Summer School 'Algebra, Topology and Fjords!'
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Exercises: Triangulated categories and cofibration categories

Exercise 1: Let k be a field. Show that the category of k-vector spaces admits a triangulation with the identity functor as shift functor and such that a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A$$

is distinguished if and only if the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A \xrightarrow{f} B$$

is exact. Characterize the class of rings for which the analogous definition makes the category of left modules into a triangulated category.

Exercise 2: Show that the category of finitely generated free modules over the ring $\mathbb{Z}/4$ has a unique triangulation with identity shift functor and such that the triangle

$$\mathbb{Z}/4 \xrightarrow{2\cdot} \mathbb{Z}/4 \xrightarrow{2\cdot} \mathbb{Z}/4 \xrightarrow{2\cdot} \mathbb{Z}/4$$

is distinguished.

Exercise 3: Let $\mathcal{T}$ be a triangulated category.

(i) Show that for every distinguished triangle (f, g, h) the following three conditions are equivalent:
- The morphism $f : A \longrightarrow B$ has a retraction, i.e., there is a morphism r such that $rf = \mathrm{Id}_A$.
- The morphism $g : B \longrightarrow C$ has a section, i.e., there is a morphism s such that $gs = \mathrm{Id}_C$.
- The morphism $h : C \longrightarrow \Sigma A$ is zero.

(ii) Let (f, g, h) be a distinguished triangle and $s : C \longrightarrow B$ a morphism such that $gs = \mathrm{Id}_C$. Show that the morphisms $f : A \longrightarrow B$ and $s : C \longrightarrow B$ make B into a coproduct of A and C.

(iii) Let $A \oplus B$ be a coproduct of two objects A and B of $\mathcal{T}$ with respect to the morphisms $i_A : A \longrightarrow A \oplus B$ and $i_B : B \longrightarrow A \oplus B$. Show that the triangle

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

is distinguished, where p_B is the morphism determined by $p_B i_A = 0$ and $p_B i_B = \mathrm{Id}_B$.

(iv) Let (f, g, h) and (f', g', h') be distinguished triangles. Show that the triangle

$$A \oplus A' \xrightarrow{f \oplus f'} B \oplus B' \xrightarrow{g \oplus g'} C \oplus C' \xrightarrow{\kappa \circ (h \oplus h')} \Sigma(A \oplus A')$$

is distinguished, where $\kappa : (\Sigma A) \oplus (\Sigma A') \longrightarrow \Sigma(A \oplus A')$ is the canonical isomorphism.

Exercise 4: (Homotopy colimit) Let $\mathcal{T}$ be a triangulated category with countably infinite sums. We consider a countably infinite sequence

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \cdots$$

of composable morphisms in $\mathcal{T}$. A *homotopy colimit* of the sequence consists of an object $\bar{X}$ together with morphisms $\varphi_n : X_n \longrightarrow \bar{X}$ satisfying $\varphi_{n+1} f_n = \varphi_n$ such that there exists a distinguished triangle

$$\bigoplus_{n \geq 0} X_n \xrightarrow{1-f} \bigoplus_{n \geq 0} X_n \xrightarrow{\oplus \varphi_n} \bar{X} \longrightarrow \Sigma \left(\bigoplus_{n \geq 0} X_n \right).$$

Here we denote by $1 - f : \bigoplus_{n \geq 0} X_n \longrightarrow \bigoplus_{n \geq 0} X_n$ the morphism whose restriction to the i-th summand is the difference of the canonical morphism $X_i \longrightarrow \bigoplus_{n \geq 0} X_n$ and the composition of $f_i : X_i \longrightarrow X_{i+1}$ with the canonical morphism $X_{i+1} \longrightarrow \bigoplus_{n \geq 0} X_n$. Given an object Y of $\mathcal{T}$, construct a short exact sequence of abelian groups

$$0 \longrightarrow \lim^1_n \mathcal{T}(\Sigma X_n, Y) \longrightarrow \mathcal{T}(\bar{X}, Y) \longrightarrow \lim_n \mathcal{T}(X_n, Y) \longrightarrow 0 .$$

