BV(0,l)$ and $\overline{\mathcal{E}}(u_j) \to \overline{\mathcal{E}}(u)$; hence by the lower semicontinuity of $\mathcal{E}''$ we get

$$\mathcal{E}''(u) \leq \liminf_{j\to+\infty} \mathcal{E}''(u_j) \leq \lim_{j\to+\infty} \overline{\mathcal{E}}(u_j) = \overline{\mathcal{E}}(u),$$

as required. $\square$

**Remark 3.4.** Suppose that $\{\psi_n\}$, $\{f_n\}$, and $\{g_n\}$ satisfy all properties considered in this section, except the almost everywhere convergence. Using the compactness properties of monotone functions and the previous theorem it is easy to prove that the sequence $\{\mathcal{E}_n\}$ $\Gamma$-converges if and only if $\{g_n(w)\}$ converges to $g(w)$ for a.e. $w > 0$, $\sigma^{\rm ult} = g(0+)$, and $\{f_n(\varepsilon) \wedge \sigma^{\rm ult}\}$ converges to $f(\varepsilon) \wedge \sigma^{\rm ult}$ for a.e. $\varepsilon \in \mathbf{R}$.

## 4. Dirichlet Boundary Conditions

In order to study the convergence of the solutions of minimum problems for the discrete energies $\mathcal{E}_n$ with prescribed displacements at the boundary points $x = 0$ and $x = l$, we have to investigate the behaviour of some functionals which take these boundary conditions into account .

Let $d \in \mathbf{R}$; we consider the functionals $\mathcal{E}_n^d \colon L^1(0,l) \to [0,+\infty]$ defined by $\mathcal{E}_n^d(u_n) = \mathcal{E}_n(u_n)$, if $u \in \mathcal{A}_n$, $u_n(0) = 0$, $u_n(l) = d$, and by $\mathcal{E}_n^d(u) = +\infty$ for every other function of $L^1(0,l)$. We assume that $\psi_n$, $f_n$, and $g_n$ satisfy all the hypotheses of the previous section and that $\{f_n\}$ converges to $f$ pointwise on $\mathbf{R}$ and $\{g_n\}$ converges to $g$ pointwise on $[0,+\infty)$. For every $u \in BV(0,l)$ and for every $x \in [0,l]$ we set $[u](x) = u(x+) - u(x-)$, where we put $u(0-) = 0$ and $u(l+) = d$. Then we define $S_u^d = \{x \in [0,l] : [u](x) \neq 0\}$ and we extend the measures $u'$ and $u'_s$ to $[0,l]$ by setting

$$u' = \dot{u}\,dx + \sum_{x \in S_u^d} [u](x)\,\delta_x + u'_c\,, \qquad u'_s = \sum_{x \in S_u^d} [u](x)\,\delta_x + u'_c\,. \tag{4.1}$$

Note that, if $v \in BV_{\rm loc}(\mathbf{R})$ is the extension of $u$ defined by $v(x) = 0$ for $x \le 0$ and $v(x) = d$ for $x \ge l$, then $u'$ and $u'_s$ are the restrictions to $[0,l]$ of the distributional derivative $v'$ and of its singular part $v'_s$. Note also that for every $u \in BV(0,l)$ we have

$$u'([0,l]) = \int_0^l \dot{u}\,dx + \sum_{S_u^d} [u] + u'_c(0,l) = d \tag{4.2}$$

and that $u$ is uniquely determined by the measure $u'$ on $[0,l]$.

Let $\mathcal{E}^d \colon L^1(0,l) \to [0,+\infty]$ be the functional defined by

$$\mathcal{E}^d(u) = \int_0^l F(\dot{u})\,dx + \sum_{S_u^d} G([u]) = \mathcal{E}(u) + G(u(0+)) + G(d - u(l-))\,, \tag{4.3}$$

if $u \in SBV(0,l)$ and $[u] \geq 0$ in $[0,l]$ (in particular, $u(0+) \geq 0$ and $u(l-) \leq d$), while $\mathcal{E}^d(u) = +\infty$ for all other functions of $L^1(0,l)$. Moreover we consider the functional $\overline{\mathcal{E}}^d : L^1(0,l) \to [0,+\infty]$ defined by

$$(4.4)\quad \overline{\mathcal{E}}^d(u) = \int_0^l \overline{F}(\dot{u})\,dx + \sigma^{\mathrm{ult}} u_c'(0,l) + \sum_{S_u^d} G([u]) = \overline{\mathcal{E}}(u) + G(u(0+)) + G(d - u(l-))\,,$$

if $u \in BV(0,l)$ and $u_s' \geq 0$ in $[0,l]$ (in particular, $u(0+) \geq 0$ and $u(l-) \leq d$), while $\overline{\mathcal{E}}^d(u) = +\infty$ for all other functions of $L^1(0,l)$. Using the arguments of [5] it would be possible to show that $\overline{\mathcal{E}}^d$ is the lower semicontinuous envelope of $\mathcal{E}^d$ in $L^1(0,l)$, but this property will never be used in this paper. In this section we show that the sequence $\{\mathcal{E}_n^d\}$ $\Gamma$-converges to $\overline{\mathcal{E}}^d$ in $L^1(0,l)$. In the next section we shall prove the convergence of minimizers. In Sections 6 and 7 we shall show that $\mathcal{E}^d$ and $\overline{\mathcal{E}}^d$ have the same minimizers, and that this is still true if we add a term corresponding to a dead load.

**Remark 4.1.** Note that also in the case $\sigma^{\mathrm{ult}} = 0$ the functionals $\mathcal{E}^d$ and $\overline{\mathcal{E}}^d$ are infinite on $BV_{\mathrm{loc}}(0,l) \setminus BV(0,l)$, in contrast with $\mathcal{E}$ and $\overline{\mathcal{E}}$. This is explained by the fact that $BV_{\mathrm{loc}}$ functions satisfying $\overline{\mathcal{E}}(u) < +\infty$, and the boundary conditions $u(0+) \geq 0$ and $u(l-) \leq d$ are indeed in $BV(0,l)$.

**Theorem 4.2.** *If $d > l\,\varepsilon^{\min}$, then the sequence $\{\mathcal{E}_n^d\}$ $\Gamma$-converges to $\overline{\mathcal{E}}^d$ in $L^1(0,l)$.*

*Proof.* Let us preliminarily note that the result stated in Theorem 3.3 holds on every interval $I \subset \mathbf{R}$. Namely, let

$$\mathcal{E}_n(u,I) = \begin{cases} \displaystyle\int_I \Psi_n(\dot{u})\,dx & \text{if } u \in \mathcal{A}_n(I), \\ +\infty & \text{if } u \in L^1(I) \setminus \mathcal{A}_n(I), \end{cases}$$

where $\mathcal{A}_n(I)$ is the space of all continuous functions $u: I \to \mathbf{R}$ which are affine on the intervals $[x_n^{i-1}, x_n^i] \cap I$, with $x_n^i = \lambda_n i$, $i \in \mathbf{Z}$. Then repeating the arguments of Theorem 3.3 we have that

$$(4.5)\qquad \overline{\mathcal{E}}(u,I) = \Gamma\text{-}\lim_{n\to\infty} \mathcal{E}_n(u,I) \qquad \text{in } L^1(I)\,,$$

where

$$\overline{\mathcal{E}}(u,I) = \int_I \overline{F}(\dot{u})\,dx + \sigma^{\mathrm{ult}} u_c'(I) + \sum_{S_u} G([u])\,,$$

if $u \in BV_{\rm loc}(I)$ and $u'_s \geq 0$ in $I$, while $\overline{\mathcal{E}}(u,I) = +\infty$ for all other functions of $L^1(I)$.

In order to prove the lower semicontinuity inequality for $\{\mathcal{E}^d_n\}$, let $u_n \in \mathcal{A}_n$ with $u_n(0) = 0$, $u_n(d) = d$, and $u_n \to u$ in $L^1(0,l)$. Define the auxiliary functions

$$v_n(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ u_n(x) & \text{for } 0 \leq x \leq l, \\ d & \text{for } l \leq x, \end{cases} \qquad v(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ u(x) & \text{for } 0 \leq x \leq l, \\ d & \text{for } l \leq x. \end{cases}$$

Let us fix two constants $\alpha$ and $\beta$ with $\alpha < 0 < l < \beta$. If we apply (4.5) with $I = (\alpha,\beta)$, we get in particular

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) = \liminf_{n\to+\infty} \int_\alpha^\beta \Psi_n(\dot v_n)\,dx \geq \overline{\mathcal{E}}(v,(\alpha,\beta)) = \overline{\mathcal{E}}^d(u)\,,$$

as required.

In order to prove the limsup inequality, we choose $u$ such that $\overline{\mathcal{E}}^d(u) < +\infty$, $u(0+) > 0$, and $u(l-) < d$. As above let us extend $u$ to the function $v$ defined in $(\alpha,\beta)$. Then by (4.5) there exists a sequence $\{u_n\}$, with $u_n \in \mathcal{A}_n(\alpha,\beta)$, which converges to $v$ in $L^1(\alpha,\beta)$, such that

$$\lim_{n\to+\infty} \int_\alpha^\beta \Psi_n(\dot u_n)\,dx = \overline{\mathcal{E}}(v,(\alpha,\beta))\,. \tag{4.6}$$

Let us fix two points $a$ and $b$ such that $0 < a < b < l$, $a \notin S_u$, $b \notin S_u$, $u_n(a) \to u(a) = u(a-) = u(a+) > 0$, and $u_n(b) \to u(b) = u(b-) = u(b+) < d$. Using the lower semicontinuity inequality given by (4.5) for the intervals $I = (\alpha,a)$ and $I = (b,\beta)$, from (4.6) we obtain by difference

$$\limsup_{n\to+\infty} \int_a^b \Psi_n(\dot u_n)\,dx \leq \int_a^b \overline{F}(\dot u)\,dx + \sum_{S_u\cap(a,b)} G([u]) + \sigma^{\rm ult} u'_c(a,b)\,. \tag{4.7}$$

Define $v_n \in \mathcal{A}_n$ by

$$v_n(x^i_n) = \begin{cases} 0 & \text{if } i < j_n - 1, \\ u_n(a) & \text{if } i = j_n - 1, \\ u_n(x^i_n) & \text{if } j_n - 1 < i \leq k_n - 1, \\ u_n(b) & \text{if } i = k_n, \\ d & \text{if } i > k_n, \end{cases}$$

where $j_n$ and $k_n$ are the indices such that $a \in [x^{j_n-1}_n, x^{j_n}_n)$ and $b \in [x^{k_n-1}_n, x^{k_n}_n)$

We have $v_n(0) = 0$ and $v_n(l) = d$ for $n$ large enough. Moreover $\{v_n\}$ converges to $u_{a,b} := u\chi_{(a,b)} + d\chi_{(b,l)}$. Note that $\Psi_n(\dot{v}_n(x)) \leq \Psi_n(\dot{u}_n(x))$ for almost every $x \in (x_n^{j_n-1}, x_n^{k_n})$.

