Filtering of a Brownian motion with respect to its local time *

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Abstract

We consider a filtering problem when the state process is a Brownian motion W_t and the observation process is its local time Λ_s, for $s \leq t$. For this model we derive an approximation scheme based on a suitable interpolation of the observation process Λ_t. The convergence of the approximating filter to the original one combined with an explicit construction of the approximating filter allows us to derive the explicit form of the original filter. Some connections with the Azéma martingale are discussed.

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1 Introduction

Let (X_t, Y_t) be a stochastic system. We assume that the *state process* X_t of the system cannot be directly observed, while the other component Y_t is completely observable and is referred to as the *observation process*. The aim of stochastic filtering is the computation of the conditional law of the state process at time t, given the observation process up to time t. Equivalently the aim is the computation of

$$E\big[g(X_t)/\mathcal{F}_t^Y\big],$$

for all functions g belonging to a determining class, i.e. the best estimate of $g(X_t)$ given the $\sigma-$algebra of the observations up to time t, $\mathcal{F}_t^Y = \sigma\{Y_s,\, s \leq t\}$.

The classical situation arises when the observation process Y_t is a noisy function of the state X_t, and when the noise can be represented via martingales. The typical example is

$$Y_t = \int_0^t h(X_s)ds + B_t,$$

with B_t a Brownian motion, and is a generalization of the Kalman filter. In this case the problem can be solved either by the "innovation" method or by the "reference probability" method to nonlinear filtering (see, for example [5], [11]). These methods are based on martingale representation properties and allow to characterize the conditional distribution of X_t given $\mathcal{F}_t^Y$ as the solution of a system of equations, usually called the filter equation.

Another situation of interest is the so-called singular filtering. It arises when the observation is a deterministic functional of the state. A first example arises when

$$Y_t = \int_0^t h(X_s)ds,$$

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or more generally when Y_t is a functional of the state X up to time t and is absolutely continuous in time.
A second example arises when the observation Y_t is a function of the state X at a fixed time t, i.e. when

$$Y_t = h(X_t).$$

In these cases the classical methods cannot be applied because the necessary martingale representation properties are not satisfied.
There is not yet a general theory to solve these kind of problems. On the other hand much work has been done in order to give good approximations both for the first kind of example ([13] and references therein) and the second one ([9], [10]).
In this paper we deal with the problem of filtering a reflected Brownian motion W_t with respect to its local time Λ_t, in the sense of the Skorohod problem (we recall the exact definition in §2). This kind of problem naturally comes into the frame of singular filtering since it is a deterministic functional of whole history of the state W up to time t as in the first kind of example, but with the non trivial difference that in our model Λ_t is singular w.r.t. the Lebesgue measure. Therefore, to our knowledge, the techniques in the frame of singular filtering do not apply.

This model is interesting since it can be viewed as the diffusive approximation, in heavy traffic conditions, of a queueing system, see, for example, [7], [8], and, more recently [3].
To be more precise, suppose we can observe, up to time t, whether a queue is busy or idle, but we cannot observe the size of the queue. Then the problem is to evaluate the size of a queue at time t, given this information. In the setup of heavy traffic limits this means to compute the conditional law of a reflected Brownian motion $W_t + \Lambda_t$ when the observation process is its local time Λ_t. Formally our aim is the computation of $E\big[g(W_t+\Lambda_t)/\mathcal{F}_t^\Lambda\big]$. One can equivalently compute

$$\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big], \tag{1}$$

since Λ_t is $\mathcal{F}_t^\Lambda$-measurable.

In this paper we provide an approximation of $\pi_t(g)$, and, by a limiting procedure, we also derive an explicit expression for $\pi_t(g)$.

Our approach consists in constructing an L^1-approximation for $\pi_t(g)$ where $g \in C_b(\mathbb{R})$, with compact support by conditioning to $\mathcal{F}_t^{\Lambda^n}$, where Λ_t^n is a suitable approximation for the observation Λ_t.
The approximating sequence Λ_t^n has the following properties.

- The process Λ_t^n is proportional to a counting process and therefore we can use the techniques of nonlinear filtering with counting observations (see, for instance [4]).
- The sequence of processes Λ_t^n converges to Λ_t for all ω in the space $D_{\mathbb{R}}[0,\infty)$ w.r.t. the topology of the uniform convergence.
- The information $\mathcal{F}_t^{\Lambda^n}$ carried by Λ^n up to time t is increasing in n and strictly contained in the information $\mathcal{F}_t^\Lambda$ carried by Λ up to time t. Moreover it can be uniquely determined just looking at the observation process Λ_t on a suitable sequence of stopping times.
- It is possible to construct an explicit version for the approximating filter

$$\pi_t^n(g) = E\big[g(W_t)/\mathcal{F}_t^{\Lambda^n}\big]. \tag{2}$$

(see Theorem 3.3) converging P-a.s. and in L^1 to the original one (see Theorem 4.4).

The paper is organized as follows. In Section 2 we construct the approximating model and we show the convergence result for the filters. In Section 3 we study the approximating filter $\pi_t^n(g)$, and in particular in Theorem 3.3 we provide its explicit expression

$$\pi_t^n(g) = \frac{1}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right) - 1} \int_{-\frac{1}{2^n}}^{+\infty} g(-\frac{j}{2^n} + x) \left[\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} - \frac{e^{-\frac{(\frac{2}{2^n}+x)^2}{2s}}}{\sqrt{2\pi s}} \right] dx \Big|_{s=t-\sigma_j^n},$$

when $\sigma_j^n < t < \sigma_{j+1}^n$, where $\{\sigma_j^n,\ j \in \mathbb{N}\}$ are the jump times of Λ^n, and where Φ is the distribution function of a standard gaussian random variable.

In Section 4 we identify the original filter $\pi_t(g)$ as the limit of the previous approximation. In Theorem 4.4 we get its explicit expression

$$E\left[g(W_t)/\mathcal{F}_t^\Lambda\right] = \int_0^\infty g\left(-l + y\sqrt{s}\right) y e^{\frac{-y^2}{2}} dy \Big|_{l=\Lambda_t, s=\zeta_t},$$

where ζ_t is the *elapsed time from the last visit to 0* for the process $W_t + \Lambda_t$, that is

$$\zeta_t = t - \sup\{u < t \text{ s.t. } W_u + \Lambda_u = 0\}, \tag{3}$$

which is $\mathcal{F}_t^\Lambda-$measurable, since

$$\sup\{u < t \text{ s.t. } W_u + \Lambda_u = 0\} = \sup\{u < t \text{ s.t. } \Lambda_u < \Lambda_t\}.$$

As a consequence, we get immediately that

$$E\left[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\right] = \int_0^\infty g\left(y\sqrt{s}\right) y e^{\frac{-y^2}{2}} dy \Big|_{s=\zeta_t}.$$

The previous steps can be extended to a general Brownian motion with drift coefficient c and diffusion coefficient a^2. In this case the approximating filters and the exact filter are modifications of the corresponding filters for the standard Brownian motion. We obtain them starting from the standard Brownian motion case, via a deterministic time change, Girsanov Theorem and Kallianpur Striebel formula (see Section 6).
Moreover, in Section 5, we point out some connections with the Azéma's martingale. Starting from these connections we provide another approach for the computation of $E\left[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\right]$.

