Irreducible Metrics and Stable Extensions

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Abstract

1. Background of cone spherical metrics
2. Developing maps
3. Viewpoint from Lie Theory
4. Viewpoint from vector bundles
5. Reducible metrics(if time permitted)

§1 Background of cone spherical metrics

Examples, definitions, open problems and some known results of cone spherical metrics.

Cone metric

A *cone (conformal) metric* ds^2 on a compact Riemann surface X consists of the following data:

- A finite subset $\{P_1, \cdots, P_n\}$ of X and $1 \neq \theta_1, \cdots, \theta_n > 0$
- A conformal metric ds^2 on the punctured surface $X \backslash \{P_1, \cdots, P_n\}$
- $ds^2 = e^{2u} |dz|^2$ near P_j such that

$$u - (\theta_j - 1) \log |z|$$

is continuous at $z(P_j) = 0$.

We say that the cone metric ds^2 *has cone angle* $2\pi\theta_j$ *at the cone singularity* P_j. We call that ds^2 *represents the real divisor* $D := \sum_{j=1}^{n} (\theta_j - 1) P_j$ *on* X.

The Picard-Poincaré Problem

Let X be a compact Riemann surface of genus g_X.

Describe all the real divisors $D = \sum_j (\theta_j - 1)P_j$ with $1 \neq \theta_j > 0$ on X such that there exists a cone metric ds^2 representing D which has constant Gauss curvature $K \in \{-1, 0, +1\}$, called *cone hyperbolic, flat and spherical metric*, respectively.

The Gauss-Bonnet formula gives a *necessary* condition that K has the same sign as the singular Euler number $\chi(X,\, D) := \big(2 - 2\, g_X + \deg\, D\big)$ for the existence of such metrics.

Cone hyperbolic and flat metrics

(Picard, Poincaré, Heins, McOwen, Troyanov) There exists a cone hyperbolic (flat) metric representing D on X iff

$$\chi(X, D) < (=) 0.$$

And the metric representing D is *unique* (up to scaling).

Existence of cone spherical metrics is open

The natural necessary condition of

$$\chi(X, D) = \big(2 - 2\, g_X\big) + \deg D > 0$$

given by Gauss-Bonnet is *not* sufficient for the existence. The cone spherical metrics rep. D are multiple in general if they exist.

Liouville equations:I

Let Δ_0, K_0 and $\mathrm{d}A_0$ be the Laplacian, the Gaussian curvature and the area element of a smooth background conformal metric g_0 on X with unit area.

Denote $G(\cdot,\, Q)$ by the normalized Green function wrt the point Q in X, i.e. it satisfies

$$-\Delta_0\, G(\cdot,\, Q) = \delta_Q - 1, \quad \int_X G(P,\, Q)\mathrm{d}A_0(P) = 0.$$

Recall that $D = \sum(\theta_j - 1)P_j$. Define

$$h_D(P) := 4\pi \sum (\theta_j - 1)\, G(P,\, P_j).$$

Liouville equations:II

Observation(Bartolucci-De Marchis-Malchiodi) Existence of a cone spherical metric rep. D is equivalent to that of a classical solution u to the Liouville equation

$$\Delta_0 u - 2K_0 - 4\pi \sum (\theta_j - 1) + 4\pi\chi(X,\, D)\frac{2e^{u-h_D}}{\int_X 2e^{u-h_D}\,\mathrm{d}A_0} = 0. \quad (1)$$

We have the C^0 a priori estimate for (1) if D satisfies the non-bubbling condition, i.e.

$$\chi(X,\, D) \notin \left\{\mu > 0 \mid \mu = 2k + 2\sum n_j\theta_j, k \in \mathbb{Z}_{\geq 0}, n_j \in \{0, 1\}\right\}.$$

We shall give an explicit class of divisors D for which (1) has no a priori estimate.

