A $1D$ macroscopic phase field model for dislocations and a second order Γ-limit

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Abstract

We study the asymptotic behaviour in terms of Γ-convergence of the following one dimensional energy

$$F_\varepsilon(u) = \mu_\varepsilon \int_I \int_I \frac{|u(x) - u(y)|^2}{|x-y|^2}\, dx\, dy + \eta_\varepsilon \int_I W\left(\frac{u(x)}{\varepsilon}\right) dx$$

where I is a given interval, W is a one-periodic potential that vanishes exactly on $\mathbf{Z}$.

Different regimes for the asymptotic behaviour of the parameter μ_ε and η_ε are considered. In a very diluted regime we get a limit defined on $BV(I)$ and proportional to the total variation of u. In this particular case we also consider the limit of a suitable boundary value problem for which we characterize the second order Γ-limit.

The study under consideration is motivated by the analysis of a variational model for a very important class of defects in crystals, the dislocations, and the derivation of macroscopic models for plasticity.

Keywords: Dislocations, Phase Transitions, Γ-convergence, Asymptotic development.

Mathematical Subjects Classification: 82B26, 49J45

1 Introduction

Dislocations are, at the mesoscopic level, line defects in crystals that lie on a slip plane and are due to plastic slips that are not compatible with the crystal structure. In a recent paper [17] a phase field model for dislocations has been proposed. Mathematically speaking, after a renormalization, the free energy consists on a 2D singular perturbation of a non linear potential with infinitely many, periodically distributed, wells, of the form

$$E_\sigma(\xi) = \int_Q \int_Q K(x-y)|\xi(x) - \xi(y)|^2\, dx\, dy + \frac{1}{\sigma} \int_Q W(\xi)\, dx\,. \tag{1.1}$$

Here Q is the unit square in $\mathbf{R}^2$, W is one-periodic, non-negative and vanishes exactly on the integers $\mathbf{Z}$, and the nonlocal part of the energy behaves like the $H^{1/2}$ semi norm (i.e. $K(t) \sim |t|^{-3}$ as $t \to 0$). In this variational model the phase field ξ can be interpreted as the slip field measured in units of the Burgers vector $\mathbf{b}$. The non local term is due to the long range elastic energy induced by the slip $\xi\mathbf{b}$; i.e., obtained by minimizing the isotropic elastic energy of the cylinder $Q \times \mathbf{R}$ subject to suitable lateral boundary conditions and the constraint that the slip along the plane $\{x_3 = 0\}$ is given, that is $U^+ - U^- = \xi\mathbf{b}$ on $Q \times \{0\}$ where $[U] := U^+ - U^-$ is the jump of the displacement $U : Q \times \mathbf{R} \to \mathbf{R}^3$ across the plane $\{x_3 = 0\}$. The non linear potential W penalizes slips that are not compatible with the crystalline structure and the small parameter σ is proportional to the lattice parameter $|\mathbf{b}|$.

In [13] the asymptotic behaviour for a scaled version of (1.1) has been studied in terms of Γ-convergence. More precisely the Γ-limit of the functional

$$\frac{1}{|\ln \sigma|} E_\sigma$$

is finite on all functions $\xi \in BV(Q, \mathbf{Z})$ and it is given by a line tension energy of the form

$$\int_{S_\xi} \gamma(\nu_\xi) |[\xi]| \, d\mathcal{H}^1 \,,$$

where S_ξ is the jump set of ξ and ν_ξ its normal vector. In terms of the application to the problem of dislocations, the quantity $|\ln \sigma|$ represents the energy of the core of a single dislocation line. As a consequence the scaling under consideration in [13] is based on the assumption that the slip fields are of the order of the Burgers vector, i.e., $|[U]| \sim |\mathbf{b}|$ or $\xi \sim 1$; and the limit can be interpreted as a mesoscopic regime at which one can still see the line structure. On the other hand in the study of the macroscopic plastic behaviour one would expect the jumps of the displacement of order larger than $|\mathbf{b}|$.

In this paper we give a complete analysis of the one dimensional problem, under the general condition $|[U]| \gg |\mathbf{b}|$.

1.1 Heuristic derivation of the scaling

We start by rewriting in scaled variables the result in [13] which can be schematically summarized as follows

$$[\xi_\sigma]^2_{H^{1/2}(Q)} + \frac{1}{\sigma} \int_Q W(\xi_\sigma) dx \sim |\ln \sigma| |D\xi|(Q),$$

where $[\xi]_{H^{1/2}(Q)}$ denotes the $H^{1/2}$ seminorm in Q and $\xi_\sigma \to \xi$ in energy (ξ_σ is a *recovery* sequence for ξ). To this end we introduce a parameter $\delta = \delta_\sigma$ of the order of the slip; i.e. $\delta_\sigma \sim |[U]|$. We will be interested in the regime $\delta_\sigma \gg \sigma$. Recalling that the slip satisfies $|[U]| \sim \xi\sigma$ we introduce a new variable $w = \frac{\sigma}{\delta_\sigma}\xi$ that represents a normalized slip in terms of δ_σ and we rewrite the energy as

$$E_\sigma(w_\sigma, Q) = \left(\frac{\delta_\sigma}{\sigma}\right)^2 [w_\sigma]^2_{H^{1/2}(Q)} + \frac{1}{\sigma} \int_Q W\left(\frac{\delta_\sigma}{\sigma} w_\sigma\right) dx \sim \frac{\delta_\sigma}{\sigma} |\ln \sigma| |Dw|(Q).$$

This asymptotic analysis can be refined taking into account the non local character of the first term in the energy (1.1). Taking a partition of Q made of sufficiently small subsets Q_i we argue that

$$E_\sigma(w_\sigma, Q) \sim \sum_i E_\sigma(w_\sigma, Q_i) + \left(\frac{\delta}{\sigma}\right)^2 [w_\sigma]^2_{H^{1/2}(Q)},$$

and from the additivity of the total variation of w we get

$$\sum_i E_\sigma(w_\sigma, Q_i) \sim \frac{\delta_\sigma}{\sigma} |\ln\sigma| |Dw|(Q).$$

In other words it is possible to show that the leading energy contribution in the result of [13] is due to the non local part of the energy (1.1) and it is concentrated in a neighbourhood of the diagonal of $Q \times Q$ (see also Corollary 3.9). As a consequence the heuristic asymptotic expansion can be refined as follows

$$\begin{aligned} E_\sigma(w) &\sim \frac{\delta_\sigma}{\sigma} |\ln\sigma| |Dw|(Q) + \left(\frac{\delta_\sigma}{\sigma}\right)^2 [w]^2_{H^{1/2}(Q)} \\ &= \frac{\delta_\sigma}{\sigma} |\ln\sigma| \left(|Dw|(Q) + \frac{\delta_\sigma}{\sigma|\ln\sigma|} [w]^2_{H^{1/2}(Q)} \right). \end{aligned} \tag{1.2}$$

Both the prefactors $\delta_\sigma|\ln\sigma|/\sigma$ and $(\delta_\sigma/\sigma)^2$ are diverging, so that depending on the different choices of δ_σ, with $\delta_\sigma/\sigma \to +\infty$, we get three different regimes.

In view of (1.2) the three regimes are identified by the value of the following limit

$$\lim_{\sigma\to 0^+} \frac{\delta_\sigma}{\sigma|\ln\sigma|} = \Xi.$$

Specifically we have the three asymptotic behaviours:

$\Xi = 0$. The leading pre-factor is the one in front of the total variation; hence, we scale the energies by $\frac{\delta_\sigma}{\sigma}|\ln\sigma|$ and get by (1.2)

$$\frac{\delta_\sigma}{\sigma|\ln\sigma|} [w]^2_{H^{1/2}(Q)} + \frac{1}{\delta_\sigma|\ln\sigma|} \int_Q W\left(\frac{\delta_\sigma}{\sigma} w\right) dx \sim |Dw|(Q).$$

$\Xi \in (0, +\infty)$. The two pre-factors are of the same order; scaling the energies by $\frac{\delta_\sigma}{\sigma}|\ln\sigma|$ in (1.2) entails

$$\frac{\delta_\sigma}{\sigma|\ln\sigma|} [w]^2_{H^{1/2}(Q)} + \frac{1}{\delta_\sigma|\ln\sigma|} \int_Q W\left(\frac{\delta_\sigma}{\sigma} w\right) dx \sim |Dw|(Q) + \Xi\, [w]^2_{H^{1/2}(Q)}.$$

$\Xi = +\infty$. The leading pre-factor is the one in front of the $H^{1/2}$ seminorm; hence, scaling the energies in (1.2) by $(\delta_\sigma/\sigma)^2$ yields

$$[w]^2_{H^{1/2}(Q)} + \frac{\sigma}{\delta_\sigma^2} \int_Q W\left(\frac{\delta_\sigma}{\sigma} w\right) dx \sim [w]^2_{H^{1/2}(Q)}.$$

1.2 The Γ-convergence results

Our results make rigorous, in the one dimensional case, the heuristic computations carried out above. More precisely we study the asymptotic behaviour in terms of Γ-convergence of the following functional

$$F_\varepsilon(u) = \begin{cases} \mu_\varepsilon [u]^2_{H^{1/2}(I)} + \eta_\varepsilon \displaystyle\int_I W\left(\frac{u(x)}{\varepsilon}\right) dx & u \in H^{1/2}(I) \\ +\infty & \text{otherwise.} \end{cases}$$

Depending on the different regimes for the parameters μ_ε and η_ε we obtain the three limits described in the previous section. In particular the first regime can be reduced to the case when $\mu_\varepsilon \to 0$, $\eta_\varepsilon \to \infty$ and $\varepsilon\mu_\varepsilon \ln \eta_\varepsilon \to K \in (0, +\infty)$ as $\varepsilon \to 0^+$. In this case the functional F_ε has the classical structure of a singular perturbation (with regularizing effects) of a multiple well potential, as studied by Modica and Mortola [21] with the regularization given by the Dirichlet integral. As in [21] the distance between the wells tends to zero, the competition between the regularizing term and the penalization force the optimal sequences to make many small jumps so that they may converge in L^1 to any function with bounded variation and in energy to its total variation. Indeed, the limiting functional is given by $2K|Du|(I)$. The role of the non local singular perturbation has been first studied by Alberti, Bouchitté and Seppecher in [2] for the case of a finite number of wells and then considered by Kurzke [18] in the case of infinitely many wells in the one dimensional case (for similar results in higher dimensions see [13], [19], [8] and references therein). Our strategy follows closely the one in [2] and [18] with the additional difficulty that the wells are not well separated.

We also consider a boundary value problem. Here the presence of the non local regularization requires some extra care in the definition of the boundary conditions. We show that the boundary data are compatible with the Γ-convergence result. In this case the limit functional turns out to be very degenerate and in order to get more pieces of information about the minimizers we perform a further analysis computing the next term in the so-called Γ-development (see [4] and [6]). In the same spirit are the results of Cabré and Consul [7] and Kurzke [19] where they compute the so-called renormalized energy of the minimizers (see also [5] and [9]).

The plan of the paper is the following. In Section 2 we state all the results that we will prove: the results corresponding to the regimes described above (see Theorems 2.2, Corollary 2.7, and Remark 2.8) and the ones concerning the second order Γ-limit for the first regime (see Theorems 2.3 and 2.4). Section 3 is devoted to the study of the asymptotic behaviour of (F_ε) in the most diluted regime (the one corresponding to $\Xi = 0$). The main part of the proofs is contained in that section, in particular a sharp lower bound that permits to deduce later in Section 4 also the second order expansion in this regime (see Proposition 4.3 and Proposition 4.4).

2 Statements of the results

The main goal of the paper is to study of the asymptotic behaviour as $\varepsilon \to 0^+$ of the functional $F_\varepsilon : L^1(I) \to [0, +\infty]$ defined by

$$F_\varepsilon(u) = \begin{cases} \mu_\varepsilon [u]^2_{H^{1/2}(I)} + \eta_\varepsilon \displaystyle\int_I W\left(\frac{u(x)}{\varepsilon}\right) dx & u \in H^{1/2}(I) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Here $[u]_{H^{1/2}(I)}$ denotes the seminorm of u in the Sobolev fractional space $H^{1/2}$ on the interval $I = (\alpha, \beta) \subset \mathbf{R}$; i.e.,

$$[u]^2_{H^{1/2}(I)} = \int_I \int_I \frac{|u(x) - u(y)|^2}{|x-y|^2}\, dx\, dy\,.$$

The nonlinear potential is given by a non-negative $\mathbf{Z}$-periodic continuous function W with $W^{-1}(0) = \mathbf{Z}$ and for which there exist $p > 0$, $\delta > 0$ and $c > 0$ such that

$$W(t) \geq c|t|^p \tag{2.2}$$

for any $|t| \leq \delta$. The positive parameters μ_ε and η_ε always satisfy the conditions

$$\lim_{\varepsilon \to 0^+} \varepsilon \mu_\varepsilon \ln \eta_\varepsilon = K \in [0, +\infty)\,, \tag{2.3}$$

$\eta_\varepsilon \to +\infty$ and $\mu_\varepsilon \to \mu$ as $\varepsilon \to 0^+$. We will get different behaviours depending on the value of $\mu = \lim_\varepsilon \mu_\varepsilon$, whether it is 0, finite or $+\infty$.

