THRESHOLD-BASED QUASI-STATIC BRITTLE DAMAGE EVOLUTION

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Abstract

We introduce models for static and quasi-static damage in elastic materials, based on a strain threshold, and then investigate the relationship between these threshold models and the energy-based models introduced in [17] and [15]. A somewhat surprising result is that, while classical solutions for the energy models are also threshold solutions, this is not the case for nonclassical solutions, i.e., solutions with microstructure. A new and arguably more physical definition of solutions with microstructure for the energy-based model is then given, in which the energy minimality property is satisfied by sequences of sets that generate the effective elastic tensors, rather than by the tensors themselves. We prove existence for this energy based problem, and show that these solutions are also threshold solutions. A byproduct of this analysis is that all local minimizers, in both the classical setting and for the new microstructure definition, are also global minimizers.

1. Introduction

Many phenomena in mechanics, such as fracture, plasticity, and damage, have been studied through variational models. These models, both static and quasi-static, are often inspired by a threshold criterion – fracture occurs where the stress has a sufficiently large singularity, plastic behavior starts where the stress reaches the yield surface, and materials undergo damage where the stress exceeds a given threshold. The main advantage of a variational formulation for these problems is the ease of showing existence of global minimizers (even if only for a relaxed energy). On the other hand, threshold criteria are local (in space), so that the correspondence between these variational approaches and the threshold becomes suspect. In particular, one would expect that if an energy based approach captures the threshold criterion, then, since threshold criteria are local, local minimizers should be threshold solutions. Hence, in order for global minimizers to correspond to threshold solutions, there should be no local minimizers besides global minimizers, when the norm defining locality is local in space (e.g., L^1, but not L^∞). Yet frequently in these variational models, there is a difference between local and global minimizers (e.g., [9], [11], [12], [18], [16]), or at least uncertainty about whether there is a difference ([17], [15]).

Additionally, if there is equivalence between the energy formulation and the threshold criterion, a new issue arises if the energy needs to be relaxed. Namely, this means that the threshold criterion also needs to be relaxed (i.e., there are no solutions to the threshold formulation, but there are approximate solutions that develop microstructure). A natural weak form of the threshold criterion then needs to be formulated. When one considers quasi-static evolution, the correspondence between the energy and threshold formulations should be maintained, so that the weak form of the threshold formulation suggests the correct relaxation for the energy based formulation. For the quasi-static damage model that we consider, what seemed to be the most natural class of competitors in the energy formulation needed to be modified in order to correspond to the threshold point of view. This is discussed further and in detail in Remark 3.3.

For these reasons (and others – see the note on dynamics at the end of this introduction), our view is that there can be substantial value in formulating models using threshold criteria explicitly, as these models can then be compared to the energy-based ones, leading to either support for the existing models, or possibly to their improvement or correction. Indeed, our unrelaxed threshold formulation, which we describe shortly, strongly supports the unrelaxed energy formulations in [17] and [15]. On the other hand, as just described, our relaxed formulation requires a refinement of the formulation in [15]. More specifically, it requires that the relaxed energy formulations be posed in terms of sequences of sets that generate microstructure, rather than in terms of the effective behavior of the microstructure. This new energy formulation appears to be more physical than the one in [15], which suggests that the monotonic G-closure as described below might not be the right object for quasi-static evolutions.

To begin, we first outline the variational model for damage proposed by Francfort and Marigo [17] and the quasi-static model by Francfort and Garroni [15]. In these models, two states, undamaged and damaged, are given by two elastic well-ordered tensors, A_s and A_w, and the energy of each displacement u is given by

$$\int_\Omega W(e(u))\,dx - \int_\Omega f\,u\,dx\,,$$

where $e(u) = \frac{\nabla u + \nabla u^T}{2}$ is the symmetrized gradient and

$$W(\varepsilon) = \min\left\{\frac{1}{2}A_s\varepsilon\,\varepsilon\,,\ \frac{1}{2}A_w\varepsilon\,\varepsilon + k\right\}\,,$$

where k can be viewed as the cost (per volume) for the material to undergo damage, or the energy dissipated (per volume) when the material undergoes damage. This energy density is not quasi-convex, thus in the minimization procedure we expect microstructure, which requires relaxation. The quasi-convex envelope of W can be represented as follows

$$QW(\varepsilon) = \min_{\Theta\in[0,1]}\ \min_{A\in G_\Theta(A_s,A_w)}\left\{\frac{1}{2}A\varepsilon\,\varepsilon + k\Theta\right\}\,,$$

where $G_\Theta(A_s, A_w)$ is the G-closure of A_s and A_w mixed with volume fractions $1-\Theta$ and Θ.

Given an external loading $f(t)$ parametrized by time, a relaxed quasi-static evolution for this model (that also includes irreversibility of damage) was constructed in [15]. There, it was proved that there exists a time parametrized family of elastic tensors $A(t, x)$ (mixture of A_s and A_w with proportion $\Theta(t, x)$) satisfying a monotonicity property (irreversibility of the damage), a minimality condition (described in detail below) and an energy balance.

In the case $\Theta(t, x) \in \{0, 1\}$, i.e., $\Theta(t, \cdot) = \chi_{D(t)}(\cdot)$, we say that $D(t)$ is a strong solution of this energy-based formulation.

We now describe the new models for damage evolution based explicitly on a strain threshold, without any reference to an energetic cost for damage. We restrict our analysis to the scalar (anti-plane deformations) isotropic case ($A_s = \alpha I$ and $A_w = \beta I$). Our formulation is based on three principles. First, damage is irreversible, so that the damaged region is increasing in time; second, there exists a damage threshold λ, such that in the undamaged region, the strain is at or below λ; and third, damage only occurs as is necessary in order to maintain the second condition. The first two conditions are straightforward, but a precise formulation of the third is not as obvious. As an illustration, consider a one-dimensional homogeneous bar being strained along its length by a boundary condition at its endpoints, so that at every time the strain is constant along the bar. When the difference in boundary conditions at the endpoints is large enough, the material cannot remain undamaged and still satisfy the threshold condition. The question then is, where does damage occur? There are really only two possibilites: i) damage occurs everywhere, and ii) damage occurs at an arbitrary location, in a region just large enough so that the second condition remains satisfied. We adopt the latter view, for the following reason: to say that, mathematically, the bar is homogeneous is to say that we are modeling a bar that is almost homogeneous, in which case the damage threshold will vary from point to point. Points with lower thresholds will undergo damage first, and once enough damage has occurred for the threshold condition to be satisfied in the undamaged region, no further damage will occur.

Still, formulating this last condition for quasi-static evolutions seems necessarily a bit technical, and we considered more than one formulation, finally settling on the following, which we found to be the most straightforward to work with. We say that $t \mapsto D(t)$ is a threshold-based quasi-static damage evolution with threshold λ if

(1) Monotonicity: $t \mapsto D(t)$ is increasing
(2) Threshold: Setting $\sigma_{D(t)} I := \alpha I \chi_{D(t)} + \beta I (1 - \chi_{D(t)})$ and $u(t)$ to be the solution of
$$-\mathrm{div}(\sigma_{D(t)} \nabla u(t)) = f(t),$$
we have $|\nabla u(t)| \le \lambda$ a.e. in $\Omega \setminus D(t)$
(3) $D(t)$ is necessary: $\forall E \subset D(t)$ with $|E| > 0$, and all Δt sufficiently small, $\exists \tau < t - \Delta t$ such that if we consider the solution v of
$$-\mathrm{div}(\sigma_{D(\tau+\Delta t)\setminus \Delta E} \nabla v) = f(\tau + \Delta t),$$
where $\Delta E := E \cap [D(\tau + \Delta t) \setminus D(\tau)]$, we have $|\nabla v| > \lambda$ on a set of positive measure in ΔE.

We will show in Theorem 4.4 that there is a correspondence between the k in [15] and the threshold λ such that if a set-valued function D is a strong solution to the variational formulation in [15], then it is a solution of the above threshold problem. We note that the other direction holds also, but we are currently unable to show this for the relaxed formulation, and so we do not discuss this direction further here.

A corresponding "microstructure"-based threshold model is more difficult to formulate, and the connection to [15] is more delicate. Because any threshold model must involve pointwise properties of deformation gradients, there are difficulties in formulating a relaxed model just in terms of weak limits of approximating sequences. Instead one can use the pointwise properties of deformation gradients corresponding to sequences of damage sets $\{D_n\}$ that generate the relaxed solution A. For the most natural microstructure version, there is a problem in making the argument that there exists a sequence $\{D_n\}$ that generates the [15] microstructure solution and is also a threshold solution. This is due to the fact that we cannot conclude that the D_n have any (reasonable) minimality property, even asymptotically. The reason is that if a sequence of mixtures of strong and weak materials (i.e., undamaged and damaged) generates an effective elasticity tensor A, and another sequence of mixtures with more weak material (in the sense of set inclusion) generates a tensor A', it is not necessarily true that A' is obtainable as a mixture of A and the weak material. Yet it is only with respect to such mixtures that [15] evolutions are minimal. This fact and its consequences on relaxed formulations are detailed in Remark 3.3. We were therefore led to define an energetic formulation explicitly in terms of the mixtures that generate the effective tensors, which has solutions that are also solutions of our relaxed threshold formulation.

