Riemann surfaces, ribbon graphs and combinatorial classes

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Contents

1 Introduction

1.1 Overview

1.1.1 Moduli space and Teichmüller space. Consider a compact oriented surface S of genus g together with a finite subset $X = \{x_1, \dots, x_n\}$, such that $2g - 2 + n > 0$.

The moduli space $\mathcal{M}_{g,X}$ is the set of all X-pointed Riemann surfaces of genus g up to isomorphism. Its universal cover (in the orbifold sense) can be identified with the Teichmüller space $\mathcal{T}(S, X)$, which parametrizes complex structures on S up to isotopy (relative to X); equivalently, $\mathcal{T}(S, X)$ parametrizes isomorphism classes of (S, X)-marked Riemann surfaces. Thus, $\mathcal{M}_{g,X}$ is the quotient of $\mathcal{T}(S, X)$ under the action of the mapping class group $\Gamma(S, X) = \mathrm{Diff}_+(S, X)/\mathrm{Diff}_0(S, X)$.

As $\mathcal{T}(S, X)$ is contractible (Teichmüller [68]), we also have that $\mathcal{M}_{g,X} \simeq B\Gamma(S, X)$. However, $\Gamma(S, X)$ acts on $\mathcal{T}(S, X)$ discontinuously but with finite stabilizers. Thus, $\mathcal{M}_{g,X}$ is naturally an orbifold and $\mathcal{M}_{g,X} \simeq B\Gamma(S, X)$ must be intended in the orbifold category.

1.1.2 Algebro-geometric point of view. As compact Riemann surfaces are complex algebraic curves, $\mathcal{M}_{g,X}$ has an algebraic structure and is in fact a Deligne-Mumford stack, which is the algebraic analogue of an orbifold. The underlying space $M_{g,X}$ (forgetting the isotropy groups) is a quasi-projective variety.

The problem of counting curves with suitable properties, a topic which is also called "enumerative geometry of curves", has always been central in algebraic geometry. The usual set-up is to describe the loci in $\mathcal{M}_{g,X}$ of curves that satisfy the wished properties and then to compute their intersection, which naturally leads to seeking for a suitable compactification of $\mathcal{M}_{g,X}$. Deligne and Mumford [15] understood that it was sufficient to consider algebraic curves with mild singularities to compactify $\mathcal{M}_{g,X}$. In fact, their compactification $\overline{\mathcal{M}}_{g,X}$ is the moduli space of X-pointed stable (algebraic) curves of genus g, where a complex projective curve C is "stable" if its only singularities are nodes (that is, in local analytic coordinates C looks like $\{(x, y) \in \mathbb{C}^2 \,|\, xy = 0\}$) and every irreducible component of the smooth locus of $C \setminus X$ has negative Euler characteristic.

The main tool to prove the completeness of $\overline{\mathcal{M}}_{g,X}$ is the stable reduction theorem, which essentially says that a smooth holomorphic family $\mathcal{C}^* \to \Delta^*$ of

X-pointed Riemann surfaces of genus g over the pointed disc can be completed to a family over Δ (after a suitable change of base $z \mapsto z^k$) using a stable curve.

The beauty of $\overline{\mathcal{M}}_{g,X}$ is that it is smooth (as an orbifold) and that its coarse space $\overline{M}_{g,X}$ is a projective variety (Mumford [56], Gieseker [21], Knudsen [38] [39], Kollár [40] and Cornalba [14]).

1.1.3 Tautological maps. The map $\overline{\mathcal{M}}_{g,X\cup\{y\}} \to \overline{\mathcal{M}}_{g,X}$ that forgets the y-marking (and then stabilizes the possibly unstable X-marked curve) can be identified to the universal family over $\overline{\mathcal{M}}_{g,X}$ and is the first example of tautological map.

Moreover, $\overline{\mathcal{M}}_{g,X}$ has a natural algebraic stratification, in which each stratum corresponds to a topological type of curve: for instance, smooth curves correspond to the open stratum $\mathcal{M}_{g,X}$. As another example: irreducible curves with one node correspond to an irreducible locally closed subvariety of (complex) codimension 1, which is the image of the (generically 2 : 1) tautological boundary map $\mathcal{M}_{g-1,X\cup\{y_1,y_2\}} \to \overline{\mathcal{M}}_{g,X}$ that glues y_1 to y_2. Thus, every stratum is the image of a (finite-to-one) tautological boundary map, and thus is isomorphic to a finite quotient of a product of smaller moduli spaces.

1.1.4 Augmented Teichmüller space. Teichmüller theorists are more interested in compactifying $\mathcal{T}(S,X)$ rather than $\mathcal{M}_{g,X}$. One of the most popular way to do this is due to Thurston (see [20]): the boundary of $\mathcal{T}(S,X)$ is thus made of projective measured laminations and it is homeomorphic to a sphere.

Clearly, there cannot be any clear link between a compactification of $\mathcal{T}(S,X)$ and of $\mathcal{M}_{g,X}$, as the infinite discrete group $\Gamma(S,X)$ would not act discontinuously on a compact boundary $\partial\mathcal{T}(S,X)$.

Thus, a $\Gamma(S,X)$-equivariant bordification of $\mathcal{T}(S,X)$ whose quotient is $\overline{\mathcal{M}}_{g,X}$ cannot be compact. A way to understand such a bordification is to endow $\mathcal{M}_{g,X}$ (and $\mathcal{T}(S,X)$) with the Weil-Petersson metric [70] and to show that its completion is exactly $\overline{\mathcal{M}}_{g,X}$ [48]. Hence, the Weil-Petersson completion $\overline{\mathcal{T}}(S,X)$ can be identified to the set of (S,X)-marked stable Riemann surfaces.

Similarly to $\overline{\mathcal{M}}_{g,X}$, also $\overline{\mathcal{T}}(S,X)$ has a stratification by topological type and each stratum is a (finite quotient of a) product of smaller Teichmüller spaces.

1.1.5 Tautological classes. The moduli space $\overline{\mathcal{M}}_{g,X}$ comes equipped with natural vector bundles: for instance, $\mathcal{L}_i$ is the holomorphic line bundle whose fiber at $[C]$ is the cotangent space T^*_{C,x_i}. Chern classes of these line bundles and their push-forward through tautological maps generate the so-called tautological classes (which can be seen in the Chow ring or in cohomology). The κ classes were defined by Mumford [57] and Morita [54] and then modified (to make them behave better under tautological maps) by Arbarello and Cornalba [5]. The ψ classes were defined by E.Miller [49] and their importance was successively rediscovered by Witten [71].

The importance of the tautological classes is due to the following facts (among others):

- their geometric meaning appears quite clear
- they behave very naturally under the tautological maps (see, for instance, [5])
- they often occur in computations of enumerative geometry; that is, Poincaré duals of interesting algebraic loci are often tautological (see [57]) but not always (see [24])!
- they are defined on $\overline{\mathcal{M}}_{g,X}$ for every g and X (provided $2g-2+|X|>0$), and they generate the stable cohomology ring over $\mathbb{Q}$ due to Madsen-Weiss's solution [47] of Mumford's conjecture (see Section 5.3)
- there is a set of generators (ψ's and κ's) which have non-negativity properties (see [4] and [57])
- they are strictly related to the Weil-Petersson geometry of $\overline{\mathcal{M}}_{g,X}$ (see [73], [76], [77] and [50]).

1.1.6 Simplicial complexes associated to a surface. One way to analyze the (co)homology of $\mathcal{M}_{g,X}$, and so of $\Gamma(S,X)$, is to construct a highly connected simplicial complex on which $\Gamma(S,X)$ acts. This is usually achieved by considering complexes of disjoint, pairwise non-homotopic simple closed curves on $S\setminus X$ with suitable properties (for instance, Harvey's complex of curves [29]).

If X is nonempty (or if S has boundary), then one can construct a complex using systems of homotopically nontrivial, disjoint arcs joining two (not necessarily distinct) points in X (or in ∂S), thus obtaining the arc complex $\mathfrak{A}(S,X)$ (see [27]). It has an "interior" $\mathfrak{A}^\circ(S,X)$ made of systems of arcs that cut $S\setminus X$ in discs (or pointed discs) and a complementary "boundary" $\mathfrak{A}^\infty(S,X)$.

An important result, which has many fathers (Harer-Mumford-Thurston [27], Penner [58], Bowditch-Epstein [12]), says that the topological realization $|\mathfrak{A}^\circ(S,X)|$ of $\mathfrak{A}^\circ(S,X)$ is $\Gamma(S,X)$-equivariantly homeomorphic to $\mathcal{T}(S,X)\times\Delta_X$ (where Δ_X is the standard simplex in $\mathbb{R}^X$). Thus, we can transfer the cell structure of $|\mathfrak{A}^\circ(S,X)|$ to an (orbi)cell structure on $\mathcal{M}_{g,X}\times\Delta_X$.

The homeomorphism is realized by coherently associating a weighted system of arcs to every X-marked Riemann surface, equipped with a decoration $\underline{p}\in\Delta_X$. There are two traditional ways to do this: using the flat structure arising from a Jenkins-Strebel quadratic differential (Harer-Mumford-Thurston) with prescribed residues at X or using the hyperbolic metric coming from the uniformization theorem (Penner and Bowditch-Epstein). Quite recently, several other ways have been introduced (see [44], [45], [53] and [52]).

1.1.7 Ribbon graphs. To better understand the homeomorphism between $|\mathfrak{A}^\circ(S,X)|$ and $\mathcal{T}(S,X)\times\Delta_X$, it is often convenient to adopt a dual point of view, that is to think of weighted systems of arcs as of metrized graphs $\mathbb{G}$, embedded in $S\setminus X$ through a homotopy equivalence.

This can be done by picking a vertex in each disc cut by the system of arcs and joining these vertices by adding an edge transverse to each arc. What we obtain is an (S,X)-marked metrized ribbon graph. Thus, points in $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)\cong\mathcal{M}_{g,x}\times\Delta_X$ correspond to metrized X-marked ribbon graphs of genus g.

This point of view is particularly useful to understand singular surfaces (see also [12], [41], [43], [62], [78], [7] and [52]). The object dual to a weighted system of arcs in $\mathfrak{A}^\infty(S,X)$ is a collection of data that we called an (S,X)-marked "enriched" ribbon graph. Notice that an X-marked "enriched" metrized ribbon graph does not carry all the information needed to construct a stable Riemann surface. Hence, the map $\overline{\mathcal{M}}_{g,X}\times\Delta_X\to|\mathfrak{A}(S,X)|/\Gamma(S,X)$ is not injective on the locus of singular curves, but still it is a homeomorphism on a dense open subset.

1.1.8 Topological results. The utility of the $\Gamma(S,X)$-equivariant homotopy equivalence $\mathcal{T}(S,X)\simeq|\mathfrak{A}^\circ(S,X)|$ is the possibility of making topological computations on $|\mathfrak{A}^\circ(S,X)|$. For instance, Harer [27] determined the virtual cohomological dimension of $\Gamma(S,X)$ (and so of $\mathcal{M}_{g,X}$) using the high connectivity of $|\mathfrak{A}^\infty(S,X)|$ and he has established that $\Gamma(S,X)$ is a virtual duality group, by showing that $|\mathfrak{A}^\infty(S,X)|$ is spherical. An analysis of the singularities of $|\mathfrak{A}(S,X)|/\Gamma(S,X)$ is in [63].

Successively, Harer-Zagier [28] and Penner [59] have computed the orbifold Euler characteristic of $\mathcal{M}_{g,X}$, where by "orbifold" we mean that a cell with stabilizer G has Euler characteristic $1/|G|$. Because of the cellularization, the problem translates into enumerating X-marked ribbon graphs of genus g and counting them with the correct sign.

Techniques for enumerating graphs and ribbon graphs (see, for instance, [9]) have been known to physicists for long time: they use asymptotic expansions of Gaussian integrals over spaces of matrices. The combinatorics of iterated integrations by parts is responsible for the appearance of (ribbon) graphs (Wick's lemma). Thus, the problem of computing $\chi^{orb}(\mathcal{M}_{g,X})$ can be reduced to evaluating a matrix integral (a quick solution is also given by Kontsevich in Appendix D of [41]).

1.1.9 Intersection-theoretical results. As $\mathcal{M}_{g,X}\times\Delta_X$ is not just homotopy equivalent to $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$ but actually homeomorphic (through a piecewise real-analytic diffeomorphism), it is clear that one can try to rephrase integrals over $\mathcal{M}_{g,X}$ as integrals over $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$, that is as sums over maximal systems of arcs of integrals over a single simplex. This approach looked promising in order to compute Weil-Petersson volumes (see Penner [60]). Kontsevich [41] used it to compute volumes coming from a "symplectic form" $\Omega = p_1^2\psi_1 + \cdots + p_n^2\psi_n$, thus solving Witten's conjecture [71] on the intersection numbers of the ψ classes.

However, in Witten's paper [71] matrix integrals entered in a different way. The idea was that, in order to integrate over the space of all conformal structures on S, one can pick a random decomposition of S into polygons, give each polygon a natural Euclidean structure and extend it to a conformal structure on S, thus obtaining a "random" point of $\mathcal{M}_{g,X}$. Refining the polygonalization of S leads to a measure on $\mathcal{M}_{g,X}$. Matrix integrals are used to enumerate these polygonalizations.

Witten also noticed that this refinement procedure may lead to different limits, depending on which polygons we allow. For instance, we can consider decompositions into A squares, or into A squares and B hexagons, and so on. Dualizing this last polygonalization, we obtain ribbon graphs embedded in S with A vertices of valence 4 and B vertices of valence 6. The corresponding locus in $|\mathfrak{A}^\circ(S,X)|$ is called a Witten subcomplex.

1.1.10 Witten classes. Kontsevich [41] and Penner [61] proved that Witten subcomplexes obtained by requiring that the ribbon graphs have m_i vertices of valence $(2m_i + 3)$ can be oriented (see also [13]) and they give cycles in $\overline{\mathcal{M}}^{comb}_{g,X} := |\mathfrak{A}(S,X)|/\Gamma(S,X) \times \mathbb{R}_+$, which are denoted by $\overline{W}_{m_*,X}$. The Ω-volumes of these $\overline{W}_{m_*,X}$ are also computable using matrix integrals [41] (see also [16]).

In [42], Kontsevich constructed similar cycles using structure constants of finite-dimensional cyclic A_∞-algebras with positive-definite scalar product and he also claimed that the classes $W_{m_*,X}$ (restriction of $\overline{W}_{m_*,X}$ to $\mathcal{M}_{g,X}$) are Poincaré dual to tautological classes.

This last statement (usually called Witten-Kontsevich's conjecture) was settled independently by Igusa [31] [32] and Mondello [51], whereas very little is known about the nature of the (non-homogeneous) A_∞-classes.

1.1.11 Surfaces with boundary. The key point of all constructions of a ribbon graph out of a surface is that X must be nonempty, so that $S \setminus X$ can be retracted by deformation onto a graph. In fact, it is not difficult to see that the spine construction of Penner and Bowditch-Epstein can be performed (even in a more natural way) on hyperbolic surfaces Σ with geodesic boundary. The associated cellularization of the corresponding moduli space is due to Luo [44] (for smooth surfaces) and by Mondello [52] (also for singular surfaces, using Luo's result).

The interesting fact (see [53] and [52]) is that gluing semi-infinite cylinders at $\partial\Sigma$ produces (conformally) punctured surfaces that "interpolate" between hyperbolic surfaces with cusps and flat surfaces arising from Jenkins-Strebel differentials.

1.2 Structure of the chapter.

In Sections 2.1 and 2.2, we carefully define systems of arcs and ribbon graphs, both in the singular and in the nonsingular case, and we explain how the duality between the two works. Moreover, we recall Harer's results on $\mathfrak{A}^{\circ}(S,X)$ and $\mathfrak{A}^{\infty}(S,X)$ and we state a simple criterion for compactness inside $|\mathfrak{A}^{\circ}(S,X)|/\Gamma(S,X)$.

In Sections 3.1 and 3.2, we describe the Deligne-Mumford moduli space of curves and the structure of its boundary, the associated stratification and boundary maps. In 3.3, we explain how the analogous bordification of the Teichmüller space $\overline{\mathcal{T}}(S,X)$ can be obtained as completion with respect to the Weil-Petersson metric.

Tautological classes and rings are introduced in 3.4 and Kontsevich's compactification of $\mathcal{M}_{g,X}$ is described in 3.5.

In 4.1, we explain and sketch a proof of Harer-Mumford-Thurston cellularization of the moduli space and we illustrate the analogous result of Penner-Bowditch-Epstein in 4.2. In 4.3, we quickly discuss the relations between the two constructions using hyperbolic surfaces with geodesic boundary.

In 5.1, we define Witten subcomplexes and Witten cycles and we prove (after Kontsevich) that Ω orients them. We sketch the ideas involved in the proof the Witten cycles are tautological in Section 5.2.

Finally, in 5.3, we recall Harer's stability theorem and we exhibit a combinatorial construction that shows that Witten cycles are stable. The fact (and probably also the construction) is well-known and it is also a direct consequence of Witten-Kontsevich's conjecture and Miller's work.

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2 Systems of arcs and ribbon graphs

Let S be a compact oriented differentiable surface of genus g with $n > 0$ distinct marked points $X = \{x_1, \dots, x_n\} \subset S$. We will always assume that the Euler characteristic of the punctured surface $S \setminus X$ is negative, that is $2 - 2g - n < 0$. This restriction only rules out the cases in which $S \setminus X$ is the sphere with less than 3 punctures.

Let $\mathrm{Diff}_+(S, X)$ be the group of orientation-preserving diffeomorphisms of S that fix X pointwise. The *mapping class group* $\Gamma(S, X)$ is the group of connected components of $\mathrm{Diff}_+(S, X)$.

In what follows, we borrow some notation and some ideas from [43].

2.1 Systems of arcs

2.1.1 Arcs and arc complex. An oriented *arc* in S is a smooth path $\overrightarrow{\alpha} : [0, 1] \to S$ such that $\overrightarrow{\alpha}([0, 1]) \cap X = \{\overrightarrow{\alpha}(0), \overrightarrow{\alpha}(1)\}$, up to reparametrization. Let $\mathcal{A}^{or}(S, X)$ be the space of oriented arcs in S, endowed with its natural topology. Define $\sigma_1 : \mathcal{A}^{or}(S, X) \to \mathcal{A}^{or}(S, X)$ to be the orientation-reversing operator and we will write $\sigma_1(\overrightarrow{\alpha}) = \overleftarrow{\alpha}$. Call α the σ_1-orbit of $\overrightarrow{\alpha}$ and denote by $\mathcal{A}(S, X)$ the (quotient) space of σ_1-orbits in $\mathcal{A}^{or}(S, X)$.

A *system of* $(k+1)$*-arcs* in S is a collection $\underline{\boldsymbol{\alpha}} = \{\alpha_0, \dots, \alpha_k\} \subset \mathcal{A}(S, X)$ of $k + 1$ unoriented arcs such that:

- if $i \neq j$, then the intersection of α_i and α_j is contained in X
- no arc in $\underline{\boldsymbol{\alpha}}$ is homotopically trivial
- no pair of arcs in $\underline{\boldsymbol{\alpha}}$ are homotopic to each other.

We will denote by $S \setminus \underline{\boldsymbol{\alpha}}$ the *complementary subsurface* of S obtained by removing $\alpha_0, \dots, \alpha_k$.

Each connected component of the space of systems of $(k+1)$-arcs $\mathcal{AS}_k(S, X)$ is clearly contractible, with the topology induced by the inclusion $\mathcal{AS}_k(S, X) \hookrightarrow \mathcal{A}(S, X)/\mathfrak{S}_k$.

Let $\mathfrak{A}_k(S, X)$ be the set of homotopy classes of systems of $k + 1$ arcs, that is $\mathfrak{A}_k(S, X) := \pi_0 \mathcal{AS}_k(S, X)$.

The *arc complex* is the simplicial complex $\mathfrak{A}(S, X) = \bigcup_{k \geq 0} \mathfrak{A}_k(S, X)$.

Notation. We will implicitly identify arc systems $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\alpha}}'$ that are homotopic to each other. Similarly, we will identify the isotopic subsurfaces $S \setminus \underline{\boldsymbol{\alpha}}$ and $S \setminus \underline{\boldsymbol{\alpha}}'$.

2.1.2 Proper simplices. An arc system $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}(S, X)$ *fills* (resp. *quasi-fills*) S if $S \setminus \underline{\boldsymbol{\alpha}}$ is a disjoint union of subsurfaces homeomorphic to discs (resp. discs and pointed discs). It is easy to check that the star of $\underline{\boldsymbol{\alpha}}$ is finite if and only if $\underline{\boldsymbol{\alpha}}$ quasi-fills S. In this case, we also say that $\underline{\boldsymbol{\alpha}}$ is a *proper* simplex of $\mathfrak{A}(S,X)$.

Denote by $\mathfrak{A}^\infty(S,X) \subset \mathfrak{A}(S,X)$ the subcomplex of non-proper simplices and let $\mathfrak{A}^\circ(S,X) = \mathfrak{A}(S,X) \setminus \mathfrak{A}^\infty(S,X)$ be the collection of proper ones.

Notation. We denote by $|\mathfrak{A}^\infty(S,X)|$ and $|\mathfrak{A}(S,X)|$ the topological realizations of $\mathfrak{A}^\infty(S,X)$ and $\mathfrak{A}(S,X)$. We will use the symbol $|\mathfrak{A}^\circ(S,X)|$ to mean the complement of $|\mathfrak{A}^\infty(S,X)|$ inside $|\mathfrak{A}(S,X)|$.

2.1.3 Topologies on $|\mathfrak{A}(S,X)|$. The realization $|\mathfrak{A}(S,X)|$ of the arc complex can be endowed with two natural topologies (as is remarked in [12], [43] and [7]).

The former (which we call *standard*) is the finest topology that makes the inclusions $|\underline{\boldsymbol{\alpha}}| \hookrightarrow |\mathfrak{A}(S,X)|$ continuous for all $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}(S,X)$; in other words, a subset $U \subset |\mathfrak{A}(S,X)|$ is declared to be open if and only if $U \cap |\underline{\boldsymbol{\alpha}}|$ is open for every $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}(S,X)$. The latter topology is induced by the path *metric* d, which is the largest metric that restricts to the Euclidean one on each closed simplex.

The two topologies are the same where $|\mathfrak{A}(S,X)|$ is locally finite, but the latter is coarser elsewhere. We will always consider all realizations to be endowed with the metric topology.

2.1.4 Visible subsurfaces. For every system of arcs $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}(S,X)$, define $S(\underline{\boldsymbol{\alpha}})_+$ to be the largest isotopy class of open subsurfaces of S such that

- every arc in $\underline{\boldsymbol{\alpha}}$ is contained in $S(\underline{\boldsymbol{\alpha}})_+$
- $\underline{\boldsymbol{\alpha}}$ quasi-fills $S(\underline{\boldsymbol{\alpha}})_+$.

The *visible subsurface* $S(\underline{\boldsymbol{\alpha}})_+$ can be constructed by taking the union of a thickening a representative of $\underline{\boldsymbol{\alpha}}$ inside S and all those connected components of $S \setminus \underline{\boldsymbol{\alpha}}$ which are homeomorphic to discs or punctured discs (this construction appears already in [12]). We will always consider $S(\underline{\boldsymbol{\alpha}})_+$ as an open subsurface (up to isotopy), homotopically equivalent to its closure $\overline{S(\underline{\boldsymbol{\alpha}})_+}$, which is an embedded surface with boundary.

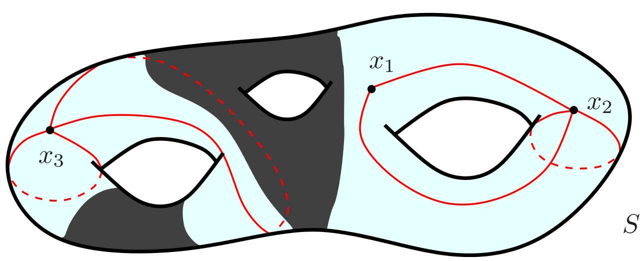

Figure 1. The invisible subsurface is the dark non-cylindrical component.

One can rephrase 2.1.2 by saying that $\underline{\boldsymbol{\alpha}}$ is proper if and only if all S is $\underline{\boldsymbol{\alpha}}$-visible.

We call *invisible subsurface* $S(\underline{\boldsymbol{\alpha}})_-$ associated to $\underline{\boldsymbol{\alpha}}$ the union of the interior of the connected components of $S \setminus \overline{S(\underline{\boldsymbol{\alpha}})_+}$ which are not unmarked cylinders. Thus, $S \setminus (S(\underline{\boldsymbol{\alpha}})_+ \cup S(\underline{\boldsymbol{\alpha}})_-)$ is a disjoint union of circles and cylinders.

We also say that a marked point x_i is (in)visible for $\underline{\boldsymbol{\alpha}}$ if it belongs to the $\underline{\boldsymbol{\alpha}}$-(in)visible subsurface.

2.1.5 Ideal triangulations. A maximal system of arcs $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}(S, X)$ is also called an *ideal triangulation* of S. In fact, it is easy to check that, in this case, each component of $S \setminus \underline{\boldsymbol{\alpha}}$ bounded by three arcs and so is a "triangle". (The term "ideal" comes from the fact that one often thinks of (S, X) as a hyperbolic surface with cusps at X and of $\underline{\boldsymbol{\alpha}}$ as a collection of hyperbolic geodesics.) It is also clear that such an $\underline{\boldsymbol{\alpha}}$ is proper.

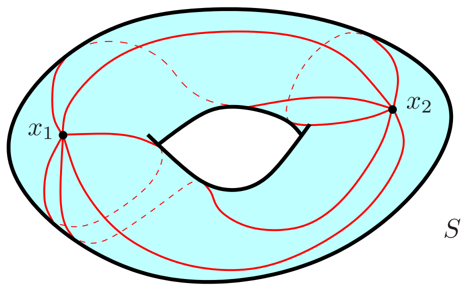

Figure 2. An example of an ideal triangulation for $(g, n) = (1, 2)$.

A simple calculation with the Euler characteristic of S shows that an ideal triangulation is made of exactly $6g - 6 + 3n$ arcs.

2.1.6 The spine of $|\mathfrak{A}^\circ(S, X)|$. Consider the baricentric subdivision $\mathfrak{A}(S, X)'$, whose k-simplices are chains $(\underline{\boldsymbol{\alpha}}_0 \subsetneq \underline{\boldsymbol{\alpha}}_1 \subsetneq \cdots \subsetneq \underline{\boldsymbol{\alpha}}_k)$. There is an obvious piecewise-affine homeomorphism $|\mathfrak{A}(S, X)'| \to |\mathfrak{A}(S, X)|$, that sends a vertex $(\underline{\boldsymbol{\alpha}}_0)$ to the baricenter of $|\underline{\boldsymbol{\alpha}}_0| \subset |\mathfrak{A}(S, X)|$.

Call $\mathfrak{A}^\circ(S, X)'$ the subcomplex of $\mathfrak{A}(S, X)'$, whose simplices are chains of simplices that belong to $\mathfrak{A}^\circ(S, X)$. Clearly, $|\mathfrak{A}^\circ(S, X)'| \subset |\mathfrak{A}(S, X)'|$ is contained in $|\mathfrak{A}^\circ(S, X)| \subset |\mathfrak{A}(S, X)|$ through the homeomorphism above.