Remark 2.1 *The asymptotic behaviour for the functional $\mathcal{F}_\sigma : L^1(I) \to [0, +\infty]$ defined by*

$$\mathcal{F}_\sigma(u) = \frac{\delta_\sigma}{\sigma|\ln \sigma|} [u]^2_{H^{1/2}(I)} + \frac{1}{\delta_\sigma |\ln \sigma|} \int_I W\left(\frac{\delta_\sigma}{\sigma} u(x)\right) dx$$

and discussed in Section 1.1 can be deduced from the study of F_ε defined in (2.1) by setting $\varepsilon = \sigma/\delta_\sigma$, $\mu_\varepsilon \sim \frac{\delta_\sigma}{\sigma|\ln \sigma|}$ and $\eta_\varepsilon = \frac{1}{\delta_\sigma|\ln \sigma|}$. With this choice

$$\lim_{\varepsilon \to 0^+} \varepsilon \mu_\varepsilon \ln \eta_\varepsilon = \lim_{\sigma \to 0^+} \frac{|\ln \delta_\sigma|}{|\ln \sigma|},$$

and the three regimes discussed in Section 1.1 are given by the three cases $\mu = 0$, $\mu \in (0, +\infty)$ and $\mu = +\infty$.

As for the first regime we have the following result (see Propositions 3.4 and 3.8 in Section 3).

Theorem 2.2 *Assume that $\mu_\varepsilon \to 0^+$ and that (2.3) is satisfied with $K \neq 0$. The family (F_ε) in (2.1) Γ-converges w.r.t. the L^1 topology to the functional $F : L^1(I) \to [0, +\infty]$ given by*

$$F(u) = \begin{cases} 2K|Du|(I) & \text{if } u \in BV(I) \\ +\infty & \text{otherwise.} \end{cases}$$

More precisely

(i) Compactness: *if* $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$*, then there exist* $u \in BV(I)$*, a subsequence* (u_{ε_k}) *and* $(a_k) \subset \mathbf{R}$ *for which* $(u_{\varepsilon_k} - a_k)$ *converges to* u *in* $L^p(I)$ *for every* $p \in [1, +\infty)$*.*

(ii) Lower bound inequality: *if* $u_\varepsilon \to u$ *in* $L^1(I)$*, we have*

$$\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \geq F(u). \tag{2.4}$$

(iii) Upper bound inequality: *for every* $u \in L^1(I)$ *there exists* $(u_\varepsilon) \subset H^{1/2}(I)$ *such that* $u_\varepsilon \to u$ *in* $L^1(I)$ *and*

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \leq F(u).$$

The result obtained above generalizes the ones contained in [2], [18] and to some extent Theorem 9 in [13] to the case of accumulating zeros. Its proof is partly inspired by those, although some non-trivial points have to be fixed.

In terms of the application to the dislocation problem, we are considering a very diluted regime, so that the total variation of the phase field represents the self interaction of the dislocation lines. In view of the heuristic performed in Section 1.1 we however expect that the regularization due to the long range interaction between dislocations should also play a role. Our aim is then to look at the next order for the energy and identify a term that accounts for this effect. To make this asymptotic expansion rigurous in terms of Γ-convergence we need to specify the type of minimum problem we are considering.

In the following we will consider the case of boundary conditions $u(\alpha) = 0$ and $u(\beta) = L$, with $L > 0$ fixed. The non local regularization and the fact that the limit energy is defined in BV require a suitable definition of the boundary condition. For a given $\delta \in (0, |I|/2)$ we consider the spaces

$$\mathcal{D}_L^\varepsilon(I) = \{u \in H^{1/2}(I) : u\big|_{(\alpha,\alpha+\delta)} \equiv 0,\ u\big|_{(\beta-\delta,\beta)} \equiv \varepsilon\,[L/\varepsilon]\}, \tag{2.5}$$

and introduce the energies $G_\varepsilon : L^1(I) \to [0, +\infty]$ given by

$$G_\varepsilon(u) = \begin{cases} F_\varepsilon(u) & \text{if } u \in \mathcal{D}_L^\varepsilon(I) \\ \\ +\infty & \text{otherwise in } L^1(I). \end{cases} \tag{2.6}$$

This stronger condition is necessary in order to take the boundary data to the limit. A more detailed explanation of this choice will be discussed in Section 4.

Slightly modifying the argument used to prove the Γ-convergence result in Theorem 2.2 we can infer an analogous statement in case boundary data are taken into account.

Theorem 2.3 *Assume that* $\mu_\varepsilon \to 0^+$ *and that (2.3) is satisfied with* $K \neq 0$*. The family of functionals* (G_ε)*,* Γ*-converges w.r.t. the* L^1 *topology to the functional* $G : L^1(I) \to [0, +\infty]$ *defined by*

$$G(u) = \begin{cases} 2K|Du|(I) & \textit{if } u \in \mathcal{D}_L \cap BV(I), \\ \\ +\infty & \textit{otherwise,} \end{cases}$$

where

$$\mathcal{D}_L(I) = \{u \in H^{1/2}(I) : u\big|_{(\alpha,\alpha+\delta)} \equiv 0,\ u\big|_{(\beta-\delta,\beta)} \equiv L\}.$$

The above theorem together with the compactness result (i) of Theorem 2.2 imply the convergence of the minimum problems $\min G_\varepsilon$ to the corresponding minimum of G, together with the convergence of the minimizing sequences to minimum points of G (see Corollary 7.17 [10]).

Clearly the minimization of G does not give many pieces of information on the minimizers of G_ε. Indeed, the minimum value of G on $L^1(I)$ is $2KL$, and it is achieved by any non decreasing function in $BV \cap \mathcal{D}_L(I)$.

Following [4], more accurate pieces of information on the minimizers of G_ε can be recovered by studying the asymptotic Γ-development of (G_ε). In order to do that, by keeping the same notation introduced in Theorem 2.3, we define for any $u \in L^1(I)$

$$G^1_\varepsilon(u) = \frac{G_\varepsilon(u) - 2[L/\varepsilon]\varepsilon^2\mu_\varepsilon \ln \eta_\varepsilon}{\mu_\varepsilon}, \tag{2.7}$$

the scaling μ_ε being suggested by the construction of the recovery sequence in the Γ-limit of F_ε (see (3.34)). In Section 4.2 (see Propositions 4.3 and 4.4) we prove

Theorem 2.4 *Assume that $\mu_\varepsilon \to 0^+$ and that (2.3) is satisfied with $K \neq 0$. Let $\varepsilon = o(\mu_\varepsilon)$, let (G^1_ε) be the family defined in (2.7) and let $G^1 : L^1(I) \to [0, +\infty]$ be given by*

$$G^1(u) = \begin{cases} [u]^2_{H^{1/2}(I)} & u \in \mathcal{D}_L \cap BV(I),\ |Du|(I) = L \\ \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

Then

(i) Compactness: *if $\sup_\varepsilon G^1_\varepsilon(u_\varepsilon) < +\infty$, then there exist $u \in \mathcal{D}_L \cap BV(I)$ such that $|Du|(I) = L$, and a subsequence (u_{ε_k}) which converges to u in $L^p(I)$ for every $p \in [1, +\infty)$.*

(ii) (G^1_ε) Γ-converges w.r.t. the L^1 topology to the functional G^1.

As remarked before, the first order Γ-limit energy arises only from the contribution of the non local part of G_ε concentrated on a neighbourhood of the diagonal. Thus, separating it from the contribution far from the diagonal we are able to prove that the effect of rescaling G_ε by μ_ε is to get the $H^{1/2}$ seminorm in the limit (see Proposition 4.3).

Remark 2.5 *From the result above we can easily deduce that if (u_ε) is a minimizing sequence for G_ε such that*

$$G_\varepsilon(u_\varepsilon) = \min G_\varepsilon + o(\mu_\varepsilon)\,,$$

then (u_ε) is a minimizing sequence also for the functional G^1_ε; hence up to a subsequence it converges to a minimum point for G^1. Futhermore we get the following asymptotic expansion for the minimizers

$$\min G_\varepsilon = 2[L/\varepsilon]\varepsilon^2\mu_\varepsilon \ln \eta_\varepsilon + \mu_\varepsilon \min G^1 + o(\mu_\varepsilon)\,.$$

Remark 2.6 *By strict convexity G^1 has a unique minimizer. By means of a qualitative analysis of the related Euler-Lagrange equation, it is possible to prove that the behaviour of the minimizer at $\alpha + \delta$ and $\beta - \delta$ is of the type*

$$\sqrt{x - \alpha - \delta} \qquad \text{and} \qquad \sqrt{\beta - \delta - x},$$

respectively. If we interpretate this result in terms of the dislocations problem we recover the well known fact (see for instance [16] and [20]) that for a pile up problem the dislocations accumulate at the obstacle with a square root rate (see Figure 1 below).

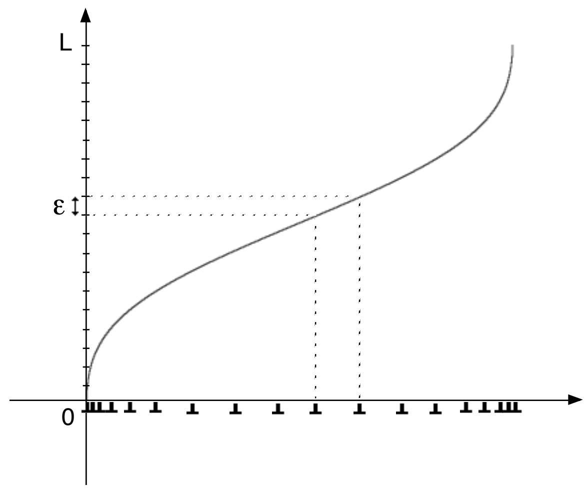

Figure 1: Pile up of dislocations.

The analysis performed in this diluted regime ($\Xi = 0$) permits to deduce straightforward the asymptotic behaviour of the remaining two cases. More precisely, in the case $\Xi \in (0, +\infty)$ we will prove that the two terms in the heuristic expansion are of the same order and both contribute to the limit energy.

Corollary 2.7 *Assume that $\mu_\varepsilon \to \mu \in (0, +\infty)$ and (2.3) hold, then the family (F_ε) in (2.1) Γ-converges w.r.t. the L^1 topology to the functional $F^\mu : L^1(I) \to [0, +\infty]$ given by*

$$F^\mu(u) = \begin{cases} \mu[u]^2_{H^{1/2}(I)} + 2K|Du|(I) & \text{if } u \in BV(I) \cap H^{1/2}(I) \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, if $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, then there exist $u \in BV \cap H^{1/2}(I)$, a subsequence (u_{ε_k}) and $(a_k) \subset \mathbf{R}$ for which $(u_{\varepsilon_k} - a_k)$ converges to u in $L^p(I)$ for every $p \in [1, +\infty)$.

Eventually, we deal with the case $\Xi = +\infty$ which corresponds to $\mu_\varepsilon \to +\infty$. In this case we have to rescale the energy in order to get a finite limit and study the Γ-limit of $F_\varepsilon/\mu_\varepsilon$. This identifies three other regimes and the corresponding asymptotic behaviours can be deduced either from Theorem 2.2 or from Corollary 2.7.

Remark 2.8 *The asymptotic behaviour of the functional $\mathcal{F}_\sigma$ for the dislocations problem in the regime $\Xi = +\infty$ corresponds to the Γ-limit of*

$$\frac{F_\varepsilon(u)}{\mu_\varepsilon} = \begin{cases} [u]^2_{H^{1/2}(I)} + \dfrac{\eta_\varepsilon}{\mu_\varepsilon} \displaystyle\int_I W\left(\frac{u(x)}{\varepsilon}\right) dx & u \in H^{1/2}(I) \\ +\infty & \text{otherwise} \end{cases}$$

in the case when $\varepsilon \ln \frac{\eta_\varepsilon}{\mu_\varepsilon} = \frac{\sigma}{\delta} \ln \frac{\sigma}{\delta^2} \to 0$. *In view of Corollary 2.7 this gives*

$$F^\infty(u) = \begin{cases} [u]^2_{H^{1/2}(I)} & \text{if } u \in H^{1/2}(I) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

3 Proof of the Γ-convergence result

In this section we prove the Γ-convergence result for the family (F_ε) stated in Theorem 2.2 as well as some refinements concerning the localization of the energy contribution (see Corollaries 3.9 and 3.10).

The proof of Theorem 2.2 is split in the subsequent Proposition 3.4 and Proposition 3.8. In particular, in the former we prove the compactness result as well as the lower bound inequality, while the latter is devoted to the proof of the upper bound inequality.

In the following we need some more notation. In particular, we denote by $F_\varepsilon(\cdot, A)$ the localized version of F_ε obtained by changing the domain of integration I into an open interval A contained in I.

3.1 The compactness and the Γ-liminf inequality

The main tool in order to prove the compactness and the Γ-liminf inequality is the optimal lower bound given by the following lemma (in the same spirit of what done in [2],[18]). Denote by $A_{u,t}$ the *t-super-level set* of u in A, i.e. $A_{u,t} = \{x \in A : u(x) > t\}$.

Lemma 3.1 *Let* $\theta_\varepsilon \in (0, \varepsilon/4)$ *and a non-constant* $u \in L^\infty \cap H^{1/2}(A)$, *A being an interval, be given. If* $r, s \in \mathbf{Z}$ *are such that* $\left[\frac{\operatorname{ess\,inf}_A u}{\varepsilon}\right] \le r < s \le \left[\frac{\operatorname{ess\,sup}_A u}{\varepsilon}\right]$[1], *then*

$$[u]^2_{H^{1/2}(A)} \ge 2(s-r)(\varepsilon - 2\theta_\varepsilon)^2 \left(\ln\left(\frac{|A_{u,\varepsilon s - \theta_\varepsilon}||A \setminus A_{u,\varepsilon r + \theta_\varepsilon}|}{|A|^2} \right) - \ln\left(\frac{|B^{r,s}_\varepsilon|}{|A|} \right) \right), \tag{3.1}$$

where $B^{r,s}_\varepsilon = \cup_{j=r}^s (A_{u,\varepsilon j + \theta_\varepsilon} \setminus A_{u,\varepsilon(j+1) - \theta_\varepsilon})$.

