CHAPTER 7

Differential graded Lie algebras

7.1. Differential graded vector spaces

Every vector space is considered over a fixed field $K$; unless otherwise specified, by the symbol $\otimes$ we mean the tensor product $\otimes_K$ over the field $K$.

The category $\text{DG}$. By a graded vector space we mean a $K$-vector spaces $V$ endowed with a $\mathbb{Z}$-graded direct sum decomposition $V = \oplus_{i \in \mathbb{Z}} V^i$. The elements of $V_i$ are called homogeneous of degree $i$.

If $V = \oplus_{n \in \mathbb{Z}} V^n \in G$ we write $\deg(a; V) = i \in \mathbb{Z}$ if $a \in V_i$; if there is no possibility of confusion about $V$ we simply denote $\overline{a} = \deg(a; V)$.

Definition 7.1.1. A DG-vector space is the data of a graded vector space $V = \oplus_{n \in \mathbb{Z}} V^n$ together a linear map $d: V \to V$, called differential, such that $d(V^n) \subseteq V^{n+1}$ for every $n$ and $d^2 = d \circ d = 0$.

A morphism $f: (V, d_V) \to (W, d_W)$ of DG-vector spaces is a linear map $f: V \to W$ such that $f(V^n) \subseteq W^n$ for every $n$ and $d_W f = f d_V$.

The category of DG-vector spaces will be denoted $\text{DG}$.

Thus, giving a morphism $f: (V, d_V) \to (W, d_W)$ of DG-vector spaces is the same of giving a sequence of linear maps $f_n: V^n \to W^n$ such that $d_W f_n = f_{n+1} d_V$ for every $n$.

Given a DG-vector space $(V, d)$ we denote as usual by $Z(V) = \ker d$ the space of cycles, by $B(V) = d(V)$ the space of boundaries and by $H(V) = Z(V)/B(V)$ the cohomology of $V$.

A morphism in $\text{DG}$ is called a quasiisomorphism, or a weak equivalence, if it induces an isomorphism in cohomology. A DG-vector space $(V, d)$ is called acyclic if $H(V) = 0$, i.e. if it is weak equivalent to 0.

Remark 7.1.2. In a completely similar way we may define dg-vector spaces, in which differentials have degree $-1$, i.e. $d(V_i) \subseteq V_{i-1}$. A differential graded vector space is either a DG-vector space or a dg-vector space.

Example 7.1.3. Every complex of vector spaces

$\ldots \to V^n \overset{d}{\to} V^{n+1} \overset{d}{\to} V^{n+2} \to \ldots$

can be trivially considered as a DG-vector space.

Given a double complex $C^{i,j}$, $i, j \in \mathbb{Z}$, of vector spaces, with differentials

$d_1: C^{i,j} \to C^{i,j+1}$, $d_2: C^{i,j} \to C^{i+1,j}$, $d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$

we define the associated total complex as the DG-vector space

$\text{Tot}(C^{*,*}) = \bigoplus_{n \in \mathbb{Z}} \text{Tot}(C^{*,*})^n$, $\text{Tot}(C^{*,*})^n = \bigoplus_{i+j=n} C^{i,j}$, $d = d_1 + d_2$.

The category $\text{DG}$ contains products: more precisely if $\{(V_i, d_i)\}$ is a family of DG-vector spaces, we have

$\prod_i V_i = \bigoplus_{n \in \mathbb{Z}} \prod_i V_i^n$, $(\prod_i V_i)^n = \prod_i V_i^n$, $d(\{v_i\}) = \{d_i(v_i)\}$, $v_i \in V_i$. 

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K"unneth formulas. Given two DG-vector spaces $V,W$ we may define their tensor product $V \otimes W \in \text{DG}$ and their internal Hom $\text{Hom}^n_{\text{DG}}(V,W) \in \text{DG}$ in the following way:

$$ V \otimes W = \bigoplus_{n \in \mathbb{Z}} (V \otimes W)^n, \text{ where } (V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes W^j, $$

$$ d(v \otimes w) = dv \otimes w + (-1)^{\deg v} v \otimes dw. $$

Hom$^n_{\text{DG}}(V,W) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(V,W), \quad \text{Hom}^n_{\text{DG}}(V,W) = \{ f : V \to W \text{ linear } | f(V^i) \subset W^{i+n} \forall i \}. $

$$ d : \text{Hom}^n_{\text{DG}}(V,W) \to \text{Hom}^{n+1}_{\text{DG}}(V,W), \quad df = dw(f(v)) - (-1)^n f(dv(v)). $$

We point out that for every $V,W,Z \in \text{DG}$ we have a natural isomorphism abelian groups

$$ \text{Hom}_{\text{DG}}(V,W) = Z^0(\text{Hom}^*_\text{DG}(V,W)), \quad \text{Hom}_{\text{DG}}(V,W) = Z^0(\text{Hom}^*_\text{DG}(V,W)). $$

\textbf{Theorem 7.1.4 (K"unneth formulas).} Given a DG-vector space $V$, consider its cohomology $H^*(V) = \bigoplus_n H^*(V)$ as a DG-vector space with trivial differential. For every pair of DG-vector spaces there exists natural isomorphisms

$$ H^*(V \otimes W) = H^*(V) \otimes H^*(W), \quad H^*(\text{Hom}^*_\text{DG}(V,W)) = \text{Hom}^*_\text{DG}(H^*(V),H^*(W)). $$

\textbf{Proof.} See e.g. the book [120]. \qed

Koszul rule of signs.

\textbf{Definition 7.1.5.} Given $V,W \in \text{DG}$, we define the twisting involution

$$ \text{tw} \in \text{Hom}_{\text{DG}}(V \otimes W, W \otimes V), \quad \text{tw}(v \otimes w) = (-1)^{\deg w} w \otimes v. $$

Using the Koszul signs convention means that we choose as natural isomorphism between $V \otimes W$ and $W \otimes V$ the twisting map $\text{tw}$ and we make every commutation rule compatible with $\text{tw}$. More informally, to “get the signs right”, whenever an “object of degree $d$ passes on the other side of an object of degree $h$, a sign $(-1)^{dh}$ must be inserted”.

\textbf{Example 7.1.6.} Assume that $f \in \text{Hom}^*_\text{DG}(V,W)$ and $g \in \text{Hom}^*_\text{DG}(H,K)$. Then the Koszul rule of signs implies that the correct definition of $f \otimes g \in \text{Hom}^*_\text{DG}(V \otimes H, W \otimes K)$ is

$$(f \otimes g)(v \otimes h) = (-1)^{\deg f} f(v) \otimes g(h).$$

Notice that $\text{tw} \circ (f \otimes g) \circ \text{tw} = (-1)^{\deg f} g \otimes f$.

\textbf{Shifting indices.} Given a DG-vector space $(V,d_V)$ and an integer $p$ we can define the DG-vector space $(V[p],d_V|_{V[p]})$ by setting

$$ V[p]^n = V^{n+p}, \quad d_V|_{V[p]} = (-1)^p d_V. $$

Sometimes it is useful to use a different notation. Let $s$ be a formal symbol of degree $+1$, so that $s^p$ becomes a formal symbol of degree $p$, for every integer $p$. Then define

$$ s^p V = \{ s^p v | v \in V \}, \quad \deg s^p v = p + \deg(v). $$

Setting $ds^p = 0$, according to Leibniz and Koszul rules we have

$$ d(s^p v) = d(s^p) v + (-1)^p s^p d(v) = (-1)^p s^p d(v). $$

Clearly $(s^p V)^n = V^{n-p}$ and then $s^p V \simeq V[-p]$. Notice that the natural map

$$ s^p : V \to s^p V, \quad v \mapsto s^p v $$

belongs to $\text{Hom}^*_\text{DG}(V,s^p V)$. Some authors call the $sV$ the suspension of $V$, $s^{-1}V$ the desuspension of $V$ and more generally $s^p V$ the $p$-fold suspension of $V$.

The Koszul rule of signs gives immediately a canonical isomorphism

$$ s^q V \otimes s^p W \to s^{p+q}(V \otimes W), \quad s^p V \otimes s^q W \to (-1)^{pq} s^{p+q}(v \otimes w), $$

Similarly we have $\text{Hom}^*_\text{DG}(s^p V,s^q W) \simeq s^{q-p} \text{Hom}^*_\text{DG}(V,W)$. 

