Conformal prescription of curvatures on manifolds with boundary

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The problem

Let (Σ, g) be an orientable compact surface with boundary. In this talk we consider the problem of prescribing the Gaussian curvature of Σ and the geodesic curvature of $\partial\Sigma$ via a conformal change of the metric.

This question leads to the boundary value problem:

$$\begin{cases} -\Delta_g u + 2K_g(x) = 2K(x)e^u, & \text{on } \Sigma; \\ \frac{\partial u}{\partial \nu} + 2h_g(x) = 2h(x)e^{u/2} & \text{on } \partial\Sigma. \end{cases}$$

Here e^u is the *conformal factor*, ν is outer unit normal;

K_g, h_g are the background Gaussian and geodesic curvatures;

K, h the Gaussian and geodesic curvatures to be prescribed.

Some history

For closed manifolds, the study was initiated by [Kazdan-Waner, '74-'75]. for the case with boundary some existence results were proved for:

- $h = 0$: [Chang-Yang, '88], ...

- $K = 0$: [Chang-Liu, '96], [Liu-Huang, '05], ...

blow-up for solutions with a-priori bounded volume has also been studied: [Da Lio-Martinazzi-Riviére, '15], [Battaglia-Medina-Pistoia, '21] ...

- constant curvatures K, h:

a parabolic flow yields metrics of constant curvatures ([Brendle, '02]);
classification of solutions in the annulus ([Jiménez, '12]);
classification in $\mathbb{R}^2_+$ ([Li-Zhu, '95], [Zhang, '03], [Gálvez-Mira, '09]);
study of *mean field equations* ([Battaglia-Lopez-Soriano, '20]).

- non-constant curvatures K, h:

in [Cherrier, '84], [Hamza, '90], but with some extra Lagrange multipliers.

Preliminaries and variational formulation

It is easy to prescribe $h = 0, K = sgn((\chi(\Sigma)))$ (by Gauss-Bonnet). Then we are reduced to study

$$(E_{K,h}) \qquad \begin{cases} -\Delta u + 2K_g(x) = 2K(x)e^u, & \text{on } \Sigma; \\ \frac{\partial u}{\partial \nu} = 2h(x)e^{u/2} & \text{on } \partial\Sigma. \end{cases}$$

with a background metric s.t. $K_g = sgn((\chi(\Sigma))) = 0, \pm 1$, and $h_g \equiv 0$.

The associated energy functional is given by $I : H^1(\Sigma) \to \mathbb{R}$,

$$I(u) = \int_\Sigma \left(\frac{1}{2}|\nabla u|^2 + 2K_g\, u + 2|K(x)|e^u \right) - 4 \oint_{\partial\Sigma} h(x)e^{u/2},$$

with $H^1(\Sigma) = \{u \,:\, \nabla u \in L^2(\Sigma)\}$.

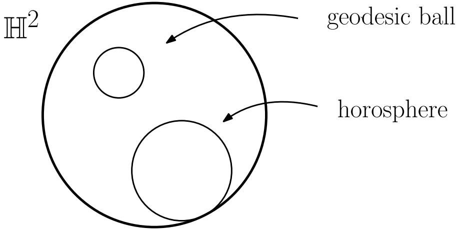
- In this talk we are interested in the case of negative K and $\chi(\Sigma) \leq 0$.

A scaling-invariant quantity

Define $\mathfrak{D} : \partial\Sigma \to \mathbb{R}$ as

$$\mathfrak{D}(x) = \frac{h(x)}{\sqrt{|K(x)|}}.$$

Notice that, in the hyperbolic space, $\mathfrak{D} > 1$ on geodesic spheres, and $\mathfrak{D} = 1$ on *horospheres*.



Such geometric objects are the candidate *blow-up* profiles for solutions of $(E_{K,h})$: one should expect none at boundary points with $\mathfrak{D} < 1$, and infinite-volume blow-up when $\mathfrak{D} = 1$.

