Part 1

Mean Field Equation, a survey

Chang-Shou Lin,

June 20-24, 2022, in Cortona, Italy

Reference

[BDM] D.Bartolucci, F.De Marchis, A.Malchiodi; IMRN (2011) 24.

[BT] D.Bartolucci and G.Tarantello; CPM 229 (2002), 3-47

[CL1] C.C.Chen and C.S.Lin ; CPAM 55 (2002), 728-771

[CL2] ____________________;CPAM 56 (2003), 1667-1727

[CL3] ____________________; Discrete and Continuous Dynamic Systems A.28, no.3 (2010), 1237-1272.

[CL4] ___________________ ; CPAM 68 (2015), no.6, 887-947

[CLY] Z.J.Chen, C.S.Lin and Y.Yang, preprint (2022).

[E] A. Eremanko ; Co-axial monodromy, Ann. Sc. Norm. Super. Pisa Cl. Sei (5) (2020) 20, 619-634.

[ET] A. Eremenko and V. Tarasov; SIGMA, (2018) 14, 12pp.

[MP1] G.Mondello and D.Panov ; IMRN (2016), no.16, 4937-4995

[MP2] G.Mondello and D.Panov ; GAFA (2019), 1110-1193.

§1-1 Non-bubbling criteria

In [PM2], Mondello-Panov studied the moduli space $\mathrm{MSPh}_{g,n}(\theta)$ of **spherical metrics** on surfaces of genus g with n conical singularities of angles $2\pi\theta = 2\pi(\theta_1,\ldots,\theta_n)$. One of the main results there is the properness of the forgetful map if the following non-bubbling criteria holds:

$$2-2g-n+\sum_{i=1}^{n}(\pm\theta_i)\notin 2\mathbb{Z}_{\geq 0} \tag{1.1}$$

for any choice of (± 1). This result can also be proved by the analytic method of PDE. Let (S,h) be Riemann surface, and $H(x)>0$ be smooth. Consider the MFE on S,

$$\Delta_h u+\rho\left(\frac{H(x)e^u}{\int_S H(x)e^u dg}-\frac{1}{|S|}\right)=4\pi\sum_{i=1}^{n}\alpha_i\left(\delta_{p_i}-\frac{1}{|S|}\right),\ \ p_i\in S \tag{1.2}$$

§1-1 Non-bubbling criteria

Suppose q_j, $j=1,2,\ldots,m$, are **blow-up points** of a sequence of solutions u_k . Near each q_j the local mass at q_j is

$$\mathrm{lm}(q_j)=\lim_{r\to 0}\lim_{k\to\infty}\rho_k\int_{B(q_j,r)}\frac{H_k(x)e^{\mathrm{u}_\mathrm{k}}}{\int_S H_k e^{\mathrm{u}_\mathrm{k}}}$$

Then we have

$$(i)\ \ \mathrm{lm}(q_j)=\begin{cases} 8\pi & if \ \ q_j\notin\{p_1,\ldots,p_n\} \\ 8\pi(1+\alpha_i) & if \ \ q_j=p_i \end{cases}$$

$$\text{(ii)}\ \ \rho_\infty=\lim_{k\to\infty}\rho_k=\sum_{j=1}^{m}\mathrm{lm}(q_j)$$

The identity (i) is due to D. Bartolucci and G. Tarantello [BT], and the (ii) follows from the so-called the phenomena of “blows-up implies concentration “.

§1-2 Blowing-up implies concentration

By cancelling out singularities, we let

$$w_k(x) = u_k(x) - 4\pi\sum_{j=1}^{N}\alpha_j G(x, p_j).$$

Assume $u_k(x)$ blows up, i.e., $\lambda_k = \max_S \; w_k(x) \to +\infty$. Then

(a) $w_k(x) \to -\infty$ uniformly in any compact set of $S \setminus \{q_1, ..., q_m\}$;

(b) $\min_S w_k(x) = -\lambda_k + O(1)$;

(c) Suppose the blow-up points $q_j \notin \{p_1, ..., p_n\}$, $j = 1, 2, ..., m$. Then $\rho_\infty = 8\pi m$ and there is $C \in \mathbb{R}$ such that

$$\rho_k - 8\pi m = (C + o(1))\lambda_k e^{-\lambda k}$$

§1-2 Blowing-up implies concentration

In case $C = 0$ (for example if the PDE is a curvature equation), we have

$$\rho_k - 8\pi m = (D + o(1))e^{-\lambda_k} \qquad (1.3)$$

for some $D \in \mathbb{R}$.

This constant D is difficult to compute. When S is a torus and (1.1) has only one singularity, (1.3) provide useful information. We will discuss it in Part 2.

Thus if

$$\rho \notin \Lambda \doteq \left\{ 8\pi l + \sum_{p_j \in I} 8\pi(1+\alpha_j) \,\middle|\, I \subseteq \{p_1, \ldots, p_n\}, l \in \mathbb{Z}_{\geq 0} \right\}$$

$$= \{ n_i \mid n_1 < n_2 < n_3 < \ldots \}$$

then all solutions of (1.1) are bounded, and then the (Leray-Schauder) topological degree can be introduced. Such ρ is called non-critical.

§1-3 Topological degree

Suppose (1.1) has finitely many solutions $u_1,\ldots,u_l$, and the linearized equation at any u_i is non-degenerate. Define the degree d_ρ by

$$d_\rho = \sum_{i=1}^{l} (-1)^{\#(u_i)},$$

where $\#(u_i)$ is the number of negative eigenvalues. This notion can be defined very well if a priori bound exists even without non-degeneracy of linearized equations. For (1.1), the degree d_ρ is well-defined if ρ is non-critical and d_ρ can be calculated by the generating function g(t).

$$g(t) = (1+t+\ldots+t^j+\ldots)^{-\chi(\mathrm{s})+\mathrm{n}} \prod_{i=1}^{n} (1-t^{\alpha_i+1}) = 1+b_1 t^{n_1}+\ldots+b_k t^{n_k}+\ldots\,,$$

where $\Lambda = \{n_1, n_2, \ldots\}$.

§1-3 Topological degree

Theorem 1.1 ([CL2,4]) *The degree* $d_\rho = \sum_{n_k < \rho} b_k$, *if* ρ *is non-critical.*

Note: If $\alpha_i \in \mathbb{N}$ then $\Lambda \subseteq \mathbb{N}$. Further, if $\chi(S) \le 0$, then

$$g(t) = (1+t+\ldots)^{-\chi(s)} \prod_{i=1}^{n} (1+t+\ldots+t^{\alpha_i}) \Rightarrow d_\rho > 0, \ \forall \rho \notin \mathbb{N}.$$

However if $\alpha_i \notin \mathbb{N}$, then the coefficients b_k could be positive or negative. So there are cancellation for the summation. In general, the computation of d_ρ is not easy.

§1-4 Curvature equations

Let (S,h) be a Riemann surface, $p_1,\ldots,p_n \in S$ and $2\pi\theta_1,\ldots,2\pi\theta_n$, $1\neq\theta_i>0$. Find a conformal conical spherical metric $ds^2=\frac{1}{2}e^{\mathrm{v}}h$ with conic singularities p_i and the angle $2\pi\theta_i$. This is equivalent to solving

$$\Delta_h \mathrm{v} + e^{\mathrm{v}} - 2K = 4\pi\sum_{j=1}^{n}\alpha_j\delta_{p_j} \quad \text{on} \quad S \tag{1.4}$$

where $\theta_j=\alpha_j+1$, and K: curvature of h.

Obviously, (1.4) is a special case of mean field equations. The parameter, denoted by $8\pi\rho$, can be computed by

$$8\pi\rho=\int_S e^{v}=2\int_S K+4\pi\sum_{j=1}^{n}\alpha_j=4\pi(\chi(S)+\sum_{j=1}^{n}\alpha_j),$$

i.e.,

$$\rho=1-g+\frac{1}{2}\sum_{j=1}^{n}\alpha_j \ .$$

§ 1-5 Projective structure and Integrability

It is known that any metric h on S determines a complex structure. By the uniformation theorem, we may further take the complex structure as a projective structure, i.e, there is an open covering $\{U_i\}$ of S such that the transition function is a Mobius transformation. In each U_i , we consider a solution u_i of

$$\begin{cases} 4u_{z\bar{z}} + e^{u} = 4\pi \sum_{p_k \in U_i} \alpha_k \delta_{p_k} \quad \text{on } U_i \text{ , and satisfies} \\ u_j = u_i + 2\log\left|\frac{dz_i}{dz_j}\right| \quad \text{on } \mathrm{U}_i \cap \mathrm{U}_j \text{ .} \end{cases} \tag{1.5}$$

We could see (1.4) and (1.5) are equivalent.

