Free discontinuity problems

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1. Introduction

The notation "free-discontinuity problems" indicates those problems in the Calculus of Variations where the unknown is a pair (u, K), with K a closed set and u a (sufficiently) smooth function on $\Omega \setminus K$ (Ω a fixed open set). The two main examples of problems of this type are the Mumford-Shah functional in computer vision (see [19], [17], [11], [2]), and models in fracture mechanics for brittle hyperelastic media (see [16], [15], [20], [10]). We will focus on this second example, the first one leading to a similar variational formulation.

If we consider a hyperelastic medium subject to brittle fracture, following Griffith's theory, it can be modeled by the introduction, besides the elastic volume energy, of a surface term which accounts for crack initiation. In its simplest formulation, the energy of a deformation u will be of the form

$$E(u, K) = \int_{\Omega \setminus K} f(\nabla u)\, dx + \lambda \mathcal{H}^{n-1}(K), \tag{1.1}$$

where ∇u is the deformation gradient, Ω the reference configuration, K is the crack surface, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional (Hausdorff) measure ($n = 2, 3$ in bi- and three-dimensional elasticity problems, respectively). The bulk energy density f accounts for elastic deformations outside the crack, while λ is a constant given by Griffith's criterion for fracture initiation. The existence of equilibria, under appropriate boundary conditions, can be deduced from the study of minimum pairs (u, K) for the energy (1.1). Note that if $E(u, K) < +\infty$ then the Lebesgue measure of K is zero, u can be regarded as a measurable function defined on all Ω, and the set K can be thought of as (a set containing) the set of discontinuity points for u. Note moreover that in general K will not be the boundary of a set (in this special case we talk of free *boundary* problems).

The presence of two unknowns, the surface K and the deformation u, can be overcome by a weak formulation of the problem in spaces of discontinuous functions.

The space of *special functions of bounded variation* $SBV(\Omega;\mathbb{R}^m)$ has been introduced by De Giorgi and Ambrosio in [12] as the subset of $\mathbb{R}^m$-valued functions of bounded variation on the open set $\Omega \subset \mathbb{R}^n$, whose measure first derivative can be written in the form

$$Du = \nabla u\, \mathcal{L}^n \llcorner \Omega + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S(u),$$

where

- ∇u is now the *approximate gradient* of u,
- $S(u)$ is the complement of the set of Lebesgue points of u (*jump set* of u),
- ν_u is the unit *normal* to $S(u)$,
- u^+, u^- are the *approximate trace values* of u on both sides of $S(u)$,
- the measures $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ are the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure, respectively.

The energy functional in (1.1) can be rewritten as

$$\mathcal{E}(u) = \int_\Omega f(\nabla u)\, dx + \lambda \mathcal{H}^{n-1}(S(u)),$$

which makes sense on $SBV(\Omega;\mathbb{R}^m)$. If f is quasiconvex or polyconvex (see [9]) and satisfies some standard growth conditions, then we can apply the direct methods of the Calculus of Variations to obtain minimum points for problems involving $\mathcal{E}$, using Ambrosio's lower semicontinuity and compactness theorems (see [1], [3], [4]). A complete regularity theory for minimum points u for $\mathcal{E}$ has not been developed yet, but in some cases it is possible to prove that the jump set $S(u)$ is $\mathcal{H}^{n-1}$-equivalent to its closure ([13]) or even more regular (see [8], [7]), and that u is smooth on $\Omega \setminus \overline{S(u)}$, thus obtaining minimizing pairs $(u, K) = (u, \overline{S(u)})$ for the functional E.

The viewpoint described above privileges the reference configuration, neglecting the effects of crack deformation. Our aim is to define a sub-class of SBV functions which allow the statement (and solution) of problems taking into account also the deformation of $S(u)$, *i.e.*, the shape of the crack surface in the deformed configuration.

