On the Convergence of Sequences of Stationary Jump Markov Processes

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Abstract. This paper presents two main results: first, a Liapunov type criterion for the existence of a stationary probability distribution for a jump Markov process; second, a Liapunov type criterion for existence and tightness of stationary probability distributions for a sequence of jump Markov processes. If the corresponding semigroups $T_N(t)$ converge, under suitable hypotheses on the limit semigroup, this last result yields the weak convergence of the sequence of stationary processes $(T_N(t), \pi_N)$ to the stationary limit one.

Keywords. Jump Markov process, stationary distribution, tightness, weak convergence.

0. Introduction

The development of mathematical models in chemistry and biology is stimulating the interest both in the qualitative analysis of discrete processes and in problems of convergence of sequences of discrete processes to continuous ones (cf. Goel and Dyn (1974), Kurtz (1971, 1980) and Malek Mansour et al. (preprint)).

In this framework, in particular the problem arises of finding conditions for the convergence of a sequence of stationary processes to a stationary limit process. In this paper we make some steps in this direction by stating some Liapunov type criteria for the existence of stationary probability distributions and for their tightness, in the case of pure jump Markov processes.

The criterion for the existence of stationary probability distributions exposed in this paper, corresponding to a well-known condition for the existence of stationary distributions for diffusion processes (cf. Has’minskij (1980)), offers remarkable advantages, with respect to the classical ones (cf. Gihman and Skorokod (1969) and Kannan (1979)): first it can be handled more easily in the applications, second it covers a wider range of situations; for instance it does not require the uniform boundedness of the waiting time parameters: thus all our results can be for example applied in the study of processes with polynomial infinitesimal parameters, such as in chemical reaction models.

By the same Liapunov type techniques we prove in Section 2 a sufficient condition both for existence and tightness of stationary distributions for a sequence of processes. Then, if the semigroups of the processes converge to a limit semigroup admitting a unique stationary distribution, the corresponding stationary processes converge to the stationary limit one. These results make it possible to take advantage of information of the stationarity properties of the limit (a diffusion process, in many applications, to which a large number of results are easily applicable) in studying the given jump processes.

Obviously, this Liapunov type approach cannot lead to a necessary condition for tightness; nevertheless it enables us to prove a sufficient condition for the ‘non-tightness’ of a sequence of stationary distributions.

The paper ends with the application of the previous results to the general model for a class of
chemical reactions (cf. Malek Mansour et al. (preprint)). In this case our approach makes apparent the connection between the stationarity of the stochastic model and the stability of the corresponding deterministic one.

1. A criterion for the existence of stationary distributions

Let $X$ be a closed set in $\mathbb{R}^d$, $\mathcal{B}(X)$ the family of its Borel sets and $\beta(x, A)$ the function given by

$$\beta(x, A) : X \times \mathcal{B}(X) \to \mathbb{R}$$

such that $\beta(\cdot, A)$ is a Borel function of $x$, for each fixed $A \in \mathcal{B}(X)$, and, for each fixed $x \in X$, $\beta(x, \cdot)$ is a finite signed measure on $\mathcal{B}(X)$, such that

$$\beta(x, X - (x)) = -\beta(x, (x)) > 0.$$

It is well known (cf. Gihman and Shokorod (1969, Chap. VII, Section 7)) that under these assumptions $\beta(x, A)$ defines a pure jump right continuous homogeneous Markov process $x(t)$, with values in $X^0 = \mathbb{R} \cup (\infty)$ (the one point compactification of $X$) such that

$$\lim_{t \to 0^+} \frac{P(t, x, A) - I_A(x)}{t} = \beta(x, A)$$

for any $x \in X, A \in \mathcal{B}(X)$

where $P(t, x, A)$ denotes the transition probabilities of $x(t)$.

Throughout this paper we shall make only one further assumption on $\beta(x, A)$, namely we shall assume that $\beta(x, (x))$ is bounded on every compact set of $X$.