For $\delta > 0$ and for $n$ large enough we obtain

$$\begin{aligned}\int_0^l \Psi_n(\dot{v}_n)\,dx &\leq \lambda_n \Psi_n\Big(\frac{u_n(a)}{\lambda_n}\Big) + \int_{x_n^{j_n-1}}^{x_n^{k_n}} \Psi_n(\dot{u}_n)\,dx + \lambda_n \Psi_n\Big(\frac{d-u_n(b)}{\lambda_n}\Big)\\ &\leq \int_{a-\delta}^{b+\delta} \Psi_n(\dot{u}_n)\,dx + 2\lambda_n F_n(\varepsilon_n^{\mathrm{ult}}) + G_n(u_n(a)) + G_n(d-u_n(b)),\end{aligned}$$

and taking the limit as $n \to +\infty$, by (4.7) and the arbitrarines of $\delta > 0$,

$$\limsup_{n\to+\infty} \int_0^l \Psi_n(\dot{v}_n)\,dx \leq \overline{\mathcal{E}}(u,(a,b)) + G(u(a)) + G(d-u(b)) = \overline{\mathcal{E}}^d(u_{a,b}).$$

We have then $\Gamma\text{-}\limsup_n \mathcal{E}_n^d(u_{a,b}) \leq \overline{\mathcal{E}}^d(u_{a,b})$; letting $a \to 0$ and $b \to l$, by the lower semicontinuity of the $\Gamma$-limsup we eventually deduce $\Gamma\text{-}\limsup_n \mathcal{E}_n^d(u) \leq \overline{\mathcal{E}}^d(u)$.

If $\overline{\mathcal{E}}^d(u) < +\infty$ and $u(0+) = 0$ or $u(l-) = d$, the inequality $d > l\,\varepsilon^{\min}$ implies that there exists a sequence $\{u_j\}$ converging to $u$ uniformly in $[0,l]$ such that $\int_0^l \overline{F}(\dot{u}_j)\,dx \to \int_0^l \overline{F}(\dot{u})\,dx$, $|u_j' - u'|(0,l) \to 0$, $u_j(0+) > 0$, and $u_j(l-) < d$. By the previous step we have $\Gamma\text{-}\limsup_n \mathcal{E}_n^d(u_j) \leq \overline{\mathcal{E}}^d(u_j)$ for every $j$. Passing to the limit as $j \to +\infty$, by the lower semicontinuity of the $\Gamma$-limsup we eventually obtain $\Gamma\text{-}\limsup_n \mathcal{E}_n^d(u) \leq \overline{\mathcal{E}}^d(u)$, as required. $\square$

## 5. Boundary Value Problems with External Forces

In this section we consider the limit, as $n$ tends to infinity, of the minimum points of the problems (2.6) involving an external force $\{\lambda_n h_n^i\}$. We assume that $\psi_n$, $f_n$, $g_n$, $f$ and $g$ satisfy all hypotheses of the Section 4. Let $\tilde{h}_n \in L^\infty(0,l)$ be the piecewise constant function defined by $\tilde{h}_n(x) = h_n^i$ for $x_n^{i-1} < x \leq x_n^i$, $i = 1,\dots,n$, where we set $h_n^n = 0$. We assume that the sequence $\{\tilde{h}_n\}$ converges in $L^1(0,l)$ to a function $h$.

We are now in a position to state our main convergence result for energy minimizing configurations with prescribed displacements at the boundary.

**Theorem 5.1.** *Assume that $d > l\,\varepsilon^{\min}$. For every $n$ let $\{u_n^i\}$ be a minimum point of problem (2.6) and let $u_n$ be the affine interpolation of $\{u_n^i\}$ on the points $\{x_n^i\}$. Then the sequence $\{u_n\}$ is bounded in $BV(0,l)$ and a subsequence of $\{u_n\}$ converges in $L^1(0,l)$ to a minimum point $u$ of the problem*

$$\min\Big\{\overline{\mathcal{E}}^d(u) - \int_0^l hu\,dx : u \in BV(0,l)\Big\}. \tag{5.1}$$

*Moreover the minimum values of (2.6) converge to the minimum value of (5.1).*

To prove the theorem we need the following lemmas.

**Lemma 5.2.** *For every $n$ let $v_n \in \mathcal{A}_n$ with $v_n(0) = 0$ and $v_n(l) = d$. Suppose that the sequence $\big\{E_n^h(\{v_n^i\})\big\}$ is bounded. Then the sequence $\{\dot{v}_n\}$ is bounded in $L^1(0,l)$.*

*Proof.* As $|v_n^i| = |\int_0^{x_n^i} \dot{v}_n dx| \le \int_0^l |\dot{v}_n|\,dx$, we have

$$\sum_{i=1}^{n-1} \lambda_n h_n^i v_n^i \le \int_0^l |\tilde{h}_n|\,dx \int_0^l |\dot{v}_n|\,dx. \tag{5.2}$$

Since $\{\tilde{h}_n\}$ is bounded in $L^1(0,l)$ and $\big\{E_n^h(\{v_n^i\})\big\}$ is bounded, it follows from (3.2), (2.5), and (5.2) that there exist two constants $A$ and $B$ such that

$$\int_0^l \Psi_*(\dot{v}_n \wedge 0)\,dx \le A \int_0^l |\dot{v}_n|\,dx + B. \tag{5.3}$$

Since $\int_0^l (\dot{v}_n)^+ dx = d + \int_0^l (\dot{v}_n)^- dx$, we have $\int_0^l |\dot{v}_n|\,dx \le d + 2\int_0^l (\dot{v}_n)^- dx$, so that, by (5.3) and (3.3) the sequences $\{(\dot{v}_n)^-\}$ and $\{\dot{v}_n\}$ are bounded in $L^1(0,l)$. ☐

**Lemma 5.3.** *For every $n$ let $v_n \in \mathcal{A}_n$. Suppose that $\{v_n\}$ converges in $L^1(0,l)$ to a function $v$ and that $\{\dot{v}_n\}$ is bounded in $L^1(0,l)$. Then*

$$\lim_{n\to+\infty} \sum_{i=1}^{n-1} \lambda_n h_n^i v_n^i = \int_0^l hv\,dx. \tag{5.4}$$

*Proof.* Let $\tilde{v}_n \in L^\infty(0,l)$ be the piecewise constant function defined by $\tilde{v}_n(x) = v_n^i$ for $x_n^{i-1} < x \le x_n^i$, $i = 1, \dots, n$. It is clear that

$$\sum_{i=1}^{n-1} \lambda_n h_n^i v_n^i = \int_0^l \tilde{h}_n \tilde{v}_n dx. \tag{5.5}$$

We want to prove that $\{\tilde v_n\}$ converges to $v$ in $L^1(0,l)$. Since $v_n$ is piecewise affine, by an elementary computation we obtain

$$\int_0^l |\tilde v_n - v_n|\,dx = \frac{\lambda_n}{2}\sum_{i=1}^n |v_n^i - v_n^{i-1}| \le \frac{\lambda_n}{2}\int_0^l |\dot v_n|\,dx\,.$$

As $\{\lambda_n\}$ tends to 0 and $\{\dot v_n\}$ is bounded in $L^1(0,l)$, we conclude that $\{\tilde v_n - v_n\}$ converges to 0 in $L^1(0,l)$, so that $\{\tilde v_n\}$ converges to $v$ in $L^1(0,l)$. Since $\{v_n\}$ is bounded in $L^\infty(0,l)$, the same property holds for $\{\tilde v_n\}$. Therefore (5.4) follows from (5.5) and from the convergence of $\{\tilde h_n\}$. □

*Proof of Theorem 5.1.* Let us prove that the sequence $\big\{E_n^h(\{u_n^i\})\big\}$ is bounded. If $d \ge 0$, we define $w_n^0 = 0$ and $w_n^i = d$ for $i = 1,\dots,n$. Then, by minimality,

$$E_n^h(\{u_n^i\}) \le E_n^h(\{w_n^i\}) \le \lambda_n F_n(\varepsilon_n^{\rm ult}) + G_n(d) - d\int_0^l \tilde h_n dx \le \sigma_*(\varepsilon_*\lambda_n + d) - d\int_0^l \tilde h_n dx\,.$$

Since the sequence $\{\tilde h_n\}$ is bounded in $L^1(0,l)$, the previous inequalities prove that $\big\{E_n^h(\{u_n^i\})\big\}$ is bounded. If $l\,\varepsilon^{\min} < d < 0$, we define $z_n^i = x_n^i d/l$. Then, by minimality,

$$E_n^h(\{u_n^i\}) \le E_n^h(\{z_n^i\}) \le l\,F_n(d/l) + d\int_0^l |\tilde h_n|\,dx\,.$$

Since $\varepsilon^{\min} < d/l < 0$, the sequence $\{F_n(d/l)\}$ converges to $F(d/l)$, and $\big\{E_n^h(\{u_n^i\})\big\}$ is bounded also in this case.

By Lemma 5.2 the sequence $\{\dot u_n\}$ is bounded in $L^1(0,l)$. Therefore there exists a subsequence, still denoted by $\{u_n\}$, which converges in $L^1(0,l)$ to a function $u$. Since $E_n^h(\{u_n^i\}) = \mathcal{E}_n^d(u_n) - \sum_i \lambda_n h_n^i u_n^i$, from Theorem 4.2 and Lemma 5.3 we obtain that

$$\overline{\mathcal{E}}^d(u) - \int_0^l hu\,dx \le \liminf_{n\to+\infty} E_n^h(\{u_n^i\})\,. \tag{5.6}$$

Let $v \in L^1(0,l)$ with $\overline{\mathcal{E}}^d(v) < +\infty$. By Theorem 4.2 there exists a sequence $\{v_n\}$ converging to $v$ in $L^1(0,l)$ such that $\overline{\mathcal{E}}^d(v) = \lim_n \mathcal{E}_n^d(v_n)$. By Lemma 5.2, applied with $h_n^i = 0$, the sequence $\{\dot v_n\}$ is bounded in $L^1(0,l)$, so that, by Lemma 5.3, $\overline{\mathcal{E}}^d(v) - \int_0^l hv\,dx = \lim_n E_n^h(\{v_n^i\})$. By minimality we have $E_n^h(\{v_n^i\}) \ge E_n^h(\{u_n^i\})$, and, consequently,

$$\overline{\mathcal{E}}^d(v) - \int_0^l hv\,dx \ge \limsup_{n\to+\infty} E_n^h(\{u_n^i\})\,. \tag{5.7}$$

From (5.6) and (5.7) we obtain that $u$ is a minimum point of (5.1) and that

$$\overline{\mathcal{E}}^d(u) - \int_0^l hu\,dx = \lim_{n\to+\infty} E_n^h(\{u_n^i\})\,.$$