**Definition 7.1.7.** For a morphism of DG-vector spaces \( f: V \to W \) we will denote by \( C_f \) the suspension of the mapping cone of \( f \). More explicitly, \( C_f = V \oplus sW \) and the differential is

\[
\delta: C_f^n = V^n \oplus W^{n-1} \to C_f^{n+1} = V^{n+1} \oplus W^n, \quad \delta(v, w) = (dv, f(v) - dw).
\]

The projection \( p: C_f \to V \) and the inclusion \( i: sW \to C_f \) are morphisms of DG-vector spaces and we have a long exact cohomology sequence

\[
\cdots \to H^i(V) \xrightarrow{f} H^i(W) \xrightarrow{i} H^{i+1}(C_f) \xrightarrow{p} H^{i+1}(V) \xrightarrow{f} H^{i+1}(W) \to \cdots
\]

In particular, given a commutative square

\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & E \\
\downarrow f & & \downarrow g \\
W & \xrightarrow{\beta} & F
\end{array}
\]

if both \( \alpha \) and \( \beta \) are quasiisomorphisms, then also the induced map \( C_f \to C_g \) is a quasiisomorphism.

### 7.2. DG-algebras

**Definition 7.2.1.** A DG-algebra (short for Differential graded commutative algebra) is the data of a DG-vector space \( A \) and a morphism of DG-vector spaces

\[
A \otimes A \to A, \quad a \otimes b \mapsto ab,
\]

called **product**, which is associative and invariant under the twisting involution.

More concretely, this means that for \( a, b, c \in A \) we have:

1. (associativity) \((ab)c = a(bc)\),
2. (graded commutativity) \(ab = (-1)^{pq}ba\),
3. (graded Leibniz) \(d(ab) = da(b) + (-1)^p ad(b)\).

A morphism of DG-algebras is simply a morphism of DG-vector spaces commuting with products. The category of DG-algebras will be denoted by \( \text{DGA} \). A DG-algebra \( A \) is called **unitary** if there exists a unit \( 1 \in A^0 \).

**Example 7.2.2.** Every commutative \( \mathbb{K} \)-algebra can be considered as a DG-algebra concentrated in degree 0.

**Example 7.2.3.** The de Rham complex of a smooth manifold, endowed with wedge product is a DG-algebra.

**Example 7.2.4 (Koszul algebras).** Let \( V \) be a vector space and consider the graded algebra

\[
A = \bigoplus_{n \geq 0} A^n, \quad A^{-n} = \bigwedge^n V,
\]

with the wedge product as a multiplication map. Given a linear map \( f: V \to \mathbb{K} \), we may define a differential \( d: A^{-i} \to A^{-i+1}, i \geq 0 \):

\[
d = f \circ \wedge: \bigwedge^i V \to \bigwedge^{i-1} V,
\]

where the contraction operator \( \circ \) is defined by the formula

\[
f \circ (v_1 \wedge \cdots \wedge v_h) = \sum_{j=1}^h (-1)^{j-1} f(v_j) v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_h.
\]

**Example 7.2.5.** The de Rham complex of algebraic differential forms on the affine line will be denoted by \( \mathbb{K}[t, dt] \). We may write

\[
\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt
\]

where \( t, dt \) are indeterminates of degrees \( \bar{t} = 0, \bar{dt} = 1 \) and the differential \( d \) is determined by the “obvious” equality \( d(t) = dt \) and therefore \( d(p(t) + q(t)dt) = p(t)'dt \). The inclusion \( \mathbb{K} \to \mathbb{K}[t, dt] \) and the evaluation maps

\[
e_s: \mathbb{K}[t, dt] \to \mathbb{K}, \quad p(t) + q(t)dt \mapsto p(s), \quad s \in \mathbb{K},
\]
are morphisms of DG-algebras.

**Lemma 7.2.6.** In characteristic 0, every evaluation morphism \( e_s : \mathbb{K}[t, dt] \to \mathbb{K} \) is a quasiisomorphism.

**Proof.** If \( i : \mathbb{K} \to \mathbb{K}[t, dt] \) is the natural inclusion, we have \( e_s \circ i = \text{Id} \) and then it is sufficient to prove that \( i \) is a quasiisomorphism. This is obvious since every cocycle of \( \mathbb{K}[t, dt] \) is of type \( a + q(t)dt \) with \( a \in \mathbb{K} \) and \( q(t)dt \) is exact, being the differential of \( \int_0^t q(s)ds \). \( \square \)

The tensor product of two DG-algebras is still a DG-algebra; clearly we need to take attention to Koszul sign convention. If \( A, B \) are DG-algebras, then the product on \( A \otimes B \) is defined as the linear extension of

\[
(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\deg(a_1) \deg(b_2)} a_1 a_2 \otimes b_1 b_2.
\]

### 7.3. Differential graded Lie algebras

In this section \( \mathbb{K} \) will be a field of characteristic 0.

**Definition 7.3.1.** A **differential graded Lie algebra** (DGLA for short) is the data of a DG-vector space \((L, d)\) with a bilinear bracket \( [\cdot, \cdot] : L \times L \to L \) satisfying the following condition:

1. \( [\cdot, \cdot] \) is homogeneous skewsymmetric: this means \([L^i, L^j] \subseteq L^{i+j}\) and \([a, b] + (-1)^{\deg(a) \deg(b)} [b, a] = 0\) for every \( a, b \) homogeneous.
2. Every triple of homogeneous elements \( a, b, c \) satisfies the (graded) Jacobi identity

\[
[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a) \deg(b)} [b, [a, c]].
\]
3. (graded Leibniz) \( d[a, b] = [da, b] + (-1)^{\deg(a) \deg(b)} [a, db] \).

We should take attention that skewsymmetry implies that \([a, a] = 0\) only if \( a \) is of even degree, while for odd degrees we have the following result.

**Lemma 7.3.2.** Let \( L \) be a DGLA and \( a \in L \) homogeneous of odd degree. Then \([a, [a, a]] = 0\).

**Proof.** By graded Jacobi and graded skewsymmetry we have

\[
[a, [a, a]] = [[a, a], a] - [a, [a, a]] = -2[a, [a, a]].
\]

\( \square \)

**Example 7.3.3.** If \( L = \bigoplus L^i \) is a DGLA then \( L^0 \) is a Lie algebra in the usual sense. Conversely, every Lie algebra can be considered as a DGLA concentrated in degree 0.

**Example 7.3.4.** Let \( A \) be a DG-algebra and \( L \) a DGLA. Then the DG-vector space \( L \otimes A \) has a natural structure of DGLA with bracket

\[
[x \otimes a, y \otimes b] = (-1)^{\deg(x) \deg(y)} [x, y] \otimes ab.
\]

**Example 7.3.5.** Let \( V \) be a DG-vector space. Then the total Hom complex \( \text{Hom}^\mathbb{K}_e(V, V) \) has a natural structure of DGLA with bracket

\[
[f, g] = fg - (-1)^{\deg(f) \deg(g)} gf.
\]

Notice that the differential on \( \text{Hom}^\mathbb{K}_e(V, V) \) is equal to the adjoint operator \([d, -]\), where \( d \) is the differential of \( V \).

**Example 7.3.6.** Let \( E \) be a holomorphic vector bundle on a complex manifold \( M \). We may define a DGLA \( L = \bigoplus L^p \), \( L^p = \Gamma(M, \mathcal{A}^{p, 0}(\mathcal{E}ud(E))) \) with the Dolbeault differential and the natural bracket. More precisely if \( e, g \) are local holomorphic sections of \( \mathcal{E}ud(E) \) and \( \phi, \psi \) differential forms we define \( d(\phi \psi) = (\overline{\partial} \phi) \psi + \phi \overline{\partial} \psi \).

**Example 7.3.7.** Let \( T_M \) be the holomorphic tangent bundle of a complex manifold \( M \). The **Kodaira-Spencer DGLA** is defined as \( KS(M) = \bigoplus \Gamma(M, \mathcal{A}^{0, p}(T_M)[−p]) \) with the Dolbeault differential; if \( z_1, \ldots, z_n \) are local holomorphic coordinates we have \([\phi dz_I, \psi dz_J] = [\phi, \psi] dz_I \wedge d\overline{z}_J \) for \( \phi, \psi \in \mathcal{A}^{0, 0}(T_M), I, J \in \{1, \ldots, n\} \).