As an example we can think of an elastic body in two dimensions subject to fracture, so that a "hole" is formed bounded by two curves Γ^+ and Γ^- which are the images of $S(u)$ by u^+ and u^-, respectively. If the traces are sufficiently smooth then the length of (the boundary of the hole) $\Gamma^+ \cup \Gamma^-$ is given by

$$E_1(u) = \int_{S(u)} \Big(\big|\frac{\partial u^+}{\partial \tau}\big| + \big|\frac{\partial u^-}{\partial \tau}\big| \Big)\, d\mathcal{H}^1,$$

where τ is the tangent to $S(u)$. Similarly, if u is bounded and we have an "opening hole" (that is, $\Gamma^+ \cup \Gamma^-$ is compactly contained in $u(\Omega)$) we can also consider the "area of the hole", given by

$$E_2(u) = \int_{\text{hole}} dy_1 dy_2 = -\int_{\Gamma^+\cup\Gamma^-} y_1 dy_2 = -\int_{S(u)} \Big(u_1^+ \frac{\partial u_2^+}{\partial \tau} - u_1^- \frac{\partial u_2^-}{\partial \tau} \Big) \, d\mathcal{H}^1,$$

which again makes sense if the tangential derivatives of $u^\pm$ exist. An analogous formulation for three dimensional elasticity is possible, taking into account the orientation of the surface $\Gamma^+ \cup \Gamma^-$.

It is clear that the crucial point in order to extend the definition of functional as E_1 and E_2 to a class wide-enough to apply the direct methods of the Calculus of Variations will be a weak definition of the tangential derivatives of u^+ and u^- on $S(u)$. Simple examples show that it is not possible to gain regularity of the traces by imposing higher integrability of the bulk gradient ∇u; hence we will require the definition of a new functional space. At first, we limit our analysis to the scalar case $m = 1$.

The starting point is a characterization of the space SBV due to Ambrosio [5]: a function u belongs to $SBV(\Omega)$ and $\mathcal{H}^{n-1}(S(u)) < +\infty$ if and only if there exist a function $a = (a_1, \ldots, a_n)$ and measures μ_i on $\Omega \times \mathbb{R}$ ($i = 1, \ldots, n$) such that

$$\int_\Omega \left(\frac{\partial \varphi}{\partial x_i}(x, u(x)) + \frac{\partial \varphi}{\partial y}(x, u(x)) a_i(x) \right) dx = \int_{\Omega\times\mathbb{R}} \varphi(x, y) \, d\mu_i \tag{1.2}$$

for all $\varphi \in C^1(\Omega \times \mathbb{R})$ with compact support. In this case $a = \nabla u$. This characterization is a consequence of the chain rule formula for function in BV.

We can interpret the formula (1.2) above as a property of the graph of u, which is given for BV functions by

$$\Gamma = \{(x, u(x)) : \ x \in \Omega, \ \exists \nabla u(x)\},$$

and is oriented by the unit co-vector

$$\eta(x, u(x)) = \frac{1}{\sqrt{1 + |\nabla u|^2}} (e_1, \frac{\partial u}{\partial x_1}) \wedge \ldots \wedge (e_n, \frac{\partial u}{\partial x_n}) \, ,$$

where $\{e_1, \ldots, e_m\}$ is the standard orthonormal basis of $\mathbb{R}^n$ (see [14]). We can define the linear functional on n-forms (n-current) "integration on the graph", by

$$T_u(\omega) = \int_\Gamma \langle \omega, \eta \rangle d\mathcal{H}^n \, ,$$

and the *boundary of* T_u as the $(n-1)$-current given by

$$\partial T_u(\omega) = T_u(d\omega).$$

We can re-read formula (1.2) as a property of ∂T_u. In fact, using the area formula, we have

$$\int_\Omega \Big(\frac{\partial \varphi}{\partial x_i}(x, u(x)) + \frac{\partial \varphi}{\partial y}(x, u(x)) \frac{\partial u}{\partial x_i} \Big) \, dx = \partial T_u(\varphi d\widehat{x}_i)$$

where

$$d\widehat{x}_i = (-1)^{i+1} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n,$$

so that (1.2) states precisely that the boundary of T_u is a measure when computed on "horizontal forms" (*i.e.*, forms with no dy). An imprecise interpretation is that the boundary of the graph admits an integral projection on the basis Ω, given precisely by $S(u)$.

2. The class $\mathbf{SBV_0}$

Intuitively, tangential derivatives of $u^\pm$ on $S(u)$ provide information about the "vertical part of the boundary of the graph of u". Following this intuition we can define a sub-class of SBV functions with $\mathcal{H}^{n-1}(S(u)) < +\infty$, called SBV_0, simply requiring that ∂T_u be a measure also when computed on $(n-1)$-forms *with* a vertical part. This is equivalent to asking that in addition to the integration by parts formulas stated above, there exist measures μ_α (α multi-index of order $n-2$) such that