The advantage is the following integrable property: $q_j = (u_j)_{zz} - \frac{1}{2}(u_j)_z^2$ is meromorphic, and satisfies

$$q_j\, dz_j^2 = q_i\, dz_i^2 \tag{1.5}$$

i.e., $q_i\, dz_i^2$ defines a globally quadratic differential on S. A pole of $q_i\, dz_i^2$ can occur only at p_k and since $u_i(z) = 2\alpha_k \log|z| + O(1)$, we have

$$q_j(z) = -2\left[\frac{\alpha_k}{2}(\frac{\alpha_k}{2}+1)\right] z^{-2} + O(z^{-1}) \quad \text{near } z(p_k)=0 \tag{1.6}$$

Moreover, there is a developing map $h_i(z)$ such that

$$u_i(z) = \log \frac{8|h_i'(z)|^2}{(1+|h_i(z)|^2)^2} \quad \text{(Liouville's theorem)} \tag{1.7}$$

By (1.7), the Schwarz derivative of $h_i(z)$ satisfies

$$q_i(z) = \{h_i(z), z\} = \left(h_i''/h_i\right)' - \frac{1}{2}\left(h_i''/h_i'\right)^2 . \tag{1.8}$$

The Schwarz derivatives have a *beautiful* connection with second order ODEs of complex variable. Consider a second order ODE

$$y''(z)+\frac{1}{2}q_i(z)y=0 \qquad (1.9)$$

and y_k, $k=1,2$, are linearly independent solutions of (1.9). Take a ratio $h(z)=y_1(z)/y_2(z)$.

Then the Schwarz derivative of h satisfies

$$\{h(z),z\}=q_i(z).$$

Set $\mathrm{z}_i=\dfrac{az_j+b}{cz_j+d}$ to be the transition function. Let $y_j(z_j)=(cz_i+d)^{-1}y_i(z_i)$.

Then $y_i''+\frac{1}{2}q_i(z_i)y_i=0$ iff $y_j(z_j)+\frac{1}{2}q_j(z_j)y_j(z_j)=0$

This transformation laws allow us to define the monodromy representation ρ from $\pi_1(\mathrm{S}\backslash\{p_1,\ldots,p_n\})$ to $\mathrm{SL}(2,\mathbb{C})$, if a fundamental solution is fixed. Due to the Liouville Theorem, the image of the representation ρ is unitariable, i.e., there is a matrix P s.t. $P\rho(\gamma)P^{-1}$ is unitary for all $\gamma\in\pi_1(\mathrm{S}\backslash\{p_1,\ldots,p_n\})$. Indeed this is also a sufficient condition to have a solution for curvature equation.

§1-5 Properness of the forgetful map

Let (S,h) be a Riemann surface. Consider

$$\Delta_h \mathrm{u} + e^{\mathrm{u}} - 2K = 4\pi\sum_{j=1}^{n}\alpha_j\delta_{p_j} \text{ on S.}$$

In the form of MFE, the parameter $8\pi\rho$ is given by

$$8\pi\rho = \int_S e^{\mathrm{u}} = 4\pi(\chi(\mathrm{S}) + \sum_{j=1}^{n}\alpha_j)$$

Suppose there is a sequence of blowing-up solutions which blow-up points are $p_i,\ i \in I \subseteq \{1,2....,n\}$ and $q_j \notin \{p_1, p_2, ..., p_n\}$, $j = 1,2,...,l$.

Then $8\pi\rho$ is equal to the sum of the local masses, i.e.,

$$\chi(\mathrm{S}) + \sum_{j=1}^{n}\alpha_j = 2l + 2\sum_{i\in I}(1+\alpha_i) \Rightarrow 2 - 2g - n + \sum_{j\in I^C}\theta_j - \sum_{j\in I}\theta_j = 2l.$$

§1-6 Calculation of the degree

Therefore, if $2-2g-n+\sum \pm\theta_j \notin 2\cdot\mathbb{Z}_{\geq 0}$ for any choice of (± 1), then *there is a priori bound for all solutions of the PDE, or in geometry, the forgetful map is proper.*

Consider the Riemann surface to be the standard sphere and $n \geq 3$.

If there are at most two non-interger angles θ_j, then the problem has been solved completely.

Hence we might assume there are at least three non-interger angles and the PDE can be written as

$$\begin{cases} \Delta \mathrm{u} + e^{\mathrm{u}} = 4\pi(\alpha_0\delta_0 + \alpha_1\delta_1 + \sum_{j=4}^{n} \alpha_j \delta_{t_j}) \text{ on } \mathbb{C}, \\ \mathrm{u}(x) = -2(2+\alpha_\infty)\log|x| + O(1) \text{ as } |x| \to \infty, \end{cases} \quad (1.10)$$

where $\alpha_0, \alpha_1, \alpha_\infty \notin \mathbb{Z}$. The angle θ_j at $p_j \in \{0, 1, \infty, t_4, ..., t_n\}$ is

$$\theta_j = \alpha_j + 1$$

From now on, we always assume the following conditions hold:

(I) Non-bubbling criteria : $2-n+\sum \pm\theta_i \notin 2\cdot\mathbb{Z}_{\geq 0}$ for all choice of (± 1).

(II) $\mathrm{d}_1(\mathbb{Z}_o^n,\alpha)>1$, where $\mathbb{Z}_o^n=\left\{(m_1,\dots,m_n)\middle| m_i\in\mathbb{Z} \text{ and } \sum m_i=\text{odd}\right\}$

and $\alpha=(\alpha_0,\alpha_1,\alpha_\infty,\alpha_4,\dots,\alpha_n)$. The distance $\mathrm{d}_1(x,y) \doteqdot \sum_{i=1}^{n}\left|x_i-y_i\right|$, $x,y\in\mathbb{R}^n$.

The notation $\mathrm{d}_1(\mathbb{Z}_o^n,\alpha)$ was introduced in (PM1), where Mandello-Panov proved that

If $\mathrm{d}_1(\mathbb{Z}_o^n,\alpha)<1$ *then the curvature equation has no solution.*

Indeed they also proved that

If $\mathrm{d}_1(\mathbb{Z}_0^n,\alpha)>1$, *then there are* $t_4,\dots,t_n$ $(t_i\neq t_j)$ *such that the monodromy group is not a commutative group.*

This theorem is a deep result, because it gives examples of PDEs with degree 0, but the equation still has solutions. Recall degree of solutions of (1.10) can be counted by

$$\deg = \sum_{n_k<\rho} b_k, \quad \rho = \frac{1}{2}(\sum \alpha_j) + 1,$$

where b_k are the coefficients

$$g(t) = (1 + t + \ldots + t^l + \ldots)^{n-2} \prod_j (1 - t^{\alpha_j+1}) = \sum b_k t^{n_k}$$

Theorem1.2 *Assume* α_0, α_1 *and* $\alpha_\infty \notin \mathbb{Z}$ *and* $\alpha_j \in \mathbb{N},\ j = 4, \ldots, n.$ *If*

$$d_1(\mathbb{Z}_0^n, \alpha) > 1, \text{ then } |\deg| = \prod_{j=4}^{n} \theta_j.$$

Consequently, there is at least a solution (1.10) with any $n-3$ distinct points $t_4, \ldots, t_n \notin \{0, 1, \infty\}$

Note that the existence of the PDE was proved by Eremanko and V.Tarasov [ET].

Next consider the conical sphere with 4 singular points with angle $2\pi(\theta_1, \theta_2, \theta_3, \theta_4)$.

Set $\alpha_i = \theta_i - 1 = l_i \pm \beta_i, \ \ \beta_i \in (0, \frac{1}{2}],$

We might assume $\beta_1 \geq \beta_2 \geq \beta_3 \geq \beta_4$, and $l_i \geq 0 \ \forall i$.

(i) $-2 + \sum \pm \theta_i \notin 2 \cdot \mathbb{Z}_{\geq 0}$ for all choices of ± 1

(ii) $d_1(\mathbb{Z}_0^4, \alpha) > 0$

We have two groups: Group 1: $\sum l_i$ is even ; Group 2: $\sum l_i$ is odd.

In the following, we will compute the Leray-Schauder degree in Group 1. The computations of Group 2 could follow from Group 1. For Group 1, we have two subgroups, i.e.,

Subgroup1: $\beta_1 + \beta_4 < \beta_2 + \beta_3$

In this subgroup, we always have $\beta_4 < \beta_3$. The degree $= -(l_4 + 1)$.

Subgroup 2: $\beta_1+\beta_4>\beta_2+\beta_3$

This subgroup yields $\beta_1>\beta_2$. The formulas of degree is more complicated than Group 1.