Set

$$\tau(\emptyset) = \inf \{ t \geq 0 : x(t) \notin \emptyset \}, \quad \emptyset \in \mathcal{B}(X),$$

$$S_R = \{ x \in X : |x| \leq R \}, \quad R \in \mathbb{R}^+.$$

Then the previous assumption implies that

$$P_x(\tau(X) < \infty, \tau(K) = \tau(X)) = 0 \quad \forall x \in X$$

for any compact set $K \subseteq X$, that is, the process $x(t)$ can reach the point $\infty$ only by actually taking an unbounded sequence of values.

**Definition 1.1.** We shall say that $x(t)$ is regular if

$$P_x(\tau(X) < \infty) = 0 \quad \forall x \in X.$$

Obviously, if $x(t)$ is regular, we can always construct a stochastically equivalent process with values in $X$. Therefore in the sequel when speaking of a regular process $x(t)$ we shall always mean its version with values in $X$.

Set

$$Af(x) := \int_X [f(y) - f(x)] \beta(x, dy)$$

for any $f : X \to \mathbb{R}$ for which the right-hand side is meaningful.

**Lemma 1.2.** For any nonnegative measurable function $V : X \to \mathbb{R}$, which is bounded on any compact set of $X$, and such that

$$\int_{X - (x)} V(y) \beta(x, dy) < +\infty \quad \forall x \in X,$$  \hspace{1cm} (1.1)

the following formula holds,

$$E_x [V(x(t \wedge \tau(K)))] - V(x) = E_x [\int_0^{t \wedge \tau(K)} AV(x(s)) \, ds],$$  \hspace{1cm} (1.2)

for any compact set $K \subseteq X$ and for any $x \in X$.

**Proof.** Hypothesis (1.1) implies that $AV(x)$ is defined and finite for any $x \in X$.

Consider, for any $n \in \mathbb{N}$, the functions $V_n(x) = V(x)I_{S_n(x)}$ for which (1.2) holds. Then by observing that

$$AV_n(x) = \int_{X - (x)} V_n(y) \beta(x, dy) + V_n(x) \beta(x, (x)),$$

the lemma follows in the limit $n \to \infty$.  \hspace{1cm} $\Box$

**Lemma 1.3.** Let $(K_i)_{i \in \mathbb{N}}$ be an increasing family of compact sets of $X$ such that $\bigcup_i K_i = X$ and $V : X \to \mathbb{R}$ a nonnegative measurable function, bounded on any compact set and such that

$$AV(x) \leq c_i V(x) \quad \forall x \in K_i, c_i > 0,$$  \hspace{1cm} (1.3)

$$\lim_{i \to \infty} \frac{e^{ct}}{\inf \{ V(x), x \in K_i^c \}} = 0, \quad t > 0.$$  \hspace{1cm} (1.4)

Then the process $x(t)$ is regular.
Proof. By (1.2) it follows that

\[
E_x \left[ e^{-c(t \wedge \tau(K_i))} V(x \wedge (t \wedge \tau(K_i))) \right] - V(x) =
\]

\[
= E_x \left[ V(x(t \wedge \tau(K_i))) e^{-c(t \wedge \tau(K_i))} - 1 \right] + E_x \left[ V(x(t \wedge \tau(K_i))) - V(x) \right]
\]

\[
= E_x \left[ -c \int_0^{t \wedge \tau(K_i)} e^{-c(s)} \left[ V(x(s \wedge \tau(K_i))) \right. \right.
\]

\[
\left. \left. - V(x(s)) \right] ds \right] + E_x \left[ -c \int_0^{t \wedge \tau(K_i)} e^{-c(s)} \left[ V(x(s \wedge \tau(K_i))) \right. \right.
\]

\[
\left. \left. - V(x(s)) \right] ds \right]
\]

\[
+ E_x \left[ -c \int_0^{t \wedge \tau(K_i)} e^{-c(s)} \left[ A V(x(s)) + c V(x(s)) \right] ds \right]
\]