$$\begin{aligned} \int_{\Omega \times \mathrm{I\!R}} \varphi(x)\psi(y) \, d\mu_\alpha &= \partial T_u(\phi(x)\psi(y) dx_\alpha \wedge dy) \\ &= \int_{S(u)} \Big(\int_{u^-(x)}^{u^+(x)} \psi(y) \, dy \Big) \Big(\frac{\partial \phi}{\partial x_{i_1}} \nu_{i_2} - \frac{\partial \phi}{\partial x_{i_2}} \nu_{i_1} \Big) d\mathcal{H}^{n-1} \end{aligned}$$

for any $\phi \in C_0^1(\Omega)$, $\psi \in C_b^1(\mathrm{I\!R})$, where i_1, i_2 are indices such that

$$dx_{i_1} \wedge dx_{i_2} \wedge dx_\alpha = dx_1 \wedge \ldots \wedge dx_n.$$

The simplest case ($n = 2$) gives $\alpha = 0$, and

$$\int_{\Omega \times \mathrm{I\!R}} \varphi(x)\psi(y) \, d\mu_\alpha = \int_{S(u)} \Big(\int_{u^-(x)}^{u^+(x)} \psi(y) \, dy \Big) \frac{\partial \phi}{\partial \tau} \, d\mathcal{H}^1,$$

(τ the tangent to $S(u)$), which is somehow the "weak version" of

$$\int_{\Omega \times \mathrm{I\!R}} \varphi(x)\psi(y) \, d\mu_\alpha = -\int_{S(u)} \Big(\psi(u^+) \frac{\partial u^+}{\partial \tau} - \psi(u^-) \frac{\partial u^-}{\partial \tau} \Big) \phi(x) \, d\mathcal{H}^1 \tag{2.1}$$

(this formula is correct if $S(u)$ and $u_{|\Omega\setminus S(u)}$ are smooth enough). Roughly speaking, this is equivalent to requiring that the traces $u^{\pm}$ be functions of bounded variation on $S(u)$ (this is not precisely so since $S(u)$ may present a very complex structure). Moreover if $u \in SBV_0(\Omega)$ then it can be proved that the *approximate differentials* $\nabla u^{\pm}$ exist $\mathcal{H}^{n-1}$-a.e. on $S(u)$, and

$$\int_{S(u)} |\nabla u^{\pm}|\, d\mathcal{H}^{n-1} < +\infty.$$

We denote by $\partial_v T_u$ the vector of the measures μ_α; *i.e.*, the components of ∂T_u corresponding to differential forms $\varphi dx_\alpha \wedge dy$. The letter v refers to the fact that we have in mind "vertical components". Note that

$$\mathcal{E}_1(u) = \|\partial_v T_u\|$$

is a (lower semicontinuous) extension of the "length functional" E_1.

The class $SBV_0(\Omega)$ has the following compactness property (see [6]).

Theorem 2.1 *Let (u_h) be a sequence in $SBV_0(\Omega) \cap L^\infty(\Omega)$, let $p > 1$ and assume that*

$$\sup_{h\in\mathbb{N}} \Big\{ \int_\Omega |\nabla u_h|^p\, dx + \mathcal{H}^{n-1}\big(S(u_h)\big) + \|u_h\|_\infty \Big\} < +\infty.$$

and that the sequence $\|\partial_v T_{u_h}\|(\Omega \times \mathbb{R})$ be bounded; then there exists a subsequence $(u_{h(k)})$ converging in $L^1_{\rm loc}(\Omega)$ to $u \in SBV_0(\Omega)$ such that $\partial T_{u_{h(k)}}$ weakly converges to ∂T_u as measures on $\Omega \times \mathbb{R}$.

3. $\mathbf{SBV_0}$-functions with Sobolev traces

As a subclass of $SBV_0(\Omega)$ (that is, "SBV-functions with BV-traces on $S(u)$") we can consider the family of "SBV-functions with Sobolev traces on $S(u)$", that is, those SBV_0 functions such that

$$\int_{S(u)} |\nabla u^{\pm}|^p\, dx < +\infty$$

for some $p \geq 1$, and such that the measure $\partial_v T_u$ is determined by the analogue of (2.1) in dimension n with $\nabla u^{\pm}$ in place of $\partial u^{\pm}/\partial\tau$. Unfortunately, this subclass is not compact: it is possible to give an example such that all hypotheses of the compactness theorem are satisfied and in addition $\nabla u^{\pm}_h$ are equi-bounded; nevertheless the limit u does not possess Sobolev traces on $S(u)$. This phenomenon is due to the fact that

$S(u_h)$ may converge only in a weak sense to $S(u)$, while it does not occur if we have strong convergence; *i.e.*, $\mathcal{H}^{n-1}(S(u_h)) \to \mathcal{H}^{n-1}(S(u))$.