Theorem1.3. *Suppose $\beta_1+\beta_4>\beta_2+\beta_3$. Then the degree is presented in the following table.*

Notation : The sign at α_i is given by $\pm$ if $\alpha_i=l_i\pm\beta_i$

Remark: The ODE associated with (1.10) is

$$y''(z)+Q(\chi)y=0 \text{ with the Riemann scheme } \begin{pmatrix} 0 & 1 & t & \infty \\ -\alpha_0/2 & -\alpha_1/2 & -\alpha_t/2 & -\alpha_\infty/2 \\ \alpha_0/2+1 & \alpha_1/2+1 & \alpha_t/2+1 & \alpha_\infty/2+1 \end{pmatrix},$$

where

$$Q(\chi)=\frac{\beta_0}{\chi^2}+\frac{\beta_1}{(\chi-1)^2}+\frac{\beta_t}{(\chi-t)^2}+\frac{d_0}{\chi}+\frac{d_1}{\chi-1}+\frac{d_t}{\chi-t},$$

$$\beta_p=\frac{\alpha_p(\alpha_p+2)}{4},\quad \sum_p d_p=0,\ \sum_p(\beta_p+pd_p)=\beta_\infty.$$

Theorem1.3 shows if deg $=0$ then the monodromy matrices are unitariable. This result seems difficult from the algebraic viewpoint.

signs $\alpha_1,\alpha_2,\alpha_3,\alpha_4$	degree
$+\ +\ +\ +$	$\frac{l_1-l_2-l_3-l_4}{2}-1$
$-\ +\ +\ +$	$\frac{l_1+l_2+l_3+l_4}{2}+2$
$+\ -\ +\ +$	$\frac{l_3+l_4-l_1-l_2}{2}$
$+\ +\ -\ +$	$\frac{l_2+l_4-l_1-l_3}{2}$
$+\ +\ +\ -$	$\frac{l_2+l_3-l_1-l_4}{2}$
$-\ -\ +\ +$	$\frac{l_2-l_1-l_3-l_4}{2}-1$
$-\ +\ -\ +$	$\frac{l_3-l_1-l_2-l_4}{2}-1$
$-\ +\ +\ -$	$\frac{l_4-l_1-l_2-l_3}{2}-1$

signs $\alpha_1,\alpha_2,\alpha_3,\alpha_4$	degree
$+\ +\ -\ -$	$\frac{l_1+l_3+l_4-l_2}{2}+1$
$+\ -\ +\ -$	$\frac{l_1+l_2+l_4-l_3}{2}+1$
$+\ -\ -\ +$	$\frac{l_1+l_2+l_3-l_4}{2}+1$
$+\ -\ -\ -$	$-(\frac{l_1+l_2+l_3+l_4}{2}+2)$
$-\ +\ -\ -$	$\frac{l_1+l_2-l_3-l_4}{2}$
$-\ -\ +\ -$	$l_1+l_3-l_2-l_4$
$-\ -\ -\ +$	$\frac{l_1+l_4-l_2-l_3}{2}$
$-\ -\ -\ -$	$\frac{l_2+l_3+l_4-l_1}{2}+1$

For the case of positive genus, Bartolucci-DeMarchis-Malchiodi [BDM] proved

Theorem(BDM). *Assume* $\alpha_i > 0,\ i = 1, 2, \ldots, n,$ *and* $\sum_{i=1}^{n} \theta_i > 2 \cdot genus > 0$. *Then the curvature equation has a solution provided that the non-bubbling criteria holds.*

The case when $\alpha_i < 0$ for some i is delicated. In particular if there is one small angle, then Theorem D of [PM2] shows the non-existence of solutions. For this case, the degree can be computed as follows.

Theorem 3. *Suppose* $\theta_2, \ldots, \theta_n$ *are fixed such that* $\sum_{i=2}^{n} \theta_i > 2g - 2 + n$. *Then there is a small positive* ε *such that if* $\theta_1 < \varepsilon$, *then* $\deg = \sum_I (-1)^{\# I} C_I$, where the summation is over subsets I of $\{2,\ldots,n\}$ such that

$2-2g-n+\sum_{i\in I^c}\theta_i-\sum_{i\in I}\theta_i=2k,\ k\in\mathbb{Z}_{\geq 0},$ and

$C_I=C(2k+n-\chi(\mathrm{s}),n-\chi(\mathrm{s})).$

Remark. If no such I exists, then $\deg=0$. In this case, Theorem D of [PM2] proved that the curvature equation has no solutions, a stronger and interesting claim.

So far, we only apply the analytic method. Here, we give an example to describe the applications of integralbility. Consider the curvature equation on $\mathbb{R}^2$ with $\theta=(\theta_0,\theta_1,\theta_\infty,\theta_t)$ (t is a singular point) such that $\theta_0,\theta_1,\theta_\infty\notin\mathbb{Z}$ and $\theta_t\in\mathbb{N}$. A solution u is called *co-axial* if the monodromy group is commutative. The existence of such a metric was obtained by Eremenko [E]. In the following, the angle $(\theta_0,\theta_1,\theta_\infty,\theta_t)$ are fixed.

Theorem 1.4. *There exists $t\notin\{0,1,\infty\}$ such that the curvature equation has a co-axial solution iff for any $p\in\{0,1,\infty\}$, there are $\varepsilon_p\in\{\pm1\}$ such that the following two conditions hold :*

(i) $k=\sum_{p\in\{0,1,\infty\}}\varepsilon_p\theta_p\in\mathbb{Z}_{\geq 0}$, *and*

(ii) $\theta_t-2-k\in 2\cdot\mathbb{Z}_{\geq 0}$.

Set $Q_\theta = Q(\theta_0, \theta_1, \theta_\infty)$. Applying the integralbility, we prove

Theorem 1.5. [CLY]
(a) The number of t such that the curvature equation has a co-axial solution is equal to $\theta_t^2 - k^2/4$, counted with multiplicity.
(b) Let $A = \{t:$ the curvature equation has a co-axial solution$\}$. Then any $t \in A$ is an algebraic number over Q_θ, and $Q_\theta(A)$ are the smallest field containing $t \in A$ and Q_θ is a Galois extension over Q_θ. The degree $[Q_\theta(A), Q_\theta] \leq C_\theta$, where C_θ depends on θ only.

For the critical case, the topological degree from the mean field equations would depend on the function $H(x)$ even when the a priori bound exists. The simplest among the critical cases is

$$\Delta \mathrm{u} + e^{\mathrm{u}} = 8\pi n \delta_0 \text{ on } E_\tau.$$

Part 2
Curvature Equation on Flat Tori

Notations

- $\tau \in H = \left\{ \tau \,\middle|\, \operatorname{Im} \tau > 0 \right\}$.
- $E_\tau = \mathbb{C} \big/ \mathbb{Z} \oplus \mathbb{Z}_\tau$; $\omega_1 = 1,\ \omega_2 = \tau,\ \omega_3 = 1 + \tau$.
- $\wp(z;\tau)(= \wp(z))$: the Weierstrass elliptic function of order 2

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}.$$

- $\zeta(z) : \zeta'(z) = -\wp(z)$ and satisfies $\zeta(-z) = -\zeta(z)$.

  quasi-periods: $\zeta(z + \omega_i) = \zeta(z) + \eta_i,\ i = 1, 2$

  Legendre relation: $\eta_2(\tau) - \tau\eta_1(\tau) = -2\pi i$
- σ(z): entire function with (simple) zeros only at 0 with

$$\sigma(z + \omega_i) = -e^{\eta_i (z + \frac{1}{2}\omega_i)} \sigma(z)$$

Reference

[CKL] Z.J.Chen, T.L.Kuo and C.S.Lin ; Proof of Dahmen and Beukers' Conjecture of Counting Integral Lame Equations with Finite Monodromy, preprint, 2021.

[CKLW] Z.J.Chen, T.L.Kuo, C.S.Lin and C.L.Wang ; JDG (2018) 108, 185-241.

[CLW] C.L.Chai, C.S.Lin and C.L.Wang ; Cambridge J.Math. (2015) 3, 127-274.

[CL1] Z.J.Chen and C.S.Lin ; JDG (2019) 113, 189-226. Critical

[CL2]____________________ ; CPM (2020) 374, 111-144.

[CL3]____________________ ; Amer.J.Math. (2020) 142, 1269-1301.

[CL4]____________________ ; JDG (2021) 118, 45-485. Exact

[CL5]____________________ ; Elliptic KdV potentials and spherical tori, in preparation.

[E] A.Eremenko ; PAMS, (2020) 148, no.9, 3957-3965.

[EGMP] A.Eremenko, A.Gabrielov, G.Mondello, D.Panov ; Comm.Contemp. Math. (2022) 24, 68pp.

[LW1] C.S.Lin and C.L.Wang, Ann. Math. (2010) 172, 911-954.

[LW2]____________________ ; IMRN (2017) 18, 5591-5608.

[LW3]____________________ ; J.E_c.Polytoch. Math. (2017) 4, 557-593.