Since this result does not depend on the subsequence, we conclude that the minimum values of (2.6) converge to the minimum value of (5.1). $\square$

Theorem 5.1 can be easily extended to the more general minimum problem

$$\min\Big\{\overline{\mathcal{E}}^d(u) + \int_0^l b(x,u)\,dx : u \in BV(0,l)\Big\}\,, \tag{5.8}$$

where $b\colon [0,l]\times\mathbf{R} \to \mathbf{R}$ is a Carathéodory function, i.e., $b(x,w)$ is measurable in $x$ and continuous in $w$, and

$$\int_0^l \max_{|w|\le r} |b(x,w)|\,dx < +\infty \qquad \text{for every } r \in [0,+\infty)\,. \tag{5.9}$$

For every integer $n > 1$ and for every $i = 1,\dots,n-1$ let $b_n^i\colon \mathbf{R} \to \mathbf{R}$ be a function such that

$$b_n^i(w) \ge -a_n^i|w| \qquad \text{for every } w \in \mathbf{R}\,, \tag{5.10}$$

where the coefficients $a_n^i \ge 0$ satisfy the inequality

$$\sum_{i=1}^{n-1} \lambda_n a_n^i \le A \tag{5.11}$$

with a constant $A < +\infty$ independent of $n$. Assume that

$$\lim_{n\to+\infty} \sum_{i=1}^{n-1} \lambda_n b_n^i(v_n^i) = \int_0^l b(x,v)\,dx \tag{5.12}$$

whenever the affine interpolations $v_n$ of $\{v_n^i\}$ converge to $v$ in $L^1(0,l)$ as $n \to +\infty$ and their derivatives $\dot v_n$ are bounded in $L^1(0,l)$.

Condition (5.12) is satisfied, for instance, when $b$ is continuous and $b_n^i(w) = b(x_n^i, w)$. Another interesting case is when $b(x,w) = |w - h(x)|^p$, with $h \in L^p(0,l)$ and $p \ge 1$. Then (5.12) is satisfied by $b_n^i(w) = |w - h_n^i|^p$, provided $h$ is the $L^p(0,l)$-limit of the piecewise constant functions $\tilde h_n$ associated with $\{h_n^i\}$ (the definition of $\tilde h_n$ is given at the beginning of the section). In both cases the proof can be obtained as in Lemma 5.3.

**Theorem 5.4.** *Assume that conditions (5.9)–(5.12) are satisfied and that* $d > l\,\varepsilon^{\min}$. *For every* $n$ *let* $\{u_n^i\}$ *be a minimum point of the problem*

$$\min\Big\{E_n(\{u_n^i\}) + \sum_{i=1}^{n-1} \lambda_n b_n^i(u_n^i) : u_n^0 = 0,\ u_n^n = d\Big\}, \tag{5.13}$$

*and let* $u_n$ *be the affine interpolation of* $\{u_n^i\}$ *on the points* $\{x_n^i\}$. *Then the sequence* $\{u_n\}$ *is bounded in* $BV(0,l)$ *and a subsequence of* $\{u_n\}$ *converges in* $L^1(0,l)$ *to a minimum point* $u$ *of problem (5.8). Moreover the minimum values of (5.13) converge to the minimum value of (5.8).*

*Proof.* It is enough to adapt the proof of Theorem 5.1. □

## 6. Stationary Configurations for the Continuous Model

In this section we study in detail the properties of the stationary points for the functionals $\mathcal{E}^d$ and $\overline{\mathcal{E}}^d$ considered in the previous sections. In order to simplify the exposition we make some additional continuity and monotonicity assumptions on the functions $f$ and $g$.

We suppose that $f\colon \mathbf{R} \to [-\infty, +\infty]$ and $g\colon [0,+\infty) \to [0,+\infty)$ are continuous and that $f(0) = 0$ and $g(0) = \sigma^{\rm ult} > 0$. Moreover we assume that $f$ is non-decreasing on $\mathbf{R}$ and increasing on $(\varepsilon^{\min}, \varepsilon^{\max}) := \{|f| < +\infty\}$, while $g$ is non-increasing on $[0,+\infty)$ and decreasing on $[0, w^{\rm frac}) := \{g > 0\}$. Finally, we assume that

$$\lim_{\varepsilon\to-\infty} f(\varepsilon) = -\infty \qquad \text{and} \qquad \lim_{\varepsilon\to+\infty} f(\varepsilon) > \sigma^{\rm ult},$$

so that there exists a unique $\varepsilon^{\rm ult} > 0$ such that $f(\varepsilon^{\rm ult}) = \sigma^{\rm ult}$. Note that, by continuity, $f(\varepsilon^{\min}) = -\infty$ if $\varepsilon^{\min}) > -\infty$.

Let $h \in L^1(0,l)$ and let $H$ be its primitive vanishing at $0$. The function $h$ plays the role of the density of an external force, while $H(x)$ is the total force acting on the segment between 0 and $x$. Using integration by parts the minimum problem (5.1) can be written as

$$\min\Big\{\overline{\mathcal{E}}^d(u) + \int_{[0,l]} Hu' - H(l)d : u \in BV(0,l)\Big\},$$

where $\int_{[0,l]} Hu'$ denotes the integral of the continuous function $H$ with respect to the measure $u'$ defined in (4.1), which takes into account the boundary conditions $u(0-) = 0$

and $u(l+) = d$. Similarly, the minimum problem

$$\min\Big\{\mathcal{E}^d(u) - \int_0^l hu\,dx : u \in SBV(0,l)\Big\},$$

can be written as

$$\min\Big\{\mathcal{E}^d(u) + \int_{[0,l]} Hu' - H(l)d : u \in SBV(0,l)\Big\}.$$

In order to study the properties of the solutions of these minimum problems we introduce the functionals $\mathcal{E}^d_H : SBV(0,l) \to (-\infty, +\infty]$ and $\overline{\mathcal{E}}^d_H : BV(0,l) \to (-\infty, +\infty]$ defined by

$$\mathcal{E}^d_H(u) = \mathcal{E}^d(u) + \int_{[0,l]} Hu', \qquad \overline{\mathcal{E}}^d_H(u) = \overline{\mathcal{E}}^d(u) + \int_{[0,l]} Hu', \tag{6.1}$$

where $\mathcal{E}^d$ and $\overline{\mathcal{E}}^d$ are the functionals defined in (4.3) and (4.4).

We recall the definition of stationary point.

**Definition 6.1.** Let $\mathcal{U}$ be a vector space, and let $\mathcal{F}: \mathcal{U} \to (-\infty, +\infty]$ be a function. We say that a point $u \in \mathcal{U}$ is a (lower) *stationary point* for $\mathcal{F}$ in $\mathcal{U}$ if $\mathcal{F}(u) < +\infty$ and

$$\liminf_{t \to 0^+} \frac{\mathcal{F}(u + tv) - \mathcal{F}(u)}{t} \geq 0 \tag{6.2}$$

for every $v \in \mathcal{U}$.

The next theorems establish the Euler conditions which are necessary and sufficient for the stationarity of $\mathcal{E}^d_H$ in $SBV(0,l)$ and of $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$.

**Theorem 6.2.** *A function $u \in SBV(0,l)$ is a stationary point for $\mathcal{E}^d_H$ in $SBV(0,l)$ if and only if there exists a constant $\sigma$ such that the following conditions are satisfied:*
(i) $-\infty \leq \sigma \leq \sigma^{\mathrm{ult}} + H$ *in* $(0,l)$*;*
(ii) $F(\dot{u}) < +\infty$ *a.e. in* $(0,l)$*;*
(iii) $[u] \geq 0$ *in* $[0,l]$*;*
(iv) $f(\dot{u}) + H = \sigma$ *a.e. in* $(0,l)$*;*
(v) $g([u]) + H = \sigma$ *in* $S^d_u$*.*
*In this case we have $\dot{u} < \varepsilon^{\mathrm{ult}}$ a.e. in $\{\sigma < \sigma^{\mathrm{ult}} + H\}$ and $\dot{u} = \varepsilon^{\mathrm{ult}}$ a.e. in $\{\sigma = \sigma^{\mathrm{ult}} + H\}$. If $\sigma > -\infty$, then there exists a constant $\varepsilon_u > \varepsilon^{\min}$ such that $\dot{u} \geq \varepsilon_u$ a.e. in $(0,l)$, and*

(ii) *follows from* (i) *and* (iv). *The case* $\sigma = -\infty$ *occurs if and only if* $F(\varepsilon^{\min}) < +\infty$, $d = l\,\varepsilon^{\min}$, *and* $u(x) = \varepsilon^{\min} x$ *for a.e.* $x \in (0,l)$.

*Proof.* Assume that $u$ is stationary point for $\mathcal{E}^d_H$ in $SBV(0,l)$. Then (ii) and (iii) follow from the fact that $\mathcal{E}^d_H(u) < +\infty$ and from (4.3).

To prove (iv), we want to show that the function $z = f(\dot u) + H$ is constant a.e. on $(0,l)$. If not, there exist two constants $c_1 < c_2$ such that both sets $E_1 := \{z \le c_1\}$ and $E_2 := \{z \ge c_2\}$ have positive measure. Note that $f(\dot u)$ is bounded from above in $E_1$, and this implies that $f(\dot u + \eta)$ is bounded from above in $E_1$ for $\eta$ small enough. In a similar way we prove that $f(\dot u - \eta)$ is bounded from below in $E_2$ for $\eta$ small enough. Let $\psi \in W^{1,1}_0(0,l)$ be the function defined by $\dot\psi = |E_2|\chi_{E_1} - |E_1|\chi_{E_2}$ a.e. in $(0,l)$. Then, using the monotonicity of $f$, for every $t > 0$ we have

$$
\begin{aligned}
\frac{\mathcal{E}^d_H(u+t\psi) - \mathcal{E}^d_H(u)}{t} &= \int_{E_1} \frac{F(\dot u + t|E_2|) - F(\dot u)}{t}\,dx \\
&+ \int_{E_2} \frac{F(\dot u - t|E_1|) - F(\dot u)}{t}\,dx + |E_2|\int_{E_1} H\,dx - |E_1|\int_{E_2} H\,dx \\
&\le |E_2|\int_{E_1}\big(f(\dot u + t|E_2|) + H\big)\,dx - |E_1|\int_{E_2}\big(f(\dot u - t|E_1|) + H\big)\,dx\,.
\end{aligned}
$$

By the unilateral boundedness of $f(\dot u \pm \eta)$ in $E_1$ and $E_2$, and by the monotone convergence theorem we obtain

$$
\limsup_{t\to 0^+} \frac{\mathcal{E}^d_H(u+t\psi) - \mathcal{E}^d_H(u)}{t} \le |E_1||E_2|(c_1 - c_2) < 0\,,
$$

which contradicts the stationarity of $u$ (Definition 6.1). Hence, $z$ is constant a.e. in $(0,l)$, i.e., there exists a constant $\sigma \in [-\infty, +\infty]$ such that (iv) holds. This implies $f(\dot u) < \sigma^{\mathrm{ult}}$ a.e. in $\{\sigma < \sigma^{\mathrm{ult}} + H\}$, so that $\dot u < \varepsilon^{\mathrm{ult}}$ in the same set by the strict monotonicity of $f$. Moreover $f(\dot u) = \sigma^{\mathrm{ult}}$ a.e. in $\{\sigma = \sigma^{\mathrm{ult}} + H\}$, so that $\dot u = \varepsilon^{\mathrm{ult}}$ a.e. in the same set.