4.Vector-valued SBV$_0$-functions

In the vector-valued case the definition of SBV_0 is the same as in the scalar case, requiring that ∂T_u be a vector measure. Notice however that now we must take into account all differential forms

$$\varphi\, dx_\alpha \wedge dy_\beta,$$

where α and β are multi-indices with $|\alpha|+|\beta| = n-1$. This means that we will have to take into account also non-linear quantities involving minors of the matrix ∇u.

As an example we illustrate the case $n = m = 2$. In this case, the orientation η of the graph Γ of u is given by

$$\begin{aligned}\eta(x,y) =& \frac{1}{\sqrt{1+|\nabla u|^2+|\det \nabla u|^2}}\Big(e_1\wedge e_2 - \frac{\partial u_1}{\partial x_1}\, e_2\wedge\varepsilon_1 - \frac{\partial u_2}{\partial x_1}\, e_2\wedge\varepsilon_2 \\ &+ \frac{\partial u_1}{\partial x_2}\, e_1\wedge\varepsilon_1 + \frac{\partial u_2}{\partial x_2}\, e_1\wedge\varepsilon_2 + \det\nabla u\, \varepsilon_1\wedge\varepsilon_2\Big),\end{aligned}$$

where (e_1, e_2) and $(\varepsilon_1, \varepsilon_2)$ denote the canonical orthonormal bases on Ω and on the target space, respectively, and $\mathcal{H}^2(\Gamma) < +\infty$ if and only if $\nabla u \in L^1$ and $\det\nabla u \in L^1$. The integration of the "vertical components" of the current ∂T_u can be expressed then by

$$\begin{aligned}\partial T_u(\varphi dy_1) &= \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial\varphi}{\partial y_2}\det\nabla u\Big)\, dx \\ \partial T_u(\varphi dy_2) &= \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial\varphi}{\partial y_1}\det\nabla u\Big)\, dx.\end{aligned}$$

We have that $u \in SBV_0$ if and only if there exist two bounded measures μ_1 and μ_2 on $\Omega\times\mathrm{I\!R}^2$, such that

$$\int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1}\Big)\, dx = \int_\Omega \frac{\partial\varphi}{\partial y_2}\det\nabla u\, dx + \int_{\Omega\times\mathrm{I\!R}^2} \varphi d\mu_1,$$

$$\int_\Omega \Big(\frac{\partial\varphi}{\partial x_2}\frac{\partial u_2}{\partial x_1} - \frac{\partial\varphi}{\partial x_1}\frac{\partial u_2}{\partial x_2}\Big)\, dx = \int_\Omega \frac{\partial\varphi}{\partial y_1}\det\nabla u\, dx + \int_{\Omega\times\mathrm{I\!R}^2} \varphi d\mu_2.$$

for all $\varphi \in C^1_0(\Omega\times\mathrm{I\!R}^2)$. In particular, if u is bounded, we get (choosing $\varphi(x,y) = y_2\phi(x)$ on the range of u)

$$\int_\Omega u_2\Big(\frac{\partial\phi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\phi}{\partial x_2}\frac{\partial u_1}{\partial x_1}\Big)\, dx = \int_\Omega \phi\,\det\nabla u\, dx + \int_{\Omega\times\mathrm{I\!R}^2} \phi y_2 d\mu_1,$$

for all $\phi \in C^1_0(\Omega)$, which can be summarized in the equality, which links the distributional and the pointwise determinant,

$$\mathrm{Det}\nabla u = \mathrm{det}\nabla u\,\mathcal{L}^2 + \pi_{\#}(y_2\,\mu_1).$$

Note that the equality $\mathrm{Det}\nabla u = \mathrm{det}\nabla u\,\mathcal{L}^n + \lambda$ may hold with non-trivial λ also when u is a Sobolev function. In the case $u : \{x \in \mathrm{I\!R}^2 :\ |x| < 1\} \to \mathrm{I\!R}^2$ given by $u(x) = x/|x|$, for example, $\mathrm{det}\nabla u = 0$, but

$$\mathrm{Det}\nabla u = \pi\delta_0.$$

Some examples by Müller [18] show that λ may also be a Hausdorff measure of fractional dimension restricted to a fractal set.