The following remark, based on a simple computation, is a hint that the energy formulation is equivalent to the threshold formulation with threshold λ.

Remark 1.1. *We consider the Dirichlet problem on $(0,1)$ with $u(0) = 0$ and $u(1) = \lambda + \delta$, and we compute the elastic energy corresponding to no damage and also the energy of the solution corresponding to the damaged set $D = (0,d)$, with d chosen so that u satisfies $u' = \lambda$ in $(d,1)$ (this corresponds to $d = \frac{\delta\alpha}{\lambda(\beta-\alpha)}$). In the first case, the energy is $E = \frac{1}{2}\beta(\lambda+\delta)^2$. In the second case $E_d = \frac{1}{2}\lambda\beta(\lambda+\delta)$. We notice that the statement*

$$E - E_d > kd \quad \iff \quad \delta > 0$$

is true exactly when the damage energy penalty $k = \frac{\lambda^2\beta(\beta-\alpha)}{2\alpha}$. That is, having a damage region of $(0,d)$ minimizes the energy exactly when this is just enough damage to bring the strain outside the damage region down to the threshold λ.

Finally, we show that all local minimizers (and even all stable states) for our energy model are global minimizers.

We end this introduction by noting that, while an energetic formulation for dynamics with damage is not so clear, with a threshold, there is a natural preliminary formulation, based on the same three principles as the quasi-static formulation: irreversibility, threshold, and necessity. Namely, we have

(1) Monotonicity: $t \mapsto D(t)$ is increasing
(2) Threshold: Setting $\sigma_{D(t)}I := \alpha I\chi_{D(t)} + \beta I(1-\chi_{D(t)})$ and u to be the solution of
$$u_{tt} - \operatorname{div}(\sigma_D \nabla u) = f(t)$$
subject to appropriate initial conditions, we have $|\nabla u(t)| \leq \lambda$ a.e. in $\Omega \setminus D(t)$, for every $t \in [0,T]$
(3) D is necessary: $\forall E \subset D(T)$ with $|E| > 0$, and all $\Delta t > 0$, the solution v of
$$v_{tt} - \operatorname{div}(\sigma_{D\setminus E}\nabla v) = f(t),$$
subject to the same initial conditions as u, satisfies $|\nabla v(x,t)| > \lambda$ for some $x \in D(t) \cap E$ and $t \in [\tau, \tau + \Delta t]$, where
$$\tau := \inf\{t : |E \cap D(t)| > 0\}.$$

2. Preliminaries and notation

2.1. **Homogenization.** Homogenization is the main tool in the relaxation of the energy-based models. We recall, for the readers' convenience, the notions of homogenization and of G-convergence (see, e.g., [13], or [36] for the more general case of nonsymmetric linear operators and H-convergence), and specialize them to the case of two-phase mixtures of linearly elastic materials.

Consider a sequence $A^n \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$, where $0<\alpha<\beta$ and

$$\begin{aligned}\mathcal{F}(\alpha,\beta) \; &:= \{\text{fourth order tensors } B \text{ with symmetries } B_{ijkh}=B_{khij}=B_{jikh} \\ &\quad \text{such that } B\varepsilon\varepsilon \in [\alpha|\varepsilon|^2, \beta|\varepsilon|^2],\; \varepsilon \text{ symmetric } \in \mathbb{R}^{N\times N}\}.\end{aligned}$$

We say that $A^n \xrightarrow{G} A$, $A \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$, iff, for every body force $f \in H^{-1}(\Omega;\mathbb{R}^N)$, the solutions u^n of the equilibrium equations

$$-\mathrm{div}(A^n e(u^n)) = f, \quad u^n \in H^1_0(\Omega;\mathbb{R}^N),$$

where the linearized strain tensor $e(u^n)$ is given by $e(u^n) := \frac{(\nabla u^n + (\nabla u^n)^t)}{2}$, satisfy

$$\begin{cases} u^n \rightharpoonup u, \text{ weakly in } H^1_0(\Omega;\mathbb{R}^N) \\ A^n e(u^n) \rightharpoonup Ae(u), \text{ weakly in } L^2(\Omega;\mathbb{R}^{N\times N}), \end{cases} \tag{2.1}$$

where u is the solution of

$$-\mathrm{div}(Ae(u)) = f.$$

Note that in the case of symmetric tensors the first property of (2.1) is enough to characterize G-convergence and the second condition, which is in turn essential in the nonsymmetric case, can be obtained as a consequence.

Now, let B and C be the stiffness tensors (Hooke's laws) of two phases, that is elements of $\mathcal{F}(\alpha,\beta)$. We look, for any mixture of those two phases – that is for any characteristic function χ of, say, phase B – at a new elastic material with stiffness

$$\sigma_\chi := \chi B + (1-\chi)C.$$

Considering a sequence of characteristic functions $\chi^n \overset{*}{\rightharpoonup} \Theta$ (which from now on we understand to mean weak-$*$ convergence in $L^\infty(\Omega)$), we investigate the possible G-limits of σ_{χ^n}. The properties of G-convergence that will be needed are

- *Compactness*: for any sequence $A^n \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$, there exists a subsequence, $A^{k(n)}$, and $A \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$ such that $A^{k(n)} \xrightarrow{G} A$;
- *Convergence of the energy*: if $A^n \xrightarrow{G} A$, then, with u^n and u defined as above,
$$\int_\Omega A^n e(u^n)e(u^n)\,dx \to \int_\Omega Ae(u)e(u)\,dx;$$
- *Metrizability*: G-convergence is associated to a metrizable topology on $L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$;
- *Ordering*: if $B^n \le A^n$ and $B^n \xrightarrow{G} B$, $A^n \xrightarrow{G} A$, then $B \le A$ (the inequalities are in the sense of quadratic forms);
- *Locality*: if $B^n \xrightarrow{G} B$, $A^n \xrightarrow{G} A$, and χ is a characteristic function on Ω, then $\chi B^n + (1-\chi)A^n \xrightarrow{G} \chi B + (1-\chi)A$;

- *Periodicity*: if $A^n(x) := A(nx)$, with $A \in L^\infty([0,1]^N; \mathcal{F}(\alpha,\beta))$ periodic, then the whole sequence A^n G-converges to A^0, which is the constant tensor given by

$$A^0 ee = \inf_{\varphi \text{ periodic}} \int_{[0,1]^N} A(y)(e+e(\varphi))(e+e(\varphi))\,dy. \tag{2.2}$$

In the case of a two-phase material, we consider $\sigma_{\chi^n}(x) = \chi(nx)B + (1 - \chi(nx))C$, with χ a characteristic function on $[0,1]^N$, and we speak of periodic mixtures with volume fraction $\Theta := \int_{[0,1]^N} \chi\,dy$ of material B.

The set of all G-limits resulting from the periodic mixture of B and C with volume fractions Θ and $1-\Theta$ is denoted by $G_\Theta(B,C)$.

The relevance of this set is clarified by a famous unpublished result of localization due to Dal Maso and Kohn (see [38] for the nonlinear case). It claims that the range of all possible mixuters of B and C is given by periodic homogenization. More precisely if $\Theta \in L^\infty(\Omega,[0,1])$ and we denote by $\mathcal{G}_\Theta(B,C) \subset L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$ the set of all possible G-limits of σ_{χ^n}, where $\chi^n \overset{*}{\rightharpoonup} \Theta$, then

$$\mathcal{G}_\Theta(B,C) = \{A \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta)) : A(x) \in \overline{G}_{\Theta(x)}(B,C), \text{ a.e. in } \Omega\}. \tag{2.3}$$

The set of all possible mixtures of B and C, as the volume fraction varies from point to point, is the G-closure of B and C and will be denoted by $\mathcal{G}(B,C)$ and as consequence of the localization result mentioned above is given by

$$\mathcal{G}(B,C) = \{A \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta)) \;:\; \exists \Theta \in L^\infty([0,1]^N;[0,1]), \text{ such that } A(x) \in \overline{G}_{\Theta(x)}(B,C), \text{ a.e. in } \Omega\}.$$

3. Energy based solutions

The idea of the formulation given in [15], which was inspired by [17], is that the G-closure of A_s and A_w is the right space for both solutions and their competitors, when describing the minimality properties of solutions. Due to irreversibility, the minimality property of damage is only with respect to adding further damage. This is naturally formulated in [15] as the minimality, for each t, of $(A(t),\Theta(t))$ with respect to competitors (A',Θ') such that $A' \in \mathcal{G}(A(t),A_w)$. Furthermore, this evolution can be approximated by a sequence of sets $D_n(t)$ (the damage sets), increasing in t, such that

$$\chi_{D_n(t)}A_w + (1-\chi_{D_n(t)})A_s \xrightarrow{G} A(t) \qquad \text{and} \qquad \chi_{D_n(t)} \overset{*}{\rightharpoonup} \Theta(t)\,.$$

However, rather surprisingly, even though $(A(t),\Theta(t))$ has this seemingly natural minimality property, the sequence $D_n(t)$ (and, indeed, any such approximating sequence) does not have good optimality properties, since if $D_n \subset D_n'$ for each $n \in \mathbb{N}$, it is not generally true that the G-limit of (a subsequence of) $\sigma_{D_n'}$ is in $\mathcal{G}(A_w,A(t))$ (see Remark 3.3).