[LW4]____________________ ; Comm. Anal.Geom. (2020) 28, 1289-1313.

In this part, I will focus on the equation,

$$\Delta u + e^{u} = 8\pi n \delta_0 \text{ on } E_\tau \tag{2.1}$$

The motivation for my study is not geometric but only purely analytic because (2.1) is simplest among all the critical case. During the earlier period of studying (2.1), to me the integrability means the Liouville theorem (and developing map).

The Liouville Theorem immediately allows us to recover the developing map by the solution u through the Schwarz derivative, and the (projective) monodromy of developing maps. This already provides a really useful tool to study (2.1), or

$$\Delta u + e^{u} = 8\pi \alpha \delta_0 \text{ on } E_\tau \tag{2.2}$$

- α: non-critical iff $\alpha \notin \mathbb{N}$
- degree $= n+1$ if $n < \alpha < n+1$.

Theorem2.1 [LW2]. *If* $\alpha \notin \mathbb{N}$, *then any solution* u *of* (2.2) *is even*, *i.e.*, $u(-z) = u(z)$.

§2-1 Multiple Green functions

The integrability is also very useful for $\alpha = n$. Here we recall a result from mean field equations. Let u_k be a sequence of blowing-up solutions of (2.2) with $\alpha_k \to n$, and $q_1, \dots, q_n$ be the blow up points. Then n-th tuple $(q_1, \dots, q_n)$ is a critical point of the n-th multiple Green function $G_n(z_1, \dots, z_n)$, where

- G is the Green function i.e., $\Delta G = 1 - \dfrac{1}{|E_\tau|}$,
- $G_n(z_1, \dots, z_n) = n\sum_{i=1}^{n} G(z_i) - \sum_{j<i} G(z_i - z_j)$.
- Let u be a soiution of (2.1) and $f(z)$ be the developing map.

Then [CLW] proves

(i) $f(z)$ is single-valued, and $f(z+\omega_j) = e^{i\theta_j} f(z)$, $\theta_j \in \mathbb{R}$, $j = 1, 2$.

(ii) For any $\lambda > 0$,

$$\mathrm{u}_\lambda(z) = \log \frac{8\lambda^2 |f'(z)|^2}{(1+\lambda^2 |f(z)|^2)^2} \quad \text{is also a solution.}$$

§2-1 Multiple Green functions

As $\lambda \to \infty$, $\mathrm{u}_\lambda(z)$ blows up at $q_1,\ldots,q_n$, the set of zeros of f,
and $\lambda \to 0$, $\mathrm{u}_\lambda(z)$ blows up at $p_1,\ldots,p_n$, the set of poles of f. Further,
$q_i \neq \pm q_j$ for any i, j and $\{-q_1,\ldots,-q_n\} = \{p_1,\ldots,p_n\}$.

(iii) $f(0) \neq 0, \infty$ and $f(-z) = \dfrac{f(0)^2}{f(z)}$. Choose $\lambda_0 = \sqrt{|f(0)|}$. Then $\mathrm{u}_{\lambda_0}(z)$ is the unique even solution among the one-parameter family $\mathrm{u}_\lambda(z)$ of solutions.

- Consider $n = 1$.

q (the only zero of $f(z)$) is a critical point of G, and by (ii) $q \neq -q$. Thus $q \notin \left\{\frac{\omega_i}{2}\right\}$.

- $G_z(z) = \zeta(z) - r\eta_1(z) - s\eta_2(z)$, $z = r + s\tau$, $r, s \in \mathbb{R}$. Thus if $q = r + s\tau$ then q satisfies

$$\zeta(q) = r\eta_1(\tau) + s\eta_2(z) \qquad (2.3)$$

§2-1 Multiple Green functions

Conversely, if a non-half period point $q = r + s\tau,\ r, s \in \mathbb{R},$ is a critical point of $G(z)$. Then we define

$f(z) = e^{cz}\dfrac{\sigma(z-q)}{\sigma(z+q)}$, where $c = 2(\eta_1(\tau)r + \eta_2(\tau)s),$ satisfies $f(z+\omega_1) = e^{4\pi si} f(z)$ and $f(z+\omega_2) = e^{-4\pi ri} f(z),$ which implies $\left|f(z+\omega_i)\right| = \left|f(z)\right|$. Since $f(0) \neq 0$ and $f'(0) = f(0)(c + 2\zeta(-q)) = 2f(0)(\eta_1 r + \eta_2 s - \zeta(q)) = 0$ by (2.3).

Thus $\mathrm{u}(z) = \log\dfrac{8\left|f'(z)\right|^2}{(1+\left|f(z)\right|^2)^2}$ has a singularity at $z = 0,$ and it is easy to see that $\mathrm{u}(z)$ is a solution of (2.1) with $n = 1$. By the above argument, for $\lambda > 0,$

$\mathrm{u}_\lambda(z) = \log\dfrac{8\lambda^2\left|f'(z)\right|^2}{(1+\lambda^2\left|f(z)\right|^2)^2}$ is a one-parameter family of solutions and there is a unique even solutions among them. So we get

$$\#\,of\ even\ solutions = \#\ of\ \pm q \notin \left\{\frac{\omega_i}{2}\right\} : critical\ points\ of\ G.$$

§2-1 Multiple Green functions

How to describe the geometry of a flat torus? One could use the Green function G(z) and the multiple Green function $G_n(z_1,\ldots,z_n)$

Theorem 2.2 [LW1,2].

(a) *G has either three critical points or five critical points* ;

(b) *If* G *has five critical point* $\{\omega_1/2,\omega_2/2,\omega_3/2,\pm q\}$ *then* $\pm q$ *are the minimum points of* G, and $\omega_i/2$, $i=1,2,3$, *are non-degenerate saddle points of* G.

Theorem 2.2 $=$ the integrability $+$ Theorem 2.3:

Theorem 2.3 [LW1]. *Suppose u is an even solution of*

$$\Delta \mathrm{u}+e^{\mathrm{u}}=8\pi\alpha\delta_0 \text{ on } E_\tau \tag{2.4}$$

If $0<\alpha\le 1$, *then the linearized equation of* (2.2) *at u is non-degenerate. Consequently,* (2.2) *has exactly one solution provided that* $0<\alpha<1$ *and has one even solution at most if* $\alpha=1$.

Theorem 2.3 is proved via the method of symmetrization, and the classical Pol's inequality (for surface with curvature 1).

§2-1 Multiple Green functions

Theorem 2.2 (b) follow from the following calculation of the Hessian of G at $\frac{1}{2}\omega_i$:

$$\det D^2 G(\frac{1}{2}\omega_i;\tau) = -\frac{1}{4\pi^2}\left|e_i+\eta_1\right|^2 \operatorname{Im}(\tau - \frac{2\pi i}{e_i+\eta_1}),\ e_i = \wp(\frac{\omega_i}{2}). \tag{2.5}$$

See [LW1,2]. The bubbling analysis has more applications. Consider a sequence of blowing-up solutions u_k of (2.2) with $\alpha = \alpha_k \to 1$, which has only one blow-up point $q \in E_\tau$ and $q \neq 0$. Then there is a constant $D(q)$ such that

$$8\pi(\alpha_k - 1) = (D(q) + o(1))e^{-\lambda_k}, \tag{2.6}$$

where λ_k is the maximum of u_k after cancelling out the singularity.

If $q \notin \{\omega_i/2\}$, then the existence of one-parameter solutions implies $D(q) = 0$.

If q is half period, the computations of $D(q)$ is not trivial.

§2-1 Multiple Green functions

To describe the constant $D(q)$ where q is a half period, we let

$\widetilde{G}(z,q)=G(z,q)+\frac{1}{2\pi}\log|z-q|,\ \phi(q)=\widetilde{G}(q,q)$. Then

$$D(q)=\lim_{r\to 0}\Big(\int_{E_\tau\setminus B_r(q)}\frac{e^{-8\pi G(z)}e^{8\pi(\widetilde{G}(z,q)-\phi(q))}+8\pi G(q)}{|z-q|^4}-\int_{\mathbb{R}^2\setminus B_r(q)}\frac{e^{-8\pi G(q)}}{|z-q|^4}\Big)\ .$$

In [LW2], we prove the following identity:

$$\text{D(q)= a positive constant} \times (-\det D^2G(q)). \qquad (2.7)$$

The identity can be extended to the n-th multiple Green function $G_n(z_1,\dots,z_n)$. We will discuss it later. The identity (2.6) can be used to study the behavior of solution $u_k(z;\tau_k)$ of (2.2) as $\alpha_k\to 1$, and $\tau_k\to\tau$.

§2-1 Multiple Green functions

The constant $D(q)$ and the Hessian $\det D^2G(q)$ are two quantities to control how many sequence of blowing-up solutions as $\alpha_k \to 1$.