The case $\chi(\Sigma) < 0$

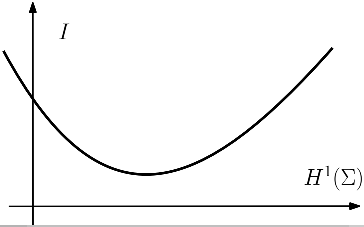
Theorem A

Assume that $\chi(\Sigma) < 0$. Let K, h be continuous functions such that $K < 0$ and $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$. Then the functional I is coercive and $(E_{K,h})$ admits a solution of minimum type.

One can show via a *trace inequality* that

$$I(u) \geq \int_\Sigma \left(\varepsilon|\nabla u|^2 + 2\varepsilon|K(x)|e^u + 2K_g\, u\right) - C.$$

Since $K_g < 0$, $\lim_{u\to\pm\infty} 2\varepsilon|K(x)|e^u + 2K_g\, u = +\infty$, so I is coercive.



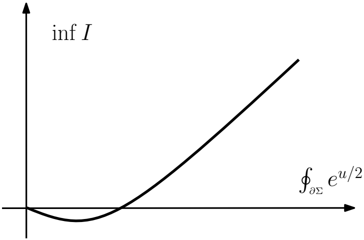
Minimizers in case $\chi(\Sigma) = 0$ (annular surface)

Theorem B

Assume that $\chi(\Sigma) = 0$. Let K, h be continuous functions such that $K < 0$, $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$ and $\oint_{\partial\Sigma} h(x) > 0$. Then the functional I is attains its infimum, and $(E_{K,h})$ admits a solution of minimum type.

Coercivity is lost, but if $u_n = -n$ then

$$I(u_n) = \int_\Sigma 2|K(x)|e^{-n} - 4\oint_{\partial\Sigma} h(x)e^{-n/2} \nearrow 0.$$

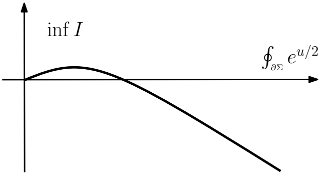


Mountain-pass structure for $\chi(\Sigma) = 0$

Theorem

Assume that $\chi(\Sigma) = 0$. Let K, h be continuous functions such that $K < 0$, $\mathfrak{D}(p) > 1$ for some $p \in \partial\Sigma$ and $\oint_{\partial\Sigma} h(x) < 0$. Then the functional I has a mountain-pass geometry.

If $\mathfrak{D}(p) > 1$, one finds functions $\varphi_{p,\lambda}$ such that $\oint_{\partial\Sigma} e^{\varphi_{p,\lambda}/2} \to +\infty$ and such that $I(\varphi_{p,\lambda}) \to -\infty$. The energy picture is as follows



• There are cases with $\chi(\Sigma) < 0$ where one can find also a mountain-pass structure, but the assumptions are not as clean as for $\chi(\Sigma) = 0$.

Some rigidity results

In view of the next results, our assumptions look rather natural.

Proposition [Jimenez, '12]

If Σ is an annulus, $K \equiv -1$, and h_1, h_2 are constant, then the problem is solvable iff

(i) $h_1 + h_2 > 0$ and both $h_i < 1$ (minimum).

(ii) $h_1 + h_2 < 0$ and some $h_i > 1$ (mountain-pass).

(iii) $h_1 = 1$, $h_2 = -1$ or viceversa.

Proposition [Rosenberg, '06], [Sun, '09]

Let Σ be a compact surface with boundary, and assume that $h(p) > \sqrt{|K^-(q)|}$ for all $p \in \partial\Sigma$, $q \in \Sigma$. Then Σ is homeomorphic to a disk.

Blow-up analysis

Min-max theory generates *Palais-Smale sequences* (approximate solutions). However these may not converge. A useful toos is *Struwe's monotonicity trick*, a sort of entropy method.

This produces a sequence of solutions to *perturbed equations*:

$$(E_n) \qquad \begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{on } \Sigma; \\ \frac{\partial u_n}{\partial \nu} = 2h_n(x)e^{u_n/2} & \text{on } \partial\Sigma. \end{cases}$$

Here $\tilde{K}_n \leq 0$, $\tilde{K}_n \to K_g$, $K_n \to K$, $h_n \to h$ in C^1, with $K < 0$. The convergence problem is still there, but we have now exact solutions.