2 Approximation results

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let W be a standard Brownian motion defined on it. The local time Λ_t at level 0 of the process W_t, in the sense of solution of the Skorohod problem (see, e.g., [12]) is defined by

$$\Lambda_t = \ell_t(W),$$

where ℓ is the deterministic functional

$$\ell : D_{\mathbb{R}}[0, +\infty) \to D_{\mathbb{R}}[0, +\infty); \qquad \ell_t(x) = -\inf_{0 \le s \le t}\big(x(s) \wedge 0\big) = -\inf_{0 \le s \le t}\big(x(s)\big) \wedge 0. \tag{4}$$

Remark 2.1. *It is easy to see that ℓ is a continuous function w.r.t. the topology of the uniform convergence on compact sets.*

Observe that $W_0 = 0$, so that

$$\Lambda_t = -\inf_{0\le s\le t}\big(W_s\big). \tag{5}$$

For each $n \in \mathbb{N}$ consider the sequence of stopping times $\sigma_j^n,\ j \ge 0$ defined as the first time the process Λ_t crosses the threshold $\frac{j}{2^n}$, that is

$$\sigma_0^n = 0, \quad \sigma_j^n = \inf\big\{t \quad \text{s.t.} \quad \Lambda_t \ge \frac{j}{2^n}\big\}, \tag{6}$$

or, equivalently, as the first time the process W_t crosses the threshold $-\frac{j}{2^n}$, that is

$$\sigma_0^n = 0, \quad \sigma_j^n = \inf\big\{t \quad \text{s.t.} \quad W_t \le -\frac{j}{2^n}\big\}. \tag{7}$$

The approximating observation process Λ^n is the $D_{\mathbb{R}}[0,+\infty)$-valued process

$$\Lambda_t^n = \sum_{j=0}^{\infty} \frac{j}{2^n}\mathbb{I}(\sigma_j^n \le t < \sigma_{j+1}^n) = \frac{1}{2^n}\sum_{j=0}^{\infty}\mathbb{I}(\sigma_j^n \le t), \tag{8}$$

where $\mathbb{I}(A)$ denotes the indicator function of A.

Lemma 2.2. *Let Λ^n be defined as in (8). Then*

$$\Lambda^n \underset{n\to\infty}{\to} \Lambda \quad a.s.$$

w.r.t. the topology of the uniform convergence.

Proof. The local time of a Brownian motion is continuous and nondecreasing almost surely. The continuity property implies $\Lambda_{\sigma_{j+1}^n} = \frac{j+1}{2^n} = \Lambda_{\sigma_j^n} + \frac{1}{2^n}$ and therefore

$$\Lambda_t^n = \Lambda_{\sigma_j^n}^n = \Lambda_{\sigma_j^n}, \text{ for any } t \in [\sigma_j^n, \sigma_{j+1}^n).$$

The nondecreasing property implies

$$\Lambda_{\sigma_j^n} \le \Lambda_t \le \Lambda_{\sigma_{j+1}^n} = \Lambda_{\sigma_j^n} + \frac{1}{2^n}, \text{ for any } t \in [\sigma_j^n, \sigma_{j+1}^n).$$

Then, almost surely,

$$\Lambda_t^n \le \Lambda_t \le \Lambda_t^n + \frac{1}{2^n}, \qquad \text{for any } t \ge 0, \tag{9}$$

that is the thesis.

□

For notational convenience from now on we set

$$\mathcal{G}_t^n = \mathcal{F}_t^{\Lambda^n} = \sigma\big(\Lambda_s^n, s \le t\big).$$

Moreover we introduce $\mathcal{G}_t^\infty = \bigvee_n \mathcal{G}_t^n$. The following theorem is the most important tool for showing the convergence result.

Lemma 2.3. *Let $t \in \mathbb{R}^+$. Then $\{\mathcal{G}_t^n, n \in \mathbb{N}\}$ is an increasing sequence of σ-algebras such that $\mathcal{G}_t^\infty = \mathcal{F}_t^\Lambda$.*

Proof. The choice of the thresholds $\frac{j}{2^n}$ yields Λ_t^n to be $\mathcal{G}_t^{n+1}$-measurable for each $n \in \mathbb{N}$. In fact (6)(or (7)) easily provides $\sigma_{2j}^{n+1} = \sigma_j^n$, for each $j \in \mathbb{N}$, and this, together with (8), implies Λ_t^n to be $\mathcal{G}_t^{n+1}$-measurable for each $n \in \mathbb{N}$. Therefore $\mathcal{G}_t^n$ is an increasing sequence of σ-algebras. To complete the proof we show successively the inclusions $\mathcal{G}_t^\infty \supseteq \mathcal{F}_t^\Lambda$ and $\mathcal{G}_t^\infty \subseteq \mathcal{F}_t^\Lambda$.

To prove the first inclusion we note that, for each $m \in \mathbb{N}$, for each choice of $t_1 \leq t_2 \leq ... \leq t_m \leq t$, and for each $\omega \in \Omega$ it occurs $\left(\Lambda_{t_1}^n, ..., \Lambda_{t_m}^n\right) \to \left(\Lambda_{t_1}, ..., \Lambda_{t_m}\right)$ so that $\left(\Lambda_{t_1}, ..., \Lambda_{t_m}\right) \in \mathcal{G}_t^\infty$.

On the other hand Λ_t^n is $\mathcal{F}_t^\Lambda$-measurable for each $n \in \mathbb{N}$. Indeed by (6) it is trivial to see that $\{\sigma_j^n, j \in \mathbb{N}\}$ is a sequence of $\mathcal{F}_t^\Lambda$-stopping times for each $n \in \mathbb{N}$. Then, by (8), $\mathcal{G}_t^n \subseteq \mathcal{F}_t^\Lambda$ for each $n \in \mathbb{N}$, and the second inclusion is achieved. $\square$

Theorem 2.4. *Let g be a bounded measurable function. Then*

$$\pi_t^n(g) \to \pi_t(g) \quad \text{a.s. and in } L^1 \tag{10}$$

Proof. Theorem 2.3 guarantees that $\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] = E\big[g(W_t)/\mathcal{G}_t^\infty\big]$. Moreover, for each $t \in \mathbb{R}^+$ and $g \in C_b(\mathbb{R})$, $\pi_t^n(g)$ coincides with $M^n = E\big[g(W_t)/\mathcal{G}_t^n\big]$. The sequence $\{M_n, n \geq 1\}$ is a bounded uniformly integrable $\mathcal{G}_t^n$-martingale in discrete time and then

$$M_n \to M_\infty = E\big[g(W_t)/\mathcal{G}_t^\infty\big] \qquad \text{a.s. and in } L^1,$$

that is (10). $\square$

3 The approximating filter

In order to derive the explicit expression of

$$\pi_t^n(g) = E\big[g(W_t)/\mathcal{G}_t^n\big]$$

we need the following preliminary result.

Proposition 3.1. *The conditional law of W_t given $\mathcal{G}_t^n$ admits the following representation P-a.s.*

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty} \frac{E\Big[g\left(-\frac{j}{2^n} + W_{s+\sigma_j^n} - W_{\sigma_j^n}\right)\mathbb{I}\left(S_{j+1}^n > s\right)\Big]_{s=t-\sigma_j^n}}{E\Big[\mathbb{I}(S_{j+1}^n > s)\Big]_{s=t-\sigma_j^n}} \mathbb{I}\{\sigma_j^n \leq t < \sigma_{j+1}^n\}, \tag{11}$$

where $\{\sigma_j^n, j \in \mathbb{N}\}$ are defined by (7), and $\{S_j^n = \sigma_j^n - \sigma_{j-1}^n, j \in \mathbb{N}\}$.
Moreover, let W_s^* be a standard Brownian motion, let Λ_s^* be its local time at level 0, then

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty} E\Big[g(-\frac{j}{2^n} + W_s^*)\Big/\Lambda_s^* < \frac{1}{2^n}\Big]_{s=t-\sigma_j^n} \mathbb{I}\{\sigma_j^n \leq t < \sigma_{j+1}^n\}. \tag{12}$$

Proof. Observe that

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty} E\Big[g(W_t)/\mathcal{G}_t^n\Big]\mathbb{I}\{\sigma_j^n \leq t < \sigma_{j+1}^n\},$$

since $\big\{\{\sigma_j^n \leq t < \sigma_{j+1}^n\}, j \in \mathbb{N}\big\}$ is a partition.