It is a general fact that there is a deformation retraction of $|\mathfrak{A}^\circ(S, X)|$ onto the spine $|\mathfrak{A}^\circ(S, X)'|$: on each simplex of $|\mathfrak{A}(S, X)'| \cap |\mathfrak{A}^\circ(S, X)|$ this is given by projecting onto the face contained in $|\mathfrak{A}^\circ(S, X)'|$. It is also clear that the retraction is $\Gamma(S, X)$-equivariant.

In the special case of $X = \{x_1\}$, a proper system contains at least $2g$ arcs; whereas a maximal system contains exactly $6g - 3$ arcs. Thus, the (real) dimension of $|\mathfrak{A}^\circ(S, X)'|$ is $(6g - 3) - 2g = 4g - 3$.

Proposition 2.1 (Harer [27]). *If $X = \{x_1\}$, the spine $|\mathfrak{A}^\circ(S,X)'|$ has dimension $4g-3$.*

2.1.7 Action of $\boldsymbol{\sigma}$-operators. For every arc system $\underline{\boldsymbol{\alpha}} = \{\alpha_0, \dots, \alpha_k\}$, denote by $E(\underline{\boldsymbol{\alpha}})$ the subset $\{\overrightarrow{\alpha_0}, \overleftarrow{\alpha_0}, \dots, \overrightarrow{\alpha_k}, \overleftarrow{\alpha_k}\}$ of $\pi_0\mathcal{A}^{or}(S,X)$. The action of σ_1 clearly restricts to $E(\underline{\boldsymbol{\alpha}})$.

For each $i = 1, \dots, n$, the orientation of S induces a cyclic ordering of the oriented arcs in $E(\underline{\boldsymbol{\alpha}})$ outgoing from x_i.

If $\overrightarrow{\alpha_j}$ starts at x_i, then define $\sigma_\infty(\overrightarrow{\alpha_j})$ to be the oriented arc in $E(\underline{\boldsymbol{\alpha}})$ outgoing from x_i that comes just *before* $\overrightarrow{\alpha_j}$. Moreover, σ_0 is defined by $\sigma_0 = \sigma_\infty^{-1}\sigma_1$.

If we call $E_t(\underline{\boldsymbol{\alpha}})$ the orbits of $E(\underline{\boldsymbol{\alpha}})$ under the action of σ_t, then

- $E_1(\underline{\boldsymbol{\alpha}})$ can be identified with $\underline{\boldsymbol{\alpha}}$
- $E_\infty(\underline{\boldsymbol{\alpha}})$ can be identified with the set of $\underline{\boldsymbol{\alpha}}$-visible marked points
- $E_0(\underline{\boldsymbol{\alpha}})$ can be identified to the set of connected components of $S(\underline{\boldsymbol{\alpha}})_+ \setminus \underline{\boldsymbol{\alpha}}$.

Denote by $[\overrightarrow{\alpha_j}]_t$ the σ_t-orbit of $\overrightarrow{\alpha_j}$, so that $[\overrightarrow{\alpha_j}]_1 = \alpha_j$ and $[\overrightarrow{\alpha_j}]_\infty$ is the starting point of $\overrightarrow{\alpha_j}$, whereas $[\overrightarrow{\alpha_j}]_0$ is the component of $S(\underline{\boldsymbol{\alpha}})_+ \setminus \underline{\boldsymbol{\alpha}}$ adjacent to α_j and which induces the orientation $\overrightarrow{\alpha_j}$ on it.

2.1.8 Action of $\boldsymbol{\Gamma(S,X)}$ on $\boldsymbol{\mathfrak{A}(S,X)}$. There is a natural right action of the mapping class group

$$\begin{aligned} \mathcal{A}(S,X) \times \Gamma(S,X) &\longrightarrow \mathcal{A}(S,X) \\ (\alpha, g) &\longmapsto \alpha \circ g \end{aligned}$$

The induced action on $\mathfrak{A}(S,X)$ preserves $\mathfrak{A}^\infty(S,X)$ and so $\mathfrak{A}^\circ(S,X)$.

It is easy to see that the stabilizer (under $\Gamma(S,X)$) of a simplex $\underline{\boldsymbol{\alpha}}$ fits in the following exact sequence

$$1 \to \Gamma_{cpt}(S \setminus \underline{\boldsymbol{\alpha}}, X) \to \mathrm{stab}_\Gamma(\underline{\boldsymbol{\alpha}}) \to \mathfrak{S}(\underline{\boldsymbol{\alpha}})$$

where $\mathfrak{S}(\underline{\boldsymbol{\alpha}})$ is the group of permutations of $\underline{\boldsymbol{\alpha}}$ and $\Gamma_{cpt}(S \setminus \underline{\boldsymbol{\alpha}}, X)$ is the mapping class group of orientation-preserving diffeomorphisms of $S \setminus \underline{\boldsymbol{\alpha}}$ *with compact support* that fix X. Define the image of $\mathrm{stab}_\Gamma(\underline{\boldsymbol{\alpha}}) \to \mathfrak{S}(\underline{\boldsymbol{\alpha}})$ to be the *automorphism group of* $\underline{\boldsymbol{\alpha}}$.

We can immediately conclude that $\underline{\boldsymbol{\alpha}}$ is proper if and only if $\mathrm{stab}_\Gamma(\underline{\boldsymbol{\alpha}})$ is finite (equivalently, if and only if $\Gamma_{cpt}(S \setminus \underline{\boldsymbol{\alpha}}, X)$ is trivial).

2.1.9 Weighted arc systems. A point $\overline{w} \in |\mathfrak{A}(S,X)|$ consists of a map $\overline{w} : \mathfrak{A}_0(S,X) \to [0,1]$ such that

- the support of $\overline{w}$ is a simplex $\underline{\boldsymbol{\alpha}} = \{\alpha_0, \dots, \alpha_k\} \in \mathfrak{A}(S,X)$
- $\displaystyle\sum_{i=0}^{k} \overline{w}(\alpha_i) = 1.$

We will call $\overline{w}$ the *(projective) weight* of $\underline{\boldsymbol{\alpha}}$. A *weight* for $\underline{\boldsymbol{\alpha}}$ is a point of $w \in |\mathfrak{A}(S,X)|_{\mathbb{R}} := |\mathfrak{A}(S,X)| \times \mathbb{R}_+$, that is a map $w : \mathfrak{A}_0(S,X) \to \mathbb{R}_+$ with support on $\underline{\boldsymbol{\alpha}}$. Call $\overline{w}$ its associated projective weight.

2.1.10 Compactness in $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$. We are going to prove a simple criterion for a subset of $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$ to be compact.

Call $\mathcal{C}(S,X)$ the set of free homotopy classes of simple closed curves on $S \setminus X$, which are neither contractible nor homotopic to a puncture.

Define the "intersection product"

$$\iota : \mathcal{C}(S,X) \times |\mathfrak{A}(S,X)| \to \mathbb{R}_{\geq 0}$$

as $\iota(\gamma,\overline{w}) = \sum_\alpha \iota(\gamma,\alpha)\overline{w}(\alpha)$, where $\iota(\gamma,\alpha)$ is the *geometric* intersection number. We will also refer to $\iota(\gamma,\overline{w})$ as to the *length* of γ at $\overline{w}$. Consequently, we will say that the *systol* at $\overline{w}$ is

$$\mathrm{sys}(\overline{w}) = \inf\{\iota(\gamma,\overline{w}) \mid \gamma \in \mathcal{C}(S,X)\}.$$

Clearly, the function sys descends to

$$\mathrm{sys} : |\mathfrak{A}(S,X)|/\Gamma(S,X) \to \mathbb{R}_+$$

Lemma 2.2. *A closed subset $K \subset |\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$ is compact if and only if $\exists \varepsilon > 0$ such that $\mathrm{sys}([\overline{w}]) \geq \varepsilon$ for all $[\overline{w}] \in K$.*

Proof. In $\mathbb{R}^N$ we easily have $d_2 \leq d_1 \leq \sqrt{N} \cdot d_2$, where d_r is the L^r-distance. Similarly, in $|\mathfrak{A}(S,X)|$ we have

$$d(\overline{w}, |\mathfrak{A}^\infty(S,X)|) \leq \mathrm{sys}(\overline{w}) \leq \sqrt{N} \cdot d(\overline{w}, |\mathfrak{A}^\infty(S,X)|)$$

where $N = 6g - 7 + 3n$. The same holds in $|\mathfrak{A}(S,X)|/\Gamma(S,X)$.

Thus, if $[\underline{\boldsymbol{\alpha}}] \in \mathfrak{A}^\circ(S,X)/\Gamma(S,X)$, then $|\underline{\boldsymbol{\alpha}}| \cap \mathrm{sys}^{-1}([\varepsilon,\infty)) \cap |\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$ is compact for every $\varepsilon > 0$. As $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$ contains finitely many cells, we conclude that $\mathrm{sys}^{-1}([\varepsilon,\infty)) \cap |\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$ is compact.

Vice versa, if $\mathrm{sys} : K \to \mathbb{R}_+$ is not bounded from below, then we can find a sequence $[\overline{w}_m] \subset K$ such that $\mathrm{sys}(\overline{w}_m) \to 0$. Thus, $[\overline{w}_m]$ approaches $|\mathfrak{A}^\infty(S,X)|/\Gamma(S,X)$ and so is divergent in $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$. □

2.1.11 Boundary weight map. Let Δ_X be the standard simplex in $\mathbb{R}^X$. The *boundary weight map* $\ell_\partial : |\mathfrak{A}(S,X)|_{\mathbb{R}} \to \Delta_X \times \mathbb{R}_+ \subset \mathbb{R}^X$ is the piecewise-linear map that sends $\{\alpha\} \mapsto [\overrightarrow{\alpha}]_\infty + [\overleftarrow{\alpha}]_\infty$. The projective boundary weight map $\frac{1}{2}\ell_\partial : |\mathfrak{A}(S,X)| \to \Delta_X$ instead sends $\{\alpha\} \mapsto \frac{1}{2}[\overrightarrow{\alpha}]_\infty + \frac{1}{2}[\overleftarrow{\alpha}]_\infty$.

2.1.12 Results on the arc complex. A few things are known about the topology of $|\mathfrak{A}(S,X)|$.

(a) The space of proper arc systems $|\mathfrak{A}^\circ(S,X)|$ can be naturally given the structure of piecewise-affine topological manifold with boundary (Hubbard-Masur [30], credited to Whitney) of (real) dimension $6g-7+3n$.

(b) The space $|\mathfrak{A}^\circ(S,X)|$ is $\Gamma(S,X)$-equivariantly homeomorphic to $\mathcal{T}(S,X)\times\Delta_X$, where $\mathcal{T}(S,X)$ is the Teichmüller space of (S,X) (see 3.1.1 for definitions and Section 4 for an extensive discussion on this result), and so is contractible. This result could also be probably extracted from [30], but it is more explicitly stated in Harer [27] (who attributes it to Mumford and Thurston), Penner [58] and Bowditch-Epstein [12]. As the moduli space of X-marked Riemann surfaces of genus g can be obtained as $\mathcal{M}_{g,X}\cong\mathcal{T}(S,X)/\Gamma(S,X)$ (see 3.1.2), then $\mathcal{M}_{g,X}\simeq B\Gamma(S,X)$ in the orbifold category.

(c) The space $|\mathfrak{A}^\infty(S,X)|$ is homotopy equivalent to an infinite wedge of spheres of dimension $2g-3+n$ (Harer [27]).

Results (b) and (c) are the key step in the following.

Theorem 2.3 (Harer [27]). $\Gamma(S,X)$ *is a virtual duality group (that is, it has a subgroup of finite index which is a duality group) of dimension* $4g-4+n$ *for* $n>0$ *(and* $4g-5$ *for* $n=0$*).*

Actually, it is sufficient to work with $X=\{x_1\}$, in which case the upper bound is given by (b) and Proposition 2.1, and the duality by (c).

2.2 Ribbon graphs

2.2.1 Graphs. A *graph* G is a triple $(E,\sim,\sigma_1)$, where E is a finite set, $\sigma_1:E\rightarrow E$ is a fixed-point-free involution and $\sim$ is an equivalence relation on E.

In ordinary language

- E is the set of *oriented edges* of the graph
- σ_1 is the orientation-reversing involution of E, so that the set of unoriented edges is $E_1:=E/\sigma_1$
- two oriented edges are equivalent if and only if they come out from the same vertex, so that the set V of vertices is $E/\sim$ and the valence of $v\in E/\sim$ is exactly $|v|$.

A *ribbon graph* $\mathbb{G}$ is a triple (E,σ_0,σ_1), where E is a (finite) set, $\sigma_1:E\rightarrow E$ is a fixed-point-free involution and $\sigma_1:E\rightarrow E$ is a permutation. Define $\sigma_\infty:=\sigma_1\circ\sigma_0^{-1}$ and call E_t the set of orbits of σ_t and $[\cdot]_t:E\rightarrow E_t$ the natural projection. A disjoint union of two ribbon graphs is defined in the natural way.

Remark 2.4. Given a ribbon graph $\mathbb{G}$, the underlying ordinary graph $G = \mathbb{G}^{ord}$ is obtained by declaring that oriented edges in the same σ_0-orbit are equivalent and forgetting about the precise action of σ_0.

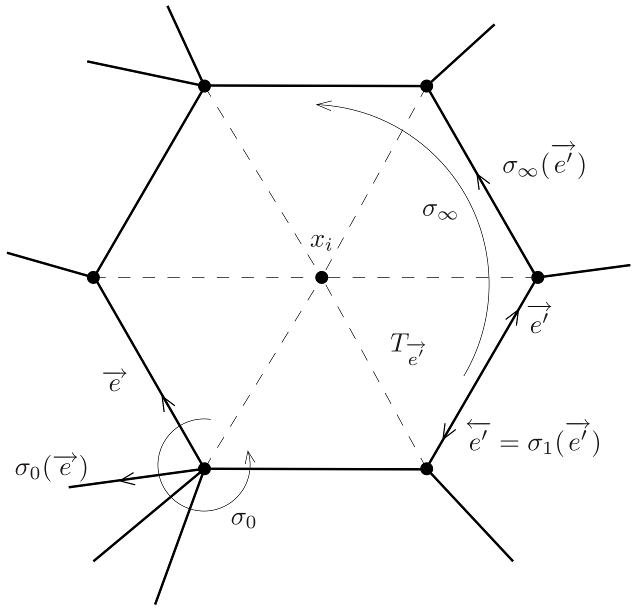

Figure 3. Geometric representation of a ribbon graph

In ordinary language, a ribbon graph is an ordinary graph endowed with a cyclic ordering of the oriented edges outgoing from each vertex.

The σ_∞-orbits are sometimes called *holes*. A *connected component* of $\mathbb{G}$ is an orbit of $E(\mathbb{G})$ under the action of $\langle\sigma_0, \sigma_1\rangle$.

The *Euler characteristic* of a ribbon graph $\mathbb{G}$ is $\chi(\mathbb{G}) = |E_0(\mathbb{G})| - |E_1(\mathbb{G})|$ and its *genus* is $g(\mathbb{G}) = 1 + \frac{1}{2}(|E_1(\mathbb{G})| - |E_0(\mathbb{G})| - |E_\infty(\mathbb{G})|)$.

A *(ribbon) tree* is a connected (ribbon) graph of genus zero with one hole.

2.2.2 Subgraphs and quotients. Let $\mathbb{G} = (E, \sigma_0, \sigma_1)$ be a ribbon graph and let $Z \subsetneq E_1$ be a nonempty subset of edges.

The *subgraph* $\mathbb{G}_Z$ is given by $(\tilde{Z}, \sigma_0^Z, \sigma_1^Z)$, where $\tilde{Z} = Z \times_{E_1} E$ and σ_0^Z, σ_1^Z are the induced operators (that is, for every $e \in \tilde{Z}$ we define $\sigma_0^Z(e) = \sigma_0^k(e)$, where $k = \min\{k > 0 \,|\, \sigma_0^k(e) \in \tilde{Z}\}$).

Similarly, the *quotient* $\mathbb{G}/Z$ is $(\mathbb{G} \setminus \tilde{Z}, \sigma_0^{Z^c}, \sigma_1^{Z^c})$, where $\sigma_1^{Z^c}$ and $\sigma_\infty^{Z^c}$ are the operators induced on $E \setminus \tilde{Z}$ and $\sigma_0^{Z^c}$ is defined accordingly. A *new vertex* of $\mathbb{G}/Z$ is a $\sigma_0^{Z^c}$-orbit of $E \setminus \tilde{Z} \hookrightarrow \mathbb{G}$, which is not a σ_0-orbit.

2.2.3 Bicolored graphs. A *bicolored graph* ζ is a finite connected graph with a partition $V = V_+ \cup V_-$ of its vertices. We say that ζ is *reduced* if no two vertices of V_- are adjacent. If not differently specified, we will always understand that bicolored graphs are reduced.

If ζ contains an edge z that joins $w_1, w_2 \in V_-$, then we can obtain a new graph ζ' *merging* w_1 and w_2 along z into a new vertex $w' \in V'_-$ (by simply forgetting $\vec{z}$ and $\overleftarrow{z}$ and by declaring that vertices outgoing from w_1 are equivalent to vertices outgoing from w_2).

If ζ comes equipped with a function $g : V_- \to \mathbb{N}$, then $g' : V'_- \to \mathbb{N}$ is defined so that $g'(w') = g(w_1) + g(w_2)$ if $w_1 \neq w_2$, or $g'(w') = g(w_1) + 1$ if $w_1 = w_2$.

As merging reduces the number of edges, we can iterate the process only a finite number of times. The result is independent of the choice of which edges to merge first and is a reduced graph ζ^{red} (possibly with a g^{red}).

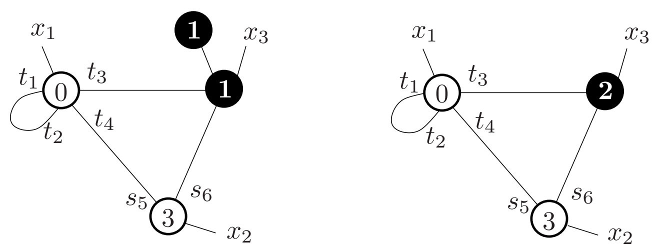

Figure 4. A non-reduced bicolored graph (on the left) and its reduction (on the right). Vertices in V_- are black. See Example 2.5.

2.2.4 Enriched ribbon graphs. An *enriched X-marked ribbon graph* $\mathbb{G}^{en}$ is the datum of

- a connected bicolored graph (ζ, V_+)
- a ribbon graph $\mathbb{G}$ plus a bijection $V_+ \to \{\text{connected components of } \mathbb{G}\}$
- an (invisible) genus function $g : V_- \to \mathbb{N}$
- a X-marking map $m : X \to V_- \cup E_\infty(\mathbb{G}) \cup E_0(\mathbb{G})$ such that the restriction $m^{-1}(E_\infty(\mathbb{G})) \to E_\infty(\mathbb{G})$ is bijective and the restriction $m^{-1}(E_0(\mathbb{G})) \to E_0(\mathbb{G})$ is injective (a vertex in the image of this last map is called *marked*)
- an injection $s_v : \{\text{oriented edges of } \zeta \text{ outgoing from } v\} \to E_0(\mathbb{G}_v)$ (vertices of $\mathbb{G}_v$ in the image of s_v are called *nodal*; a vertex is called *special* if it is either marked or nodal)

that satisfy the following properties:

- for every $v \in V_+$ and $y \in E_0(\mathbb{G}_v)$ we have $|m^{-1}(y) \cup s_v^{-1}(y)| \leq 1$ (i.e. no more than one marking or one node at each vertex of $\mathbb{G}_v$)
- $2g(v) - 2 + |\{\text{oriented edges of } \zeta \text{ outgoing from } v\}| +$
  $+ |\{\text{marked points on } v\}| > 0$ for every $v \in V$ (stability condition)

- every non-special vertex of $\mathbb{G}_v$ must be at least trivalent for all $v \in V_+$.

We say that $\mathbb{G}^{en}$ is *reduced* if ζ is.

If the graph ζ is not reduced, then we can merge two vertices of ζ along an edge of ζ and obtain a new enriched X-marked ribbon graph. $\mathbb{G}_1^{en}$ and $\mathbb{G}_2^{en}$ are considered equivalent if they are related by a sequence of merging operations. It is clear that each equivalence class can be identified to its reduced representative. Unless differently specified, we will always refer to an enriched graph as the canonical reduced representative.

The total *genus* of $\mathbb{G}^{en}$ is $g(\mathbb{G}^{en}) = 1-\chi(\zeta)+\sum_{v\in V_+} g(\mathbb{G}_v)+\sum_{w\in V_-} g(w)$.

Example 2.5. In Figure 4, the genus of each vertex is written inside, x_1 and x_2 are marking the two holes of $\mathbb{G}$ (sitting in different components), whereas x_3 is an invisible marked point. Moreover, t_1, t_2, t_3, t_4 (resp. s_5, s_6) are distinct (special) vertices of the visible component of genus 0 (resp. of genus 3): in particular, t_4 is marking the oriented edge that goes from the genus 0 component to the genus 3 component, whereas s_5 is marking the same edge with the opposite orientation. (Note that, if x_i marked a vertex s of some visible component, then we would have written "s" close the tail that joined v to s.) The total genus of the associated $\mathbb{G}^{en}$ is 7.

Remark 2.6. If an edge z of ζ joins $v \in V_+$ and $w \in V_-$ and this edge is marked by the special vertex $y \in E_0(\mathbb{G}_v)$, then we will say, for brevity, that z joins w and y.

An enriched X-marked ribbon graph is *nonsingular* if ζ consists of a single visible vertex. Equivalently, an enriched nonsingular X-marked ribbon graph consists of a connected ribbon graph $\mathbb{G}$ together with an injection $X \hookrightarrow E_\infty(\mathbb{G}) \cup E_0(\mathbb{G})$, whose image is exactly $E_\infty(\mathbb{G}) \cup \{\text{special vertices}\}$, such that non-special vertices are at least trivalent and $\chi(\mathbb{G}) - |\{\text{marked vertices}\}| < 0$.

2.2.5 Category of nonsingular ribbon graphs. A *morphism* of nonsingular X-marked ribbon graphs $\mathbb{G}_1 \to \mathbb{G}_2$ is an injective map $f : E(\mathbb{G}_2) \hookrightarrow E(\mathbb{G}_1)$ such that

- f commutes with σ_1, σ_∞ and respects the X-marking
- $\mathbb{G}_{1,Z}$ is a disjoint union of trees, where $Z = E_1(\mathbb{G}_1) \setminus E_1(\mathbb{G}_2)$.

Notice that, as f preserves the X-markings (which are *injections* $X \hookrightarrow E_\infty(\mathbb{G}_i) \cup E_0(\mathbb{G}_i)$), then each component of Z may contain at most one special vertex.

Vice versa, if $\mathbb{G}$ is a nonsingular X-marked ribbon graph and $\emptyset \neq Z \subsetneq E_1(\mathbb{G})$ such that $\mathbb{G}_Z$ is a disjoint union of trees (each one containing at most a special vertex), then the inclusion $f : E_1(\mathbb{G}) \setminus \tilde{Z} \hookrightarrow E_1(\mathbb{G})$ induces a morphism of nonsingular ribbon graphs $\mathbb{G} \to \mathbb{G}/Z$.

Remark 2.7. A morphism is an *isomorphism* if and only if f is bijective.

$\mathfrak{RG}_{X,ns}$ is the small category whose objects are nonsingular X-marked ribbon graphs $\mathbb{G}$ (where we assume that $E(\mathbb{G})$ is contained in a fixed countable set) with the morphisms defined above. We use the symbol $\mathfrak{RG}_{g,X,ns}$ to denote the full subcategory of ribbon graphs of genus g.

2.2.6 Topological realization of nonsingular ribbon graphs. The *topological realization* $|G|$ of the graph $G = (E, \sim, \sigma_1)$ is the one-dimensional CW-complex obtained from $I \times E$ (where $I = [0, 1]$) by identifying

- $(t, \overrightarrow{e}) \sim (1 - t, \overleftarrow{e})$ for all $t \in I$ and $\overrightarrow{e} \in E$
- $(0, \overrightarrow{e}) \sim (0, \overrightarrow{e'})$ whenever $e \sim e'$.

The *topological realization* $|\mathbb{G}|$ of the nonsingular X-marked ribbon graph $\mathbb{G} = (E, \sigma_0, \sigma_1)$ is the oriented surface obtained from $T \times E$ (where $T = I \times [0, \infty]/I \times \{\infty\}$) by identifying

- $(t, 0, \overrightarrow{e}) \sim (1 - t, 0, \overleftarrow{e})$ for all $\overrightarrow{e} \in E$
- $(1, y, \overrightarrow{e}) \sim (0, y, s_\infty(\overrightarrow{e}))$ for all $\overrightarrow{e} \in E$ and $y \in [0, \infty]$.

If G is the ordinary graph underlying $\mathbb{G}$, then there is a natural embedding $|G| \hookrightarrow |\mathbb{G}|$, which we call the *spine*.

The points at infinity in $|\mathbb{G}|$ are called *centers* of the holes and can be identified to $E_\infty(\mathbb{G})$. Thus, $|\mathbb{G}|$ is naturally an X-marked surface.

Notice that a morphism of nonsingular X-marked ribbon graphs $\mathbb{G}_1 \to \mathbb{G}_2$ induces an isotopy class of orientation-preserving homeomorphisms $|\mathbb{G}_1| \to |\mathbb{G}_2|$ that respect the X-marking.

2.2.7 Nonsingular (S, X)-markings. An (S, X)*-marking* of the nonsingular X-marked ribbon graph $\mathbb{G}$ is an isotopy class of orientation-preserving homeomorphisms $f : S \to |\mathbb{G}|$, compatible with $X \hookrightarrow E_\infty(\mathbb{G}) \cup E_0(\mathbb{G})$.

Define $\mathfrak{RG}_{ns}(S, X)$ to be the category whose objects are (S, X)-marked nonsingular ribbon graphs $(\mathbb{G}, f)$ and whose morphisms $(\mathbb{G}_1, f_1) \to (\mathbb{G}_2, f_2)$ are morphisms $\mathbb{G}_1 \to \mathbb{G}_2$ such that $S \xrightarrow{f_1} |\mathbb{G}_1| \to |\mathbb{G}_2|$ is isotopic to $f_2 : S \to |\mathbb{G}_2|$.

As usual, there is a right action of the mapping class group $\Gamma(S, X)$ on $\mathfrak{RG}_{ns}(S, X)$ and the quotient category $\mathfrak{RG}_{ns}(S, X)/\Gamma(S, X)$ is obtained from $\mathfrak{RG}_{ns}(S, X)$ by adding an (iso)morphism $[f : S \to \mathbb{G}] \to [f \circ g : S \to \mathbb{G}]$ for each $g \in \Gamma(S, X)$ and each object $[f : S \to \mathbb{G}]$. It can be shown that the functor $\mathfrak{RG}_{ns}(S, X)/\Gamma(S, X) \to \mathfrak{RG}_{g,X,ns}$ that forgets the S-marking is an equivalence.