By integration:

$$\int_\Sigma K_n(x)e^{u_n} + \oint_{\partial\Sigma} h_n(x)e^{u_n/2} \to 2\pi\chi(\Sigma).$$

If volumes stay bounded, the analysis is more or less standard. Otherwise there could be compensation of diverging masses!! (examples later)

A classification result in the half-plane

The following result is useful to understand the limiting behaviour of solutions to (E_n) in case of loss of compactness.

Theorem ([Zhang, '03], [Gálvez-Mira, '09])

Let $K_0 < 0$, and consider the problem ($\mathfrak{D}_0 = \frac{h_0}{\sqrt{|K_0|}}$)

$$\begin{cases} -\Delta u = 2\, K_0\, e^u, & \text{on } \mathbb{R}^2_+; \\ \frac{\partial u}{\partial \nu} = 2\, h_0\, e^{u/2} & \text{on } \partial \mathbb{R}^2_+ \end{cases} \quad \Longrightarrow \quad \begin{cases} -\Delta u = -2\, e^u, & \text{on } \mathbb{R}^2_+; \\ \frac{\partial u}{\partial \nu} = 2\, \mathfrak{D}_0\, e^{u/2} & \text{on } \partial \mathbb{R}^2_+. \end{cases}$$

- If $\mathfrak{D}_0 < 1$ there is no solution.

- If $\mathfrak{D}_0 = 1$ the only solutions are one-dimensional (of infinite volume)

- If $\mathfrak{D}_0 > 1$ then solutions can either have infinite volume or finite volume. In the latter case, they arise from Möbius maps (*bubbles*).

Limits of blowing-up sequences

Let us define the *singular set*

$$S = \{p \in \Sigma \ : \ \exists y_n \in \Sigma, y_n \to p, u_n(y_n) \to +\infty\} .$$

Proposition

Let $p \in S$. Then there exists $x_n \to p$ such that, after a suitable rescaling, we obtain a solution of the problem in the half-plane in the limit. In particular $S \subseteq \{p \in \Sigma \ : \ \mathfrak{D} \geq 1\}$.

If $\int_\Sigma e^{u_n} \to +\infty$, there exists a positive unit measure σ on $\partial\Sigma$ s.t.:

$$\frac{|K_n|e^{u_n}}{\int_\Sigma |K_n|e^{u_n}} \rightharpoonup \sigma.$$

The above analysis implies that $supp(\sigma) \subseteq S \subseteq \{p \in \partial\Sigma \ : \ \mathfrak{D}(p) \geq 1\}$.

On the support of σ

We have indeed more precise information on the concentration set:

Proposition

If $\int_\Sigma e^{u_n} \to +\infty$, the support of σ is contained in $\{p \in \partial\Sigma \ : \ \mathfrak{D}_\tau(p) = 0\}$, where $\mathfrak{D}_\tau$ is the tangential derivative of $\mathfrak{D}$ on $\partial\Sigma$.

The proof uses *holomorphic domain variations* and integration by parts.

However the argument is tricky since we do not have information on the asymptotic behavior of the term $\int_\Sigma |\nabla u_n|^2$. We need to use cancellation properties from the C-R equations for holomorphic vector fields.

This is all the information we have without further hypotheses on u_n. Further information can be derived from Morse index bounds.

Information from the Morse index

From now on we assume that the sequence of solutions u_n has bounded Morse index. This is verified, for instance, for solutions constructed from (standard) min-max variational methods.

Estimates via under Morse index bounds have been derived in several contexts, but not much for this class of PDEs.

One advantage of this information is that if u_n has bounded Morse index, then the bound is retained for blow-up limits (by scaling arguments), and some of them can be excluded in this way.

Morse index of the limit problem

Theorem

Let u be a solution of the problem:

$$\begin{cases} -\Delta u = -2e^u, & \text{on } \mathbb{R}^2_+; \\ \frac{\partial u}{\partial \nu} = 2\,\mathfrak{D}_0\, e^{u/2} & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

Define

$$Q(\psi) = \int_{\mathbb{R}^2_+} |\nabla \psi|^2 + 2\int_{\mathbb{R}^2_+} e^u \psi^2 - \mathfrak{D}_0 \oint_{\partial\mathbb{R}^2_+} e^{u/2}\psi^2;$$

$$ind(u) = \sup\left\{dim(E) \;:\; E \subseteq C^\infty_0(\mathbb{R}^2_+) \text{ subspace}, Q(\psi) < 0 \quad \forall \psi \in E\right\}.$$

Then one has

If $\mathfrak{D}_0 = 1$, then $ind(u) = 0$, namely u is stable.