Moreover (see [4] chapter III T5)

$$\mathcal{G}_t^n \cap \{\sigma_j^n \le t < \sigma_{j+1}^n\} = \sigma\Big\{\sigma_0^n, ..., \sigma_j^n\Big\} \cap \{\sigma_j^n \le t < \sigma_{j+1}^n\} = \mathcal{G}_{\sigma_j^n}^n \cap \{\sigma_j^n \le t < \sigma_{j+1}^n\}$$

so that, for any $\mathcal{G}_t^n$-measurable random variable Y, there exists a $\mathcal{G}_{\sigma_j^n}^n$-measurable random variable Y' s.t.

$$Y\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\} = Y'\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}.$$

In particular

$$E\Big[g(W_t)/\mathcal{G}_t^n\Big]\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\} = Y'\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}$$

and, by conditioning both sides on $\mathcal{G}_{\sigma_j^n}^n$,

$$E\Big[E\big[g(W_t)/\mathcal{G}_t^n\big]\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}/\mathcal{G}_{\sigma_j^n}^n\Big] = Y'E\Big[\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}/\mathcal{G}_{\sigma_j^n}^n\Big],$$

that is

$$E\Big[E\big[g(W_t)\mathbb{I}\{\sigma_{j+1}^n > t\}/\mathcal{G}_t^n\big]/\mathcal{G}_{\sigma_j^n}^n\Big]\mathbb{I}\{\sigma_j^n \le t\} = Y'\mathbb{I}\{\sigma_j^n \le t\}P\big(\sigma_{j+1}^n > t/\mathcal{G}_{\sigma_j^n}^n\big).$$

Then, recalling that $\sigma_{j+1}^n = \sigma_j^n + S_{j+1}^n$,

$$Y' = \frac{E\Big[g(W_t)\mathbb{I}\left(S_{j+1}^n > t - \sigma_j^n\right)/\mathcal{G}_{\sigma_j^n}^n\Big]}{P\Big(S_{j+1}^n > t - \sigma_j^n/\mathcal{G}_{\sigma_j^n}^n\Big)}\mathbb{I}\{\sigma_j^n \le t\}.$$

Moreover on $\{\sigma_j^n \le t < \sigma_{j+1}^n\}$, $W_{\sigma_j^n} = -\frac{j}{2^n}$ and $W_t = -\frac{j}{2^n} + W_t - W_{\sigma_j^n}$. So

$$\frac{E\Big[g(W_t)\mathbb{I}\left(S_{j+1}^n > t - \sigma_j^n\right)/\mathcal{G}_{\sigma_j^n}^n\Big]}{P\Big(S_{j+1}^n > t - \sigma_j^n/\mathcal{G}_{\sigma_j^n}^n\Big)}\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\} =$$

$$= \frac{E\Big[g\left(-\frac{j}{2^n} + W_{s+\sigma_j^n} - W_{\sigma_j^n}\right)\mathbb{I}\left(S_{j+1}^n > s\right)|_{s=t-\sigma_j^n}/\mathcal{G}_{\sigma_j^n}^n\Big]}{E\Big[\mathbb{I}(S_{j+1}^n > s)\big|_{s=t-\sigma_j^n}/\mathcal{G}_{\sigma_j^n}^n\Big]}\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}.$$

The process $W_{s+\sigma_j^n} - W_{\sigma_j^n}$ is a standard Brownian motion independent of $\mathcal{G}_{\sigma_j^n}^n$, S_{j+1}^n is a random variable independent of $\mathcal{G}_{\sigma_j^n}^n$ and $t - \sigma_j^n$ is $\mathcal{G}_{\sigma_j^n}^n$-measurable. Then (11) follows since

$$\frac{E\Big[g\left(-\frac{j}{2^n} + W_{s+\sigma_j^n} - W_{\sigma_j^n}\right)\mathbb{I}\left(S_{j+1}^n > s\right)|_{s=t-\sigma_j^n}/\mathcal{G}_{\sigma_j^n}^n\Big]}{E\Big[\mathbb{I}(S_{j+1}^n > s)\big|_{s=t-\sigma_j^n}/\mathcal{G}_{\sigma_j^n}^n\Big]}\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\} =$$

$$= \frac{E\Big[g\left(-\frac{j}{2^n} + W_{s+\sigma_j^n} - W_{\sigma_j^n}\right)\mathbb{I}\left(S_{j+1}^n > s\right)\Big]_{s=t-\sigma_j^n}}{E\Big[\mathbb{I}(S_{j+1}^n > s)\Big]_{s=t-\sigma_j^n}}\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}.$$

Finally (12) follows by noting that

$$\begin{aligned} E\Big[g(-\frac{j}{2^n} + W_{s+\sigma_j^n} - W_{\sigma_j^n})\mathbb{I}\left(S_{j+1}^n > s\right)\Big] &= E\Big[g(-\frac{j}{2^n} + W_s^*)\mathbb{I}\left(\sigma_1^{*n} > s\right)\Big] = \\ &= E\Big[g(-\frac{j}{2^n} + W_s^*)\mathbb{I}\left(\Lambda_s^* < \frac{1}{2^n}\right)\Big], \end{aligned} \tag{13}$$

where σ_1^{*n} is first exit time of W_t^* from the region $(-\frac{1}{2^n}, \infty)$. $\square$

Our next step is to give a more explicit expression of (12). First of all note that (12) can be rewritten as

$$\begin{aligned} E\Big[g(W_t)/\mathcal{G}_t^n\Big] &= E\Big[g(-l+W_s^*)\Big/\Lambda_s^* < \frac{1}{2^n}\Big]\Big|_{l=\Lambda_t^n,\ s=\zeta_t^n} \\ &= E\Big[g(-l+W_s^*)\Big/\Lambda_s^{*n} = 0\Big]\Big|_{l=\Lambda_t^n,\ s=\zeta_t^n}, \end{aligned} \tag{14}$$

where Λ_t^n is defined in (8),

$$\zeta_t^n = t - \sup\{u \le t \quad \text{s.t.} \quad \Lambda_u^n < \Lambda_t^n\}, \tag{15}$$

and Λ^{*n} is obtained from W^* as Λ^n from W. Then, denoting

$$\Pi^n(s,l;g) = E\Big[g(-l+W_s^*)\Big/\Lambda_s^* < \frac{1}{2^n}\Big] = E\Big[g(-l+W_s^*)\Big/\Lambda_s^{*n} = 0\Big], \tag{16}$$

the conditional expectation (12) can be shortly rewritten as

$$E\Big[g(W_t)/\mathcal{G}_t^n\Big] = \Pi^n(\zeta_t^n, \Lambda_t^n; g). \tag{17}$$

In order to give the explicit expression of the above conditional expectation, and taking into account the definition of $\Pi^n(s,l)$, we need to compute the conditional law of W_s^* given $\{\Lambda_s^* < k\}$, for $k > 0$, where (W_s^*, Λ_s^*) are the processes of Proposition 3.1.