\[
= E_x \left[ -c \int_0^{t \wedge \tau(K_i)} e^{-c(s)} \left[ A V(x(s)) - c V(x(s)) \right] ds \right]
\]

\[
+ E_x \left[ -c \int_0^{t \wedge \tau(K_i)} e^{-c(s)} \left[ A V(x(s)) - c V(x(s)) \right] ds \right]
\]

\[
= E_x \left[ -c \int_0^{t \wedge \tau(K_i)} e^{-c(s)} \left[ A V(x(s)) - c V(x(s)) \right] ds \right]
\]

\[
\leq 0
\]

by (1.3). Then the nonnegativity of \( V \) implies

\[
P_x(\tau(K_i) \leq t) \leq \frac{V(x)}{\inf(V(x), x \in K_i)},
\]

which implies, by (1.4),

\[
P_x(\tau(X) < \infty) = \lim_{t \to \infty} \lim_{t \to \infty} P_x(\tau(K_i) \leq t) = 0.
\]

\( \square \)

Remark 1.4. The hypotheses of Lemma 1.3 are verified, in particular if \( AV(x) \leq c V(x), c > 0 \) \( \forall x \in X \), and \( \lim_{|x| \to \infty} V(x) = + \infty \) when \( X \) is not bounded.

Theorem 1.5. Assume that the process \( x(t) \) is regular and Feller (that is, \( E_x f(x(t)) \) is a continuous bounded function of \( x \), for any positive \( t \), every time that \( f(x) \) is).

Let \( (K_i)_{i \in \mathbb{N}} \) be an increasing family of compact sets of \( X \) such that \( \bigcup_i K_i = X \) and \( V: X \to \mathbb{R} \) a nonnegative measurable function, bounded on any compact set and such that

\[
AV(x) \leq c \quad \forall x \in X, c > 0,
\]

\[
AV(x) \leq -a_i \quad \forall x \in K_i, a_i > 0,
\]

\[
\lim_{i \to \infty} a_i = + \infty,
\]

then there exists a stationary probability distribution for the process \( x(t) \).

Proof. By (1.2), (1.5) and (1.6) we get for any closed sphere \( S_n \)

\[
E_x \left[ V(x(t \wedge \tau(S_n))) \right] - V(x) =
\]

\[
= E_x \left[ \int_0^{t \wedge \tau(S_n)} AV(x(s)) ds \right]
\]

\[
\leq E_x \left[ \int_0^{t \wedge \tau(S_n)} I_{K_i}(x(s)) AV(x(s)) ds \right]
\]

\[
+ E_x \left[ \int_0^{t \wedge \tau(S_n)} I_{K_i}(x(s)) AV(x(s)) ds \right]
\]

\[
\leq c E_x \left[ \int_0^{t \wedge \tau(S_n)} I_{K_i}(x(s)) ds \right].
\]

Then the nonnegativity of \( V \) implies

\[
E_x \left[ \int_0^{t \wedge \tau(S_n)} I_{K_i}(x(s)) ds \right] \leq \frac{V(x)}{a_i} + \frac{c E_x (t \wedge \tau(S_n))}{a_i}
\]

and by letting \( n \to \infty \), since the process is regular

\[
\frac{1}{t} \int_0^t P(s, x, K_i^c) ds \leq \frac{V(x)}{ta_i} + \frac{c}{a_i}.
\]

Thus by (1.7) it follows that

\[
\lim_{i \to \infty} \lim_{t \to \infty} \frac{1}{t} \int_0^t P(s, x, K_i^c) ds = 0
\]

and, therefore, the theorem (cf. Has'minskij (1980, Theorem 3, Section 2.1)).

As we mentioned in Section 0, the proof of Theorem 1.5 is analogous to the one of Theorem 3.5.1 in Has'minskij (1980) for diffusion processes. A similar result for diffusion processes (which as well can be extended to jump processes) is also proved in Zakai (1969) following Benes (1968). We have preferred to employ the techniques of
Has'minskij (1980) because they do not require the processes to be $\hat{S}$ (condition (6) in Zakai (1969) corresponding to condition (2) in Benes (1968)). This allows us to deal with a much larger class of processes, including for example models of chemical reactions such as the one developed in Section 3: in fact, in this case, for instance, the processes considered do not satisfy condition (2) in Benes (1968), while Theorem 1.5 is still applicable.

