GRADIENT FLOWS AND RATE-INDEPENDENT EVOLUTIONS: A VARIATIONAL APPROACH

GIUSEPPE SAVARÉ

SEVENTH SUMMER SCHOOL IN ANALYSIS AND APPLIED MATHEMATICS
ROME, JUNE 17-21, 2013

The course will give a brief overview of the variational theory for gradient flows and rate-independent evolutions, trying to focus on the most important aspects: variational approximations, convergence results, energy-dissipation inequalities and metric characterization for gradient flows; energetic descriptions, BV solutions and optimal jump transitions, viscous approximations in the case of rate-independent evolutions. Some useful tools of convex and metric analysis will also be recalled.

Here is a more detailed description of the topics and a few references.

A setting for gradient flows and rate-independent evolutions. The simplest variational setting for gradient flows and rate-independent evolutions consists in

- a state space X (here, for simplicity, a Banach space),
- a convex "dissipation functional" $\Psi : X \to [0, \infty)$,
- a time dependent energy $\mathcal{E} : X \times [0, T] \to \mathbb{R} \cup \{+\infty\}$.

One looks for curves $u : [0, T] \to X$ which follow the direction of maximal descending slope of the energy $\mathcal{E}$. Such a direction is characterized in terms of the dissipation potential Ψ and of the differential $\mathrm{D}\mathcal{E}$ of the energy; when everything is smooth, it amounts to solve the doubly nonlinear evolution equation [3, 10]

$$\mathrm{D}\Psi(\dot{u}(t)) = -\mathrm{D}\mathcal{E}(t, u(t)) \quad \text{in } X^*, \quad t \in [0, T], \quad u(0) = u_0. \tag{1}$$

(1) should be carefully formulated in nonsmooth/infinite-dimensional setting in order to cover interesting PDE problems.

Gradient flows: superlinear dissipations. Gradient-flows correspond to *superlinear* dissipation functions Ψ: perhaps the most common example is when X is an Hilbert space, Ψ is its squared norm and (1) takes the simpler form [2, 12]

$$\dot{u}(t) = -\nabla\mathcal{E}(t, u(t)), \tag{2}$$

where one usually does not distinguish between X and X^* thanks to the identification given by the Riesz isomorphism $\mathrm{D}\Psi$.

The Minimizing Movement scheme. One of the most powerful method to prove existence of a solution to (1) consists in the so-called *Minimizing movement* approximation [4, 1] scheme: denoting by $\tau > 0$ a time step size, under quite general coercivity-lower semicontinuity assumptions on the energy $\mathcal{E}$, one can construct a sequence $(U^n_\tau)_n$, $n = 0, 1, \ldots, N$, $N\tau \geq T$, such that

$$U^n_\tau \quad \text{minimizes the functional} \quad U \mapsto \tau\Psi\Big(\frac{U - U^{n-1}_\tau}{\tau}\Big) + \mathcal{E}(n\tau, U) \tag{3}$$

Denoting by U_τ the piecewise constant interpolant of the discrete values U^n_τ on each interval $((n-1)\tau, n\tau]$, it is possible to prove that $U_{\tau_k} \to u$ uniformly in $[0, T]$ for a suitable vanishing sequence $\tau_k \downarrow 0$.

Energy-dissipation inequality. It is remarkable that the limit curve can be characterized by a single scalar energy-dissipation inequality [5, 1, 10]

$$\mathcal{E}(T, u(T)) + \int_0^T \Big(\Psi(\dot u(t)) + \Psi^*(-\mathrm{D}\mathcal{E}(t, u(t))\Big)\,\mathrm{d}t \le \mathcal{E}(0, u_0), \tag{4}$$

that encode the full information of the differential equation (1) in the smooth cases and provides a very useful tool in more general frameworks, as metric spaces without a linear structure.

1-homogeneous dissipations: rate-independent evolutions. In this rough scheme, a rate-independent evolution corresponds, formally, to *one homogeneous dissipations* Ψ, so that (1) becomes invariant by monotone time rescalings. The typical example is $\Psi(v) := \|v\|_X$, the norm of the Banach space X.

Even if the formal structure of the equation looks similar, the loss of superlinearity introduces many new problems and difficulties with respect to the gradient flow case. In particular, the energy inequality (4) provides just a BV estimate in time for the solutions, so that jumps can typically occur during the evolution, at least when the energy is not convex. Therefore, even for very regular energies and data, one has to deal with a non-smooth setting and possibly discontinuous solutions.

We will address two different strategies to construct a solution to the rate-independent evolution system.