There is an obvious notion of morphism of differential graded Lie algebras: it is a morphism of DG-vector spaces commutating with brackets. The category of differential graded Lie algebras will be denoted DGLA.
Example 7.3.8. The fiber product \( L \times_H M \) of two morphisms \( f: L \to H, g: M \to H \) of DGLA is a DGLA with bracket
\[
[(a, x), (b, y)] = ([a, b], [x, y]).
\]

Definition 7.3.9. A quasiisomorphism of DGLAs is a morphism of DGLAs which is a quasiisomorphism of DG-vector spaces. Two DGLAs are said to be quasiisomorphic if they are equivalent under the equivalence relation generated by quasiisomorphisms.

Example 7.3.10. Denote by \( \mathbb{K}[t, dt] \) the differential graded algebra of polynomial differential forms over the affine line and, for every differential graded Lie algebra \( L \) denote \( L[t, dt] = L \otimes \mathbb{K}[t, dt] \). As a graded vector space \( L[t, dt] \) is generated by elements of the form \( aq(t) + bp(t)dt \), for \( p, q \in \mathbb{K}[t] \) and \( a, b \in L \). The differential and the bracket on \( L[t, dt] \) are
\[
d(aq(t) + bp(t)dt) = (da)q(t) + (-1)^{\deg(a)}aq(t)'dt + (db)p(t)dt,
\]
\[
[aq(t), ch(t)] = [a, c]q(t)h(t), \quad [aq(t), ch(t)dt] = [a, c]q(t)h(t)dt.
\]
For every \( s \in \mathbb{K} \), the evaluation morphism
\[
e_s: L[t, dt] \to L, \quad e_s(aq(t) + bp(t)dt) = q(s)a
\]
is a morphism of differential graded Lie algebras. According to Lemma 7.2.6 and Künneth formulas, it is also a quasiisomorphism of DGLA.

Example 7.3.11. Let \( f: L \to H, g: M \to H \) be two morphisms of differential graded Lie algebras. Their homotopy fiber product is defined as
\[
L \times^h_L M := \{ (l, m, h(t)) \in L \times M \times H[t, dt] \mid h(0) = f(l), h(1) = g(m) \},
\]
where for every \( s \in \mathbb{K} \) we denote for simplicity \( h(s) = e_s(h(t)) \). It is immediate to verify that it is a differential graded Lie Algebra and that the natural projections
\[
L \times^h_L M \to L, \quad L \times^h_L M \to M,
\]
are surjective morphisms of DGLAs.

Remark 7.3.12. In the notation of Example 7.3.11, it is an easy exercise to prove that, if \( f: L \to H \) is a quasiisomorphism, then the projection \( L \times^h_L M \to M \) is a quasiisomorphism. This is a consequence of a more general results that we will prove in ??.

Using this fact it is immediate to observe that two differential graded Lie algebras \( L, M \) are quasiisomorphic if and only if there exists a DGLA \( K \) and two quasiisomorphisms \( K \to L, K \to M \).

The cohomology of a DGLA is itself a differential graded Lie algebra with the induced bracket and zero differential:

Definition 7.3.13. A DGLA \( L \) is called formal if it is quasiisomorphic to its cohomology DGLA \( H^*(L) \).

We will see later on, that there exists differential graded Lie algebras that are not formal.

Lemma 7.3.14. For every DG-vector space \( V \), the differential graded Lie algebra \( \text{Hom}^ *(V, V) \) is formal.

Proof. For every index \( i \) we choose a vector subspace \( H^i \subseteq Z^i(V) \) such that the projection \( H^i \to H^i(V) \) is bijective. The graded vector space \( H = \oplus H^i \) is a quasiisomorphic subcomplex of \( V \). The subspace \( K = \{ f \in \text{Hom}^ *(V, V) \mid f(H) \subseteq H \} \) is a differential graded Lie subalgebra and there exists a commutative diagram of complexes with exact rows
\[
0 \to K \xrightarrow{\alpha} \text{Hom}^ *(V, V) \to \text{Hom}^ *(H, V/H) \to 0
\]
\[
0 \to \text{Hom}^ *(H, H) \xrightarrow{\beta} \text{Hom}^ *(H, V) \to \text{Hom}^ *(H, V/H) \to 0
\]
The maps \( \alpha \) and \( \beta \) are morphisms of differential graded Lie algebras. The complex \( \text{Hom}^ *(H, V/H) \) is acyclic and \( \gamma \) is a quasiisomorphism, therefore also \( \alpha \) and \( \beta \) are quasiisomorphisms. \( \square \)
7.4. Further examples of differential graded Lie algebras

Given a graded vector space $V$ and a bilinear map $\bullet : V \times V \to V$ such that $V^i \bullet V^j \subset V^{i+j}$, the vector

$$A(x, y, z) = (x \bullet y) \bullet z - x \bullet (y \bullet z)$$

is called the associator of the triple $x, y, z$: the product $\bullet$ is associative if and only if $A(x, y, z) = 0$ for every $x, y, z$.

**Lemma 7.4.1.** Assume that the associator is graded symmetric in the last two variables, i.e. $A(x, y, z) = (-1)^{y}A(x, z, y)$. Then the graded commutator

$$\begin{align*}
[x, y] &= x \bullet y - (-1)^{y} y \bullet x
\end{align*}$$

satisfies the graded Jacobi identity.

**Proof.** Straightforward. \(\square\)

**Example 7.4.2** (The Gerstenhaber bracket). Let $A$ be a vector space and, for every integer $n \geq 0$ let $V^n = \text{Hom}_\mathbb{K}(\bigotimes^{n+1} A, A)$ be the space of multilinear maps

$$f : A \times \cdots \times A \to A.$$**

The Gerstenhaber product is defined as

$$\bullet : V^n \times V^m \to V^{n+m},$$

$$(f \bullet g)(a_0, \ldots, a_{n+m}) = \sum_{i=0}^{n} (-1)^{im} f(a_0, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+m}), a_{i+m+1}, \ldots, a_{n+m}).$$

It is easy to verify that the associator is graded symmetric in the last two variables and then the graded commutator

$$[x, y] = x \bullet y - (-1)^{y} y \bullet x,$$

called Gerstenhaber bracket satisfies the graded Jacobi identity. Notice that for an element $m \in V^1$ we have $[m, m] = 2m \bullet m$ and

$$m \bullet m(a, b, c) = m(m(a, b), c) - m(a, m(b, c)).$$

Therefore $[m, m] = 0$ if and only if $m : A \times A \to A$ is an associative product.

**Example 7.4.3** (The Hochschild DGLA). Let $A$ be an associative $\mathbb{K}$-algebra and denote by $m : A \times A \to A$, $m(a, b) = ab$, the multiplication map. We have seen that the graded vector space

$$Hoch^*(A) = \bigoplus_{n \geq 0} \text{Hom}_\mathbb{K}(\bigotimes^{n+1} A, A),$$

endowed with Gerstenhaber bracket is a graded Lie algebra. The Hochschild differential is defined as the linear map

$$d : Hoch^n(A) \to Hoch^{n+1}(A), \quad d(f) = -[f, m].$$

In a more explicit form, for $f \in Hoch^n(A)$ we have

$$df(a_0, \ldots, a_{n+1}) = a_0f(a_1, \ldots, a_{n+1}) + (-1)^n f(a_0, \ldots, a_n)a_{n+1}$$

$$- \sum_{i=0}^{n} (-1)^i f(a_0, \ldots, a_{i-1}, a_ia_{i+1}, a_{i+2}, \ldots, a_{n+1}).$$

Setting

$$\delta : Hoch^n(A) \to Hoch^{n+1}(A), \quad \delta(f) = [m, f] = (-1)^n d(f),$$

Jacobi identity gives:

1. $\delta^2(f) = [m, [m, f]] = \frac{1}{2}[[m, m], f] = 0$ since $m$ is associative and then $[m, m] = 0$,
2. $\delta[f, g] = [\delta f, g] + (-1)^f [f, \delta g].$

Therefore the triple $(Hoch^*(A), \delta, [\cdot, \cdot])$ is a differential graded Lie algebra.