Relevant results of cone spherical metrics I:

- A sufficient condition by Troyanov for the sub-critical case:

$$0 < \chi(X, D) < \min\left\{2,\ 2\min_{1\le j\le n} \theta_j\right\}.$$

- A sufficient condition by Bartolucci-De Marchis-Malchiodi for the super-critical case:
  1. $g_X > 0$ and $\theta_j > 1$ for all $1 \le j \le n$;
  2. $\chi(X, D)$ is greater than 2 and satisfies the non-bubbling condition.

Note that (1) in the above two items have C^0 a priori estimate.

Chen-Lin computed the Leray-Schauder degree of their solutions.

Relevant results of cone spherical metrics:II

- Mondello-Panov, Chen-Li-Song-X, Eremenko: Angle constraint for cone spherical metrics on compact Riemann surfaces, consisting of the two cases of $g_X = 0$ and $g_X > 0$.
- Lin et al proved many deep and interesting results for $n = 1$ and $g_X = 1$. In particular on rectangular tori.
- Mazzeo-Zhu and Karpukhin-Zhu: Deformation properties and bounded 2-eigenfunctions

The authors above used also other techniques besides Analysis of PDEs so that the non-bubbling condition was not an obstruction for them.

Some open questions

- No general conjecture for the existence problem.
- Finitely or infinitely many if the metrics exist?
- Little known if $g_X \geq 2$ and D violates the non-bubbling condition.
- $\cdots\cdots$

Li-Song-X obtained some progresses on the latter two questions if $g_X \geq 2$ and D is an effective divisor $\sum(\theta_j - 1)P_j$, $\theta_j \in \mathbb{Z}_{>1}$, by using stable bundles.

Main results (Li-Song-X)

Let $g_X \geq 2$.

- If $\text{odd} = \deg\, D > 2g_X - 2$, for almost every $D \in \mathrm{Sym}^d(X)$, there exists finitely many cone spherical metrics representing D. Note that D satisfies the non-bubbling condition and is realizable by Troyanov and Bartolucci-De Marchis-Malchiodi.
- Let $\text{even} = \deg\, D > 2g_X - 2$. Then D violates the non-bubbling condition. There exists an effective divisor D' such that D' is linearly equivalent to D, i.e. $D - D' = (f)$ for some meromorphic function f on X, and D' is realizable. Note that effective divisors linearly equivalent to D form a projective space $|D|$ of dimension $\deg\, D - g_X \geq g_X$, called the complete linear system of D.

§2 Developing maps of spherical metrics

Following Liouville, we introduce the developing maps of a cone spherical metrics. **These maps are locally univalent, multi-valued meromorphic functions on $X \backslash \operatorname{supp} D$.**

Orientation-preserving isometries of $\mathbb{S}^2(1)$

Consider the unit 2-sphere $\mathbb{S}^2(1)$ centered at 0 in $\mathbb{R}^3$. Orientation-preserving isometries of $\mathbb{S}^2(1)$ are exactly rotations around lines through $0 \in \mathbb{R}^3$. All of them form the Lie group $\mathrm{SO}(3)$.

As a Riemannian manifold, $\mathbb{S}^2(1)$ is the same as the Riemann sphere $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$ endowed with the standard conformal metric $g_{\mathrm{st}} = \dfrac{4|dw|^2}{(1+|w|^2)^2}$.

The orientation-preserving isometry group of g_{st} coincides with

$$\mathrm{PSU}(2) := \left\{ z \mapsto \frac{az+b}{-\overline{b}z+\overline{a}} : |a|^2+|b|^2 = 1 \right\} \subset \mathrm{PGL}(2, \mathbb{C}).$$

Hence $\mathrm{SO}(3) \cong \mathrm{PSU}(2)$ and we do not distinguish them later on.

Projective functions on Riemann surfaces

Let S be a Riemann surface. Call a multi-valued meromorphic function $f : S \to \mathbb{P}^1$ *projective* if the monodromy representation of f forms a group homomorphism

$$\mathcal{M}_f : \pi_1(S, B) \to \mathrm{PGL}(2, \mathbb{C}).$$

Then the Schwarzian derivative of f

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

becomes a single valued meromorphic function on each complex coordinate chart of S.