If $\mathfrak{D}_0 > 1$, and u is a bubble, then $ind(u) = 1$. Otherwise, $ind(u) = +\infty$.

An idea of the proof

We only comment on $\mathfrak{D}_0 > 1$. It was proven in [Galvez-Mira, '09] that

$$u = 2\log\left(\frac{2|f'(z)|}{1-|f(z)|^2}\right),$$

with $f : \mathbb{R}^2_+ \to D_{\mathfrak{D}_0}$ holomorphic, locally univalent. The conformal volume is finite if and only if f is a Möbius map.

It is easy to show that the index is one for Möbius maps, working on the hyperbolic disk $D_{\mathfrak{D}_0}$ in polar coordinates.

If f is not globally univalent, the contribution to the index is one for any pre-image, and these are infinitely-many. It is then possible to make the variations compact by using cut-offs ϕ such that

$$\phi \equiv 1 \text{ on } B_R^+, \quad R \text{ large };\qquad \int_{\mathbb{R}^2} |\nabla\phi|^2 dx \le C.$$

Theorem

Let u_n be a blowing-up sequence of solutions to the problem:

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{on } \Sigma; \\ \frac{\partial u_n}{\partial \nu} = 2h_m(x)e^{u_n/2} & \text{on } \partial\Sigma. \end{cases}$$

Here $\tilde{K}_n \to \tilde{K} \leq 0$, $K_n \to K < 0$ and $h_n \to h$ in C^1. Then

- $S \subseteq \{p \in \partial\Sigma \, : \, \mathfrak{D}(p) \geq 1, \mathfrak{D}_\tau(p) = 0\}$.
- Moreover, if $ind(u_n)$ is bounded then

$$S \subseteq \{p \in \partial\Sigma \, : \, \mathfrak{D}(p) = 1, \mathfrak{D}_\tau(p) = 0\}.$$

Proof of the last property (sketch)

We need to show that if $ind(u_n) \leq C$ and $p \in S$, then $\mathfrak{D}(p) = 1$.

Admissible blow-up profiles with bounded index are either 1-D solutions ($\mathfrak{D}(p) = 1$) or *bubbles* ($\mathfrak{D}(p) > 1$). It remains to exclude the latter ones.

Bubbling solutions have the following asymptotic profile near points $p \in S$

$$u_n(x) \simeq 4 \log \frac{\mu_n}{1 + \mu_n^2 d(x,p)}; \qquad \mu_n \to +\infty.$$

Some work is needed to show that this holds until $d(x,p) \simeq O(1)$.

However if $\chi(\Sigma) \leq 0$ one has the bound $\int_\Sigma |\nabla u_n^-|^2 \leq C$, which is violated by the above asymptotic profile. So no bubbles can form.

Remark. Bubbles can instead form on the disk. Blow-ups with finite volume were characterized in [Jevnikar-Lopez-Soriano-Medina-Ruiz, '20], and constructed in [Battaglia-Medina-Pistoia, '21].

Existence of min-max solutions for $\chi(\Sigma) = 0$

Theorem C

Assume that $\chi(\Sigma) = 0$. Let K, h (smooth) be such that $K < 0$ and:

(i) $\mathfrak{D}(p) > 1$ for some $p \in \partial\Sigma$;

(ii) $\fint_{\partial\Sigma} h < 0$;

(iii) $\mathfrak{D}_\tau(p) \neq 0$ for any $p \in \partial\Sigma$ with $\mathfrak{D}(p) = 1$.

Then I has a mountain-pass critical point, which is a solution of $(E_{K,h})$.

We also need to exclude *blow-down*, i.e. $u_n \to -\infty$ everywhere on Σ: this follows from energetic reasons and estimates on the min-max value.