If $\sigma = -\infty$, then (iv) gives $f(\dot u) = -\infty$ a.e. in $(0,l)$; since $\mathcal{E}^d_H(u) < +\infty$, this implies $\dot u = \varepsilon^{\min}$ a.e. in $(0,l)$ and $F(\varepsilon^{\min}) < +\infty$. As $H$ is bounded, (iv) implies that $f(\dot u)$ is bounded if $\sigma \in \mathbf{R}$; consequently $f(\dot u + \eta)$ is bounded if $\eta$ is a small constant and $\dot u \ge \varepsilon_u$ a.e. in $(0,l)$ for a suitable a constant $\varepsilon_u > \varepsilon^{\min}$.

To prove (i) and (v) we fix $x_0 \in [0,l]$ and we consider the function $\psi \in SBV(0,l)$ defined by $\psi(x) = -x/l$, if $x < x_0$, and by $\psi(x) = (l-x)/l$, if $x > x_0$. Using $v = u + \psi$

in Definition 6.1, by the monotone convergence theorem we obtain from (iv)

$$
\begin{aligned}
0 &\le \lim_{t\to 0^+} \frac{\mathcal{E}^d_H(u+t\psi)-\mathcal{E}^d_H(u)}{t} \\
&= \int_0^l (f(\dot u)+H)\dot\psi\,dx + g([u](x_0)) + H(x_0) \\
&= \sigma \int_0^l \dot\psi\,dx + g([u](x_0)) + H(x_0) \\
&= g([u](x_0)) + H(x_0) - \sigma\,.
\end{aligned}
\tag{6.3}
$$

If $x_0 \notin S^d_u$, then $g([u](x_0)) = \sigma^{\rm ult}$, so that $\sigma \le \sigma^{\rm ult} + H(x_0)$. By the arbitrariness of $x_0$ and the continuity of $H$ this implies (i).

If $x_0 \in S^d_u$, we can take $v = u - \psi$ in Definition 6.1. Since $[u - t\psi](x_0) > 0$ for $t > 0$ small enough, (6.3) gives $0 \le -g([u](x_0)) - H(x_0) + \sigma$, which, together with the opposite inequality, proves (v). This shows also that $S^d_u = \emptyset$ if $\sigma = -\infty$. Since in this case we have $\dot u = \varepsilon^{\min}$ a.e. in $(0,l)$, we conclude that $u(x) = \varepsilon^{\min}x$ for a.e. $x \in (0,l)$ and that $d = l\,\varepsilon^{\min}$.

If $\sigma > -\infty$, by (i) and (iv) we have $\varepsilon_u \le \dot u \le \varepsilon^{\rm ult}$ a.e. in $(0,l)$ for some constant $\varepsilon_u > \varepsilon^{\min}$, and this implies (ii).

Conversely, let us assume that $u$ satisfies (i)–(v) and let $v \in SBV(0,l)$. Using (iv) and the convexity of $F$ we get

$$
\begin{aligned}
\mathcal{E}^d_H(u+t(v-u)) - \mathcal{E}^d_H(u) \ge{}& t\int_0^l (f(\dot u)+H)(\dot v - \dot u)\,dx \\
&+ \sum_{S^d_u\cup S^d_v} \big(G([u]+t[v-u]) - G([u]) + tH[v-u]\big)\,.
\end{aligned}
\tag{6.4}
$$

By (iv) we have $f(\dot u) + H = \sigma$ a.e. in $(0,l)$, while (i) and (iv) give $g([u]) + H = \sigma$ in $S^d_u$ and $g([u]) + H \ge \sigma$ in $[0,l]$. If $\mathcal{E}^d_H(u+t(v-u)) < +\infty$ for some $t > 0$, then $[v-u] = (1/t)[u+t(v-u)] \ge 0$ in $S^d_v \setminus S^d_u$. Therefore $\big(g([u])+H\big)[v-u] \ge \sigma[v-u]$ in $S^d_u \cup S^d_v$. By the dominated convergence theorem we obtain from (6.4)

$$
\liminf_{t\to 0^+} \frac{\mathcal{E}^d_H(u+t(v-u))-\mathcal{E}^d_H(u)}{t} \ge \sigma\int_0^l (\dot v - \dot u)\,dx + \sigma \sum_{S^d_u\cup S^d_v}[v-u] = 0\,,
$$

where the last equality follows from (4.2). $\square$

**Theorem 6.3.** *A function $u \in BV(0,l)$ is a stationary point for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$ if and only if there exists a constant $\sigma$ such that the following conditions are satisfied:*

(i) $-\infty \leq \sigma \leq \sigma^{\rm ult} + H$ *in* $(0,l)$*;*

(ii) $\overline{F}(\dot u) < +\infty$ *a.e. in* $(0,l)$*;*

(iii) $u'_s \geq 0$ *in* $[0,l]$*;*

(iv) $\big(f(\dot u) \wedge \sigma^{\rm ult}\big) + H = \sigma$ *a.e. in* $(0,l)$*;*

(v) $g([u]) + H = \sigma$ *in* $S^d_u$*;*

(vi) $\operatorname{supp} u'_c \subset \{\sigma = \sigma^{\rm ult} + H\}$*.*

*In this case we have $\dot u < \varepsilon^{\rm ult}$ and $f(\dot u) + H = \sigma$ a.e. in $\{\sigma < \sigma^{\rm ult} + H\}$, while $\dot u \geq \varepsilon^{\rm ult}$ a.e. in $\{\sigma = \sigma^{\rm ult} + H\}$. If $\sigma > -\infty$, then there exists a constant $\varepsilon_u > \varepsilon^{\min}$ such that $\dot u \geq \varepsilon_u$ a.e. in $(0,l)$, and* (ii) *follows from* (i) *and* (iv)*. The case $\sigma = -\infty$ occurs if and only if $F(\varepsilon^{\min}) < +\infty$, $d = l\,\varepsilon^{\min}$, and $u(x) = \varepsilon^{\min} x$ for a.e. $x \in (0,l)$.*

**Remark 6.4.** Let $u$ be a stationary point for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$. If $\sigma < \sigma^{\rm ult} + \min H$, then $\{\sigma < \sigma^{\rm ult} + H\} = [0,l]$ and $\{\sigma = \sigma^{\rm ult} + H\} = \emptyset$, so that $u'_c = 0$ and, consequently, $u \in SBV(0,l)$. If $\sigma = \sigma^{\rm ult} + \min H$, then $\{\sigma = \sigma^{\rm ult} + H\} = \{H = \min H\}$, so that $\operatorname{supp} u'_c \subset \{H = \min H\}$; moreover, since $g(w) < \sigma^{\rm ult}$ for $w > 0$, condition (vi) implies that $S^d_u \cap \{H = \min H\} = \emptyset$.

*Proof of Theorem 6.3.* Assume that $u$ is a stationary point for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$. Then (ii) and (iii) follow from the fact that $\overline{\mathcal{E}}^d_H(u) < +\infty$ and from (4.4). Conditions (iv) and (v) can be proved as in Theorem 6.2, replacing $f(\varepsilon)$ by $f(\varepsilon) \wedge \sigma^{\rm ult}$. Condition (i) follows from (iv) and from the continuity of $H$.

In order to prove (vi), we define $U = \{\sigma < \sigma^{\rm ult} + H\}$ and $\psi(x) = u'_c(U)(x/l) - u'_c(U \cap (0,x))$. Using $v = u + \psi$ in Definition 6.1, from (iv) we obtain

$$
\begin{aligned}
0 &\leq \lim_{t \to 0^+} \frac{\overline{\mathcal{E}}^d_H(u + t\psi) - \overline{\mathcal{E}}^d_H(u)}{t} \\
&= \frac{u'_c(U)}{l} \int_0^l \Big(\big(f(\dot u) \wedge \sigma^{\rm ult}\big) + H\Big)\, dx - \sigma^{\rm ult} u'_c(U) - \int_U H u'_c \\
&= (\sigma - \sigma^{\rm ult}) u'_c(U) - \int_U H u'_c = \int_U (\sigma - \sigma^{\rm ult} - H) u'_c \leq 0\,.
\end{aligned}
$$

Since $\sigma - \sigma^{\rm ult} - H < 0$ on $U$ and $u'_c$ is a non-negative measure by (iii), we conclude that $u'_c(U) = 0$, which implies (vi).

Conversely, if $u$ satisfies (i)–(vi), the arguments used in Theorem 6.2 prove that $u$ is a stationary point for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$ and that the final assertions of the theorem are true. $\square$

**Remark 6.5.** From Theorems 6.2, 6.3, and Remark 6.4 it follows that every stationary point for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$ with $\sigma < \sigma^{\rm ult} + \min H$ is stationary for $\mathcal{E}^d_H$ in $SBV(0,l)$. Conversely, every stationary point for $\mathcal{E}^d_H$ in $SBV(0,l)$ is stationary for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$.

**Remark 6.6.** In the model described by the energy $\mathcal{E}^d_H$ the term $f(\dot u)$ represents the stress due to the deformation gradient $\dot u$ in the undamaged region $[0,l]\setminus S^d_u$, while for $x\in S^d_u$ the term $g([u](x))$ represents the stress transmitted through the damaged region that in the reference configuration is represented by the point $x$, and in the deformed configuration is given by the interval $[u(x-),u(x+)]$ (strain concentrated at $x$). This shows that $\sigma^{\rm ult}$ is the maximum possible stress for an equilibrium configuration, i.e., the ultimate tensile stress.

In the model described by the relaxed energy $\overline{\mathcal{E}}^d_H$, which is the energy obtained as $\Gamma$-limit of the discrete energies, the term $f(\dot u)\wedge\sigma^{\rm ult}$ represents the stress due to the macroscopic deformation gradient $\dot u$, while $g([u])$ represents the stress transmitted through the damaged regions where we have concentration of the strain (which are represented by $S^d_u$ in the reference configuration). Condition (vi) in Theorem 6.3 leads to consider the Cantor part $u'_c$ as a singular strain, which is not concentrated on points in the reference configuration, but lives on a set of Lebesgue measure zero through which the stress $\sigma^{\rm ult}$ is transmitted. Also for this model $\sigma^{\rm ult}$ is the maximum possible stress for the elastic part of the bar (described by $\dot u$), for the damaged part with concentrated strain (described by $[u]$), and for the part with a singular, but non-concentrated, strain (described by $u'_c$).