2.2.8 Nonsingular arcs/graph duality. Let $\underline{\boldsymbol{\alpha}} = \{\alpha_0, \ldots, \alpha_k\} \in \mathfrak{A}^\circ(S, X)$ be a proper arc system and let $\sigma_0, \sigma_1, \sigma_\infty$ the corresponding operators on the set of oriented arcs $E(\underline{\boldsymbol{\alpha}})$. The *ribbon graph dual to* $\underline{\boldsymbol{\alpha}}$ is $\mathbb{G}_{\underline{\boldsymbol{\alpha}}} = (E(\underline{\boldsymbol{\alpha}}), \sigma_0, \sigma_1)$, which comes naturally equipped with an X-marking (see 2.1.7).

Define the (S,X)-marking $f : S \to |\mathbb{G}|$ in the following way. Fix a point c_v in each component v of $S \setminus \underline{\boldsymbol{\alpha}}$ (which must be exactly the marked point, if the component is a pointed disc) and let f send it to the corresponding vertex v of $|\mathbb{G}|$. For each arc $\alpha_i \in \underline{\boldsymbol{\alpha}}$, consider a transverse path β_i from $c_{v'}$ to $c_{v''}$ that joins the two components v' and v'' separated by α_i, intersecting α_i exactly once, in such a way that the interiors of β_i and β_j are disjoint, if $i \neq j$. Define f to be a homeomorphism of β_i onto the oriented edge in $|\mathbb{G}|$ corresponding to α_i that runs from v' to v''.

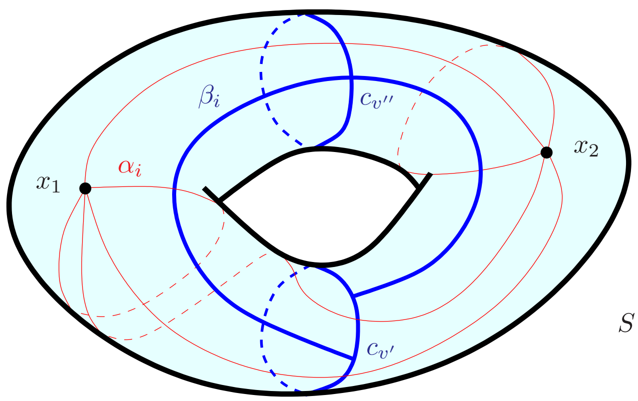

Figure 5. Thick curves represent $f^{-1}(|G|)$ and thin ones their dual arcs.

Because all components of $S \setminus \underline{\boldsymbol{\alpha}}$ are discs (or pointed discs), it is easy to see that there is a unique way of extending f to a homeomorphism (up to isotopy).

Proposition 2.8. *The association above defines a* $\Gamma(S,X)$*-equivariant equivalence of categories*

$$\widehat{\mathfrak{A}^\circ}(S,X) \longrightarrow \mathfrak{RG}_{ns}(S,X)$$

where $\widehat{\mathfrak{A}^\circ}(S,X)$ *is the category of proper arc systems on* (S,X)*, whose morphisms are reversed inclusions.*

In fact, an inclusion $\underline{\boldsymbol{\alpha}} \hookrightarrow \underline{\boldsymbol{\beta}}$ of proper systems induces a morphism $\mathbb{G}_{\underline{\boldsymbol{\beta}}} \to \mathbb{G}_{\underline{\boldsymbol{\alpha}}}$ of nonsingular (S,X)-marked ribbon graphs.

A pseudo-inverse is constructed as follows. Let $f : S \to |\mathbb{G}|$ be a nonsingular (S,X)-marked ribbon graph and let $|G| \hookrightarrow |\mathbb{G}|$ be the spine. The graph $f^{-1}(|G|)$ decomposes S into a disjoint union of one-pointed discs. For each edge e of $|G|$, let α_e be the simple arc joining the points in the two discs separated by e. Thus, we can associate the system of arcs $\{\alpha_e \,|\, e \in E_1(\mathbb{G})\}$ to $(\mathbb{G}, f)$ and this defines a pseudo-inverse $\mathfrak{RG}_{ns}(S,X) \longrightarrow \widehat{\mathfrak{A}^\circ}(S,X)$.

2.2.9 Metrized nonsingular ribbon graphs. A *metric* on a ribbon graph $\mathbb{G}$ is a map $\ell : E_1(\mathbb{G}) \to \mathbb{R}_+$. Given a simple closed curve $\gamma \in \mathcal{C}(S,X)$ and an (S,X)-marked nonsingular ribbon graph $f : S \to |\mathbb{G}|$, there is a unique simple closed curve $\tilde{\gamma} = |e_{i_1}| \cup \cdots \cup |e_{i_k}|$ contained inside $|G| \subset |\mathbb{G}|$ such that $f^{-1}(\tilde{\gamma})$ is freely homotopic to γ.

If $\mathbb{G}$ is metrized, then we can define the *length* $\ell(\gamma)$ to be $\ell(\tilde{\gamma}) = \ell(e_{i_1}) + \cdots + \ell(e_{i_k})$. Consequently, the *systol* is given by $\inf\{\ell(\gamma) \,|\, \gamma \in \mathcal{C}(S,X)\}$.

Given a proper weighted arc system $w \in |\mathfrak{A}^\circ(S,X)|_{\mathbb{R}}$, supported on $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}^\circ(S,X)$, we can endow the corresponding ribbon graph $\mathbb{G}_{\underline{\boldsymbol{\alpha}}}$ with a *metric* , by simply setting $\ell(\alpha_i) := w(\alpha_i)$. Thus, one can extend the correspondence to proper weighted arc systems and metrized (S,X)-marked nonsingular ribbon graphs. Moreover, the notions of length and systol agree with those given in 2.1.10.

Notice the similarity between Lemma 2.2 and Mumford-Mahler criterion for compactness in $\mathcal{M}_{g,n}$.

2.2.10 Category of enriched ribbon graphs. An *isomorphism* of enriched X-marked ribbon graphs $\mathbb{G}_1^{en} \to \mathbb{G}_2^{en}$ is the datum of compatible isomorphisms of their (reduced) graphs $c : \zeta_1 \to \zeta_2$ and of the ribbon graphs $\mathbb{G}_1 \to \mathbb{G}_2$, such that $c(V_{1,+}) = V_{2,+}$ and they respect the rest of the data.

Let $\mathbb{G}^{en}$ be an enriched X-marked ribbon graph and let $e \in E_1(\mathbb{G}_v)$, where $v \in V_+$. Assume that $|V_+| > 1$ or that $|E_1(\mathbb{G}_v)| > 1$. We define $\mathbb{G}^{en}/e$ in the following way.

(a) If e is the only edge of $\mathbb{G}_v$, then we just turn v into an invisible component and we define $g(v) := g(\mathbb{G}_v)$ and $m(x_i) = v$ for all $x_i \in X$ that marked a hole or a vertex of $\mathbb{G}_v$. In what follows, suppose that $|E_1(\mathbb{G}_v)| > 1$.

(b) If $[\overrightarrow{e}]_0$ and $[\overleftarrow{e}]_0$ are distinct and not both special, then we obtain $\mathbb{G}^{en}/e$ from $\mathbb{G}^{en}$ by simply replacing $\mathbb{G}_v$ by $\mathbb{G}_v/e$.

(c) If $[\overrightarrow{e}]_0 = [\overleftarrow{e}]_0$ is not special, then replace $\mathbb{G}_v$ by $\mathbb{G}_v/e$. If $\{\overrightarrow{e}\}$ was a hole marked by x_j, then mark the new vertex of $\mathbb{G}_v/e$ by x_j. Otherwise, add an edge to ζ that joins the two new vertices of $\mathbb{G}_v/e$ (which may or may not split into two visible components).

(d) In case $[\overrightarrow{e}]_0$ and $[\overleftarrow{e}]_0$ are both special vertices (whether or not they are distinct), add a new invisible component w of genus 0 to ζ, replace $\mathbb{G}_v$ by $\mathbb{G}_v/e$ (if $\mathbb{G}_v/e$ is disconnected, the vertex v splits) and join w to the new vertices (one or two) of $\mathbb{G}_v/e$ and to the old edges $s_v^{-1}([\overrightarrow{e}]_0) \cup s_v^{-1}([\overleftarrow{e}]_0)$. Moreover, if $\{\overrightarrow{e}\}$ was a hole marked by x_j, then mark w by x_j.

Notice that $\mathbb{G}^{en}/e$ can be not reduced, so we may want to consider the reduced enriched graph $\widetilde{\mathbb{G}^{en}/e}$ associated to it. We define $\mathbb{G}^{en} \to \widetilde{\mathbb{G}^{en}/e}$ to be an *elementary contraction*.

X-marked enriched ribbon graphs form a (small) category $\mathfrak{RG}_X$, whose morphisms are compositions of isomorphisms and elementary contractions.

Call $\mathfrak{RG}_{g,X}$ the full subcategory of $\mathfrak{RG}_X$ whose objects are ribbon graphs of genus g.

Remark 2.9. Really, the automorphism group of an enriched ribbon graph must be defined as the product of the automorphism group as defined above by $\prod_{v \in V_-} \mathrm{Aut}(v)$, where $\mathrm{Aut}(v)$ is the group of automorphisms of the generic Riemann surface of type $(g(v), n(v))$ (where $n(v)$ is the number of oriented edges of ζ outgoing from v and of marked points on v). Fortunately, $\mathrm{Aut}(v)$ is almost always trivial, except if $g(v) = n(v) = 1$, when $\mathrm{Aut}(v) \cong \mathbb{Z}/2\mathbb{Z}$.

2.2.11 Topological realization of enriched ribbon graphs. The *topological realization* of the enriched X-marked ribbon graph $\mathbb{G}^{en}$ is the nodal X-marked oriented surface $|\mathbb{G}^{en}|$ obtained as a quotient of

$$\left(\coprod_{v \in V_+} |\mathbb{G}_v| \right) \coprod \left(\coprod_{w \in V_-} S_w \right)$$

by a suitable equivalence relation, where S_w is a compact oriented surface of genus $g(w)$ with marked points given by $m^{-1}(w)$ and by the oriented edges of ζ outgoing from w. The equivalence relation identifies couples of points (two special vertices of $\mathbb{G}$ or a special vertex on a visible component and a point on an invisible one) corresponding to the same edge of ζ.

As in the nonsingular case, for each $v \in V_+$ the positive component $|\mathbb{G}_v|$ naturally contains an embedded *spine* $|G_v|$. Notice that there is an obvious correspondence between edges of ζ and nodes of $|\mathbb{G}^{en}|$.

Moreover, the elementary contraction $\mathbb{G}^{en} \to \mathbb{G}^{en}/e$ to the non-reduced $\mathbb{G}^{en}/e$ defines a unique homotopy class of maps $|\mathbb{G}^{en}| \to |\mathbb{G}^{en}/e|$, which may shrink a circle inside a positive component of $|\mathbb{G}^{en}|$ to a singular point (only in cases (c) and (d)), and which are homeomorphisms elsewhere.

If $\widetilde{\mathbb{G}^{en}/e}$ is the reduced graph associated to $\mathbb{G}^{en}/e$, then we also have a map $|\widetilde{\mathbb{G}^{en}/e}| \to |\mathbb{G}^{en}/e|$ that shrinks some circles inside the invisible components to singular points and is a homeomorphism elsewhere.

$$|\mathbb{G}^{en}| \longrightarrow |\mathbb{G}^{en}/e| \longleftarrow |\widetilde{\mathbb{G}^{en}/e}|$$

2.2.12 (S,X)-markings of $\mathbb{G}^{en}$. An (S,X)-*marking* of an enriched X-marked ribbon graph $\mathbb{G}^{en}$ is a map $f : S \to |\mathbb{G}^{en}|$ compatible with $X \hookrightarrow$

$E_\infty(\mathbb{G}) \cup E_0(\mathbb{G})$ such that $f^{-1}(\{\text{nodes}\})$ is a disjoint union of circles and f is an orientation-preserving homeomorphism elsewhere, up to isotopy. The subsurface $S_+ := f^{-1}(|\mathbb{G}| \setminus \{\text{special points}\})$ is the *visible subsurface*.

An *isomorphism* of (S, X)-marked (reduced) enriched ribbon graphs is an isomorphism $\mathbb{G}_1^{en} \to \mathbb{G}_2^{en}$ such that $S \xrightarrow{f_1} |\mathbb{G}_1^{en}| \to |\mathbb{G}_2^{en}|$ is homotopic to $f_2 : S \to |\mathbb{G}_2^{en}|$.

Given (S, X)-markings $f : S \to |\mathbb{G}^{en}|$ and $f' : S \to |\widetilde{\mathbb{G}^{en}/e}|$ such that $S \xrightarrow{f} |\mathbb{G}^{en}| \to |\mathbb{G}^{en}/e|$ is homotopic to $S \xrightarrow{f'} |\widetilde{\mathbb{G}^{en}/e}| \to |\mathbb{G}^{en}/e|$, then we define $(\mathbb{G}^{en}, f) \to (\widetilde{\mathbb{G}^{en}/e}, f')$ to be an *elementary contraction* of (S, X)-marked enriched ribbon graphs.

Define $\mathfrak{RG}(S, X)$ to be the category whose objects are (equivalence classes of) (S, X)-marked enriched ribbon graphs $(\mathbb{G}^{en}, f)$ and whose morphisms are compositions of isomorphisms and elementary contractions.

Again, the mapping class group $\Gamma(S, X)$ acts on $\mathfrak{RG}(S, X)$ and the quotient $\mathfrak{RG}(S, X)/\Gamma(S, X)$ is equivalent to $\mathfrak{RG}_{g,X}$.

2.2.13 Arcs/graph duality. Let $\underline{\boldsymbol{\alpha}} = \{\alpha_0, \ldots, \alpha_k\} \in \mathfrak{A}^\circ(S, X)$ be an arc system and let $\sigma_0, \sigma_1, \sigma_\infty$ the corresponding operators on the set of oriented arcs $E(\underline{\boldsymbol{\alpha}})$.

Define V_+ to be the set of connected components of $S(\underline{\boldsymbol{\alpha}})_+$ and V_- the set of components of $S(\underline{\boldsymbol{\alpha}})_-$. Let ζ be a graph whose vertices are $V = V_+ \cup V_-$ and whose edges correspond to connected components of $S \setminus (S(\underline{\boldsymbol{\alpha}})_+ \cup S(\underline{\boldsymbol{\alpha}})_-)$, where an edge connects v and w (possibly $v = w$) if the associated component bounds v and w.

Define $g : V_- \to \mathbb{N}$ to be the genus function associated to the connected components of $S(\underline{\boldsymbol{\alpha}})_-$.

Call S_v the subsurface associated to $v \in V_+$ and let $\hat{S}_v$ be the quotient of $\overline{S}_v$ obtained by identifying each component of ∂S_v to a point. We denote by $\underline{\boldsymbol{\alpha}} \cap \hat{S}_v$ the system of arcs induced on $\hat{S}_v$ by $\underline{\boldsymbol{\alpha}}$.

As $\underline{\boldsymbol{\alpha}} \cap \hat{S}_v$ quasi-fills $\hat{S}_v$, we can construct a dual ribbon graph $\mathbb{G}_v$ and a homeomorphism $\hat{S}_v \to |\mathbb{G}_v|$ by sending ∂S_v to nodal vertices of $|\mathbb{G}_v|$ and marked points on $\hat{S}_v$ to centers or marked vertices of $|\mathbb{G}_v|$. These homeomorphisms glue to give a map $S \to |\mathbb{G}^{en}|$ that shrinks circles and cylinders in $S \setminus (S(\underline{\boldsymbol{\alpha}})_+ \cup S(\underline{\boldsymbol{\alpha}})_-)$ to nodes and is a homeomorphism elsewhere, which is thus homotopic to a marking of $|\mathbb{G}^{en}|$.

We have obtain an enriched (S, X)-marked (reduced) ribbon graph $\mathbb{G}_{\underline{\boldsymbol{\alpha}}}^{en}$ *dual to* $\underline{\boldsymbol{\alpha}}$.

Proposition 2.10. *The above construction defines a* $\Gamma(S, X)$*-equivariant equivalence of categories*

$$\widehat{\mathfrak{A}}(S, X) \longrightarrow \mathfrak{RG}(S, X)$$

where $\widehat{\mathfrak{A}}(S,X)$ *is the category of arc systems on* (S,X)*, whose morphisms are reversed inclusions.*

As before, an inclusion $\underline{\boldsymbol{\alpha}} \hookrightarrow \underline{\boldsymbol{\beta}}$ of systems of arcs induces a morphism $\mathbb{G}^{en}_{\underline{\boldsymbol{\beta}}} \to \mathbb{G}^{en}_{\underline{\boldsymbol{\alpha}}}$ of (S,X)-marked enriched ribbon graphs.

To construct a pseudo-inverse, start with $(\mathbb{G}^{en}, f)$ and call $\hat{S}_v$ the surface obtained from $f^{-1}(|\mathbb{G}_v|)$ by shrinking each boundary circle to a point. By nonsingular duality, we can construct a system of arcs $\underline{\boldsymbol{\alpha}}_v$ inside $\hat{S}_v$ dual to $f_v : \hat{S}_v \to |\mathbb{G}_v|$. As the arcs miss the vertices of $f_v^{-1}(|G_v|)$ by construction, $\underline{\boldsymbol{\alpha}}_v$ can be lifted to S. The wanted arc system on S is $\underline{\boldsymbol{\alpha}} = \bigcup_{v \in V_+} \underline{\boldsymbol{\alpha}}_v$.

2.2.14 Metrized enriched ribbon graphs. A metric on $\mathbb{G}^{en}$ is a map $\ell : E_1(\mathbb{G}) \to \mathbb{R}_+$. Given $\gamma \in \mathcal{C}(S,X)$ and an (S,X)-marking $f : S \to |\mathbb{G}^{en}|$, we can define $\gamma_+ := \gamma \cap S_+$. As in the nonsingular case, there is a unique $\tilde{\gamma}_+ = |e_{i_1}| \cup \cdots \cup |e_{i_k}|$ inside $|G| \subset |\mathbb{G}|$ such that $f^{-1}(\tilde{\gamma}_+) \simeq \gamma_+$.

Hence, we can define $\ell(\gamma) := \ell(\gamma_+) = \ell(e_{i_1}) + \cdots + \ell(e_{i_k})$. Clearly, $\ell(\gamma) = i(\gamma, w)$, where w is the weight function supported on the arc system dual to $(\mathbb{G}^{en}, f)$. Thus, the arc/graph duality also establishes a correspondence between weighted arc systems on (S,X) and metrized (S,X)-marked enriched ribbon graphs.

3 Differential and algebro-geometric point of view

3.1 The Deligne-Mumford moduli space

3.1.1 The Teichmüller space. Fix a compact oriented surface S of genus g and a subset $X = \{x_1, \dots, x_n\} \subset S$ such that $2g - 2 + n > 0$.

A *smooth family* of (S,X)-marked Riemann surfaces is a commutative diagram

$$\begin{array}{ccc} B \times S & \xrightarrow{\ f\ } & \mathcal{C} \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

where f is an oriented diffeomorphism, $B \times S \to B$ is the projection on the first factor and the fibers $\mathcal{C}_b$ of π are Riemann surfaces, whose complex structure varies smoothly with $b \in B$.

Two families (f_1, π_1) and (f_2, π_2) over B are *isomorphic* if there exists a continuous map $h : \mathcal{C}_1 \to \mathcal{C}_2$ such that

- $h_b \circ f_{1,b} : (S,X) \to (\mathcal{C}_{2,b}, h_b \circ f_{1,b}(X))$ is homotopic to $f_{2,b}$ for every $b \in B$

- $h_b : (\mathcal{C}_{1,b}, f_{1,b}(X)) \to (\mathcal{C}_{2,b}, f_{2,b}(X))$ is biholomorphic for every $b \in B$.

The functor $\mathcal{T}(S,X)$: (manifolds) $\to$ (sets) defined by

$$B \mapsto \left\{ \begin{matrix} \text{smooth families of } (S,X)\text{-marked} \\ \text{Riemann surfaces over } B \end{matrix} \right\} /\text{iso}$$

is represented by the *Teichmüller space* $\mathcal{T}(S,X)$.

It is a classical result that $\mathcal{T}(S,X)$ is a complex-analytic manifold of (complex) dimension $3g-3+n$ (Ahlfors [1], Bers [8] and Ahlfors-Bers [3]) and is diffeomorphic to a ball (Teichmüller [68]).

3.1.2 The moduli space of Riemann surfaces. A *smooth family* of X-marked Riemann surfaces of genus g is

- a submersion $\pi : \mathcal{C} \to B$
- a smooth embedding $s : X \times B \to \mathcal{C}$

such that the fibers $\mathcal{C}_b$ are Riemann surfaces of genus g, whose complex structure varies smoothly in $b \in B$, and $s_{x_i} : B \to \mathcal{C}$ is a section for every $x_i \in X$.

Two families (π_1, s_1) and (π_2, s_2) over B are isomorphic if there exists a diffeomorphism $h : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\pi_2 \circ h = \pi_1$, the restriction of h to each fiber $h_b : \mathcal{C}_{1,b} \to \mathcal{C}_{2,b}$ is a biholomorphism and $h \circ s_1 = s_2$.

The existence of Riemann surfaces with nontrivial automorphisms (for $g \geq 1$) prevents the functor

$$\begin{aligned} \mathcal{M}_{g,X} : \ (\text{manifolds}) &\longrightarrow (\text{sets}) \\ B &\longmapsto \left\{ \begin{matrix} \text{smooth families of } X\text{-marked} \\ \text{Riemann surfaces over } B \end{matrix} \right\} /\text{iso} \end{aligned}$$

from being representable. However, Riemann surfaces with $2g-2+n > 0$ have finitely many automorphisms and so $\mathcal{M}_{g,X}$ is actually represented by an orbifold, which is in fact $\mathcal{T}(S,X)/\Gamma(S,X)$ (in the orbifold sense). In the algebraic category, we would rather say that $\mathcal{M}_{g,X}$ is a Deligne-Mumford stack with quasi-projective coarse space. In any case, we will always refer to $\mathcal{M}_{g,X}$ as the *moduli space* of X-marked Riemann surfaces of genus g.

3.1.3 Stable curves. Enumerative geometry is traditionally reduced to intersection theory on suitable moduli spaces. In our case, $\mathcal{M}_{g,X}$ is not a compact orbifold. To compactify it in an algebraically meaningful way, we need to look at how algebraic families of complex projective curves can degenerate.

In particular, given a holomorphic family $\mathcal{C}^* \to \Delta^*$ of algebraic curves over the punctured disc, we must understand how to complete the family over Δ.

Example 3.1. Consider the family $\mathcal{C}^* = \{(b, [x:y:z]) \in \Delta^* \times \mathbb{CP}^2 \,|\, y^2 z = x(x-bz)(x-2z)\}$ of curves of genus 1 with the marked point $[2:0:1] \in \mathbb{CP}^2$, parametrized by $b \in \Delta^*$. Notice that the projection $\mathcal{C}_b^* \to \mathbb{CP}^1$ given by

$[x:y:z] \mapsto [x:z]$ (where $[0:1:0] \mapsto [1:0]$) is a $2:1$ cover, branched over $\{0, b, 2, \infty\}$. Fix a $\overline{b} \in \Delta^*$ and consider a closed curve $\gamma \subset \mathbb{CP}^1$ that separates $\{\overline{b}, 2\}$ from $\{0, \infty\}$ and pick one of the two (simple closed) lifts $\tilde{\gamma} \subset \mathcal{C}^*_{\overline{b}}$.

This $\tilde{\gamma}$ determines a nontrivial element of $H_1(\mathcal{C}^*_{\overline{b}})$. A quick analysis tells us that the endomorphism $T : H_1(\mathcal{C}^*_{\overline{b}}) \to H_1(\mathcal{C}^*_{\overline{b}})$ induced by the monodromy around a generator of $\pi_1(\Delta^*, \overline{b})$ is nontrivial. Thus, the family $\mathcal{C}^* \to \Delta^*$ cannot be completed over Δ as smooth family (because it would have trivial monodromy).

If we want to compactify our moduli space, we must allow our curves to acquire some singularities. Thus, it makes no longer sense to ask them to be submersions. Instead, we will require them to be *flat*.

Given an open subset $0 \in B \subset \mathbb{C}$, a flat family of connected projective curves $\mathcal{C} \to B$ may typically look like (up to shrinking B)

- $\Delta \times B \to B$ around a smooth point of $\mathcal{C}_0$

- $\{(x, y) \in \mathbb{C}^2 \,|\, xy = 0\} \times B \to B$ around a node of $\mathcal{C}_0$ that persists on each $\mathcal{C}_b$

- $\{(b, x, y) \in B \times \mathbb{C}^2 \,|\, xy = b\} \to B$ around a node of $\mathcal{C}_0$ that does not persist on the other curves $\mathcal{C}_b$ with $b \neq 0$

in local analytic coordinates.

Notice that, in the above cases, the (arithmetic) genus of each fiber $g_b = 1 - \frac{1}{2}[\chi(\mathcal{C}_b) - \nu_b]$ is constant in b, where ν_b is the number of nodes in $\mathcal{C}_b$.

To prove that allowing nodal curves is enough to compactify $\mathcal{M}_{g,X}$, one must show that it is always possible to complete any family $\mathcal{C}^* \to \Delta^*$ to a family over Δ. However, because nodal curves may have nontrivial automorphisms, we shall consider also the case in which $0 \in \Delta$ is an orbifold point. Thus, it is sufficient to be able to complete not exactly the family $\mathcal{C}^* \to \Delta^*$ but its pull-back under a suitable map $\Delta^* \to \Delta^*$ given by $z \mapsto z^k$. This is exactly the *semi-stable reduction theorem*.

One can observe that it is always possible to avoid producing genus 0 components with 1 or 2 nodes. Thus, we can consider only *stable curves*, that is nodal projective (connected) curves such that all irreducible components have finitely many automorphisms (equivalently, no irreducible component is a sphere with less than three nodes/marked points).

The *Deligne-Mumford compactification* $\overline{\mathcal{M}}_{g,X}$ of $\mathcal{M}_{g,X}$ is the moduli space of X-marked stable curves of genus g, which is a compact orbifold (algebraically, a Deligne-Mumford stack with projective coarse moduli space).

Its underlying topological space is a projective variety of complex dimension $3g - 3 + n$.

3.2 The system of moduli spaces of curves

3.2.1 Boundary maps. Many facts suggest that one should not look separately at each of the moduli spaces of X-pointed genus g curves $\overline{\mathcal{M}}_{g,X}$, but one must consider the whole system $(\overline{\mathcal{M}}_{g,X})_{g,X}$. An evidence is given by the existence of three families of maps that relate different moduli spaces.

(1) The *forgetful map* is a projective flat morphism

$$\pi_q : \overline{\mathcal{M}}_{g,X\cup\{q\}} \longrightarrow \overline{\mathcal{M}}_{g,X}$$

that forgets the point q and stabilizes the curve (i.e. contracts a possible two-pointed sphere). This map can be identified to the universal family and so is endowed with *tautological sections*

$$\vartheta_{0,\{x_i,q\}} : \overline{\mathcal{M}}_{g,X} \to \overline{\mathcal{M}}_{g,X\cup\{q\}}$$

for all $x_i \in X$.

(2) The *boundary map* corresponding to irreducible curves is the finite map

$$\vartheta_{irr} : \overline{\mathcal{M}}_{g-1,X\cup\{x',x''\}} \longrightarrow \overline{\mathcal{M}}_{g,X}$$

(defined for $g > 0$) that glues x' and x'' together. It is generically $2:1$ and its image sits in the boundary of $\overline{\mathcal{M}}_{g,X}$.