Roughly speaking the proof goes as follows. After reducing to monotone functions, we have to estimate the transition cost between two wells $k\varepsilon$ far away, it is then possible to show that it is more convenient to make k-transitions of height ε rather than one of height $k\varepsilon$ (see inequality (3.8)). Then an optimal lower bound follows essentially by a rearrangement of all the terms involved.

As in [2] a lower bound for the energy of a function u on a transition between two wells can be obtained by estimating the non local term with a double integral on suitably chosen sublevels of u. In this direction we will often use the following remark.

Remark 3.2 *Let* $(\alpha_i, \beta_i) \subset I$, $i = 1, 2$, *be disjoint with* $\beta_1 \le \alpha_2$, *then*

$$\int_{(\alpha_1,\beta_1)\times(\alpha_2,\beta_2)} \frac{1}{|x-y|^2} dxdy \le \frac{(\beta_2 - \alpha_2)(\beta_1 - \alpha_1)}{(\alpha_2 - \beta_1)^2}, \tag{3.2}$$

[1]Note that r and s satisfying the assumption of Lemma 3.1, always exist if ε is small enough.

more precisely a direct computation yields

$$\int_{(\alpha_1,\beta_1)\times(\alpha_2,\beta_2)} \frac{1}{|x-y|^2}dxdy = \ln\frac{(\beta_2-\beta_1)(\alpha_2-\alpha_1)}{(\beta_2-\alpha_1)(\alpha_2-\beta_1)}. \tag{3.3}$$

Proof of Lemma 3.1. Let us first assume u to be non-decreasing and consider the sets $A^i_\varepsilon := A_{u,\varepsilon i-\theta_\varepsilon} \setminus A_{u,\varepsilon i+\theta_\varepsilon}$ for $i \in \{r+1,\dots,s-1\}$. With a slight abuse of notation we also denote $A^r_\varepsilon := A \setminus A_{u,\varepsilon r+\theta_\varepsilon}$ and $A^s_\varepsilon := A_{u,\varepsilon s-\theta_\varepsilon}$. Then the sets A^i_ε, for $i \in \{r,\dots,s\}$, and $A_{u,\varepsilon j+\theta_\varepsilon} \setminus A_{u,\varepsilon(j+1)-\theta_\varepsilon}$ for $j \in \{r,\dots,s\}$ provide a disjoint ordered partition of A into intervals.

By using the notations introduced above and by taking into account (3.3) we get

$$\begin{aligned}
[u]^2_{H^{1/2}(A)} &\geq 2\sum_{r\leq i<j\leq s}\int_{A^i_\varepsilon\times A^j_\varepsilon}\frac{((j-i)\varepsilon-2\theta_\varepsilon)^2}{|x-y|^2}dxdy \\
&= \sum_{r\leq i<j\leq s}\Lambda^\varepsilon_{j-i}\ln\frac{(\sup A^j_\varepsilon-\sup A^i_\varepsilon)(\inf A^j_\varepsilon-\inf A^i_\varepsilon)}{(\sup A^j_\varepsilon-\inf A^i_\varepsilon)(\inf A^j_\varepsilon-\sup A^i_\varepsilon)},
\end{aligned} \tag{3.4}$$

where

$$\Lambda^\varepsilon_k = 2(\varepsilon k-2\theta_\varepsilon)^2 \tag{3.5}$$

for every $k \geq 1$ and $\Lambda^\varepsilon_0 = 0$.

Let us denote now $a^i_\varepsilon = |A_{u,\varepsilon i-\theta_\varepsilon} \setminus A_{u,\varepsilon i+\theta_\varepsilon}|$ and $b^i_\varepsilon = |A_{u,\varepsilon i+\theta_\varepsilon} \setminus A_{u,\varepsilon(i+1)-\theta_\varepsilon}|$. Since u is non decreasing and belongs to $H^{1/2}(A)$, it is continuous and then a^i_ε and b^i_ε are strictly positive for every $i \in \{r,\dots,s\}$. Moreover, let $a^{j,k}_\varepsilon = \sum_{i=j}^k a^i_\varepsilon$ if $k \geq j$ and 0 otherwise, and define $b^{j,k}_\varepsilon$ analogously. With this notation we have

$$\begin{aligned}
&\sup A^j_\varepsilon - \sup A^i_\varepsilon = a^{i+1,j}_\varepsilon + b^{i,j-1}_\varepsilon\ , \ \inf A^j_\varepsilon - \inf A^i_\varepsilon = a^{i,j-1}_\varepsilon + b^{i,j-1}_\varepsilon, \\
&\sup A^j_\varepsilon - \inf A^i_\varepsilon = a^{i,j}_\varepsilon + b^{i,j-1}_\varepsilon\ , \ \inf A^j_\varepsilon - \sup A^i_\varepsilon = a^{i+1,j-1}_\varepsilon + b^{i,j-1}_\varepsilon.
\end{aligned}$$

Then we can rewrite (3.4) as

$$\begin{aligned}
[u]^2_{H^{1/2}(A)} &\geq \sum_{r\leq i<j\leq s}\Lambda^\varepsilon_{j-i}\left[\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,j-1}_\varepsilon+b^{i,j-1}_\varepsilon}\right)-\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,j}_\varepsilon+b^{i,j-1}_\varepsilon}\right)\right] \\
&= \sum_{i=r}^{s-1}\sum_{k=1}^{s-i}\Lambda^\varepsilon_k\left[\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,i+k-1}_\varepsilon+b^{i,i+k-1}_\varepsilon}\right)-\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,i+k}_\varepsilon+b^{i,i+k-1}_\varepsilon}\right)\right] \\
&\geq \sum_{i=r}^{s-1}\sum_{k=1}^{s-i}\Lambda^\varepsilon_k\left[\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,i+k-1}_\varepsilon+|B^{r,s}_\varepsilon|}\right)-\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,i+k}_\varepsilon+|B^{r,s}_\varepsilon|}\right)\right],
\end{aligned} \tag{3.6}$$

where the last inequality follows from the fact that the functions

$$t\in[0,+\infty)\to\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,i+k-1}_\varepsilon+t}\right)-\ln\left(1+\frac{a^i_\varepsilon}{a^{i+1,i+k}_\varepsilon+t}\right) \tag{3.7}$$

are non-negative and decreasing for every i,k.

By the convexity of $t \to 2t^2$ we deduce $2\Lambda^\varepsilon_{k+1} < \Lambda^\varepsilon_{k+2} + \Lambda^\varepsilon_k$, which in turn, by induction, implies that the sequence $(k^{-1}\Lambda^\varepsilon_k)_{k\geq 1}$ is increasing, and thus

$$k\Lambda^\varepsilon_1 \leq \Lambda^\varepsilon_k. \tag{3.8}$$

Inequality (3.8), together with the fact that the function in (3.7) is positive and a rearrangement of the terms in (3.6), gives

$$\begin{aligned}[u]^2_{H^{1/2}(A)} \geq{}& \Lambda^\varepsilon_1 \sum_{i=r}^{s-1} \ln\left(1 + \frac{a^i_\varepsilon}{|B^{r,s}_\varepsilon|}\right) \\ &-\Lambda^\varepsilon_1 \sum_{i=r}^{s-1}(s-i)\ln\left(1 + \frac{a^i_\varepsilon}{a^{i+1,s}_\varepsilon + |B^{r,s}_\varepsilon|}\right) + \Lambda^\varepsilon_1 \sum_{i=r}^{s-2}\sum_{k=1}^{s-i-1} \ln\left(1 + \frac{a^i_\varepsilon}{a^{i+1,i+k}_\varepsilon + |B^{r,s}_\varepsilon|}\right).\end{aligned} \tag{3.9}$$

Set $L^{i,j}_\varepsilon = \ln\big((|B^{r,s}_\varepsilon| + a^{i,j}_\varepsilon)/|A|\big)$, then by its very definition $L^{i,j}_\varepsilon \leq 0$ for every i, j, $L^{r,s}_\varepsilon = 0$, and $L^{i,j}_\varepsilon = \ln(|B^{r,s}_\varepsilon|/|A|)$ for $i > j$. By using this notation, the right hand side of (3.9) can be rewritten as

$$\begin{aligned}[u]^2_{H^{1/2}(A)} &\geq \Lambda^\varepsilon_1 \sum_{i=r}^{s-1} L^{i,i}_\varepsilon - (s-r)\Lambda^\varepsilon_1 \ln\left(\frac{|B^{r,s}_\varepsilon|}{|A|}\right) \\ &\quad -\Lambda^\varepsilon_1 \sum_{i=r}^{s-1}(s-i)\left(L^{i,s}_\varepsilon - L^{i+1,s}_\varepsilon\right) + \Lambda^\varepsilon_1 \sum_{i=r}^{s-2}\sum_{k=1}^{s-i-1}\left(L^{i,i+k}_\varepsilon - L^{i+1,i+k}_\varepsilon\right) \\ &= \Lambda^\varepsilon_1 \sum_{i=r}^{s-1}\left(L^{r,i}_\varepsilon + L^{i+1,s}_\varepsilon - \ln\left(\frac{|B^{r,s}_\varepsilon|}{|A|}\right)\right).\end{aligned} \tag{3.10}$$

Eventually, since $L^{r,i}_\varepsilon \geq L^{r,r}_\varepsilon$, $L^{i+1,s}_\varepsilon \geq L^{s,s}_\varepsilon$ for every $i \in \{r, \dots, s-1\}$, we get by (3.10)

$$[u]^2_{H^{1/2}(A)} \geq \Lambda^\varepsilon_1 (s-r)\left(L^{r,r}_\varepsilon + L^{s,s}_\varepsilon - \ln\left(\frac{|B^{r,s}_\varepsilon|}{|A|}\right)\right),$$

from which the conclusion follows.

In order to remove the additional assumption on u to be non-decreasing we notice that (3.1) involves only the measure of sub-/super-level sets of u. Then, to prove (3.1) in the general case it is enough to apply the previous argument to the non-decreasing rearrangement u^* of u, since the two functions are equi-distributed and moreover $[u]_{H^{1/2}(A)} \geq [u^*]_{H^{1/2}(A)}$ (see [14] or Theorem 5.8 [1]). □

By combining (3.1) and the potential term we are now able to get a pointwise lower bound for the energies F_ε.

Corollary 3.3 *Fix $\theta_\varepsilon = \varepsilon^{1+1/p}$. Under the same assumptions and keeping the same notations of Lemma 3.1 it holds* [2]

$$\begin{aligned}F_\varepsilon(u, A) \geq{}& 2\mu_\varepsilon(s-r)(\varepsilon - 2\theta_\varepsilon)^2 \ln\left(\frac{|A_{u,\varepsilon s-\theta_\varepsilon}||A \setminus A_{u,\varepsilon r+\theta_\varepsilon}|}{|A|^2}\right) \\ &+2\mu_\varepsilon(s-r)(\varepsilon-2\theta_\varepsilon)^2 \ln\left(\frac{\omega_\varepsilon\eta_\varepsilon|A|}{2\mu_\varepsilon(\varepsilon-2\theta_\varepsilon)^2(s-r)}\right)\end{aligned} \tag{3.11}$$

[2] As in Lemma 3.1 the assumptions of Corollary 3.3 are satisfied if ε is small enough.

where $\omega_\varepsilon = \inf_{d(y,\mathbf{Z})\geq\theta_\varepsilon/\varepsilon} W$.

Proof. By the very definition of $B_\varepsilon^{r,s}$ it holds

$$\int_A W\left(\frac{u(x)}{\varepsilon}\right) dx \geq \omega_\varepsilon |B_\varepsilon^{r,s}|,$$

thus by applying (3.1) one gets

$$\begin{aligned} F_\varepsilon(u,A) &\geq 2\mu_\varepsilon(s-r)(\varepsilon-2\theta_\varepsilon)^2 \ln\left(\frac{|A_{u,\varepsilon s-\theta_\varepsilon}||A\setminus A_{u,\varepsilon r+\theta_\varepsilon}|}{|A|^2}\right) \\ &-2\mu_\varepsilon(s-r)(\varepsilon-2\theta_\varepsilon)^2 \ln\left(\frac{|B_\varepsilon^{r,s}|}{|A|}\right) + \eta_\varepsilon\omega_\varepsilon|B_\varepsilon^{r,s}|. \end{aligned}$$

By optimizing the right hand side above as a function of $|B_\varepsilon^{r,s}|$ it is easy to see that the minimum value is attained in $\frac{2\mu_\varepsilon(\varepsilon-2\theta_\varepsilon)^2(s-r)}{\eta_\varepsilon\omega_\varepsilon}$, which gives

$$\begin{aligned} F_\varepsilon(u,A) &\geq 2\mu_\varepsilon(s-r)(\varepsilon-2\theta_\varepsilon)^2 \ln\left(\frac{|A_{u,\varepsilon s-\theta_\varepsilon}||A\setminus A_{u,\varepsilon r+\theta_\varepsilon}|}{|A|^2}\right) \\ &+2\mu_\varepsilon(s-r)(\varepsilon-2\theta_\varepsilon)^2 \ln\left(\frac{\omega_\varepsilon\eta_\varepsilon|A|}{2\mu_\varepsilon(\varepsilon-2\theta_\varepsilon)^2(s-r)}\right) + \eta_\varepsilon\omega_\varepsilon\frac{2\mu_\varepsilon(\varepsilon-2\theta_\varepsilon)^2(s-r)}{\eta_\varepsilon\omega_\varepsilon} \end{aligned}$$

and implies (3.11). □

We are now in a position to prove the compactness and the lower bound results stated in Theorem 2.2.

