Spherical surfaces and related topics

INdAM meeting (Cortona, 20-24 June 2022)

Mini-courses

Chang-Shou Lin (National Taiwan University) - ONLINE
Curvature equation from the aspect of integrability

The curvature equation for finding conical spherical metrics in a conformal class of punctured Riemann surfaces is an integrable system. From analytic point of view, it is a special case of the so-called mean field equation in PDE theory. When the case is noncritical, there are some deep results concerning the bubbling phenomana. The simplest one among them is the existence of a priori bound, and equivalent to the properness of the forget map considered by a GAFA paper by Mondello-Panov. But the analytic works yield stronger results. We will discuss those results in the first lecture. The integrability could associate the curvature equation with a second order Fuchsian ODE, and the existence of solution of the PDE is equivalent to that the monodromy group of the complex ODE is unitarizable. For the second and third lecture, I will concentrate on the case when the surface is a flat torus and the equation contains only one singular point with the angle $2\pi(2n+1)$, $n \in \mathbb{N}$. In this case, the associated ODE is an integral Lamé equation, and a theory of premodular form has been developpped by C.L. Wang and myself. One main result is any zero of premodular form is simple. This is proved by applying the Painlevé sixth equation. Lamé potentials is a KdV elliptic potential. We will also show how to apply this fact to obtain solution of some curvature equations, which flows from the Lamé potential.

Feng Luo (Rutgers University)
Conformal geometry of polyhedral surfaces

The purpose of the talks is to introduce some work on conformal geometry of polyhedral (PL) surfaces, especially on discrete uniformization theorem. We will also discuss several open problems relating discrete uniformization to the Weyl problem on convex surfaces, the Koebe circle domain conjecture and Cauchy-Pogorelov rigidity theorem.

Lecture 1.
Basic polyhedral surfaces and their curvatures. Derivative cosine law, variational principles associated to PL metrics. Rigidity theorems of Thurston and others.

Lecture 2.
Discrete conformal equivalence of PL metrics, discrete cross ratio, Delaunay triangulation, and convex hull in hyperbolic 3-space. Discrete uniformization theorem.

Lecture 3.
Sketch of the proof the discrete uniformization theorem. Some open problems relating to Weyl problem, Koebe conjecture and Cauchy-Pogorelov rigidity.

This is a join work with David Gu, Jian Sun, and Tianqi Wu.

Dmitri Panov (King's College of London)
Spherical surfaces and their moduli spaces

A spherical surface is a compact surface (S, h) with marked points $x_1, \ldots, x_n$ with a curvature 1 metric on the complement $\dot{S}$ to x_i's and with conical singularities at x_i's. Such a surface admits a geodesic triangulation, i.e. it can be obtained by gluing together a collection of spherical triangles.

The mini-course will focus on the basic properties of moduli spaces of spherical surfaces with prescribed angles. Here, we fix the genus g of S, prescribe conical angles $2\pi\vartheta_i$ at x_i and ask when the corresponding moduli space is non-empty, whether it is connected, what is its topology in the simplest cases. Each spherical surface is a Riemann surface, and the nature of the forgetful map to the moduli space of marked Riemann surfaces is of considerable interest.

The course will be mainly based on papers `arXiv:1505.01994`, `arXiv:1807.04373`, `arXiv:2008.02772` (the first two joint with Gabriele Mondello and the last with Alexandre Eremenko and Gabriele Mondello).

Research talks

Daniele Bartolucci (Università di Roma "Tor Vergata")
On the uniqueness of spherical polyhedra

After a short introduction concerning Riemann surfaces with conical singularities and the best pinching constants related to the case of the sphere, we will discuss a recent pde proof of the Luo-Tian uniqueness result of spherical polyhedra, that is, the Riemann sphere with a metric of positive constant Gaussian curvature and $N \geq 3$ conical singularities, where all the angles are less than 2π.

Joint work with C. Gui, A. Jevnikar, A. Moradifam.

Martin de Borbon (King's College of London)
Parabolic bundles and spherical metrics

The existence and uniqueness of spherical metrics with cone angles $< 2\pi$ at a prescribed configuration of points in the Riemann sphere is well understood since Troyanov (1991) and Luo-Tian (1992). I will present a new proof that uses the Kobayashi-Hitchin/Mehta-Seshadri correspondence for parabolic bundles.