The point is that while the material damage being modeled can occur on a very small scale, it is always a finite one. Hence, the relaxed formulation needs to be viewed as an approximation for damage occurring on a very small scale, but without microstructure. The energy formulation with relaxation therefore needs to be consistent with the formulation for finite, albeit small, scales. Below we give a definition of energy minimization based on this idea, but first we introduce the space of competitors that we will use.

From now on, with a little abuse of notation, we define

$$\sigma_E := \chi_E A_w + (1-\chi_E)A_s.$$

Given $A \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$ and $\Theta \in L^\infty(\Omega,[0,1])$, with $A(x) \in \overline{G}_{\Theta(x)}(A_w,A_s)$ a.e. $x \in \Omega$, and a sequence of sets D_n such that

$$\begin{cases} \chi_{D_n} \overset{*}{\rightharpoonup} \Theta \\ \sigma_{D_n} \xrightarrow{G} A\,, \end{cases}$$

we give the following definition.

Definition 3.1 (Constrained G-closure). *We denote by $\hat{\mathcal{G}}(\{D_n\},A_w,A_s)$, which we call the constrained G-closure, the subset of $\mathcal{G}(A_w,A_s)$ given by all symmetric tensors A' that are the G-limit of a subsequence of $\sigma_{D'_n}$ satisfying $D'_n \supseteq D_n$. If $\chi_{D'_n} \overset{*}{\rightharpoonup} \Theta'$ we say that $A' \in \hat{\mathcal{G}}_{\Theta'}(\{D_n\},A_w,A_s)$.*

In the sequel, since A_s and A_w are fixed, the constrained G-closure will be simply denoted by $\hat{\mathcal{G}}(\{D_n\})$ (and $\hat{\mathcal{G}}_{\Theta'}(\{D_n\})$).

Definition 3.2 (Energy Minimizing Evolution). *Given $f \in W^{1,1}([0,T],H^{-1}(\Omega))$, we say that $(A(t),\Theta(t))$ is a quasi-static evolution for the Energy Minimization Problem (EMP) if for each $t\in[0,T]$, we have $\Theta(t) \in L^\infty(\Omega)$, $A(t) \in \overline{G}_{\Theta(t)}(A_w,A_s)$, and they satisfy the following properties*

1. *Monotonicity: $A(t)$ is decreasing and $\Theta(t)$ is increasing as functions of t;*
2. *Energy balance: The total energy*

$$\mathcal{E}(t) := \int_\Omega \frac{1}{2}A(t)e(u(t))e(u(t))\,dx - \langle f(t),u(t)\rangle + k\int_\Omega \Theta(t)\,dx$$

satisfies

$$\mathcal{E}(t) = \mathcal{E}(0) - \int_0^t \langle \dot{f}(\sigma),u(\sigma)\rangle\,d\sigma,$$

where $u(t)$ is the solution in $H^1_0(\Omega)$ of

$$-\mathrm{div}(A(t)e(u(t))) = f(t) \quad \text{in } \Omega\,;$$

3. *Minimality: There exists a time-indexed family of sequences of sets $D_n(t)$, monotonically increasing in t, such that for every $t\in[0,T]$,*

$$\begin{cases} \chi_{D_n(t)} \overset{*}{\rightharpoonup} \Theta(t) \\ \sigma_{D_n(t)} \xrightarrow{G} A(t)\,, \end{cases} \tag{3.4}$$

and for every (A',Θ') such that $A' \in \hat{\mathcal{G}}_{\Theta'}(\{D_n\})$ we have

$$\begin{aligned} &\int_\Omega \frac{1}{2}A(t)e(u(t))e(u(t))\,dx - \langle f(t),u(t)\rangle + k\int_\Omega \Theta(t)\,dx \\ &\qquad\le \int_\Omega \frac{1}{2}A'e(v)e(v)\,dx - \langle f(t),v\rangle + k\int_\Omega \Theta'\,dx, \end{aligned} \tag{3.5}$$

among all $v \in H^1_0(\Omega;\mathbb{R}^N)$.

Remark 3.3. *Note that the minimality condition in Definition 3.2 of quasi-static evolutions is stronger than the one given on* [15]*:*

$$\int_\Omega \frac{1}{2}A(t)e(u(t))e(u(t))\,dx - \langle f(t),u(t)\rangle \leq \int_\Omega \frac{1}{2}A'e(v)e(v)\,dx - \langle f(t),v\rangle + k\int_\Omega (1-\Theta(t))\theta\,dx, \tag{3.6}$$

among all $v \in H^1_0(\Omega;\mathbb{R}^N)$*,* $\theta \in L^\infty(\Omega,[0,1])$ *and* $A'(x) \in \overline{G}_{\theta(x)}(A_w,A(t))$ *a.e.* $x \in \Omega$*. Indeed for every* $t \in (0,T]$ *and any* $E \subseteq \Omega$ *we can apply the minimality condition of Definition 3.2 to the pair* (A',Θ') *given by* $A' = \chi_E A_w + (1-\chi_E)A(t)$ *and* $\Theta' = (1-\Theta(t))\chi_E + \Theta(t)$*. By the locality of the G-convergence this pair is an admissible competitor for condition (3) of Definition 3.2. Indeed* $A' \in \hat{\mathcal{G}}(\{D_n(t)\})$ *(*A' *being the G-limit of* $\sigma_{D_n(t)\cup E}$ *for any sequence* $D_n(t)$ *satisfying (3.4)) and* $(1-\Theta(t))\chi_E + \Theta(t)$ *the weak* limit of* $\chi_{D_n(t)\cup E}$*. On the other hand we have that, a.e.* $x \in \Omega$*,* $A'(x) \in G_{\theta(x)}(A_w,A(t))$ *with* $\theta = \chi_E$*. As a consequence of (3.5) we deduce*

$$\int_\Omega \frac{1}{2}A(t)e(u(t))e(u(t))\,dx - \langle f(t),u(t)\rangle$$
$$\leq \min_{v\in H^1_0(\Omega)} \int_\Omega \frac{1}{2}(\chi_E A_w + (1-\chi_E)A(t))e(v)e(v)\,dx - \langle f(t),v\rangle + k\int_E (1-\Theta(t))\,dx.$$

Thus we deduce (3.6) using the fact that for every A' *such that* $A'(x) \in \overline{G}_{\theta(x)}(A_w,A(t))$*, with* $\theta \in L^\infty(\Omega,[0.1])$*, we have* $A' \in \mathcal{G}_\theta(A_w,A(t))$ *and there exists a sequence of sets* E_h *such that* $\chi_{E_h}A_w + (1-\chi_{E_h})A(t) \xrightarrow{G} A'$ *and* $\chi_{E_h} \overset{*}{\rightharpoonup} \theta$*, hence*

$$\lim_{h\to\infty} \min_{v\in H^1_0(\Omega)} \int_\Omega \frac{1}{2}(\chi_{E_h}A_w + (1-\chi_{E_h})A(t))e(v)e(v)\,dx - \langle f(t),v\rangle + k\int_{E_h}(1-\Theta(t))$$
$$= \min_{v\in H^1_0(\Omega)} \int_\Omega \frac{1}{2}A'e(v)e(v)\,dx - \langle f(t),v\rangle + k\int_\Omega (1-\Theta(t))\theta\,dx.$$

Note that the argument in Remark 3.3 can in particular be used to show that

$$\hat{\mathcal{G}}(\{D_n\}) \supset \mathcal{G}(A_w,A) = \cup_\Theta \mathcal{G}_\Theta(A_w,A)\,. \tag{3.7}$$

This inclusion can be strict as shown in the following example, and this implies that Definition 3.2 is strictly stronger than the one given in [15] (and that [15] solutions cannot be shown to be threshold solutions). A natural interpretation of this difference is that the only admissible competitors for $(A(t),\Theta(t))$ in [15] are states with at least as much damage, and all additional damage must occur on a larger scale than the relaxed damage represented by $(A(t),\Theta(t))$.

Example 3.4. We show with an explicit example that given $A \in \mathcal{G}(A_w,A_s)$ and a sequence $\{D_n\}$ of sets such that σ_{D_n} G-converges to A, the set $\hat{\mathcal{G}}(\{D_n\})$ can be strictly larger than $\mathcal{G}(A_w,A)$. In other words we show that we can construct a sequence $D'_n \supset D_n$ such that $\sigma_{D'_n}$ G-converges to a tensor A' with $A' \notin \mathcal{G}(A_w,A)$.

We give the example in two dimensions for scalar problems, but the same construction can be also done in general. Consider $A_s = \beta I$ and $A_w = \alpha I$, where $\beta > \alpha$ and I denotes the 2×2 identity matrix. Consider also a ball B of radius r, $0 < r < \frac{1}{2}$, centered in the unit cube and then consider the sequence of periodic sets D_n such that $\chi_{D_n}(x) = \chi_B(n(x-[x]))$, where $[x]$ denotes, with a little abuse

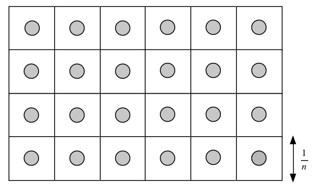

FIGURE 1. The periodic set D_n

of notation, the vector whose components are the integer part of the components of x (see Figure 1).