• Suppose $\mathrm{u}_1^k(z;\tau)$ and $\mathrm{u}_2^k(z;\tau)$ are two sequence of blowing-up solutions of (2.2) with the same $\alpha_k \to 1$, and assume q is the blow-up point. Lin-Yan proved.

• **Local uniqueness**. *If both $D(q)$ and $\det D^2G(q)$ are not zero, then $u_1^k = u_2^k$ for large k.*

• **Constructing blow-up solutions**. *If q is a critical point and both $D(q)$ and the Hessian at q are not zero, then there exists a sequence of blowing-up solutions of (2.2) with some $\alpha_k \to 1$.*

We denote $\Omega_5 = \{\tau \in H \mid G(z;\tau) \text{ has five critical points}\}$, and

$$\Omega_3 = \{\tau \in H \mid G(z;\tau) \text{ has three critical points only}\}.$$

Clearly, Ω_5 is open and Ω_3 is closed.

§2-1 Multiple Green functions

Theorem 2.4 [LW2].

(a) Suppose $\alpha_k < 1$. Then $\mathrm{u}_k(z;\tau)$ blows up iff $\tau \in \Omega_3$.

In this case, the blow-up point q (a half period) is the minimum point. The constant $D(q) < 0$ iff $\tau \in \Omega_3^{\mathrm{int}}$, and $D(q) = 0$ if $\tau \in \partial\Omega_3$. The other two half periods are non degenerate saddle points with $D > 0$.

Consequently, there is a small $\varepsilon > 0$ such that (2.2) has two solutions if $1 < \alpha < 1+\varepsilon$. The two sequences of solutions must blow up as $\alpha \to 1$.

Proof : Let q be the blow-up point of $\mathrm{u}_k(z;\tau)$. Then

- q is a half period because $\mathrm{u}_k(z;\tau)$ is even.
- $D(q) \leq 0$ because $\alpha_k < 1$ and $\det D^2G(q;\tau) \geq 1$. Thus q is the minimum point, which implies $\tau \in \Omega_3$. This proves the sufficient part.

Conversely, if $\tau \in \Omega_3$ then (2.1) has no solutions. This implies $\mathrm{u}_k(z;\tau)$ blows up. Together we prove the first part.

For the second part, we note that $G(z;\tau)$ has the other two half periods q_1 and q_2. By Theorem 2.2 (b) and (2.6), $\det D^2G(q_i) < 0$ and $D(q_i) > 0$. By the local uniqueness and the constructing of blowing-up solution, the second part is proved. □

§2-1 Multiple Green functions

(b) *If* $\tau \in \Omega_5$, *then there is a small* $\varepsilon > 0$ *such that* (2.2) *has exactly four solutions if* $1 < \alpha < 1 + \varepsilon_0$. *Among them, there are three sequences of solutions blows up at one of the three half periods respectively. The remainder converges to an even solution of* (2.1) *as* $\alpha \to 1$.

proof : By the same arguement as (a), there are three blow-up solutions of (2.2) with $\alpha_k > 1$, which blows up at half period (all are saddle). There is another solution of (2.2) which does not blow up as $\alpha \to 1$, because of Theorem 2.1 and Theorem 2.3. □

(c) If $\tau_k \in \Omega_5$ and the unique even solution $\mathrm{u}(z;\tau_k)$ of (2.1) blows up as $\tau_k \to \tau$, then τ belongs to $\partial\Omega_3$, and the blow-up point is the unique minimum point of $G(z;\tau)$.

proof : Suppose q is the blow-up point. Then q is a half period and $D(q) = 0$ by (2.6) because $\alpha_k = 1$ for all k. This implies $\tau \in \Omega_3$. Since τ is the limiting point of τ_k, we have $\tau \in \partial\Omega_3$. □

§2-1 Multiple Green functions

Next we discuss equation (2.1) with $n \geq 1$ and the multiple Green function. Recall that (i) the developing map $f(z)$: there are one-parameter solutions,

$$\mathrm{u}_\lambda(z) = \log \frac{8\lambda^2 |f'(z)|^2}{(1+\lambda^2 |f(z)|^2)^2}, \quad \lambda > 0.$$

(ii) $f(z+\omega_j) = e^{i\theta_j} f(z),\ \theta_j \in \mathbb{R}.$

(iii) $f(0) \neq 0$. Wlog, f can be normalized by $f(0)=1$. Then $f(-z) = \dfrac{1}{f(z)}$.

(iv) f has zeros $q_1,..,q_n$ and all are simple zeros. Also f has poles $\mathrm{p}_1,..., p_n$ and all are simple poles. Further $q_i \neq -q_i,\ q_i \neq \pm q_j$ and $\{q_1,...,q_n\} = \{-p_1,...,-p_n\}$.

Part (i) follows from (ii) and (iv) follows from (iii). It is not difficult to prove (ii) and (iii). See [CLW].

How to find the relations between q_j? These are two mthods:

§2-1 Multiple Green functions

(1) Consider the logarithmic derivative $g(z)=f'(z)/f(z)$. Then g(z) is an even elliptic function with (simple) poles at $q_1,\dots,q_n$ (residue 1) and $p_1,\dots,p_n$ (residue -1). This implies 0 is a zero of order $2n$. Therefore, g(z) satisfies

- $g(z)=\dfrac{\wp'(q_1)}{\wp(z)-\wp(q_1)}+\dots+\dfrac{\wp'(q_n)}{\wp(z)-\wp(q_n)}.$
- $g^{(r)}(0)=0,\ r=0,1,2,\dots,2n-1,\ g^{(2\mathrm{n})}(0)\neq 0.$

Near $z=0$, the expansion:

$$g(z)=\sum_{j=1}^{n}\frac{\wp'(q_j)}{\wp(z)(1-\wp(q_j)/\wp(z))}=\sum_{m=0}^{\infty}\Big(\sum_{j=1}^{n}\wp'(q_j)\wp(q_j)^m\Big)\wp(z)^{-m-1}\Rightarrow$$

$$\sum_{j=1}^{n}\wp'(q_j)\wp(q_j)^m=0,\quad m=0,1,2,\dots,n-2. \tag{2.8}$$

The set $\{q_1,\dots,q_n\}$ uniquely determines g and $f(z)$ $(=\exp\int_0^z g(s)ds)$.

However, conditions (ii) and (iii) require $q_1,\dots,q_n$ to satisfy

$$\sum_{i=1}^{n}G_z(q_i)=0. \tag{2.9}$$

Theorem 2.5 [CLW]. The curvature equation has an even function if and only if there are a n-tuple $(q_1,\dots,q_n)$ satisfying (2.7) and (2.8) such that $q_i\neq -q_i$ and $q_i\neq \pm q_j\ \ \forall i\neq j$.

§2-1 Multiple Green functions

The second method :

As $\lambda \to \infty$, $u_\lambda(z)$ blows up at $q_1,\dots,q_n$. Then the n-tuple is a critical point of $G_n(z_1,\dots,z_n)$ i.e.,

$$nG_z(q_i) - \sum_{j\neq i} G_z(q_i - q_j) = 0,\ 1 \le i \le n,$$

which is equivalent to the following system:

$$\sum_{j\neq i} \zeta(q_i) - \zeta(q_j) - \zeta(q_i - q_j) = 0,\quad 1 \le i \le n. \qquad (2.10)$$

if we apply the following identity:

$$G_z(r+s\tau) = \zeta(r+s\tau) - r\eta_1 - s\eta_2,\quad r,s \in \mathbb{R}.$$

Theorem 2.6 [CLW] *The curvature equation* (2.1) *has a even solution iff there is a critical point* $(q_1,\dots,q_n)$ *of* G (i.e., (2.10)) *satisfying* (2.9), $q_i \neq -q_i$, $q_i \neq \pm q_j$.

proof : Write $\sum q_j = r + s\tau$, $r,s \in \mathbb{R}$. (2.10) yields: $\sum \zeta(q_j) = r\eta_1 + s\eta_2$,

Define $f(z) = e^{2z\sum_j \zeta(q_j)} \dfrac{\prod_i \sigma(z-q_i)}{\prod_j \sigma(z+q_j)}$. The transformation law for σ yields $f(z+w_i) = e^{zd_i} f(z)$,

where by (2.9) $d_i = \sum_j \zeta(q_j)\omega_i - \eta_i \sum q_j = \begin{cases} -2\pi is & \text{if } i=1, \\ 2\pi ir & \text{if } i=2, \end{cases}$

Thus $u(z) = \log \dfrac{8|f'(z)|^2}{(1+|f(z)|^2)^2}$ yields a solution. The converse part be already done. □

§2-1 Multiple Green functions

So, (2.8) and (2.9) together are equivalent to (2.9) and (2.10). Naturally, we guess that (2.8) and (2.10) should be equivalent. [CLW] confirmed it.