**Example 7.4.4** (Derivations). Let $A$ be a DG-algebra over the field $\mathbb{K}$. 

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7.4. FURTHER EXAMPLES OF DIFFERENTIAL GRADED LIE ALGEBRAS 6
Definition 7.4.5. The DGLA of derivations of a DG-algebra $A$ is $\text{Der}_K^*(A, A) = \bigoplus_n \text{Der}_K^n(A, A)$, where $\text{Der}_K^n(A, A)$ is the space of derivations of degree $n$ defined as

$$\text{Der}_K^n(A, A) = \{ \phi \in \text{Hom}_K^n(A, A) \mid \phi(ab) = \phi(a)b + (-1)^{|a||b|}\phi(b) \}.$$ 

In particular the differential of $A$ is a derivation of degree $+1$. It is easy to prove that derivations are closed under graded commutator and then $\text{Der}_K^*(A, A)$ is a DG-Lie subalgebra of $\text{Hom}_K^*(A, A)$.

Similarly, if $L$ is a DGLA, then $\text{Der}_K^*(L, L) = \bigoplus_n \text{Der}_K^n(L, L)$, where

$$\text{Der}_K^n(L, L) = \{ \phi \in \text{Hom}_K^n(L, L) \mid \phi[a, b] = [\phi(a), b] + (-1)^{|a||b|}[a, \phi(b)] \}$$

is a DG-Lie subalgebra of $\text{Hom}_K^*(L, L)$.

Example 7.4.6 (Differential operators). Let $A$ be a DG-algebra over the field $\mathbb{K}$ with unit $1 \in A^0$. We may consider $A$ as an abelian DG-Lie subalgebra of $\text{Hom}_K^*(A, A)$, where every $a \in A$ is identified with the operator

$$a : A \to A, \quad a(b) = ab.$$ 

For every integer $k$ we will denote by

$$\text{Diff}_k(A) = \bigoplus_{n \in \mathbb{Z}} \text{Diff}_K^n(A) \subset \text{Hom}_K^n(A, A)$$

the graded subspace of differential operators of order $\leq k$: it is defined recursively by setting $\text{Diff}_k(A) = 0$ for $k < 0$ and

$$\text{Diff}_k(A) = \{ f \in \text{Hom}_K^n(A, A) \mid [f, a] \in \text{Diff}_{k-1}(A) \forall a \in A \}$$

for $k \geq 0$. Notice that $f \in \text{Diff}_0(A)$ if and only if $f(1)a = f(a)$ and every derivation belongs to $\text{Diff}_1(A)$.

A very simple induction on $h + k$ gives that

$$\text{Diff}_k(A) \text{Diff}_h(A) \subset \text{Diff}_{h+k}(A), \quad [\text{Diff}_k(A), \text{Diff}_h(A)] \subset \text{Diff}_{h+k-1}(A).$$

In particular, the spaces $\text{Diff}_1(A)$ and $\text{Diff}(A) = \bigcup_k \text{Diff}_k(A)$ are DG-Lie subalgebras of $\text{Hom}_K^*(A, A)$.

7.5. Maurer-Cartan equation and gauge action

Definition 7.5.1. The Maurer-Cartan equation (also called the deformation equation) of a DGLA $L$ is

$$da + \frac{1}{2}[a, a] = 0, \quad a \in L^1.$$ 

The solutions of the Maurer-Cartan equation are called the Maurer-Cartan elements of the DGLA $L$. The set of such solutions will be denoted $MC(L) \subset L^1$.

It is plain that Maurer-Cartan equation commutes with morphisms of differential graded Lie algebras.

The notion of nilpotent Lie algebra extends naturally to the differential graded case; in particular for every DGLA $L$ and every proper ideal $I$ of a local artinian $\mathbb{K}$-algebra the DGLA $L \otimes I$ is nilpotent.

Assume now that $L$ is a nilpotent DGLA, in particular $L^0$ is a nilpotent Lie algebras and we can consider its exponential group $\exp(L^0)$. By Jacobi identity, for every $a \in L^0$ the corresponding adjoint operator

$$ad a : L \to L, \quad (ad a)b = [a, b],$$

is a nilpotent derivation of degree $0$ and then its exponential

$$e^{ad a} : L \to L, \quad e^{ad a}(b) = \sum_{n \geq 0} \frac{(ad a)^n}{n!}(b),$$

is an isomorphism of graded Lie algebras, i.e. for every $b, c \in L$ we have

$$e^{ad a}([b, c]) = [e^{ad a}(b), e^{ad a}(c)].$$

In particular the quadratic cone $\{ b \in L^1 \mid [b, b] = 0 \}$ is stable under the adjoint action of $\exp(L^0)$. 

The **gauge action** is a derived from the adjoint action via the next construction. Given a DGLA \((L, [\cdot, \cdot], d)\) we can construct a new DGLA \((L', [\cdot', \cdot'], d')\) by setting \((L')^i = L^i\) for every \(i \neq 1\), \((L')^1 = L^1 \oplus \mathbb{K} d\) (here \(d\) is considered as a formal symbol of degree 1) with the bracket and the differential

\[
[a + vd, b + ud]' = [a, b] + vd(b) + (-1)^p ud(a), \quad d'(a + vd) = [d, a + vd]' = d(a).
\]

Since \((L')^n \subset L^{(n-1)} + dL^{[n-1]}\) for every \(n \geq 3\), if \(L\) is nilpotent, then also \(L'\) is nilpotent.

The natural inclusion \(L \subset L'\) is a morphism of DGLA; denote by \(\phi\) the affine embedding \(\phi: L' \to (L')^1\), \(\phi(x) = x + d\). The image of \(\phi\) is stable under the adjoint action and then it makes sense the following definition.

**Definition 7.5.2.** Let \(L\) be a nilpotent DGLA. The **gauge action** of \(\exp(L^0)\) on \(L^1\) is defined as

\[
e^a \ast x = \phi^{-1}(e^{ad_a}(\phi(x))) = e^{ad_a}(x + d) - d.
\]

Explicitely

\[
e^a \ast x = \sum_{n \geq 0} \frac{1}{n!} (ad_a)^n(x) + \sum_{n \geq 1} \frac{1}{n!} (ad_a)^n(d)
\]

\[
= \sum_{n \geq 0} \frac{1}{n!} (ad_a)^n(x) - \sum_{n \geq 1} \frac{1}{n!} (ad_a)^{n-1}(da)
\]

\[
= w + \sum_{n \geq 0} \frac{(ad_a)^n}{(n + 1)!} (a, x) - da.
\]

**Lemma 7.5.3.** Let \(L\) be a nilpotent DGLA, then:

1. the set of Maurer-Cartan solutions is stable under the gauge action;
2. \(e^a \ast x = x\) if and only if \([x, a] + da = 0\);
3. for every \(x \in \text{MC}(L)\) and every \(u \in L^{-1}\) we have \(e^{[x,u]}du \ast x = x\).

**Proof.** [1] For an element \(a \in L^1\) we have

\[
d(a) + \frac{1}{2} [a, a] = 0 \iff [\phi(a), \phi(a)]' = 0
\]

and the quadratic cone \([b + d \in (L^1)', [b + d, b + d]' = 0\) is stable under the adjoint action of \(\exp(L^0)\).

[2] Since \(ad_a\) is nilpotent, the operator \(\sum_{n \geq 0} \frac{(ad_a)^n}{(n + 1)!} = \frac{e^{ad_a} - 1}{ad_a}\) is invertible.