Exercise 5: Let $\mathcal{T}$ be a triangulated category with infinite sums. An object A of $\mathcal{T}$ is called *compact* if for every family $\{X^i\}_{i \in I}$ of objects the canonical map

$$\bigoplus_{i \in I} \mathcal{T}(A, X^i) \longrightarrow \mathcal{T}(A, \bigoplus_{i \in I} X^i)$$

is an isomorphism. Let $f_n : X_n \longrightarrow X_{n+1}$ a sequence of composable morphisms and $(\bar{X}, \varphi_n)$ a homotopy colimit of the sequence $\{f_n\}$. Show that for every compact object A of $\mathcal{T}$ the map

$$\mathrm{colim}_n\, \mathcal{T}(A, X_n) \longrightarrow \mathcal{T}(A, \bar{X})$$

induced by $\mathcal{T}(A, \varphi_n)$ is an isomorphism.

Exercise 6: Show that in the triangulated category of vector spaces over a field (compare Exercise 1) every categorical colimit of a sequences is also a homotopy colimit. When is a vector space compact as an object of this triangulated category?

Exercise 7: (Splitting idempotents) Let $\mathcal{T}$ be a triangulated category with countable sums. Let X be any object of $\mathcal{T}$ and $e : X \longrightarrow X$ an idempotent endomorphism. Show that e splits in the following sense: there are objects eX and $(1-e)X$ and an isomorphism between X and the sum $eX \oplus (1-e)X$ under which $e : X \longrightarrow X$ corresponds to the endomorphism

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : eX \oplus (1-e)X \longrightarrow eX \oplus (1-e)X .$$

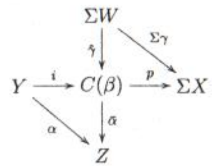
(Hint: construct eX as the homotopy colimit of the sequence of e's).

Exercise 8: Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms in a triangulated category $\mathcal{T}$ such that the composite $gf : X \longrightarrow Z$ zero and the group $\mathcal{T}(\Sigma X, Z)$ is trivial. Show that there is at most one morphism $h : Z \longrightarrow \Sigma X$ such that (f, g, h) is a distinguished triangle.

Exercise 9: (Toda brackets) Let $\mathcal{T}$ be a triangulated category and $\alpha : Y \longrightarrow Z$, $\beta : X \longrightarrow Y$ and $\gamma : W \longrightarrow X$ three composable morphisms that satisfy $\alpha\beta = 0 = \beta\gamma$. We define the *Toda bracket* $\langle \alpha, \beta, \gamma \rangle$, a subset of the morphism group $\mathcal{T}(\Sigma W, Z)$, as follows. We choose a distinguished triangle

$$X \xrightarrow{\beta} Y \xrightarrow{i} C(\beta) \xrightarrow{p} \Sigma X .$$

Since $\alpha\beta = 0$ there exists a morphism $\bar{\alpha} : C(\beta) \longrightarrow Z$ such that $\bar{\alpha} i = \alpha$; since $\beta\gamma = 0$ there exists a morphism $\bar{\gamma} : \Sigma W \longrightarrow C(\beta)$ such that $p\bar{\gamma} = \Sigma\gamma$, compare the commutative diagram:



The bracket $\langle \alpha, \beta, \gamma \rangle$ then consists of all morphisms of the form $\bar{\alpha}\bar{\gamma} : \Sigma W \longrightarrow Z$ for varying $\bar{\alpha}$ and $\bar{\gamma}$.

(i) Show that the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is independent of the choice of distinguished triangle.

(ii) Show that the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is a coset of the subgroup

$$(\alpha \circ \mathcal{T}(\Sigma W, Y)) + (\mathcal{T}(\Sigma X, Z) \circ \Sigma\gamma) \quad \text{of} \quad \mathcal{T}(\Sigma W, X) .$$

(iii) Let $\delta : V \longrightarrow W$ be another morphism such $\gamma\delta = 0$. Show the relation

$$\alpha \circ \langle \beta, \gamma, \delta \rangle = \langle \alpha, \beta, \gamma \rangle \circ (\Sigma\delta)$$

as subsets of $\mathcal{T}(\Sigma V, Z)$.