Since each set D_n is invariant by 90° rotations, it follows from the homogenization formula (2.2) that the effective matrix corresponding to this microstructure is isotropic, i.e., there exists $\beta' \in (\alpha, \beta)$ such that σ_{D_n} G-converges to $\beta' I$.

Now let $E = \{x \ : \ |x_1 - \frac{1}{2}| < R\}$, with $r \leq R < \frac{1}{2}$. Clearly $B \subseteq E$ and the corresponding periodic sequence E_n (the lamination satisfying $\chi_{E_n}(x) = \chi_E(n(x - [x]))$) contains D_n.

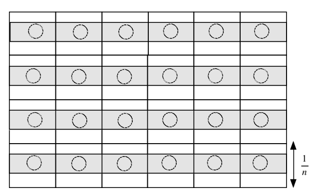

FIGURE 2. The periodic set $E_n \supset D_n$, a lamination

It is well known that the sequence σ_{E_n} G-converges to a 2×2 matrix A', that by construction belongs to $\hat{\mathcal{G}}(\{D_n\})$ and also belongs to the boundary of $G(\alpha I, \beta I) := \cup_\Theta G_\Theta(\alpha I, \beta I)$. On the other hand the set $G(\alpha I, \beta' I)$ is strictly contained in $G(\alpha I, \beta I)$ (except for the point αI). We know in particular that

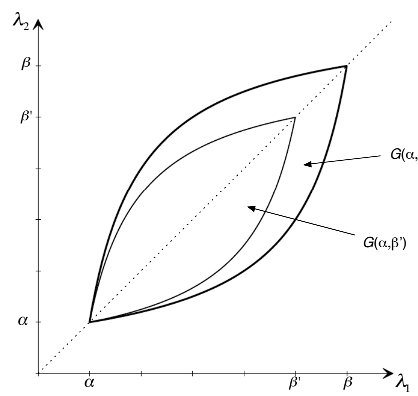

FIGURE 3. $G(\alpha I, \beta' I) \subset G(\alpha I, \beta I)$

$\partial G(\alpha I, \beta I) \cap G(\alpha I, \beta' I) = \alpha I$, as shown in Figure 3 where the sets $G(\alpha I, \beta I)$ and $G(\alpha I, \beta' I)$ are represented in terms of their eigenvalues λ_1 and λ_2.

Hence $A' \notin G(\alpha I, \beta' I)$ and this shows that $\hat{\mathcal{G}}(\{D_n\}) \supset \mathcal{G}(\alpha I, \beta' I)$.

Remark 3.5. *Notice that in* [15] *the solution constructed by a discrete time approximation satisfies the extra property, not explicitly stated in the definition, that* $A(t) \in G_{1-\frac{1-\Theta(t)}{1-\Theta(s)}}(A_w, A(s))$ *for every* $s < t$. *This may be not true for a solution of Definition 3.2 as constructed in Theorem 3.7. On the other hand, solutions of Definition 3.2 satisfy* $A(t) \in \hat{\mathcal{G}}_{\Theta(t)}(\{D_n(s)\})$ *for every* $s < t$.

Definition 3.6 (Strong Energy Minimizing Evolution)**.** *We say that the set function* $D(t)$ *(the damage set at time* t*) is a quasi-static evolution for the Strong Energy Minimization Problem (SEMP) if the pair* $(\sigma_{D(t)}, \chi_{D(t)})$ *is a solution of (EMP).*

In the following theorem we prove the existence of a solution to Definition 3.2.

Theorem 3.7. *For every* $f \in W^{1,1}([0,T], H^{-1}(\Omega))$*, there exists a solution* $(A(t), \Theta(t))$ *of (EMP).*

Proof. We follow the proof obtained by discretization in time in [15]. There, the issue was the existence of a solution for a weaker formulation than the one stated in Definition 3.2, where the minimality condition (3) is satisfied for a smaller class of competitors, i.e., $A' \in G_{\Theta'}(A_w, A(t))$ (see Remark 3.3). This difference does not change dramatically the strategy of the proof, but requires a slightly different definition of the incremental problems for the discrete approximation.

Given the interval $[0,T]$ and given $n \in \mathbb{N}$ we consider a partition $\{t_i^n\}_{i=0\ldots m}$ with $t_0^n = 0$, $t_m^n = T$ and $t_i^n - t_{i-1}^n \leq \frac{1}{n}$ (and $m \sim nT$). We now construct a piecewise constant approximation of the solution starting from the almost minimizers of an appropriate incremental variational problem, that is defined as follows.

Given t_i^n we denote $f_i^n(\cdot) = f(t_i^n, \cdot)$.

Step 1:The first time step $t_0^n = 0$.
At the first time step, we wish to almost minimize, over $(v, \chi_D) \in H_0^1(\Omega) \times L^\infty(\Omega;\{0,1\})$,

$$E_{tot}(v, D, f_0) := \int_\Omega \left[\frac{1}{2}\Big(\chi_D A_w + (1-\chi_D)A_s\Big)e(v)e(v) + k\chi_D\right]\, dx - \langle f_0, v\rangle,$$

with $f_0 := f(0)$. We choose a sequence of subsets $D_0^k \subset \Omega$ such that

$$\min_{v \in H_0^1(\Omega)} E_{tot}(v, D_0^k, f_0) \leq \inf_D \min_{v \in H_0^1(\Omega)} E_{tot}(v, D, f_0) + \frac{1}{2k}\,.$$

Step 2: The subsequent time steps.
Given $n \in \mathbb{N}$, for every $i \in \{1, ..., m\}$ we then can choose the sequence $\{D_{i,n}^k\}$ with $D_{i,n}^k \supseteq D_{i-1,n}^k$ (where $D_{0,n}^k := D_0^k$), such that

$$\min_{v \in H_0^1(\Omega)} E_{tot}(v, D_{i,n}^k, f_i^n) \leq \inf_{D \supseteq D_{i-1,n}^k} \min_{v \in H_0^1(\Omega)} E_{tot}(v, D, f_i^n) + \frac{1}{2^{i+1}k}\,.$$

Step 3: The discrete approximation.
For every $n \in \mathbb{N}$ we can define a piecewise constant in time sequence of sets $D_n^k(t)$, nondecreasing in t, by

$$D_n^k(t) := D_{i,n}^k \qquad \text{if } \ t_i^n \le t < t_{i+1}^n \quad \text{and } i = 0, ..., m-1\,.$$

We then extract a subsequence in k, still denoted by $D_n^k(\cdot)$, such that for every $t \in [0,T]$ we have

$$\begin{cases} \chi_{D_n^k(t)} \overset{*}{\rightharpoonup} \Theta_n(t) \\ \\ \sigma_{D_n^k(t)} := \chi_{D_n^k(t)} A_w + (1 - \chi_{D_n^k(t)}) A_s \xrightarrow{G} A_n(t) \end{cases} \qquad \text{as } k \to \infty$$

for some $\Theta_n(t) \in L^\infty(\Omega, [0,1])$ and $A_n(t) \in L^\infty(\Omega; \mathcal{F}(\alpha, \beta))$, with $A(t,x)$ belonging to $G_{\Theta_n(t,x)}(A_w, A_s)$ a.e. in Ω, both monotonic in t. Thus (see for instance [15] Theorem 3.1 and Remark 3.3) we can also extract a subsequence of n such that there exists $\Theta(t) \in L^\infty(\Omega, [0,1])$ and $A(t) \in L^\infty(\Omega; \mathcal{F}(\alpha, \beta))$, with $A(t,x) \in G_{\Theta(t,x)}(A_w, A_s)$ a.e. in Ω, satifying

$$\begin{cases} \Theta_n(t) \overset{*}{\rightharpoonup} \Theta(t) \\ \\ A_n(t) \xrightarrow{G} A(t) \end{cases} \qquad \text{as } n \to \infty$$

for every $t \in [0,T]$. Finally by a diagonal argument we can find a sequence $k(n) \to \infty$ as $n \to \infty$ such that the sequence of sets $D_n(t) := D_n^{k(n)}(t)$ satisfies for every $t \in [0,T]$

$$\begin{cases} \chi_{D_n(t)} \overset{*}{\rightharpoonup} \Theta(t) \\ \\ \sigma_{D_n(t)} \xrightarrow{G} A(t) \end{cases} \qquad \text{as } n \to \infty\,.$$

Note that the monotonicity of $D_n(\cdot)$ and $\Theta(\cdot)$ are guaranteed by their construction. It remains to show that $(\Theta(t), A(t))$ satisfies the energy balance (property (2) of Definition 3.2) and the minimality (property (3) of Definition 3.2).