Proposition 3.4 *Assume that* $\mu_\varepsilon \to 0^+$ *and that (2.3) is satisfied with* $K \neq 0$. *Let* $(u_\varepsilon) \subset H^{1/2}(I)$, *then*

(i) Compactness: *if* $\sup F_\varepsilon(u_\varepsilon) < +\infty$, *then there exist* $u \in BV(I)$, *a subsequence* (u_{ε_k}) *and* $(z_k) \subset \mathbf{Z}$ *for which* $(u_{\varepsilon_k} - \varepsilon_k z_k)$ *converges to* u *in* $L^p(I)$ *for every* $p \in [1,+\infty)$.

(ii) Lower bound inequality: *if* $u_\varepsilon \to u$ *in* $L^1(I)$, *we have*

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \geq F(u). \tag{3.12}$$

The proof of the compactness follows in part the strategy developed in [18] for the case in which W has countably but well separated zeros. Further, in the same paper an optimal estimate in the Orlicz space e^L is proved (see Remark 3.6). This result is based on a fine estimate of the decay of the super-levels of the sequence u_ε combined with the well known Trudinger's embedding for fractional Sobolev spaces (see [15], [23]) that we recall below for the reader's convenience.

Theorem 3.5 *There are constants* $C, C' > 0$ *such that for every* $u \in H_0^{1/2}(A)$, *the following estimate holds*

$$\int_A \exp\left(\frac{C|u(x)|^2}{[u]^2_{H^{1/2}(A)}}\right) dx \leq C'|A|,$$

where $H_0^{1/2}(A)$ *denotes the closure of* $C_0^\infty(A)$ *w.r.t. the seminorm* $[\cdot]_{H^{1/2}(A)}$.

Proof of Proposition 3.4. *Compactness.* Let $M > 0$ be such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) \le M$, and denote by $\mathrm{med}(u_\varepsilon, I)$ a median of u_ε in I, i.e., $|I_{-u_\varepsilon,-t}| \le |I|/2$ for $t < \mathrm{med}(u_\varepsilon, I)$ and $|I_{u_\varepsilon,t}| \le |I|/2$ for $t > \mathrm{med}(u_\varepsilon, I)$, and let $z_\varepsilon = [\mathrm{med}(u_\varepsilon, I)/\varepsilon] \in \mathbf{Z}$.

If we define $v_\varepsilon = u_\varepsilon - \varepsilon z_\varepsilon$, it is clear from the definition of z_ε that $|I \setminus I_{v_\varepsilon,\varepsilon}| \ge |I|/2$ and $|I_{v_\varepsilon,-\varepsilon}| \ge |I|/2$.

We claim that (v_ε) has a subsequence pre-compact in $L^p(I)$ for every $p \in [1, +\infty)$. The proof of the claimed assertion will be split into several steps.

Step 1. *An estimate on the super level sets: We show that there exist $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0)$ and $t \ge 16M/K$ it holds*

$$|I_{|v_\varepsilon|,t}| \le 2[(2\mu_\varepsilon \varepsilon t) \vee 1]|I|e^{-\frac{M}{2\mu_\varepsilon \varepsilon t}}. \tag{3.13}$$

In order to prove (3.13) it is clear that we can assume $|I_{|v_\varepsilon|,t}| > 0$, being otherwise the inequality trivial. Furthermore, since $I_{|v_\varepsilon|,t} = I_{v_\varepsilon,t} \cup I_{-v_\varepsilon,t}$ we can estimate the measures of the latter two sets separately. Moreover, we choose M large enough such that $16M/K > 1$.

Fix $\varepsilon_0 \in (0, 1/2)$ such that $2\mu_\varepsilon \varepsilon \ln(\omega_\varepsilon \eta_\varepsilon |I|) \ge K$ for $\varepsilon \in (0, \varepsilon_0)$, then by applying Corollary 3.3 with $r = 1$ and $s = [\frac{t}{\varepsilon}]$ we get

$$\begin{aligned} M &\ge F_\varepsilon(u_\varepsilon) = F_\varepsilon(v_\varepsilon) \\ &\ge 2\mu_\varepsilon \left(\left[\frac{t}{\varepsilon} \right] - 1 \right) (\varepsilon - 2\theta_\varepsilon)^2 \left[\ln \left(\frac{|I_{v_\varepsilon,\varepsilon[\frac{t}{\varepsilon}]-\theta_\varepsilon}||I \setminus I_{v_\varepsilon,\varepsilon+\theta_\varepsilon}|}{|I|^2} \right) + \ln \left(\frac{\omega_\varepsilon \eta_\varepsilon |I|}{2\mu_\varepsilon \varepsilon t} \right) \right] \\ &\ge 2\mu_\varepsilon \left(\left[\frac{t}{\varepsilon} \right] - 1 \right) (\varepsilon - 2\theta_\varepsilon)^2 \left[\ln \left(\frac{|I_{v_\varepsilon,t}|}{2|I|} \right) - (\ln(2\mu_\varepsilon \varepsilon t) \vee 0) + \frac{K}{2\mu_\varepsilon \varepsilon} \right]. \end{aligned} \tag{3.14}$$

To obtain the last inequality we used $|I_{v_\varepsilon,\varepsilon[\frac{t}{\varepsilon}]-\theta_\varepsilon}| \ge |I_{v_\varepsilon,t}|$, $|I \setminus I_{v_\varepsilon,\varepsilon+\theta_\varepsilon}| \ge |I \setminus I_{v_\varepsilon,\varepsilon}| \ge |I|/2$ and that, thanks to the choice of ε_0, $2\mu_\varepsilon \varepsilon \ln(\omega_\varepsilon \eta_\varepsilon |I|) \ge K$. Since $\theta_\varepsilon \in (0, \varepsilon/4)$, the fact that $\varepsilon < 1/2$ and $t > 16M/K > 1$ implies that $\frac{\varepsilon}{4}(t - 2\varepsilon) \le \left(\left[\frac{t}{\varepsilon}\right] - 1\right)(\varepsilon - 2\theta_\varepsilon)^2 \le \varepsilon t$ and that $t - 2\varepsilon > \frac{t}{2}$. Thus from (3.14) we get

$$M \ge t \left(2\mu_\varepsilon \varepsilon \ln \left(\frac{|I_{v_\varepsilon,t}|}{2|I|} \right) - 2\mu_\varepsilon \varepsilon (\ln(2\mu_\varepsilon \varepsilon t) \vee 0) + \frac{K}{8} \right)$$

and so

$$|I_{v_\varepsilon,t}| \le 2|I| \exp \left(\frac{M - \frac{K}{8}t}{2\mu_\varepsilon \varepsilon t} + \ln(2\mu_\varepsilon \varepsilon t) \vee 0 \right).$$

To derive the desired inequality for $|I_{v_\varepsilon,t}|$ notice that $\exp(\ln(2\mu_\varepsilon \varepsilon t) \vee 0) = (2\mu_\varepsilon \varepsilon t) \vee 1$ and that $M - \frac{K}{8}t \le -M$ on $(16M/K, +\infty)$.

Obviously, the very same proof applies to $-v_\varepsilon$ and t in order to estimate $|I_{-v_\varepsilon,t}|$, so that (3.13) follows.

Step 2. *(v_ε) is weakly pre-compact in $L^1(I)$.*
In this step we improve the pointwise bound for the measure of the super-level sets of $|v_\varepsilon|$ obtained in Step 1 to a uniform bound in ε. From this we deduce the equi-integrabilty of the sequence v_ε, which in turn gives the desired pre-compactness property by the Dunford-Pettis' criterion (see Proposition 1.3 [22]). More precisely, we show that there exist two positive constants c_1, c_2 such that for every $t \ge 16M/K$ and for all ε small enough we have

$$|I_{|v_\varepsilon|,t}| \le c_1 e^{-c_2 t}. \tag{3.15}$$

As before we estimate $|I_{v_\varepsilon,t}|$ and then deduce the same inequality for $|I_{-v_\varepsilon,t}|$. Furthermore, without loss of generality we may assume v_ε to be non-decreasing (see [14] or Theorem 5.8 [1]).

In order to improve estimate (3.13) the idea is to make use of the Trudinger's inequality stated in Theorem 3.5. This requires to modify v_ε. Set $\tilde{I} = (\alpha, 2\beta - \alpha)$ and let $\tilde{v}_\varepsilon : \tilde{I} \to \mathbf{R}$ be the function

$$\tilde{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & x \in (\alpha,\beta) \\ v_\varepsilon(2\beta - x) & x \in (\beta, 2\beta-\alpha). \end{cases}$$

It is easy to check that $\tilde{v}_\varepsilon \in H^{1/2}(\tilde{I})$ with $[\tilde{v}_\varepsilon]^2_{H^{1/2}(\tilde{I})} \le 4[v_\varepsilon]^2_{H^{1/2}(I)}$[3]. Then we define $\hat{v}_\varepsilon = (\tilde{v}_\varepsilon - N)_+$, with $N = 16M/K$. Hence $\hat{v}_\varepsilon \in H^{1/2}_0(\tilde{I}_{\tilde{v}_\varepsilon,N})$

By applying Theorem 3.5 above to $\hat{v}_\varepsilon$ on the interval $A = \tilde{I}_{\tilde{v}_\varepsilon,N}$, Chebychev's inequality yields, for every $t > N$,

$$|\tilde{I}_{\tilde{v}_\varepsilon,t}| \le C'|\tilde{I}_{\tilde{v}_\varepsilon,N}| \exp\left(-\frac{Ct^2}{[\hat{v}_\varepsilon]^2_{H^{1/2}(\tilde{I}_{\tilde{v}_\varepsilon,N})}}\right) \le C'|\tilde{I}_{\tilde{v}_\varepsilon,N}|e^{-C\mu_\varepsilon t^2/4M}\,, \tag{3.16}$$

where we also used that $\mu_\varepsilon[\hat{v}_\varepsilon]^2_{H^{1/2}(\tilde{I}_{\tilde{v}_\varepsilon,N})} \le 4M$. Moreover, since $|\tilde{I}_{\tilde{v}_\varepsilon,s}| = 2|I_{v_\varepsilon,s}|$ for any s, we have by (3.16), Step 1

$$|I_{v_\varepsilon,t}| \le C'|I_{v_\varepsilon,N}|e^{-\frac{C\mu_\varepsilon t^2}{4M}} \le c[(2\mu_\varepsilon\varepsilon N)\vee 1]e^{-\frac{M}{2\varepsilon\mu_\varepsilon N} - \frac{C\mu_\varepsilon t^2}{4M}}.$$

If ε is small enough to guarantee that $2\mu_\varepsilon\varepsilon N < 1$ and we apply Young's inequality to the previous estimate we get

$$|I_{v_\varepsilon,t}| \le c_1 e^{-c_2 t}$$

for some positive constants c_1 and c_2. Obviously, a similar inequality holds for $|I_{-v_\varepsilon,t}|$ if $t \ge 16M/K$, so that the claim follows.

Step 3. *Ruling-out oscillations: (v_ε) is pre-compact w.r.t. the convergence in measure.*
By Step 2 we may apply the Fundamental Theorem of Young Measures (see Theorem 6.2 [22]) and extract a sub-sequence (v_{ε_k}) which generates a Young measure (ν_x) on I.

For every open subset $A \subseteq I$, let

$$L_A = \sup\left\{t \in \mathbf{R} : \liminf_{k\to+\infty} |A_{v_{\varepsilon_k},t}| > 0\right\},$$
$$l_A = \inf\left\{t \in \mathbf{R} : \liminf_{k\to+\infty} |A \setminus A_{v_{\varepsilon_k},t}| > 0\right\},$$

and note that $l_A \le L_A$ and $l_A, L_A \in (-16M/K, 16M/K)$ by (3.13). Moreover, for every $t \in \mathbf{R}$ we have (as follows easily from Theorem 6.2 [22])

$$\begin{aligned}\liminf_{k\to+\infty} |A_{v_{\varepsilon_k},t}| &= \liminf_{k\to+\infty} \int_A \chi_{(t,+\infty)}(v_{\varepsilon_k}(x))dx \\ &\ge \int_A \int_{\mathbf{R}} \chi_{(t,+\infty)}(\lambda)d\nu_x(\lambda)dx = \int_A \nu_x((t,+\infty))dx,\end{aligned}$$

[3]Indeed, we have $[\tilde{v}_\varepsilon]^2_{H^{1/2}(\tilde{I})} = [v_\varepsilon]^2_{H^{1/2}(I)} + [\tilde{v}_\varepsilon]^2_{H^{1/2}(\tilde{I}\setminus I)} + 2\int_{I\times(\tilde{I}\setminus I)} \left|\frac{v_\varepsilon(x)-\tilde{v}_\varepsilon(y)}{x-y}\right|^2 dxdy$. Then, an easy change of variables gives $[\tilde{v}_\varepsilon]^2_{H^{1/2}(\tilde{I}\setminus I)} = [v_\varepsilon]^2_{H^{1/2}(I)}$, and $\int_{I\times(\tilde{I}\setminus I)} \left|\frac{v_\varepsilon(x)-\tilde{v}_\varepsilon(y)}{x-y}\right|^2 dxdy \le [v_\varepsilon]^2_{H^{1/2}(I)}$.

and analogously

$$\liminf_{k\to+\infty} |A \setminus A_{v_{\varepsilon_k},t}| \geq \int_A \nu_x((-\infty,t))dx.$$

From this, one infers spt$\nu_x \subseteq [l_A, L_A]$ for a.e. $x \in A$ and, being $u(x) = \langle Id, \nu_x\rangle$ the barycenter of ν_x, we have $u(x) \in [l_A, L_A]$ for a.e. $x \in A$.