**Remark 6.7.** When $H=0$ it is easy to give an elementary description of all stationary points of $\mathcal{E}^d$ in $SBV(0,l)$ based on Theorem 6.2. In this case $\sigma$ represents the constant tension of the bar at equilibrium.

(a) If $d < l\,\varepsilon^{\min}$, then $\mathcal{E}^d(u) = +\infty$ for every $u\in SBV(0,l)$, so that there are no stationary points.

(b) If $d = l\,\varepsilon^{\min}$, then $u$ is stationary if and only if $F(\varepsilon^{\min}) < +\infty$ and $u(x)=\varepsilon^{\min}x$ for a.e. $x\in(0,l)$; in this case $\sigma=-\infty$.

(c) If $l\,\varepsilon^{\min} < d \le 0$, then $u$ is stationary if and only if $u(x)=\varepsilon_0 x$ for a.e. $x\in(0,l)$, with $\varepsilon_0 = d/l$; in this case $\sigma = f(\varepsilon_0)$ and $-\infty<\sigma\le 0$ (with $\sigma<0$ for $d<0$).

(d) If $d>0$, then $u$ is stationary if and only if $u\in SBV(0,l)$ and one of the following conditions is satisfied, where $\#S^d_u$ denotes the number of elements of $S^d_u$:

(d1) $u(x) = \varepsilon_0 x$ for a.e. $x \in (0,l)$, with $\varepsilon_0 = d/l \leq \varepsilon^{\mathrm{ult}}$; in this case $\sigma = f(\varepsilon_0)$ and $0 < \sigma \leq \sigma^{\mathrm{ult}}$ (with $\sigma = \sigma^{\mathrm{ult}}$ only if $d = l\,\varepsilon^{\mathrm{ult}}$);

(d2) $\#S_u^d = k$, $1 \leq k < +\infty$, and there exist two constants $\varepsilon_0$ and $w_0$ such that $0 < \varepsilon_0 < \varepsilon^{\mathrm{ult}}$, $0 < w_0 < w^{\mathrm{frac}}$, $f(\varepsilon_0) = g(w_0)$, $\dot{u} = \varepsilon_0$ a.e. in $(0,l)$, $[u] = w_0$ in $S_u^d$, and $l\varepsilon_0 + kw_0 = d$; in this case $\sigma = f(\varepsilon_0) = g(w_0)$ and $0 < \sigma < \sigma^{\mathrm{ult}}$

(d3) $1 \leq \#S_u^d < +\infty$, $\dot{u} = 0$ a.e. in $(0,l)$, $[u] \geq w^{\mathrm{frac}}$ in $S_u^d$, and $\sum_{S_u^d}[u] = d$; in this case $\sigma = 0$.

It is clear that case (d3) may occur only if $d \geq w^{\mathrm{frac}}$. In case (d2) the point $(\varepsilon_0, w_0)$ is a stationary for the function $lF(\varepsilon) + kG(w)$ on the manifold $\{(\varepsilon, w) \in \mathbf{R}^2 : w > 0,\ l\varepsilon + kw = d\}$.

Conversely, if $d > 0$, $k = 1, 2, \ldots$, and $(\varepsilon_0, w_0)$ is a stationary point of the function $lF(\varepsilon) + kG(w)$ on that manifold, then by the Lagrange multipliers rule we have $f(\varepsilon_0) = g(w_0)$, and by Theorem 6.2 any function $u \in SBV(0,l)$ with $\dot{u} = \varepsilon_0$ a.e. in $(0,l)$, $\#S_u^d = k$, and $[u] = w_0$ in $S_u^d$ is stationary for $\mathcal{E}^d$ in $SBV(0,l)$.

## 7. Minimum Energy Configurations for the Continuous Model

In this section we study in detail the properties of local and absolute minima for the functionals $\mathcal{E}_H^d$ and $\overline{\mathcal{E}}_H^d$ introduced in (6.1). We always assume that $f$, $g$, $h$, and $H$ satisfy all conditions of the previous section. We recall the definition of local minimum with respect to the strong topology of $BV(0,l)$.

**Definition 7.1.** Let $\mathcal{U}$ be a subset of $BV(0,l)$, and let $\mathcal{F}\colon\mathcal{U} \to [-\infty, +\infty]$ be a functional. We say that a function $u \in \mathcal{U}$ is a *local minimum* for $\mathcal{F}$ in $\mathcal{U}$ if $\mathcal{F}(u) < +\infty$ and there exists $\eta > 0$ such that $\mathcal{F}(u) \leq \mathcal{F}(v)$ for every $v \in \mathcal{U}$ satisfying $|u' - v'|([0,l]) < \eta$, where $u'$ and $v'$ are the measures on $[0,l]$ defined by (4.1) taking the boundary conditions into account. We say that $u$ is a *strict local minimum* if we have $\mathcal{F}(u) < \mathcal{F}(v)$ if $v$ is as above and $v \neq u$.

**Theorem 7.2.** *A function $u$ is a local minimum (respectively, a strict local minimum, or an absolute minimum) for $\overline{\mathcal{E}}_H^d$ in $BV(0,l)$ if and only if it is a local minimum (respectively, a strict local minimum, or an asolute minimum) for $\mathcal{E}_H^d$ in $SBV(0,l)$. In this case we have $\dot{u} \leq \varepsilon^{\mathrm{ult}}$ a.e. in $(0,l)$ and $\overline{\mathcal{E}}_H^d(u) = \mathcal{E}_H^d(u)$. In particular, all local minima of $\overline{\mathcal{E}}_H^d$ belong to $SBV(0,l)$.*

To prove the theorem we need the following lemma.

**Lemma 7.3.** *If $u$ is a local minimum for $\mathcal{E}^d_H$ in $SBV(0,l)$ and $\#S^d_u > 1$, then $[u] > w^{\mathrm{frac}}$ in $S^d_u$ and $H$ is constant in $S^d_u$.*

*Proof.* Suppose that $x_1 \neq x_2$ are two points in $S_u$, denote $w_i = [u](x_i)$, and suppose that $w_2 \le w^{\mathrm{frac}}$. Since $g$ is decreasing on $[0, w^{\mathrm{frac}}]$ we have that

$$\int_{w_1}^{w_1+\lambda} g(s)\,ds - \int_{w_2-\lambda}^{w_2} g(s)\,ds < \big(g(w_1) - g(w_2)\big)\lambda \tag{7.1}$$

for $0 < \lambda < w_2$. Let $\psi$ be the function defined by $\psi = \chi_{(x_1,x_2)}$, if $x_1 < x_2$, and by $\psi = -\chi_{(x_2,x_1)}$, if $x_2 < x_1$, and let $v = u + \lambda\psi$. As $u$ is a local minimum, we can suppose that $\lambda$ is small enough as to have $\mathcal{E}^d_H(u) \le \mathcal{E}^d_H(v)$. This implies

$$\begin{aligned} &G(w_1) + G(w_2) + H(x_1)w_1 + H(x_2)w_2 \\ &\qquad \le G(w_1+\lambda) + G(w_2-\lambda) + H(x_1)(w_1+\lambda) + H(x_2)(w_2-\lambda)\,. \end{aligned}$$

By Theorem 6.2(v) we get

$$\big(G(w_1+\lambda) - G(w_1)\big) + \big(G(w_2-\lambda) - G(w_2)\big) \ge \lambda\big(g(w_1) - g(w_2)\big)\,,$$

which contradicts (7.1). This proves that $[u] > w^{\mathrm{frac}}$ in $S^d_u$. Therefore $g([u]) = 0$ in $S^d_u$, and Theorem 6.2(v) implies that $H$ is constant in $S^d_u$. □

*Proof of Theorem 7.2.* Let $u$ be a local minimum for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$. Then $u$ is a stationary point for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$. We want to prove that $u \in SBV(0,l)$. Suppose, by contradiction, that $u'_c \neq 0$. By Theorem 6.3 and Remark 6.4 we have $\sigma = \sigma^{\mathrm{ult}} + \min H$, $\dot u < \varepsilon^{\mathrm{ult}}$ a.e. in $\{H > \min H\}$, $\operatorname{supp} u'_c \subset \{H = \min H\}$, and $S^d_u \cap \{H = \min H\} = \emptyset$.

Let $0 < \lambda < 1$ and let $x_0 \in \{H = \min H\}$. Let $v \in BV(0,l)$ be such that $v' = u' - \lambda u'_c + \lambda u'_c(0,l)\,\delta_{x_0}$ in $[0,l]$. Then the concavity of $G$, which is strict on $[0, w^{\mathrm{frac}}]$, gives $\overline{\mathcal{E}}^d_H(v) < \overline{\mathcal{E}}^d_H(u)$, so that the definition of local minimum is violated for $\lambda$ small enough. This contradition shows that $u \in SBV(0,l)$. In the same way we can prove that $\dot u \le \varepsilon^{\mathrm{ult}}$ a.e. in $(0,l)$, by choosing $w \in SBV(0,l)$ such that

$$v' = u' - \lambda(\dot u - \varepsilon^{\mathrm{ult}})^+ dx + \lambda\Big(\int_0^l (\dot u - \varepsilon^{\mathrm{ult}})^+ dx\Big)\delta_{x_0} \qquad \text{in } [0,l]\,,$$

with $\lambda > 0$ small enough. Hence $u \in SBV(0,l)$, $\mathcal{E}^d_H(u) = \overline{\mathcal{E}}^d_H(u)$, and consequently $u$ is a local minimum for $\mathcal{E}^d_H$ in $SBV(0,l)$.

Conversely, let $u$ be a local minimum for $\mathcal{E}^d_H$ in $SBV(0,l)$. From Theorem 6.2 we obtain $\dot u \le \varepsilon^{\rm ult}$ a.e. in $(0,l)$, hence $\overline{\mathcal{E}}^d_H(u) = \mathcal{E}^d_H(u)$. Let $v \in BV(0,l)$ with $\overline{\mathcal{E}}^d_H(v) < +\infty$. We construct a function $z \in SBV(0,l)$ such that $z$ is close to $v$ and $\mathcal{E}^d_H(z) \le \overline{\mathcal{E}}^d_H(v)$. The function $z$ is defined by

$$z' = v' - (\dot v - \varepsilon^{\rm ult})^+ dx - v'_c + \alpha\delta_{x_0} = (\dot v \wedge \varepsilon^{\rm ult})dx + \sum_{x\in S^d_v} [v](x)\delta_x + \alpha\delta_{x_0}$$

in $[0,l]$, where $\alpha = \int_0^l (\dot v - \varepsilon^{\rm ult})^+ dx + v'_c(0,l)$ and $x_0$ is a point in $(0,l)\setminus S_v$ such that $H(x_0)\alpha \le \int_0^l H(\dot v - \varepsilon^{\rm ult})^+ dx + \int_0^l H v'_c$. After elementary calculation we obtain

$$\mathcal{E}^d_H(z) \le \overline{\mathcal{E}}^d_H(v) + G(\alpha) - \sigma^{\rm ult}\alpha + H(x_0)\alpha - \int_0^l H(\dot v - \varepsilon^{\rm ult})^+ dx - \int_0^l H v'_c\,,$$

and the right hand side is non-positive by the choice of $x_0$ and by the (strict) concavity of $G$, which leads to the inequality $G(\alpha) \le \sigma^{\rm ult}\alpha$ (which is strict for $\alpha > 0$). Therefore $\mathcal{E}^d_H(z) \le \overline{\mathcal{E}}^d_H(v)$. If $|v' - u'|([0,l]) < \eta$, it is easy to see that $|z' - v'|([0,l]) < 2\eta$, since $\dot u \le \varepsilon^{\rm ult}$ a.e. in $(0,l)$ and $u'_c = 0$ in $(0,l)$. Therefore $|z' - u'|([0,l]) < 3\eta$. Since $u$ is a local minimum for $\mathcal{E}^d_H$ in $SBV(0,l)$, if $\eta$ is small we have $\overline{\mathcal{E}}^d_H(u) = \mathcal{E}^d_H(u) \le \mathcal{E}^d_H(z) \le \overline{\mathcal{E}}^d_H(v)$, which proves that $u$ is a local minimum for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$.