Suppose $(q_1 \dots q_n)$ *is non-trivial. Then the n-tuple* $(q_1,\dots,q_n)$, $q_i \neq 0$ *and* $q_i \neq q_j$, *is a critical point of* G_n *iff* $\sum_{i=1}^{n} \wp'(q_i)\wp(q_i)^m = 0, \quad 0 \le m \le n-2.$ (2.11)

Applying the equivalence, both Theorem 2.5 and Theorem 2.6 are identical.

Definition. *A critical point* $(q_1,\dots,q_n)$ *of* G_n *is called trivial if* $\{q_1,\dots,q_n\} = \{-q_1,\dots,-q_n\}$. *Otherwise, it is called non-trivial.*

If $q = (q_1,\dots,q_n)$ is a non-trivial critical point then $\{q_1,\dots,q_n\} \cap \{-q_1,\dots,-q_n\} = \phi.$

We summarize:

The curvature equation has an (even) solution iff there is a ***non-trivial*** *critical point* $(q_1,\dots,q_n)$ *of* G_n *such that* (2.9) *holds.*

§2-1 Multiple Green functions

The set of trivial critical points is finite. Indeed we have

Theorem 2.7 [CLW]. *G_n has exactly $2n+1$ trivial critical points, counted with multiplicity.*

In subsection 2.3, we will introduce the Lame equation and its spectral polynomial $\ell_n(B)$.

The spectral polynomial $\ell_n(B)$ is a monic polynomial of degree $2n+1$. For any critical point

$q=(q_1,\ldots,q_n)$ of G_n, set $B=(2n-1)\sum_{i=1}^{n}\wp(q_i)$. Then in subsection 2.3, we prove

$$q \text{ is trivial iff } B \text{ is a zero of } \ell_n(B).$$

Theorem 2.7 follows from this claim.

Theorem 2.8 [CLW]. *The set of critical points of G_n forms a hyper-elliptic curve and the genus is n.*

Indeed, the map $(q_1,\ldots,q_n)\to(2n-1)\sum_{i=1}^{n}\wp(q_i)$ defines a covering map from the set of critical points onto $\mathbb{C}\cup\{\infty\}$ of degree 2. The branch points of this covering map are exactly those trivial critical points and the point $\infty(=(0,\ldots,0))$. Since the degree is two and each ramified index $=2$, the Riemann-Hurwitz formula: $2g-2=2\cdot(-2)+2n+2\Rightarrow g=n.$

Remark A.Eremenko [E] proved that the number of even solutions of (2.1) is finite.

By Theorem 2.5, this number is equal to the number of non-trivial critical points of G_n such that (2.9) holds.

§2-1 Multiple Green functions

For a sequence of blowing-up solutions of (2.2) with $\alpha_k \to n,$ the asymptotic formulas (2.5) also holds:

$$8\pi(\alpha_k - n) = (D(q) + o(1))e^{-\lambda_k}, \tag{2.12}$$

where $q = (q_1, \ldots, q_n)$ and $q_1, \ldots, q_n$ are blow-up point. Although $D(q)$ is a sum of integrations with non-trivial expressions, the identity similar to (2.6) still holds if q is a trivial critical point.

$$\text{Det } D^2 G_n(q) = (-1)^n c_q D(q) \tag{2.13}$$

where Det $D^2 G_n(q)$ is the Hessian of G_n (as a function of $2n$ real variables) and $c_q \geq 0$. Further, $c_q = 0$ iff $B_q := (2n-1)\sum_{i=1}^{n} \wp(q_i)$ is not a multiple zero of $\ell_n(B)$.

§2-1 Multiple Green functions

The computation of the Hessian of G_n is not easy, since it is the determinant of $2n\times 2n$ matrix. If n is small, the computation is possible.

For example $n=2$, it is not difficult to find that G_2 have five critical point $\left\{(\frac{1}{2}\omega_i,\frac{1}{2}\omega_j)\ |i\neq j\right\}$ and $\left\{(q_\pm,-q_\pm)\ |\wp(q_\pm)=\pm\sqrt{g_2/12}\right\}$, where $\wp'^2=4\wp^3-g_2\wp-g_3$.

Then we have $\det D^2G_2(q_\pm,-q_\pm)=\dfrac{3|g_2(\tau)|}{4\pi^2\operatorname{Im}\tau}\left|\eta_1(z)\pm\sqrt{g_2(\tau)/12}\right|\cdot\operatorname{Im}\phi_\pm(\tau)$,

$$\det D^2G_2(\frac{1}{2}\omega_i,\frac{1}{2}\omega_j)=\frac{4|G_k(\tau)|^2}{(2\pi)^4\operatorname{Im}\tau}\cdot\operatorname{Im}\phi_k(\tau),\ \ \{i,j,k\}=\{1,2,3\},$$

where $e_k=\wp(\frac{\omega_k}{2})$,

$$\phi_\pm(\tau)=\tau-\frac{2\pi i}{\eta_1(\tau)\pm\sqrt{g_2/12}};\quad G_k(\tau)=\frac{1}{2}g_2+3\eta_1(\tau)e_k-3e_k{}^2$$

$$\phi_k(\tau)=\tau-\frac{6\pi i e_k(\tau)}{\frac{g_2(\tau)}{2}+3\eta_1e_k-3e_k{}^2}.$$

§2-1 Multiple Green functions

When $\tau = ib$, we can further obtain

$$\det D^2 G_2(q_\pm, -q_\pm) > 0 \ ;$$

$$\det D^2 G_2(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) = \begin{cases} > 0 \ \text{ if } \{i,j\} = \{1,2\} \\ < 0 \ \text{ if } \{i,j\} = \{1,3\} \ \text{ or } \{2,3\} \end{cases}.$$

So, $G_2(z;\tau)$ has two trivial critical point with negative Hessian and three trivial critical point with positive Hessian. For $n \geq 3$, this result still holds, however we need another new ideas.

- Suppose there is a sequence of blowing-up even solutions u_k of (2.1) on E_{τ_k}, $\tau_k \to \tau$.

  We claim $G_n(z_1,\ldots,z_n;\tau)$ has a degenerate trivial critical point.

proof : Suppose $q_1,\ldots,q_n$ are the blow-up points. Since u_k is even, $q=(q_1,\ldots,q_n)$ is a trivial critical point of $G_n(q;\tau)$. Since $\alpha_k = n$, the identity (2.12),

$$0 = 8\pi(\alpha_k - n) = (D(q)+o(1))^{-\lambda_k},$$

yields $D(q)=0$. By (2.13), $\det D^2 G_n(q) = (-1)^n c_q D(q) = 0$. This prove the claim. □

Following [EGMP], we set $\mathrm{LW_n}$ = the quotient space of

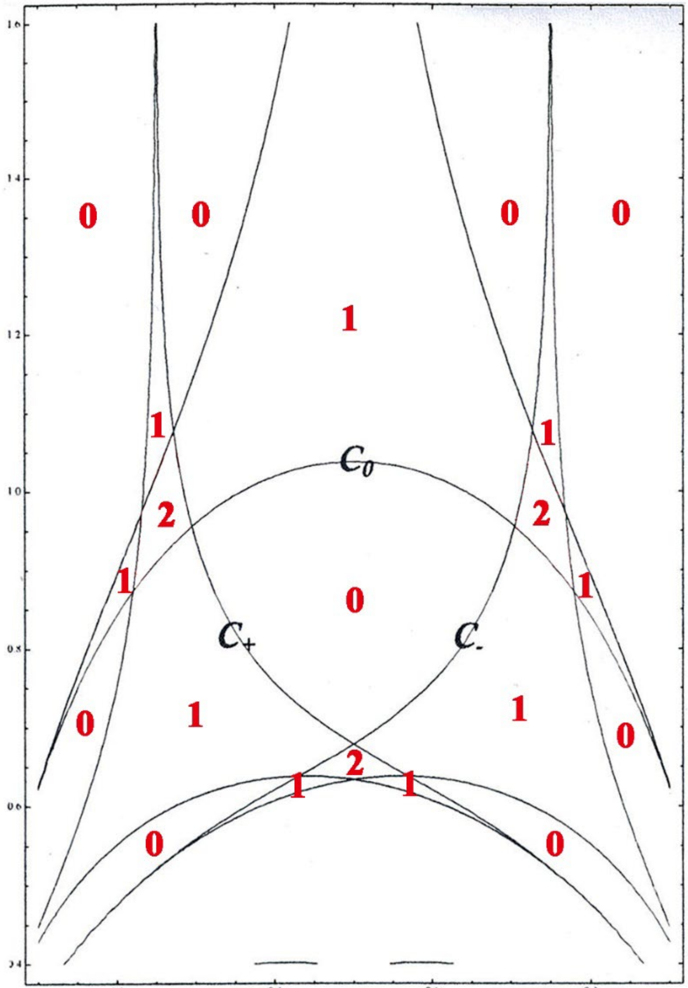
$\{\tau \mid G_n(z_1,\ldots;\tau)$ has a degenerate trivial critical point$\}$ by $\mathrm{SL}(2,\mathbb{Z})$.