Developing map (Liouville, Chen-Wu-X)

Let ds^2 be a cone spherical metric rep. $D = \sum_{j=1}^{n} (\theta_j - 1)P_j$ on X. Then there exists a projective function $f : X\backslash \mathrm{supp}\, D \to \mathbb{P}^1$ such that

- (*Pull-back*) $ds^2 = f^* g_{\mathrm{st}}$.
- (*Monodromy*) The monodromy of f becomes a homo. $\mathcal{M}_f : \pi_1(X\backslash \mathrm{supp}\, D,\, B) \to \mathrm{PSU}(2) \subset \mathrm{PGL}(2,\, \mathbb{C})$. f extends to a projective function on X if D is effective.
- (*Singularities*) In each complex coordinate $(U,\, z)$ centered at P_j, the Schwarzian derivative $\{f,\, z\}$ of f has the principal singular part of $\frac{1-\theta_j^2}{2z^2}$.

Here f is called a *developing map* of ds^2 which is unique up to a pre-composition with a Möbius transformation in $\mathrm{PSU}(2)$.

And vice versa.

Example: pull-back metric by branched cover

- Let $f : X \to \mathbb{P}^1$ be a branched cover. Then the pull-back cone sphericl metric f^*g_{st} represents the ramification divisor R_f such that

$$\deg\, R_f = 2\deg\, f - 2 + 2g_X = \text{even}.$$

- (Lin-Wei-Ye, Chen-Wu-X, Eremenko-Gabrielov) A cone spherical metric ds^2 with integral cone angles on $\mathbb{P}^1$ must be such a pull-back metric by some branched cover $f : \mathbb{P}^1 \to \mathbb{P}^1$. Moreover, an effective divisor $D = \sum_{j=1}^{n} m_j P_j$ on $\mathbb{P}^1$ is realizable iff

$$\sum_j m_j = \text{even} = 2d - 2, \quad m_j < d.$$

Equation (1) for ramification divisors D of branched covers have no a priori estimate since the space of solutions to (1) is non-compact.

Associated Lie group of cone spherical metric

Let ds^2 be a cone spherical metric rep. D on X and $f : X \setminus \mathrm{supp}\, D \to \mathbb{P}^1$ its developing map.

The image of the monodromy homomorphism of f

$$\mathcal{M}_f : \pi_1(X \setminus \mathrm{supp}\, D) \to \mathrm{PSU}(2)$$

is *not* a Lie subgroup of $\mathrm{PSU}(2)$ in general. We call its closure the *associated Lie group* of ds^2, which is unique up to conjugacy in $\mathrm{PSU}(2)$.

Definition of irreducible metric

(Umehara-Yamada) A cone spherical metric ds^2 is called *reducible=coaxial* if its associated Lie group is contained in

$$\mathrm{U}(1) = \left\{ z \mapsto e^{\sqrt{-1}t} z : t \in [0, 2\pi) \right\}.$$

Otherwise, it is called *irreducible=non-coaxial*.

Note that the pull-back metrics by branched covers are all reducible.

Stable vector bundles: definitions

§4 Viewpoint from vector bundles

Definition (Mumford)

- The slope $\mu(E)$ of a holomorphic vector bundle $E \to X$ is defined to be $\mu(E) = \frac{\deg \det E}{\mathrm{rk}\, E}$.
- We call E *stable* iff $\mu(F) < \mu(E)$ for each proper sub-bundle F of E.
- We call an extension $0 \to L \to E \to M \to 0$ of a line bundle M by another line bundle L stable iff E is a rank 2 stable vector bundle. Then

$$\deg L < \deg M.$$

Stable vector bundles: properties

Theorem (Narasimhan-Seshadri)

- A rank 2 vector bundle $E \to X$ is stable iff the projective bundle $\mathbb{P}^1 \to \mathbb{P}(E) \to X$ comes from an irreducible representation $\rho : \pi_1(X) \to \mathrm{PSU}(2)$, i.e. $\mathrm{Im}\,\rho$ is *not* contained in $\mathrm{U}(1) \subset \mathrm{PSU}(2)$.
- A rank 2 stable bundle E with $\det E = \mathcal{O}_X$ comes from an irreducible unitary representation $\pi_1(X) \to \mathrm{SU}(2)$.