(iv) Let $\mathcal{T}$ be an algebraic triangulated category, n an integer and $f : X \longrightarrow Y$ a morphism in $\mathcal{T}$ such that $n \cdot f = 0$. Show that the Toda bracket $\langle n \cdot \mathrm{Id}_Y, f, n \cdot \mathrm{Id}_X \rangle$ contains 0. Give an example of a triangulated category and a Toda bracket of the form $\langle n \cdot \mathrm{Id}_Y, f, n \cdot \mathrm{Id}_X \rangle$ that does not contain 0. (Hint: Exercise 2)

Exercise 10: (Exact functors, I) Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. An *exact functor* is a pair (F,τ) consisting of a functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$ and a natural isomorphism $\tau : F \circ \Sigma \cong \Sigma \circ F$ such that for every distinguished triangle (f,g,h) in $\mathcal{T}$ the triangle

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{\tau_A \circ Fh} \Sigma(FA)$$

is distinguished in $\mathcal{T}'$. Show that every exact functor between triangulated categories is additive.

Exercise 11: For a cofibration category $\mathcal{C}$ we let $\mathrm{Cof}(\mathcal{C})$ be the category of cofibrations in $\mathcal{C}$: the objects of $\mathrm{Cof}\,\mathcal{C}$ are the cofibrations in $\mathcal{C}$ and a morphism from a cofibration $i : A \longrightarrow B$ to a cofibration $i' : A' \longrightarrow B'$ is a pair $(\alpha : A \longrightarrow A', \beta : B \longrightarrow B')$ of morphisms such that $\beta i = i'\alpha$. A morphism (α,β) is a weak equivalence in $\mathrm{Cof}\,\mathcal{C}$ if α and β are weak equivalences in $\mathcal{C}$, and $(\alpha.\beta)$ is a cofibration in $\mathrm{Cof}\,\mathcal{C}$ if α and $i' \cup \beta : A' \cup_A B \longrightarrow B'$ are cofibrations in $\mathcal{C}$. Show that these definitions make $\mathrm{Cof}(\mathcal{C})$ into a cofibration category.

Exercise 12: Let $\mathcal{C}$ be a pointed cofibration category. We denote by $\mathrm{Cone}(\mathcal{C})$, the *category of cones* in $\mathcal{C}$, the full subcategory of $\mathrm{Cof}\,\mathcal{C}$ spanned by those cofibrations $i : A \longrightarrow C$ whose target C is weakly contractible.

(i) Show that $\mathrm{Cone}(\mathcal{C})$ is a cofibration category by restriction of the cofibrations structure on $\mathrm{Cof}\,\mathcal{C}$.
(ii) Show that the forgetful functor

$$U : \mathrm{Cone}(\mathcal{C}) \longrightarrow \mathcal{C}\ , \quad U(i : A \longrightarrow C) = A$$

is exact and induces an equivalence of homotopy categories $\mathrm{Ho}(\mathrm{Cone}(\mathcal{C})) \cong \mathrm{Ho}(\mathcal{C})$.
(iii) Let $F : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathrm{Cone}(\mathcal{C}))$ be a quasi-inverse to the equivalence categories from (ii). Show that the composite

$$\mathrm{Ho}(\mathcal{C}) \xrightarrow[\cong]{F} \mathrm{Ho}(\mathrm{Cone}(\mathcal{C})) \xrightarrow{(i:A\longrightarrow C)\mapsto C/A} \mathrm{Ho}(\mathcal{C})$$

is naturally isomorphic to the suspension functor. Conclude that $\mathcal{C}$ is stable if and only if the functor $\mathrm{Cone}(\mathcal{C}) \longrightarrow \mathcal{C}$ that sends $(i : A \longrightarrow C)$ to the quotient C/A induces an equivalence of homotopy categories from $\mathrm{Ho}(\mathrm{Cone}(\mathcal{C}))$ to $\mathrm{Ho}(\mathcal{C})$.