Besides, as far as the assumptions on the function $V$ are concerned, we do not require condition (B) in Zakai (1969) to be verified a priori.

2. Tightness of a family of stationary distributions

We shall now consider a sequence $(X_N(t))_{N \in \mathbb{N}}$ of regular pure jump, right continuous, Feller, homogeneous Markov processes with values in a family of closed sets $X_N \subseteq X$.

**Theorem 2.1.** Let $(K_i)_{i \in \mathbb{N}}$ be an increasing family of compact sets such that $\bigcup_i K_i = X$ and $(V_N)$ a sequence of nonnegative measurable functions bounded on any compact set of $X$ and such that

\begin{align}
A_N V_N(x) &\leq C_N \quad \forall x \in X_N, C_N > 0, \\
A_N V_N(x) &\leq -a_{N_i} \quad \forall x \in K_i^c \cap X_N, \\
a_{N_i} &> 0 \quad \forall N \geq N_0 \quad \forall i, \\
\lim_{i \to \infty} a_{N_i} &\rightarrow +\infty \quad \forall N, \\
\lim_{i \to \infty} \lim_{N \to \infty} \frac{C_N}{a_{N_i}} &\rightarrow 0.
\end{align}

Then each process $X_N$ has a stationary probability distribution. Let $\hat{\pi}_N$ be a stationary distribution for $X_N$ and set $\pi_N(\mathcal{B}) := \hat{\pi}_N(\mathcal{B} \cap X_N)$, $\forall \mathcal{B} \in \mathcal{B}(x)$: the family $(\pi_N)_{N \in \mathbb{N}}$ is tight.

**Proof.** From the proof of Theorem 1.5 it follows that for $J$ sufficiently large

$$\lim_{i \to +\infty} \frac{1}{t} \int_0^t P_N(s, x, K_i^c \cap X_N) \, ds \leq \frac{C_N}{a_{N_i}}.$$ 

By (2.3) this proves the first statement, and implies that

$$\lim_{J \to \infty} \lim_{N \to \infty} \pi_N(K_i^c) = 0$$

and therefore the theorem by the Prohorov criterion. \(\square\)

The following corollary easily follows by our Theorem 2.1 and Theorem 7.1 in Kurtz (1980).

**Corollary 2.2.** Under the same hypotheses of the previous theorem if there exists a right continuous, Feller, homogeneous Markov process $x$ with values in $X$ such that

$$\lim_{N \to \infty} \sup_{x \in K \cap X_N} |E_x f(X_N(t)) - E_x f(x(t))| = 0$$

for any compact set $K \subseteq X$ and for any bounded continuous function on $X$, then the process $x$ has a stationary probability distribution $\pi$. Moreover, if such a distribution is unique, the sequence $\pi_N$ converges to $\pi$. In particular, if the processes $X_N$ converge weakly to $x$ every time their initial distributions converge weakly to the initial distribution of $x$, the sequence $(X_N, \pi_N)$ converges weakly to $(x, \pi)$.

A related result is proved in Blankenship and Papanicolau (1978, Section 6) for a sequence of wide-band noise perturbed differential systems converging to a diffusion process. Beyond the dissimilarity of the context, the main difference between the result in Blankenship and Papanicolau (1978) and Theorem 2.1 with Corollary 2.2 is that the former requires the coefficients of the infinitesimal generators of the processes to be bounded or grow sublinearly. Such an assumption is not only in principle restrictive, because it excludes a nontrivial class of processes, including many chemical reactions and population dynamics models (cf. for instance Section 3).

Obviously the previous theorem does not supply any necessary condition for the tightness of the family $\pi_N$. Nevertheless one can prove the following sufficient criterion for the ‘non-tightness’.

