Convex integration for embeddings and fluid mechanics

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Certain nonlinear systems of PDE seem to have the following property: at a sufficiently high "classical" degree of regularity the equations behave well-posed, solutions are rather "rigid" and the solution-space is very small. However, if one relaxes the regularity assumption, the solutions become highly "flexible" and the solution space becomes huge. Moreover, typical solutions will have very pathological behaviour. This passage from rigidity to flexibility has been first observed in the 1950s by Nash and Kuiper: whereas the only C^2 isometric embedding of S^2 into $\mathbb{R}^3$ is the standard embedding modulo rigid motion, there exist many C^1 isometric embeddings which can "wrinkle" S^2 into arbitrarily small regions.

In these lectures I will treat two examples of this phenomenon: (1) the theorem of Nash and Kuiper on C^1 isometric embeddings; and (2) the recent application of the same method, obtained in joint work with Camillo De Lellis, to the incompressible Euler equations of fluid mechanics, extending the work of Scheffer and Shnirelman.

An interesting question is to find the borderline regularity between flexibility and rigidity. This turns out to be highly relevant for both examples and has indeed in both cases generated a lot of interesting activity. I will survey the known results and open problems in this area.