The proof for strict local minima and for absolute minima is similar. □

**Remark 7.4.** Since the existence of an absolute minumum for $\overline{\mathcal{E}}^d_H$ in $BV(0,l)$ was proved in Theorem 5.1 when $d > l\,\varepsilon^{\rm min}$ (and is trivial when $d = l\,\varepsilon^{\rm min}$), we deduce that, under the same conditions, the functional $\mathcal{E}^d_H$ has an absolute minimum in $SBV(0,l)$, even though it is not lower semicontinuous.

**Theorem 7.5.** *If $u$ is a strict local minimum of $\mathcal{E}^d_H$ in $SBV(0,l)$, then $\#S^d_u \le 1$.*

*Proof.* If $u$ is a local minimum for $\mathcal{E}^d_H$ and $\#S_u > 1$, Lemma 7.3 shows that $[u] > w^{\rm frac}$ and $H$ is constant in $S^d_u$, therefore $\mathcal{E}^d_H(u) = \mathcal{E}^d_H(v)$ for every function of the form $v = u + \lambda\chi_{(x_1,x_2)}$, where $x_1, x_2 \in S^d_u$, $x_1 < x_2$, and $\lambda > 0$ is sufficiently small. This shows that $u$ is not a strict local minimum. □

We conclude this section by studying the properties of the absolute minimizers of $\mathcal{E}^d_H$.

**Theorem 7.6.** *Assume that $d > l\,\varepsilon^{\min}$ (or $d = l\,\varepsilon^{\min}$ and $F(\varepsilon^{\min}) < +\infty$). Then the functional $\mathcal{E}^d_H$ attains its absolute minimum on $SBV(0,l)$, and for each minimum point $u$ we have $\# S^d_u \leq 1$, $S_u \subset \{H = \min H\}$, and $\dot u \leq \varepsilon^{\rm ult}$ a.e. in $(0,l)$. Moreover, given $x_0 \in \{H = \min H\}$, there exists a minimum point $u$ for which $S^d_u \subset \{x_0\}$, i.e., $u \in W^{1,1}((0,x_0)\cup(x_0,l))$.*

*Proof.* Let $x_0$ be a minimum point of $H$ in $[0,l]$. For every $u \in SBV(0,l)$, with $\mathcal{E}^d_H(u) < +\infty$, we consider the function $v \in SBV(0,l)$ which satisfies $\dot v = \dot u \wedge \varepsilon^{\rm ult}$ a.e. in $(0,l)$ and $v'_s = \alpha\delta_{x_0}$ in $[0,l]$, where $\alpha = \sum_{S^d_u}[u] + \int_0^l (\dot u - \varepsilon^{\rm ult})^+ dx$. By the subadditivity of $G$ and by the inequality

$$\int_{\{\dot u > \varepsilon^{\rm ult}\}} (F(\dot u) - F(\varepsilon^{\rm ult}))\,dx = \sigma^{\rm ult}\int_0^l (\dot u - \varepsilon^{\rm ult})^+ dx \geq G\Big(\int_0^l (\dot u - \varepsilon^{\rm ult})^+ dx\Big),$$

it is easy to check that

$$\mathcal{E}^d_H(v) \leq \mathcal{E}^d_H(u) + \int_{\{\dot u > \varepsilon^{\rm ult}\}} (F(\varepsilon^{\rm ult}) - F(\dot u))\,dx + G\Big(\int_0^l (\dot u - \varepsilon^{\rm ult})^+ dx\Big) \leq \mathcal{E}^d_H(u).$$

Note that the first inequality is strict if $\# S^d_u > 1$, or if $S^d_u \setminus \{H = \min H\} \neq \emptyset$, or if $\dot u > \varepsilon^{\rm ult}$ on a set of positive measure. This shows that every minimum point satisfies $\# S^d_u \leq 1$, $S_u \subset \{H = \min H\}$, and $\dot u \leq \varepsilon^{\rm ult}$ a.e. in $(0,l)$. Moreover it proves that for every minimum point $v$ there exists a minimum point $u$ with $S_u \subset \{x_0\}$.

We now give a new proof of the existence of the minimum, which is independent of and easier than the proof given in Theorem 5.1 and Remark 7.4. If $d = l\,\varepsilon^{\min}$ and $F(\varepsilon^{\min}) < +\infty$, the function $u(x) = \varepsilon^{\min}x$ is the only function for which $\mathcal{E}^d_H(u) < +\infty$, so it is the unique minimizer of $\mathcal{E}^d_H$.

Let us assume that $d > l\,\varepsilon^{\min}$ and let us fix $x_0 \in \{H = \min H\}$. It is sufficient to prove the existence of a minimum of $\mathcal{E}^d_H$ in the space of functions $u \in SBV(0,l)$ with $S_u \subset \{x_0\}$ and $\dot u \leq \varepsilon^{\rm ult}$ a.e. in $(0,l)$, i.e., in the space of functions $u \in W^{1,1}((0,x_0)\cup(x_0,l))$ such that $\dot u \leq \varepsilon^{\rm ult}$ a.e. in $(0,l)$. To prove the compactness of a minimizing sequence $\{u_n\}$ it is therefore sufficient to remark that the sequence $\{v_n\} \subset W^{1,1}(0,l)$ defined by $v(0) = 0$ and $\dot v_n = \dot u_n$ is weakly pre-compact in $W^{1,1}(0,l)$ by (3.12) and (3.3), and the sequence $[u_n](x_0) = d - \int_0^l \dot v_n\,dx$ is bounded. The lower semicontinuity of the functional $\mathcal{E}^d_H$ on $W^{1,1}((0,x_0)\cup(x_0,l))$ implies that the limit of every minimizing sequence is an absolute minimum. □

In the case $H = 0$ it is easy to give an elementary description of all local minima of $\mathcal{E}^d$ in $SBV(0,l)$.

**Remark 7.7.** When $d \leq 0$, by Remark 6.7 the function $u(x) = \varepsilon_0 x$, with $\varepsilon_0 = d/l$, is the unique stationary point, provided $F(\varepsilon_0) < +\infty$. By the existence result (Theorem 7.6) we conclude that this function $u$ is the unique absolute minimum. As a consequence of these facts, $u$ is the only strict local minimum.

We consider now the case $d > 0$.

**Theorem 7.8.** *Let $d > 0$ and let $u \in SBV(0,l)$. Then $u$ is a local minimum of $\mathcal{E}^d$ in $SBV(0,l)$ if and only if one of the following conditions is satisfied:*
(a) *there exist three constants $\varepsilon_0$, $w_0$, and $x_0$ such that $0 \leq \varepsilon_0 \leq \varepsilon^{\text{ult}}$, $w_0 \geq 0$, $0 \leq x_0 \leq l$, $u(x) = \varepsilon_0 x + w_0 \chi_{(x_0,l)}(x)$ for a.e. $x \in (0,l)$, and $(\varepsilon_0, w_0)$ is a local minimum of the function $lF(\varepsilon) + G(w)$ on the manifold $K(d) = \{(\varepsilon, w) \in \mathbf{R}^2 : w \geq 0,\ l\varepsilon + w = d\}$;*
(b) *$\dot{u} = 0$ a.e. in $(0,l)$, $1 < \# S_u^d < +\infty$, $[u] > w^{\text{frac}}$ in $S_u^d$, and $\sum_{S_u^d} [u] = d$.*
*The function $u$ is an absolute (respectively, a strict local) minimum of $\mathcal{E}^d$ in $SBV(0,l)$ if and only if* (a) *holds and $(\varepsilon_0, w_0)$ is an absolute (respectively, a strict local) minimum of the function $lF(\varepsilon) + G(w)$ on $K(d)$.*

*Proof.* Suppose that $u$ is a local minimum of $\mathcal{E}^d$ in $SBV(0,l)$. By Remark 6.7 the set $S_u^d$ is finite or empty, and there exists a constant $\varepsilon_0$ such that $0 \leq \varepsilon_0 \leq \varepsilon^{\text{ult}}$ and $\dot{u} = \varepsilon_0$ a.e. in $(0,l)$. If $S_u^d$ has more than one point, by Lemma 7.3 we have $[u] > w^{\text{frac}}$ in $S_u^d$, hence $\sigma = 0$ in Theorem 6.2, so that $\dot{u} = 0$ a.e. in $(0,l)$ and (b) is proved. If $S_u^d$ has at most one point $x_0$, we have $u(x) = \varepsilon_0 x + w_0 \chi_{(x_0,l)}(x)$ for a.e. $x \in (0,l)$, with $w_0 = [u](x_0)$, so that $\mathcal{E}^d(u) = lF(\varepsilon_0) + G(w_0)$. If $(\varepsilon, w)$ satisfies $l\varepsilon + w = d$ and $w \geq 0$, then the function $v(x) = \varepsilon\, x + w\, \chi_{(x_0,l)}(x)$ belongs to $SBV(0,l)$ and $\mathcal{E}^d(v) = lF(\varepsilon) + G(w)$, therefore the local minimality of $u$ implies the local minimality of $(\varepsilon_0, w_0)$ and (a) is proved.