(3) The *boundary maps* corresponding to reducible curves are the finite maps

$$\vartheta_{g',I} : \overline{\mathcal{M}}_{g',I\cup\{x'\}} \times \overline{\mathcal{M}}_{g-g',I^c\cup\{x''\}} \longrightarrow \overline{\mathcal{M}}_{g,X}$$

(defined for every $0 \leq g' \leq g$ and $I \subseteq X$ such that the spaces involved are nonempty) that take two curves and glue them together identifying x' and x''. They are generically $1-1$ (except in the case $g = 2g'$ and $X = \emptyset$, when the map is generically $2:1$) and their images sit in the boundary of $\overline{\mathcal{M}}_{g,X}$ too.

Let $\delta_{0,\{x_i,q\}}$ be the Cartier divisor in $\overline{\mathcal{M}}_{g,X\cup\{q\}}$ corresponding to the image of the tautological section $\vartheta_{0,\{x_i,q\}}$ and call $D_q := \sum_i \delta_{0,\{x_i,q\}}$.

3.2.2 Stratification by topological type. We observe that $\overline{\mathcal{M}}_{g,X}$ has a natural *stratification* by topological type of the complex curve. In fact, we can attach to every stable curve Σ its *dual graph* ζ_Σ, whose vertices V correspond to irreducible components and whose edges correspond to nodes of Σ. Moreover, we can define a *genus function* $g : V \to \mathbb{N}$ such that $g(v)$ is the genus of the normalization of the irreducible component corresponding to v and a *marking function* $m : X \to V$ (determined by requiring that x_i is marking a point on the irreducible component corresponding to $m(x_i)$). Equivalently, we will also say that the vertex $v \in V$ is *labelled* by $(g(v), X_v := m^{-1}(v))$. Call Q_v the singular points of Σ_v.

For every such labeled graph ζ, we can construct a boundary map

$$\vartheta_\zeta : \prod_{v \in V} \overline{\mathcal{M}}_{g_v, X_v \cup Q_v} \longrightarrow \overline{\mathcal{M}}_{g,X}$$

which is a finite morphism.

3.3 Augmented Teichmüller space

3.3.1 Bordifications of $\mathcal{T}(S,X)$. Fix a compact oriented surface S of genus g and let $X = \{x_1, \dots, x_n\} \subset S$ such that $2g - 2 + n > 0$.

It is natural to look for natural *bordifications* of $\mathcal{T}(S,X)$: that is, we look for a space $\overline{\mathcal{T}}(S,X) \supset \mathcal{T}(S,X)$ that contains $\mathcal{T}(S,X)$ as a dense subspace and such that the action of the mapping class group $\Gamma(S,X)$ extends to $\overline{\mathcal{T}}(S,X)$.

A remarkable example is given by Thurston's compactification $\overline{\mathcal{T}}^{Th}(S,X) = \mathcal{T}(S,X) \cup \mathbb{P}\mathcal{ML}(S,X)$, in which points at infinity are (isotopy classes of) projective measured laminations with compact support in $S \setminus X$. Thurston showed that $\mathbb{P}\mathcal{ML}(S,X)$ is compact and homeomorphic to a sphere. As $\Gamma(S,X)$ is infinite and discrete, this means that the quotient $\overline{\mathcal{T}}^{Th}(S,X)/\Gamma(S,X)$ cannot be too good and so this does not sound like a convenient way to compactify $\mathcal{M}_{g,X}$.

We will see in Section 4 that $\mathcal{T}(S,X)$ can be identified to $|\mathfrak{A}^\circ(S,X)|$. Thus, another remarkable example will be given by $|\mathfrak{A}(S,X)|$.

A natural question is how to define a bordification $\overline{\mathcal{T}}(S,X)$ such that $\overline{\mathcal{T}}(S,X)/\Gamma(S,X) \cong \overline{\mathcal{M}}_{g,X}$.

3.3.2 Deligne-Mumford augmentation. A *(continuous) family of stable (S,X)-marked curves* is a commutative diagram

$$\begin{array}{ccc} B \times S & \xrightarrow{\ f\ } & \mathcal{C} \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

where $B \times S \to B$ is the projection on the first factor and

- the family π is obtained as a pull-back of a flat stable family of X-marked curves $\mathcal{C}' \to B'$ through a continuous map $B \to B'$
- if $N_b \subset \mathcal{C}_b$ is the subset of nodes, then $f^{-1}(\nu)$ is a smooth loop in $S \times \{b\}$ for every $\nu \in N_b$
- for every $b \in B$ the restriction $f_b : S \setminus f^{-1}(N_b) \to \mathcal{C}_b \setminus N_b$ is an orientation-preserving homeomorphism, compatible with the X-marking.

Isomorphisms of such families are defined in the obvious way.

Example 3.2. Start with a flat family $\mathcal{C}' \to \Delta$ such that $\mathcal{C}'_b$ are all homeomorphic for $b \neq 0$. Then consider the path $B = [0, \varepsilon) \subset \Delta$ and call $\mathcal{C} := \mathcal{C}' \times_\Delta B$. Over $(0, \varepsilon)$, the family $\mathcal{C}$ is topologically trivial, whereas $\mathcal{C}_0$ may contain some new nodes.

Consider a marking $S \to \mathcal{C}_{\varepsilon/2}$ that pinches circles to nodes, is an oriented homeomorphism elsewhere and is compatible with X. The homeomorphism $S \times (0, \varepsilon) \to \mathcal{C}_{\varepsilon/2} \times (0, \varepsilon) \xrightarrow{\sim} \mathcal{C}$ extends over 0 to a map $S \times [0, \varepsilon) \hookrightarrow \mathrm{Bl}_{\mathcal{C}_0}\mathcal{C} \twoheadrightarrow \mathcal{C}$, where $\mathrm{Bl}_{\mathcal{C}_0}\mathcal{C}$ is the real-oriented blow-up of $\mathcal{C}$ along $\mathcal{C}_0$. This is our wished (S, X)-marking.

The *Deligne-Mumford augmentation* of $\mathcal{T}(S, X)$ is the topological space $\overline{\mathcal{T}}^{DM}(S, X)$ that classifies families of stable (S, X)-marked curves.

It follows easily that $\overline{\mathcal{T}}^{DM}(S, X)/\Gamma(S, X) = \overline{\mathcal{M}}_{g,X}$ as topological spaces. However, $\overline{\mathcal{T}}^{DM}(S, X) \to \overline{\mathcal{M}}_{g,X}$ has infinite ramification at $\partial^{DM}\mathcal{T}(S, X)$, due to the Dehn twists around the pinched loops.

3.3.3 Hyperbolic length functions. Let $[f : S \to \Sigma]$ be a point of $\mathcal{T}(S, X)$. As $\chi(S \setminus X) = 2 - 2g - n < 0$, the uniformization theorem provides a universal cover $\mathbb{H} \to \Sigma \setminus f(X)$, which endows $\Sigma \setminus f(X)$ with a hyperbolic metric of finite volume, with cusps at $f(X)$.

In fact, we can interpret $\mathcal{T}(S, X)$ as the classifying space of (S, X)-marked families of hyperbolic surfaces. It is clear that continuous variation of the complex structure corresponds to continuous variation of the hyperbolic metric (uniformly on the compact subsets, for instance), and so to continuity of the holonomy map $H : \pi_1(S \setminus X) \times \mathcal{T}(S, X) \to \mathrm{PSL}_2(\mathbb{R})$.

In particular, for every $\gamma \in \pi_1(S \setminus X)$ the function $\ell_\gamma : \mathcal{T}(S, X) \to \mathbb{R}$ that associates to $[f : S \to \Sigma]$ the *length* of the unique geodesic in the free homotopy class $f_*\gamma$ is continuous. As $\cosh(\ell_\gamma/2) = |\mathrm{Tr}(H_\gamma/2)|$, one can check that H can be reconstructed from sufficiently (but finitely) many length functions. So that the continuity of these is equivalent to the continuity of the family.

3.3.4 Fenchel-Nielsen coordinates. Let $\underline{\gamma} = \{\gamma_1, \ldots, \gamma_N\}$ be a maximal system of disjoint simple closed curves of $S \setminus X$ (and so $N = 3g - 3 + n$) such that no γ_i is contractible in $S \setminus X$ or homotopic to a puncture and no couple γ_i, γ_j bounds a cylinder contained in $S \setminus X$.

The system $\underline{\gamma}$ induces a *pair of pants decomposition* of S, that is $S \setminus (\gamma_1 \cup \cdots \cup \gamma_N) = P_1 \cup P_2 \cup \cdots \cup P_{2g-2+n}$, and each P_i is a pair of pants (i.e. a surface of genus 0 with $\chi(P_i) = -1$).

Given $[f : S \to \Sigma] \in \mathcal{T}(S, X)$, we have *lengths* $\ell_i(f) = \ell_{\gamma_i}(f)$ for $i = 1, \ldots, N$, which determine the hyperbolic type of all pants $P_1, \ldots, P_{2g-2+n}$. The information about how the pants are glued together is encoded in the *twist parameters* $\tau_i = \tau_{\gamma_i} \in \mathbb{R}$, which are well-defined up to some choices. What is important is that, whatever choices we make, the difference $\tau_i(f_1) - \tau_i(f_2)$ is the same and it is well-defined.

The *Fenchel-Nielsen coordinates* $(\ell_i, \tau_i)_{i=1}^N$ exhibit a real-analytic diffeomorphism $\mathcal{T}(S,X) \xrightarrow{\sim} (\mathbb{R}_+ \times \mathbb{R})^N$ (which clearly depends on the choice of $\underline{\gamma}$).

3.3.5 Fenchel-Nielsen coordinates around nodal curves. Points of $\partial^{DM}\mathcal{T}(S,X)$ are (S,X)-marked stable curves or, equivalently (using the uniformization theorem componentwise), (S,X)-marked hyperbolic surfaces with nodes, i.e. homotopy classes of maps $f : S \to \Sigma$, where Σ is a hyperbolic surface with nodes $\nu_1, \dots, \nu_k$, the fiber $f^{-1}(\nu_j)$ is a simple closed curve γ_j and f is an oriented diffeomorphism outside the nodes.

Complete $\{\gamma_1, \dots, \gamma_k\}$ to a maximal set $\underline{\gamma}$ of simple closed curves in (S,X) and consider the associated Fenchel-Nielsen coordinates (ℓ_j, τ_j) on $\mathcal{T}(S,X)$. As we approach the point $[f]$, the holonomies $H_{\gamma_1}, \dots, H_{\gamma_k}$ tend to parabolics and so the lengths $\ell_1, \dots, \ell_k$ tend to zero. In fact, the hyperbolic metric on the surface Σ has a pair of cusps at each node ν_j.

This shows that the lengths functions $\ell_1, \dots, \ell_k$ continuously extend to zero at $[f]$. On the other hand, the twist parameters $\tau_1(f), \dots, \tau_k(f)$ make no longer sense.

If we look at what happens on $\overline{\mathcal{M}}_{g,X}$, we may notice that the couples $(\ell_j, \tau_j)_{j=1}^k$ behave like polar coordinate around $[\Sigma]$, so that it seems natural to set $\vartheta_m = 2\pi\tau_m/\ell_m$ for all $m = 1, \dots, N$ and define consequently a map $F_{\underline{\gamma}} : (\mathbb{R}^2)^N \to \overline{\mathcal{M}}_{g,X}$, that associates to $(\ell_1, \vartheta_1, \dots, \ell_N, \vartheta_N)$ the surface with Fenchel-Nielsen coordinates $(\ell_m, \tau_m = \ell_m\vartheta_m/2\pi)$. Notice that the map is well-defined, because a twist along γ_j by ℓ_j is a diffeomorphism of the surface (a Dehn twist).

The map $F_{\underline{\gamma}}$ is an orbifold cover $F_{\underline{\gamma}} : \mathbb{R}^{2N} \to F_{\underline{\gamma}}(\mathbb{R}^{2N}) \subset \overline{\mathcal{M}}_{g,X}$ and its image contains $[\Sigma]$. Varying $\underline{\gamma}$, we can cover the whole $\overline{\mathcal{M}}_{g,X}$ and thus give it a *Fenchel-Nielsen smooth structure*.

The bad news, analyzed by Wolpert [75], is that the Fenchel-Nielsen smooth structure is different (at $\partial\mathcal{M}_{g,X}$) from the Deligne-Mumford one. In fact, if a boundary divisor is locally described by $\{z_1 = 0\}$, then the length ℓ_γ of the corresponding vanishing geodesic is related to z_1 by $|z_1| \approx \exp(-1/\ell_\gamma)$, which shows that the identity map $\overline{\mathcal{M}}_{g,X}^{FN} \to \overline{\mathcal{M}}_{g,X}^{DM}$ is Lipschitz, but its inverse it not Hölder-continuous.

3.3.6 Weil-Petersson metric. Let Σ be a Riemann surface of genus g with marked points $X \hookrightarrow \Sigma$ such that $2g-2+n > 0$. First-order deformations of the complex structure can be rephrased in terms of $\overline{\partial}$ operator as $\overline{\partial} + \varepsilon\mu\partial + o(\varepsilon)$, where the *Beltrami differential* $\mu \in \Omega^{0,1}(T_\Sigma(-X))$ can be locally written as

$\mu(z)\dfrac{d\overline{z}}{dz}$ with respect to some holomorphic coordinate z on Σ and $\mu(z)$ vanishes at X.

Given a smooth vector field $V = V(z)\dfrac{\partial}{\partial z}$ on Σ that vanishes at X, the deformations induced by μ and $\mu+\overline{\partial}V$ differ only by an isotopy of Σ generated by V (which fixes X).

Thus, the *tangent space* $T_{[\Sigma]}\mathcal{M}_{g,X}$ can be identified to $H^{0,1}(\Sigma, T_\Sigma(-X))$. As a consequence, the *cotangent space* $T^*_{[\Sigma]}\mathcal{M}_{g,X}$ identifies to the space $\mathcal{Q}(\Sigma, X)$ of *integrable holomorphic quadratic differentials* on $\Sigma \setminus X$, that is, which are allowed to have a simple pole at each $x_i \in X$. The duality between $T_{[\Sigma]}\mathcal{M}_{g,X}$ and $T^*_{[\Sigma]}\mathcal{M}_{g,X}$ is given by

$$\begin{aligned} H^{0,1}(\Sigma, T_\Sigma(-X)) \times H^0(\Sigma, K_\Sigma^{\otimes 2}(X)) &\longrightarrow \mathbb{C} \\ (\mu, \varphi) &\longmapsto \int_\Sigma \mu\varphi \end{aligned}$$

If $\Sigma \setminus X$ is given the hyperbolic metric λ, then elements in $H^{0,1}(\Sigma, T_\Sigma(-X))$ can be identified to the space of *harmonic Beltrami differentials* $\mathcal{H}(\Sigma, X) = \{\overline{\varphi}/\lambda \,|\, \varphi \in \mathcal{Q}(\Sigma, X)\}$.

The *Weil-Petersson Hermitean metric* $h = g + i\omega$ (defined by Weil [70] using Petersson's pairing of modular forms) is

$$h(\mu, \nu) := \int_\Sigma \mu\overline{\nu} \cdot \lambda$$

for $\mu, \nu \in \mathcal{H}(\Sigma, X) \cong T_\Sigma \mathcal{M}_{g,X}$.

This metric has a lot of properties: it is Kähler (Weil [70] and Ahlfors [2]) and it is mildly divergent at $\partial\mathcal{M}_{g,X}$, so that the Weil-Petersson distance extends to a non-degenerate distance on $\overline{\mathcal{M}}_{g,X}$ and all points of $\partial\mathcal{M}_{g,X}$ are at finite distance (Masur [48], Wolpert [72]).

Because $\overline{\mathcal{M}}_{g,X}$ is compact and so WP-complete, the lifting of the Weil-Petersson metric to $\overline{\mathcal{T}}(S, X)$ is also complete. Thus, $\overline{\mathcal{T}}(S, X)$ can be seen as the *Weil-Petersson completion* of $\mathcal{T}(S, X)$.

3.3.7 Weil-Petersson form. We should emphasize that the Weil-Petersson symplectic form ω_{WP} depends more directly on the hyperbolic metric on the surface than on its holomorphic structure.

In particular, Wolpert [74] has shown that

$$\omega_{WP} = \sum_i d\ell_i \wedge d\tau_i$$

on $\mathcal{T}(S, X)$, where (ℓ_i, τ_i) are Fenchel-Nielsen coordinates associated to any pair of pants decomposition of (S, X).

On the other hand, if we identify $\mathcal{T}(S,X)$ with an open subset of $\mathrm{Hom}(\pi_1(S\setminus X),\mathrm{SL}_2(\mathbb{R}))/\mathrm{SL}_2(\mathbb{R})$, then points of $\mathcal{T}(S,X)$ are associated $\mathfrak{g}$-local systems ρ on $S\setminus X$ (with parabolic holonomies at X and hyperbolic holonomies otherwise), where $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{R})$ is endowed with the symmetric bilinear form $\langle\alpha,\beta\rangle = \mathrm{Tr}(\alpha\beta)$.

Goldman [23] has proved that, in this description, the tangent space to $\mathcal{T}(S,X)$ at ρ is naturally $H^1(S,X;\mathfrak{g})$ and that ω_{WP} is given by $\omega(\mu,\nu) = \mathrm{Tr}(\mu\smile\nu)\cap[S]$.

Remark 3.3. Another description of ω in terms of shear coordinates and Thurston's symplectic form on measured laminations is given by Bonahon-Sözen [65].

One can feel that the complex structure J on $\mathcal{T}(S,X)$ inevitably shows up whenever we deal with the Weil-Petersson metric, as $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$. On the other hand, the knowledge of ω is sufficient to compute volumes and characteristic classes.

3.4 Tautological classes

3.4.1 Relative dualizing sheaf. All the maps between moduli spaces we have defined are in some sense tautological as they are very naturally constructed and they reflect intrinsic relations among the various moduli spaces. It is evident that one can look at these as classifying maps to the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,X}$ (which obviously descend to maps between coarse moduli spaces). Hence, we can consider all the cycles obtained by pushing forward or pulling back via these maps as being "tautologically" defined.

Moreover, there is an ingredient we have not considered yet: it is the *relative dualizing sheaf* of the universal family $\pi_q:\overline{\mathcal{M}}_{g,X\cup\{q\}}\to\overline{\mathcal{M}}_{g,X}$. One expects that it carries many informations and that it can produce many classes of interest.

The relative dualizing sheaf ω_{π_q} is the sheaf on $\overline{\mathcal{M}}_{g,X\cup\{q\}}$, whose local sections are (algebraically varying) Abelian differentials that are allowed to have simple poles at the nodes, provided the two residues at each node are opposite. The local sections of $\omega_{\pi_q}(D_q)$ (the logarithmic variant of ω_{π_q}) are sections of ω_{π_q} that may have simple poles at the X-marked points.

3.4.2 MMMAC classes. The *Miller classes* are

$$\psi_{x_i} := c_1(\mathcal{L}_i)\in CH^1(\overline{\mathcal{M}}_{g,X})_{\mathbb{Q}}$$

where $\mathcal{L}_i := \vartheta^*_{0,\{x_i,q\}}\omega_{\pi_q}$ and the *Mumford-Morita classes* (suitably modified by Arbarello-Cornalba) are

$$\kappa_j := (\pi_q)_*(\psi_q^{j+1}) \in CH^j(\overline{\mathcal{M}}_{g,X})_{\mathbb{Q}}.$$

One could moreover define the l-th *Hodge bundle* as $\mathbb{E}_l := (\pi_q)_*(\omega_{\pi_q}^{\otimes l})$ and consider the Chern classes of these bundles (for example, the λ classes $\lambda_i := c_i(\mathbb{E}_1)$). However, using Grothendieck-Riemann-Roch, Mumford [57] and Bini [10] proved that $c_i(\mathbb{E}_j)$ can be expressed as a linear combination of Mumford-Morita classes up to elements in the boundary, so that they do not introduce anything really new.

When there is no risk of ambiguity, we will denote in the same way the classes ψ and κ belonging to different $\overline{\mathcal{M}}_{g,X}$'s as it is now traditional.

Remark 3.4. Wolpert has proven [73] that, on $\overline{\mathcal{M}}_g$, we have $\kappa_1 = [\omega_{WP}]/\pi^2$ and that the amplitude of $\kappa_1 \in A^1(\overline{\mathcal{M}}_g)$ (and so the projectivity of $\overline{\mathcal{M}}_g$) can be recovered from the fact that $[\omega_{WP}/\pi^2]$ is an integral Kähler class [76]. He also showed that the cohomological identity $[\omega_{WP}/\pi^2] = \kappa_1 = (\pi_q)_*\psi_q^2$ admits a clean pointwise interpretation [77].

3.4.3 Tautological rings. Because of the natural definition of κ and ψ classes, as explained before, the subring $R^*(\mathcal{M}_{g,X})$ of $CH^*(\mathcal{M}_{g,X})_{\mathbb{Q}}$ they generate is called the *tautological ring* of $\mathcal{M}_{g,X}$. Its image $RH^*(\mathcal{M}_{g,X})$ through the cycle class map is called cohomology tautological ring.

From an axiomatic point of view, the *system of tautological rings* $(R^*(\overline{\mathcal{M}}_{g,X}))$ is the minimal system of subrings of $(CH^*(\overline{\mathcal{M}}_{g,X}))$ is the minimal system of subrings such that

- every $R^*(\overline{\mathcal{M}}_{g,X})$ contains the fundamental class $[\overline{\mathcal{M}}_{g,X}]$
- the system is closed under push-forward maps π_*, $(\vartheta_{irr})_*$ and $(\vartheta_{g',I})_*$.

$R^*(\mathcal{M}_{g,X})$ is defined to be the image of the restriction map $R^*(\overline{\mathcal{M}}_{g,X}) \to CH^*(\mathcal{M}_{g,X})$. The definition for the rational cohomology is analogous (where the role of $[\overline{\mathcal{M}}_{g,X}]$ is here played by its Poincaré dual $1 \in H^0(\overline{\mathcal{M}}_{g,X};\mathbb{Q})$).

It is a simple fact to remark that all tautological rings contain ψ and κ classes and in fact that $R^*(\mathcal{M}_{g,X})$ is generated by them. Really, this was the original definition of $R^*(\mathcal{M}_{g,X})$.

3.4.4 Faber's formula. The ψ classes interact reasonably well with the forgetful maps. In fact

$$(\pi_q)_*(\psi_{x_1}^{r_1}\cdots\psi_{x_n}^{r_n}) = \sum_{\{i|r_i>0\}} \psi_{x_1}^{r_1}\cdots\psi_{x_i}^{r_i-1}\cdots\psi_{x_n}^{r_n}$$

$$(\pi_q)_*(\psi_{x_1}^{r_1}\cdots\psi_{x_n}^{r_n}\psi_q^{b+1}) = \psi_{x_1}^{r_1}\cdots\psi_{x_n}^{r_n}\kappa_b$$

where the first one is the so-called *string equation* and the second one for $b = 0$ is the *dilaton equation* (see [71]). They have been generalized by Faber for maps that forget more than one point: *Faber's formula* (which we are going to describe below) can be proven using the second equation above and the relation $\pi_q^*(\kappa_j) = \kappa_j - \psi_q^j$ (proven in [5]).

Let $Q := \{q_1, \dots, q_m\}$ and let $\pi_Q : \overline{\mathcal{M}}_{g,X\cup Q} \to \overline{\mathcal{M}}_{g,X}$ be the forgetful map. Then

$$(\pi_Q)_*(\psi_{x_1}^{r_1} \cdots \psi_{x_n}^{r_n} \psi_{q_1}^{b_1+1} \cdots \psi_{q_m}^{b_m+1}) = \psi_{x_1}^{r_1} \cdots \psi_{x_n}^{r_n} K_{b_1 \cdots b_m}$$

where $K_{b_1 \cdots b_m} = \sum_{\sigma \in \mathfrak{S}_m} \kappa_{b(\sigma)}$ and $\kappa_{b(\sigma)}$ is defined in the following way. If $\gamma = (c_1, \dots, c_l)$ is a cycle, then set $b(\gamma) := \sum_{j=1}^{l} b_{c_j}$. If $\sigma = \gamma_1 \cdots \gamma_\nu$ is the decomposition in disjoint cycles (including 1-cycles), then we let $k_{b(\sigma)} := \prod_{i=1}^{\nu} \kappa_{b(\gamma_i)}$. We refer to [36] for more details on Faber's formula, to [5] and [6] for more properties of tautological classes and to [19] (and [55]) for a conjectural description (which is now partially proven) of the tautological rings.

3.5 Kontsevich's compactification

3.5.1 The line bundle $\mathbb{L}$. It has been observed by Witten [71] that the intersection theory of κ and ψ classes can be reduced to that of ψ classes only by using the push-pull formula with respect to the forgetful morphisms. Moreover recall that

$$\psi_{x_i} = c_1(\omega_{\pi_{x_i}}(D_{x_i}))$$

on $\overline{\mathcal{M}}_{g,X}$, where $D_{x_i} = \sum_{j \neq i} \delta_{0,\{x_i,x_j\}}$ (as shown in [71]). So, in order to find a "minimal" projective compactification of $\mathcal{M}_{g,X}$ where to compute the intersection numbers of the ψ classes, it is natural to look at the maps induced by the linear system $\mathbb{L} := \sum_{x_i \in X} \omega_{\pi_{x_i}}(D_{x_i})$. It is well-known that $\mathbb{L}$ is nef and big (Arakelov [4] and Mumford [57]), so that the problem is to decide whether $\mathbb{L}$ is semi-ample and to determine its exceptional locus $Ex(\mathbb{L}^{\otimes d})$ for $d \gg 0$.

It is easy to see that $\mathbb{L}^{\otimes d}$ pulls back to the trivial line bundle via the boundary map $\overline{\mathcal{M}}_{g',\{x'\}} \times \{C\} \longrightarrow \overline{\mathcal{M}}_{g,X}$, where C is a fixed curve of genus $g - g'$ with a $X \cup \{x''\}$-marking and the map glues x' with x''. Hence the map induced by the linear system $\mathbb{L}^{\otimes d}$ (if base-point-free) should restrict to the projection $\overline{\mathcal{M}}_{g,\{x'\}} \times \overline{\mathcal{M}}_{g-g',X\cup\{x''\}} \longrightarrow \overline{\mathcal{M}}_{g-g',X\cup\{x''\}}$ on these boundary components.

Whereas $\mathbb{L}$ is semi-ample in characteristic $p > 0$, it is not so in characteristic 0 (Keel [37]). However, one can still topologically contract the exceptional (with respect to $\mathbb{L}$) curves to obtain Kontsevich's map

$$\xi' : \overline{\mathcal{M}}_{g,X} \longrightarrow \overline{\mathcal{M}}_{g,X}^{K}$$

which is a proper continuous surjection of orbispaces. A consequence of Keel's result is that the coarse $\overline{M}^K_{g,P}$ cannot be given a scheme structure such that the contraction map is a morphism. This is in some sense unexpected, because the morphism behaves as if it were algebraic: in particular, the fiber product $\overline{M}_{g,X} \times_{\overline{M}^K_{g,X}} \overline{M}_{g,X}$ is projective.