Explicit examples of blow-up

Let $A(0;r,1)$ be the annulus with radii $r, 1$, $0 < r < 1$. Consider

$$\begin{cases} -\Delta u = -2e^u, & \text{in } A(0;r,1), \\ \frac{\partial u}{\partial \nu} + 2 = 2h_1 e^{u/2} & \text{on } |x| = 1, \\ \frac{\partial u}{\partial \nu} - 2/r = 2h_2 e^{u/2} & \text{on } |x| = r. \end{cases}$$

All solutions were classified in [Jiménez, 2012].

For example, the function:

$$u(x) = \log\left(\frac{4}{|x|^2(\lambda + 2\log|x|)^2}\right), \qquad \text{for any } \lambda < 0,$$

is a solution with $h_1 = 1$ and $h_2 = -1$. Observe that if λ tends to 0 then u blows up at a whole component of the boundary.

The singular set $S = \{x = 1\}$ is not finite, and indeed it is a whole curve.

A second example

Given any $h_1 > 1$, $\gamma \in \mathbb{N}$, there exists an explicit solution of the form:

$$u_\gamma(z) = 2\log\left(\frac{\gamma|z|^{\gamma-1}}{h_1 + Re(z^\gamma)}\right),$$

where $h_2 = -h_1 r^{-\gamma}$.

Those solutions blow up as $\gamma \to +\infty$, keeping $h + 1 > 1$ fixed. Also here $S = \{|z| = 1\}$, but now $h_1 > 1$.

The asymptotic profile is:

$$u(s,t) = 2\log\left(\frac{e^{-t}}{h_1 + e^{-t}\cos s}\right),$$

defined in the half-plane $\{t \geq 0\}$. This is indeed a solution to the limit problem in $\mathbb{R}^2_+$ with $K = -1$ and $h_1 > 1$, with infinite Morse index.

The higher-dimensional case

Consider a manifold with boundary (M^n, g), $n \geq 3$. The analogue of the above problem consists in prescribing conformally the scalar curvature and the mean curvature at the boundary. If the background ones are S_g and m_g, one needs to find positive solutions to

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g u + S_g\, u = K(x) u^{\frac{n+2}{n-2}}, & \text{on } M; \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + m_g u = H(x) u^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases}$$

In the following incomplete list of papers, there are contributions, especially concerning constant curvatures (often one is zero): [Cherrier, '84], [Escobar, '92, '96], [Han-Li, '00], [Djadli-M.-O.-Ahmedou, '03], [Marques, '05], [Almaraz, '10], [Mayer-Ndiaye, '17], [Ahmedou-Ben Ayed, '20], ...

Some extensions of the previous existence results

In the recent paper [Cruz-Blàzquez-M.-Ruiz, '21] we proved analogues of the previous existence results. In particular, we recover Theorems A and B, about minimal solutions, in all dimensions. Concerning the analogue of Theorem C, we are able to prove it only in three dimensions.

The reason for this difference relies in the decay properties of bubble functions, tending to zero as the fundamental solution, i.e. like $\frac{1}{|x|^{n-2}}$. For $n = 3$ one has the strongest interaction, which allows to prove that blow-ups are *simple* (only one bubble at a time).

The situation in the limit case (half-space) is more rigid, but one lacks the useful complex-analytic tools of the 2D case. It would be useful to analyze the stability properties of *singular solutions* in $\mathbb{R}^n_+$. Without boundary, such solutions were studied in [Caffarelli-Gidas-Spruck, '89], [Chen-Lin, '95], [Korevaar-Mazzeo-Pacard-Schoen, '99], ...

Some open problems

- Understand the case of the disk, where the Möbius group acts. Can one have coexistence of finite-mass and infinite-mass blow-ups?

- (hard) Construct explicitly blowing-up solutions for non-constant curvature cases. They should exist, since one can play with deforming parameters and the obstruction criteria ([Borer-Galimberti-Struwe, '15]).

- Can one improve the blow-up analysis without Morse index bounds?

- Study the problem on conical surfaces: some particular cases are considered in [Jost-Wang-Zhou, '09], [Battaglia-Jevnikar-Wang-Yang, '20].

Thanks for your attention!