[3] Setting \(a = [x, u] + du\) we have

\[
[x, a] + da = [x, [x, u]] + [x, du] + d[x, u] = \frac{1}{2} [[x, x], u] + [dx, u] = 0.
\]

**Remark 7.5.4.** For every \(a \in L^0, x \in L^1\), the polynomial \(\gamma(t) = e^{\alpha} \ast x \in L^1 \otimes \mathbb{K}[t]\) is the solution of the “Cauchy problem”

\[
\begin{cases}
\frac{d\gamma(t)}{dt} = [a, \gamma(t)] - da \\
\gamma(0) = x
\end{cases}
\]

### 7.6. Deformation functors associated to a DGLA

In order to introduce the basic ideas of the use of DGLA in deformation theory we begin with an example where technical difficulties are reduced at minimum. Consider a finite complex of vector spaces

\[
(V, \overrightarrow{\partial}) : \quad 0 \longrightarrow V^0 \longrightarrow V^1 \longrightarrow \cdots \longrightarrow \overrightarrow{\partial} V^n \longrightarrow 0.
\]

Given a local artinian \(\mathbb{K}\)-algebra \(A\) with maximal ideal \(m_A\) and residue field \(\mathbb{K}\), we define a deformation of \((V, \overrightarrow{\partial})\) over \(A\) as a complex of \(A\)-modules of the form

\[
0 \longrightarrow V^0 \otimes A \longrightarrow V^1 \otimes A \longrightarrow \cdots \longrightarrow \overrightarrow{\partial} V^n \otimes A \longrightarrow 0
\]
such that its residue modulo $m_A$ gives the complex $(V, \partial)$. Since, as a $\mathbb{K}$ vector space, $A = \mathbb{K} \oplus m_A$, this last condition is equivalent to say that

$$\overline{\partial}_A = \partial + \xi, \quad \text{where} \quad \xi \in \text{Hom}^1(V, V) \otimes m_A.$$ 

The “integrability” condition $\overline{\partial}_A^2 = 0$ becomes

$$0 = (\partial + \xi)^2 = \partial \xi + \xi \partial + \xi^2 = d\xi + \frac{1}{2} [\xi, \xi],$$

where $d$ and $[ , ]$ are the differential and the bracket on the differential graded Lie algebra $\text{Hom}^*(V, V) \otimes m_A$.

Two deformations $\overline{\partial}_A, \overline{\partial}_A'$ are isomorphic if there exists a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & V^0 \otimes A & \xrightarrow{\overline{\partial}_A} & V^1 \otimes A & \xrightarrow{\overline{\partial}_A} & \cdots & \xrightarrow{\overline{\partial}_A} & V^n \otimes A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \vdots & & \downarrow & & \downarrow \\
0 & \rightarrow & V^0 \otimes A & \xrightarrow{\overline{\partial}_A'} & V^1 \otimes A & \xrightarrow{\overline{\partial}_A'} & \cdots & \xrightarrow{\overline{\partial}_A'} & V^n \otimes A & \rightarrow & 0 \\
\end{array}
$$

such that every $\phi_i$ is an isomorphism of $A$-modules whose specialization to the residue field is the identity. Therefore we can write $\phi := \sum_i \phi_i = \text{Id} + \eta$, where $\eta \in \text{Hom}^0(V, V) \otimes m_A$ and, since $\mathbb{K}$ is assumed of characteristic $0$ we can take the logarithm and write $\phi = e^a$ for some $a \in \text{Hom}^1(V, V) \otimes m_A$. The commutativity of the diagram is therefore given by the equation

$$\overline{\partial}_A' = e^{ad} \circ \overline{\partial}_A \circ e^{-ad}.$$ 

Writing $\overline{\partial}_A = \partial + \xi, \overline{\partial}_A' = \partial + \xi'$ and using the relation $e^{ad} \circ b \circ e^{-ad} = e^{ad(b)}$ we get

$$\xi' = e^{ad} (\partial + \xi) - \partial = \xi + \frac{e^{ad} - 1}{ad} ([a, \xi] + [a, \xi]) = \xi + \sum_{n=0}^{\infty} \frac{ad^n}{(n+1)!} ([a, \xi] - da).$$

In particular both the integrability condition and isomorphism are entirely written in terms of the DGLA structure of $\text{Hom}^*(V, V) \otimes m_A$. This leads to the following general construction.

Denote by $\text{Art}$ the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ and by $\text{Set}$ the category of sets (we ignore all the set-theoretic problems, for example by restricting to some universe). Unless otherwise specified, for every objects $A \in \text{Art}$ we denote by $m_A$ its maximal ideal.

Let $L = \oplus L^i$ be a DGLA over $\mathbb{K}$, we can define the following three functors:

1. The exponential functor $\exp L: \text{Art}_\mathbb{K} \rightarrow \text{Grp},$

$$\exp L(A) = \exp(L^0 \otimes m_A).$$

It is immediate to see that $\exp L$ is smooth.

2. The Maurer-Cartan functor $MC_L: \text{Art}_\mathbb{K} \rightarrow \text{Set}$ defined by

$$MC_L(A) = MC(L \otimes m_A) = \left\{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2} [x, x] = 0 \right\}.$$

3. The gauge action of the group $\exp(L^0 \otimes m_A)$ on the set $MC(L \otimes m_A)$ is functorial in $A$ and gives an action of the group functor $\exp L$ on the Maurer-Cartan functor $MC_L$. The quotient functor $\text{Def}_L = MC_L/G_L$ is called the deformation functor associated to the DGLA $L$. For every $A \in \text{Art}_\mathbb{K}$ we have

$$\text{Def}_L(A) = \frac{MC(L \otimes m_A)}{\exp(L^0 \otimes m_A)} = \left\{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2} [x, x] = 0 \right\} / \text{gauge action}.$$ 

Both $MC_L$ and $\text{Def}_L$ are deformation functors in the sense of Definition 4.1.5; in fact $MC_L$ commutes with fiber products, while, according to Proposition 4.2.7 $\text{Def}_L$ is a deformation functor and the projection $MC_L \rightarrow \text{Def}_L$ is smooth.

The reader should make attention to the difference between the deformation functor $\text{Def}_L$ associated to a DGLA $L$ and the functor of deformations of a DGLA $L$.

**Lemma 7.6.1.** If $L \otimes m_A$ is abelian then $\text{Def}_L(A) = H^1(L) \otimes m_A$. In particular

$$T^1 \text{Def}_L = \text{Def}_L(\mathbb{K} [\epsilon]) = H^1(L) \otimes \mathbb{K} \epsilon, \quad \epsilon^2 = 0.$$
Proof. The Maurer-Cartan equation reduces to $dx = 0$ and then $\text{MC}_L(A) = Z^1(L) \otimes m_A$. If $a \in L^0 \otimes m_A$ and $x \in L^1 \otimes m_A$ we have

$$e^a \ast x = x + \sum_{n \geq 0} \frac{\text{ad}(a)^n}{(n+1)!} ([a,x] - da) = x - da$$

and then $\text{Def}_L(A) = \frac{Z^1(L) \otimes m_A}{d(L^0 \otimes m_A)} = H^1(L) \otimes m_A$. \hfill $\square$

It is clear that every morphism $\alpha: L \to N$ of DGLA induces morphisms of functors $G_L \to G_N$, $\text{MC}_L \to \text{MC}_N$. These morphisms are compatible with the gauge actions and therefore induce a morphism between the deformation functors $\text{Def}_L: \text{Def}_L \to \text{Def}_N$.

**Obstructions for $\text{MC}_L$ and $\text{Def}_L$.** Let $L$ be a differential graded Lie algebra. We want to show that $\text{MC}_L$ has a “natural” obstruction theory $(H^2(L), v_e)$.

Let’s consider a small extension in $\textbf{Art}$

$$e: \quad 0 \longrightarrow M \longrightarrow A \longrightarrow B \longrightarrow 0$$

and let $x \in \text{MC}_L(B) = \{x \in L^1 \otimes m_B \mid dx + \frac{1}{2} [x,x] = 0\}$; we define an obstruction $v_e(x) \in H^2(L \otimes M) = H^2(L) \otimes M$ in the following way: first take a lifting $\tilde{x} \in L^1 \otimes m_A$ of $x$ and consider $h = d\tilde{x} + \frac{1}{2} [[\tilde{x}, \tilde{x}], \tilde{x}]$.

Since $[L^2 \otimes M, L^1 \otimes m_A] = 0$ we have $[h, \tilde{x}] = 0$, by Jacobi identity $[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$ and then $dh = 0$. Define $v_e(x)$ as the class of $h$ in $H^2(L \otimes M) = H^2(L) \otimes M$; the first thing to prove is that $v_e(x)$ is independent from the choice of the lifting $\tilde{x}$; every other lifting is of the form $y = \tilde{x} + z$, $z \in L^1 \otimes M$ and then

$$d\tilde{y} + \frac{1}{2} [y, y] = h + dz.$$

It is evident from the above computation that $(H^2(L), v_e)$ is a complete obstruction theory for the functor $\text{MC}_L$.