Exercise 13: Let $\mathcal{C}$ be a cofibration category and let $\varphi : A \longrightarrow B$ be any $\mathcal{C}$-morphism. Show that $\gamma(\varphi)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ if and only if there are cofibrations $f : B \longrightarrow B'$ and $f' : B' \longrightarrow B''$ such that $f'\varphi$ and $f''f'$ are weak equivalences.

Exercise 14: (Exact functors, II) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between cofibration categories is *exact* if it preserves weak equivalences, cofibrations, initial objects and pushouts along cofibrations. Let $\mathcal{C}$ and $\mathcal{D}$ be pointed cofibration categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor. Construct a natural isomorphism $\tau : \mathrm{Ho}\,F \circ \Sigma_{\mathrm{Ho}(\mathcal{C})} \longrightarrow \Sigma_{\mathrm{Ho}(\mathcal{D})} \circ \mathrm{Ho}\,F$ of functors $\mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ such that for every distinguished triangle (f,g,h) in $\mathrm{Ho}(\mathcal{C})$ the triangle

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{\tau_C \circ Fh} \Sigma FA$$

is distinguished. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are stable cofibration categories, then the induced functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ is an exact functor of triangulated categories.

Exercise 15: (Additive categories with translation) Let $\mathcal{A}$ be additive category equipped with an additive endofunctor $T : \mathcal{A} \longrightarrow \mathcal{A}$. A *differential object* in $(\mathcal{A},T)$ is a pair (X,d) consisting of an object X of $\mathcal{A}$ and a morphism $d : X \longrightarrow TX$, the *differential*. (Beware that there is no condition on the composite $(Td)d : X \longrightarrow TTX$, i.e., we are *not* asking for a 'complex'). A morphism $f : (X,d) \longrightarrow (X',d')$ of differential objects is an $\mathcal{A}$-morphism $f : X \longrightarrow X'$ satisfying $d'f = (Tf)d : X \longrightarrow TX'$. Two morphisms $f,g : (X,d) \longrightarrow (X',d')$ are *homotopic* if there exists an $\mathcal{A}$-morphism $s : TX \longrightarrow X'$ (the *homotopy*) such that $d's + (Ts)(Td) = Tf - Tg$ as morphisms $TX \longrightarrow TX'$. A morphism $f : (X,d) \longrightarrow (X',d')$ of differential objects is a *homotopy equivalence* if there is a morphism $g : (X',d') \longrightarrow (X,d)$ of differential objects such that fg and gf are homotopic to the respective identity maps. A morphism f of differential objects is a *cofibration* if the underlying map in $\mathcal{A}$ is a split monomorphism, i.e., if there is an $\mathcal{A}$-morphism $g : C \longrightarrow X'$ such that $f + g : X \oplus C \longrightarrow X'$ is an isomorphism.

(i) Show that the cofibrations and homotopy equivalences make the category of differential object in $(\mathcal{A},T)$ into a cofibration category in which every object is fibrant.
(ii) Show that the notion of 'homotopy' is an additive equivalence relation compatible with composition. Let $\mathbf{K}(\mathcal{A},T)$ denote the category whose objects are differential objects in $(\mathcal{A},T)$ and whose morphisms are homotopy classes of morphisms. Show that $\mathbf{K}(\mathcal{A},T)$ is a homotopy category of $(\mathcal{A},T)$ for the cofibration structure in (i).
(iii) The *shift* of a differential object is given by $(X,d)[1] = (TX,-Td)$ on objects and by $f[1] = Tf$ on morphisms. Show that the shift functor on the category of differential object passes to a shift functor on the homotopy category $\mathbf{K}(\mathcal{A},T)$. Show that this induced shift functor on $\mathbf{K}(\mathcal{A},T)$ is naturally isomorphic to the suspension functor of the cofibration structure from (i).
(iv) The *mapping cone* of a morphism $f : (X,d) \longrightarrow (X',d')$ of differential objects is the object

$$Cf = \left(X \oplus TX',\ \begin{pmatrix} d & f \\ 0 & -Td' \end{pmatrix}\right)\ .$$