Step 4: Minimality property.
Fix $t \in [0,T]$ and consider $D_n' \supset D_n(t)$ such that $\sigma_{D_n'}$ G-converges to some A' and $\chi_{D_n'}$ weakly $*$ converges to some Θ'. By the definition of $D_n(t)$ we have

$$\min_{v \in H_0^1(\Omega)} E_{tot}(v, D_n(t), f_n(t)) \le \min_{v \in H_0^1(\Omega)} E_{tot}(v, D_n', f_n(t)) + o(1)$$

where $f_n(t)$ is the piecewise constant in time approximation of f given by $f_n(t,\cdot) := f(t_i^n, \cdot)$ if $t_i^n \le t < t_{i+1}^n$, and $o(1) \to 0$ as $n \to \infty$. By the definition of G-convergence and using the fact that $f_n(t) \to f(t)$ strongly in $H^{-1}(\Omega)$ we get

$$\min_{v \in H_0^1(\Omega)} E_{tot}(v, A(t), \Theta(t), f(t)) \le \min_{v \in H_0^1(\Omega)} E_{tot}(v, A', \Theta', f(t))\,, \tag{3.8}$$

where with a little abuse of notation we denoted

$$E_{tot}(v, A, \Theta, f) := \int_\Omega \left[\frac{1}{2} A e(v) e(v) + k\Theta \right] dx - \langle f, v \rangle\,.$$

From (3.8) we deduce (3.5), which concludes the proof of minimality.

Step 5: Energy Balance.
From the definition of $D^k_{i,n}$ in Step 2 and the appropriate choice of each $v^k_{i,n}$ we get

$$\begin{aligned}
E_{tot}(v^k_{i,n}, D^k_{i,n}, f^n_i) &= \min_{v\in H^1_0(\Omega)} E_{tot}(v, D^k_{i,n}, f^n_i) \\
&\leq \inf_{D\supseteq D^k_{i-1,n}} \min_{v\in H^1_0(\Omega)} E_{tot}(v, D, f^n_i) + \frac{1}{2^{i+1}k} \\
&\leq E_{tot}(v^k_{i-1,n}, D^k_{i-1,n}, f^n_i) + \frac{1}{2^{i+1}k} \\
&\leq \min_{v\in H^1_0(\Omega)} E_{tot}(v, D^k_{i-1,n}, f^n_{i-1}) + \langle f^n_{i-1} - f^n_i, v^k_{i-1,n}\rangle + \frac{1}{2^{i+1}k}\,.
\end{aligned}$$

Then iterating the above relation from $j \in \{0, ..., m\}$ to 0 we obtain

$$\min_{v\in H^1_0(\Omega)} E_{tot}(v, D^k_{j,n}, f^n_j) \leq \min_{v\in H^1_0(\Omega)} E_{tot}(v, D^k_0, f_0) + \sum_{i=0}^{j} \langle f^n_{i-1} - f^n_i, v^k_{i-1,n}\rangle + \frac{1}{k}\,.$$

Then by the definition of $D_n(t)$ for every $t \in [t^n_j, t^n_{j+1})$ we get

$$E_{tot}(v_n(t), D_n(t), f_n(t)) \leq E_{tot}(v_n(0), D_n(0), f(0)) - \sum_{i=0}^{j} \langle f^n_i - f^n_{i-1}, v_n(t^n_i)\rangle + o(1)$$

as $n \to \infty$, where $v_n(t)$ denotes the solution in $H^1_0(\Omega)$ of

$$-\mathrm{div}\,(\sigma_{D_n(t)} e(v)) = f_n(t) \quad \text{in} \quad \Omega\,.$$

Then using that

$$\frac{f(\tau + \Delta t) - f(\tau)}{\Delta t} \xrightarrow{H^{-1}} \dot f(\tau) \qquad \text{as } \Delta t \to 0 \text{ for a.e. } \tau$$

and taking the limit as $\Delta t \to 0$ we obtain

$$\mathcal{E}(t) \leq \mathcal{E}(0) - \int_0^t \langle \dot f(\tau), u(\tau)\rangle d\tau\,.$$

The proof of the inverse inequality is standard and based on the fact that from the construction of $A(t)$ it is easy to check that $(A(t), \Theta(t))$ is an admissible competitor for the minimality condition at time $s < t$. A detailed proof can be obtained as a particular case of the proof of the more involved inequality (4.16) (see also [15]).
□

4. Threshold Formulation

We consider now the case of A_s and A_w isotropic: $A_s = \beta I$ and $A_w = \alpha I$. We also restrict our attention to the scalar case (i.e., functions that take values in $\mathbb{R}$ rather than in $\mathbb{R}^N$).

Given $D \subset \Omega$, from now on, with a little abuse of notation, σ_D will denote the scalar function

$$\sigma_D := \alpha\chi_D + \beta(1 - \chi_D)\,.$$

First we formulate a definition for the classical situation in which the damage at time t occurs in a set $D(t)$, rather than needing to represent the damage region as a limit of a sequence of sets.

Definition 4.1. *$D(t)$ is a solution of the Strong Threshold Problem (STP) with threshold λ if the following hold:*

(1) *Monotonicity: $t \mapsto D(t)$ is increasing;*
(2) *Threshold: Setting $u(t)$ to be the solution of*
$$-\mathrm{div}(\sigma_{D(t)}\nabla u(t)) = f(t),$$
we have $|\nabla u(t)| \leq \lambda$ a.e. in $\Omega \setminus D(t)$;
(3) *D is necessary:*
- *$\forall E \subset D(T)$ with $|E| > 0$ and all Δt small enough, $\exists \tau < t - \Delta t$ such that if we consider the solution v of*
$$-\mathrm{div}(\sigma_{D(\tau+\Delta t)\setminus \Delta E}\nabla v) = f(\tau + \Delta t),$$
where $\Delta E := E \cap [D(\tau + \Delta t) \setminus D(\tau)]$, we have $|\nabla v| > \lambda$ in a subset of ΔE with positive measure.
- *If D is not continuous at T, then we also require that $\forall E \subset D(T) \setminus D(T^-)$ with $|E| > 0$ and $D(T^-) := \cup_{t<T} D(t)$, the solution v of*
$$-\mathrm{div}(\sigma_{D(T)\setminus E}\nabla v) = f(T)$$
satisfies $|\nabla v(x)| > \lambda$ in a subset of E with positive measure.

Theorem 4.2. *If $D(t)$ is a solution of (SEMP), then it is a solution of (STP) with threshold λ satisfying $k = \frac{\lambda^2\beta(\beta-\alpha)}{2\alpha}$.*

The above result can be obtained as a special case of the more general Theorem 4.4 below, which is not restricted to the classical situation of damage occuring in sets. We first need the following definition for threshold-based damage for the weaker setting in which there is damage microstructure.

Definition 4.3. *$(A(t), \Theta(t))$ is a solution of the Threshold Problem (TP) with threshold λ if for every $t \in [0, T]$ there exists a sequence $\{D_n(t)\}$ such that $\sigma_{D_n(t)} I \xrightarrow{G} A(t)$ and $\chi_{D_n(t)} \overset{*}{\rightharpoonup} \Theta(t)$ in L^∞, and the following hold*

(1) *Monotonicity: $D_n(\cdot)$ is increasing;*
(2) *Threshold: For the solution u_n of*
$$-\mathrm{div}(\sigma_{D_n(t)}\nabla u_n) = f(t),$$
we have that $\forall \delta > 0$, the sets in which there is no damage but the threshold is exceeded by at least δ,
$$U_n := \{x \notin D_n(t) : |\nabla u_n(x)| > \lambda + \delta\},$$
satisfy
$$|U_n| \to 0 \,;$$
(3) *Necessity of the damage:*
- *For all $E_n \subset D_n(T)$ with $\liminf |E_n| > 0$, we have that $\forall \delta > 0$ and $\forall$ $\Delta t > 0$ small enough, there exists $\tau < T - \Delta t$ such that, setting v_n to be the solution of*
$$-\mathrm{div}(\sigma_{D_n(\tau+\Delta t)\setminus \Delta E_n}\nabla v_n) = f(\tau + \Delta t),$$
where $\Delta E_n := E_n \cap [D_n(\tau + \Delta t) \setminus D_n(\tau)]$, we have that the subsets of ΔE_n in which the threshold is almost exceeded,
$$\Delta E_n^\delta := \{x \in \Delta E_n : \ |\nabla v_n(x)| > \lambda - \delta\},$$

satisfy

$$\liminf_{n\to\infty} |\Delta E_n^\delta| > 0.$$

- *If $\int_\Omega \Theta(\cdot)\,dx$ is not continuous at T, we have the following additional requirement: $\forall t_n \nearrow T$ and $\forall E_n \subset D_n(T) \setminus D_n(t_n)$ with $\liminf |E_n| > 0$, and for every $\delta > 0$, the solution v_n of*

$$-\mathrm{div}(\sigma_{D_n(T)\setminus E_n} \nabla v_n) = f(T),$$

satisfies

$$\liminf_{n\to\infty} |\{x \in E_n : \ |\nabla v_n(x)| > \lambda - \delta\}| > 0.$$

Theorem 4.4. *If $(A(t), \Theta(t))$ is a solution of the (EMP), then it is a solution of (TP) with threshold λ satisfying $k = \frac{\lambda^2\beta(\beta-\alpha)}{2\alpha}$.*

We will need the following two simple estimates in the proof.