Lemma 3.2. *When $s > 0$, the conditional law of W_s^* given $\{\Lambda_s^* < k\}$, has density*

$$f_{W_s^*}(x/\Lambda_s^* < k) = \frac{1}{2\Phi\left(\frac{k}{\sqrt{s}}\right) - 1}\left[\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} - \frac{e^{-\frac{(2k+x)^2}{2s}}}{\sqrt{2\pi s}}\right]\mathbb{I}(-x < k), \tag{18}$$

where Φ is the distribution function of a normal standard random variable.
When $s = 0$ the conditional law of W_0^* given $\{\Lambda_0^* < k\}$ is the Dirac measure $\delta_{\{0\}}(dx)$ in 0.

Proof. The case $s = 0$ is trivial. When $s > 0$, set $M_s^* = \max_{u \le s} W_u^*$. By the reflection principle one can compute the joint law of (W_s^*, M_s^*) (see [6], for example)

$$P\left(W_s^* \le x, M_s^* \le k\right) = \begin{cases} \Phi\left(\dfrac{x}{\sqrt{s}}\right) - \Phi\left(\dfrac{x-2k}{\sqrt{s}}\right) & \text{if } x \le k \\ P\left(M_s^* \le k\right) = \Phi\left(\dfrac{k}{\sqrt{s}}\right) - \Phi\left(\dfrac{-k}{\sqrt{s}}\right) & \text{if } x > k. \end{cases}$$

Then, by the symmetry of W, the joint law of (W_s^*, Λ_s^*) is

$$P\left(W_s^* \le x, \Lambda_s^* \le k\right) = \begin{cases} 0 & \text{if } -x \ge k \text{ and } k < 0 \\ 2\Phi\left(\dfrac{k}{\sqrt{s}}\right) - \Phi\left(\dfrac{-x}{\sqrt{s}}\right) - \Phi\left(\dfrac{x+2k}{\sqrt{s}}\right) & \text{if } -x < k \text{ and } k \ge 0. \end{cases}$$

Then the thesis trivially follows by observing that

$$P\left(W_s^* \le x/\Lambda_s^* < k\right) = \frac{2\Phi\left(\frac{k}{\sqrt{s}}\right) - \Phi\left(\frac{-x}{\sqrt{s}}\right) - \Phi\left(\frac{x+2k}{\sqrt{s}}\right)}{2\Phi\left(\frac{k}{\sqrt{s}}\right) - 1}\mathbb{I}(-x < k).$$

$\square$

Now we have all the tools to give a more explicit expression for $E\big[g(W_t)/\mathcal{G}_t^n\big]$.

Theorem 3.3. *Let*

$$f_n(x,s) = \frac{1}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}\left[\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} - \frac{e^{-\frac{(\frac{2}{2^n}+x)^2}{2s}}}{\sqrt{2\pi s}}\right]\mathbb{I}(x > -\frac{1}{2^n}), \qquad \text{for } s > 0. \tag{19}$$

Then

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty}\left\{\int_{-\frac{1}{2^n}}^{+\infty} g(-\frac{j}{2^n}+x)f_n(x,s)dx\right\}\Bigg|_{s=t-\sigma_j^n}\mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}, \tag{20}$$

with the convention that $f_n(x,0) = \delta_0(x)$, the Dirac function at the point 0.
Equivalently, the conditional law of W_t given $\mathcal{G}_t^n$ admits the representation (17) P-a.s., and

$$\Pi^n(s,l;g) = \begin{cases} \int_{-\infty}^{+\infty} g(-l+x)f_n(x,s)dx & \text{if} \quad s>0 \\ g(-l) \quad \text{if} \quad s=0. \end{cases} \tag{21}$$

Proof. From Lemma 3.2 we get that, for $s > 0$,

$$E\Big[g(-\frac{j}{2^n}+W_s^*)/\Lambda_s^* < \frac{1}{2^n}\Big] = \int_{-\infty}^{+\infty} g(-\frac{j}{2^n}+x)f_{W_s^*}(x/\Lambda_s^* < \frac{1}{2^n})dx.$$

Comparing (19) and (18) we see that

$$f_n(x,s) = f_{W_s^*}(x/\Lambda_s^* < \frac{1}{2^n}),$$

then (20) follows by substituting the previous expression into (12), and (21) is an immediate consequence of (16). □

4 The original filter as an L^1 limit

In this section we compute explicitly the L^1-limit of the approximating filter $\pi_t^n(g) = E\big[g(W_t)/\mathcal{G}_t^n\big]$. Thanks to the convergence result stated in Theorem 2.4, this limit is a version of the original filter

$$\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^{\Lambda}\big].$$

By (17) and by Theorem 3.3 we know that

$$\pi_t^n(g) = \Pi^n(\zeta_t^n, \Lambda_t^n; g),$$

where $\Pi^n(s,l;g)$ is defined in (16), its expression is given in (21), ζ_t^n is defined in (15), and Λ_t^n in (8).
We have already shown in Lemma 2.2 the uniform convergence of Λ_t^n to Λ_t. The idea is now to show that ζ_t^n is converging to ζ_t, the elapsed time from last visit to 0 for the process $W_t+\Lambda_t$ (see (3)), and that $\Pi^n(s_n,l;g)$ is converging to a limit $\Pi(s,l;g)$, to be computed explicitly, whenever $s_n \to s$. Taking into account (16), the natural candidate for $\Pi(s,l;g)$ is

$$E\left[g(-l+W_s^*)/\Lambda_s^*=0\right] := \int_{\mathbb{R}} g(-l+x)f_{W_s^*|\Lambda_s^*}(x|0)dx$$

This is the case, as explained in Remark 4.2 below by using the following Lemma.

Lemma 4.1. *Denote by*

$$\Pi(s,l;g) = \int_0^\infty g(-l+y\sqrt{s})ye^{-\frac{y^2}{2}}dy \quad s \geq 0. \tag{22}$$

Let g be a continuous function with compact support, then

$$\Pi^n(s,l;g) \underset{n\to\infty}{\to} \Pi(s,l;g) \qquad \textit{for } s \geq 0. \tag{23}$$

Moreover if $s_n \underset{n\to\infty}{\to} s$ and $s>0$, then

$$\Pi^n(s_n,l;g) \underset{n\to\infty}{\to} \Pi(s,l;g). \tag{24}$$

Proof. The proof of (23) when $s=0$ is trivial. Let s be strictly positive, and let s_n converge to s. Note that $\Pi^n(s_n,l;g)$ is the integral of the product of $g(-l+x)$ and $f_n(x,s_n)$, where f_n is defined in (19), so that

$$f_n(x,s_n) = \frac{1}{\sqrt{2\pi s_n}} \frac{1}{2\left(\Phi\left(\frac{1}{2^n\sqrt{s_n}}\right)-\Phi(0)\right)}\left[e^{-\frac{x^2}{2s_n}} - e^{-\frac{(\frac{2}{2^n}+x)^2}{2s_n}}\right]\mathbb{I}\left(x > -\frac{1}{2^n}\right).$$