With fixed $x \in I$, for every $r \in (0, d(x, \partial I))$ we claim that

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(u_{\varepsilon_k}, I_r(x)) \geq 2K \left(L_{I_r(x)} - l_{I_r(x)}\right) , \tag{3.17}$$

where $I_r(x) = (x-r, x+r)$. Taking the latter inequality for granted it is easy to prove the conclusion of Step 3. Indeed, since $\sup_\varepsilon F_\varepsilon(u_\varepsilon) \leq M$ the set $\{x \in I : \lim_r (\liminf_k F_{\varepsilon_k}(u_{\varepsilon_k}, I_r(x))) \geq \frac{1}{n}\}$ must be finite for every $n \in \mathbf{N}$, and thus estimate (3.17) implies the existence of a countable set J such that for every $x \in I \setminus J$

$$\lim_{r\to 0^+} \left(L_{I_r(x)} - l_{I_r(x)}\right) = 0.$$

Hence, $\nu_x = \delta_{u(x)}$ for $x \in I \setminus J$, and so the sequence (v_{ε_k}) converges in measure to u (see Lemma 6.3 [22]).

In the following we justify (3.17). We may assume that $L_{I_r(x)} > l_{I_r(x)}$, being otherwise the statement trivial. Fix $l_{I_r(x)} < t < T < L_{I_r(x)}$, and apply Corollary 3.3 to u_ε with reference interval $A = I_r(x)$, $r_\varepsilon = \left[\frac{t}{\varepsilon}\right]$, $s_\varepsilon = \left[\frac{T}{\varepsilon}\right]$ and ε small enough to ensure $r_\varepsilon < s_\varepsilon$, to get

$$\begin{aligned} F_\varepsilon(u_\varepsilon, A) &\geq 2\mu_\varepsilon(s_\varepsilon - r_\varepsilon)(\varepsilon - 2\theta_\varepsilon)^2 \ln\left(\frac{|A_{u_\varepsilon,\varepsilon s_\varepsilon - \theta_\varepsilon}||A \setminus A_{u_\varepsilon,\varepsilon r_\varepsilon+\theta_\varepsilon}|}{|A|^2}\right) \\ &+2\mu_\varepsilon(s_\varepsilon - r_\varepsilon)(\varepsilon - 2\theta_\varepsilon)^2 \ln\left(\frac{\omega_\varepsilon\eta_\varepsilon|A|}{2\mu_\varepsilon(s_\varepsilon - r_\varepsilon)(\varepsilon - 2\theta_\varepsilon)^2}\right). \end{aligned} \tag{3.18}$$

Notice that the choice of t, T implies that the first term on the right hand side of (3.18) above is infinitesimal as $\varepsilon \to 0^+$. Moreover, $\varepsilon \ln \omega_\varepsilon$ is infinitesimal as $\varepsilon \to 0^+$ by the continuity assumption on W, (2.2) and the choice of θ_ε. Thus, from (3.18) we may conclude that

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(u_{\varepsilon_k}, I_r(x)) \geq 2K (T - t) ,$$

and then (3.17) follows as $t \to l^+_{I_r(x)}$ and $T \to L^-_{I_r(x)}$.

Step 4. *Conclusion: (v_ε) is pre-compact in $L^p(I)$ for any $p \in [1, +\infty)$.*
Step 2 and Step 3 imply that (v_ε) is pre-compact in $L^1(I)$, the conclusion is then straightforward thanks to (3.15).

We are now in a position to prove the lower bound inequality essentially by localizing (3.11) and arguing as in Step 3 above.

Lower bound inequality. Without loss of generality we may assume $u_\varepsilon \to u$ a.e. in I, and

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \leq M < +\infty. \tag{3.19}$$

Moreover, remark that [4]

$$F_\varepsilon(u_\varepsilon) \geq F_\varepsilon\left(u_\varepsilon \wedge \varepsilon\left[\frac{k}{\varepsilon}\right] \vee \left(-\varepsilon\left[\frac{k}{\varepsilon}\right]\right)\right)$$

and that, by Step 3, $u \in L^\infty(I)$, with $\|u\|_{L^\infty} \leq 16M/K$; hence we may assume (u_ε) equi-bounded in $L^\infty(I)$.

For a given partition $(I_j)_{j=1}^r$ of I into pairwise disjoint intervals, fix $1 \leq j \leq r$ for which u is not constant on I_j, so that we may also assume u_ε not constant on I_j.

We can argue as in (3.18) of Step 3 with $A = I_j$, noticing that in this case $L_{I_j} = \operatorname{ess\,sup}_{I_j} u$ and $l_{I_j} = \operatorname{ess\,inf}_{I_j} u$ as follows from the convergence of (u_ε) to u in $L^1(I)$. Thus, (3.17) rewrites as

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I_j) \geq 2K \operatorname{osc}_{I_j} u\,, \tag{3.20}$$

where $\operatorname{osc}_A u = \operatorname{ess\,sup}_A u - \operatorname{ess\,inf}_A u$ denotes the essential oscillation of u on A. Actually, the latter estimate holds for every $1 \leq j \leq r$, being trivial in case u is constant on I_j. Then, the sub-additivity of the inferior limit operator implies

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \geq 2K \sum_{j=1}^r \operatorname{osc}_{I_j} u,$$

from which, by passing on the supremum over the partitions, we deduce $u \in BV(I)$ and

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \geq 2K|Du|(I).$$

□

Remark 3.6 *As a consequence of the estimate (3.15) in the proof of the compactness result we get an apriori estimate in the Orlicz space* e^L *for any sequence with equi-bounded energy. As in [18] one can give an example of a sequence with equi-bounded energy that is not bounded in* L^∞. *This in essence shows that the estimate in* e^L *is optimal.*

Remark 3.7 *Note that the argument performed in Step 3 in the above proof that gives (3.20) is still valid if* $\mu_\varepsilon \to \mu \in [0, +\infty)$.

3.2 The Γ-limsup inequality

In this subsection we establish the upper bound inequality which completes the proof of Theorem 2.2.

As common we first prove the Γ-limsup inequality fo a subclass of functions dense in the L^1 convergence and in a topology for which the limit energy is continuous.

We choose piecewise affine and piecewise strictly monotone functions and provide for those functions a recovery sequence. This class satisfies the requirements above and also provide the upper bound for the second order development of Section 4.2 (see Theorem 2.4 and Proposition

[4] This follows by taking into account that by truncation the oscillation of u_ε is reduced and so is the $H^{1/2}$ semi-norm, and that $W \geq 0$, while $\pm\left[\frac{k}{\varepsilon}\right] \in W^{-1}(0)$.

4.4). To this aim we keep track of all the vanishing quantities appearing in the computations below, with their exact infinitesimal order.

To simplify the calculations we need some more notation: For any $u \in H^{1/2}(I)$ and any $\mathcal{L}^2$-measurable set $\Omega \subset I \times I$ consider the *locality defect* of F_ε

$$D(u, \Omega) = \int_\Omega \left| \frac{u(x) - u(y)}{x - y} \right|^2 dxdy\,.$$

The terminology, introduced in [1], is justified since given two disjoint intervals $A, B \subseteq I$, it holds

$$F_\varepsilon(u, A \cup B) = F_\varepsilon(u, A) + F_\varepsilon(u, B) + 2\mu_\varepsilon D(u, A \times B). \tag{3.21}$$

According to Lemma 3.1 and Corollary 3.3 we build up a sequence which lies as much as possible in the ε-wells of the potential and has all the transitions of height ε.

Proposition 3.8 *Assume that $\mu_\varepsilon \to 0^+$ and that (2.3) is satisfied with $K \neq 0$. For every $u \in L^1(I)$ there exists $(u_\varepsilon) \subset H^{1/2}(I)$ such that $u_\varepsilon \to u$ in $L^1(I)$ and*

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \leq F(u).$$

Proof. Without loss of generality we assume $u \in BV(I)$. Moreover, the further reduction to piecewise affine functions can be done since such a class is dense in $BV(I)$ with respect to BV strict convergence, namely both the convergence of the functions in L^1 and of their total variations on I. Actually, it is not restrictive to also assume such functions being piecewise strictly monotone. The conclusion then follows by a standard diagonal argument.

Let (x_ε^i) be an increasing ordering of $u^{-1}(\varepsilon\mathbf{Z})$, and set $x_\varepsilon^0 = \inf I$, $x_\varepsilon^{N_\varepsilon+1} = \sup I$, where $N_\varepsilon = \# u^{-1}(\varepsilon\mathbf{Z})$. Let γ_ε be a positive infinitesimal (which we will choose appropriately later on) satisfying $\gamma_\varepsilon = o(\varepsilon)$. Define the piecewise affine functions $u_\varepsilon : I \to \mathbf{R}$ by $u_\varepsilon = P_\varepsilon(u)$ where

$$P_\varepsilon(u)(x) := \begin{cases} u(x_\varepsilon^i) + (u(x_\varepsilon^{i+1}) - u(x_\varepsilon^i)) \left(\dfrac{x - x_\varepsilon^i}{\gamma_\varepsilon} \wedge 1 \right) & x \in [x_\varepsilon^i, x_\varepsilon^{i+1}], \\ & i \in \{1, \dots, N_\varepsilon - 1\} \\ u(x_\varepsilon^1) & x \in [x_\varepsilon^0, x_\varepsilon^1] \\ u(x_\varepsilon^{N_\varepsilon}) & x \in [x_\varepsilon^{N_\varepsilon}, x_\varepsilon^{N_\varepsilon+1}]. \end{cases} \tag{3.22}$$

Notice that $u_\varepsilon \in C^0 \cap H^{1/2}(I)$ and

$$\|u_\varepsilon - u\|_{L^\infty(I)} \leq \varepsilon.$$

For ε sufficiently small, $x_\varepsilon^i + \gamma_\varepsilon \in (x_\varepsilon^i, x_\varepsilon^{i+1})$ for any $i \in \{1, \dots, N_\varepsilon - 1\}$. Thus, we may define the sets $A_\varepsilon^i = [x_\varepsilon^i, x_\varepsilon^i + \gamma_\varepsilon]$, $C_\varepsilon^i = [x_\varepsilon^i + \gamma_\varepsilon, x_\varepsilon^{i+1}]$, $C_\varepsilon^0 = [x_\varepsilon^0, x_\varepsilon^1]$, and $C_\varepsilon^{N_\varepsilon} = [x_\varepsilon^{N_\varepsilon}, x_\varepsilon^{N_\varepsilon+1}]$. By construction $u_\varepsilon\big|_{C_\varepsilon^i} \equiv u(x_\varepsilon^{i+1})$ for each $i \in \{1, \dots, N_\varepsilon - 1\}$, and $u_\varepsilon\big|_{C_\varepsilon^{N_\varepsilon}} \equiv u(x_\varepsilon^{N_\varepsilon})$. Moreover note that for any $x \in C_\varepsilon^i$ and $y \in C_\varepsilon^j$, $|u_\varepsilon(x) - u_\varepsilon(y)| = |u(x_\varepsilon^{i+1}) - u(x_\varepsilon^{j+1})| \leq |j - i|\varepsilon$ and we can

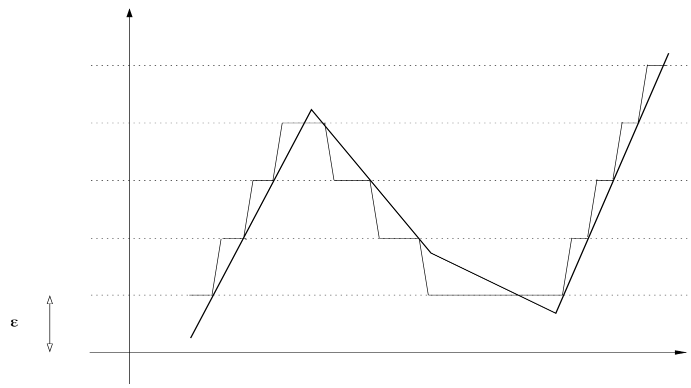

Figure 2: u is represented by the thick line, u_ε by the thinner one.

estimate the oscillation of u as follows $|u(x)-u(y)| \le (|j-i|+2)\varepsilon$. Thus we have

$$\begin{aligned}
&|u_\varepsilon(x)-u_\varepsilon(y)|^2 - |u(x)-u(y)|^2 \\
&\quad = (|u_\varepsilon(x)-u_\varepsilon(y)| + |u(x)-u(y)|)(|u_\varepsilon(x)-u_\varepsilon(y)| - |u(x)-u(y)|) \\
&\quad \le 2(|j-i|+1)\varepsilon \Big| |u_\varepsilon(x)-u_\varepsilon(y)| - |u(x)-u(y)| \Big| \\
&\quad \le 2(|j-i|+1)\varepsilon(|u_\varepsilon(x)-u(x)| + |u_\varepsilon(y)-u(y)|) \le 4(|j-i|+1)\varepsilon^2,
\end{aligned}$$

and thus

$$|u_\varepsilon(x)-u_\varepsilon(y)|^2 \le |u(x)-u(y)|^2 + 4(|j-i|+1)\varepsilon^2 \tag{3.23}$$

for any $x \in C_\varepsilon^i$ and $y \in C_\varepsilon^j$.