Lange-type theorem

Theorem (Li-Song-X) Let L, M be two line bundles on X with $g_X \geq 2$ such that deg $M >$ deg L. Then the stable extensions of M by L form a Zariski open subset in $\mathrm{Ext}^1_X(M, L) \cong H^1(X, L \otimes M^{-1})$.
Denote this Zariski open subset by

$$\{0 \to L \to E \to M \to 0\}^{\mathrm{stable}}.$$

Since there is a $\mathbb{C}^*$-action on it, we have the projective version

$$\mathbb{P}\Big(\{0 \to L \to E \to M \to 0\}^{\mathrm{stable}}\Big).$$

Remark We proved this theorem by using the algebraic method in the 1983 Math Ann paper by Lange-Narasimhan.

Integral irreducible metrics and stable extensions

- Let $\mathrm{d}s^2$ be an irr. metric rep. D on X. Then its dev. map $f: X \to \mathbb{P}^1$ is a projective function such that $R_f = D$ and the monodromy representation $\mathcal{M}_f : \pi_1(X) \to \mathrm{PSU}(2)$ is irreducible.
- f defines a flat bundle $\mathbb{P}^1$-bundle $P \to X$ corresponding to M_f such that f is a section of P.
- There exists a rank 2 stable bundle E such that $\mathbb{P}(E) = P$ since the monodromy of f lies in $\mathrm{PSU}(2)$.
- There exists a stable extension $0 \to L \to E \to M = E/L \to 0$ such that the embedding $L \to E$ corresponds to the pre-image of the section $f : X \to P = \mathbb{P}(E)$ under $E \to \mathbb{P}(E)$.

And vice versa.

Surjective map between two moduli spaces

X=compact Riemann surface of genus $g_X \geq 2$

$\mathcal{SE}(X)$=moduli space of stable extensions $0 \to L \to E \to M \to 0$ of two line bundles over X moduli the process of tensoring line bundles

$\mathcal{IM}(X, \mathbb{Z})$=moduli space of integral irreducible metrics on X

Theorem (Li-Song-X) There exists a canonical surjective map $\sigma : \mathcal{SE}(X) \to \mathcal{IM}(X, \mathbb{Z})$.

- The metric given by stable extension $0 \to L \to E \to M \to 0$ represents an effective divisor lying in $|L^{-1} \otimes M \otimes K_X|$.
- Each metric has at most 2^{2g_X} preimages under σ.

Specifying the divisor by a stable extension

- A stable extension $\mathbb{E}:\ 0 \to L \to E \xrightarrow{p} M \to 0$ gives a metric rep. $D \in |L^{-1} \otimes M \otimes K_X|$.
- Write the Chern connection D_E of the Hermitian-Einstein metric h on E as $D_E = \partial_E + \bar{\partial}_E$, where $\bar{\partial}_E$ is the complex structure of E and $\partial_E : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$ the $(1,0)$-part of D_E.
- (Li-Song-X) We could obtain an $\mathcal{O}_X$-linear map $L \xrightarrow{\theta_{\mathbb{E}}} M \otimes K_X$: $L \hookrightarrow E \xrightarrow{\partial_E} E \otimes K_X \xrightarrow{p} M \otimes K_X$. Then $\theta_{\mathbb{E}} \in H^0\Big(\mathrm{Hom}\big(L, M \otimes K_X\big)\Big)$, $(\theta_{\mathbb{E}}) = D \in |K_X \otimes L^{-1} \otimes M|$, and the ramification divisor map (RDM) $\mathfrak{R} : \mathbb{E} \mapsto \mathrm{Div}(\theta_{\mathbb{E}}) = D$,

$$\mathbb{P}\big(\mathrm{Ext}^1(M, L)\big) \supset \mathbb{P}\Big(\{0 \to L \to E \to M \to 0\}^{\mathrm{stable}}\Big) \xrightarrow{\mathfrak{R}} |K_X \otimes L^{-1} \otimes M|,$$

which is real analytic by the Kobayashi-Hitchin correspondence.