**Theorem 2.3.** Assume that each of the processes $X_N(t)$ admits a stationary probability distribution,

1 Sufficient conditions for (2.5) can be found in Kurtz (1969, 1975) in case the family $(X_N)$ is such that $\lim_{N \to \infty} \sup_{x \in X_N} |f(x)| = 0$ for any bounded continuous function on $X$.\footnote{Sufficient conditions for (2.5) can be found in Kurtz (1969, 1975) in case the family $(X_N)$ is such that $\lim_{N \to \infty} \sup_{x \in X_N} |f(x)| = 0$ for any bounded continuous function on $X$.}
and let $\pi_N$ be the corresponding distribution on $X$. If there exists an increasing sequence $(K_N)_{N \in \mathbb{N}}$ of compact sets such that $\bigcup_{N} K_N = X$, and a sequence of measurable functions $V_N$, bounded from above and bounded on any compact set, such that

$$A_N V_N(x) \geq a_N > 0 \quad \forall \ x \in K_N \cap X_N,$$

$$A_N V_N(x) \leq -b_N, \quad b_N > 0 \quad \forall \ x \in K_N \cap X_N,$$

then if

$$\lim_{N \to \infty} \frac{b_N}{a_N} < 1,$$

the family $(\pi_N)$ is not tight, and if

$$\lim_{N \to \infty} \frac{b_N}{a_N} < 1,$$

every subsequence of the family $(\pi_N)$ is not tight.

**Proof.** By (1.2) and (2.6) it can easily be seen that

$$\lim_{t \to +\infty} \lim_{N \to \infty} \mathbb{E}[P_N(s, x, K_N)] ds \leq \frac{b_N}{a_N},$$

which implies the theorem by the necessity of Prohorov criterion. \[\square\]

### 3. A chemical model

The results of this paper immediately apply to the following class of homogeneous chemical reactions (cf. Malek Mansour et al. (preprint) and Costantini and Nappo (1982)),

$$A_i + iY \xrightarrow{i+1} (i+1)Y, \quad i = 0, 1, \ldots, n,$$

where the concentrations $a_i$ of the substances $A_i$ are kept constant. This system of reactions is usually modelled by a sequence $y_N(t)$ of birth-and-death processes, describing the densities of $Y$, when the reactions take place in a volume $N$. Their infinitesimal parameters are $\lambda_{1N}(x), \lambda_{-1N}(x)$, with

$$\lambda_{1N}(x) = k_0 + \sum_{i=1}^n k_i \left(x - \frac{1}{N}\right) \cdots \times \left(x - \frac{i-1}{N}\right) \lambda_{i/N, \infty}(x), \quad N, x \in \mathbb{Z}^+,$$

$$\lambda_{-1N}(x) = \sum_{i=1}^{n+1} l_i \left(x - \frac{i-1}{N}\right) \lambda_{i/N, \infty}(x),$$

$$k_i = \hat{k}_i a_i, \quad k_0 > 0, \quad l_{n+1} > 0.$$

Set

$$F_N(x) = \beta_{1N}(x) - \beta_{-1N}(x),$$

$$Q_N(x) = \beta_{1N}(x) + \beta_{-1N}(x),$$

$$F(x) = \sum_{i=0}^{n+1} (k_i - l_i)x^i,$$

$$Q(x) = \sum_{i=0}^{n+1} (k_i + l_i)x^i,$$

$$k_{n+1} = 0, \quad l_0 = 0.$$

It is well known (cf. Kurtz (1971)) that the sequence $(y_N)$ converges to a deterministic limit which is a solution of

$$\dot{z} = F(z), \quad z > 0.$$

This dynamical system has always an equilibrium point, $\hat{x}$. In case $\hat{x}$ is $h$-asymptotically stable (that is $F^{(i)}(\hat{x}) = 0$ for $i < h$, $F^{(h)}(\hat{x}) < 0$, $h$ an odd number), the sequence of rescaled fluctuations,