If $S(u)$, the restriction of u to $\Omega \setminus S(u)$, and its traces on $S(u)$ are smooth enough to justify the application of the Gauss-Green formula, then the measures μ_i are easily characterized. In fact, we get

$$\begin{aligned}
0 &= \int_\Omega \varphi(x,u)\,\mathrm{div}\Big(\frac{\partial u_1}{\partial x_2}, -\frac{\partial u_1}{\partial x_1}\Big)\,dx \\
&= \int_{S(u)} \Big(\varphi(x,u^+)\Big(\frac{\partial u_1^+}{\partial x_1}\nu_2 - \frac{\partial u_1^+}{\partial x_2}\nu_1\Big) - \varphi(x,u^-)\Big(\frac{\partial u_1^-}{\partial x_1}\nu_2 - \frac{\partial u_1^-}{\partial x_2}\nu_1\Big)\Big)\,d\mathcal{H}^1 \\
&\quad - \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial\varphi}{\partial y_2}\mathrm{det}\nabla u\Big)\,dx, \\
&= -\int_{S(u)} \Big(\varphi(x,u^+)\frac{\partial u_1^+}{\partial\tau} - \varphi(x,u^-)\frac{\partial u_1^-}{\partial\tau}\Big)\,d\mathcal{H}^1 \\
&\quad - \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial\varphi}{\partial y_2}\mathrm{det}\nabla u\Big)\,dx,
\end{aligned}$$

so that

$$\int_{\Omega\times\mathrm{I\!R}^2} \varphi d\mu_1 = -\int_{S(u)} \Big(\varphi(x,u^+)\frac{\partial u_1^+}{\partial\tau} - \varphi(x,u^-)\frac{\partial u_1^-}{\partial\tau}\Big)\,d\mathcal{H}^1\,.$$

In the same way we obtain

$$\int_{\Omega\times\mathrm{I\!R}^2} \varphi d\mu_2 = -\int_{S(u)} \Big(\varphi(x,u^+)\frac{\partial u_2^+}{\partial\tau} - \varphi(x,u^-)\frac{\partial u_2^-}{\partial\tau}\Big)\,d\mathcal{H}^1\,.$$

In particular, the total variation of μ represents the length of the images of $S(u)$ by u^+ and u^-:

$$|\mu|(\Omega\times\mathrm{I\!R}^2) = \int_{S(u)} \Big(|\frac{\partial u^+}{\partial\tau}| + |\frac{\partial u^-}{\partial\tau}|\Big)\,d\mathcal{H}^1.$$

In the physical case $n = m = 3$ the integration by parts formulas characterize the distributional and Jacobian determinants of ∇u and its (2-dimensional) adjoint matrices.

As in the scalar case, we denote by $\partial_v T_u$ the vector of the measures $\mu_{\alpha\beta}$ related to integration of "non-horizontal" forms $\varphi dx_\alpha \wedge dy_\beta$, with $|\alpha| + |\beta| = n - 1$ and $|\alpha| < n - 1$. We have then the following compactness result.

Theorem 4.1 *Let (u_h) be a sequence in SBV_0 such that*

$$\sup_{h\in\mathbb{N}} \Big(\|u_h\|_\infty + \mathcal{H}^1(S(u_h)) + \int_\Omega |\nabla u|^q\, dx + \|\partial_v T_{u_h}\| \Big)$$

is finite, where $q \geq \min\{n, m\}$, and let

$$\det \frac{\partial (u_h)_\beta}{\partial x_\gamma}$$

be a equi-integrable sequence for every pair of multi-indices β, γ of order $\min\{n, m\}$ (in the case $n = m$ it means that $(\det \nabla u_h)$ is equi-integrable). Then, there exists a subsequence $(u_{h(k)})$ converging in $L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ to $u \in SBV_0$, such that

$$\nabla u_{h(k)} \to \nabla u \text{ weakly in } L^q(\Omega, \mathbb{R}^{nm}),$$

$$\det \frac{\partial (u_{h(k)})_\beta}{\partial x_\gamma} \to \det \frac{\partial u_\beta}{\partial x_\gamma} \text{ weakly in } L^1(\Omega)$$

for every pair of multi-indices β, γ of equal order not greater than $\min\{n, m\}$, and $\partial T_{u_{h(k)}}$ converges weakly to ∂T_u. In particular $\partial_v T_{u_{h(k)}}$ converges weakly to $\partial_v T_u$ in the sense of measures.