Remark 3.5. $\overline{\mathcal{M}}^K_{g,X}$ can be given the structure of a stratified orbispace, where the stratification is again by topological type of the generic curve in the fiber of ξ'. Also, the stabilizer of a point s in $\overline{\mathcal{M}}^K_{g,X}$ will be the same as the stabilizer of the generic point in $(\xi')^{-1}(s)$.

3.5.2 Visibly equivalent curves. So now we leave the realm of algebraic geometry and proceed topologically to construct and describe this different compactification. In fact we introduce a slight modification of Kontsevich's construction (see [41]). We realize it as a quotient of $\overline{\mathcal{M}}_{g,X} \times \Delta_X$ by an equivalence relation, where Δ_X is the standard simplex in $\mathbb{R}^X$.

If $(\Sigma, \underline{p})$ is an element of $\overline{\mathcal{M}}_{g,X} \times \Delta_X$, then we say that an irreducible component of Σ (and so the associated vertex of the dual graph ζ_Σ) is *visible* with respect to $\underline{p}$ if it contains a point $x_i \in X$ such that $p_i > 0$.

Next, we declare that $(\Sigma, \underline{p})$ is equivalent to $(\Sigma', \underline{p}')$ if $\underline{p} = \underline{p}'$ and there is a homeomorphism of pointed surfaces $\Sigma \xrightarrow{\sim} \Sigma'$, which is biholomorphic on the visible components of Σ. As this relation would not give back a Hausdorff space we consider its closure, which we are now going to describe.

Consider the following two moves on the dual graph ζ_Σ:

(1) if two invisible vertices w and w' are joined by an edge e, then we can build a new graph discarding e, merging w and w' along e, thus obtaining a new vertex w'', which we label with $(g_{w''}, X_{w''}) := (g_w + g_{w'}, X_w \cup X_{w'})$

(2) if an invisible vertex w has a loop e, we can make a new graph discarding e and relabeling w with $(g_w + 1, X_w)$.

Applying these moves to ζ_Σ iteratively until the process ends, we end up with a *reduced dual graph* $\zeta^{red}_{\Sigma,\underline{p}}$. Call $V_-(\Sigma, \underline{p})$ the subset of invisible vertices and $V_+(\Sigma, \underline{p})$ the subset of visible vertices of $\zeta^{red}_{\Sigma,\underline{p}}$.

For every couple $(\Sigma, \underline{p})$ denote by $\overline{\Sigma}$ the quotient of Σ obtained collapsing every invisible component to a point.

We say that $(\Sigma, \underline{p})$ and $(\Sigma', \underline{p}')$ are *visibly equivalent* if $\underline{p} = \underline{p}'$ and there exist a homeomorphism $\overline{\Sigma} \xrightarrow{\sim} \overline{\Sigma}'$, whose restriction to each component is analytic, and a compatible isomorphism $f^{red} : \zeta^{red}_{\Sigma,p} \xrightarrow{\sim} \zeta^{red}_{\Sigma',p'}$ of reduced dual graphs.

Remark 3.6. In other words, $(\Sigma, \underline{p}), (\Sigma', \underline{p}')$ are visibly equivalent if and only if: $\underline{p} = \underline{p}'$ and there exists a third stable Σ'' and maps $h : \Sigma'' \to \Sigma$ and

$h' : \Sigma'' \to \Sigma'$ such that h, h' are biholomorphic on the visible components and are a stable marking on the invisible components of $(\Sigma'', \underline{p})$ (that is, they may shrink some disjoint simple closed curves to nodes and are homeomorphisms elsewhere).

Finally call

$$\xi : \overline{\mathcal{M}}_{g,X} \times \Delta_X \longrightarrow \overline{\mathcal{M}}^{\Delta}_{g,X} := \overline{\mathcal{M}}_{g,X} \times \Delta_X / \sim$$

the quotient map and remark that $\overline{\mathcal{M}}^{\Delta}_{g,X}$ is compact and that ξ commutes with the projection onto Δ_X.

Similarly, one can say that two (S, X)-marked stable surfaces $([f : S \to \Sigma], \underline{p})$ and $([f' : S \to \Sigma'], \underline{p}')$ are visibly equivalent if there exists a third stable (S, X)-marked surface $[f'' : S \to \Sigma'']$ and maps $h : \Sigma'' \to \Sigma$ and $h' : \Sigma'' \to \Sigma'$ such that $h \circ f'' \simeq f$, $h' \circ f'' \simeq f'$ and $(\Sigma, \underline{p}), (\Sigma', \underline{p}')$ are visibly equivalent through h, h' (see the remark above). Consequently, we can define $\overline{\mathcal{T}}^{\Delta}(S, X)$ as the quotient of $\overline{\mathcal{T}}(S, X) \times \Delta_X$ obtained by identifying visibly equivalent (S, X)-marked surfaces.

For every $\underline{p}$ in Δ_X, we will denote by $\overline{\mathcal{M}}^{\Delta}_{g,X}(\underline{p})$ the subset of points of the type $[\Sigma, \underline{p}]$. Then it is clear that $\overline{\mathcal{M}}^{\Delta}_{g,X}(\Delta^{\circ}_X)$ is in fact homeomorphic to a product $\overline{\mathcal{M}}^{\Delta}_{g,X}(\underline{p}) \times \Delta^{\circ}_X$ for any given $\underline{p} \in \Delta^{\circ}_X$. Observe that $\overline{\mathcal{M}}^{\Delta}_{g,X}(\underline{p})$ is isomorphic to $\overline{\mathcal{M}}^{K}_{g,X}$ for all $\underline{p} \in \Delta^{\circ}_X$ in such a way that

$$\xi_{\underline{p}} : \overline{\mathcal{M}}_{g,X} \cong \overline{\mathcal{M}}_{g,X} \times \{\underline{p}\} \longrightarrow \overline{\mathcal{M}}^{\Delta}_{g,X}(\underline{p})$$

identifies to ξ'.

Notice, by the way, that the fibers of ξ are isomorphic to moduli spaces. More precisely consider a point $[\Sigma, \underline{p}]$ of $\overline{\mathcal{M}}^{\Delta}_{g,X}$. For every $w \in V_{-}(\Sigma, \underline{p})$, call Q_v the subset of edges of $\zeta^{red}_{\Sigma,\underline{p}}$ outgoing from w. Then we have the natural isomorphism

$$\xi^{-1}([\Sigma, \underline{p}]) \cong \prod_{w \in V_{-}(\Sigma, \underline{p})} \overline{\mathcal{M}}_{g_w, X_w \cup Q_w}$$

according to the fact that $\overline{M}_{g,X} \times_{\overline{M}^{K}_{g,X}} \overline{M}_{g,X}$ is projective.

4 Cell decompositions of the moduli space of curves

4.1 Harer-Mumford-Thurston construction

One traditional way to associate a weighted arc system to a Riemann surface endowed with weights at its marked points is to look at critical trajectories of Jenkins-Strebel quadratic differentials. Equivalently, to decompose the punctured surface into a union of semi-infinite flat cylinders with lengths assigned to their circumference.

4.1.1 Quadratic differentials. Let Σ be a compact Riemann surface and let φ be a *meromorphic quadratic differential*, that is $\varphi = \varphi(z)dz^2$ where z is a local holomorphic coordinate and $\varphi(z)$ is a meromorphic function. Being a quadratic differential means that, if $w = w(z)$ is another local coordinate, then $\varphi = \varphi(w)\left(\dfrac{dz}{dw}\right)^2 dw^2$.

Regular points of Σ for φ are points where φ has neither a zero nor a pole; *critical points* are zeroes or poles of φ.

We can attach a metric to φ, by simply setting $|\varphi| := \sqrt{\varphi\overline{\varphi}}$. In coordinates, $|\varphi| = |\varphi(z)|dz\,d\overline{z}$. The metric is well-defined and flat at the regular points and it has conical singularities (with angle $\alpha = (k+2)\pi$) at simple poles ($k = -1$) and at zeroes of order k. Poles of order 2 or higher are at infinite distance.

If P is a regular point, we can pick a local holomorphic coordinate z at $P \in U \subset \Sigma$ such that $z(P) = 0$ and $\varphi = dz^2$ on U. The choice of z is unique up sign. Thus, $\{Q \in U \mid z(Q) \in \mathbb{R}\}$ defines a real-analytic curve through P on Σ, which is called a *horizontal trajectory* of φ. Similarly, $\{Q \in U \mid z(Q) \in i\mathbb{R}\}$ defines the *vertical trajectory* of φ through P.

Horizontal (resp. vertical) trajectories τ are intrinsically defined by asking that the restriction of φ to τ is a positive-definite (resp. negative-definite) symmetric bilinear form on the tangent bundle of τ.

If φ has at worst double poles, then the local aspect of horizontal trajectories is as in Figure 6 (horizontal trajectories through q are drawn thicker).

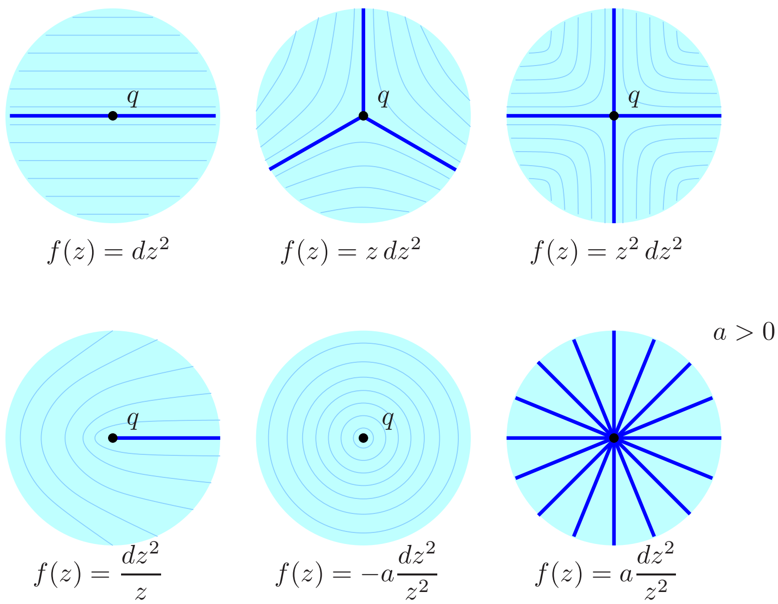

Figure 6. Local structure of horizontal trajectories.

Trajectories are called *critical* if they meet a critical point. It follows from the general classification (see [67]) that

- a trajectory is *closed* if and only if it is either periodic or it starts and ends at a critical point;
- if a horizontal trajectory τ is *periodic*, then there exists a maximal open annular domain $A \subset \Sigma$ and a number $c > 0$ such that
$$\left(A, \varphi\Big|_A\right) \xrightarrow{\sim} \left(\{z \in \mathbb{C} \,|\, r < |z| < R\}, -c\frac{dz^2}{z^2}\right)$$
and, under this identification, $\tau = \{z \in \mathbb{C} \,|\, h = |z|\}$ for some $h \in (r, R)$;
- if all horizontal trajectories are closed of finite length, then φ has at worst double poles and there it has negative quadratic residue (i.e. at a double pole, φ looks like $-a\dfrac{dz^2}{z^2}$, with $a > 0$).

4.1.2 Jenkins-Strebel differentials. There are many theorems about existence and uniqueness of quadratic differentials φ with specific behaviors of their trajectories and about their characterization using extremal properties of the associated metric $|\varphi|$ (see Jenkins [35]). The following result is the one we are interested in.

Theorem 4.1 (Strebel [66]). *Let Σ be a compact Riemann surface of genus g and $X = \{x_1, \dots, x_n\} \subset \Sigma$ such that $2g - 2 + n > 0$. For every $(p_1, \dots, p_n) \in \mathbb{R}_+^X$ there exists a unique quadratic differential φ such that*

(a) *φ is holomorphic on $\Sigma \setminus X$*

(b) *all horizontal trajectories of φ are closed*

(c) *it has a double pole at x_i with quadratic residue $-\left(\dfrac{p_i}{2\pi}\right)^2$*

(d) *the only annular domains of φ are pointed discs at the x_i's.*

Moreover, φ depends continuously on Σ and on $\underline{p} = (p_1, \dots, p_n)$.

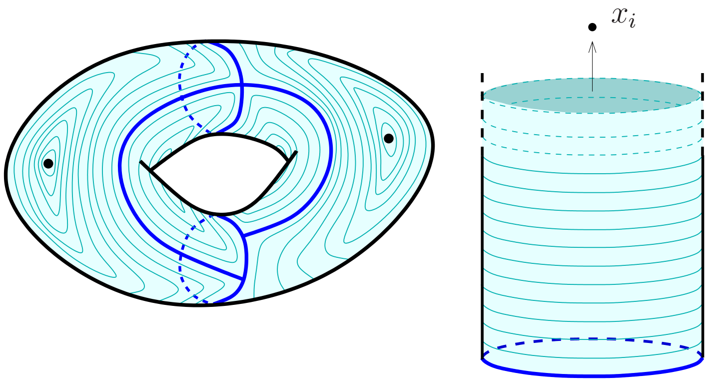

Figure 7. Example of horizontal foliation of a Jenkins-Strebel differential.

Remark 4.2. Notice that the previous result establishes the existence of a continuous map

$$\mathbb{R}_+^X \longrightarrow \{\text{continuous sections of } \mathcal{Q}(S, 2X) \to \mathcal{T}(S, X)\}$$

where $\mathcal{Q}(S, 2X)$ is the vector bundle whose fiber over $[f : S \to \Sigma]$ is the space of quadratic differentials on Σ, which can have double poles at X and are holomorphic elsewhere. Hubbard and Masur [30] proved (in a slightly different case, though) that the sections of $\mathcal{Q}(S, 2X)$ are piecewise real-analytic and gave precise equations for their image.

Quadratic differentials that satisfy (a) and (b) are called *Jenkins-Strebel differentials*. They are particularly easy to understand because their critical trajectories form a graph $G = G_{\Sigma,\underline{p}}$ embedded inside the surface Σ and G decomposes Σ into a union of cylinders (with respect to the flat metric $|\varphi|$), whose circumferences are horizontal trajectories.

Property (d) is telling us that $\Sigma \setminus X$ retracts by deformation onto G, flowing along the vertical trajectories out of X.

Remark 4.3. It can be easily seen that Theorem 4.1 still holds for $p_1, \dots, p_n \geq 0$ but $\underline{p} \neq 0$. Condition (d) can be rephrased by saying that every annular domain corresponds to some x_i for which $p_i > 0$, and that $x_j \in G$ if $p_j = 0$. It is still true that $\Sigma \setminus X$ retracts by deformation onto G.

We sketch the traditional existence proof of Theorem 4.1.

Definition 4.4. The *modulus* of a standard annulus $A(r, R) = \{z \in \mathbb{C} \,|\, r < |z| < R\}$ is $m(A(r,R)) = \dfrac{1}{2\pi}\log(R/r)$ and the modulus of an annulus A is defined to be that of a standard annulus biholomorphic to A. Consider a simply connected domain $0 \in U \subset \mathbb{C}$ and let z be a holomorphic coordinate at 0. The *reduced modulus* of the annulus $U^* = U \setminus \{0\}$ is $m(U^*, z) = m(U^* \cap \{|z| > \varepsilon\}) + \frac{1}{2\pi}\log(\varepsilon)$, which is independent of the choice of a sufficiently small $\varepsilon > 0$.

Notice that the *extremal length* E_γ of a circumference γ inside $A(r, R)$ is exactly $1/m(A(r,R))$.

Existence of Jenkins-Strebel differential. Fix holomorphic coordinates $z_1, \dots, z_n$ at $x_1, \dots, x_n$. A *system of annuli* is a holomorphic injection $s : \Delta \times X \hookrightarrow \Sigma$ such that $s(0, x_i) = x_i$, where Δ is the unit disc in $\mathbb{C}$. Call $m_i(s)$ the reduced modulus $m(s(\Delta \times \{x_i\}), z_i)$ and define the functional

$$F: \{\text{systems of annuli}\} \longrightarrow \mathbb{R}$$
$$s \longmapsto \sum_{i=1}^{n} p_i^2 m_i(s)$$

which is bounded above, because $\Sigma \setminus X$ is hyperbolic. A maximizing sequence s_n converges (up to extracting a subsequence) to a system of annuli s_∞. Let $D_i = s_\infty(\Delta \times \{x_i\})$. Notice that the restriction of s_∞ to $\Delta \times \{x_i\}$ is injective if $p_i > 0$ and is constantly x_i if $p_i = 0$.

Clearly, s_∞ is maximizing for every choice of $z_1, \dots, z_n$ and so we can assume that, whenever $p_i > 0$, z_i is the coordinate induced by s_∞.

Define the L^1_{loc}-quadratic differential φ on $\Sigma \setminus X$ as $\varphi := \left(-\dfrac{p_i^2}{4\pi^2}\dfrac{dz_i^2}{z_i^2}\right)$ on D_i (if $p_i > 0$) and $\varphi = 0$ elsewhere. Notice that $F(s_\infty) = \|\varphi\|_{red}$, where the *reduced norm* is given by

$$\|\varphi\|_{red} := \int_\Sigma \left[|\varphi|^2 - \sum_{i:p_i>0} \frac{p_i^2}{4\pi^2}\frac{dz_i\, d\overline{z_i}}{|z_i|^2}\chi(|z_i| < \varepsilon_i)\right] + \sum_{i=1}^{n} \frac{p_i^2}{2\pi}\log(\varepsilon_i)$$

which is independent of the choice of sufficiently small $\varepsilon_1, \dots, \varepsilon_n > 0$.

As s_∞ is a stationary point for F, so is for $\|\cdot\|_{red}$. Thus, for every smooth vector field $V = V(z)\partial/\partial z$ on Σ, compactly supported on $\Sigma \setminus X$, the first-order variation of

$$\|f_t^*(\varphi)\|_{red} = \|\varphi\|_{red} + 2t \int_S \mathrm{Re}(\varphi\overline{\partial}V) + o(t)$$

must vanish, where $f_t = \exp(tV)$. Thus, φ is holomorphic on $\Sigma \setminus X$ by Weyl's lemma and it satisfies all the requirements. □

4.1.3 The nonsingular case. Using the construction described above, we can attach to every $(\Sigma, X, \underline{p})$ a graph $G_{\Sigma,\underline{p}} \subset \Sigma$ (and thus an (S, X)-marked ribbon graph $\mathbb{G}_{\Sigma,\underline{p}}$) which is naturally metrized by $|\varphi|$. By arc/graph duality (in the nonsingular case, see 2.2.8), we also have a weighted proper system of arcs in Σ. Notice that, because of (c), the boundary weights are exactly $p_1, \dots, p_n$.

If $[f : S \to \Sigma]$ is a point in $\mathcal{T}(S, X)$ and $\underline{p} \in (\mathbb{R}^X_{\geq 0}) \setminus \{0\}$, then the previous construction (which is explicitly mentioned by Harer in [27], where he attributes it to Mumford and Thurston) provides a point in $|\mathfrak{A}^\circ(S, X)| \times \mathbb{R}_+$. It is however clear that, if $a > 0$, then the Strebel differential associated to $(\Sigma, a\underline{p})$ is $a\varphi$. Thus, we can just consider $\underline{p} \in \mathbb{P}(\mathbb{R}^X_{\geq 0}) \cong \Delta_X$, so that the corresponding weighted arc system belongs to $|\mathfrak{A}^\circ(S, X)|$ (after multiplying by a factor 2).

Because of the continuous dependence of φ on Σ and $\underline{p}$, the map

$$\Psi_{JS} : \mathcal{T}(S, X) \times \Delta_X \longrightarrow |\mathfrak{A}^\circ(S, X)|$$

is *continuous*.

We now show that a point $\overline{w} \in |\mathfrak{A}^\circ(S, X)|$ determines exactly one (S, X)-marked surface, which proves that Ψ_{JS} is *bijective*.

By 2.2.9, we can associate a metrized (S, X)-marked nonsingular ribbon graph $\mathbb{G}_{\underline{\boldsymbol{\alpha}}}$ to each $w \in |\mathfrak{A}^\circ(S, X)|_\mathbb{R}$ supported on $\underline{\boldsymbol{\alpha}}$. However, if we realize $|\mathbb{G}_{\underline{\boldsymbol{\alpha}}}|$ by gluing semi-infinite tiles $T_{\vec{\alpha_i}}$ of the type $[0, w(\alpha_i)]_x \times [0, \infty)_y \subset \hat{\mathbb{C}}_z$, which naturally come together with a complex structure and a quadratic differential dz^2, then $|\mathbb{G}_{\underline{\boldsymbol{\alpha}}}|$ becomes a Riemann surface endowed with the (unique) Jenkins-Strebel quadratic differential φ determined by Theorem 4.1. Thus, $\Psi_{JS}^{-1}(w) = ([f : S \to |\mathbb{G}_{\underline{\boldsymbol{\alpha}}}|], \underline{p})$, where p_i is obtained from the quadratic residue of φ at x_i. Moreover, the length function defined on $|\mathfrak{A}^\circ(S, X)|_\mathbb{R}$ exactly corresponds to the $|\varphi|$-length function on $\mathcal{T}(S, X) \times \Delta_X \times \mathbb{R}_+$.

Notice that Ψ_{JS} is $\Gamma(S, X)$-equivariant by construction and so induces a continuous bijection $\overline{\Psi}_{JS} : \mathcal{M}_{g,X} \times \Delta_X \to |\mathfrak{A}^\circ(S, X)|/\Gamma(S, X)$ on the quotient. If we prove that $\overline{\Psi}_{JS}$ is *proper*, then $\overline{\Psi}_{JS}$ is a homeomorphism. To conclude that Ψ_{JS} is a homeomorphism too, we will use the following.

Lemma 4.5. *Let Y and Z be metric spaces acted on discontinuously by a discrete group of isometries G and let $h : Y \to Z$ be a G-equivariant continuous* *injection such that the induced map* $\overline{h} : Y/G \to Z/G$ *is a homeomorphism. Then* h *is a homeomorphism.*

Proof. To show that h is surjective, let $z \in Z$. Because $\overline{h}$ is bijective, $\exists ! [y] \in Y/G$ such that $\overline{h}([y]) = [z]$. Hence, $h(y) = z \cdot g$ for some $g \in G$ and so $h(y \cdot g^{-1}) = z$.

To prove that h^{-1} is continuous, let $(y_m) \subset Y$ be a sequence such that $h(y_m) \to h(y)$ as $m \to \infty$ for some $y \in Y$. Clearly, $[h(y_m)] \to [h(y)]$ in Z/G and so $[y_m] \to [y]$ in Y/G, because $\overline{h}$ is a homeomorphism. Let $(v_m) \subset Y$ be a sequence such that $[v_m] = [y_m]$ and $v_m \to y$ and call $g_m \in G$ the element such that $y_m = v_m \cdot g_m$. By continuity of h, we have $d_Z(h(v_m), h(y)) \to 0$ and by hypothesis $d_Z(h(v_m) \cdot g_m, h(y)) \to 0$. Hence, $d_Z(h(y), h(y) \cdot g_m) \to 0$ and so $g_m \in \mathrm{stab}(h(y)) = \mathrm{stab}(y)$ for large m, because G acts discontinuously on Z. As a consequence, $y_m \to y$ and so h^{-1} is continuous. □

The final step is the following.

Lemma 4.6. $\overline{\Psi}_{JS} : \mathcal{M}_{g,X} \times \Delta_X \to |\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$ *is proper.*

Proof. Let $([\Sigma_m], \underline{p}_m)$ be a diverging sequence in $\mathcal{M}_{g,X} \times \Delta_X$ and call λ_m the hyperbolic metric on $\Sigma \setminus X$. By the Mumford-Mahler criterion, there exist simple closed hyperbolic geodesics $\gamma_m \subset \Sigma_m$ such that $\ell_{\lambda_m}(\gamma_m) \to 0$. Because the hyperbolic length and the extremal length are approximately proportional for short curves, we conclude that the extremal length $E(\gamma_m) \to 0$.

Consider now the metric $|\varphi_m|$ induced by the Jenkins-Strebel differential φ_m uniquely determined by $(\Sigma_m, \underline{p}_m)$. Call $\ell_\varphi(\gamma_m)$ the length of the unique geodesic $\tilde{\gamma}_m$ with respect to the metric $|\varphi_m|$, freely homotopic to $\gamma_m \subset \Sigma_m$. Notice that $\tilde{\gamma}_m$ is a union of critical horizontal trajectories.

Because $|\varphi_m|$ has infinite area, define a modified metric g_m on Σ_m in the same conformal class as $|\varphi_m|$ as follows.

- g_m agrees with $|\varphi_m|$ on the critical horizontal trajectories of φ_m
- Whenever $p_{i,m} > 0$, consider a coordinate z at x_i such that the annular domain of φ_m at x_i is exactly $\Delta^* = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$ and $\varphi_m = -\dfrac{p_{i,m}^2 dz^2}{4\pi^2 z^2}$. Then define g_m to agree with $|\varphi_m|$ on $\exp(-2\pi/p_{i,m}) \leq |z| < 1$ (which becomes isometric to a cylinder of circumference $p_{i,m}$ and height 1, so with area $p_{i,m}$) and to be the metric of a flat Euclidean disc of circumference $p_{i,m}$ centered at $z = 0$ (so with area $\pi p_{i,m}^2$) on $|z| < \exp(-2\pi/p_{i,m})$.

Notice that the total area $A(g_m)$ is $\pi(p_{1,m}^2 + \cdots + p_{n,m}^2) + (p_{1,m} + \cdots + p_{n,m}) \leq \pi + 1$.

Call $\ell_g(\gamma_m)$ the length of the shortest g_m-geodesic $\hat{\gamma}_m$ in the class of γ_m. By definition, $\ell_g(\gamma_m)^2/A(g_m) \leq E(\gamma_m) \to 0$ and so $\ell_g(\gamma_m) \to 0$. As a g_m-

geodesic is either longer than 1 or contained in the critical graph of φ, then $\hat{\gamma}_m$ coincides with $\tilde{\gamma}_m$ for $m \gg 0$.

Hence, $\ell_\varphi(\gamma_m) \to 0$ and so $\mathrm{sys}(\overline{w}_m) \to 0$. By Lemma 2.2, we conclude that $\overline{\Psi}_{JS}(\Sigma_m, \underline{p}_m)$ diverges in $|\mathfrak{A}^\circ(S,X)|/\Gamma(S,X)$. □

Remark 4.7. Suppose that $([f_m : S \to \Sigma_m], \underline{p}_m)$ converges to $([f : S \to \Sigma], \underline{p}) \in \overline{\mathcal{T}}_{g,X} \times \Delta_X$ and let $\Sigma' \subset \Sigma$ be an invisible component, that is a component of Σ with no positively weighted marked points. Then, $S' = f^{-1}(\Sigma')$ is bounded by simple closed curves $\gamma_1, \dots, \gamma_k \subset S$ and $\ell_{f_m^* \varphi_m}(\gamma_i) \to 0$ for $i = 1, \dots, k$. Just analyzing the shape of the critical graph of φ_m, one can check that $\ell_{\varphi_m}(\gamma) \leq \sum_{i=1}^k \ell_{\varphi_m}(\gamma_i)$ for all $\gamma \subset S'$. Hence, $\ell_{f_m^*\varphi_m}(\gamma) \to 0$ uniformly in γ, and so $f_m^* \varphi_m$ tends to zero uniformly on the compact subsets of $(S')^\circ$.