Lemma 4.5. *Let $E \subset \Omega$ and $S \subset \Omega \setminus E$ be measurable. Consider the solution $u_E \in H_0^1(\Omega)$ of the equation*

$$-\mathrm{div}\,(\sigma_E \nabla u_E) = f \quad \text{in } \Omega\,, \tag{4.9}$$

where $f \in H^{-1}(\Omega)$. Then if we set

$$E_{el}(E, f) := \frac{1}{2}\int_\Omega \sigma_E |\nabla u_E|^2 dx - \langle f, u_E\rangle$$

we have

$$\Delta E_{el} := E_{el}(E, f) - E_{el}(E \cup S, f) \leq \frac{(\beta-\alpha)\beta}{2\alpha}\|\nabla u_E\|^2_{L^2(S)}.$$

Proof. We follow a similar lemma in [27]. Setting $\phi := u_{E\cup S} - u_E$, we have from (4.9) that $\int_\Omega \sigma_E \nabla u_E \cdot \nabla\phi\,dx = \int_\Omega f\phi\,dx$. Then

$$\Delta E_{el} = \frac{1}{2}(\beta-\alpha)\int_S |\nabla u_E|^2 dx + (\beta-\alpha)\int_S \nabla u_E \cdot \nabla\phi\,dx - \frac{1}{2}\int_\Omega \sigma_{E\cup S}|\nabla\phi|^2 dx.$$

It follows that

$$\Delta E_{el} \leq \frac{1}{2}(\beta-\alpha)\|\nabla u_E\|^2_{L^2(S)} + (\beta-\alpha)\|\nabla u_E\|_{L^2(S)}\|\nabla\phi\|_{L^2(S)} - \frac{1}{2}\alpha\|\nabla\phi\|^2_{L^2(S)}.$$

Viewed as a function of $\|\nabla\phi\|_{L^2(S)}$, the right-hand side is of the form $f(x) = \frac{1}{2}ca^2 + cax - \frac{1}{2}\alpha x^2$, which has a maximum at $x = \frac{ca}{\alpha}$. Therefore, substituting $\frac{\beta-\alpha}{\alpha}\|\nabla u_E\|_{L^2(S)}$ for $\|\nabla\phi\|_{L^2(S)}$ and simplifying, we get

$$\Delta E_{el} \leq \frac{(\beta-\alpha)\beta}{2\alpha}\|\nabla u_E\|^2_{L^2(S)}. \tag{4.10}$$

□

Next, we have

Remark 4.6. *If Q_1 is the usual unit cube in $\mathbb{R}^N$ and if the sequences $\{u_n\}, \{v_n\}$ are each bounded in $H^1(Q_1)$ and are such that $u_n - v_n \to 0$ in $L^2(Q_1)$, then a straightforward computation shows that there exists $0 < R < 1$ such that, if for each $0 < \gamma < R$ we define w_n by*

$$w_n := \phi u_n + (1 - \phi) v_n$$

with ϕ the cutoff function equal to 1 in $Q_1 \setminus Q_R$ and equal to zero in $Q_{R-\gamma}$ (where Q_R denotes the cube with the same orientation and center as Q_1, but side R) and such that $|\nabla \phi| = \frac{1}{\gamma}$ on $Q_R \setminus Q_{R-\gamma}$, then

$$\lim_{\gamma \to 0} \lim_{n \to \infty} \int_{Q_R \setminus Q_{R-\gamma}} |\nabla w_n|^2 dx = 0$$

and also

$$\lim_{\gamma \to 0} \lim_{n \to \infty} \int_{Q_R} |\nabla w_n - \nabla v_n|^2 dx = 0. \tag{4.11}$$

The only issue in the computation is to choose R so as to avoid concentrations in $|\nabla u_n|^2$ and $|\nabla v_n|^2$ on ∂Q_R.

Proof of Theorem 4.4. By Definition 3.2 we have that there exists a sequence $\{D_n(t)\}$ (monotonically increasing in t), such that $\sigma_{D_n(t)} I \xrightarrow{G} A(t)$ and $\chi_{D_n(t)} \overset{*}{\rightharpoonup} \Theta(t)$ in $L^\infty(\Omega)$ and for which the Minimality property (3) is satisfied. We will show that $\{D_n(t)\}$ also satisfies properties (2) and (3) of Definition 4.3. Both conditions are proved by contradiction, using the following strategies.

If property (2) is false, then there exists $\delta > 0$ such that the sets U_n do not eventually have small measure. We can then localize to nice points in U_n and add laminates of damage corresponding to the damage sets in Remark 1.1. This creates a competitor D'_n to D_n, whose energy is lower on the order of $\limsup_{n\to\infty} |U_n|$, contradicting the minimality of D_n.

Proving condition (3) is somewhat more subtle. If (3) is not satisfied, then there exists a sequence of sets $E_n \subseteq D_n(T)$ with $\liminf_{n\to\infty} |E_n| = \gamma > 0$, and there exist $\delta > 0$ and $\Delta t \searrow 0$ such that for all τ small enough, the sequence of sets ΔE_n^δ satisfy $|\Delta E_n^\delta| \to 0$ as $n \to \infty$. This suggests that the sets $E_n \subset D_n$ were not necessary at time τ in order for the strain in the undamaged region to remain below the threshold, and so it should not have been worth the cost to add the slices $\Delta E_n = E_n \cap [D_n(\tau + \Delta t) \setminus D_n(\tau)]$ to $D_n(\tau)$. A difficulty is that as $\Delta t \searrow 0$, these slices in general disappear, so we need a way to keep account of this sub-optimality even as $\Delta t \searrow 0$. Using the fact that in ΔE_n we are below the threshold by δ, we show that if we consider a new dissipation coefficient k_δ on E_n corresponding to the lower threshold $\lambda - \delta$, then we have an energy balance with this new coefficient on E_n, contradicting the energy balance with the original coefficient.

We first prove that property (2) is satisfied. Suppose it is false, so that there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} |\{x \notin D_n(t) : |\nabla u_n(x)| > \lambda + \delta\}| = \gamma > 0\,. \tag{4.12}$$

Set $U_n := \{x \notin D_n(t) : |\nabla u_n(x)| > \lambda + \delta\}$. Note that for every $\varepsilon > 0$ the set $\mathcal{Q}_\varepsilon^n$ of all cubes Q that satisfy the following conditions

i) The center x_0 is in U_n and is a Lebesgue point for ∇u_n, u_n, and χ_{U_n}

ii) Two sides of Q are orthogonal to $\nabla u_n(x_0)$, which we label η_n

iii) Setting $u^{x_0}_{\eta_n}(x) := \eta_n \cdot (x - x_0) + u_n(x_0)$, we have

$$\|u_n - u^{x_0}_{\eta_n}\|_{H^1(Q)} \leq \varepsilon |Q|$$

and

iv) $|D_n \cap Q| \leq \varepsilon |Q|$

is a fine covering of U_n (except possibly for a set of measure zero). Therefore for every given ε and n we can choose a countable collection of disjoint cubes in $\mathcal{Q}^n_\varepsilon$, $\{Q_i\}$, such that

$$|E_n \setminus \cup_i Q_i| = 0\,.$$

Step 1.

In each cube we will perform a construction that lowers the energy of D_n, which we now illustrate for a single cube Q that we assume to be centered at the origin. We also assume, without loss of generality, that $u_n(0) = 0$ and we set $\eta := \eta_n$ and $u_\eta := u^0_{\eta_n}$. We also define for each open set $S \subseteq \Omega$

$$E_{tot}(u, D, f, S) := \frac{1}{2}\int_S \sigma_D |\nabla u|^2 dx - \langle f, u\rangle_S + k|D|\,;$$

as above when $S = \Omega$ we will omit the dependence on S in the notation and with $\langle \cdot, \cdot \rangle_S$, with a little abuse of notation, we denote the localization of the reprentative of f to S. More precisely, for $f \in H^{-1}(\Omega)$, there exists $g \in L^2(\Omega, \mathbb{R}^n)$ such that $f = \mathrm{div} g$, i.e., $\langle f, u\rangle = \int_\Omega g\nabla u\, dx$ for all $u \in H^1_0(\Omega)$. Then

$$\langle f, u\rangle_S = \int_S g\nabla u\, dx \qquad \forall u \in H^1(S)\,.$$

We first consider E_{tot} for functions that agree with u_η on ∂Q, rather than with u_n. We claim that

$$\inf\left\{\frac{1}{2}\int_Q \sigma_{D'} |\nabla w|^2 dx - \langle f(t), w\rangle_Q + k|D'| \,:\, w - u_\eta \in H^1_0(Q) \text{ and } D' \subseteq Q\right\}$$

$$\leq E_{tot}(u_\eta, \emptyset, f(t), Q) - \frac{1}{2}\beta\delta^2|Q|\,. \tag{4.13}$$

This can be seen by first considering the one dimensional function

$$z(y) := \begin{cases} \frac{\beta}{\alpha}\lambda y & \text{if } y \in (0, d) \\ \lambda y + \frac{\beta}{\alpha}\lambda d & \text{if } y \in [d, 1), \end{cases}$$

where $d := \frac{\delta\alpha}{\lambda(\beta - \alpha)}$. We extend z to all of $\mathbb{R}$ by periodicity and to $\mathbb{R}^N$ by

$$v_\varepsilon(x) := z\left(\frac{x}{\varepsilon} \cdot \frac{\eta}{|\eta|}\right)$$

for $x \in \mathbb{R}^N$. Notice that the sequence v_ε converges to u_η in $L^2(Q)$ and is bounded in $H^1(Q)$. Hence, we can apply Remark 4.6 to $(Q, u_\eta, v_\varepsilon)$, producing w that agrees with u_η on ∂Q and whose total energy is arbitrarily close to

$$\frac{1}{2}|Q|\beta\lambda(\lambda + \delta) - \langle f(t), u_\eta\rangle_Q + \frac{1}{2}\beta\lambda\delta|Q|$$

(see also Remark 1.1). Then a straightforward calculation gives (4.13).