Show that the image of the sequence

$$(X,d) \xrightarrow{f} (X',d') \xrightarrow{(1,0)} Cf \xrightarrow{\binom{0}{1}} (X,d)[1]$$

is a distinguished triangle in the homotopy category $\mathbf{K}(\mathcal{A},T)$.
(v) Suppose now that the translation functor T is an auto-equivalence. Show that the cofibration structure in (i) is then stable. Conclude that the homotopy category $\mathbf{K}(\mathcal{A},T)$ is a triangulated category. Show that a triangle in $\mathbf{K}(\mathcal{A},T)$ is distinguished if and only if it is isomorphic to the image of a diagram as in part (iv) for some morphism $f : (X,d) \longrightarrow (X',d')$ of differential objects.
(vi) Let $\mathcal{A}$ be the category of vector spaces over a field k and let T be the identity functor. Show that then $\mathbf{K}(\mathcal{A},T) = \mathbf{K}(k\text{-vector spaces}, \mathrm{Id})$ is equivalent, as a triangulated category, to the triangulated category of Exercise 1.

Exercise 16: (Cochain complexes in additive categories) For any additive category $\mathcal{A}$ we denote by $C(\mathcal{A})$ the category of chain complex in $\mathcal{A}$; the homotopy category $\mathbf{K}(C(\mathcal{A}))$ of chain complexes in $\mathcal{A}$ has as objects the chain complexes and as morphisms the chain homotopy classes of chain morphisms. Define a structure of stable cofibration category on $C(\mathcal{A})$ in which the weak equivalences are the chain homotopy equivalences. Show that the homotopy category $\mathbf{K}(C(\mathcal{A}))$ has a triangulated structure with suspension functor giving by the shift of a complex. (Hint: reduce to Exercise 15)

Exercise 17: (Frobenius categories) A *(right) Frobenius ring* is a ring R such that the class of projective right R-modules coincides with the class of injective right R-modules. In this exercise, all R-modules are right R-modules. Two morphisms of R-modules $f,g : M \longrightarrow N$ are *homotopic* if the difference $f-g$ factors through a projective R-module. A morphism $f : M \longrightarrow N$ of R-modules is a *stable equivalence* if there is a morphism $g : N \longrightarrow M$ such that fg and gf are homotopic to the respective identity maps.

(i) Show that the monomorphisms and stable equivalences make the category of right R-modules into a stable cofibration category in which every object is fibrant.
(ii) Show that the notion of 'homotopy' is an additive equivalence relation compatible with composition. Let $\mathbf{S}(\mathrm{Mod}R)$ denote the *stable category* whose objects are the right R-modules and whose morphisms are homotopy classes of R-linear maps. Show that $\mathbf{S}(\mathrm{Mod}R)$ is a homotopy category of the cofibration structure in (i). Conclude that the stable category $\mathbf{S}(\mathrm{Mod}R)$ is a triangulated category.
(iii) Let M be an R-module and $i : M \longrightarrow I$ an injective hull, i.e., a monomorphism with injective target. Show that the quotient I/M can be taken as the suspension of M in $\mathbf{S}(\mathrm{Mod}R)$.
(iv) Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} Q \longrightarrow 0$$

be a short exact sequence of R-modules and let $i : M \longrightarrow I$ be an injective hull. We choose an extension $j : N \longrightarrow I$, i.e., a homomorphism such that $j \circ f = i$. Then we define a map $\delta : Q \longrightarrow I/M$ by

$$\delta(q) = j(\bar{q}) + i(M)\ ,$$

where $\bar{q} \in N$ satisfies $g(\bar{q}) = q$. Show that δ is well-defined and R-linear. Show that the image of the morphism δ in $\mathbf{S}(\mathrm{Mod}R)$ is independent of the extension j. Show that the triangle

$$M \xrightarrow{f} N \xrightarrow{g} Q \xrightarrow{\delta} I/M = \Sigma M$$

in the stable category $\mathbf{S}(\mathrm{Mod}R)$ is distinguished. Show that a triangle in $\mathbf{S}(\mathrm{Mod}R)$ is distinguished if and only if is isomorphic to a triangle arising in this way from a short exact sequence of R-modules.