4.1.4 The case of stable curves. We want to extend the map Ψ_{JS} to Deligne-Mumford's augmentation and, by abuse of notation, we will still call $\Psi_{JS} : \overline{\mathcal{T}}(S,X) \times \Delta_X \to |\mathfrak{A}(S,X)|$ this extension.

Given $([f : S \to \Sigma], \underline{p})$, we can construct a Jenkins-Strebel differential φ on each visible component of Σ, by considering nodes as marked points with zero weight. Extend φ to zero over the invisible components. Clearly, φ is a holomorphic section of $\omega_\Sigma^{\otimes 2}(2X)$ (the square of the logarithmic dualizing sheaf on Σ): call it the *Jenkins-Strebel differential associated to* $(\Sigma, \underline{p})$. Notice that it clearly maximizes the functional F, used in the proof of Theorem 4.1.

As φ defines a metrized ribbon graph for each visible component of Σ, one can easily see that thus we have an (S,X)-marked enriched ribbon graph $\mathbb{G}^{en}$ (see 2.2.4), where ζ is the dual graph of Σ and V_+ is the set of visible components of $(\Sigma, \underline{p})$, m is determined by the X-marking and s by the position of the nodes.

By arc/graph duality (see 2.2.13), we obtain a system of arcs $\underline{\boldsymbol{\alpha}}$ in (S,X) and the metrics provide a system of weights $\overline{w}$ with support on $\underline{\boldsymbol{\alpha}}$. This defines the set-theoretic extension of Ψ_{JS}. Clearly, it is still $\Gamma(S,X)$-equivariant and it identifies visibly equivalent (S,X)-marked surfaces. Thus, it descends to a bijection $\Psi_{JS} : \overline{\mathcal{T}}^\Delta(S,X) \to |\mathfrak{A}(S,X)|$ and we also have

$$\overline{\Psi}_{JS} : \overline{\mathcal{M}}_{g,X}^\Delta \longrightarrow |\mathfrak{A}(S,X)|/\Gamma(S,X)$$

where $|\mathfrak{A}(S,X)|/\Gamma(S,X)$ can be naturally given the structure of an orbispace (essentially, forgetting the Dehn twists along curves of S that are shrunk to points, so that the stabilizer of an arc system just becomes the automorphism group of the corresponding enriched X-marked ribbon graph).

The only thing left to prove is that Ψ_{JS} is continuous. In fact, $\overline{\mathcal{M}}_{g,X}^\Delta$ is compact and $|\mathfrak{A}(S,X)|/\Gamma(S,X)$ is Hausdorff: hence, $\overline{\Psi}_{JS}$ would be (continuous and) automatically proper, and so a homeomorphism. Using Lemma 4.5 again (using a metric pulled back from $\overline{\mathcal{M}}^{\Delta}_{g,X}$), we could conclude that Ψ_{JS} is a homeomorphism too.

Continuity of Ψ_{JS}. Consider a differentiable stable family

$$\begin{array}{ccc} S \times [0,\varepsilon] & \xrightarrow{f} & \mathcal{C} \\ & \searrow & \downarrow g \\ & & [0,\varepsilon] \end{array}$$

of (S, X)-marked curves (that is, obtained restricting to $[0, \varepsilon]$ a smooth family over the unit disc Δ), such that g is topologically trivial over $(0, \varepsilon]$ with fiber a curve with k nodes. Let also $\underline{p} : [0, \varepsilon] \to \Delta_X$ be a differentiable family of weights.

We can assume that there are disjoint simple closed curves $\gamma_1, \dots, \gamma_k, \eta_1, \dots, \eta_h \subset S$ such that $f(\gamma_i \times \{t\})$ is a node for all t, that $f(\eta_j \times \{t\})$ is a node for $t = 0$ and that $\mathcal{C}_t$ is smooth away from these nodes.

Fix a nonempty open relatively compact subset K of $S \setminus (\gamma_1 \cup \dots \cup \gamma_k \cup \eta_1 \cup \dots \cup \eta_h)$ that intersects every connected component. Define a reduced L^1 norm of a section ψ_t of $\omega_{\mathcal{C}_t}^{\otimes 2}(2X)$ to be $\|\psi\|_{red} = \int_{f_t(K)} |\psi|$. Notice that L^1 convergence of holomorphic sections ψ_t as $t \to 0$ implies uniform convergence of $f_t^* \psi_t$ on the compact subsets of $S \setminus (\gamma_1 \cup \dots \cup \gamma_k \cup \eta_1 \cup \dots \cup \eta_h)$.

Call φ_t the Jenkins-Strebel differential associated to $(\mathcal{C}_t, \underline{p}_t)$ with annular domains $D_{1,t}, \dots, D_{n,t}$.

As all the components of $\mathcal{C}_t$ are hyperbolic, $\|\varphi_t\|_{red}$ is uniformly bounded and we can assume (up to extracting a subsequence) that φ_t converges to a holomorphic section φ'_0 of $\omega_{\mathcal{C}_0}^{\otimes 2}(2X)$ in the reduced norm. Clearly, φ'_0 will have double poles at x_i with prescribed residue.

Remark 4.7 implies that φ'_0 vanishes on the invisible components of $\mathcal{C}_0$, whereas it certainly does not on the visible ones.

For all those $(i, t) \in \{1, \dots, n\} \times [0, \varepsilon]$ such that $p_{i,t} > 0$, let $z_{i,t}$ be the coordinate at x_i (uniquely defined up to phase) given by $z_{i,t} = u_{i,t}^{-1}\Big|_{D_{i,t}}$ and

$$u_{i,t} : \overline{\Delta} \longrightarrow \overline{D}_{i,t} \subset \mathcal{C}_t$$

is continuous on $\overline{\Delta}$ and biholomorphic in the interior for all $t > 0$ and $\varphi_t\Big|_{D_{i,t}} = -\dfrac{p_{i,t}^2 dz_{i,t}^2}{4\pi^2 z_{i,t}^2}$ for $t \geq 0$. Whenever $p_{i,t} = 0$, choose $z_{i,t}$ such that $\varphi_t\Big|_{D_{i,t}} = z^k\, dz^2$, with $k = \mathrm{ord}_{x_i} \varphi_t$. When $p_{i,t} > 0$, we can choose the phases of $u_{i,t}$ in such a way that $u_{i,t}$ vary continuously with $t \geq 0$.

If $p_{i,0} = 0$, then set $D_{i,0} = \emptyset$. Otherwise, $p_{i,0} > 0$ and so $D_{i,0}$ cannot shrink to $\{x_i\}$ (because F_t would go to $-\infty$ as $t \to 0$). In this case, call $D_{i,0}$ the region $\{|z_{i,0}|<1\}\subset \mathcal{C}_0$. Notice that φ_0' has a double pole at x_i with residue $p_{i,0}>0$ and clearly $\varphi_0'\Big|_{D_{i,0}} = -\dfrac{p_{i,0}^2 dz_{i,0}^2}{4\pi^2 z_{i,0}^2}$.

We want to prove that the visible subsurface of $\mathcal{C}_0$ is covered by $\bigcup_i \overline{D}_{i,0}$ and so φ_0' is a Jenkins-Strebel differential on each visible component of $\mathcal{C}_0$. By uniqueness, it must coincide with φ_0.

Consider a point y in the interior of $f_0^{-1}(\mathcal{C}_{0,+})\setminus X$. For every $t>0$ there exists an $y_t\in S$ such that $f_t(y_t)$ does not belong to the critical graph of φ_t and the $f_t^*|\varphi_t|$-distance $d_t(y,y_t)<t$. As $\varphi_t\to\varphi_0$ in reduced norm and $y,y_t\notin X$, then $d_0(y,y_t)\to 0$ as $t\to 0$.

We can assume (up to discarding some t's) that $f_t(y_t)$ belongs to $D_{i,t}$ for a fixed i and in particular that $f_t(y_t)=u_{i,t}(c_t)$ for some $c_t\in\Delta$. Up to discarding some t's, we can also assume that $c_t\to c_0\in\overline{\Delta}$. Call y_t' the point given by $f_0(y_t')=u_{i,0}(c_t)$.

$$\begin{aligned} d_0(y_t',y) &\leq d_0(y_t,y)+d_0(y_t',y_t)\leq d_0(y_t,y)+d_0(f_0^{-1}u_{i,0}(c_t),f_t^{-1}u_{i,t}(c_t))\leq\\ &\leq d_0(y_t,y)+d_0(f_0^{-1}u_{i,0}(c_t),f_0^{-1}u_{i,0}(c_0))+\\ &\qquad + d_0(f_0^{-1}u_{i,0}(c_0),f_t^{-1}u_{i,t}(c_0))+d_0(f_t^{-1}u_{i,t}(c_0),f_t^{-1}u_{i,t}(c_t)) \end{aligned}$$

and all terms go to zero as $t\to 0$. Thus, every point in the smooth locus $\mathcal{C}_{0,+}\setminus X$ is at $|\varphi_0|$-distance zero from some $D_{i,0}$. Hence, φ_0 is a Jenkins-Strebel differential on the visible components.

With a few simple considerations, one can easily conclude that

- the zeroes of φ_t move continuously as $t\in[0,\varepsilon]$
- if e_t is an edge of the critical graph of φ_t which starts at $y_{1,t}$ and ends at $y_{2,t}$, and if $y_{i,t}\to y_{i,0}$ for $i=1,2$, then $e_t\to e_0$ the corresponding edge of the critical graph of φ_0 starting at $y_{1,0}$ and ending at $y_{2,0}$; moreover, $\ell_{|\varphi_t|}(e_t)\to\ell_{|\varphi_0|}(e_0)$
- the critical graph of φ_t converges to that of φ_0 for the Gromov-Hausdorff distance.

Thus, the associated weighted arc systems $\overline{w}_t\in|\mathfrak{A}(S,X)|$ converge to $\overline{w}_0$ for $t\to 0$. □

Thus, we have proved the following result, claimed by Kontsevich in [41] (see Looijenga's [43] and Zvonkine's [78]).

Proposition 4.8. *The map defined above*

$$\Psi_{JS}:\overline{\mathcal{T}}^{\Delta}(S,X)\longrightarrow|\mathfrak{A}(S,X)|$$

is a $\Gamma(S,X)$-equivariant homeomorphism, which commutes with the projection onto Δ_X. Hence, $\overline{\Psi}_{JS}:\overline{\mathcal{M}}^{\Delta}_{g,X}\to|\mathfrak{A}(S,X)|/\Gamma(S,X)$ is a homeomorphism of orbispaces too.

A consequence of the previous proposition and of 2.2.13 is that the realization $B\mathfrak{RG}_{g,X,ns}$ is the classifying space of $\Gamma(S,X)$ and that $B\mathfrak{RG}_{g,X} \to \overline{\mathcal{M}}_{g,X}$ is a homotopy equivalence (in the category of orbispaces).

4.2 Penner-Bowditch-Epstein construction

The other traditional way to obtain a weighted arc system out of a Riemann surface with weighted marked points is to look at the spine of the truncated surface obtained by removing horoballs of prescribed circumference. Equivalently, to decompose the surface into a union of hyperbolic cusps.

4.2.1 Spines of hyperbolic surfaces. Let $[f : S \to \Sigma]$ be an (S,X)-marked hyperbolic surface and let $\underline{p} \in \Delta_X$. Call $H_i \subset \Sigma$ the horoball at x_i with circumference p_i (as $p_i \leq 1$, the horoball is embedded in Σ) and let $\Sigma_{tr} = \Sigma \setminus \bigcup_i H_i$ be the *truncated surface*. The datum $(\Sigma, \partial H_1, \dots, \partial H_n)$ is also called a *decorated surface*.

For every $y \in \Sigma \setminus X$ at finite distance from $\partial\Sigma_{tr}$, let the *valence* $\mathrm{val}(y)$ be the number of paths that realize $\mathrm{dist}(y, \partial\Sigma_{tr})$, which is generically 1. We will call a *projection* of y a point on $\partial\Sigma_{tr}$ which is at shortest distance from y: clearly, there are $\mathrm{val}(y)$ of them.

Let the *spine* $\mathrm{Sp}(\Sigma, \underline{p})$ be the locus of points of Σ which are at finite distance from $\partial\Sigma_{tr}$ and such that $\mathrm{val}(y) \geq 2$ (see Figure 8). In particular, $\mathrm{val}^{-1}(2)$ is a disjoint union of finitely many geodesic arcs (the *edges*) and $\mathrm{val}^{-1}([3,\infty))$ is a finite collection of points (the *vertices*). If $p_i = 0$, then we include x_i in $\mathrm{Sp}(\Sigma, \underline{p})$ and we consider it a vertex. Its valence is defined to be the number of half-edges of the spine incident at x_i.

There is a deformation retraction of $\Sigma_{tr} \cap \Sigma_+$ (where Σ_+ is the visible subsurface) onto $\mathrm{Sp}(\Sigma, \underline{p})$, defined on $\mathrm{val}^{-1}(1)$ simply flowing away from $\partial\Sigma_{tr}$ along the unique geodesic that realizes the distance from $\partial\Sigma_{tr}$.

This shows that $\mathrm{Sp}(\Sigma, \underline{p})$ defines an (S,X)-marked enriched ribbon graph $\mathbb{G}^{en}_{sp}$. By arc/graph duality, we also have an associated *spinal arc system* $\underline{\boldsymbol{\alpha}}_{sp} \in \mathfrak{A}(S,X)$.

4.2.2 Horocyclic lengths and weights. As Σ is a hyperbolic surface, we could metrize $\mathrm{Sp}(\Sigma, \underline{p})$ by inducing a length on each edge. However, the relation between this metric and $\underline{p}$ would be a little involved.

Instead, for every edge e of $\mathbb{G}^{en}_{sp}$ (that is, of $\mathrm{Sp}(\Sigma, \underline{p})$), consider one of its two projections $pr(e)$ to $\partial\Sigma_{tr}$ and define $\ell(e)$ to be the *horocyclic length* of e, that is the hyperbolic length of $pr(e)$, which clearly does not depend on the chosen projection. Thus, the boundary weights vector ℓ_∂ is exactly $\underline{p}$.

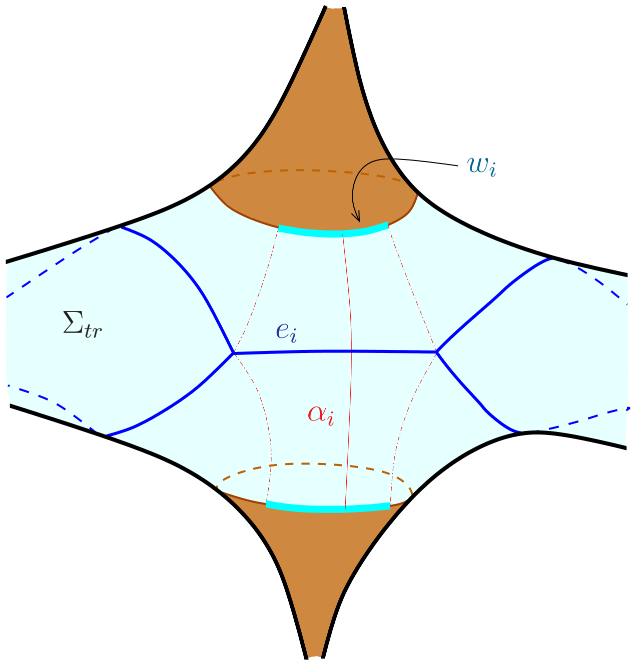

Figure 8. Weights come from lengths of horocyclic arcs.

This endows $\mathbb{G}_{sp}^{en}$ with a metric and so $\underline{\boldsymbol{\alpha}}_{sp}$ with a projective weight $\overline{w}_{sp} \in |\mathfrak{A}(S,X)|$. Notice that visibly equivalent surfaces are associated to the same point of $|\mathfrak{A}(S,X)|$.

This defines a $\Gamma(S,X)$-equivariant map

$$\Phi_0 : \overline{\mathcal{T}}^{\Delta}(S,X) \longrightarrow |\mathfrak{A}(S,X)|$$

that commutes with the projection onto Δ_X.

Penner [58] proved that the restriction of Φ_0 to $\mathcal{T}(S,X) \times \Delta^\circ$ is a homeomorphism; the statement that the whole Φ_0 is a homeomorphism appears in Bowditch-Epstein's [12] (and a very detailed treatment will appear in [7]). We refer to these papers for a proof of this result.

4.3 Hyperbolic surfaces with boundary

The purpose of this informal subsection is to briefly illustrate the bridge between the cellular decomposition of the Teichmüller space obtained using Jenkins-Strebel differentials and that obtained using spines of decorated surfaces.

4.3.1 Teichmüller and moduli space of hyperbolic surfaces. Fix a compact oriented surface S as before and $X=\{x_1,\dots,x_n\}\subset S$ a nonempty subset.

A *stable hyperbolic surface* Σ is a nodal surface such that $\Sigma\setminus\{\text{nodes}\}$ is hyperbolic with geodesic boundary and/or cusps. Notice that, by convention, $\partial\Sigma$ includes the cusps but it does not include the possible nodes of Σ.

An X-marking of a (stable) hyperbolic surface Σ is a bijection $X\to\pi_0(\partial\Sigma)$.

An (S,X)-marking of the (stable) hyperbolic surface Σ is an isotopy class of maps $f: S\setminus X\to\Sigma$, that may shrink disjoint simple closed curves to nodes and are homeomorphisms onto $\Sigma\setminus(\partial\Sigma\cup\{\text{nodes}\})$ elsewhere.

Let $\overline{\mathcal{T}}^\partial(S,X)$ be the Teichmüller space of (S,X)-marked stable hyperbolic surfaces. There is a natural map $\ell_\partial:\overline{\mathcal{T}}^\partial(S,X)\to\mathbb{R}^X_{\geq 0}$ that associates to $[f: S\to\Sigma]$ the boundary lengths of Σ, which thus descends to $\overline{\ell}_\partial:\overline{\mathcal{M}}^\partial_{g,X}\to\mathbb{R}^X_{\geq 0}$. Call $\overline{\mathcal{T}}^\partial(S,X)(\underline{p})$ (resp. $\overline{\mathcal{M}}^\partial_{g,X}(\underline{p})$) the leaf $\ell_\partial^{-1}(\underline{p})$ (resp. $\overline{\ell}_\partial^{-1}(\underline{p})$).

There is an obvious identification between $\overline{\mathcal{T}}^\partial(S,X)(0)$ (resp. $\overline{\mathcal{M}}^\partial_{g,X}(0)$) and $\overline{\mathcal{T}}(S,X)$ (resp. $\overline{\mathcal{M}}_{g,X}$).

Call $\widehat{\mathcal{M}}_{g,X}$ the blow-up of $\overline{\mathcal{M}}^\partial_{g,X}$ along $\overline{\mathcal{M}}^\partial_{g,X}(0)$: the exceptional locus can be naturally identified to the space of (projectively) decorated surfaces with cusps (which is homeomorphic to $\overline{\mathcal{M}}_{g,X}\times\Delta_X$). Define similarly $\widehat{\mathcal{T}}(S,X)$.

4.3.2 Tangent space to the moduli space. The conformal analogue of a hyperbolic surface with geodesic boundary Σ is a Riemann surface with real boundary. In fact, the double of Σ is a hyperbolic surface with no boundary and an orientation-reversing involution, that is a Riemann surface with an anti-holomorphic involution. As a consequence, $\partial\Sigma$ is a real-analytic submanifold.

This means that first-order deformations are determined by Beltrami differentials on Σ which are real on $\partial\Sigma$, and so $T_{[\Sigma]}\overline{\mathcal{M}}^\partial_{g,X}\cong H^{0,1}(\Sigma,T_\Sigma)$, where T_Σ is the sheaf of tangent vector fields $V=V(z)\partial/\partial z$, which are real on $\partial\Sigma$.

Dually, the cotangent space $T^*_{[\Sigma]}\overline{\mathcal{M}}^\partial_{g,X}$ is given by the space $\mathcal{Q}(\Sigma)$ of holomorphic quadratic differentials that are real on $\partial\Sigma$. If we call $\mathcal{H}(\Sigma)=\{\overline{\varphi}/\lambda\,|\,\varphi\in\mathcal{Q}(\Sigma)\}$, where λ is the hyperbolic metric on Σ, then $H^{0,1}(\Sigma,T_\Sigma)$ is identified to the space of harmonic Beltrami differentials $\mathcal{H}(\Sigma)$.

As usual, if Σ has a node, then quadratic differentials are allowed to have a double pole at the node, with the same quadratic residue on both branches.

If a boundary component of Σ collapses to a cusp x_i, then the cotangent cone to $\overline{\mathcal{M}}^\partial_{g,X}$ at $[\Sigma]$ is given by quadratic differentials that may have at worst a double pole at x_i with positive residue. The phase of the residue being zero corresponds to the fact that, if we take Fenchel-Nielsen coordinates on the double of Σ which are symmetric under the real involution, then the twists along $\partial\Sigma$ are zero.

4.3.3 Weil-Petersson metric. Mimicking what is done for surfaces with cusps, we can define Hermitean pairings on $\mathcal{Q}(\Sigma)$ and $\mathcal{H}(\Sigma)$, where Σ is a hyperbolic surface with boundary. In particular,

$$h(\mu,\nu) = \int_\Sigma \mu\,\overline{\nu}\cdot\lambda$$
$$h^*(\varphi,\psi) = \int_\Sigma \frac{\varphi\,\overline{\psi}}{\lambda}$$

where $\mu,\nu \in \mathcal{H}(\Sigma)$ and $\varphi,\psi \in \mathcal{Q}(\Sigma)$.

Thus, if $h = g + i\omega$, then g is the Weil-Petersson Riemannian metric and ω is the Weil-Petersson form. Write similarly $h^* = g^* + i\omega^*$, where g^* is the cometric dual to g and ω^* is the Weil-Petersson bivector field.

Notice that ω and ω^* are degenerate. This can be easily seen, because Wolpert's formula $\omega = \sum_i d\ell_i \wedge d\tau_i$ still holds. We can also conclude that the symplectic leaves of ω^* are exactly the fibers of the boundary length map ℓ_∂.

4.3.4 Spines of hyperbolic surfaces with boundary. The spine construction can be carried on, even in a more natural way, on hyperbolic surfaces with geodesic boundary.

In fact, given such a Σ whose boundary components are called $x_1,\dots,x_n$, we can define the distance from $\partial\Sigma$ and so the valence of a point in Σ and consequently the spine $\mathrm{Sp}(\Sigma)$, with no need of further information.

Similarly, if Σ has also nodes (that is, some holonomy degenerates to a parabolic element), then $\mathrm{Sp}(\Sigma)$ is embedded inside the *visible components of* Σ, i.e. those components of Σ that contain a boundary circle of positive length.

The weight of an arc $\alpha_i \in \underline{\boldsymbol{\alpha}}_{sp}$ dual to the edge e_i of $\mathrm{Sp}(\Sigma)$ is still defined as the hyperbolic length of one of the two projections of e_i to $\partial\Sigma$. Thus, the above construction gives a point $w_{sp} \in |\mathfrak{A}(S,X)| \times (0,\infty)$.

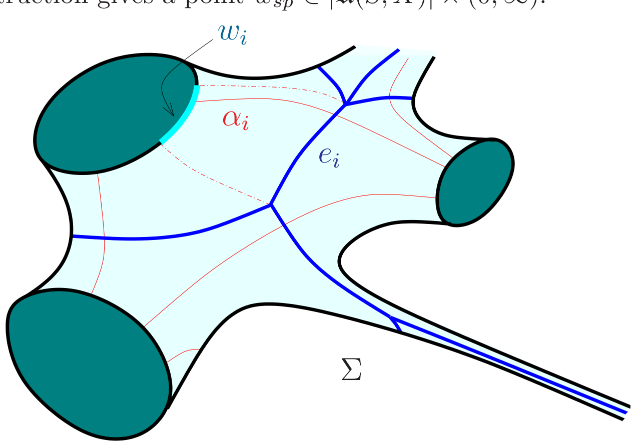

Figure 9. Weights come from lengths of geodesic boundary arcs.

It is easy to check (see [53] or [52]) that w_{sp} converges to the $\overline{w}_{sp}$ defined above when the hyperbolic surface with boundary converges to a decorated surface with cusps in $\widehat{\mathcal{T}}(S,X)$. Thus, the $\Gamma(S,X)$-equivariant map

$$\Phi : \widehat{\mathcal{T}}(S,X) \longrightarrow |\mathfrak{A}(S,X)| \times [0,\infty)$$

reduces to Φ_0 for decorated surfaces with cusps.

Theorem 4.9 (Luo [44]). *The restriction of Φ to smooth surfaces with no boundary cusps gives a homeomorphism onto its image.*

The continuity of the whole Φ is proven in [52], using Luo's result.

The key point of Luo's proof is the following. Pick a generic hyperbolic surface with geodesic boundary Σ and suppose that the spinal arc system is the ideal triangulation $\underline{\boldsymbol{\alpha}}_{sp} = \{\alpha_1, \dots, \alpha_M\} \in \mathfrak{A}^\circ(\Sigma, X)$ with weight w_{sp}. We can define the length ℓ_{α_i} as the hyperbolic length of the shortest geodesic $\tilde{\alpha}_i$ in the free homotopy class of α_i.

The curves $\{\tilde{\alpha}_i\}$ cut Σ into hyperbolic hexagons, which are completely determined by $\{\ell_{\beta_1}, \dots \ell_{\beta_{2M}}\}$, where the β_j's are the sides of the hexagons lying on $\partial\Sigma$. Unfortunately, going from the ℓ_{β_j}'s to w_{sp} is much easier than the converse. In fact, $w_{\alpha_1}, \dots, w_{\alpha_M}$ can be written as explicit linear combinations of the ℓ_{β_j}'s: in matrix notation, $B = (\ell_{\beta_j})$ is a solution of the system $W = RB$, where R is a fixed $(M \times 2M)$-matrix (that encodes the combinatorics is $\underline{\boldsymbol{\alpha}}_{sp}$) and $W = (w_{\alpha_i})$. Clearly, there is a whole affine space E_W of dimension M of solutions of $W = RB$. The problem is that a random point in E_W would determine hyperbolic structures on the hexagons of $\Sigma \setminus \underline{\boldsymbol{\alpha}}_{sp}$ that do not glue, because we are not requiring the two sides of each α_i to have the same length.

Starting from very natural quantities associated to hyperbolic hexagons with right angles, Luo defines a functional on the space $(b_1, \dots, b_{2M}) \in \mathbb{R}^{2M}_{\geq 0}$. For every W, the space E_W is not empty (which proves the surjectivity of Φ) and the restriction of Luo's functional to E_W is strictly concave and achieves its (unique) maximum exactly when $B = (\ell_{\beta_j})$ (which proves the injectivity of Φ).

The geometric meaning of this functional is still not entirely clear, but it seems related to some volume of a three-dimensional hyperbolic manifold associated to Σ. Quite recently, Luo [45] (see also [25]) has introduced a modified functional F_c, which depends on a parameter $c \in \mathbb{R}$, and he has produced other realizations of the Teichmüller space as a polytope, and so different systems of "simplicial" coordinates.