Theorem 2.10 [EGMP]. $\mathrm{LW_n}$ *consists of* $\dfrac{n(n+1)}{2}$ *real analytic curves.*

This result is proved recently by Chen and I, by using analytic methods.

§2-1 Multiple Green functions

LW_n plays an important role to determine the number of even solutions, which is the main purposes for studying (2.1). For $n=1$, $\mathrm{LW}_1=\partial\Omega_3$. For $n=2$, LW_2 is the curve (see next page) quotiented by $\Gamma_0(2)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}) \;|c \text{ is even}\right\}$.


0
0
0
0
1
1
1
C_0
2
2
1
1
0
C_+
C_-
0
1
1
0
2
1
1
0
0

§2-1 Multiple Green functions

Let $F_0 = \left\{ \tau \in H \ |0 \le \operatorname{Re}\tau \le 1, \ \ |\tau - \frac{1}{2}| \ge \frac{1}{2} \right\}$ is the closure of a fundamental domain of $\Gamma_0(2)$. The figure shows $F\backslash LW_2$ is decomposed into a union of disjoint regions.

Theorem A. The number of even solutions of (2,1) with n=2 is shown as in the Figure of p.46

Theorem A is recently proved in a joint work with Z. J. Chen. The conjecture is to extend Theorem A to $n \ge 3$. In particular, the number of even solutions of (2,1) is bounded by n.

We remark those curves are also related to other problem. For example, the curves $C_0, \ C_\pm$ are related to the distribution of critical points of $E_2(\tau)$ (see [CL1]), where $E_2(\tau)$ is the classical Eisenstein series of weight 2 :

$$E_2(\tau) = \sum_{m \in \tau} \sum_{\substack{n \in \mathbb{Z} \\ n \ne 0 \text{ if } m=0}} \frac{1}{m\tau + n}.$$

§2-1 Multiple Green functions

In general, it is difficult to find the location of LW_n curves. However, [CL4] showed

Theorem 2.11 [CL4].

(a) For any n, $\mathrm{LW}_n \cap \{ \tau \mid \tau = ib, b > 0 \} = \phi$.

proof : The proof of (a) depends on another identity which relates the Hessan and the monodromy data C (see the subsection 2-4) of the Lame equation: If q is a trivial critical point,

$$\det D^2 G_n(q;\tau) = (-1)^n c_q P_2(q) \operatorname{Im} C, \tag{2.14}$$

where $P_2(q) = 0$ if and only if the monodromy data $C = \infty$. If this happen, then $\det D^2 G_n(q;\tau) = 0$.

Thus, (a) follows (2.14) and the following claim :

Remark : (a) is equivalent to : *All trivial critical points are non-degenerate.*

Thus $D(q;\tau) \neq 0$ by (2.13). Indeed a stronger result was obtained in [CL4].

(b) *There are* n (*or* $n+1$) *trivial critical points with negative* (*or positive*) $D(q)$.

Applying the local uniqueness and constructing blowing-up solution when both $\det D^2 G_n(q)$ and $D(q)$ are non-zero, (b) implies

Theorem 2.12 [LW4]. *Suppose* $\tau = ib$. *There is a small* $\varepsilon > 0$ *such that* (2.2) *has* n *even solutions with* $n - \varepsilon < \alpha < n$ *and* $n+1$ *even solutions for* $n < \alpha < n + \varepsilon$.

Conjecture: *There are exactly* $n+1$ (*even*) *solutions of* (2.2) *if* $\tau = ib$ *and* $n < \alpha < n+1$.

The conjecture is proved only when $n = 0$.

§2-2 Elliptic KdV potential and curvature equations

We present an application of our theory. Suppose $q=(q_1,\dots,q_n)$ is a critical point of G_n and consider

$$\Delta \mathrm{u}+e^{\mathrm{u}}=8\pi(n-1)\delta_0+8\pi\Big(\sum_{j=1}^{n}\delta_{q_j}\Big). \qquad (2.15)$$

The following result seems very surprising.

Theorem 2.13 [CL5]. *Suppose* (2.1) *has a solution. Then* (2.15) *also has a solution.*

The ODE associated with a solution u of (2.1) is the integral Lame equation

$$y''(z)=(n(n+1)\wp(z)+B)y.$$

The potential $Q(z)=n(n+1)\wp(z)$ is well-known as KdV potential. An elliptic function $Q(z)$ is called an *elliptic* KdV *potential* if there is a $(2m+1)$-th order ODE $P(\frac{d}{dz})$ such that

$$[P(\frac{d}{dz}),\ \frac{d^2}{dz^2}-Q(z)]=0$$

In [CL5], we find a connection of elliptic KdV potential and the curvature equation. The proof of Theorem 2.13 is based on this connection.

§2-3 Lame equation

Suppose u is a solution of (2.1), and $f(z)$ is the developing map. Then we have

$$u_{zz} - \frac{1}{2}u_z^2 = \{f, z\} = -2[n(n+1)\wp(z) + B],\ B \in \mathbb{C}.$$

The complex ODE

$$y'' = [n(n+1)\wp(z) + B]y \tag{2.14}$$

is called the Lame equatim, and the potential $Q(z) = n(n+1)\wp(z)$ is called a Lame equation.

- Let $y_i,\ i = 1, 2,$ are two linearly independent solutions of (2.14) and set $f(z) = y_2(z)/y_1(z)$. Then

$$\{f(z), z\} = -2[n(n+1)\wp(z) + B].$$

- **Monodromy matrices**: We note that any solution $y(z)$ is single-valued and moromorphic in $z \in \mathbb{C}$. Choose any system of fundamental solution $y_i, i = 1, 2,$

$$\begin{pmatrix} y_1(z+\omega_i) \\ y_2(z+\omega_i) \end{pmatrix} = S_i \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix},\ S_i \in SL(2, \mathbb{C}),\ i = 1, 2.$$

§2-3 Lame equation

Obviously S_1 and S_2 commutes. So, there are two cases :

(i) Both S_1 and S_2 can be diagonalized (The case is called completely reducible).

We might choose $y_\pm$ as a base such that both S_1 and S_2 are diagonal matrics with eigenvalues $\exp \pm 2\pi si$ and $\exp \mp 2\pi ri$, $i=\sqrt{-1}$, respectively. In this case, we can prove that $y_\pm(z)=e^{\pm cz}\prod_{i=1}^{n}\sigma(z\pm q_i)/\sigma(z)^n$, where $q_i,\ i=1,2,\ldots,n,$ satisfy the system

(a) $\sum_{j\neq i}(\zeta(q_i)-\zeta(q_j)+\zeta(q_j-q_i))=0,\ \forall i,$ that is,

$q=(q_1,\ldots,q_n)$ is a non-trivial critical point of G_n.

(b) $B=(2n-1)\sum_{i=1}^{n}\wp(q_i), and\ \mathrm{c}=\sum_{i=1}^{n}\zeta(q_i)$

(c) The monodromy data (r,s) satisfies $\sum_{j=1}^{n}q_j=r+s\tau,$ and $c=r\eta_1+s\eta_2$.

(d) The pair $(r,s) \in (\mathbb{C} \setminus \frac{1}{2}\mathbb{Z})^2$. Moreover, the pair $(r,s) \in \mathbb{R}^2 \setminus (1/2 \cdot \mathbb{Z})^2$ iff the Lame equation is derived from a solution of (2.1).

Remark : In literature, $y_{\pm}(z)$ are also called Baker-Akhiezer function after normalization.

(ii) Both S_1 and S_2 can not diagonalized simultaneously (called not completely reducible). In this case $q = (q_1, \ldots, q_n)$ is a trivial critical point, we can choose $y_+(z) = e^{cz} \prod \sigma(z - q_i) / \sigma(z)^n$ as $y_2(z)$, and the other solution $y_1(z)$ can be chosen so that

$$S_1 = \varepsilon_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S_2 = \varepsilon_2 \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}, \quad \varepsilon_i \in \{\pm 1\}.$$

If $C = \infty$, then S_1 and S_2 are understood as

$$S_1 = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } S_2 = \varepsilon_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

See [CLW] and [LW3].

§2-3 Lame equation

C in case (ii) is called *the monodromy data*. Case (ii) could happen iff the acessary parameter B of (b) is a zero of the *spectral polynomial*. The sequence $y_2^2(z)$ is elliptic, which is known as the *Lame equation*.