Conversely, assume (a). Let $v \in SBV(0,l)$ with $\mathcal{E}^d(v) < +\infty$. We set $\varepsilon = (1/l) \int_0^l \dot{v}\, dx$, $w = \sum_{S_v} [v]$, and $z(x) = \varepsilon\, x + w\, \chi_{(x_0,l)}(x)$. We have then, using Jensen's inequality and the subadditivity of $G$,

$$lF(\varepsilon) + G(w) \leq \int_0^l F(\dot{v})\, dx + \sum_{S_v} G([v]) = \mathcal{E}^d(v)\,.$$

If $|v' - u'|([0,l]) < \eta$, it is easy to see that $|\varepsilon - \varepsilon_0| < \eta/l$ and $|w - w_0| < \eta$. Since $(\varepsilon_0, w_0)$ is a local minimum of the function $lF(\varepsilon) + G(w)$ on $K(d)$, if $\eta > 0$ is small we have

$$\mathcal{E}^d(u) = lF(\varepsilon_0) + G(w_0) \leq lF(\varepsilon) + G(w) \leq \mathcal{E}^d(v)\,,$$

which proves that $u$ is a local minimum for $\mathcal{E}^d$ in $SBV(0,l)$.

If (b) holds, there exists a constant $\eta > 0$ such that $[u] > w^{\rm frac} + \eta$ in $S_u^d$. Then for every $v \in SBV(0,l)$ with $|v' - u'|(0,l) < \eta$ we have $S_u^d \subset S_v^d$ and $[v] > w^{\rm frac}$ in $S_u^d$. As $G(w) = G(w^{\rm frac})$ for every $w \geq w^{\rm frac}$ we obtain

$$\mathcal{E}^d(u) = \#S_u^d\, G(w^{\rm frac}) \leq \int_0^l F(\dot v)\, dx + \sum_{S_v^d} G([v]) = \mathcal{E}^d(v)\,,$$

so that $u$ is a local minimum for $\mathcal{E}^d$ in $SBV(0,l)$.

The proof for absolute minima and for strict local minima is analogous. The only difference is that we use Theorems 7.5 and 7.6 to exclude (b). □

**Corollary 7.9.** *If $0 < d < l\,\varepsilon^{\rm ult}$, then the function $u(x) = \varepsilon_0 x$, with $\varepsilon_0 = d/l$, is a strict local minimum.*

*Proof.* Since $G'(0) = g(0) = \sigma^{\rm ult} = f(\varepsilon^{\rm ult}) > f(\varepsilon_0)$, we have $G(w) > f(\varepsilon_0)w$ for $w > 0$ small enough. If $l\,\varepsilon + w = d$ and $w > 0$ is small, by convexity we obtain $lF(\varepsilon) + G(w) > lF(\varepsilon_0) + lf(\varepsilon_0)(\varepsilon - \varepsilon_0) + f(\varepsilon_0)w = lF(\varepsilon_0)$. This shows that $(\varepsilon_0, 0)$ is a strict local minimum of the function $lF(\varepsilon) + G(w)$ on $K(d)$, and the conclusion follows from Theorem 7.8. □

**Corollary 7.10.** *If $d > w^{\rm frac}$, then for every $x_0 \in [0,l]$ the function $u(x) = d\,\chi_{(x_0,l)}(x)$ is a strict local minimum.*

*Proof.* As $F(\varepsilon) > 0$ for $\varepsilon \neq 0$ and $G(w) = G(w^{\rm frac})$ for $w \geq w^{\rm frac}$, if $d > w^{\rm frac}$ it is easy to see that $(0,d)$ is a strict local minimum of the function $lF(\varepsilon) + G(w)$ on $K(d)$. The conclusion follows from Theorem 7.8. □

## 8. Scale Effects

In this section we examine some assumptions which provide an easy characterization of all local minima of $\mathcal{E}^d$ in $SBV(0,l)$. These conditions are influenced by the length $l$ of the bar. For the sake of clarity we limit our analysis to the pure displacement problem $H = 0$.

We still assume that $f$ and $g$ satisfy all properties of Section 6. Let $f^{-1}\colon [0, \sigma^{\rm ult}] \to [0, \varepsilon^{\rm ult}]$ and $g^{-1}\colon [0, \sigma^{\rm ult}] \to [0, w^{\rm frac}]$ be the inverse functions of the restrictions $f|_{[0,\varepsilon^{\rm ult}]}$

and $g|_{[0,w^{\rm frac}]}$ respectively. Since all strict local minima have at most one jump point (Theorem 7.5), it is useful to determine the stationary points without jumps and those with exactly one jump. By Remark 6.7 the only stationary point without jumps corresponding to the stress $\sigma \in [0,\sigma^{\rm ult}]$ is the function $u(x)=f^{-1}(\sigma)x$, so that $d=lf^{-1}(\sigma)$, while the stationary points with one jump at $x_0\in[0,l]$ are characterized by $\dot u = f^{-1}(\sigma)$ a.e. in $[0,l]$ and $[u](x_0)=g^{-1}(\sigma)$, so that $d=lf^{-1}(\sigma)+g^{-1}(\sigma)$.

Since, usually, only the elongation $d$ is given and the stress $\sigma$ is unknown, the analysis of the problem is simplified if there is only one $\sigma$ such that $lf^{-1}(\sigma)+g^{-1}(\sigma)=d$. This happens if and only if the function $\sigma\mapsto lf^{-1}(\sigma)+g^{-1}(\sigma)$ is stricty monotone. In the next propositions we shall see that in this case we have an elementary description of all strict local minima of $\mathcal{E}^d$ in $SBV(0,l)$ for all admissible values of $d$.

**Proposition 8.1.** *Assume that the function $\sigma\mapsto lf^{-1}(\sigma)+g^{-1}(\sigma)$ is decreasing on $[0,\sigma^{\rm ult}]$ and, consequently, $l\,\varepsilon^{\rm ult}<w^{\rm frac}$.*

(i) *If $l\,\varepsilon^{\rm min}<d\le l\,\varepsilon^{\rm ult}$ (or if $d=l\,\varepsilon^{\rm min}$ and $F(\varepsilon^{\rm min})<+\infty$), then the function $u_0(x)=\varepsilon_0 x$, with $\varepsilon_0=d/l$, is the unique local minimum of $\mathcal{E}^d$ in $SBV(0,l)$; moreover $u_0$ is the only stationary point and, consequently, it is the strict absolute minimum.*

(ii) *If $l\,\varepsilon^{\rm ult}<d\le w^{\rm frac}$, let $\sigma_0$ be the unique solution in $[0,\sigma^{\rm ult})$ of the equation $lf^{-1}(\sigma_0)+g^{-1}(\sigma_0)=d$, and let $\varepsilon_0=f^{-1}(\sigma_0)$ and $w_0=g^{-1}(\sigma_0)$ (in particular $0\le\varepsilon_0<\varepsilon^{\rm ult}$ and $0<w_0\le w^{\rm frac}$, with equality only if $d=w^{\rm frac}$); then $u$ is a local minimum of $\mathcal{E}^d$ in $SBV(0,l)$ if and only if there exists $x_0\in[0,l]$ such that $u(x)=\varepsilon_0x+w_0\chi_{(x_0,l)}(x)$ for a.e. $x\in(0,l)$; all these functions are absolute minima and strict local minima, and they are the only stationary points with $\#S^d_u\le 1$.*

(iii) *If $d>w^{\rm frac}$, then $u$ is a strict local minimum of $\mathcal{E}^d$ in $SBV(0,l)$ if and only if there exists $x_0\in[0,l]$ such that $u(x)=d\,\chi_{(x_0,l)}(x)$ for a.e. $x\in(0,l)$. These functions are the only absolute minima; moreover they are the only stationary points with $\#S^d_u\le 1$.*

*Proof.* The case $d\le 0$ is considered in Remark 7.7. Assume that $d>0$, let $u$ be a stationary point of $\mathcal{E}^d$ in $SBV(0,l)$, and let $\sigma\ge 0$ be the constant given by Theorem 6.2.

(a) If $S^d_u=\emptyset$, by Remark 6.7 we have $u(x)=\varepsilon_0 x$, with $0<\varepsilon_0=d/l\le\varepsilon^{\rm ult}$, so that $d\le l\,\varepsilon^{\rm ult}<w^{\rm frac}$.

(b) If $\#S^d_u=k\ge 1$, then two cases are possible: either $0<\sigma<\sigma^{\rm ult}$ or $\sigma=0$.

(b1) If $0<\sigma<\sigma^{\rm ult}$, then $u$ satisfies condition (d2) of Remark 6.7 with $\varepsilon_0=f^{-1}(\sigma)$

and $w_0 = g^{-1}(\sigma)$, so that $lf^{-1}(\sigma) + kg^{-1}(\sigma) = d$; under our monotonicity assumptions, each function $\sigma \mapsto lf^{-1}(\sigma) + kg^{-1}(\sigma)$ is decreasing on $(0, \sigma^{\text{ult}})$ and its image is $(l\,\varepsilon^{\text{ult}}, k\,w^{\text{frac}})$, so that $l\,\varepsilon^{\text{ult}} < d < k\,w^{\text{frac}}$.

(b2) If $\sigma = 0$, then $u$ satisfies condition (d3) of Remark 6.7, so that $d \geq w^{\text{frac}} > l\,\varepsilon^{\text{ult}}$.

From (a) and (b) it follows that, if $0 < d \leq l\,\varepsilon^{\text{ult}}$, then the only stationary point is the function $u_0(x) = \varepsilon_0 x$, with $\varepsilon_0 = d/l$. By the existence result (Theorem 7.6) this stationary point is the unique (strict) absolute minimum.

If $l\,\varepsilon^{\text{ult}} < d < w^{\text{frac}}$, we deduce from (a) and (b) that $0 < \sigma < \sigma^{\text{ult}}$ and that every stationary point $u$ has at least a jump point and $[u] < w^{\text{frac}}$ in $S_u^d$, so that $\#S_u^d = 1$ if $u$ is a local minimum (Lemma 7.3). Conversely, if $u$ is a stationary point with $\#S_u^d \leq 1$, then (a) gives $\#S_u^d = 1$ and Remark 6.7 provides three constants $\varepsilon_0$, $w_0$, and $x_0$, with $0 < \varepsilon_0 < \varepsilon^{\text{ult}}$, $0 < w_0 < w^{\text{frac}}$, $0 \leq x_0 \leq l$, such that $f(\varepsilon_0) = g(w_0) = \sigma$, $l\,\varepsilon_0 + w_0 = d$, and $u(x) = \varepsilon_0 x + w_0 \chi_{(x_0,l)}(x)$. This shows that $\sigma = \sigma_0$. Moreover $(\varepsilon_0, w_0)$ is a stationary point of the function $lF(\varepsilon) + G(w)$ on the manifold $K(d) = \{(\varepsilon, w) \in \mathbf{R}^2 : w \geq 0,\ l\varepsilon + w = d\}$. From our monotonicity assumptions we deduce that $f(\varepsilon) - g(d - l\varepsilon)$ is positive if $\varepsilon_0 < \varepsilon \leq d/l$ and negative if $\varepsilon < \varepsilon_0$. Therefore $\varepsilon_0$ is the unique absolute minimum of of the function $lF(\varepsilon) + G(d - l\,\varepsilon)$ on $[0, d/l]$, and this implies that $(\varepsilon_0, w_0)$ is the strict absolute minimum of the function $lF(\varepsilon) + G(w)$ on the manifold $K(d)$. By Theorem 7.8 we conclude that $u$ is a strict local minimum and an absolute minimum of $\mathcal{E}^d$ in $SBV(0, l)$.