4.3.5 Surfaces with large boundary components. To close the circle, we must relate the limit of Φ for surfaces whose boundary lengths diverge to

Ψ_{JS}. This is the topic of [52]. Here, we only sketch the main ideas. To simplify the exposition, we will only deal with smooth surfaces.

Consider an X-marked hyperbolic surface with geodesic boundary Σ. Define $\mathrm{gr}_\infty(\Sigma)$ to be the surface obtained by gluing semi-infinite flat cylinders at $\partial\Sigma$ of lengths $(p_1, \dots, p_n) = \ell_\partial(\Sigma)$.

Thus, $\mathrm{gr}_\infty(\Sigma)$ has a hyperbolic core and flat ends and the underlying conformal structure is that of an X-punctured Riemann surface. This *infinite grafting* procedure defines a map

$$(\mathrm{gr}_\infty, \ell_\partial) : \mathcal{T}^\partial(S, X) \longrightarrow \mathcal{T}(S, X) \times \mathbb{R}^N_{\geq 0}$$

For more details about (finite) grafting, see [18].

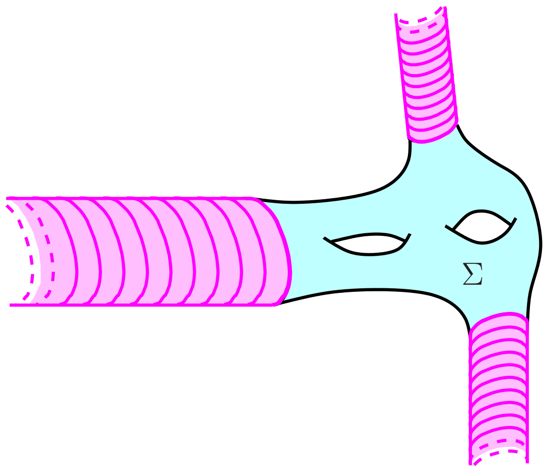

Figure 10. A grafted surface $\mathrm{gr}_\infty(\Sigma)$.

Proposition 4.10 ([52]). *The map* $(\mathrm{gr}_\infty, \ell_\partial)$ *is a* $\Gamma(S, X)$*-equivariant homeomorphism.*

The proof is a variation of Scannell-Wolf's [64] that finite grafting is a self-homeomorphism of the Teichmüller space.

Thus, the composition of $(\mathrm{gr}_\infty, \ell_\partial)^{-1}$ and Φ gives (after blowing up the locus $\{\ell_\partial = 0\}$) the homeomorphism

$$\Psi : \mathcal{T}(S, X) \times \Delta_X \times [0, \infty) \longrightarrow |\mathfrak{A}^\circ(S, X)| \times [0, \infty)$$

Proposition 4.11 ([52]). *The map* Ψ *extends to a* $\Gamma(S, X)$*-equivariant homeomorphism*

$$\Psi : \mathcal{T}(S, X) \times \Delta_X \times [0, \infty] \longrightarrow |\mathfrak{A}^\circ(S, X)| \times [0, \infty]$$

and Ψ_∞ *coincides with Harer-Mumford-Thurston's* Ψ_{JS}*.*

The main point is to show that a surface Σ with large boundaries and with spine $\mathrm{Sp}(\Sigma)$ is very close in $\mathcal{T}(S,X)$ to the flat surface whose Jenkins-Strebel differential has critical graph isomorphic to $\mathrm{Sp}(\Sigma)$ (as metrized ribbon graphs).

To understand why this is reasonable, consider a sequence of hyperbolic surfaces Σ_m whose spine has fixed isomorphism type $\mathbb{G}$ and fixed projective metric and such that $\ell_\partial(\Sigma_m) = c_m(p_1,\dots,p_n)$, where c_m diverges as $m \to \infty$. Consider the grafted surfaces $\mathrm{gr}_\infty(\Sigma_m)$ and rescale them so that $\sum_i p_i = 1$. The flat metric on the cylinders is naturally induced by a holomorphic quadratic differential, which has negative quadratic residue at X. Extend this differential to zero on the hyperbolic core.

Because of the rescaling, the distance between the flat cylinders and the spine goes to zero and the differential converges in L^1_{red} to a Jenkins-Strebel differential.

Dumas [17] has shown that an analogous phenomenon occurs for closed surfaces grafted along a measured lamination $t\lambda$ as $t \to +\infty$.

4.3.6 Weil-Petersson form and Penner's formula. Using Wolpert's result and hyperbolic geometry, Penner [60] proved that the pull-back of the Weil-Petersson form on the space of decorated hyperbolic surfaces with cusps, which can be identified to $\mathcal{T}(S,X) \times \Delta_X$, can be neatly written in the following way. Fix a triangulation $\underline{\boldsymbol{\alpha}} = \{\alpha_1,\dots,\alpha_M\} \in \mathfrak{A}^\circ(S,X)$. For every $([f : S \to \Sigma], \underline{p}) \in \mathcal{T}(S,X) \times \Delta_X$, let $\tilde{\alpha}_i$ be the geodesic representative in the class of $f_*(\alpha_i)$ and call $a_i := \ell(\tilde{\alpha}_i \cap \Sigma_{tr})$, where Σ_{tr} be the truncated hyperbolic surface. Then

$$\pi^*\omega_{WP} = \sum_{t\in T}(da_{t_1} \wedge da_{t_2} + da_{t_2} \wedge da_{t_3} + da_{t_3} \wedge da_{t_1})$$

where $\pi : \mathcal{T}(S,X) \times \Delta_X \to \mathcal{T}(S,X)$ is the projection, T is the set of ideal triangles in which the $\tilde{\alpha}_i$'s decompose Σ, and the sides of t are $(\alpha_{t_1}, \alpha_{t_2}, \alpha_{t_3})$ in the cyclic order induced by the orientation of t (see Figure 11).

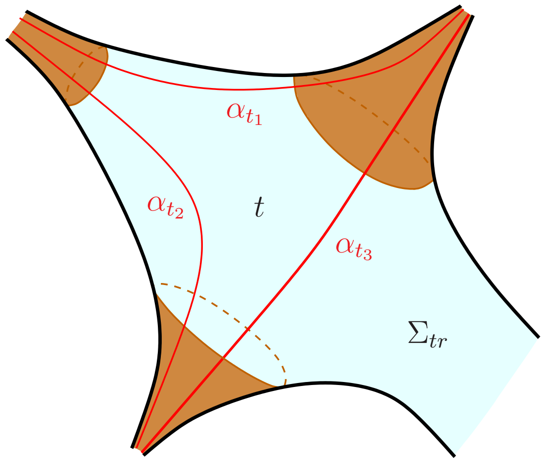

Figure 11. An ideal triangle in T.

To work on $\mathcal{M}_{g,X} \times \Delta_X$ (for instance, to compute Weil-Petersson volumes), one can restrict to the interior of the cells $\Phi_0^{-1}(|\underline{\boldsymbol{\alpha}}|)$ whose associated system of arcs $\underline{\boldsymbol{\alpha}}$ is a triangulation and write the pull-back of ω_{WP} with respect to $\underline{\boldsymbol{\alpha}}$.

4.3.7 Weil-Petersson form for surfaces with boundary. Still using methods of Wolpert [74], one can generalize Penner's formula to hyperbolic surfaces with boundary. The result is better expressed using the Weil-Petersson bivector field than the 2-form.

Proposition 4.12 ([53]). *Let Σ be a hyperbolic surface with boundary components $C_1, \dots, C_n$ and let $\underline{\boldsymbol{\alpha}} = \{\alpha_1, \dots, \alpha_M\}$ be a triangulation. Then the Weil-Petersson bivector field can be written as*

$$\omega^* = \frac{1}{4} \sum_{b=1}^{n} \sum_{\substack{y_i \in \alpha_i \cap C_b \\ y_j \in \alpha_j \cap C_b}} \frac{\sinh(p_b/2 - d_b(y_i, y_j))}{\sinh(p_b/2)} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

where $a_i = \ell(\alpha_i)$ and $d_b(y_i, y_j)$ is the length of the geodesic arc running from y_i to y_j along C_b in the positive direction (according to the orientation induced by Σ on C_b).

The idea is to use Wolpert's formula $\omega^* = -\sum_i \partial_{\ell_i} \wedge \partial_{\tau_i}$ on the double $d\Sigma$ of Σ with the pair of pants decomposition induced by doubling the arcs $\{\alpha_i\}$. Then one must compute the (first-order) effect on the a_i's of twisting $d\Sigma$ along α_j.

Though not immediate, the above formula can be shown to reduce to Penner's, when the boundary lengths go to zero, as we approximate $\sinh(x) \approx x$ for small x. Notice that Penner's formula shows that ω linearizes (with constant coefficients!) in the coordinates given by the a_i's.

More interesting is to analyze what happens for $(\Sigma, t\underline{p})$ with $\underline{p} \in \Delta_X$, as $t \to +\infty$. Assume the situation is generic and so $\Psi_{JS}(\Sigma)$ is supported on a triangulation, whose dual graph is $\mathbb{G}$.

Once again, the formula dramatically simplifies as we approximate $2\sinh(x) \approx \exp(x)$ for $x \gg 0$. Under the rescalings $\tilde{\omega}^* = c^2\omega^*$ and $\tilde{w}_i = w_i/c$ with $c = \sum_b p_b/2$, we obtain that

$$\lim_{t\to\infty} \tilde{\omega}^* = \omega^*_\infty := \frac{1}{2}\sum_{v\in E_0(\mathbb{G})} \left(\frac{\partial}{\partial \tilde{w}_{v_1}} \wedge \frac{\partial}{\partial \tilde{w}_{v_2}} + \frac{\partial}{\partial \tilde{w}_{v_2}} \wedge \frac{\partial}{\partial \tilde{w}_{v_3}} + \frac{\partial}{\partial \tilde{w}_{v_3}} \wedge \frac{\partial}{\partial \tilde{w}_{v_1}} \right)$$

where $v = \{v_1, v_2, v_3\}$ and $\sigma_0(v_j) = v_{j+1}$ (and $j \in \mathbb{Z}/3\mathbb{Z}$).

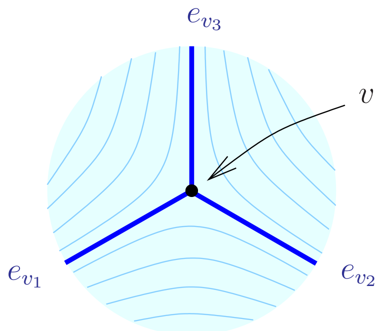

Figure 12. A trivalent vertex v of $\mathbb{G}$.

Thus, the Weil-Petersson symplectic structure is again linearized (and with constant coefficients!), but in the system of coordinates given by the w_j's, which are in some sense dual to the a_i's.

It would be nice to exhibit a clear geometric argument for the perfect symmetry of these two formulae.

5 Combinatorial classes

5.1 Witten cycles

Fix as usual a compact oriented surface S of genus g and a subset $X = \{x_1, \ldots, x_n\} \subset S$ such that $2g - 2 + n > 0$.

We introduce some remarkable $\Gamma(S, X)$-equivariant subcomplexes of $\mathfrak{A}(S, X)$, which define interesting cycles in the homology of $\overline{\mathcal{M}}^K_{g,X}$ as well as in the Borel-Moore homology of $\mathcal{M}_{g,X}$ and so, by Poincaré duality, in the cohomology of $\mathcal{M}_{g,X}$ (that is, of $\Gamma(S, X)$).

These subcomplexes are informally defined as the locus of points of $|\mathfrak{A}^\circ(S,X)|$, whose associated ribbon graphs have prescribed odd valences of their vertices. It can be easily shown that, if we assign even valence to some vertex, the subcomplex we obtain is not a cycle (even with $\mathbb{Z}/2\mathbb{Z}$ coefficients!).

We follow Kontsevich ([41]) for the orientation of the combinatorial cycles, but an alternative way is due to Penner [61] and Conant and Vogtmann [13].

Later, we will mention a slight generalization of the combinatorial classes by allowing some vertices to be marked.

Notice that we are going to use the cellularization of the moduli space of curves given by Ψ_{JS}, and so we will identify $\overline{\mathcal{M}}^{\Delta}_{g,X}$ with the orbispace $|\mathfrak{A}(S,X)|/\Gamma(S,X)$. As the arguments will be essentially combinatorial/topological, any of the decompositions described before would work.

5.1.1 Witten subcomplexes. Let $m_* = (m_0, m_1, \dots)$ be a sequence of non-negative integers such that

$$\sum_{i\geq 0}(2i+1)m_i = 4g-4+2n$$

and define $(m_*)! := \prod_{i\geq 0} m_i!$ and $r := \sum_{i\geq 0} i\, m_i$.

Definition 5.1. The *combinatorial subcomplex* $\mathfrak{A}_{m_*}(S,X) \subset \mathfrak{A}(S,X)$ is the smallest simplicial subcomplex that contains all proper simplices $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}^\circ(S,X)$ such that $S \setminus \underline{\boldsymbol{\alpha}}$ is the disjoint union of exactly m_i polygons with $2i+3$ sides.

It is convenient to set $|\mathfrak{A}_{m_*}(S,X)|_{\mathbb{R}} := |\mathfrak{A}_{m_*}(S,X)| \times \mathbb{R}_+$. Clearly, this subcomplex is $\Gamma(S,X)$-equivariant. Hence, if we call $\overline{\mathcal{M}}^{comb}_{g,X} := \overline{\mathcal{M}}^{\Delta}_{g,X} \times \mathbb{R}_+ \cong |\mathfrak{A}(S,X)|_{\mathbb{R}}/\Gamma(S,X)$, then we can define $\overline{\mathcal{M}}^{comb}_{m_*,X}$ to be the subcomplex of $\overline{\mathcal{M}}^{comb}_{g,X}$ induced by $\mathfrak{A}_{m_*}(S,X)$.

Remark 5.2. We can introduce also univalent vertices by allowing $m_{-1} > 0$. It is still possible to define the complexes $\mathfrak{A}_{m_*}(S,X)$ and $\mathfrak{A}^\circ_{m_*}(S,X)$, just allowing (finitely many) contractible loops (i.e. unmarked tails in the corresponding ribbon graph picture). However, $\mathfrak{A}_{m_*}(S,X)$ would no longer be a subcomplex of $\mathfrak{A}(S,X)$. Thus, we should construct an associated family of Riemann surfaces over $\overline{\mathcal{M}}^{comb}_{m_*,X}$ (which can be easily done) and consider the classifying map $\overline{\mathcal{M}}^{comb}_{m_*,X} \to \overline{\mathcal{M}}^{comb}_{g,X}$, whose existence is granted by the universal property of $\overline{\mathcal{M}}_{g,X}$, but which would no longer be cellular.

For every $\underline{p} \in \Delta_X \times \mathbb{R}_+$ call $\overline{\mathcal{M}}^{comb}_{g,X}(\underline{p}) := \bar{\ell}_{\partial}^{-1}(\underline{p}) \subset \overline{\mathcal{M}}^{comb}_{g,X}$ and define $\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p}) := \overline{\mathcal{M}}^{comb}_{m_*,X} \cap \overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})$.

Notice that the dimensions of the slices are the expected ones because in every cell they are described by n independent linear equations.

5.1.2 Combinatorial ψ classes. Define L_i as the space of couples $(\mathbb{G}, y)$, where $\mathbb{G}$ is an X-marked metrized ribbon graph in $\overline{\mathcal{M}}^{comb}_{g,X}(\{p_i > 0\})$ and y is a ray that joins x_i to a point of $|G| \subset |\mathbb{G}|$ that bounds the x_i-th hole.

Clearly $L_i \longrightarrow \overline{\mathcal{M}}^{comb}_{g,X}(\{p_i > 0\})$ is a topological bundle with fiber homeomorphic to S^1. It is easy to see that, for a fixed $\underline{p} \in \Delta_X \times \mathbb{R}_+$ such that $p_i > 0$, the pull-back of L_i via

$$\xi_{\underline{p}} : \overline{\mathcal{M}}_{g,X} \longrightarrow \overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})$$

is isomorphic (as a topological bundle) to the sphere bundle associated to $\mathcal{L}_i^*$.

Lemma 5.3 ([41]). *Fix x_i in X and $\underline{p} \in \Delta_X \times \mathbb{R}_+$ such that $p_i > 0$. Then on every simplex $|\underline{\boldsymbol{\alpha}}|(\underline{p}) \in \overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})$ define*

$$\overline{\eta}_i\Big|_{|\underline{\boldsymbol{\alpha}}|(\underline{p})} := \sum_{1 \le s < t \le k-1} d\tilde{e}_s \wedge d\tilde{e}_t$$

where $\tilde{e}_j = \dfrac{\ell(e_j)}{p_i}$ and x_i marks a hole with cyclically ordered sides $(e_1, \dots, e_k)$. These 2-forms glue to give a piecewise-linear 2-form $\overline{\eta}_i$ on $\overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})$, that represents $c_1(L_i)$. Hence, the pull-back class $\xi^*_{\underline{p}}[\overline{\eta}_i]$ is exactly $\psi_i = c_1(\mathcal{L}_i)$ in $H^2(\overline{\mathcal{M}}_{g,X})$.

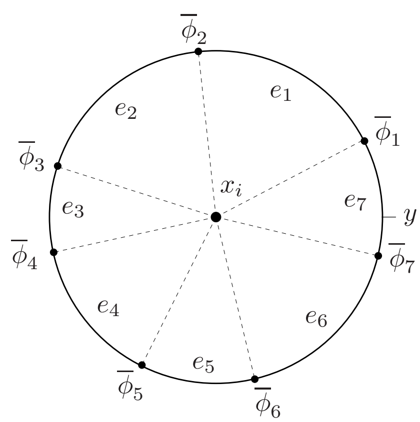

Figure 13. A fiber of the bundle L_i over a hole with 7 sides.

The proof of the previous lemma is very easy.

5.1.3 Orientation of Witten subcomplexes. The following lemma says that the η forms can be assembled in a piecewise-linear "symplectic form", that can be used to orient maximal cells of Witten subcomplexes.

Lemma 5.4 ([41]). *For every* $\underline{p} \in \Delta_X \times \mathbb{R}_+$ *the restriction of*

$$\overline{\Omega} := \sum_{i=1}^{n} p_i^2 \overline{\eta}_i$$

to the maximal simplices of $\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p})$ *is a non-degenerate symplectic form. Hence,* $\overline{\Omega}^r$ *defines an orientation on* $\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p})$*. Also,* $\overline{\Omega}^r \wedge \bar{\ell}^*_\partial \mathrm{Vol}_{\mathbb{R}^X}$ *is a volume form on* $\overline{\mathcal{M}}^{comb}_{m_*,X}$*.*

Proof. Let $|\underline{\boldsymbol{\alpha}}|(\underline{p})$ be a cell of $\overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})$, whose associated ribbon graph $\mathbb{G}_{\underline{\boldsymbol{\alpha}}}$ has only vertices of odd valence.

On $|\underline{\boldsymbol{\alpha}}|(\underline{p})$, the differentials de_i span the cotangent space. As the p_i's are fixed, we have the relation $dp_i = 0$ for all $i = 1, \dots, n$. Hence

$$T^*\overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})\Big|_{|\underline{\boldsymbol{\alpha}}|(\underline{p})} \cong |\underline{\boldsymbol{\alpha}}|(\underline{p}) \times \bigoplus_{e \in E_1(\underline{\boldsymbol{\alpha}})} \mathbb{R} \cdot de \Bigg/ \left(\sum_{[\vec{e}]_0 = x_i} de \,\Big|\, i = 1, \dots, n \right)$$

On the other hand the tangent bundle is

$$T\overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})\Big|_{|\underline{\boldsymbol{\alpha}}|(\underline{p})} \cong |\underline{\boldsymbol{\alpha}}|(\underline{p}) \times \left\{ \sum_{e \in E_1(\underline{\boldsymbol{\alpha}})} b_e \frac{\partial}{\partial e} \,\Big|\, \sum_{[\vec{e}]_0 \in x_i} b_e = 0 \quad \text{for all } i = 1, \dots, n \right\}.$$

In order to prove that $\overline{\Omega}|_{\underline{\boldsymbol{\alpha}}} : T|\underline{\boldsymbol{\alpha}}|(\underline{p}) \longrightarrow T^*|\underline{\boldsymbol{\alpha}}|(\underline{p})$ is non-degenerate, we construct its right-inverse. Define $B : T^*|\underline{\boldsymbol{\alpha}}|(\underline{p}) \longrightarrow T|\underline{\boldsymbol{\alpha}}|(\underline{p})$ as

$$B(de) = \sum_{i=1}^{2s} (-1)^i \frac{\partial}{\partial[\sigma_0^i(\overrightarrow{e})]_1} + \sum_{j=1}^{2t} (-1)^j \frac{\partial}{\partial[\sigma_0^j(\overleftarrow{e})]_1}$$

where $\vec{e}$ is any orientation of e, while $2s+1$ and $2t+1$ are the cardinalities of $[\overrightarrow{e}]_0$ and $[\overleftarrow{e}]_0$ respectively. We want to prove that $\overline{\Omega}B(de) = 4de$ for every $e \in E_1(\underline{\boldsymbol{\alpha}})$.

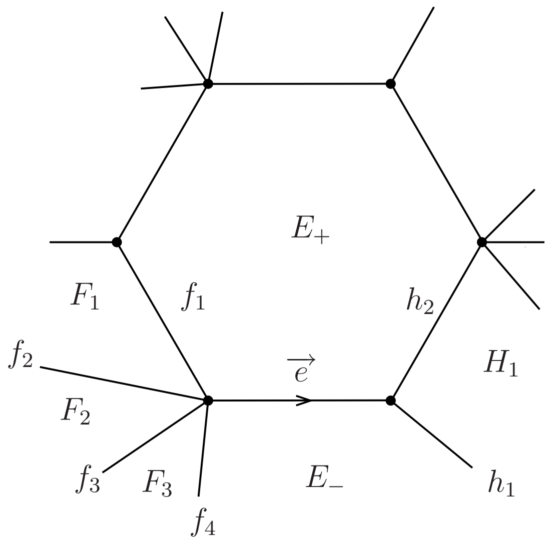

Figure 14. An example with $s = 2$ and $t = 1$.

To shorten the notation, set $f_i := [\sigma_0^i(\overrightarrow{e})]_1$ and $h_j := [\sigma_0^j(\overleftarrow{e})]_1$ and call $F_i := [\sigma_0^i(\overrightarrow{e})]_\infty$ for $i = 1, \dots, 2s-1$ and $H_j := [\sigma_0^j(\overleftarrow{e})]_\infty$ for $j = 1, \dots, 2t-1$ the holes bordered respectively by $\{f_i, f_{i+1}\}$ and $\{h_j, h_{j+1}\}$. Finally call E_+ and E_- the holes adjacent to e as in Figure 14. Remark that neither the edges f and h nor the holes F and H are necessarily distinct. This however has no importance in the following computation.

$$B(de) = -\sum_{i=1}^{2s}(-1)^i \frac{\partial}{\partial f_i} - \sum_{j=1}^{2t}(-1)^j \frac{\partial}{\partial h_j}$$

It is easy to see (using that the perimeters are constant) that

$$p_{F_i}^2 \overline{\eta}_{F_i} \left(\frac{\partial}{\partial f_i} - \frac{\partial}{\partial f_{i+1}} \right) = df_i + df_{i+1}$$

and analogously for the h's. Moreover

$$p_{E_+}^2 \overline{\eta}_{E_+} \left(\frac{\partial}{\partial h_{2s}} - \frac{\partial}{\partial f_1} \right) = dh_{2s} + df_1 + 2de$$

and similarly for E_-. Finally, we obtain $\overline{\Omega} B(de) = 4de$. □

Remark 5.5. Notice that B is the piecewise-linear extension of the restriction of the Weil-Petersson bivector field $2\tilde{\omega}_\infty^*$ to the open maximal simplices. Thus, $\overline{\Omega}$ is the piecewise-linear extension of $2\tilde{\omega}_\infty$.

Finally, we can show that the (cellular) chain obtained by adding maximal simplices of Witten subcomplexes (with the orientation determined by Ω) is in fact a cycle.

Lemma 5.6 ([41]). *With the given orientation $\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p})$ is a cycle for all $\underline{p} \in \Delta_X \times \mathbb{R}_+$ and $\overline{\mathcal{M}}^{comb}_{m_*,X}(\mathbb{R}^X_+)$ is a cycle with non-compact support.*

Proof. Given a top-dimensional cell $|\underline{\boldsymbol{\alpha}}|(\underline{p})$ in $\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p})$, each face in the boundary $\partial|\underline{\boldsymbol{\alpha}}|(\underline{p})$ is obtained by shrinking one edge of $\mathbb{G}_{\underline{\boldsymbol{\alpha}}}$. This contraction may merge two vertices as in Fig. 15.

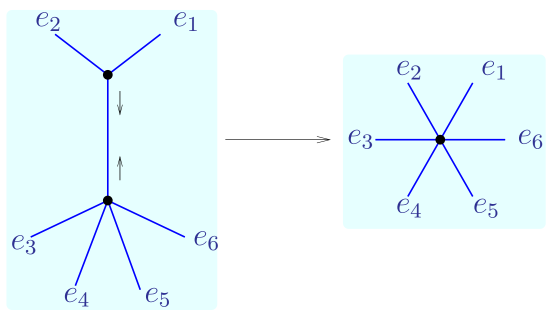

Figure 15. A contraction that merges a 3-valent and a 5-valent vertex.

Otherwise the shrinking produces a node, as in Fig. 16.

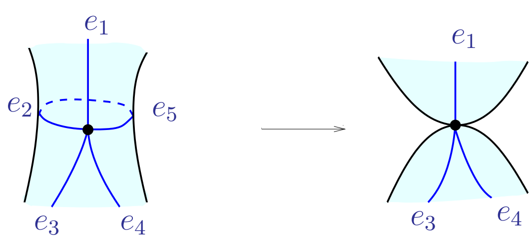

Figure 16. A contraction produces a node.

Let $|\underline{\boldsymbol{\alpha}}'|(\underline{p}) \in \partial|\underline{\boldsymbol{\alpha}}|(\underline{p})$ be the face of $|\underline{\boldsymbol{\alpha}}|(\underline{p})$ obtained by shrinking the edge e. Then $\Lambda^{6g-7+2n-2r}T|\underline{\boldsymbol{\alpha}}'|(\underline{p}) = \Lambda^{6g-6+2n-2r}T|\underline{\boldsymbol{\alpha}}|(\underline{p}) \otimes N^*_{|\underline{\boldsymbol{\alpha}}'|/|\underline{\boldsymbol{\alpha}}|}$ and so the dual of the orientation form induced by $|\underline{\boldsymbol{\alpha}}|(\underline{p})$ on $|\underline{\boldsymbol{\alpha}}'|(\underline{p})$ is $\iota_{de}(B^{6g-6+2n-2r}_{\underline{\boldsymbol{\alpha}}}) = (6g-6+2n-2r)\iota_{de}(B_{\underline{\boldsymbol{\alpha}}}) \wedge B^{6g-8+2n-2r}_{\underline{\boldsymbol{\alpha}}}$, where $B_{\underline{\boldsymbol{\alpha}}}$ is the bivector field on $|\underline{\boldsymbol{\alpha}}|(\underline{p})$ defined in Lemma 5.4.