Moreover note that, since $s>0$ we may rewrite

$$\Pi(s,l;g) = \int_0^\infty g(-l+x)\frac{x}{s}e^{-\frac{x^2}{2s}}dx. \tag{25}$$

If we prove that $f_n(x,s_n)$ is a bounded sequence and is pointwise converging to $\frac{x}{s}e^{-\frac{x^2}{2s}}$, then the dominated convergence theorem yields (24), and therefore (23), since $g(-l+x)$ is integrable. The *bp*-convergence of $f_n(x,s_n)$ is easy to prove. Indeed, first of all

$$\Phi\left(\frac{1}{2^n\sqrt{s_n}}\right) - \Phi(0) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\theta_n^2}{2}}\, \frac{1}{2^n\sqrt{s_n}},$$

where

$$\theta_n^2 \in \left(0, \frac{1}{2^n\sqrt{s_n}}\right),$$

and therefore

$$\theta_n^2 \underset{n\to\infty}{\to} 0.$$

Moreover

$$\begin{aligned} e^{-\frac{x^2}{2s_n}} - e^{-\frac{(\frac{2}{2^n}+x)^2}{2s_n}} &= e^{-y_n(x)}\frac{1}{2s_n}\left[(\frac{2}{2^n}+x)^2 - x^2\right] \\ &= e^{-y_n(x)}\frac{2}{2^n s_n}\left[x+\frac{1}{2^n}\right], \end{aligned}$$

where

$$y_n(x) \in \left(\frac{1}{2s_n}x^2, \frac{1}{2s_n}(\frac{1}{2^{n-1}}+x)^2\right),$$

and therefore

$$y_n(x) \underset{n\to\infty}{\to} \frac{x^2}{2s}.$$

Then

$$f_n(x, s_n) = e^{\frac{\theta_n^2}{2}} e^{-y_n(x)} \frac{1}{s_n} \left[x + \frac{1}{2^n} \right] \mathbb{I} \left(x > -\frac{1}{2^n} \right),$$

so that $f_n(x, s_n)$ is bounded uniformly in x, and

$$\lim_{n\to\infty} f_n(x, s_n) = \frac{x}{s} e^{-\frac{x^2}{2s}} \mathbb{I}\,(x \geq 0)\,, \tag{26}$$

and the thesis follows.

□

Remark 4.2. *The joint density of* (W_s^*, Λ_s^*) *is*

$$f_{W_s^*,\Lambda_s^*}(x, k) = \frac{2}{s\sqrt{2\pi s}}(x + 2k) \exp\left(-\frac{(x + 2k)^2}{2s} \right) \; \mathbb{I}(k \geq (-x) \vee 0), \qquad \textit{for } s > 0,$$

and then

$$f_{W_s^*|\Lambda_s^*}(x|k) = \frac{1}{s}(x + 2k) \exp\left(-\frac{(x + 2k)^2}{2s} + \frac{k^2}{2s} \right) \; \mathbb{I}(k \geq (-x) \vee 0).$$

As noted in the proof of Theorem 3.3, $f_n(x, s) = f_{W_s^*}(x/\Lambda_s^* < \frac{1}{2^n})$*, and therefore (26) is equivalent to*

$$f_{W_{s_n}^*}(x/\Lambda_{s_n}^* < \frac{1}{2^n}) \underset{n\to\infty}{\to} f_{W_s^*|\Lambda_s^*}(x|0) = \frac{x}{s} e^{-\frac{x^2}{2s}} \mathbb{I}\,(x \geq 0)\,.$$

Taking into account (25) we get the announced interpretation

$$\Pi(s, l; g) = \int_0^\infty g(-l + x) f_{W_s^*|\Lambda_s^*}(x|0) dx.$$

It is also interesting to note that when g *is differentiable, then*

$$\int_0^\infty g(-l + x) f_{W_s^*|\Lambda_s^*}(x|0) dx = g(-l) + \int_0^\infty g'(-l + x) e^{-\frac{x^2}{2s}} dx \tag{27}$$

since $-\frac{x}{s} e^{-\frac{x^2}{2s}} = \frac{d}{dx} e^{-\frac{x^2}{2s}}$.

The last observation is a key point in the alternative probabilistic proof of the result stated in Lemma 4.1, and that we postpone to the Appendix to this section.

The next result concerns the limit of the processes ζ_t^n.

Lemma 4.3. *Let* ζ_t^n *be defined by (15) and let* ζ_t *be the elapsed time from last visit to* 0 *for the process* $W_t + \Lambda_t$ *(see (3)). Then, for all* $\omega \in \Omega$

$$\zeta_t^n(\omega) \underset{n\to\infty}{\searrow} \zeta_t(\omega). \tag{28}$$

Proof. Note that $\zeta_t = t - \eta_t$ and $\zeta_t^n = t - \eta_t^n$, where

$$\eta_t = \sup\{u < t \;\text{ s.t. } \Lambda_u < \Lambda_t\}$$

and

$$\eta_t^n = \sup\{u < t \;\text{ s.t. } \Lambda_u^n < \Lambda_t^n\} \; = \; \sup\{u < t \text{ s.t. } \Lambda_u^n = \Lambda_t^n - \frac{1}{2^n}\},$$

with the convention that $\sup\{\emptyset\} = 0$.
Observe that

$$\eta^n_t = \sigma^n_j \ \text{ if and only if } \quad -\frac{j+1}{2^n} < \inf_{s\le t} W_s \le -\frac{j}{2^n}$$

and

$$\eta^{n+1}_t = \begin{cases} \sigma^{2n}_{2j} \ \text{ if and only if } \quad -\frac{2j+1}{2^{n+1}} < \inf_{s\le t} W_s \le -\frac{2j}{2^{n+1}} = -\frac{j}{2^n} \\ \sigma^{2n}_{2j+1} \ \text{ if and only if } \ \frac{j+1}{2^n} = -\frac{2j+2}{2^{n+1}} < \inf_{s\le t} W_s \le -\frac{2j+1}{2^{n+1}}. \end{cases}$$

This observation implies η^n_t to be increasing in n. Moreover $\eta^n_t \le t$ and so $\eta^n_t \uparrow \eta^\infty_t$.

To complete the proof we show successively the inequalities $\eta^\infty_t \le \eta_t$ and $\eta^\infty_t \ge \eta_t$.

The first inequality is equivalent to $\eta^n_t \le \eta_t$, for any $n \in \mathbb{N}$. In case $t < \sigma^n_1$ this inequality is trivial since $\eta^n_t = 0 \le \eta_t$. When $t \ge \sigma^n_1$ observe that $\eta^n_t \in \{\sigma^n_j, j \in \mathbb{N}\}$ and $\Lambda^n_{\sigma^n_j} = \Lambda_{\sigma^n_j}$ for all $n, j \in \mathbb{N}$ and so

$$\Lambda_{\eta^n_t} = \Lambda^n_{\eta^n_t} = \Lambda^n_t - \frac{1}{2^n}.$$

Moreover, by (9), $\Lambda^n_t \le \Lambda_t$, so that $\Lambda_{\eta^n_t} \le \Lambda_t - \frac{1}{2^n}$. The last inequality implies $\eta^n_t \le \eta_t$ for all $n \in \mathbb{N}$.

On the other hand, if $\eta^\infty_t < s \le t$ then $\eta^n_t \le \eta^\infty_t < s \le t$ for all $n \in \mathbb{N}$, and therefore $\Lambda^n_s = \Lambda^n_t$ for all n. Passing to the limit in the previous equality we get $\Lambda_s = \Lambda_t$ and so $s \ge \eta_t$. Therefore $\eta^\infty_t \ge \eta_t$. □

Finally we can state our main result

Theorem 4.4. *Let ζ_t, Λ_t and $\Pi(s,l)$ be defined in (3), (5) and (22), respectively. Then, for all continuous functions g, with compact support,*

$$\int_0^T E\Big|\pi^n_t(g) - \Pi(\zeta_t, \Lambda_t; g)\Big| dt \underset{n\to\infty}{\longrightarrow} 0,$$

and the process $\Pi(\zeta_t, \Lambda_t; g)$ is a cadlag version of $\pi_t(g) = E\big[g(W_t)/\mathcal{F}^\Lambda_t\big]$, for all bounded measurable functions g.