**Lemma 7.6.2.** The complete obstruction theory described above for the functor $\text{MC}_L$ is invariant under the gauge action and then it is also a complete obstruction theory for $\text{Def}_L$.

**Proof.** Since the projection $\text{MC}_L \to \text{Def}_L$ is smooth, it is sufficient to apply the general properties of universal obstruction theories. It is instructive to give also a direct and elementary proof of this lemma. Let $x, y$ be two gauge equivalent solutions of the Maurer-Cartan equation in $L \otimes m_B$ and let $\tilde{x} \in L^1 \otimes m_A$ be a lifting of $x$. It is sufficient to prove that there exists a lifting $\hat{y}$ of $y$ such that

$$h := d\tilde{x} + \frac{1}{2} [[\tilde{x}, \tilde{x}], \tilde{x}] = d\hat{y} + \frac{1}{2} [[\hat{y}, \hat{y}], \hat{y}].$$

Let $a \in L^0 \otimes m_B$ be such that $e^a \ast x = y$, choose a lifting $\tilde{a} \in L^0 \otimes m_A$ and define $e^{\tilde{a}} \ast \tilde{x} = \hat{y}$. We have

$$d\hat{y} + \frac{1}{2} [[\hat{y}, \hat{y}], \hat{y}] = [[\hat{y} + d, \hat{y} + d]' = [e^{ad} (\tilde{x} + d), e^{ad} (\tilde{x} + d)]' = e^{ad} [\tilde{x} + d, \tilde{x} + d]' = e^{ad} (h) = h.$$

Finally, it is clear that every morphism of differential graded Lie algebras $f: L \to M$ induces natural transformations of functors

$$f: \text{MC}_L \to \text{MC}_M, \quad f: \text{Def}_L \to \text{Def}_M.$$
7.7. Semicosimplicial groupoids

For reader convenience we recall some basic notion of category theory. For simplicity of notation, if $C$ is a category we shall write $x \in C$ if $x$ is an object and $f \in \text{Mor}_C$ if $f$ is a morphism.

**Definition 7.7.1.** A **small category** is a category whose class of objects is a set.

**Example 7.7.2.** The category of finite ordinals $\Delta$ is a small category.

**Definition 7.7.3.** A **groupoid** is a small category such that every morphism is an isomorphism. We will denote by $\text{Grpd}$ the category of groupoids.

Notice that, for a groupoid $G$ and every object $g \in G$ the set $\text{Mor}_G(g, g)$ is a group.

**Example 7.7.4.** The fundamental groupoid $\pi_{\leq 1}(X)$ of a topological space $X$ is defined in the following way: the set of objects is $X$, while the morphisms are the continuous path up to homotopy of paths.

**Definition 7.7.5.** For a groupoid $G$ we well denote by $\pi_0(G)$ the set of isomorphism classes of objects of $G$ and, for every $g \in G$ we will denote $\pi_1(G, g) = \text{Mor}_G(g, g)$.

It is clear that every equivalence of groupoids $f : G \to E$ induce a bijection $f : \pi_0(G) \to \pi_0(E)$.

**Definition 7.7.6.** Let $G^\Delta : \begin{array}{ccc} G_0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & \cdots \\ \partial_0 & \longrightarrow & \partial_1 & \longrightarrow & \partial_2 & \longrightarrow & \cdots \\ \partial_0 \partial_0 l & \longrightarrow & \partial_0 \partial_1 l & \longrightarrow & \partial_2 \partial_0 l \\ \partial_1 \partial_0 l & \longrightarrow & \partial_1 \partial_1 l & \longrightarrow & \partial_2 \partial_1 l \\ \partial_2 \partial_0 l & \longrightarrow & \partial_2 \partial_1 l & \longrightarrow & \cdots \\ \partial_3 \partial_1 l & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\ \end{array}$

be a semicosimplicial groupoid with face maps $\partial_l : G_n \to G_{n+1}$: $\partial_l \partial_k = \partial_{k+1} \partial_l$, for any $l \leq k$.

The **total groupoid** $\text{Tot}(G^\Delta)$ is defined in the following way (cf. $[25, 46, 55]$):

1. The objects of $\text{Tot}(G^\Delta)$ are the pairs $(l, m)$ with $l \in G_0$ and $m$ is a morphism in $G_1$ between $\partial_0 l$ and $\partial_1 l$ such that the diagram

\[
\begin{array}{ccc}
\partial_0 l & \longrightarrow & \partial_1 l \\
\partial_0 \partial_0 l & \longrightarrow & \partial_0 \partial_1 l & \longrightarrow & \partial_2 \partial_0 l \\
\partial_1 \partial_0 l & \longrightarrow & \partial_1 \partial_1 l & \longrightarrow & \partial_2 \partial_1 l \\
\partial_2 \partial_0 l & \longrightarrow & \partial_2 \partial_1 l & \longrightarrow & \cdots \\
\partial_3 \partial_1 l & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\end{array}
\]

is commutative in $G_2$.

2. The morphisms between $(l_0, m_0)$ and $(l_1, m_1)$ are morphisms $a$ in $G_0$ between $l_0$ and $l_1$ making the diagram

\[
\begin{array}{ccc}
\partial_0 l_0 & \longrightarrow & \partial_1 l_0 \\
\partial_0 a & \longrightarrow & \partial_1 a \\
\partial_0 l_1 & \longrightarrow & \partial_1 l_1 \\
\partial_2 a & \longrightarrow & \cdots \\
\partial_3 a & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\end{array}
\]

commutative in $G_1$.

**Example 7.7.7.** caso delle SCLA. Da scrivere

7.8. Deligne groupoids

**Definition 7.8.1.** Let $L$ be a nilpotent differential graded Lie algebra. The **Deligne groupoid** of $L$ is the groupoid $\text{Del}(L)$ defined as follows:

1. the set of objects is $\text{MC}(L)$,
2. the morphisms are

$$\text{Mor}_{\text{Del}(L)}(x, y) = \{ e^n \in \exp(L^0) \mid e^n \ast x = y \}, \quad x, y \in \text{MC}(L).$$

**Definition 7.8.2.** Let $L$ be a nilpotent differential graded Lie algebra. The **irrelevant stabilizer** of a Maurer-Cartan element $x \in \text{MC}(L)$ is defined as the subgroup (see Lemma 7.5.3):

$$I(x) = \{ e^{du + [x, u]} \mid u \in L^{-1} \} \subset \text{Mor}_{\text{Del}(L)}(x, x).$$
Lemma 7.8.3. Let $L$ be a nilpotent differential graded Lie algebra, $a \in L^0$ and $x \in \text{MC}(L)$. Then

$$e^a I(x)e^{-a} = I(y), \quad \text{where} \quad y = e^a \star x.$$  

In particular $I(x)$ is a normal subgroup of $\text{Mor}_{\text{Def}(L)}(x, x)$ and there exists a natural isomorphism

$$\frac{\text{Mor}_{\text{Def}(L)}(x, y)}{I(x)} = \frac{\text{Mor}_{\text{Def}(L)}(x, y)}{I(y)},$$

with $I(x)$ and $I(y)$ acting in the obvious way.

Proof. Recall that for every $a, b \in L^0$ we have

$$e^a e^b e^{-a} = e^{\text{ad}_a(e^b)} = e^c,$$

where $c = e^{\text{ad}_a}(b)$. And then $e^{[a, u] + du} e^{-a} = e^c$, where, setting $v = e^{\text{ad}_a}(u)$, we have

$$c = e^{\text{ad}_a}([a, u] + du) = e^{\text{ad}_a}([a + d, u]) = [e^{\text{ad}_a}(a + d), v] = [y, v] + dv.$$

Definition 7.8.4 ([55, 68, 125]). The reduced Deligne groupoid of a nilpotent differential graded Lie algebra $L$ is the groupoid $\text{Def}(L)$ having as objects the Maurer-Cartan elements of $L$ and as morphisms

$$\text{Mor}_{\text{Def}(L)}(x, y) := \frac{\text{Mor}_{\text{Def}(L)}(x, y)}{I(x)} = \frac{\text{Mor}_{\text{Def}(L)}(x, y)}{I(y)}, \quad x, y \in \text{MC}(L).$$

In order to verify that $\text{Def}(L)$ is a groupoid we only need to verify that the (associative) multiplication map

$$\text{Mor}_{\text{Def}(L)}(y, z) \times \text{Mor}_{\text{Def}(L)}(x, y) \to \text{Mor}_{\text{Def}(L)}(x, z), \quad (e^a, e^b) \mapsto e^a e^b,$$

factors to a morphism

$$\frac{\text{Mor}_{\text{Def}(L)}(y, z)}{I(y)} \times \frac{\text{Mor}_{\text{Def}(L)}(x, y)}{I(x)} \to \frac{\text{Mor}_{\text{Def}(L)}(x, z)}{I(x)},$$

and this follows immediately from Lemma 7.8.3.