Consider the graph $\mathbb{G}_{\underline{\boldsymbol{\alpha}}'}$ that occurs in the boundary of a top-dimensional cell of $\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p})$. Suppose it is obtained merging two vertices of valences $2t_1+3$ and $2t_2+3$ in a vertex v of valence $2(t_1+t_2)+4$. Then $|\underline{\boldsymbol{\alpha}}'|(\underline{p})$ is in the boundary of exactly $2(t_1+t_2)+4$ cells of $\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p})$ or t_1+t_2+2 ones in the case $t_1=t_2$. In any case, the number of cells $|\underline{\boldsymbol{\alpha}}'|(\underline{p})$ is bordered by are even: we need to prove that half of them induces on $|\underline{\boldsymbol{\alpha}}'|(\underline{p})$ an orientation and the other half induces the opposite one. If $\mathbb{G}_{\underline{\boldsymbol{\alpha}}'}$ is obtained from some $\mathbb{G}_{\underline{\boldsymbol{\alpha}}}$ contracting an edge e, then we just have to compute the vector field $\iota_{de}(B_{\underline{\boldsymbol{\alpha}}})$, which turns to be

$$\iota_{de}(B_{\underline{\boldsymbol{\alpha}}}) = \pm \sum_{i=1}^{2(t_1+t_2)+4} (-1)^i \frac{\partial}{\partial f_i}$$

where $f_1, \dots, f_{2(t_1+t_2)+4}$ are the edges of $\mathbb{G}_{\underline{\boldsymbol{\alpha}}'}$ outgoing from v. It is a straightforward computation to check that one obtains in half the cases a plus and in half the cases a minus.

When $\mathbb{G}^{en}_{\underline{\boldsymbol{\alpha}}'}$ has a node with $2t_1+2$ edges on one side (which we will denote by $f_1, \dots, f_{2t_1+2}$) and $2t_2+3$ edges on the other side, the computation is similar. The cell occurs as boundary of exactly $(2t_1+2)(2t_2+3)$ top-dimensional cells and, if $\mathbb{G}_{\underline{\boldsymbol{\alpha}}'}$ is obtained by $\mathbb{G}_{\underline{\boldsymbol{\alpha}}}$ contracting the edge e, then

$$\iota_{de}(B_{\underline{\boldsymbol{\alpha}}}) = \pm 2 \sum_{i=1}^{2t_1+2} (-1)^i \frac{\partial}{\partial f_i}.$$

A quick check ensures that the signs cancel. □

Define the *Witten classes* $\overline{W}_{m_*,X}(\underline{p}) := [\overline{\mathcal{M}}^{comb}_{m_*,X}(\underline{p})]$ and let $W_{m_*,X}(\underline{p})$ be its restriction to $\mathcal{M}^{comb}_{g,X}(\underline{p})$, which defines (by Poincaré duality) a cohomology class in $H^{2r}(\mathcal{M}_{g,X})$, independent of $\underline{p}$.

5.1.4 Generalized Witten cycles. It is possible to define a slight generalization of the previous classes, prescribing that some markings hit vertices with assigned valence.

These *generalized Witten classes* are related to the previous $W_{m_*,X}$ in an intuitively obvious way, because forgetting the markings of some vertices will map them onto one another. We will omit the details and refer to [51].

5.2 Witten cycles and tautological classes

In this subsection, we will sketch the proof of the following result, due to K. Igusa [31] and [32] (see also [33]) and Mondello [51] independently.

Theorem 5.7. *Witten cycles $W_{m_*,X}$ on $\mathcal{M}_{g,X}$ are Poincaré dual to polynomials in the κ classes and vice versa.*

In [51], the following results are also proven:

- Witten generalized cycles on $\mathcal{M}_{g,X}$ are Poincaré dual to polynomials in the ψ and the κ classes
- ordinary and generalized Witten cycles on $\overline{\mathcal{M}}^{comb}_{g,X}(\underline{p})$ are push-forward of (the Poincaré dual of) tautological classes from $\overline{\mathcal{M}}_{g,X}$; an explicit recipe to produce such tautological classes is given.

5.2.1 The case with one special vertex. We want to consider a combinatorial cycle on $\mathcal{M}_{g,X}$ supported on ribbon graphs, whose vertices are generically all trivalent except one, which is $(2r+3)$-valent (and $r \geq 1$). To shorten the notation, call this Witten cycle W_{2r+3}.

We also define a generalized Witten cycle on the universal curve $\mathcal{C}_{g,X} \subset \overline{\mathcal{M}}_{g,X\cup\{y\}}$ supported on the locus of ribbon graphs, which have a $(2r+3)$-valent vertex marked by y and all the other vertices are trivalent and unmarked. Call W^y_{2r+3} this cycle.

We would like to show that $\mathrm{PD}(W^y_{2r+3}) = c(r)\psi_y^{r+1}$, where $c(r)$ is some constant. As a consequence, pushing the two hand-sides down through the proper map $\pi_y : \mathcal{C}_{g,X} \to \mathcal{M}_{g,X}$, we would obtain $\mathrm{PD}(W_{2r+3}) = c(r)\kappa_r$.

Lemma 5.3 gives us the nice piecewise-linear 2-form η_y, that is pulled back to ψ_y through ξ. The only problem is that η_y is defined only for $p_y > 0$, whereas W^y_{2r+3} is exactly contained in the locus $\{p_y = 0\}$.

To compare the two, one can look at the blow-up $\mathrm{Bl}_{p_y=0}\overline{\mathcal{M}}^{comb}_{g,X\cup\{y\}}$ of $\overline{\mathcal{M}}^{comb}_{g,X\cup\{y\}}$ along the locus $\{p_y = 0\}$. Points in the exceptional locus E can be identified with metrized (nonsingular) ribbon graphs $\mathbb{G}$, in which y marks a vertex, plus *angles* ϑ between consecutive oriented edges outgoing from y. One must think of these angles as of *infinitesimal edges*.

It is clear now that η_y extends to E by

$$\eta_y\big|_{|\underline{\boldsymbol{\alpha}}|(\underline{p})} := \sum_{1\leq s<t\leq k-1} d\tilde{e}_s \wedge d\tilde{e}_t$$

where $\tilde{e}_j = \dfrac{\vartheta_j}{2\pi}$, y marks a vertex with cyclically ordered outgoing edges $(\overrightarrow{e}_1, \ldots, \overrightarrow{e}_k)$ and ϑ_j is the angle between $\overrightarrow{e}_j$ and $\overrightarrow{e}_{j+1}$ (with $j \in \mathbb{Z}/k\mathbb{Z}$).

Thus, pushing forward η_y^{r+1} through $E \to \overline{\mathcal{M}}^{comb}_{g,X}(p_y = 0)$, we obtain $c(r)\overline{W}^y_{2r+3}$ plus other terms contained in the boundary, and the coefficient $c(r)$ is exactly the integral of η_y^{r+1} on a fiber (that is, a simplex), which turns out to be $c(r) = \dfrac{(r+1)!}{(2r+2)!}$. Thus, W^y_{2r+3} is Poincaré dual to $2^{r+1}(2r+1)!!\psi_y^{r+1}$.

5.2.2 The case with many special vertices. To mimic what done for one non-trivalent vertex, let's consider combinatorial classes with two non-trivalent vertices. Thus, we look at the class $\psi_y^{r+1}\psi_z^{s+1}$ (with $r, s \geq 1$) on $\mathcal{C}^2_{g,X} := \mathcal{C}_{g,X} \times_{\mathcal{M}_{g,X}} \mathcal{C}_{g,X}$.

Look at the blow-up $\mathrm{Bl}_{p_y=0,p_z=0}\overline{\mathcal{M}}^{comb}_{g,X\cup\{y\}}$ of $\overline{\mathcal{M}}^{comb}_{g,X\cup\{y,z\}}$ along the locus $\{p_y = 0\} \cup \{p_z = 0\}$ and let $E = E_y \cap E_z$, where E_y and E_z are the exceptional loci.

As before, we can identify $E \cap \{y \neq z\}$ with the set of metrized ribbon graphs $\mathbb{G}$, with angles at the vertices y and z. Thus, pushing $\eta_y^{r+1}\eta_z^{s+1}$ forward through the blow-up map (which forgets the angles at y and z), we obtain a multiple of the generalized combinatorial cycles given by y marking a $(2r+3)$-valent vertex and z marking a $(2s+3)$-valent (distinct) vertex. The coefficient $c(r, s)$ will just be $\dfrac{(r+1)!(s+1)!}{(2r+2)!(2s+2)!}$.

Points in $E \cap \{y = z\}$ can be thought of as metrized ribbon graphs $\mathbb{G}$ with two infinitesimal holes (respectively marked by y and z) adjacent to each other. If we perform the push-forward of $\eta_y^{r+1}\eta_z^{s+1}$ forgetting first the angles at z and then the angles at y, then we obtain some contribution only from the loci in which the infinitesimal z-hole has $(2s + 3)$ edges and the infinitesimal y-hole has $(2r + 4)$ edges (including the common one). Thus, we obtain the same contribution for each of the $b(r, s)$ configurations of two adjacent holes of valences $(2s + 3)$ and $(2r + 4)$.

Thus, we obtain a cycle supported on the locus of metrized ribbon graphs $\mathbb{G}$ in which $y = z$ marks a $(2r+2s+3)$-valent vertex, with coefficient $b(r, s)c(r, s)$.

Hence, $\psi_y^{r+1}\psi_z^{s+1}$ is Poincaré dual to a linear combination of generalized combinatorial cycles. As before, using the forgetful map, the same holds for the Witten cycles obtained by deleting the y and the z markings.

One can easily see that the transformation laws from ψ classes to combinatorial classes are invertible (because they are "upper triangular" in a suitable sense).

Clearly, in order to deal with many ψ classes (that is, with many non-trivalent marked vertices), one must compute more and more complicated combinatorial factors like $b(r, s)$.

We refer to [33] and [51] for two (complementary) methods to calculate these factors.

5.3 Stability of Witten cycles

5.3.1 Harer's stability theorem. The (co)homologies of the mapping class groups have the remarkable property that they stabilize when the genus of the surface increases. This was proven by Harer [26], and the stability bound was then improved by Ivanov [34] (and successively again by Harer for homology with rational coefficients, in an unpublished paper). We now want to recall some of Harer's results.

Let $S_{g,n,b}$ be a compact oriented surface of genus g with n marked points and b boundary components $C_1, \ldots, C_b$. Call $\Gamma(S_{g,n,b})$ the group of isotopy classes of diffeomorphisms of S that fix the marked points and ∂S pointwise.

Call also $P = S_{0,0,3}$ a fixed pair of pants and denote by B_1, B_2, B_3 its boundary components.

Consider the following two operations:

(y) gluing $S_{g,n,b}$ and P by identifying C_b with B_1, thus producing an oriented surface of genus g with n marked points and $b+1$ boundary components

(v) identify C_{b-1} with C_b of $S_{g,n,b}$, thus producing an oriented surface of genus $g+1$ with n marked points and $b-2$ boundary components.

Clearly, they induce homomorphism at the level of mapping class groups

$$\mathcal{Y} : \Gamma(S_{g,n,b}) \longrightarrow \Gamma(S_{g,n,b+1})$$

when $b \geq 1$ (by extending the diffeomorphism as the identity on P) and

$$\mathcal{V} : \Gamma(S_{g,n,b}) \longrightarrow \Gamma(S_{g+1,n,b-2})$$

when $b \geq 2$.

Theorem 5.8 (Harer [26]). *The induced maps in homology*

$$\mathcal{Y}_* : H_k(\Gamma(S_{g,n,b})) \longrightarrow H_k(\Gamma(S_{g,n,b+1}))$$
$$\mathcal{V}_* : H_k(\Gamma(S_{g,n,b})) \longrightarrow H_k(\Gamma(S_{g+1,n,b-2}))$$

are isomorphisms for $g \geq 3k$.

The exact bound is not important for our purposes. We only want to stress that the theorem implies that $H_k(\Gamma(S_{g,n,b}))$ stabilizes for large g. In particular, fixed $n \geq 0$, the rational homology of $\mathcal{M}_{g,n}$ stabilizes for large g.

Remark 5.9. We have $B\Gamma(S_{g,n,b}) \simeq \mathcal{M}_{g,X,T}$, where $\mathcal{M}_{g,X,T}$ is the moduli space of Riemann surfaces of genus g with $X \cup T$ marked points ($X = \{x_1, \ldots, x_n\}$ and $T = \{t_1, \ldots, t_b\}$) and a nonzero tangent vector at each point of T. If $b \geq 1$, then $\mathcal{M}_{g,X,T}$ is a smooth variety: in fact, an automorphism of a Riemann surface that fixes a point and a tangent direction at that point is the identity (this follows from uniformization and Schwarz lemma).

5.3.2 Mumford's conjecture. Call $\Gamma_{\infty,n} = \lim_{g\to\infty} \Gamma(S_{g,n,1})$, where the map $\Gamma(S_{g,n,1}) \to \Gamma(S_{g+1,n,1})$ corresponds to gluing a torus with two holes at the boundary component of $S_{g,n,1}$.

Then, $H^k(\Gamma_{\infty,n})$ coincides with $H^k(\Gamma_{g,n})$ for $g \gg k$.

Mumford conjectured that $H^*(\Gamma_\infty; \mathbb{Q})$ is the polynomial algebra on the κ classes. Miller [49] showed that $H^*(\Gamma_\infty; \mathbb{Q})$ is a Hopf algebra that contains $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$.

Recently, after works of Tillmann (for instance, [69]) and Madsen-Tillmann [46], Madsen and Weiss [47] proved a much stronger statement of homotopy theory, which in particular implies Mumford's conjecture.

Thanks to a result of Bödigheimer-Tillmann [11], it follows that $H^*(\Gamma_{\infty,n};\mathbb{Q})$ is a polynomial algebra on $\psi_1,\dots,\psi_n$ and the κ classes.

Thus, generalized Witten classes, being polynomials in ψ and κ, are also stable. In what follows, we would like to prove this stability in a direct way.

5.3.3 Ribbon graphs with tails. One way to cellularize the moduli space of curves with marked points and tangent vectors at the marked points is to use ribbon graphs with tails (see, for instance, [22]).

Consider Σ a compact Riemann surface of genus g with marked points $X \cup T = \{x_1,\dots,x_n\} \cup \{t_1,\dots,t_b\}$ and nonzero tangent vectors $v_1,\dots,v_b$ at $t_1,\dots,t_b$.

Given $p_1,\dots,p_n \geq 0$ and $q_1,\dots,q_b > 0$, we can construct the ribbon graph $\mathbb{G}$ associated to $(\Sigma,\underline{p},\underline{q})$, say using the Jenkins-Strebel differential φ.

For every $j = 1,\dots,b$, move from the center t_j along a vertical trajectory γ_j of φ determined by the tangent vector v_j, until we hit the critical graph. Parametrize the opposite path γ_j^* by arc-length, so that $\gamma_j^* : [0,\infty] \to \Sigma$, $\gamma_j^*(0)$ lies on the critical graph and $\gamma_j^*(\infty) = t_j$. Then, construct a new ribbon graph out of $\mathbb{G}$ by "adding" a new vertex (which we will call $\tilde{v}_j$) and a new edge e_{v_j} of length $|v_j|$ (a *tail*), whose realization is $\gamma_j^*([0,|v_j|])$ (see Figure 17).

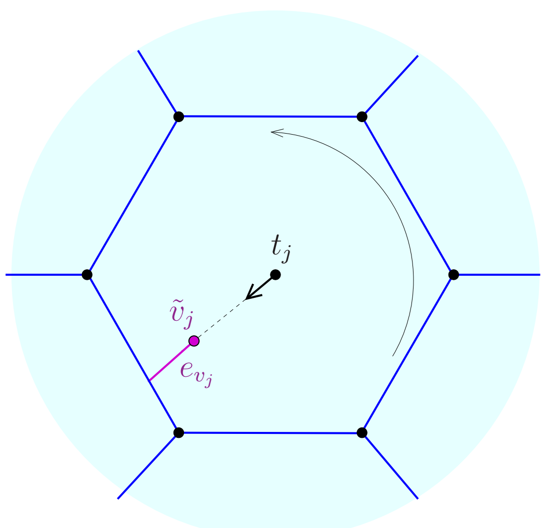

Figure 17. Correspondence between a tail and a nonzero tangent vector.

Thus, we have realized an embedding of $\mathcal{M}_{g,X,T}\times\mathbb{R}^X_{\geq 0}\times\mathbb{R}^T_+$ inside $\mathcal{M}^{comb}_{g,X\cup T\cup V}$, where $V=\{\tilde{v}_1,\dots,\tilde{v}_b\}$. If we call $\mathcal{M}^{comb}_{g,X,T}$ its image, we have obtained the following.

Lemma 5.10. $\mathcal{M}^{comb}_{g,X,T}\simeq B\Gamma(S_{g,n,b})$.

Notice that the embedding $\mathcal{M}^{comb}_{g,X,T}\hookrightarrow\mathcal{M}^{comb}_{g,X\cup T\cup V}$ allows us to define (generalized) Witten cycles $W_{m_*,X,T}$ on $\mathcal{M}^{comb}_{g,X,T}$ simply by restriction.

5.3.4 Gluing ribbon graphs with tails. Let $\mathbb{G}'$ and $\mathbb{G}''$ be two ribbon graphs with tails $\overrightarrow{e'}$ and $\overrightarrow{e''}$, i.e. $\overrightarrow{e'}\in E(\mathbb{G}')$ and $\overrightarrow{e''}\in E(\mathbb{G}'')$ with the property that $\sigma'_0(\overrightarrow{e'})=\overrightarrow{e'}$ and $\sigma''_0(\overrightarrow{e''})=\overrightarrow{e''}$.

We produce a third ribbon graph $\mathbb{G}$ by *gluing* $\mathbb{G}'$ *and* $\mathbb{G}''$ in the following way.

We set $E(\mathbb{G})=(E(\mathbb{G}')\cup E(\mathbb{G}''))\,/\sim$, where we declare that $\overrightarrow{e'}\sim\overleftarrow{e''}$ and $\overleftarrow{e'}\sim\overrightarrow{e''}$. Thus, we have a natural σ_1 induced on $E(\mathbb{G})$. Moreover, we define σ_0 acting on $E(\mathbb{G})$ as

$$\sigma_0([\overrightarrow{e}])=\begin{cases}[\sigma'_0(\overrightarrow{e})] & \text{if } \overrightarrow{e}\in E(\mathbb{G}') \text{ and } \overrightarrow{e}\neq\overrightarrow{e'}\\ [\sigma''_0(\overrightarrow{e})] & \text{if } \overrightarrow{e}\in E(\mathbb{G}'') \text{ and } \overrightarrow{e}\neq\overrightarrow{e''}\end{cases}$$

If $\mathbb{G}'$ and $\mathbb{G}''$ are metrized, then we induce a metric on $\mathbb{G}$ in a canonical way, declaring the length of the new edge of $\mathbb{G}$ to be $\ell(e')+\ell(e'')$.

Suppose that $\mathbb{G}'$ is marked by $\{x_1,\dots,x_n,t'\}$ and e' is a tail contained in the hole t' and that $\mathbb{G}''$ is marked by $\{y_1,\dots,y_m,t''\}$ and if e'' is a tail contained in the hole t'', then $\mathbb{G}$ is marked by $\{x_1,\dots,x_n,y_1,\dots,y_m,t\}$, where t is a new hole obtained *merging* the holes centered at t' and t''.

Thus, we have constructed a *combinatorial gluing map*

$$\mathcal{M}^{comb}_{g',X',T'\cup\{t'\}}\times\mathcal{M}^{comb}_{g'',X'',T''\cup\{t''\}}\longrightarrow\mathcal{M}^{comb}_{g'+g'',X'\cup X''\cup\{t\},T'\cup T''}$$

5.3.5 The combinatorial stabilization maps. Consider the gluing maps in two special cases which are slightly different from what we have seen before.

Call $S_{g,X,T}$ a compact oriented surface of genus g with boundary components labelled by T and marked points labeled by X.

Fix a trivalent ribbon graph $\mathbb{G}_j$, with genus 1, one hole and j tails for $j=1,2$ (for instance, $j=2$ in Figure 18).

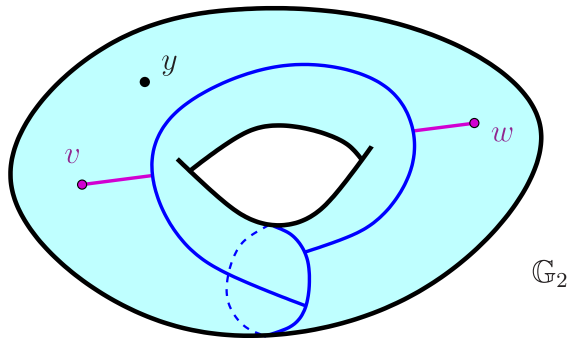

Figure 18. Example of a fixed torus.

Consider the combinatorial gluing maps

$$\mathcal{S}_1^{comb} : \mathcal{M}^{comb}_{g,X,\{t\}} \longrightarrow \mathcal{M}^{comb}_{g+1,X\cup\{t\}}$$
$$\mathcal{S}_2^{comb} : \mathcal{M}^{comb}_{g,X,\{t\}} \longrightarrow \mathcal{M}^{comb}_{g+1,X,\{t\}}$$

where $\mathcal{S}_j^{comb}$ is obtained by simply gluing a graph $\mathbb{G}$ in $\mathcal{M}^{comb}_{g,X,\{t\}}$ with the fixed graph $\mathbb{G}_j$, identifying the unique tail of $\mathcal{M}^{comb}_{g,X,\{t\}}$ with the v-tail of $\mathbb{G}_j$ and renaming the new hole by t.

It is easy to see that $\mathcal{S}_2^{comb}$ incarnates a stabilization map (obtained by composing twice $\mathcal{Y}$ and once $\mathcal{V}$).

On the other hand, consider the map $\mathcal{S}_1 : B\Gamma(S_{g,X,\{t\}}) \to B\Gamma(S_{g+1,X\cup\{t\}})$, that glues a torus $S_{1,\{y\},\{t'\}}$ with one puncture and one boundary component to the unique boundary component of $S_{g,X,\{t\}}$, by identifying t and t', and relabels the y-puncture by t.

The composition of $\mathcal{S}_1$ followed by the map π_t that forgets the t-marking

$$B\Gamma(S_{g,X,\{t\}}) \xrightarrow{\mathcal{S}_1} B\Gamma(S_{g+1,X\cup\{t\}}) \xrightarrow{\pi_t} B\Gamma(S_{g+1,X})$$

induces an isomorphism on H_k for $k \gg g$, because it can also be obtained composing $\mathcal{Y}$ and $\mathcal{V}$.

Notice that $\pi_t : B\Gamma(S_{g+1,X\cup\{t\}}) \to B\Gamma(S_{g+1,X})$ can be realized as a *combinatorial forgetful map* $\pi_t^{comb} : \mathcal{M}^{comb}_{g+1,X\cup\{t\}}(\mathbb{R}^X_+ \times \{0\}) \to \mathcal{M}^{comb}_{g+1,X}(\mathbb{R}^X_+)$ in the following way.

Let $\mathbb{G}$ be a metrized ribbon graph in $\mathcal{M}^{comb}_{g+1,X\cup\{t\}}(\mathbb{R}^X_+ \times \{0\})$. If t is marking a vertex of valence 3 or more, then just forget the t-marking. If t is marking a vertex of valence 2, then forget the t marking and merge the two edges outgoing from t in one new edge. Finally, if t is marking a univalent vertex of $\mathbb{G}$ lying on an edge e, then replace $\mathbb{G}$ by $\mathbb{G}/e$ and forget the t-marking.

5.3.6 Behavior of Witten cycles. The induced homomorphism on Borel-Moore homology

$$(\pi_t^{comb})^* : H_*^{BM}(\mathcal{M}_{g+1,X}^{comb}(\mathbb{R}_+^X)) \longrightarrow H_*^{BM}(\mathcal{M}_{g+1,X\cup\{t\}}^{comb}(\mathbb{R}_+^X \times \{0\}))$$

pulls $W_{m_*,X}$ back to the combinatorial class $W_{m_*+\delta_0,X}^t$, corresponding to (the closure of the locus of) ribbon graphs with one univalent vertex marked by t and $m_i + \delta_{0,i}$ vertices of valence $(2i+3)$ for all $i \geq 0$.

We now use the fact that, for X nonempty, there is a homotopy equivalence

$$E : \mathcal{M}_{g+1,X\cup\{t\}}^{comb}(\mathbb{R}_+^X \times \mathbb{R}_+) \xrightarrow{\sim} \mathcal{M}_{g+1,X\cup\{t\}}^{comb}(\mathbb{R}_+^X \times \{0\})$$

and that $E^*(W_{m_*+\delta_0,X}^t) = W_{m_*+2\delta_0,X\cup\{t\}}$.

This last phenomenon can be understood by simply observing that E^{-1} corresponds to opening the (generically univalent) t-marked vertex to a small t-marked hole, thus producing an extra trivalent vertex.

Finally, $(\mathcal{S}_1^{comb})^*(W_{m_*+2\delta_0,X\cup\{t\}}) = W_{m_*-\delta_0,X,\{t\}}$, because $\mathbb{G}_1$ has exactly 3 trivalent vertices.

As a consequence, we have obtained that

$$(\pi_t^{comb} \circ E \circ \mathcal{S}_1^{comb})^* : H_*^{BM}(\mathcal{M}_{g+1,X}^{comb}(\mathbb{R}_+^X)) \longrightarrow H_*^{BM}(\mathcal{M}_{g,X,\{t\}}^{comb}(\mathbb{R}_+^X \times \mathbb{R}_+))$$

is an isomorphism for $g \gg *$ and pulls $W_{m_*,X}$ back to $W_{m_*-\delta_0,X,\{t\}}$.

The other gluing map is much simpler: the induced

$$(\mathcal{S}_2^{comb})^* : H_*^{BM}(\mathcal{M}_{g+1,X,\{t\}}^{comb}(\mathbb{R}_+^X \times \mathbb{R}_+)) \longrightarrow H_*^{BM}(\mathcal{M}_{g,X,\{t\}}^{comb}(\mathbb{R}_+^X \times \mathbb{R}_+))$$

carries $W_{m_*,X,\{t\}}$ to $W_{m_*-4\delta_0,X,\{t\}}$, because $\mathbb{G}_2$ has 4 trivalent vertices.

We recall that a class in $H^k(\Gamma_{\infty,X})$ (i.e. a stable class) is a sequence of classes $\{\beta_g \in H^k(\mathcal{M}_{g,X}) \mid g \geq g_0\}$, which are compatible with the stabilization maps, and that two sequences are equivalent (i.e. they represent the same stable class) if they are equal for large g.

Proposition 5.11. *Let $m_* = (m_0, m_1, \dots)$ be a sequence of nonnegative integers such that $m_N = 0$ for large N and let $|X| = n > 0$. Define*

$$c(g) = 4g - 4 + 2n - \sum_{j\geq 1}(2j+1)m_j$$

and call $g_0 = \inf\{g \in \mathbb{N} \mid c(g) \geq 0\}$. Then, the collection $\{W_{m_*+c(g)\delta_0,X} \in H^{2k}(\mathcal{M}_{g,X}) \mid g \geq g_0\}$ is a stable class, where $k = \sum_{j>0} j\, m_j$.

It is clear that an analogous statement can be proven for generalized Witten cycles. Notice that Proposition 5.11 implies Miller's result [49] that ψ and κ classes are stable.

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Index