If $d > w^{\text{frac}}$, by Theorem 7.5 every strict local minimum is a stationary point with at most one jump. From (a) and (b) we deduce that every stationary point $u$ with $\#S_u^d \leq 1$ has exactly one jump and $\sigma = 0$, so by condition (d3) of Remark 6.7 there exists $x_0 \in [0, l]$ such that $u(x) = d\,\chi_{(x_0,l)}(x)$. Since these functions have the same energy, they are absolute minima by Theorem 7.6. □

**Remark 8.2.** If $w^{\text{frac}} < d \leq 2w^{\text{frac}}$, from the proof of the previous proposition and from Theorem 7.8 it follows that the functions $u(x) = d\,\chi_{(x_0,l)}(x)$ are the only local minima of $\mathcal{E}^d$ in $SBV(0, l)$. If $d > 2w^{\text{frac}}$, there are other local minima. For instance, the function $u(x) = (d/2)\chi_{(x_1,l)}(x) + (d/2)\chi_{(x_2,l)}(x)$ is a local minimum for every $x_1$, $x_2 \in [0, l]$.

**Proposition 8.3.** *Assume that the function $\sigma \mapsto lf^{-1}(\sigma) + g^{-1}(\sigma)$ is increasing on $[0, \sigma^{\text{ult}}]$ and, consequently, $w^{\text{frac}} < l\,\varepsilon^{\text{ult}}$.*

(i) *If* $l\,\varepsilon^{\min} < d < w^{\rm frac}$ *(or if* $d = l\,\varepsilon^{\min}$ *and* $F(\varepsilon^{\min}) < +\infty$*), then the function* $u_0(x) = \varepsilon_0 x$*, with* $\varepsilon_0 = d/l$*, is the unique local minimum of* $\mathcal{E}^d$ *in* $SBV(0,l)$*; moreover* $u_0$ *is the only stationary point and, consequently, it is the strict absolute minimum.*

(ii) *If* $d = w^{\rm frac}$*, then the function* $u_0(x) = \varepsilon_0 x$*, with* $\varepsilon_0 = d/l$*, is the unique local minimum of* $\mathcal{E}^d$ *in* $SBV(0,l)$ *and the strict absolute minimum. The other stationary points are the functions* $v_{x_0}(x) = d\,\chi_{(x_0,l)}(x)$*, with* $x_0 \in [0,l]$*.*

(iii) *If* $w^{\rm frac} < d < l\,\varepsilon^{\rm ult}$*, then the functions* $u_0(x) = \varepsilon_0 x$*, with* $\varepsilon_0 = d/l$*, and* $v_{x_0}(x) = d\,\chi_{(x_0,l)}(x)$*, with* $x_0 \in [0,l]$*, are the only strict local minima of* $\mathcal{E}^d$ *in* $SBV(0,l)$*. Let* $\sigma_0$ *be the unique solution in* $[0,\sigma^{\rm ult}]$ *of the equation* $l f^{-1}(\sigma_0) + g^{-1}(\sigma_0) = d$*, let* $\varepsilon_0 = f^{-1}(\sigma_0)$*,* $w_0 = g^{-1}(\sigma_0)$*, and let* $x_0 \in [0,l]$*; then the function* $z_{x_0}(x) = \varepsilon_0 x + w_0\chi_{(x_0,l)}(x)$ *is a stationary point which is not a local minimum. There are no other stationary points with* $\#S_u^d \le 1$*.*

(iv) *If* $d = l\,\varepsilon^{\rm ult}$*, then the functions* $v_{x_0}(x) = d\,\chi_{(x_0,l)}(x)$*, with* $x_0 \in [0,l]$*, are the only strict local minima of* $\mathcal{E}^d$ *in* $SBV(0,l)$ *and the only absolute minima. The function* $u_0(x) = \varepsilon^{\rm ult} x$ *is the only other stationary point with* $\#S_u^d \le 1$*.*

(v) *If* $d > l\,\varepsilon^{\rm ult}$*, then the functions* $v_{x_0}(x) = d\,\chi_{(x_0,l)}(x)$*, with* $x_0 \in [0,l]$*, are the only strict local minima of* $\mathcal{E}^d$ *in* $SBV(0,l)$ *and the only absolute minima; there are no other stationary points with* $\#S_u^d \le 1$*.*

*Proof.* The characterization of all stationary points with $\#S_u^d \le 1$ is proved as in the previous proposition. This leads immediately to (i) and (v).

The minimality of $u_0$ in (ii) and (iii) follows from Corollary 7.9, while the minimality of $v_{x_0}$ in (iii) and (iv) follows from Corollary 7.10.

To prove that $v_{x_0}$ is not a local minimum in (ii) we observe that our monotonicity assumptions imply that $f(\varepsilon) < g(w^{\rm frac} - l\,\varepsilon)$ for $0 < \varepsilon < w^{\rm frac}/l$, so that $G(w^{\rm frac}) > lF(\varepsilon) + G(w^{\rm frac} - l\,\varepsilon)$ for $0 < \varepsilon < w^{\rm frac}/l$ and, consequently, $v_{x_0}$ is not a local minimum by Theorem 7.8.

To prove that $z_{x_0}$ is not a local minimum in (iii) we observe that our monotonicity assumptions imply that $f(\varepsilon) - g(d - l\,\varepsilon)$ is positive for $0 < \varepsilon < \varepsilon_0$ and is negative for $\varepsilon_0 < \varepsilon < d/l$. Therefore $lF(\varepsilon_0) + G(d - l\,\varepsilon_0) > lF(\varepsilon) + G(d - l\,\varepsilon)$ for $0 < \varepsilon < d/l$ and, consequently, $z_{x_0}$ is not a local minimum by Theorem 7.8.

To prove that $u_0$ is not a local minimum in (iv) we observe that our monotonicity assumptions imply that $f(\varepsilon) > g(l\,\varepsilon^{\rm ult} - l\,\varepsilon)$ for $0 < \varepsilon < \varepsilon^{\rm ult}$, so that $lF(\varepsilon^{\rm ult}) > lF(\varepsilon) + G(d - l\,\varepsilon)$ for $0 < \varepsilon < \varepsilon^{\rm ult}$ and, consequently, $u_0$ is not a local minimum by

Theorem 7.8.

The statements about absolute minima in (ii) and (iv) follow easily from the existence result (Theorem 7.6). □

**Remark 8.4.** From (ii) and (iv) we have $lF(w^{\rm frac}/l) < G(w^{\rm frac}) = G(l\,\varepsilon^{\rm ult}) < lF(\varepsilon^{\rm ult})$. Therefore there exists a unique critical value $d^{\rm cr}$, with $w^{\rm frac} < d^{\rm cr} < l\,\varepsilon^{\rm ult}$, such that $lF(d^{\rm cr}/l) = G(w^{\rm frac})$. Using Theorem 7.8, we can improve (iii) as follows: for $w^{\rm frac} < d < d^{\rm cr}$ the function $u_0$ is the strict absolute minimum, while the functions $v_{x_0}$ are not absolute minima; for $d^{\rm cr} < d < l\,\varepsilon^{\rm ult}$ the functions $v_{x_0}$ are absolute minima, while $u_0$ is not an absolute minimum; for $d = d^{\rm cr}$ both functions $u_0$ and $v_{x_0}$ are absolute minima.

**Remark 8.5.** Under the assumptions of the previous proposition, if $w^{\rm frac} < d \le 2w^{\rm frac} < l\,\varepsilon^{\rm ult}$, then one can prove, using Theorem 7.8, that there are no local minima different from the strict local minima mentioned in (iii). If $2w^{\rm frac} < d < l\,\varepsilon^{\rm ult}$, the example given in Remark 8.2 shows that there are other local minima.

Let us define

$$\begin{aligned} l_* &= \sup\{l \ge 0 : \sigma \mapsto lf^{-1}(\sigma) + g^{-1}(\sigma) \text{ is decreasing}\}\,, \\ l^* &= \inf\{l > 0 : \sigma \mapsto lf^{-1}(\sigma) + g^{-1}(\sigma) \text{ is increasing}\}\,. \end{aligned}$$

As $f^{-1}$ is increasing and $g^{-1}$ is decreasing, we have $0 \le l_* \le l^* \le +\infty$, and the function $\sigma \mapsto lf^{-1}(\sigma) + g^{-1}(\sigma)$ is decreasing for $l < l_*$ and increasing for $l > l^*$. It is easy to see that $l_* = l^*$ if and only if $l_* = l^* = w^{\rm frac}/\varepsilon^{\rm ult}$ and $g(w) = f(\varepsilon^{\rm ult} - (\varepsilon^{\rm ult}/w^{\rm frac})w)$ for every $w \in [0, w^{\rm frac}]$. Note that this condition is always satisfied if $f|_{[0,\varepsilon^{\rm ult}]}$ and $g|_{[0,w^{\rm frac}]}$ are affine. If $f|_{[0,\varepsilon^{\rm ult}]}$ and $g|_{[0,w^{\rm frac}]}$ are continuously differentiable and their derivatives are never zero, then $0 < l_* \le l^* < +\infty$.

We have seen in Propositions 8.1 and 8.3 that the behaviour of the local minima is different in the two cases $l < l_*$ and $l > l^*$. Notice that, for given constitutive relations $f$ and $g$, the alternative depends only on the length $l$ of the bar. This fact explains some scale effects observed in fracture mechanics. In particular it explains the observation that the fracture tends to be brittle if the length of the specimen is large enough. Figures 1 and 2 show the qualitative dependence of the tension $\sigma$ on the elongation $d$ in the cases $l < l_*$ and $l > l^*$ respectively.

**Fig. 1. The case $0 < l < l_*$, i.e., $lf^{-1} + g^{-1}$ decreasing.**

**Fig. 2. The case $l^* < l < +\infty$, i.e., $lf^{-1} + g^{-1}$ increasing.**

Note that, if we increase $d$ continuously, in the case $l < l_*$ the stress $\sigma$ corresponding to the unique stable solution varies continuously. In the case $l > l^*$ it is not possible to select, for every elongation $d$, a stable solution whose stress $\sigma$ depends on $d$ continuously. This phenomenon was called the cusp catastrophe in [8] and is used to explain the behaviour of brittle fractures.

**Remark 8.6.** In the discrete case, a description of the strict local minimizers of $E_n$ similar to Propositions 8.1 and 8.3 can be easily obtained if we assume the strict monotonicity of the function $\sigma \mapsto (l-\lambda_n)f_n^{-1}(\sigma) + g_n^{-1}(\sigma)$. In the spirit of Proposition 2.4, the role of $S_u^d$ is played by the set of indices $i$ such that $\dot{u}_n^i > \varepsilon_n^{\rm ult}$.

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