Notice that $\tau_0(\text{Def}(L)) = \pi_0(\text{Def}(L)) = \text{Def}(L)$.

Every cosimplicial nilpotent DGLA gives a semicosimplicial reduced Deligne groupoid and then a total groupoid. Here we are interested to a particular case of this construction.

Definition 7.8.5. Given a pair of morphisms $h, g: L \to M$ of nilpotent differential graded Lie algebras define

$$\text{Def}(h, g) = \text{Tot} \left( \xymatrix{
\text{Def}(L) \ar[r]^h & \text{Def}(M) \ar[r]^g & 0 \cdots}
\right), \quad \text{Def}(h, g) = \pi_0(\text{Def}(h, g)).$$

Lemma 7.8.6. For any pair of morphisms $h, g: L \to M$ of nilpotent differential graded Lie algebras we have

$$\text{Def}(h, g) = \left\{(x, e^a) \in \text{MC}(L) \times \exp(M^0) \mid e^a h(x) = g(x) \right\},$$

where $(x, e^a) \sim (y, e^b)$ if there exists $\alpha \in L^0$ such that $e^\alpha \star x = y$ and the diagram

$$\xymatrix{
h(x) \ar[r]^{e^a} \ar[d]^{h(x)} & g(x) \ar[d]^{g(x)}
\ar[r]_{h(x)} & \ar[d]_{g(x)}
h(y) \ar[r]^{e^b} & g(y)}$$

is commutative in the reduced Deligne groupoid of $M$.

Proof. Immediate from definition. \qed

Notice that for $M = 0$ we reobtain the usual set $\text{Def}(L)$.

Given a pair of morphisms $h, g: L \to M$ of differential graded Lie algebras, for every $A \in \text{Art}$ we get a pair of morphisms of nilpotent DGLA $h, g: L \otimes M \to M \otimes M$ and therefore we are in the position to define in the obvious way the functor

$$\text{Def}(h, g): \text{Art} \to \text{Set}.$$
Proposition 7.8.7. In the notation above, Def_{(h,g)} is a deformation functor with tangent and obstruction spaces equal to $H^1(C_{h-g})$ and $H^2(C_{h-g})$ respectively, being $C_{h-g}$ the suspended mapping cone of the morphism of DG-vector spaces $h - g: L \to M$.

Proof. The fact that Def_{(h,g)} is a deformation functor follows from Proposition 4.2.7, while it is straightforward to prove the equality $T^1\text{Def}_{(h,g)} = H^1(C_{h-g})$. We now compute obstructions using the description given in Lemma 7.8.6. Let

$$0 \to I \to A \to B \to 0$$

be a small extension and $(\hat{x}, e^a) \in MC(L \otimes m_B) \times \exp(M^0 \otimes m_B)$ be such that $e^a \ast h(\hat{x}) = g(\hat{x})$. Choose a lifting $(x, e^a) \in L^1 \otimes m_A \times \exp(M^0 \otimes m_A)$ and consider the elements

$$r = dx + \frac{1}{2}[x, x] \in L^2 \otimes I, \quad s = e^a \ast h(x) - g(x) \in M^1 \otimes I, \quad t = (r, s) \in C^2_{h-g} \otimes I.$$ 

We first prove that $dt = 0$; we already know that $dr = 0$; since

$$g(r) = dg(x) + \frac{1}{2}[g(x), g(x)] = d(e^a \ast h(x)) - ds + \frac{1}{2}[e^a \ast h(x), e^a \ast h(x)]$$

$$= \frac{1}{2}[e^a \ast h(x) + d, e^a \ast h(x) + d'] - ds = \frac{1}{2}[e^{ad}a(h(x) + d), e^{ad}a(h(x) + d)]' - ds$$

$$= \frac{1}{2}e^{ad}a[h(x) + d, h(x) + d'] - ds = e^{ad}a h(r) + ds = h(r) - ds.$$

we have $(h - g)r - ds = 0$ and then $t$ is a cocycle in $C_{h-g}$.

If $x$ is replaced with $x + u$, $u \in L^1 \otimes I$ and $a$ is replaced with $a + v$, $v \in M^0 \otimes I$, the element $(r, s)$ will be replaced with $(r + du, s + (h - g)u - dv)$. This implies that the cohomology class of $t$ in $H^2(C_{h-g}) \otimes I$ is well defined and is a complete obstruction. 

\[ \square \]

7.9. Homotopy invariance of deformation functors

We shall say that a functor $F: \text{DGLA} \to \text{C}$ is homotopy invariant if for every quasisisomorphism $f$ of DGLA the morphism $F(f)$ is an isomorphism in the category $\text{C}$. The main theme of this chapter is to prove that the functor

$$\text{Def}: \text{DGLA} \to \{\text{Def} \text{formation functors}\}$$

is homotopy invariant.

We have already pointed out that every morphism $f: L \to N$ of DGLA induces a morphism of associated deformation functors $f: \text{Def}_L \to \text{Def}_N$.

**Theorem 7.9.1.** Let $f: L \to N$ be a morphism of differential graded Lie algebras. Assume that the morphism $f: H^i(L) \to H^i(N)$ is:

1. surjective for $i = 0$,
2. bijective for $i = 1$,
3. injective for $i = 2$.

Then $f: \text{Def}_L \to \text{Def}_N$ is an isomorphism of functors.

**Corollary 7.9.2.** Let $L \to N$ be a quasisisomorphism of DGLA. Then the induced morphism $\text{Def}_L \to \text{Def}_N$ is an isomorphism.

In this chapter we give a proof of the above results that uses obstruction theory and the standard smoothness criterion for deformation functors (Theorem 4.5.12). Before doing this we need some preliminary results of independent interest.

\[ \checkmark \]

One of the most frequent wrong interpretations of Corollary 7.9.2 asserts that if $L \to N$ is a quasisisomorphism of nilpotent DGLA then $MC(L)/\exp(L^0) \to MC(N)/\exp(N^0)$ is a bijection. This is false in general: consider for instance $L = 0$ and $N = \oplus N^i$ with $N^i = C$ for $i = 1, 2$, $N^i = 0$ for $i \neq 1, 2$, $d: N^1 \to N^2$ the identity and $[a, b] = ab$ for $a, b \in N^1 = C$. 

\[ \checkmark \]
Lemma 7.9.3. Let $f : L \to N$ be a morphism of differential graded Lie algebras. If $f : H^1(L) \to H^1(N)$ is surjective and $f : H^2(L) \to H^2(N)$ is injective, then the morphism $f : \text{Def}_L \to \text{Def}_N$ is smooth.

Proof. Since $H^1(L)$ is the tangent space of $\text{Def}_L$ and $H^2(L)$ is a complete obstruction space, it is sufficient to apply the standard smoothness criterion.

Example 7.9.4. Let $L = \oplus L^i$ be a DGLA such that $[L^1, L^1] \cap Z^2(L) \subset B^2(L)$. Then $\text{Def}_L$ is smooth. In fact, consider the differential graded Lie subalgebra $N = \oplus N^i \subset L$ where:

1. $N^i = 0$ for every $i \leq 0$,
2. $N^1 = L^1$,
3. $N^2 = [L^1, L^1] + B^2(L)$,
4. $N^i = L^i$ for every $i > 2$.

By assumption $H^2(N) = 0$ and then $\text{Def}_N$ is smooth. Since $H^1(N) \to H^1(L)$ is surjective, the morphism $\text{Def}_N \to \text{Def}_L$ is smooth.