In the sequel c denotes a positive constant, which may vary from line to line, independent from ε.

Step 1. *Estimate of the $H^{1/2}$ seminorms. We prove that*

$$[u_\varepsilon]^2_{H^{1/2}(I)} \le -2\varepsilon \ln \gamma_\varepsilon |Du|(I) + [u]^2_{H^{1/2}(I)} + o(1), \tag{3.24}$$

the infinitesimal $o(1)$ being uniform for all functions u such that $0 < a \le |u'| \le b$.

To get estimate (3.24) consider the decomposition

$$\begin{aligned}
[u_\varepsilon]^2_{H^{1/2}(I)} &= \sum_{i,j=0}^{N_\varepsilon} D\left(u_\varepsilon, C_\varepsilon^i \times C_\varepsilon^j\right) \\
&+ \sum_{i,j=1}^{N_\varepsilon-1} D\left(u_\varepsilon, A_\varepsilon^i \times A_\varepsilon^j\right) + \sum_{\substack{i=0,\dots,N_\varepsilon \\ j=1,\dots,N_\varepsilon-1}} D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^j\right).
\end{aligned} \tag{3.25}$$

In the following we will treat separately each sum above and show that the main contribution is given by the first, while the latter are infinitesimal.

To deal with the first term in (3.25) we further separate the interactions of height ε, for which we take into account the sharp computation of (3.3), from the others, for which we use the following estimate

$$D\left(u_\varepsilon, C_\varepsilon^i \times C_\varepsilon^j\right) \le D\left(u, C_\varepsilon^i \times C_\varepsilon^j\right) + \frac{c|j-i|\varepsilon^4}{\left(|j-i|\varepsilon + \gamma_\varepsilon\right)^2}, \tag{3.26}$$

where $i, j \in \{0, \dots, N_\varepsilon\}$, with $i \neq j$. The latter can be deduced straightforward by (3.2), (3.23) and taking into account that $|C_\varepsilon^i| = O(\varepsilon)$ for every i, thanks to the piecewise strict monotonicity of u.

In view of (3.3) and (3.26) we get

$$\begin{aligned}
\sum_{i,j=0}^{N_\varepsilon} D\left(u_\varepsilon, C_\varepsilon^i \times C_\varepsilon^j\right) &= \sum_{h=1}^{N_\varepsilon} \sum_{|j-i|=h} D\left(u_\varepsilon, C_\varepsilon^i \times C_\varepsilon^j\right) \\
&\le 2\varepsilon^2 \sum_{i=1}^{N_\varepsilon} \ln\left(\frac{(x_\varepsilon^{i+1} - x_\varepsilon^i)(x_\varepsilon^i - x_\varepsilon^{i-1})}{(x_\varepsilon^{i+1} - x_\varepsilon^{i-1} - \gamma_\varepsilon)\gamma_\varepsilon}\right) + \sum_{h=2}^{N_\varepsilon} \sum_{|j-i|=h} D\left(u_\varepsilon, C_\varepsilon^i \times C_\varepsilon^j\right) \\
&\le 2(-\varepsilon^2 \ln\gamma_\varepsilon + c\varepsilon^2 \ln\varepsilon)N_\varepsilon + \sum_{h=2}^{N_\varepsilon} \sum_{|j-i|=h} \left(D\left(u, C_\varepsilon^i \times C_\varepsilon^j\right) + \frac{ch\varepsilon^2}{\left(h + \frac{\gamma_\varepsilon}{\varepsilon}\right)^2}\right) \\
&\le -2\varepsilon^2 N_\varepsilon \ln\gamma_\varepsilon + [u]^2_{H^{1/2}(I)} + c\varepsilon^2 \sum_{h=2}^{N_\varepsilon - 1} \sum_{|j-i|=h} \frac{1}{h} \\
&\le -2\varepsilon^2 N_\varepsilon \ln\gamma_\varepsilon + [u]^2_{H^{1/2}(I)} + c\varepsilon^2 N_\varepsilon \ln N_\varepsilon \\
&\le -2\varepsilon \ln\gamma_\varepsilon |Du|(I) + [u]^2_{H^{1/2}(I)} + o(1).
\end{aligned} \tag{3.27}$$

In the last inequality we have used that $\varepsilon N_\varepsilon \le |Du|(I)$.

For what the second term in (3.25) is concerned, fix $i, j \in \{1, \dots, N_\varepsilon - 1\}$ and recall that u_ε is either constant or affine with slope $\varepsilon/\gamma_\varepsilon$ on each A_ε^i and A_ε^j, so that

- for $i = j$, if u_ε is constant the corresponding term gives a null contribution, while if u_ε is affine the double integral is nothing but $|A_i^\varepsilon \times A_i^\varepsilon|\varepsilon^2/\gamma_\varepsilon^2$, which reduces to ε^2;
- for $i \neq j$, $|x - y| \ge c|j - i|\varepsilon$ and $|u_\varepsilon(x) - u_\varepsilon(y)| \le (|j - i| + 1)\varepsilon$ for any $x \in A_\varepsilon^i$ and $y \in A_\varepsilon^j$.

Taking this into account, we get

$$\begin{aligned}
\sum_{i,j=1}^{N_\varepsilon - 1} D\left(u_\varepsilon, A_\varepsilon^i \times A_\varepsilon^j\right) &\le c\gamma_\varepsilon^2 \sum_{\substack{i,j=1,\dots,N_\varepsilon-1 \\ i\neq j}} \left(\frac{|j-i|+1}{|j-i|}\right)^2 + 2\varepsilon^2 N_\varepsilon \\
&\le c(\varepsilon N_\varepsilon)^2 \left(\frac{\gamma_\varepsilon}{\varepsilon}\right)^2 + 2\varepsilon^2 N_\varepsilon \le c|Du|^2(I) \left(\frac{\gamma_\varepsilon}{\varepsilon}\right)^2 + 2\varepsilon|Du|(I) = o(1).
\end{aligned} \tag{3.28}$$

Now consider the following further splitting of the third term in (3.25)

$$\sum_{\substack{i=0,\dots,N_\varepsilon \\ j=1,\dots,N_\varepsilon-1}} D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^j\right) = \sum_{\substack{|j-i|\geq 1 \\ j\neq i-1}} D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^j\right)$$

$$+ \sum_{i=1}^{N_\varepsilon-1} D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^i\right) + \sum_{i=2}^{N_\varepsilon} D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^{i-1}\right). \tag{3.29}$$

The first sum on the right hand side above can be dealt with as the one in (3.28), so that

$$\sum_{\substack{|j-i|\geq 1 \\ j\neq i-1}} D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^j\right) = o(1). \tag{3.30}$$

Moreover, a direct integration yields

$$D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^i\right) \leq \varepsilon^2,$$

and of course a similar computation holds for $D\left(u_\varepsilon, C_\varepsilon^{i-1} \times A_\varepsilon^i\right)$, so that

$$\sum_{i=1}^{N_\varepsilon-1} \left(D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^i\right) + D\left(u_\varepsilon, C_\varepsilon^{i-1} \times A_\varepsilon^i\right)\right) \leq 2\varepsilon^2 N_\varepsilon. \tag{3.31}$$

By collecting (3.30), and (3.31) we have by (3.29)

$$\sum_{\substack{i=0,\dots,N_\varepsilon \\ j=1,\dots,N_\varepsilon-1}} D\left(u_\varepsilon, C_\varepsilon^i \times A_\varepsilon^j\right) = o(1). \tag{3.32}$$

Eventually, (3.27), (3.28) and (3.32) give (3.24).

Step 2. *Estimate of the potential term.*
The one-periodicity of W and the very definition of u_ε yield

$$\int_I W\left(\frac{u_\varepsilon(x)}{\varepsilon}\right) dx = \sum_{i=1}^{N_\varepsilon-1} \int_{x_\varepsilon^i}^{x_\varepsilon^i+\gamma_\varepsilon} W\left(\frac{u_\varepsilon(x)}{\varepsilon}\right) dx \tag{3.33}$$

$$= \gamma_\varepsilon \sum_{i=1}^{N_\varepsilon-1} \int_{u(x_\varepsilon^i)/\varepsilon}^{u(x_\varepsilon^{i+1})/\varepsilon} W(t)dt \leq \gamma_\varepsilon N_\varepsilon \int_0^1 W(t)dt \leq \frac{\gamma_\varepsilon}{\varepsilon}|Du|(I) \int_0^1 W(t)dt.$$

Step 3. *Conclusion.*
From Step 1 and Step 2 it follows

$$F_\varepsilon(u_\varepsilon) \leq \left(-2\mu_\varepsilon \varepsilon \ln \gamma_\varepsilon + \eta_\varepsilon \frac{\gamma_\varepsilon}{\varepsilon} \int_0^1 W(t)dt\right) |Du|(I) + \mu_\varepsilon [u]^2_{H^{1/2}(I)} + o(\mu_\varepsilon).$$

Eventually, the choice $\gamma_\varepsilon = \mu_\varepsilon \varepsilon^2/\eta_\varepsilon$ is such that $\gamma_\varepsilon = o(\varepsilon)$ and

$$F_\varepsilon(u_\varepsilon) \leq 2\mu_\varepsilon \varepsilon \ln\left(\frac{\eta_\varepsilon}{\mu_\varepsilon}\right) |Du|(I) + \mu_\varepsilon [u]^2_{H^{1/2}(I)} + o(\mu_\varepsilon). \tag{3.34}$$

Taking the limit as $\varepsilon \to 0^+$ we conclude the proof. □

3.3 Some related results

In the following two corollaries we refine (ii) of Proposition 3.4, and prove that the energy of any sequence is concentrated on a neighbourhood of the diagonal set Δ in $I \times I$. To show that, we exploit the independence of the Γ-limit from the particular choice of the potential W.

Corollary 3.9 *Assume that $\mu_\varepsilon \to 0^+$ and that (2.3) is satisfied with $K \neq 0$. For any $u \in BV(I)$ and any sequence (u_ε) converging to u in $L^1(I)$ with $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, we have*

$$\liminf_{\varepsilon \to 0^+} \mu_\varepsilon D(u_\varepsilon, \Delta_{\gamma_\varepsilon}) \geq 2K|Du|(I), \tag{3.35}$$

where γ_ε is any positive infinitesimal such that $\mu_\varepsilon = o(\gamma_\varepsilon^2)$, and for every $\gamma > 0$

$$\Delta_\gamma = \{(x, y) \in I \times I : |x - y| \leq \gamma\}.$$

Proof. We first notice that the results of Proposition 3.4 do not depend on the particular choice of W, but only on the qualitative assumptions on it. More precisely, fixed $\sigma > 0$, (ii) in Proposition 3.4 yields

$$\liminf_{\varepsilon \to 0^+} \left[\mu_\varepsilon [u_\varepsilon]^2_{H^{1/2}(I)} + \eta_\varepsilon \sigma \int_I W\left(\frac{u_\varepsilon(x)}{\varepsilon}\right) dx \right] \geq 2K|Du|(I),$$

which, by letting $\sigma \to 0^+$ and taking into account the upper bound for the energy of u_ε, implies

$$\liminf_{\varepsilon \to 0^+} \mu_\varepsilon [u_\varepsilon]^2_{H^{1/2}(I)} \geq 2K|Du|(I). \tag{3.36}$$

Moreover, given $\gamma > 0$, if $(I_j)_j$ is a partition of I such that $|I_j| \leq \sqrt{2}\gamma$ and $|Du|(\partial I_j) = 0$ for every j, from the lower bound inequality (Proposition 3.4, (ii)) we infer

$$\begin{aligned} &\liminf_{\varepsilon \to 0^+} \left[\mu_\varepsilon D(u_\varepsilon, \Delta_\gamma) + \eta_\varepsilon \int_I W\left(\frac{u_\varepsilon(x)}{\varepsilon}\right) dx \right] \\ &\geq \sum_j \liminf_{\varepsilon \to 0^+} \left[\mu_\varepsilon [u_\varepsilon]^2_{H^{1/2}(I_j)} + \eta_\varepsilon \int_{I_j} W\left(\frac{u_\varepsilon(x)}{\varepsilon}\right) dx \right] \geq 2K|Du|(I). \end{aligned} \tag{3.37}$$

By combining the arguments leading to (3.36) and (3.37), it follows

$$\liminf_{\varepsilon \to 0^+} \mu_\varepsilon D(u_\varepsilon, \Delta_\gamma) \geq 2K|Du|(I). \tag{3.38}$$

In addition, the compactness result in (i) of Proposition 3.4 entails that (u_ε) (up to ε-integer translations) actually converges to u in $L^2(I)$ and thus

$$\lim_{\varepsilon \to 0^+} \mu_\varepsilon D(u_\varepsilon, I \times I \setminus \Delta_{\gamma_\varepsilon}) = 0 \tag{3.39}$$

provided that $\mu_\varepsilon = o(\gamma_\varepsilon^2)$. Hence, (3.35) follows by (3.38) and (3.39). □

Corollary 3.10 *For any $u \in BV(I)$ and any sequence (u_ε) converging to u in $L^1(I)$ with $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, any cluster point ν of the family of measures*

$$\nu_\varepsilon = \mu_\varepsilon \left| \frac{u_\varepsilon(x) - u_\varepsilon(y)}{x - y} \right|^2 d\mathcal{L}^2 \lfloor (I \times I), \tag{3.40}$$

is concentrated on the diagonal Δ, and satisfies $\pi_\# \nu \geq 2K|Du|$ where π denotes the projection on the first coordinate.