Express RDM in terms of HE metric

- The HE metric h on E induces Hermitian metrics on L and M such that $M \cong L^{\perp}$ as complex vector bundles.
- Write the Chern connection D_E of the HE metric h on E as

$$D_E = \begin{pmatrix} D_L & -\beta \\ \beta^{*_h} & D_M \end{pmatrix},$$

where β is a harmonic $(0,1)$-form with value in $\mathrm{Hom}(M, L)$ and β^{*_h} is a holomorphic one-form with value in $\mathrm{Hom}(L, M)$.

- RDM $\mathfrak{R}$ before taking quotient $/\mathbb{C}^*$ expressed by

$$\beta \mapsto \beta^{*_h}, \quad \left(\mathcal{H}^{(0,1)}_{\bar\partial}(X,\, L \otimes M^{-1})\right)^{\text{stable}} \to H^0(X,\, L^{-1} \otimes M \otimes K_X).$$

RDM is well defined in the sense β^{*_h} does not depend on the representative of the extension class $0 \to L \to E \to M \to 0$.

Even/odd spherical metrics

Definition Call an integral spherical metric *even (odd)* iff the degree of its effective divisor is even (odd).

- (Troyanov, Bartolucci et al) Let $g_X \geq 1$. Each odd effective divisor D with degree $> 2g_X - 2$ could be represented by a cone spherical metric on X, which must be irreducible.
- Each integral reducible metric is even.
- (Li-Song-X) Let $g_X \geq 2$. Each even irreducible metric comes from an stable extension with form $0 \to L \to E \to L^{-1} \to 0$, where E is flat and comes from an irreducible representation $\pi_1(X) \to \mathrm{SU}(2)$.

Existence of even irreducible metrics

Theorem (Li-Song-X) *Let D be an effective divisor of degree* $\mathrm{even} > 2g_X - 2$ *on X with $g_X \geq 2$. There exists $D' \in |D|$ such that D' is realizable. As a consequence, we have*

$$\dim_{\mathbb{R}} \left\{ D \in \mathrm{Sym}^d(X) : D \text{ is realizable} \right\} \geq 2g_X.$$

Proof. Choose line bundle L s.t. $L^{-2} \otimes K_X = \mathcal{O}_X(D)$. Then each D' lying in the image of the ramification divisor map

$$\mathbb{P}\Big(\{0 \to L \to E \to L^{-1} \to 0\}^{\text{stable}}\Big) \xrightarrow{\mathfrak{R}} |K_X \otimes L^{-2}| = |D|$$

could be represented by an irreducible metric.

Remark I speculate that there exists in $\mathrm{Sym}^d(X)$ an Euclidean open subset whose effective divisors are realizable.

Finiteness of odd spherical metrics

Example (Li-Song-X) *Let $g_X \geq 2$. For almost every $D \in \mathrm{Sym}^{2g_X-1}(X)$, there exists finitely many spherical metrics representing D.*

- Since $\deg\ D = 2g_X - 1 = \mathrm{odd}$, each metric g rep. D is irreducible.
- There exists stable extension $0 \to \mathcal{O}_X \to E \to M \to 0$ defining g such that $\deg\ M = 1$ and $|D| = |K_X \otimes M|$.
- Since each non-trivial extension of M by $\mathcal{O}_X$ is stable, the ramification divisor map $\mathfrak{R} : \mathbb{P}\big(H^1(X, M^{-1})\big) \cong \mathbb{P}^{g_X-1} \to |D| \cong \mathbb{P}^{g_X-1}$ is surjective and smooth. It holds true for a.e. divisors in $|D|$ by the Sard theorem. RDM in this case is both beautiful and mysterious to us!
- It follows from Fubini and the fibration arising from Riemann-Roch $\mathbb{P}^{g_X-1} \to \mathrm{Sym}^{2g_X-1}(X) \to \mathrm{Pic}^{2g_X-1}(X)$.