Proof. Taking into account (2) and (17), our first aim is to show that, for all continuous functions g with compact support,

$$\int_0^T E\Big|\Pi^n(\zeta^n_t, \Lambda^n_t; g) - \Pi(\zeta_t, \Lambda_t; g)\Big| dt \underset{n\to\infty}{\longrightarrow} 0.$$

Note that

$$\begin{aligned} E\Big|\Pi^n(\zeta^n_t, \Lambda^n_t; g) - \Pi(\zeta_t, \Lambda_t; g)\Big| \le \\ \le E\Big|\Pi^n(\zeta^n_t, \Lambda^n_t; g) - \Pi^n(\zeta^n_t, \Lambda_t; g)\Big| + E\Big|\Pi^n(\zeta^n_t, \Lambda_t; g) - \Pi(\zeta_t, \Lambda_t; g)\Big|. \end{aligned}$$

Consider separately the terms on the right hand side. Recall (9), so that
$\Big|\Lambda^n_t - \Lambda_t\Big| \le \frac{1}{2^n}$. Then, by (21), a direct computation yields

$$\Big|\Pi^n(\zeta_t^n,\Lambda_t^n;g)-\Pi^n(\zeta_t^n,\Lambda_t;g)\Big|\leq\omega_g\left(\frac{1}{2^n}\right),$$

where $\omega_g(\delta)=\sup_{|s-u|<\delta}|g(s)-g(u)|$.

As far as the second term is concerned, (24) and (28) imply that for each (t,ω) satisfying the condition $\zeta_t(\omega)>0$

$$\Big|\Pi^n(\zeta_t^n,\Lambda_t;g)-\Pi(\zeta_t,\Lambda_t;g)\Big|\underset{n\to\infty}{\longrightarrow}0.$$

The observation that $\{(t,\omega)\in[0,T]\times\Omega$ such that $\zeta_t(\omega)=0\}$ is a zero measure set with respect to $dt\times dP$, and an easy application of the dominated convergence theorem provide

$$E\big[g(W_t)/\mathcal{G}_t^n\big]\underset{L^1([0,T]\times\Omega)}{\longrightarrow}\Pi(\zeta_t,\Lambda_t;g).$$

This first result, combined with Theorem 2.4, yields that $\pi_t(g)=\Pi(\zeta_t,\Lambda_t;g)$ $dt\times dP$-a.e., for continuous functions with compact support. Then the probability measures π_t and $\Pi(\zeta_t,\Lambda_t)$ coincide $dt\times dP$-a.e., and the second statement is achieved.

□

We end this section by observing that the cadlag version of the filter $\pi_t(g)$ can be also written as follows (see Remark 4.2)

$$\Pi(\zeta_t,\Lambda_t;g)=g(-\Lambda_t)\mathbb{I}(\zeta_t=0)+\int_0^\infty g(-\Lambda_t+x)f_{W_s^*|\Lambda_s^*}(x|0)dx\Big|_{s=\zeta_t}\mathbb{I}(\zeta_t>0),\tag{29}$$

and consequently we can write down the cadlag version for the filter of $W_t+\Lambda_t$ w.r.t. $\mathcal{F}_t^\Lambda$

$$E\big[g(W_t+\Lambda_t)/\mathcal{F}_t^\Lambda\big]=\int_0^\infty g(x)f_{W_s^*|\Lambda_s^*}(x|0)dx\Big|_{s=\zeta_t}=\int_0^\infty g(y\sqrt{s})ye^{-\frac{y^2}{2}}dy\Big|_{s=\zeta_t}.\tag{30}$$

Appendix to Section 4

We now give a different derivation of the limit $\Pi(s,l;g)$, under the hypothesis that g has continuous derivative and is uniformly continuous. For the sake of simplicity we consider only the case $s_n=s>0$, but the general case when s_n is converging to $s>0$ can be managed in the same manner. By the definition (19) of $f_n(x,s)$, we have

$$\Pi^n(s,l;g)=\frac{E\Big[g(-l+W_s^*)\mathbb{I}(W_s^*>-\frac{1}{2^n})\Big]-E\Big[g(-l+W_s^*-2\frac{1}{2^n})\mathbb{I}(W_s^*-2\frac{1}{2^n}>-\frac{1}{2^n})\Big]}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}.$$

Note that the above expression could also be derived directly from (13) by means of the reflection principle.

Assume that g is differentiable, and observe that

$$g(-l+x)\mathbb{I}(x>-\frac{1}{2^n})-g(-l+x-2\frac{1}{2^n})\mathbb{I}(x-2\frac{1}{2^n}>-\frac{1}{2^n})=$$

$$g(-l+x)\mathbb{I}(-\frac{1}{2^n}<x\leq\frac{1}{2^n})+\left(g(-l+x)-g(-l+x-2\frac{1}{2^n}))\mathbb{I}(x>\frac{1}{2^n}\right)=$$

$$g(-l+x)\mathbb{I}(-\frac{1}{2^n}<x\leq\frac{1}{2^n})+\left(g'(-l+x-\vartheta^n(x)2\frac{1}{2^n}))2\frac{1}{2^n}\mathbb{I}(x>\frac{1}{2^n}\right),$$

where $\vartheta^n(x)\in(0,1)$.

Then

$$E\Big[g(-l+W_s^*)\mathbb{I}(W_s^*>-\frac{1}{2^n})-g(-l+W_s^*-2\frac{1}{2^n})\mathbb{I}(W_s^*-2\frac{1}{2^n}>-\frac{1}{2^n})\Big]=$$

$$=E\Big[g(-l+W_s^*)\mathbb{I}(-\frac{1}{2^n}<W_s^*\leq\frac{1}{2^n})+g'(-l+W_s^*-\vartheta^n(W_s^*)2\frac{1}{2^n})\;2\frac{1}{2^n}\mathbb{I}(W_s^*>\frac{1}{2^n})\Big],$$

and therefore

$$\begin{aligned}\Pi^n(s,l;g) &= \frac{E\Big[g(-l+W_s^*)\mathbb{I}(-\frac{1}{2^n}<W_s^*\leq\frac{1}{2^n})\Big]}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}\\ &\quad+\frac{E\Big[g'(-l+W_s^*-\vartheta^n(W_s^*)2\frac{1}{2^n})\mathbb{I}(W_s^*>\frac{1}{2^n})\Big]\;2\frac{1}{2^n}}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}.\end{aligned}$$

The first addend is converging to $g(-l)$ since $2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1=P(-\frac{1}{2^n}<W_s^*\leq\frac{1}{2^n}))$ and $|g(-l+x)-g(-l)|\leq\omega_g(\frac{1}{2^n})$ for $x\in(-\frac{1}{2^n},\frac{1}{2^n}]$. For the second addend observe that the expectation in the numerator is converging to $E\Big[g'(-l+W_s^*)\mathbb{I}(W_s^*>0)\Big]$, while

$$\lim_{n\to\infty}\frac{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}{2\frac{1}{2^n}}=\frac{1}{\sqrt{2\pi s}}\,.$$

The expression (25) of $\Pi(s,l;g)$ is achieved again since

$$E\Big[g'(-l+W_s^*)\mathbb{I}(W_s^*>0)\Big]=\frac{1}{\sqrt{2\pi s}}\int_0^\infty g'(-l+x)e^{-\frac{x^2}{2s}}dx$$

and since the following integration by part formula, which can be thought as the counterpart of (27), yields

$$E\Big[g'(-l+W_s^*)\mathbb{I}(W_s^*>0)\Big]=\frac{1}{\sqrt{2\pi s}}\left\{g(-l+x)e^{-\frac{x^2}{2s}}\Big|_0^\infty+\int_0^\infty g(-l+x)\frac{x}{s}e^{-\frac{x^2}{2s}}dx\right\}=$$

$$=\frac{1}{\sqrt{2\pi s}}\left\{-g(-l)+\int_0^\infty g(-l+x)\frac{x}{s}e^{-\frac{x^2}{2s}}dx\right\}.$$

5 Connections with the Azéma's martingale

Let B_t be a standard Brownian motion. Consider the process $L_t(B)$ defined by

$$L_t(B)=|B_t|-\int_0^t \mathrm{sgn}(B_s)dB_s \tag{32}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1 \quad \text{if} \quad x > 0 \\ -1 \quad \text{if} \quad x \leq 0 \end{cases}$$

The process $L_t(B)$ is the usual definition of the local time of a standard Brownian motion B_t via the Tanaka formula.