Step 2.
We next claim that in each cube Q in $\mathcal{Q}^n_\varepsilon$ we have

$$\inf\left\{\frac{1}{2}\int_Q \sigma_{D'}|\nabla w|^2dx - \langle f(t), w\rangle_Q + k|D'| \,:\, w - u_n(t) \in H^1_0(Q) \text{ and } D' \subseteq Q\right\}$$
$$\leq E_{tot}(u_n(t), D_n(t), f(t), Q) - \frac{1}{2}\beta\delta^2|Q| + o(1)|Q| \tag{4.14}$$

where $o(1) \to 0$ as $\varepsilon \to 0$, uniformly with respect to n. This follows from Step 1 once we note that by properties (iii) and (iv) of Q we have

$$|E_{tot}(u_n(t), D_n(t), f(t), Q) - E_{tot}(u_\eta, \emptyset, f(t), Q)| \leq o(1)|Q|$$

and that we can make minimizing sequences for the inf in Step 1 admissible for the above inf by adding $u_n(t) - u_\eta$, introducing an error $o(1)|Q|$, with $o(1) \to 0$ as $\varepsilon \to 0$ independent of n.

Step 3.
We return now to our disjoint family of cubes $\{Q_i\}$ and, using Step 2, we construct a competitor (w_n, D'_n) for $(u_n(t), D_n(t))$ in Ω, with $D'_n \supset D_n(t)$, that agrees with $(u_n(t), D_n(t))$ outside $\cup Q_i$ and is such that

$$E_{tot}(w_n, D'_n, f(t)) \leq E_{tot}(u_n(t), D_n(t), f(t)) - \frac{1}{2}\beta\delta^2\sum_i |Q_i| + o(1)\sum_i |Q_i|\,.$$

Now up to a subsequence $\sigma_{D'_n}$ G-converges to some $A' \in G_{\Theta'}(\alpha I, \beta I)$, with $\chi_{D'_n} \overset{*}{\rightharpoonup} \Theta'$. In particular $A' \in \hat{\mathcal{G}}_{\Theta'}(\{D_n\})$. Then taking the limit as $n \to \infty$ we get

$$\min\{E_{tot}(w, A', \Theta', f(t)) \,:\, w \in H^1_0(\Omega)\} \leq E_{tot}(u(t), A(t), \Theta(t), f(t)) - \frac{1}{2}\beta\delta^2\gamma + o(1)\gamma$$

which contradicts the minimality of $(u(t), A(t), \Theta(t))$ for ε small enough.

We now prove that property (3) is satisfied. Assume first that $\int_\Omega \Theta(\cdot)\,dx$ is continuous from below at T, in which case we assume by contradiction that there exist:

i) a sequence of sets $E_n \subseteq D_n(T)$, with $\lim_{n\to\infty}|E_n| = \gamma > 0$,
ii) $\delta > 0$,
iii) a sequence $\Delta t \searrow 0$

such that for all $\tau < T$ and for each term in iii) satisfying $\Delta t < T - \tau$, we have that the function v_n in Definition 4.3 (3) satisfies

$$\liminf_{n\to\infty} |\Delta E^\delta_n| = 0\,, \tag{4.15}$$

where

$$\Delta E^\delta_n := \{x \in \Delta E_n \,:\, |\nabla v_n(x)| > \lambda - \delta\}$$

and $\Delta E_n := E_n \cap (D_n(\tau + \Delta t) \setminus D_n(\tau))$.

We set k_δ to be the dissipation coefficient corresponding to the threshold $\lambda - \delta$, so that $k_\delta = (\lambda - \delta)^2 \frac{\beta(\beta-\alpha)}{2\alpha}$.

Claim: u satisfies

$$\mathcal{E}(T) + \gamma(k_\delta - k) \geq \mathcal{E}(0) - \int_0^T \langle \dot{f}(\sigma), u(\sigma)\rangle\, d\sigma, \tag{4.16}$$

where

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega A(t)\nabla u(t)\nabla u(t)\,dx - \langle f(t), u(t)\rangle + k\int_\Omega \Theta(t)\,dx$$

.

This together with the equality in Definition 3.6 implies that $\gamma = 0$.

Proof of claim: For each fixed Δt we consider u at discrete-times $t_i = i\Delta t$.
Define

$$E_{el}(D_n(t_i), f(t_i)) := \min_{v\in H_0^1(\Omega)} \left[\frac{1}{2}\int_\Omega \sigma_{D_n(t_i)}|\nabla v|^2 dx - \langle f(t_i), v\rangle\right]$$

and set $\Delta E_n(t_i) := E_n \cap \Delta D_n(t_i)$ and $\Delta D_n(t_i) := D_n(t_i) \setminus D_n(t_{i-1})$. We then have, from Lemma 4.5 and from (4.15), that

$$E_{el}(D_n(t_i), f(t_i)) + k_\delta|\Delta E_n(t_i)| + o(1) \geq E_{el}(D_n(t_i) \setminus \Delta E_n(t_i), f(t_i))\,,$$

with $o(1) \to 0$ as $n \to \infty$. We take the limit as $n \to \infty$ (and take subsequences as necessary) so that

$$\chi_{\Delta E_n(t_i)} \stackrel{*}{\rightharpoonup} \Theta_E(t_i)$$

and we get

$$E_{el}(A(t_i), f(t_i)) + k_\delta\left(\int_\Omega \Theta_E(t_i)\,dx\right) \geq E_{el}(A_{\Delta t}(t_i), f(t_i))\,,$$

where

$$E_{el}(A, f) := \min_{v\in H_0^1(\Omega)} \left[\frac{1}{2}\int_\Omega A\nabla v\nabla v\,dx - \langle f, v\rangle\right] \tag{4.17}$$

and $A_{\Delta t}(t_i)$ is the G-limit of $\sigma_{D_n(t_i)\setminus\Delta E_n(t_i)}I$. Then we use the minimality property in Definition 3.2, noting that since $D_n(t_i) \setminus \Delta E_n(t_i) \supseteq D_n(t_{i-1})$, the pair $(A_{\Delta t}(t_i), \Theta(t_i) - \Theta_E(t_i))$ is a competitor for $(A(t_{i-1}), \Theta(t_{i-1}))$, and from (4.17) we get

$$\begin{aligned} &E_{el}(A(t_i), f(t_i)) + k\left(\int_\Omega \Theta(t_i)\,dx - \int_\Omega \Theta_E(t_i)\,dx\right) + k_\delta\left(\int_\Omega \Theta_E(t_i)\,dx\right) \\ &\quad\geq E_{el}(A(t_{i-1}), f(t_{i-1})) - \langle f(t_i) - f(t_{i-1}), u^{\Delta t}(t_i)\rangle + k\int_\Omega \Theta(t_{i-1})\,dx\,, \end{aligned}$$

where $u^{\Delta t}(t_i)$ is the minimizer for $E_{el}(A_{\Delta t}(t_i), f(t_i))$. Since

$$\sum_i \int_\Omega \chi_{\Delta E_n(t_i)} \text{ converges to both } \sum_i \int_\Omega \Theta_E(t_i)\,dx \text{ and } \gamma_{\Delta t},$$

where $\gamma_{\Delta t} := \lim_{n\to\infty} |E_n \cap D_n(t_{\Delta t})|$ (again dropping to a subsequence as necessary) and $t_{\Delta t} = \left[\frac{T}{\Delta t}\right]\Delta t$ (i.e. it is the last element in the partition of $[0, T]$ such that $0 \leq T - t_{\Delta t} \leq \Delta t$), summing the last inequality over i gives

$$\begin{aligned} &E_{el}(A(t_{\Delta t}), f(t_{\Delta t})) + k\left(\int_\Omega \Theta(t_{\Delta t})\,dx - \gamma_{\Delta t}\right) + k_\delta\gamma_{\Delta t} \\ &\geq E_{el}(A(0), f(0)) + k\int_\Omega \Theta(0)\,dx - \sum_i \langle f(t_i) - f(t_{i-1}), u^{\Delta t}(t_i)\rangle\,. \end{aligned}$$

Since $A(t_i) \leq A_{\Delta t}(t_i) \leq A(t_{i-1})$ and $A(t)$, being monotonic, is continuous at a.e. t, and since $f \in W^{1,1}([0,T], H^{-1}(\Omega))$, by the continuous dependence of u on A, we get

$$u^{\Delta t}(\tau) \xrightarrow{L^2} u(\tau) \qquad \text{for a.e. } \tau\,.$$

Moreover

$$\frac{f(\tau+\Delta t)-f(\tau)}{\Delta t} \xrightarrow{H^{-1}} \dot f(\tau) \qquad \text{for a.e. } \tau$$

and hence

$$\sum_i \langle f(t_i^n) - f(t_{i-1}^n), u^{\Delta t}(t_i^n)\rangle \longrightarrow \int_0^T \langle \dot f(\tau), u(\tau)\rangle\, d\tau\,.$$

As a consequence of the energy balance we get that the total energy $\mathcal{E}$ is continuous and then $\mathcal{E}(t_{\Delta t}) \to \mathcal{E}(T)$ as $\Delta t \to 0$, which, together with the fact that $\gamma_{\Delta t} \to \gamma$ as $\Delta t \to 0$, gives (4.16).