I speculate that the metric representing such an effective divisor is unique.

Unitary one-forms

Definition Call a meromorphic one-form ω on a Riemann surface S *unitary one-form* iff ω has at most simple poles and all its periods lie in $\sqrt{-1}\mathbb{R}$. In particular, all the residues of ω are real. We call

$$D_\omega := (\omega)_0 + \sum_{P\in\{\text{poles of }\omega\}} \left(|\mathrm{Res}_P(\omega)| - 1\right) P$$

the spherical divisor associated with ω.

Theorem (H. Weyl) Given $(m+2) \geq 2$ distinct points $P_1, \cdots, P_{m+2}$ on a compact Riemann surface X and $(m+2)$ nonzero real numbers $a_1, \cdots, a_{m+2}$ which sum up to zero, there exists a unique unitary one-form ω which is holomorphic on $X\backslash\{P_1, \cdots, P_{m+2}\}$ and has residues $a_1, \cdots, a_{m+2}$ at $P_1, \cdots, P_{m+2}$, resp.

Remark We know neither the positions nor the multiplicities of the zeros of ω.

Degree of residue vector

Definition Call a vector $\alpha = (a_1, \cdots, a_{m+2}) \in \mathbb{R}^{m+2}$ a *residue vector* iff all its components are nonzero and sum up to zero. The degree of α is defined to be

$$\deg \alpha = \begin{cases} \sum_{\lambda a_j > 0} \lambda a_j & \text{if } \exists\, \lambda \neq 0 \text{ s.t. } \lambda a_1, \cdots, \lambda a_{m+2} \text{ are coprime integers} \\ \infty & \text{Otherwise} \end{cases}$$

Existence theorem of unitary one-forms

Theorem (Chen-Li-Song-X(2016), Wei-Wu-X (2022)) Given a residue vector $\alpha = (a_1, \cdots, a_{m+2}) \in \mathbb{R}^{m+2}$, $m \in \mathbb{Z}_{>0}$ and a partition $(m_1, \cdots, m_\ell)$ of m, there exists on the Riemann sphere a unitary one-form ω with residue vector α and ℓ zeros of multiplicities $m_1, \cdots, m_\ell$ iff

$$\deg \alpha > \max(m_1, \cdots, m_\ell).$$

Corollary (Chen-Li-Song-X(2016), Eremenko(2017), Wei-Wu-X(2022)) We found the angle constraint of reducible metrics on the Riemann sphere.

Proof (Chen-Li-Song-X) The 2015 Pacific paper of Chen-Wu-X established a correspondence between reducible metrics and unitary one-forms on general compact Riemann surfaces. In particular, they showed that the divisor represented by a reducible metric coincides with the spherical divisor associated with the unitary one-form corresponding to the metric. QED

Liouville equation for a spherical divisor

Observation (X) Let D be the spherical divisor of a unitary one-form ω on a compact Riemann surface X. Then D must violate the non-bubbling condition. Moreover,

$$f_\lambda(z) = \lambda \exp\left(\int^z \omega \, \mathrm{d}z\right), \quad z \in X \backslash \{\text{poles of } \omega\}, \quad \lambda \in (0, \infty)$$

form the one-parameter family of developing maps corresponding to a one-parameter family $\{\mathrm{d}s_\lambda^2 := f_\lambda^* g_{\mathrm{st}}\}$ of reducible metrics, all of which represent D. Conformal factors of reducible metrics in this a family solves (1) associated with D, but they have no uniform C^0 bound.

I speculate that (1) always has C^0 a priori estimate except D is the spherical divisor of a unitary one-form.

Thank you for your attention!