Example 7.9.5. Let $L = \oplus L^i$ be a DGLA and choose a vector space decomposition $N^1 \oplus B^1(L) = L^1$.

Consider the DGLA $N = \oplus N^i$ where $N^i = 0$ if $i < 1$ and $N^1 = L^1$ if $i > 1$ with the differential and bracket induced by $L$. The natural inclusion $N \to L$ gives isomorphisms $H^i(N) \to H^i(L)$ for every $i \geq 1$. In particular the morphism $\text{Def}_N \to \text{Def}_L$ is smooth and induce an isomorphism on tangent spaces $T^1 \text{Def}_N = T^1 \text{Def}_L$.

Let now $f : L \to M$ be a fixed morphism of differential graded Lie algebras and denote by $p_0, p_1 : L \times L \to L$ the projections.

The commutative diagram of differential graded Lie algebras

\[
\begin{array}{ccc}
L & \xrightarrow{p_0} & L \\
\downarrow f & & \downarrow f \\
M & \xrightarrow{f p_0} & L \\
\end{array}
\]

induce a natural transformation of functors $\eta : \text{Def}(p_0, p_1) \to \text{Def}(fp_0, fp_1)$.

Lemma 7.9.6. In the above set-up, if $f : H^0(L) \to H^0(M)$ is surjective and $f : H^1(L) \to H^1(M)$ is injective, then the morphism $\eta$ is smooth.

Proof. According to Proposition 7.8.7 and standard smoothness criterion it is sufficient to prove that $f : H^1(C_{p_0-p_1}) \to H^1(C_{fp_0-fp_1})$ is surjective and $f : H^2(C_{p_0-p_1}) \to H^2(C_{fp_0-fp_1})$ is injective. This follows by a straightforward diagram chasing on the morphism of exact sequences

\[
\begin{array}{cccccccc}
H^0(L) & \to & H^1(C_{p_0-p_1}) & \to & H^1(L \times L) & \to & H^1(C_{p_0-p_1}) & \to & H^2(L \times L) \\
\downarrow f & & \downarrow f & & \downarrow f & & \downarrow f & & \\
H^0(M) & \to & H^1(C_{fp_0-fp_1}) & \to & H^1(L \times L) & \to & H^1(M) & \to & H^2(C_{fp_0-fp_1}) \\
\end{array}
\]

Proof of Theorem 7.9.1. Using the notation introduced above, we have already proved that the morphisms

\[
f : \text{Def}_L \to \text{Def}_M, \quad \eta : \text{Def}(p_0, p_1) \to \text{Def}(fp_0, fp_1)
\]

are smooth. Given $A \in \mathcal{A}$ we need to prove that if $x, y \in MC_L(A)$ and there exists $b \in M^0 \otimes m_A$ such that $\eta^B \star f(x) = f(y)$, then $x$ is gauge equivalent to $y$. Using the notation of Lemma 7.8.6, since $(x, y, e^b) \in \text{Def}(fp_0, fp_1)(A)$ and $\eta$ is smooth, there exists $(u, v, e^\alpha) \in \text{Def}(p_0, p_1)(A)$ such that

\[
(u, v, e^{f(\alpha)}) \sim (x, y, e^b)
\]

and this implies in particular that there exists $\alpha \in (L^0 \times L^0) \otimes m_A$ such that

\[
e^\alpha \star u = v, \quad e^\alpha \star (u, v) = (x, y),
\]

and then $x, y$ are gauge equivalent. □
**Definition 7.9.7.** Let $L$ be a DGLA and $x,y \in \text{MC}(L)$. We shall say that $x$ and $y$ are homotopy equivalent if there is some $\xi \in \text{MC}(L[t,dt])$ such that $e_0(\xi) = x$ and $e_1(\xi) = y$. Here $L[t,dt] = L \otimes \mathbb{K}[t,dt]$ and $e_0,e_1 : L[t,dt] \to L$ are the evaluation maps at $t = 0$ and $t = 1$ respectively.

We will denote by $\pi_0(\text{MC}_\bullet(L))$ the quotient of $\text{MC}(L)$ under the equivalence relation generated by homotopy.\footnote{Using this notation we have implicitly assumed that there exists a groupoid $\text{MC}_\bullet(L)$ having $\text{MC}(L)$ as objects and $\text{MC}(L[t,dt])$ as morphisms. This is almost true, in the sense that there exists a natural structure of $\infty$-groupoid on $\text{MC}_\bullet$; we will give the precise definition later on.}

The construction of $\pi_0(\text{MC}_\bullet)$ is functorial and then we may define a functor

$$\pi_0(\text{MC}_\bullet)_L : \text{Art}_{\mathbb{K}} \to \text{Set}, \quad \pi_0(\text{MC}_\bullet)_L(A) = \pi_0(\text{MC}(L \otimes m_A)).$$

**Corollary 7.9.8.** For every differential graded Lie algebra $L$, the projection $\text{MC}_L \to \pi_0(\text{MC}_\bullet)_L$ factors to an isomorphism of functors $\text{Def}_L \to \pi_0(\text{MC}_\bullet)_L$.

**Proof.** Let $L$ be be DGLA, since the inclusion $L \to L[t,dt]$ is a quasismorphism the natural transformation $\text{Def}_L \to \text{Def}_{L[t,dt]}$ is an isomorphism. Given $A \in \text{Art}$ and $x,y \in \text{MC}(L \otimes m_A)$, it is sufficient to prove that $x,y$ are gauge equivalent if and only if they are homotopy equivalent. Recall that $x$ is homotopy equivalent to $y$ if there is some $a \in L^0 \otimes m_A$ such that $e^a \ast x = y$, whereas $x$ is homotopy equivalent to $y$ if there is some $z(t) \in \text{MC}(L[t,dt] \otimes m_A)$ such that $z(0) = x$ and $z(1) = y$.

So first, assume $e^a \ast x = y$; then we can consider $x \in \text{MC}(L \otimes m_A) \subset \text{MC}(L[t,dt] \otimes m_A)$. Since $e^a \otimes m_A$ acts by gauge on $\text{MC}(L[t,dt] \otimes m_A)$, for every $t$ we can set $z(t) = e^{ta} \ast x$. Then $z(0) = x$ and $z(1) = y$.

On the other hand, notice that $\text{MC}_L = \text{MC}_{L \geq 1}$ (in fact Maurer-Cartan only depends on $L^1$ and $L^2$), so $\text{Def}_L = \text{Def}_{L \geq 1}$ and $\text{MC}_{L \geq 1} \subset \text{MC}_{L \geq 0} \otimes [t,dt]$. So it is not restrictive to assume that $L = \bigoplus_{n \geq 0} L^n$. In this case, $L[t,dt]^0 = L^0[t]$.

Now let $z(t) \in \text{MC}(L[t,dt] \otimes m_A)$. Then, as we have a smooth morphism $\iota : \text{Def}_L(A) \to \text{Def}_{L[t,dt]}(A)$, we must have some $x \in \text{MC}_L(A)$ which is gauge equivalent to $z(t)$ in $L[t,dt] \otimes m_A$. So we have $a(t) \in L[t,dt]^0 \otimes m_A = L^0[t] \otimes m_A$ such that $e^{a(t)} \ast x = z(t)$. Now $z(0) = e^{a(0)} \ast x$ and $z(1) = e^{a(1)} \ast x$, and this imply that $z(0)$ is gauge equivalent to $z(1)$.

**Remark 7.9.9.** The first consequences of Corollary 7.9.8 is that the bifunctor $\pi_0(\text{MC}_\bullet) : \text{Def} : \text{DGLA} \times \text{Art} \to \text{Set}$ is completely determined by the Maurer-Cartan bifunctor

$$\text{MC} : \text{DGLA} \times \text{Art} \to \text{Set}.$$

### 7.10. Exercises

**Exercise 7.10.1.** Let

$$G^\Delta : \quad G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

be a semicosimplicial groupoid. Assume that for every $i$ the natural map $G_i \to \pi_0(G_i)$ is an equivalence, i.e., every $G_i$ is equivalent to a set. Then also $\text{tot}(G^\Delta)$ is equivalent to a set and, more precisely, to the equalizer of the diagram of sets

$$\pi_0(G_0) \longrightarrow \pi_0(G_1).$$