- *Spectral polynomial*. Let $\hat{y}_\pm(z) = (\prod_j \sigma(q_j))^{-1} \cdot y_\pm(z)$ and $w(z;B) = \hat{y}_+(z;B)\hat{y}_-(z;B)$. Then

$w(z;B)$ is an even elliptic function.

$$w(z;B) = \prod_{i=1}^{n} \frac{\sigma(z-q_i)\sigma(z+q_i)}{\sigma^2(z)\sigma^2(q_i)} = \prod_{i=1}^{n} (\wp(z) - \wp(q_i)) = \sum_{j=0}^{n} (-1)^j s_j(B) \wp(z)^{n-j}.$$

↑

the addition theorem

Classically, it is known that w satisfies the *second symmetric power* of the Lame:

(*) $$w''' - 4(n(n+1)\wp(z) + B)w' - 2n(n+1)\wp' w = 0.$$

From the third order ODE, we could derive that $s_j(B)$ is a polynomial of degree j. Thus, $w(z;B)$ is also a polynomial in B.

§2-3 Lame equation

Let $C = \dfrac{\hat{y}_-\hat{y}'_+ - \hat{y}_+\hat{y}_-}{2}$ be half of the Wronskian of $\hat{y}_\pm$. Then we claim

$$(**) \qquad C^2 = \left(\frac{w'}{2}\right)^2 - \frac{w''w}{2} + (n(n+1)\wp(z) + B)w^2 \doteqdot \ell_{\mathrm{n}}(B).$$

We note that the RHS of (**) is independent of z because of (*). Therefore $\ell_{\mathrm{n}}(B)$ is a monic polynomial of degree $2n+1$. This polynomial is called the spectral polynomial of the Lame.

The identity (**) can be proved. From $\dfrac{w'}{w} = \dfrac{\hat{y}'_+}{\hat{y}_+} + \dfrac{\hat{y}'_-}{\hat{y}_-}$ and

$C/w^2 = \dfrac{1}{2}(\dfrac{\hat{y}'_+}{\hat{y}_+} - \dfrac{\hat{y}'_-}{\hat{y}_-})$, we have $\dfrac{\hat{y}'_+}{\hat{y}_+} = \dfrac{w' + 2C}{2w}$. Thus

$$\left(\frac{w'+2C}{2w}\right)' = \left(\frac{\hat{y}'_+}{\hat{y}_+}\right)' = \frac{\hat{y}''_+}{\hat{y}_+} - \left(\frac{\hat{y}'_+}{\hat{y}_+}\right)^2 = n(n+1)\wp(z) + B - \left(\frac{w'+2C}{2w}\right)' \Rightarrow (**)$$

§2-3 Lame equation

This computation implies

$C = 0$ *iff* y_+ *and* y_- *are linearly dependent iff* $q = (q_1, \ldots, q_n)$ *is trivial.*

• *hyper-elliptic curve.* The polynomial $\ell_n(B)$ defines a hyper-elliptic curve $\{(B,C)\ |C^2 = \ell_n(B)\} \cup \{\infty, \infty\}$. Let Y_n be the union of (0,0,...,0) and the set of critical point of G_n. Then $Y_n \approx \{(B,C)\ |C^2 = \ell_n(B)\}$ by the mapping $q = \{q_1, \ldots, q_n\} \in Y_n \to (B,C)$,

where $B = (2n-1)\sum_j \wp(q_j)$ and $C = \dfrac{\hat{y}_- \hat{y}_+' - \hat{y}_+ \hat{y}_-'}{2}$.

$*$*Premodular form.* Following [LW3], we define

$Z_{r,s}(\tau) \doteqdot \zeta(r + s\tau; \tau) - r\eta_1(\tau) - s\,\eta_2(\tau)$ (Hecke form), and we start with the following result. $\sigma_n(q) = \sum_{j=1}^{n} q_j$: *a map from* Y_n *to* E_τ.

Theorem 2.13 [LW3]. *The map σ_n has degree* $\frac{1}{2}n(n+1)$.

By Theorem 2.13, σ_n induces an embedding of $K(E_\tau)$ into $K(Y_n)$ such that $[K(Y_n):K(E_\tau)]=\frac{1}{2}n(n+1)$. This means there is a polynomial $W_n(T)$ in $Q\,[g_2, g_3, \wp(\sigma), \wp'(\sigma)][T]$ of degree $\frac{1}{2}n(n+1)$ to define the field extension of $K(Y_n)$ over $K(E_\tau)$.

Define : $z_n(q) \doteq \zeta(\sum_{j=1}^{n} q_j) - \sum_{j=1}^{n} \zeta(q_j)$. It is not difficult to see $z_n(q)$ is a meromorphic function on Y_n, i.e., $z_n \in K(Y_n)$.

Theorem 2.14 [LW3] *The meromorphic function z_n is a primitive generator for the field extension $K(Y_n)$ over $K(E_\tau)$.*

This means that $W_n(z_n)=0$. Then we define the premodular form $Z_{r,s}^{(n)}(\tau)$:

$$Z_{r,s}^{(n)}(\tau) \doteq W_n(Z_{r.s}(\tau)), (r,s)\in\mathbb{C}^2\setminus(\frac{1}{2}\mathbb{Z})^2.$$

Remark: If (r,s) is a N-torsion, i.e., $(r,s)=\left(\frac{k_1}{N},\frac{k_2}{N}\right)$ with $(k_1,k_2,N)=1$, then $Z_{r,s}^{(n)}(\tau)$ is a modular form of weight $\frac{1}{2}n(n+1)$ w.r.t. $\Gamma(\mathbb{N})$, where

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \text{ mod N}\right\}.$$

The reason we introduced the Heck form is that if (2.9) holds,

$$0=\sum\zeta(r_j+s_j\tau)-\sum(r_j\eta_1+s_j\eta_2)=Z_{r,s}(\tau)-\mathrm{z}_n(q),\ \text{where } q_j=r_j+s_j\tau,\ (r_j,s_j)\in\mathbb{R}^2.$$

Due to the above identity, we have

§2-3 Lame equation

- *Curvature equation* (2.1) *has a solution on* E_τ *iff* *there is* $(r,s)\in\mathbb{R}^2\setminus(\frac{1}{2}\mathbb{Z})^2$ *such that* $Z_{r,s}^{(n)}(\tau)=0$.

The most important property $Z_{r,s}^{(n)}(\tau)$ has is

Theorem 2.14 [CKL]. *Suppose* $(r,s)\in\mathbb{R}^2\setminus(\frac{1}{2}\mathbb{Z})^2$. *Then any zero* τ *of* $Z_{r,s}^{(n)}(\tau)$ *is simple.*

This non-trivial result is proved by applying Painlev'eVI equation, an unexpected connection. Theorem 2.14 yields that a local function $(r,s)\to\tau$ can be defined. There are some of my recent works, jointly with Z.J.Chen, to discuss the map and related problems.

§2-4 Floquet theory

The classical Floquet theory can be applied to prove

Theorem 2.15 [CL3]. *If* $\tau = ib,\ b > 0,$ *then* (2.1) *has no solutions for any* $n \geq 1.$

The Floquet theory is applied for a periodic (say1) potential

$$y''(x) + q(x)y(x) = E\,y(x),\ x \in \mathbb{R}.$$

In this theory, there is the Hill descriminant $\Delta(E)$ by

$$\Delta(E) \doteqdot \mathrm{tr}M(E),$$

where $(y_1(x+1), y_2(x+1)) = (y_1(x), y_2(x))M(E),$ and the curve S

$$S \doteqdot \Delta^{-1}([-2,2]) = \{E \in \mathbb{C} \mid -2 \leq \Delta(E) \leq 2\}.$$

When the potential $q(z) = -n(n+1)\wp(z)$ along $z = x_0 + iy,\ x_0 \neq 0,$ is a smooth periodic function. It is easy to see $\Delta(E)$ is independent of $x_0,$ and then we have $S_1^{(n)}(\tau)$. Similarly, along $z = x_0 + \tau t$ we have the second Hill discriminant and $S_2^{(n)}(\tau)$.

§2-4 Floquet theory

Obviously if $E \in S_1^{(n)}(\tau) \cap S_2^{(n)}(\tau)$ with $[\Delta_1(E), \Delta_2(E)] \neq (\pm 2, \pm 2),$ then (2.1) has a solution on E_τ.

It is very interesting to study $S_i(\tau),\ i = 1, 2,$ when τ is deformed in $\mathbb{H}$.

Conjecture: Suppose $E \in S_1^{(n)}(\tau) \cap S_2^{(n)}(\tau)$ with $[\Delta_1(E), \Delta_2(E)] \neq (\pm 2, \pm 2)$. Then the intersection of $S_1^{(n)}(\tau)$ *and* $S_2^{(n)}(\tau)$ at E.

If $n = 1,$ conjecture is proved.