Finally, we consider the case that $\int_\Omega \Theta(t)\,dx$ is not continuous from below at T. We assume by contradiction that there exist sequences $t_n \nearrow T$ and $E_n \subseteq D_n(T) \setminus D_n(t_n)$ with $\liminf_{n\to\infty} |E_n| = \gamma > 0$, together with $\delta > 0$, such that the minimizer v_n for $E_{el}(D_n(T)\setminus E_n, f(T))$, i.e., the solution in $H_0^1(\Omega)$ of

$$-\mathrm{div}(\sigma_{D_n(T)\setminus E_n}\nabla v_n) = f(T),$$

satisfies

$$\liminf_{n\to\infty} |\{x \in E_n :\ |\nabla v_n(x)| > \lambda - \delta\}| = 0.$$

From Lemma 4.5 we see that, as above,

$$\begin{aligned} &E_{el}(D_n(T),T) + k|D_n(T)\setminus E_n| + k_\delta|E_n| + \langle f(T)-f(t_n), u_n(T)\rangle + O(1) \\ &\geq E_{el}(D_n(t_n),t_n) + k|D_n(t_n)|. \end{aligned} \tag{4.18}$$

Taking the limit as $n \to \infty$, we get

$$\mathcal{E}(T) - (k-k_\delta)\gamma \geq \mathcal{E}(T),$$

a contradiction. □

5. Implications for Local Minimality and Stability

We conclude the paper discussing the notions of local minimality and stability for the energetic model consider in Section 3. We will see that our threshold point of view can also be used to show that there are no local minimizers or stable configurations besides global minimizers.

The most natural norm for defining "small" increments of damage is the L^1 norm on characteristic functions, since this is the same as the measure of the increment. Of course, the L^∞ norm would not allow any "small" increments at all. So, we will consider the L^1 norm in defining local minimality and stability of damage.

Given $A \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$ and $\Theta \in L^\infty(\Omega,[0,1])$, with $A(x) \in \overline{G}_{\Theta(x)}(A_w, A_s)$ a.e. $x \in \Omega$, and a sequence of sets D_n such that

$$\begin{cases} \chi_{D_n} \overset{*}{\rightharpoonup} \Theta \\ \sigma_{D_n} \xrightarrow{G} A\,, \end{cases}$$

we give the following definition.

Definition 5.1. *We say that* $(\{D_n\}, A, \Theta)$ *is a local minimizer corresponding to* $f \in H^{-1}(\Omega)$ *if there exists* $\varepsilon > 0$ *such that for every* $A' \in \hat{\mathcal{G}}_{\Theta'}(\{D_n\})$ *with* $\Theta' \in B(\Theta, \varepsilon)$ *(the ball in* L^1 *centered at* Θ *with radius* ε*), we have*

$$E_{el}(A, \Theta, f) + k \int_\Omega \Theta \, dx \leq E_{el}(A', \Theta', f) + k \int_\Omega \Theta' \, dx \, .$$

Furthermore, given $f \in H^{-1}(\Omega)$*, we say that* $(\{D_n\}, A, \Theta)$ *is stable if*

$$\limsup_{\varepsilon \to 0} \sup_{\substack{\Theta' \in B(\Theta,\varepsilon) \\ A' \in \hat{\mathcal{G}}_{\Theta'}(\{D_n\})}} \frac{E_{el}(A, \Theta, f) + k \int_\Omega \Theta \, dx - E_{el}(A', \Theta', f) - k \int_\Omega \Theta' \, dx}{\varepsilon} \leq 0.$$

We now show that all local minimizers, and even all stable damage "sets," must be global minimizers.

Theorem 5.2. *If* $(\{D_n\}, A, \Theta)$ *is a local minimizer corresponding to some* f*, then* (A, Θ) *is a global minimizer. Furthermore, even if the triple is only stable then* (A, Θ) *is a global minimizer.*

Proof. We suppose (A, Θ) is not an (EMP) global minimizer, and show that $(\{D_n\}, A, \Theta)$ is not a local minimizer. Since it is not a global minimizer, there exists $\Theta' \in L^\infty(\Omega, [0, 1])$, with $\Theta' \geq \Theta$, and $A' \in \hat{\mathcal{G}}_{\Theta'}(\{D_n\})$ such that

$$E_{el}(A, f) + k \int_\Omega \Theta \, dx > E_{el}(A', f) + k \int_\Omega \Theta' \, dx \, . \tag{5.19}$$

We first show that $\Theta' > \Theta$. From the definition of $\hat{\mathcal{G}}(\{D_n\})$ there exists a sequence of sets $D'_n \supseteq D_n$ such that $\chi_{D'_n} \stackrel{*}{\rightharpoonup} \Theta'$ and $\sigma_{D'_n} \stackrel{G}{\longrightarrow} A'$.

Take u_n to be the minimizer of $E_{tot}(\cdot, D_n, f)$, and similarly u'_n the minimizer of $E_{tot}(\cdot, D'_n, f)$. Then from (5.19) and the fact that $\Theta' \geq \Theta$ we have

$$\lim_{n \to \infty} \left[\frac{1}{2} \int_\Omega \sigma_{D_n} |\nabla u_n|^2 - \int_\Omega f u_n - \frac{1}{2} \int_\Omega \sigma_{D'_n} |\nabla u'_n|^2 + \int_\Omega f u'_n \right] > 0,$$

while from the minimality of u_n we have

$$\lim_{n \to \infty} \left[\frac{1}{2} \int_\Omega \sigma_{D_n} |\nabla u_n|^2 - \int_\Omega f u_n - \frac{1}{2} \int_\Omega \sigma_{D_n} |\nabla u'_n|^2 + \int_\Omega f u'_n \right] \leq 0 \, .$$

But if $\lim_{n \to \infty} |D'_n \setminus D_n| = 0$, since $\{|\nabla u'_n|^2\}$ is equi-integrable (see, e.g., [14]), then

$$0 < \lim_{n \to \infty} \left[\int_\Omega \sigma_{D_n} |\nabla u'_n|^2 - \int_\Omega \sigma_{D'_n} |\nabla u'_n|^2 \right] = \lim_{n \to \infty} (\beta - \alpha) \int_{D'_n \setminus D_n} |\nabla u'_n|^2 = 0 \, .$$

Hence, there is a contradiction unless $\Theta' > \Theta$.

As a consequence of (5.19) we also get that $|\nabla u_n|$ exceeds λ somewhere in $D'_n \setminus D_n$. More precisely we have that for some $\delta > 0$, the sets $G_n^\delta := \{x \in D'_n \setminus D_n : |\nabla u_n(x)| > \lambda + \delta\}$, satisfy

$$\limsup_{n \to \infty} |G_n^\delta| > 0.$$

In fact if for every $\delta > 0$ we had

$$\limsup_{n \to \infty} |G_n^\delta| = 0 \, , \tag{5.20}$$

using (4.10) we have

$$E_{el}(D_n, f) - E_{el}(D'_n, f) \le k_\delta |D'_n \setminus D_n| + \frac{(\beta - \alpha)\beta}{2\alpha} \|\nabla u_n\|^2_{L^2(G_n^\delta)} .$$

This, taking the limit as $n \to \infty$, using (5.20), the equintegrability of ∇u_n and the arbitrariness of δ, contradicts (5.19).

So, dropping to a subsequence, we have $\chi_{G_n^\delta} \overset{*}{\rightharpoonup} \theta > 0$ in $L^\infty(\Omega)$. We choose a point $x_0 \in \Omega$ that is a Lebesgue point for θ and $\theta(x_0) > 0$. Now, given any $\varepsilon > 0$, we can choose a ball B, with $|B| \le \varepsilon$, such that $\int_B \theta \, dx > \frac{|B|}{2}\theta(x_0)$, i.e., $\lim_{n\to\infty} |G_n^\delta \cap B| > \frac{|B|}{2}\theta(x_0) > 0$. Then, just as in the proof of condition (2) in Theorem 4.4, by adding laminates of damage within $G_n^\delta \cap B$, the energy of D_n can be lowered in the limit by at least $\frac{1}{2}\beta\delta^2\frac{|B|}{2}\theta(x_0)$. The triple $(\{\tilde{D}_n\}, \tilde{A}, \tilde{\Theta})$ generated by the union of D_n and these laminates then has lower energy than $(\{D_n\}, A, \Theta)$ and $\tilde{\Theta} \in B(\Theta, \varepsilon)$, and so $(\{D_n\}, A, \Theta)$ is not a local minimizer.

In fact, this also shows that

$$\limsup_{\varepsilon \to 0} \sup_{\substack{\Theta' \in B(\Theta,\varepsilon) \\ A' \in \hat{\mathcal{G}}_{\Theta'}(\{D_n\})}} \frac{E_{el}(A, \Theta, f) + k\int_\Omega \Theta \, dx - E_{el}(A', \Theta', f) - k\int_\Omega \Theta' \, dx}{\varepsilon} \ge \frac{1}{2}\beta\delta^2,$$

and so $(\{D_n\}, A, \Theta)$ is not stable. □

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