It is well-known that the processes $(W_t+\Lambda_t, \Lambda_t)$ and $(|B_t|, L_t(B))$ have the same law on $D_{\mathbb{R}^2}[0,\infty)$ endowed with the Skorohod topology (see e.g. [14]).
This result implies that the filter of $W_t + \Lambda_t$ given the history generated by the process Λ_t and the filter of $|B_t|$ given the history generated by the process $L_t(B)$ have the same law and that these filters are obtained just applying the same functional to the processes $\Lambda_\cdot$ and $L_\cdot(B)$ respectively.

In the sequel we see that the filter of $|B_t|$ given the history generated by the process $L_t(B)$ can be directly obtained just starting from some results related to the Azéma martingale.

Azéma and Yor ([2]) define the process

$$\mu_t = \mathrm{sgn}(B_t)\sqrt{t - g_t(B)}, \tag{33}$$

where $g_t(B) = \sup\{s < t : B_s = 0\}$. They show that μ_t is a martingale w.r.t. a filtration containing $\mathcal{F}_t^{\mathrm{sgn}(B)} = \sigma\{\mathrm{sgn}(B_s), \quad s \leq t\}$. Observe that μ_t is adapted to the filtration $\mathcal{F}_t^{\mathrm{sgn}(B)}$ and therefore is an $\mathcal{F}_t^{\mathrm{sgn}(B)}$-martingale. The process μ_t coincides, up to a constant, with the martingale

$$E\big[B_t/\mathcal{F}_t^{\mathrm{sgn}(B)}\big], \tag{34}$$

which is known in the literature as the *Azéma martingale* (see, e.g. [14]).

From Lemma 1 and Proposition 4 in [2] one can also draw that

$$E\big[f(|B_t|)/\mathcal{F}_t^{\mathrm{sgn}(B)}\big] = \int_0^\infty f(y\sqrt{t - g_t(B)})ye^{-\frac{y^2}{2}}\,dy\ . \tag{35}$$

Observe now that $g_t(B)$ is measurable w.r.t. the filtration $\mathcal{F}_t^{L(B)}$ generated by $L(B)$ up to time t since

$$g_t(B) = \sup\{s < t \quad \text{s.t.} \quad |B_s| = 0\} = \sup\{s < t \quad \text{s.t.} \quad L_s(B) < L_t(B)\} \tag{36}$$

Finally, recalling that $\mathcal{F}_t^{L(B)} \subseteq \mathcal{F}_t^{\mathrm{sgn}(B)}$ (see e.g. [14]), from (35)

$$E\big[f(|B_t|)/\mathcal{F}_t^{L(B)}\big] = \int_0^\infty f(y\sqrt{t - g_t(B)})ye^{-\frac{y^2}{2}}\,dy\ . \tag{37}$$

Then the connection with our filtering problem is evident by comparing the above equality with (30). Indeed one can obtain one from the other just by changing $(W_t+\Lambda_t, \Lambda_t)$ with $(|B_t|, L_t(B))$. Indeed, if we define

$$\gamma : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty);\ x \to \gamma(x),\ \text{s.t.}\ \gamma(x)(t) = t - \sup\{s < t, \quad \text{s.t.} \quad x_s < x_t\}$$

the process $t - g_t(B)$ in (36) and the elapsed time process ζ_t defined in Lemma 4.3 are respectively defined by

$$t - g_t(B) = \gamma\left(L(B)\right)(t),$$

$$\zeta_t = \gamma\left(\Lambda\right)(t).$$

6 The case of a general Brownian motion

In this Section we derive the approximating and exact filters of a general Brownian motion W_t with diffusion coefficient a^2 and drift $c \in \mathbb{R}$, with respect to its local time Λ_t.

Remark 6.1. *The results and the proof of Proposition 3.1 are still valid for a general Brownian motion, since also in the general case the process $W^*_s = W_{s+\sigma^n_j} - W_{\sigma^n_j}$ is a Brownian motion independent of $\mathcal{G}^n_{\sigma^n_j}$, with the same law as the process W_s.*
Therefore

$$E_{(a,c)}\left[g(W_t)/\mathcal{G}^n_t\right] = \sum_{j=0}^{\infty} E_{(a,c)}\left[g(-\frac{j}{2^n} + W^*_s)\Big/\Lambda^*_s < \frac{1}{2^n}\right]_{s=t-\sigma^n_j} \mathbb{I}\{\sigma^n_j \le t < \sigma^n_{j+1}\}, \tag{38}$$

where we use the symbol $E_{(a,c)}$ to recall the diffusion and the drift coefficient, and we use the same notations of Section 2.

The above remark and the representation (38) allow us to write the approximating filter as

$$E_{(a,c)}\left[g(-l + W^*_s)/\Lambda^*_s < \frac{1}{2^n}\right]\Big|_{(s,l)=(\zeta^n_t,\Lambda^n_t)} = E_{(a,c)}\left[g(-l + W^*_s)/\Lambda^{*n}_s = 0\right]\Big|_{(s,l)=(\zeta^n_t,\Lambda^n_t)}, \tag{39}$$

where Λ^n_t and ζ^n_t are obtained as in Section 2, and Λ^{*n} is obtained from W^* as Λ^n from W.

It is immediate to see that $c = 0$ corresponds to the case of a standard Brownian motion up to the deterministic time change $s \to a^2 s$. Indeed

$$E_{(a,0)}\left[g(-l + W^*_s)/\Lambda^{*n}_s = 0\right] = E_{(1,0)}\left[g(-l + W^*_{a^2 s})/\Lambda^{*n}_{a^2 s} = 0\right] = \Pi^n(a^2 s, l; g), \tag{40}$$

where W^*_s is a standard Brownian motion under the expectation $E_{(1,0)} = E$, and $\Pi^n(s, l; g)$ is defined in (16). Then by the results of Section 4

$$E_{(a,0)}\left[g(W_t)/\mathcal{G}^n_t\right] = \Pi^n(a^2\zeta^n_t, \Lambda^n_t; g), \tag{41}$$

and

$$\Pi^n(a^2\zeta^n_t, \Lambda^n_t; g) \underset{L^1([0,T]\times\Omega)}{\longrightarrow} \Pi(a^2\zeta_t, \Lambda_t; g), \tag{42}$$

where ζ_t is defined as in (3) and $\Pi^n(s, l; g)$ is defined as in (22), and therefore

$$\Pi(a^2 s, l; g) = E\Big[g(-l + W^*_{a^2 s})/\Lambda^*_{a^2 s} = 0\Big] = \int_0^{\infty} g\left(-l + ya\sqrt{s}\right) y \exp\left(-\frac{y^2}{2}\right) dy \quad s \ge 0. \tag{43}$$