The numerical conditions on the cone angles are interpreted as stability of a suitable parabolic structure on a holomorphic rank 2 vector bundle over the Riemann sphere. The Mehta-Seshadri theorem produces a unitary logarithmic connection adapted to the parabolic structure. The projectivized bundle (a Hirzebruch surface) admits a unique section of negative self-intersection. The metric is obtained as the pull-back (by this rigid section) of the spherical metrics on the fibres (given by the unitary logarithmic connection). The talk will focus on the parabolic bundles part, and it is based on joint work with Dmitri Panov.

Alexandre Eremenko (Purdue University)
Moduli spaces of Lamé functions

We describe topology of the moduli space of Lame functions of degree m. It is a Riemann surface which consists of two irreducible components when $m > 1$, and we determine the Euler characteristics and the numbers of punctures for each component. One application is a proof of a conjecture of Maier about degrees of Cohn's polynomials, and another is the description of the the degeneracy locus of spherical metrics on tori with one conic singularity with angle $2\pi(2m + 1)$.

Based on the joint work with A. Gabrielov, G. Mondello and D. Panov.

Gianluca Faraco (Max Planck Institute for Mathematics - Bonn)
Complex projective structures with unitary holonomy

Let S be a punctured surface of finite type (g, n) and negative Euler characteristic. We provide necessary and sufficient conditions for a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ to appear as the monodromy of the Schwarzian equation on S with regular singularities at the punctures. Equivalently, we determine which representations appears as the holonomy of some complex projective structures on S. As a corollary, we determine the representations that arise as the holonomy of spherical metrics on S with cone-points at the punctures. This is a joint work with Subhojoy Gupta.

Andrei Gabrielov (Purdue University)
Classification of generic spherical quadrilaterals

Generic spherical quadrilaterals are classified up to isometry. Condition of genericity consists in the requirement that the images of the sides under the developing map belong to four distinct circles which have no triple intersections. Under this condition, it is shown that the space of quadrilaterals with prescribed angles consists of finitely many open curves. Degeneration at the endpoints of these curves is also determined.

Matthew Gursky (University of Notre Dame)
Uniformizing conformal classes in higher dimensions and fully nonlinear equations

I will give an overview of a "uniformization" problem in dimensions greater than two that involves a Monge-Ampère type equation. In the case of manifolds with boundary I will explain some new obstructions to existence and possible applications.

Michael Kapovich (University of California - Davis)
Tensor product decompositions, crossingless matchings, arboreal polygons and branched coverings of spheres

Abstract TBA.

Andrea Malchiodi (Scuola Normale Superiore di Pisa)
Conformal prescription of curvatures on manifolds with boundary

We consider the classical Kazdan-Warner problem of prescribing the scalar curvature of a manifold by conformal deformations. On manifolds with boundary, a natural counterpart consists in prescribing at the same time the geodesic or mean curvature. The problem becomes then equivalent to solving a PDE of critical type in the interior under a nonlinear Neumann boundary condition, also of critical type.

We exploit the variational structure of the problem finding existence of solutions both of minimum and mountain-pass type in two and three dimensions. A crucial step is to tackle the possible blow-up of solutions, which might be of infinite volume and with the profile of horospheres. To rule these out, complex-analytic tools joint with Morse index estimates are employed.

This is joint work with S. Cruz-Blázquez, R. Lopez-Soriano and D. Ruiz.

Alessia Mandini (Universidade Federal Fluminense)
Hyperpolygons spaces and moduli spaces of parabolic Higgs bundles

Hyperpolygons spaces are a family of hyperkähler manifolds that can be obtained from coadjoint orbits by hyperkähler reduction. Jointly with L. Godinho, we showed that these spaces are isomorphic to certain families of parabolic Higgs bundles, when a suitable condition between the parabolic weights and the coadjoint orbits is satisfied.
In this talk I will describe this construction and give an overview of results in collaboration with L. Godinho, C. Florentino and I. Biswas, including an explicit characterization of the fixed loci of some natural involutions defined on these moduli spaces.

Gabriele Mondello (Sapienza Università di Roma)
Spherical surfaces with conical points misbehave (especially in genus 0)

Spherical surfaces of genus g with conical points of angles $2\pi\boldsymbol{\vartheta} = 2\pi(\vartheta_1, \ldots, \vartheta_n)$ behave very differently than flat or hyperbolic one; and so do their moduli spaces $\mathcal{MS}_{g,n}(\boldsymbol{\vartheta})$, even if $\boldsymbol{\vartheta}$ does not sit on certain "critical" hyperplanes where bubbling phenomena can occur.