$$x_N(t) = N^{1/(h+1)} \left[y_N\left(N^{(h-1)/(h+1)}t\right) - \hat{x}\right],$$

$$x_N(0) = x^0,$$

converges weakly (cf. Costantini and Nappo (1982)) to a diffusion process $x(t)$, satisfying

$$dx(t) = \frac{1}{h!} F^{(h)}(\hat{x})[x(t)]^h dt + Q(\hat{x})^{1/2} dw(t),$$

$$x(0) = x^0$$

if the sequence $(x^0_N)$ converges to $x^0$.

We now show that the sequence $(x_N)$ has the following properties.

1. The processes $x_N$ are regular and admit stationary distributions $\pi_N$.

2. If $\hat{x}$ is the only equilibrium point, and is then necessarily $h$-asymptotically stable, $h \geq 1$, the sequence of stationary processes $(x_N, \pi_N)$ converges weakly.

3. If $\hat{x}$ is not stable, the family $\pi_N$ may be not tight. For instance, if $F$ has exactly three zeros
$\dot{y} < \dot{x} < \dot{z}$ (then $\dot{y}$ and $\dot{z}$ are stable and $\dot{x}$ is unstable) the family $(\pi_N)$ does not admit any tight subsequence.

To see (1) we can apply Lemma 1.3 and Theorem 1.5 with $V_N(x) = x^2 + c_N$ where $c_N$ is a suitable positive constant. To verify (2) we can apply Theorem 2.1 with the same functions $V_N$. The constants $c_N$ can be chosen independent of $N$ while the constants $a_{NJ}$ calculated with respect to the family $S$, turn out to be

$$a_{NJ} = \min(-\lambda + \mu N, -\lambda_1 + \mu_1 J^{h+1}, N), \quad N > N_0,$$

with $\lambda$, $\mu$, $\lambda_1$, $\mu_1$ positive constants. Moreover, since the sequence $(x_N)$ verifies (2.5) (cf. Costantini and Nappo (1982) and Kurtz (1980)) and the limit diffusion has a unique stationary distribution $\pi$, the sequence $((x_N, \pi_N))$, converges to $(x, \pi)$ by Corollary 2.2.

Finally, we apply Theorem 2.3, to verify (3) with the functions

$$V_N(x) = N^{2/(h+1)}[\dot{x} + \frac{x}{N^{1/(h+1)}}],$$

$$V(y) = -\int_{y_0}^y \psi(t) e^{\theta(t)} \, dt, \quad y_0 > 0,$$

$$\psi(y) = (y - \dot{y})(y - \dot{x})(y - \dot{z}),$$

$$\phi(y) = \alpha y^2 + \beta y + \gamma$$

($\alpha$, $\beta$, $\gamma$ suitable constants, $\alpha < 0$) and with the compact sets,

$$K_N = [-N^{1/(h+1)}\delta, N^{1/(h+1)}\delta]$$

where $\delta$ is suitably chosen.

Then, with the notations of Theorem 2.3, the constants $a_N$ and $b_N$ turn out to be

$$a_N = \frac{1}{8} V''(\dot{x}) Q(\dot{x}) - \frac{K_1}{N} + \frac{K_2}{N},$$

$$b_N = \max\left\{\frac{1}{8} V''(\dot{y}) Q(\dot{y}) + \frac{K_1}{N} + \frac{K_2}{N}, \right.\left.\frac{1}{8} V''(\dot{z}) Q(\dot{z}) + \frac{K_1}{N} + \frac{K_2}{N}\right\}$$

$$- \inf_{y > 0} \left\{N^2 \left[ (V(y + 1/N) - V(y)) \beta_{1N}(y) + (V(y - 1/N) - V(y)) \beta_{-1N}(y) \right]\right\}$$

$$[y - \dot{x}, |y - \dot{z}|, |y - \dot{y}|] > \delta$$

with $K_1$ and $K_2$ suitably chosen positive numbers.

References