In particular, in case (u_ε) is a recovery sequence for u, the family (ν_ε) weakly $*$ converges to $\overline{\nu}$ satisfying

$$\pi_\# \overline{\nu} = 2K|Du|.$$

Proof. The proof is a straightforward consequence of the compactness result (Proposition 3.4 (i)), (3.39) and Corollary 3.9 localized on open subsets of I. □

Eventually, we briefly give the proof of Corollary 2.7. It can be worked out using the same arguments of Theorem 2.2, the only warning being that now we have also to take into account the non-vanishing contribution given by the $H^{1/2}$ seminorm in the asymptotic of (F_ε). In particular in the proof of the Γ-liminf and the Γ-limsup inequalities we follow the notation introduced in Proposition 3.4 and Proposition 3.8, respectively.

Proof of Corollary 2.7. *Compactness.* The statement trivially follows by using the compact embedding of $H^{1/2}$ functions with zero mean in $L^p(I)$ for every $p \in [1, +\infty)$.

Lower bound inequality. Fix $\gamma > 0$. From inequality (3.20), applied to a partition $(I_j)_j$ of I satisfying $\sup_j |I_j| \leq \sqrt{2}\gamma$, we easily deduce (see Remark 3.7)

$$\begin{aligned} \liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon \to 0^+} \mu_\varepsilon D(u_\varepsilon, I \times I \setminus \Delta_\gamma) + 2K \sum_{j=1}^{r} \operatorname{osc}_{I_j} u \\ &\geq \mu D(u, I \times I \setminus \Delta_\gamma) + 2K \sum_{j=1}^{r} \operatorname{osc}_{I_j} u. \end{aligned}$$

The lower bound inequality then follows by passing on the supremum on the partition and then letting $\gamma \to 0^+$.

Upper bound inequality. We repeat the construction of Proposition 3.8. The proof follows from estimate (3.34) and a standard density argument. □

4 Boundary Data

In this section we consider the case in which a boundary condition is imposed. As already discussed in Section 2 the non local regularization and the fact that the limit energy is defined in BV requires to give the boundary conditions in a strong version; i.e. restricting our functionals to the space

$$\mathcal{D}_L^\varepsilon(I) = \{u \in H^{1/2}(I) : u\big|_{(\alpha,\alpha+\delta)} \equiv 0,\ u\big|_{(\beta-\delta,\beta)} \equiv \varepsilon\,[L/\varepsilon]\}.$$

Namely we introduce the energies $G_\varepsilon : L^1(I) \to [0,+\infty]$ given by

$$G_\varepsilon(u) = \begin{cases} F_\varepsilon(u) & \text{if } u \in \mathcal{D}^\varepsilon_L(I) \\ \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

In general for functionals defined on BV, as the case of our Γ-limit, in order to impose the boundary condition it is enough to assign the outer trace; i.e. to require $u(x) = 0$ if $x < a$ and $u(x) = L$ if $x > b$. This condition is relaxed in a term that penalizes the non attainment of the boundary data. On the other hand, in terms of the functionals F_ε this condition disappears in the limit. For instance, one can approximate any constant function with a sequence (u_ε) satisfying the outer boundary condition and such that $F_\varepsilon(u_\varepsilon) \to 0$. This problem is overcome by imposing the inner trace as in the space $\mathcal{D}^\varepsilon_L(I)$ (see Remark 4.1 and Remark 4.5). Finally, the condition $u = \varepsilon\,[L/\varepsilon]$ on the set $(b-\delta, b)$ is necessary for the Γ-limit of the whole family F_ε to exist.

4.1 First Order Limit

The proof of Theorem 2.3 is a consequence of Theorem 2.2 once we show that it is possible to construct the recovery sequence matching the boundary data.

Proof of Theorem 2.3. Compactness and lower bound are an immediate consequence of Theorem 2.2.

As for the recovery sequence for a given function u in

$$\mathcal{D}_L(I) = \{u \in H^{1/2}(I) : u\big|_{(\alpha,\alpha+\delta)} \equiv 0,\ u\big|_{(\beta-\delta,\beta)} \equiv L\}$$

we only need to modify the construction done in Proposition 3.8 for piecewise affine and piecewise strictly monotone functions on I.

In fact it is not restrictive to assume $u \in \mathcal{D}_L(I)$ to be piecewise affine on I and piecewise strictly monotone on $J = (\alpha+\delta, \beta-\delta)$.

Set $J_1 = (\alpha, \alpha+\delta)$, $J_2 = (\beta-\delta, \beta)$ and $\lambda_\varepsilon = [\frac{L}{\varepsilon}]\frac{\varepsilon}{L}$. Then, using (3.22), we define

$$u_\varepsilon := \begin{cases} P_\varepsilon(\lambda_\varepsilon u) & \text{on } J, \\ 0 & \text{on } J_1, \\ [\frac{L}{\varepsilon}]\varepsilon & \text{on } J_2. \end{cases}$$

Then, $\|u_\varepsilon - u\|_{L^\infty(I)} \le \varepsilon + \frac{\varepsilon}{L}\|u\|_{L^\infty(I)}$ and in particular $u_\varepsilon \to u$ in $L^1(I)$. Since $I = J_1 \cup J \cup J_2$ by (3.21) it follows

$$G_\varepsilon(u_\varepsilon) = F_\varepsilon(u_\varepsilon, J) + 2\mu_\varepsilon\left(D(u_\varepsilon, J_1 \times J) + D(u_\varepsilon, J_1 \times J_2) + D(u_\varepsilon, J \times J_2)\right),$$

taking into account that $F_\varepsilon(u_\varepsilon, J_1) = F_\varepsilon(u_\varepsilon, J_2) = 0$. Moreover, by Step 1 of Proposition 3.8, (3.33), and using that $1 - \varepsilon/L \le \lambda_\varepsilon \le 1$, we get

$$\begin{aligned} G_\varepsilon(u_\varepsilon) \le & -2\mu_\varepsilon\varepsilon \ln\gamma_\varepsilon |Du|(I) + \mu_\varepsilon[u]^2_{H^{1/2}(J)} + o(\mu_\varepsilon) \\ & +2\mu_\varepsilon\left(D(u_\varepsilon, J_1 \times J) + D(u_\varepsilon, J_1 \times J_2) + D(u_\varepsilon, J \times J_2)\right). \end{aligned} \tag{4.1}$$

We claim that the latter estimate entails

$$G_\varepsilon(u_\varepsilon) \le -2\mu_\varepsilon\varepsilon \ln\gamma_\varepsilon |Du|(I) + \mu_\varepsilon [u]^2_{H^{1/2}(I)} - 4\varepsilon^2\mu_\varepsilon \ln\gamma_\varepsilon + o(\mu_\varepsilon). \tag{4.2}$$

Indeed, by Lebesgue Dominated Convergence Theorem we have

$$\limsup_{\varepsilon\to 0^+} D(u_\varepsilon, J_1 \times J_2) \le D(u, J_1 \times J_2),$$

thus to prove the claim it suffices to show that for $i \in \{1,2\}$ it holds

$$D(u_\varepsilon, J_i \times J) \le D(u, J_i \times J) - \varepsilon^2 \ln\gamma_\varepsilon + o(1). \tag{4.3}$$

The proof of the above estimate, as in Step 1 of Proposition 3.8, is based on the splitting

$$D\left(u_\varepsilon, J_i \times J\right) = \sum_{j=0}^{[L/\varepsilon]-1} D\left(u_\varepsilon, J_i \times C_\varepsilon^j\right) + \sum_{j=0}^{[L/\varepsilon]-1} D\left(u_\varepsilon, J_i \times A_\varepsilon^j\right), \tag{4.4}$$

and follows by similar arguments.

Eventually, recalling that $\gamma_\varepsilon = \mu_\varepsilon\varepsilon^2/\eta_\varepsilon$ (see Proposition 3.8) we deduce from (4.2)

$$\limsup_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon) \le 2K|Du|(I).$$

□

Remark 4.1 *Refining the argument used above one can realize that it is possible to characterize the* Γ*-limit also in the case* $\delta \to 0^+$ *with* $\ln\delta \sim \ln\varepsilon$*, obtaining*

$$G(u) = 2K\left(|Du|(I) + |u(\alpha+)| + |L - u(\beta-)|\right)$$

for any $u \in BV(I)$ *(*$u(\alpha+)$ *and* $u(\beta-)$ *denote the inner traces of* u *at* α *and* β*, respectively). In other words in the limit the boundary condition is substituted by the penalization given by the last two terms of the energy.*

4.2 Second Order Limit

In this subsection we select among the minimizers of G those which better describe the asymptotic behaviour of the minimum points of G_ε by proving the second order expansion in terms of Γ-convergence (see [4]) stated in Theorem 2.4 (see also Remark 2.5). In order to do that, we consider the functionals $G_\varepsilon^1 : L^1(I) \to [0, +\infty]$ defined by

$$G_\varepsilon^1(u) = \frac{G_\varepsilon(u) - 2[L/\varepsilon]\varepsilon^2\mu_\varepsilon \ln\eta_\varepsilon}{\mu_\varepsilon},$$

if $u \in \mathcal{D}_L^\varepsilon(I)$ and equal to $+\infty$ otherwise in $L^1(I)$.

Remark 4.2 *If we assume that* $\frac{\mu_\varepsilon\varepsilon\ln\eta_\varepsilon - K}{\mu_\varepsilon} \to 0$ *as* $\varepsilon \to 0^+$*, which is admissible in terms of the scaling argument given in Section 1.1, we can define* G_ε^1 *alternatively as*

$$G_\varepsilon^1(u) = \frac{G_\varepsilon(u) - 2K\varepsilon[L/\varepsilon]}{\mu_\varepsilon}.$$

The proof of Theorem 2.4 is carried out in Proposition 4.3 and Proposition 4.4 below.

Proposition 4.3 *Assume that $\mu_\varepsilon \to 0^+$, that (2.3) is satisfied with $K \neq 0$, and that $\varepsilon = o(\mu_\varepsilon)$. Let $(u_\varepsilon) \subset L^1(I)$, then*

(i) Compactness: *if $\sup_\varepsilon G^1_\varepsilon(u_\varepsilon) < +\infty$, then there exist $u \in \mathcal{D}_L \cap BV(I)$ such that $|Du|(I) = L$, and a subsequence (u_{ε_k}) which converges to u in $L^p(I)$ for every $p \in [1, +\infty)$.*

(ii) Lower bound inequality: *if $u_\varepsilon \to u$ in $L^1(I)$, we have*

$$\liminf_{\varepsilon \to 0^+} G^1_\varepsilon(u_\varepsilon) \geq G^1(u). \tag{4.5}$$

Notice that any function in $\mathcal{D}_L \cap BV(I)$ has total variation greater than or equal to L. In case the total variation is exactly L then the function is necessarily non-decreasing, and thus a minimizer for G on $\mathcal{D}_L(I)$. So that, in particular, Proposition 4.3 implies that G^1 is finite only on the minimizers of G.

Proof of Proposition 4.3. *Compactness.* By taking into account (i) of Proposition 3.4 and the boundary conditions, we obtain a subsequence (u_{ε_k}) converging in $L^p(I)$ for every $p > 1$. Moreover, if u denotes the limit point, the energy bound $\sup_\varepsilon G^1_\varepsilon(u_\varepsilon) < +\infty$ and (ii) of Proposition 3.4 implies that

$$2K|Du|(I) = G(u) \leq \liminf_{k \to +\infty} G_{\varepsilon_k}(u_{\varepsilon_k}) \leq 2KL,$$

so that $u \in BV(I)$. Furthermore, the convergence of (u_{ε_k}) to u in $L^p(I)$ implies $u \in \mathcal{D}_L \cap BV(I)$ and thus $|Du|(I) = L$.

Lower bound inequality. Let us consider $u \in L^1(I)$ and a family (u_ε) converging to u in $L^1(I)$ such that $\liminf_\varepsilon G^1_\varepsilon(u_\varepsilon) < +\infty$. In particular, we may assume that $\sup_\varepsilon G^1_\varepsilon(u_\varepsilon) < +\infty$. The compactness result proved above implies that u is a minimizer for G on $\mathcal{D}_L(I)$, and then it is non-decreasing. Hence, it is not restrictive to assume that u_ε is also non-decreasing up to passing to its non-decreasing rearrangement.

As already pointed out after the statement of Theorem 2.4 the strategy of the proof is to isolate the energy contribution outside the set Δ_γ, for an arbitrary $\gamma > 0$, as follows

$$G^1_\varepsilon(u_\varepsilon) \geq D(u_\varepsilon, I \times I \setminus \Delta_\gamma) + \left(D(u_\varepsilon, \Delta_\gamma) + \frac{\eta_\varepsilon}{\mu_\varepsilon} \int_I W\left(\frac{u_\varepsilon}{\varepsilon}\right) dx - 2[L/\varepsilon]\varepsilon^2 \ln \eta_\varepsilon \right), \tag{4.6}$$

and to prove that the second term of the right hand side above is positive in the limit as $\varepsilon \to 0^+$. This given, Fatou's Lemma and the arbitrariness of $\gamma > 0$ give the required lower bound inequality.

The proof is based on the following claim.