In this talk we will show examples that witness an array of surprising and amusing phenomena, such as: entire components of the moduli space escaping to infinity even if $\boldsymbol{\vartheta}$ does not approach the critical hyperplanes; forgetful maps from $\mathcal{MS}_{g,n}(\boldsymbol{\vartheta})$ to the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points that are not surjective, fold and so cannot be holomorphic; non-uniqueness of metrics is a given conformal class even for subcritical $\boldsymbol{\vartheta}$.

This is joint work with D. Panov.

Guillaume Tahar (Weizmann Institute of Science)
Cone spherical metrics with dihedral monodromy

Up to biholomorphic change of variable, local invariants of a quadratic differential at some point of a Riemann surface are the order and the residue if the point is a pole of even order. Using the geometric interpretation in terms of flat surfaces, we solve the Riemann-Hilbert type problem of characterizing the sets of local invariants that can be realized by a pair (X, q), where X is a compact Riemann surface and q is a meromorphic quadratic differential. As an application to geometry of surfaces with positive curvature, we give a complete characterization of the distributions of conical angles that can be realized by a cone spherical metric with dihedral monodromy.

Gabriella Tarantello (Università di Roma "Tor Vergata")
On a Donaldson functional for CMC-immersions of surfaces into Hyperbolic 3-manifolds

We discuss a parametrization for the moduli space of Constant Mean Curvature (CMC) immersions of a closed surface (orientable and of genus larger than 1) into hyperbolic 3-manifolds by elements of the tangent bundle of the Teichmueller space. Such tangent bundle is identified by pairs formed by a conformal structure on the surface and a Dolbeault cohomology class of $(0,1)$-forms valued in the corresponding holomorphic tangent bundle. Thus, for any such pair, we establish the unique solvability of the Gauss -Codazzi equations, expressing the pullback metric and the second fundamental form of the immersion in terms of the given element of the tangent bundle. The Gauss-Codazzi equations can be viewed as the Hitchin's selfduality equations for a suitable nilpotent $\mathrm{SL}(2;\mathbb{C})$-Higgs bundle, but also as the Euler-Lagrange equation of a suitable functional, introduced by Gonsalves-Uhlenbeck in 2007, and therefore referred as the Donaldson Functional. Thus we show that the given Donaldson functional admits a unique critical point corresponding to its global minimum. Actually, such uniqueness result extends to a more general version of the Donaldson functional and it permits to recover known results (also about minimal Lagrangian immersions) previously obtained via a Higgs-bundle approach. However, with the Donaldson functional in hand, we are able to describe the asymptotic behavior of its minimum when the co-homology classes vary according to some geometrical pursuit . For example for CMC immersions we shall let the constant of the mean curvature approach its limiting value and describe when the immersed CMC surfaces develop "branched" singularities. Similarly, we will analyze the behavior of minima along a whole ray of cohomology classes. Such an investigation is based on the detailed blow-up analysis developed over the years in the context of Liouville type equations. Here however, we encounter new difficulties as blow-up can occur at a point of "collapsing" zeros of the holomorphic quadratic differentials identified by the given cohomolgy classes.

Bin Xu (University of Science and Technology of China) - ONLINE
Irreducible cone spherical metrics and stable extensions of two line bundles

In this talk we introduce a correspondence between irreducible cone spherical metrics with integral angles and stable extensions of line bundles. Furthermore, by using the theory of indigenous bundles, we find a natural surjective map from the moduli space of stable extensions to that of irreducible metrics representing effective divisors, which is generically injective on a compact Riemann surfaces X of genus greater than one. Moreover, we could specify on X the effective divisor represented by the irreducible metric corresponding to a given stable extension

$$0 \to L \to E \to M \to 0$$

in terms of the Hermitian-Einstein metric on E. Precisely speaking, we obtain a real analytic map , called the ramification divisor map, from the moduli space of stable extensions of M by L to the complete linear system such that its image contains exactly all the effective divisors in this system which could be represented by some irreducible metric. As an application, we prove some new existence results about irreducible metrics.

This is a joint work with Lingguang Li (Tongji University) and Jijian Song (Tianjin University).

Xuwen Zhu (Northeastern University) - ONLINE
Spectral properties of spherical conical metrics

This talk will focus on the recent works on the spectral properties of spherical conical metrics. The motivation comes from earlier works joint with Rafe Mazzeo on the study of deformation of such spherical metrics with large cone angles, which suggests that there is a deep connection between the geometric properties of the moduli space and the analytical properties of the associated singular Laplace operator. In a joint work with Bin Xu we give spectral characterizations of the monodromy of such metrics, and in joint work with Mikhail Karpukhin we discuss the relation of spectral properties with harmonic maps.