Energetic solutions by the minimization scheme. The first one, introduced by A. Mielke and his collaborators, [11, 6, 7] is based on the same discretization method (3) and it leads to the so-called *energetic solutions.* In this approach (4) is replaced by the energy-dissipation inequality involving the total variation of u

$$\mathcal{E}(T, u(T)) + \mathrm{Var}(u; [0, T]) \le \mathcal{E}(0, u_0), \tag{5}$$

and a global stability condition

$$\mathcal{E}(t, u(t)) \le \mathcal{E}(t, v) + \|v - u(t)\| \quad \text{for every } v \in X,\ t \in [0, T], \tag{6}$$

which reflects the global character of each minimization step of the approximation scheme (notice that the small penalization parameter τ has no effect in (3) since $\tau\Psi(v/\tau) = \Psi(v)$ due to 1-homogeneity).

BV solutions by viscous regularization. In order to obtain a localized condition, various viscosity-type corrections have been proposed. The simplest ones consist to add a (asymptotically small) superlinear (e.g. quadratic) perturbation to the dissipation term, obtaining a family of superlinear dissipation potentials Ψ_ε that approximate the linearly growing Ψ as $\varepsilon \downarrow 0$. One can then apply the theory of "viscous" gradient flow to find a family of solutions u_ε corresponding to Ψ_ε and reduces the problem to study its limit u as $\varepsilon \downarrow 0$ [9].

This procedure gives raise to the notion of viscous BV solutions to the rate-independent system $(X, \Psi, \mathcal{E})$ [8], a class that it is in general different from the previous energetic one. u can be characterized by a modified energy dissipation inequality

$$\mathcal{E}(T, u(T)) + \mathcal{VAR}(u; [0, T]) \le \mathcal{E}(0, u_0), \tag{7}$$

and a local stability condition

$$\Psi^*(\mathrm{D}\mathcal{E}(t, u(t))) \le 1 \quad \text{for every } v \in X,\ t \in [0, T]. \tag{8}$$

The bigger variation $\mathcal{VAR}$ differs from the usual one Var in the contribution of the jump points: at each jump time t the solution keeps trace of a fast transition connecting the left and the right limit $u_\pm(t)$ along a gradient flow trajectory of $\mathcal{E}(\cdot, t)$. The use of the energy-dissipation inequality (4) is crucial to recover the precise description of this trajectory.

References

[1] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, second ed., 2008.

[2] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[3] P. Colli and A. Visintin, *On a class of doubly nonlinear evolution equations*, Comm. Partial Differential Equations, 15 (1990), pp. 737–756.

[4] E. De Giorgi, *New problems on minimizing movements*, in Boundary Value Problems for PDE and Applications, C. Baiocchi and J. L. Lions, eds., Masson, 1993, pp. 81–98.

[5] E. De Giorgi, A. Marino, and M. Tosques, *Problems of evolution in metric spaces and maximal decreasing curve*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 68 (1980), pp. 180–187.

[6] A. Mielke, *Evolution in rate-independent systems (Ch. 6)*, in Handbook of Differential Equations, Evolutionary Equations, vol. 2, C. Dafermos and E. Feireisl, eds., Elsevier B.V., Amsterdam, 2005, pp. 461–559.

[7] ———, *Differential, energetic, and metric formulations for rate-independent processes*, in Nonlinear PDE's and applications, vol. 2028 of Lecture Notes in Math., Springer, Heidelberg, 2011, pp. 87–170.

[8] A. Mielke, R. Rossi, and G. Savaré, *BV solutions and viscosity approximations of rate-independent systems*, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 36–80.

[9] ———, *Variational convergence of gradient flows and rate-independent evolutions in metric spaces*, Milan J. Math., 80 (2012), pp. 381–410.

[10] ———, *Nonsmooth analysis of doubly nonlinear evolution equations*, Calc. Var. Partial Differential Equations, 46 (2013), pp. 253–310.

[11] A. Mielke, F. Theil, and V. I. Levitas, *A variational formulation of rate-independent phase transformations using an extremum principle*, Arch. Ration. Mech. Anal., 162 (2002), pp. 137–177.

[12] R. Rossi and G. Savaré, *Gradient flows of non convex functionals in Hilbert spaces and applications*, ESAIM Control Optim. Calc. Var., 12 (2006), pp. 564–614 (electronic).

Dipartimento di Matematica, Università di Pavia. Via Ferrata, 1 – 27100 Pavia, Italy.
E-mail address: `giuseppe.savare@unipv.it`
URL: `http://www.imati.cnr.it/~savare`