Claim. *Fix $N \in \mathbf{N}$. There exists a constant $\sigma_N > 0$ such that for every $\varepsilon > 0$ there exists a partition of I into intervals $I^i_\varepsilon = (x^i_\varepsilon, x^{i+1}_\varepsilon)$, $i = 1, \ldots, N-1$, satisfying:*

(i) $\dfrac{|I|}{2N} \leq |I^i_\varepsilon| \leq \dfrac{3|I|}{2N}$;

(ii) $|\{x \in I_\varepsilon^i : u_\varepsilon(x) \le \varepsilon r_\varepsilon^i + \varepsilon^3\}| > \sigma_N \varepsilon$, where $r_\varepsilon^1 = 0$ and for $i \in \{2, \dots, N-1\}$

$$r_\varepsilon^i := \operatorname{argmin}\{|\varepsilon z - \inf_{I_\varepsilon^i} u_\varepsilon|,\ z \in \mathbf{Z}\};$$

(iii) $|\varepsilon r_\varepsilon^i - \inf_{I_\varepsilon^i} u_\varepsilon| < \varepsilon^3$ if $i \in \{1, \dots, N\}$;

(iv) $|\{x \in I_\varepsilon^i : u_\varepsilon(x) \ge \varepsilon s_\varepsilon^i - \varepsilon^3\}| > \sigma_N \varepsilon$, where $s_\varepsilon^N = [L/\varepsilon]$ and for $i \in \{1, \dots, N-1\}$

$$s_\varepsilon^i := \operatorname{argmin}\{|\varepsilon z - \sup_{I_\varepsilon^i} u_\varepsilon|,\ z \in \mathbf{Z}\};$$

(v) $|\varepsilon s_\varepsilon^i - \sup_{I_\varepsilon^i} u_\varepsilon| < \varepsilon^3$ if $i \in \{1, \dots, N\}$.

Note that in the above definitions $s_\varepsilon^i = r_\varepsilon^{i+1}$ for every $i \in \{1, \dots, N-1\}$.

The rough idea is that the potential term in the energy forces the sequence u_ε to have almost all ε-levels of the order ε. Thus, starting from an arbitrary partition, we can always slightly move the extremes of each interval in order to satisfy conditions (ii)–(v).

Take a partition of I given by the intervals (y_i, y_{i+1}), $i \in \{1, \dots, N-1\}$, of length at least $\frac{|I|}{N}$. For any $i \in \{1, \dots, N-1\}$, consider $I_N(y_i) = \left(y_i - \frac{|I|}{4N}, y_i + \frac{|I|}{4N}\right) \cap I$ and for any $s \in \mathbf{Z}$ define the intervals

$$J_\varepsilon^{i,s} = \{x \in I_N(y_i) : |u_\varepsilon(x) - \varepsilon s| < \varepsilon^3\}.$$

Since $\sup_\varepsilon G_\varepsilon^1(u_\varepsilon) < +\infty$, there exists a positive constant M for which $\sup_\varepsilon G_\varepsilon(u_\varepsilon) \le M$, and thus

$$\omega_\varepsilon \eta_\varepsilon \sum_{s \in \mathbf{Z}} |I_N(y_i) \setminus J_\varepsilon^{i,s}| \le \eta_\varepsilon \int_{I_N(y_i)} W\left(\frac{u_\varepsilon}{\varepsilon}\right) dx \le M.$$

This in turn implies

$$\sum_{s \in \mathbf{Z}} |J_\varepsilon^{i,s}| > \frac{|I|}{4N} - \frac{M}{\omega_\varepsilon \eta_\varepsilon},$$

and then there exists an integer s_ε^i, with $[\inf_{I_N(y_i)} u_\varepsilon/\varepsilon] \le s_\varepsilon^i \le [\sup_{I_N(y_i)} u_\varepsilon/\varepsilon]$, such that

$$|J_\varepsilon^{i,s_\varepsilon^i}| > \left(\frac{|I|}{4N} - \frac{M}{\omega_\varepsilon \eta_\varepsilon}\right) \frac{1}{\left[\frac{\operatorname{osc} u_\varepsilon}{\varepsilon}\right] + 2}.$$

By the fact that u_ε is monotone and belongs to $\mathcal{D}_L^\varepsilon(I)$, we have $0 \le [\operatorname{osc} u_\varepsilon/\varepsilon] \le [L/\varepsilon]$, from which we infer that

$$|J_\varepsilon^{i,s_\varepsilon^i}| > \left(\frac{|I|}{4N} - \frac{M}{\omega_\varepsilon \eta_\varepsilon}\right) \frac{\varepsilon}{L + 2\varepsilon}.$$

Note that since W satisfies (2.2), $\omega_\varepsilon \sim \varepsilon^{2p}$ and then $\omega_\varepsilon \eta_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$. Thus there exists $\varepsilon_0 > 0$ and $\sigma_N > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$|J_\varepsilon^{i,s_\varepsilon^i}| > 2\sigma_N \varepsilon. \tag{4.7}$$

Finally, we can define the partition required by the claim taking the points x_ε^i to be the middle points of the intervals $J_\varepsilon^{i,s_\varepsilon^i}$ if $i = 2, \dots, N-1$, $x_\varepsilon^1 = y_1$ and $x_\varepsilon^N = y_N$. With this choice

$I_\varepsilon^i = (x_\varepsilon^i, x_\varepsilon^{i+1})$ satisfy (i)–(v). Indeed, (i) holds true since $x_\varepsilon^i \in I_N(y_i)$ and $|y_{i+1} - y_i| \geq \frac{|I|}{N}$, (ii) and (iv) are satisfied by (4.7), while (iii) and (v) by construction; so that the claim is proved.

The partition provided by the claim satisfies $\cup_i I_\varepsilon^i \times I_\varepsilon^i \subseteq \Delta_{\frac{3\sqrt{2}}{4N}}$, and then using (4.6) we can estimate $G_\varepsilon^1(u_\varepsilon)$ for every $N > 0$ by

$$G_\varepsilon^1(u_\varepsilon) \geq D(u_\varepsilon, I \times I \setminus \Delta_{\frac{3\sqrt{2}}{4N}}) + \sum_{i=1}^N \left([u_\varepsilon]^2_{H^{1/2}(I_\varepsilon^i)} + \frac{\eta_\varepsilon}{\mu_\varepsilon} \int_{I_i^\varepsilon} W\left(\frac{u_\varepsilon}{\varepsilon}\right) dx \right) - 2[L/\varepsilon]\varepsilon^2 \ln \eta_\varepsilon.$$

Taking into account that $r_\varepsilon^i = s_\varepsilon^{i-1}$, $r_\varepsilon^1 = 0$, $s_\varepsilon^N = [L/\varepsilon]$ we get

$$\begin{aligned} G_\varepsilon^1(u_\varepsilon) &\geq D(u_\varepsilon, I \times I \setminus \Delta_{\frac{3\sqrt{2}}{4N}}) \\ &+ \sum_{i=1}^N \left([u_\varepsilon]^2_{H^{1/2}(I_\varepsilon^i)} + \frac{\eta_\varepsilon}{\mu_\varepsilon} \int_{I_i^\varepsilon} W\left(\frac{u_\varepsilon}{\varepsilon}\right) dx - 2(s_\varepsilon^i - r_\varepsilon^i)\varepsilon^2 \ln \eta_\varepsilon \right). \end{aligned} \tag{4.8}$$

The conclusion follows from (4.8) by the arbitrariness of N provided we show

$$\liminf_{\varepsilon \to 0^+} \sum_{i=1}^N \left([u_\varepsilon]^2_{H^{1/2}(I_\varepsilon^i)} + \frac{\eta_\varepsilon}{\mu_\varepsilon} \int_{I_i^\varepsilon} W\left(\frac{u_\varepsilon}{\varepsilon}\right) dx - 2(s_\varepsilon^i - r_\varepsilon^i)\varepsilon^2 \ln \eta_\varepsilon \right) \geq 0. \tag{4.9}$$

This is accomplished by the lower bound of Corollary 3.3 applied on the set I_ε^i with $s = s_\varepsilon^i$ and $r = r_\varepsilon^i$ (which can be done thanks to (iii) and (v)). Indeed, with these choices we get

$$\begin{aligned} &[u_\varepsilon]^2_{H^{1/2}(I_\varepsilon^i)} + \frac{\eta_\varepsilon}{\mu_\varepsilon} \int_{I_i^\varepsilon} W\left(\frac{u_\varepsilon}{\varepsilon}\right) dx - 2(s_\varepsilon^i - r_\varepsilon^i)\varepsilon^2 \ln \eta_\varepsilon \\ &\geq 2(s_\varepsilon^i - r_\varepsilon^i)(\varepsilon - 2\varepsilon^3)^2 \left(\ln\left(\frac{|A_\varepsilon^i||B_\varepsilon^i|}{|I_\varepsilon^i|^2} \right) + \ln\left(\frac{\omega_\varepsilon |I_\varepsilon^i|}{2\mu_\varepsilon (s_\varepsilon^i - r_\varepsilon^i)(\varepsilon - 2\varepsilon^3)^2} \right) \right) \\ &+ 2(s_\varepsilon^i - r_\varepsilon^i)(4\varepsilon^6 - 4\varepsilon^4) \ln \eta_\varepsilon, \end{aligned}$$

where we denote $A_\varepsilon^i = \{x \in I_\varepsilon^i : u_\varepsilon(x) > \varepsilon s_\varepsilon^i - \varepsilon^3\}$, $B_\varepsilon^i = \{x \in I_\varepsilon^i : u_\varepsilon(x) \leq \varepsilon r_\varepsilon^i + \varepsilon^3\}$, and $\omega_\varepsilon = \inf_{d(y,\mathbf{Z}) \geq \varepsilon^2} W$. Since by construction $|A_\varepsilon^i| > \sigma_N \varepsilon$, $|B_\varepsilon^i| > \sigma_N \varepsilon$ and $\frac{|I|}{2N} \leq |I_\varepsilon^i| \leq \frac{3|I|}{2N}$, it is easy to see that the right hand side tends to zero as $\varepsilon \to 0^+$ and thus (4.9) is established.

□

To complete the proof of Theorem 2.4 we etablish the upper bound inequality, which is a direct consequence of the construction performed in Theorem 2.3.

Proposition 4.4 *Assume that $\mu_\varepsilon \to 0^+$, that (2.3) is satisfied with $K \neq 0$, and that $\varepsilon = o(\mu_\varepsilon)$. For every $u \in L^1(I)$ there exists a sequence $(u_\varepsilon) \subset \mathcal{D}_L^\varepsilon(I)$ such that $u_\varepsilon \to u$ in $L^1(I)$ and*

$$\limsup_{\varepsilon \to 0^+} G_\varepsilon^1(u_\varepsilon) \leq G^1(u).$$

Proof. Without loss of generality we may assume $G^1(u) < +\infty$. Moreover, by a density argument the further reduction to piecewise affine and piecewise strictly monotone functions on

$(\alpha+\delta, \beta-\delta)$ can be done. Thus, if we consider the sequence (u_ε) constructed in the proof of Theorem 2.3, by (4.2) we have

$$G^1_\varepsilon(u_\varepsilon) \leq -2\varepsilon \ln \mu_\varepsilon |Du|(I) + [u]^2_{H^{1/2}(I)} - 4\varepsilon^2 \ln \gamma_\varepsilon + o(1).$$

The thesis then follows by passing to the limit on $\varepsilon \to 0^+$ since $\varepsilon = o(\mu_\varepsilon)$, $\gamma_\varepsilon = \mu_\varepsilon \varepsilon^2/\eta_\varepsilon$, $\lambda_\varepsilon = \left[\frac{L}{\varepsilon}\right] \frac{\varepsilon}{L}$, and $|Du|(I) = L$. □

Remark 4.5 *The proof given above is also compatible with the case $\ln \delta \sim \ln \varepsilon$ and produces as second order Γ-limit the $H^{1/2}$ seminorm in I without any constraint on the boundary values. So that the minimum point selected by this asymptotic expansion would be given by any constant function, neglecting completely the boundary conditions.*

5 Conclusions

In this paper we studied in certain regimes the asymptotic behaviour in terms of Γ-convergence of a scaled variational model for dislocations of Nabarro-Peierls type, as proposed by Koslowski-Cuitiño and Ortiz in [17].

In this model the slip is assumed to occur only on one slip plane and the dislocations are interpreted as integer level sets of a phase field (proportional to the jump of the displacement across the slip plane). In the regimes under consideration the slip is assumed to be much larger than the lattice spacing or, in other words, the amount of dislocations on a large box tends to infinity as the size of the box increases. In this respect we regard our asymptotics as a step in the direction of describing the macroscopic plastic behaviour due to the presence of a large number of dislocations.

We performed a rigorous analysis in the one dimensional case (the two dimensional case being probably very similar, but mathematically much more involved). We identify three regimes (very diluted, critical and more dense). In the critical regime we show the coexistence of two effects: a self interaction that gives rise to a line tension energy term and a long range interaction between dislocations given by a non local energy. Those effects are also present in the other two regimes at different order (as shown in the diluted case by means of an asymptotic expansion in terms of Γ-convergence). In particular in the dense regime the leading term of the energy is the non local, specifically it is given by the $H^{1/2}$ seminorm of the phase field.

In our opinion this result validates, in the case of a large number of dislocations (larger than $\log L$ in a box of side length L), the working assumption in the simulations in [17]; i.e. the idea that the overall distribution of dislocations is given by (the level sets of) a profile that minimizes the $H^{1/2}$ seminorm.

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