5. An existence result

As an application of the compactness results stated above we can give an existence result for weak minima with smooth traces on $S(u)$ for the Mumford and Shah functional of computer vision (see [19]). The strong formulation for such a problem takes into account the functional

$$F(u, K) = \int_{\Omega\setminus K} |\nabla u|^2\, dx + \lambda \int_K \Big(\sqrt{1 + |\frac{\partial u^+}{\partial \tau}|^2} + \sqrt{1 + |\frac{\partial u^-}{\partial \tau}|^2} \Big) d\mathcal{H}^1\,,$$

where $\Omega \subset \mathbb{R}^2$, K is a piecewise C^1 closed subset of Ω, and $u \in C^1(\Omega \setminus K)$ has tangential derivatives $\mathcal{H}^1$-a.e. on K. The last line integral is simply the length of the graphs of $u^\pm$ in $\mathbb{R}^3$. The weak formulation of F is given by

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^2\, dx + \lambda \|\partial T_u\|, \qquad u \in SBV_0(\Omega).$$

Example 5.1. Let $g \in L^\infty(\Omega)$. Then there exists a solution to the minimum problem

$$\min\Big\{\mathcal{F}(u) + \int_\Omega |u-g|^2\,dx :\ u \in SBV_0(\Omega)\Big\}.$$

In fact, it suffices to notice that there is no restriction in supposing that $\|u\|_\infty \leq \|g\|_\infty$, so that we can find a minimizing sequence (u_h) satisfying the hypotheses of Theorem 2.1. We obtain then, possibly passing to a subsequence, a minimizing sequence, that we still call (u_h), converging to a function $u \in SBV_0(\Omega)$ strongly in $L^2(\Omega)$, such that $\nabla u_h \to \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^n)$, and $\partial T_{u_h} \to \partial T_u$ weakly as measures on $\Omega \times \mathbb{R}$. In particular we have

$$\int_\Omega |\nabla u|^2\,dx \leq \liminf_h \int_\Omega |\nabla u_h|^2\,dx, \quad \|\partial T_u\| \leq \liminf_h \|\partial T_{u_h}\|,$$

so that

$$\mathcal{F}(u) + \int_\Omega |u-g|^2\,dx \leq \lim_h \Big(\mathcal{F}(u_h) + \int_\Omega |u_h-g|^2\,dx\Big),$$

and u is a minimum point as required.

References

[1] L. Ambrosio, A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857-881.

[2] L. Ambrosio, Variational problems in SBV and image segmentation, *Acta Appl. Math.* **17** (1989), 1-40.

[3] L. Ambrosio, Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291-322.

[4] L. Ambrosio, On lower semicontinuity of quasiconvex integrals in $SBV(\Omega;\mathbb{R}^k)$, *Nonlinear Anal.* **23** (1994), 405-425.

[5] L. Ambrosio, A new proof of the SBV compactness theorem, *Calc. Var.* **3** (1995), 127-137.

[6] L. Ambrosio, A. Braides and A. Garroni, Special functions with bounded variation and with weakly differentiable traces on the jump set, (to appear).

[7] L. Ambrosio, N. Fusco and D. Pallara, Partial regularity of free discontinuity sets II, Preprint, Pisa, 1994.

[8] L. Ambrosio and D. Pallara, Partial regularity of free discontinuity sets I, Preprint, Pisa, 1994.

[9] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337-403.

[10] A. Braides and A. Coscia, A singular perturbation approach to problems in fracture mechanics, *Math. Mod. Meth. Appl. Sci.* **3** (1993), 302-340.

[11] G. Dal Maso, J. M. Morel and S. Solimini, A variational method in image segmentation: existence and approximation results, *Acta Math.* **168** (1992), 89-151.

[12] E. De Giorgi and L. Ambrosio, Un nuovo funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199-210.

[13] E. De Giorgi, G. Carriero, and A. Leaci, Existence theorem for a minimum problem with free discontinuity set, *Arch. Rational Mech. Anal.* **108** (1989), 195-218.

[14] M. Giaquinta, G. Modica and J. Soucek, Area and the area formula, *Rend. Semin. Mat Fis. Milano* **62** (1992), 53-87.

[15] A. A. Griffith, The phenomenon of rupture and flow in solids, *Phil. Trans. Royal Soc. London A* **221** (1920), 163-198.

[16] H. Liebowitz, *Fracture: an Advanced Treatise*, Academic Press, 1969.

[17] J. M. Morel and S. Solimini, *Variational Models in Image Segmentation*, Birkhäuser, Boston, 1995.

[18] S. Müller: On the singular support of the distributional determinant, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (1993), 657-696.

[19] D. Mumford and J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989), 577-685.

[20] I.N. Sneddon and M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity.* Wiley, 1969.

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