Theorem 6.2. *Let W_t be a Brownian motion with diffusion coefficient a^2 and drift coefficient c. Then the conditional law of W_t given $\mathcal{G}_t^n = \mathcal{F}_t^{\Lambda^n}$ admits the following representation P-a.s.*

$$E_{(a,c)}\big[g(W_t)/\mathcal{G}_t^n\big] = \left.\frac{\Pi^n\big(a^2 s, l; g(\cdot)\exp(\frac{c}{a^2}\cdot)\big)}{\Pi^n\big(a^2 s, l; \exp(\frac{c}{a^2}\cdot)\big)}\right|_{(s,l)=(\zeta_t^n,\Lambda_t^n)} \tag{44}$$

Proof. The proof can be given in two ways. The first one is via Kallianpur-Striebel formula and the exponential martingale $Z_t = \exp\{\frac{c}{a^2}W_t - \frac{1}{2}\frac{c^2}{a^2}t\}$

$$E_{(a,c)}\big[g(W_t)/\mathcal{G}_t^n\big] = \frac{E_{(a,0)}\big[g(W_t)Z_t/\mathcal{G}_t^n\big]}{E_{(a,0)}\big[Z_t/\mathcal{G}_t^n\big]}$$

$$= \frac{E_{(a,0)}\big[g(W_t)\exp\{\frac{c}{a^2}W_t - \frac{1}{2}\frac{c^2}{a^2}t\}/\mathcal{G}_t^n\big]}{E_{(a,0)}\big[\exp\{\frac{c}{a^2}W_t - \frac{1}{2}\frac{c^2}{a^2}t\}/\mathcal{G}_t^n\big]} = \frac{E_{(a,0)}\big[g(W_t)\exp\{\frac{c}{a^2}W_t\}/\mathcal{G}_t^n\big]}{E_{(a,0)}\big[\exp\{\frac{c}{a^2}W_t\}/\mathcal{G}_t^n\big]}.$$

Then we apply (41) to $g(x)\exp(\frac{c}{a^2}x)$ and to $\exp(\frac{c}{a^2}x)$.

On the other hand, an elementary proof is based on the following observations. When W_s is a Brownian motion with diffusion coefficient a^2 and drift c, the joint density $f^{(a,c)}_{W_s,\Lambda_s}(x,k)$ of (W_s,Λ_s) is equal to $f^{(a,c)}_{W_s}(x)\cdot f^{(a,c)}_{\Lambda_s|W_s}(k|x)$, and the conditional density $f^{(a,c)}_{\Lambda_s|W_s}(k|x)$ of Λ_s given W_s does not depend on c (see, for instance, [1]), i.e.

$$f^{(a,c)}_{\Lambda_s|W_s}(k|x) = f^{(a,0)}_{\Lambda_s|W_s}(k|x) = f^{(1,0)}_{\Lambda_{a^2 s}|W_{a^2 s}}(k|x).$$

Then

$$\begin{aligned} f^{(a,c)}_{W_s,\Lambda_s}(x,k) =& f^{(a,c)}_{W_s}(x)\cdot f^{(a,0)}_{\Lambda_s|W_s}(k|x) = \frac{1}{a\sqrt{2\pi s}}\exp\left(-\frac{(x-cs)^2}{2a^2 s}\right)\cdot f^{(a,0)}_{\Lambda_s|W_s}(k|x) = \\ =& f^{(a,0)}_{W_s}(x)\cdot f^{(a,0)}_{\Lambda_s|W_s}(k|x)\cdot\exp\left(-\frac{2csx+c^2s^2}{2a^2 s}\right) = \\ =& f^{(a,0)}_{W_s,\Lambda_s}(x,k)\exp\left(\frac{c}{a^2}x\right)\exp\left(-\frac{c^2 s}{2a^2}\right)\mathbb{I}(x+k>0)\mathbb{I}(k>0). \end{aligned}$$

The general term in (39) can be written as

$$\begin{aligned} E_{(a,c)}\left[g(-l+W_s)\Big/\Lambda_s < \frac{1}{2^n}\right] &= \frac{\int_0^{\frac{1}{2^n}} dk\int_{-k}^{\infty} g(-l+x) f^{(a,0)}_{W_s,\Lambda_s}(x,k)\exp\big(\frac{c}{a^2}x\big)dx}{\int_0^{\frac{1}{2^n}} dk\int_{-k}^{\infty} f^{(a,0)}_{W_s,\Lambda_s}(x,k)\exp\big(\frac{c}{a^2}x\big)dx} \\ &= \frac{\int_0^{\frac{1}{2^n}} dk\int_{-k}^{\infty} g(-l+x) f^{(a,0)}_{W_s,\Lambda_s}(x,k)\exp\big(\frac{c}{a^2}(-l+x)\big)dx}{\int_0^{\frac{1}{2^n}} dk\int_{-k}^{\infty} f^{(a,0)}_{W_s,\Lambda_s}(x,k)\exp\big(\frac{c}{a^2}(-l+x)\big)dx}. \end{aligned}$$

Since $f^{(a,0)}_{W_s,\Lambda_s}(x,k) = f^{(1,0)}_{W_{a^2 s},\Lambda_{a^2 s}(x,k)}$, the thesis follows by (40). □

Following the same lines as in Section 4, and observing that the convergence result for the filter as stated in Theorem 2.4 holds either for standard Brownian motion or for general Brownian motion we can finally state the main result of this Section.

Theorem 6.3. *For all g continuous, with compact support*

$$E_{(a,c)}\big[g(W_t)/\mathcal{G}_t^n\big] \underset{L^1([0,T]\times\Omega)}{\longrightarrow} \frac{\Pi\big(a^2 s, l; g(\cdot)\exp(\frac{c}{a^2}\cdot)\big)}{\Pi\big(a^2 s, l; \exp(\frac{c}{a^2}\cdot)\big)}\Bigg|_{(s,l)=(\zeta_t,\Lambda_t)}.$$

Moreover, for all g measurable and bounded, a cadlag version of $E_{(a,c)}\big[g(W_t)/\mathcal{F}^{\Lambda_t}\big]$ is given by and then by (43),

$$E_{(a,c)}\big[g(W_t)/\mathcal{F}^{\Lambda_t}\big] = \frac{\int_0^\infty g\left(-l+ya\sqrt{s}\right)\exp(\frac{c}{a^2}\left(-l+ya\sqrt{s}\right))y\exp\left(-\frac{y^2}{2}\right)dy}{\int_0^\infty \exp(\frac{c}{a^2}\left(-l+ya\sqrt{s}\right))y\exp\left(-\frac{y^2}{2}\right)dy}\Bigg|_{(s,l)=(\zeta_t,\Lambda_t)}. \quad (45)$$

Before giving the proof, we note that the above expression (45) can be obtained directly from Theorem 4.4, via the Kallianpur-Striebel formula and a deterministic time change as in the case of the conditional law given $\mathcal{G}_n$.

Proof. Using the representation (44) instead of (17), the proof of Theorem 4.4 can be repeated verbatim for a general Brownian motion, since Lemma 4.3 still holds true, that is
1. $\zeta_t^n(\omega) \searrow \zeta_t(\omega)$ $P_{(a,c)} - a.s.$,
2. the set $\{(t,\omega) \in [0,T]\times\Omega$ such that $\zeta_t(\omega) = 0\}$ is a zero measure set with respect to the measure $dt \times dP_{(a,c)}$.
Indeed, by Girsanov theorem, there exists a probability measure $P^{(a,c)}$, equivalent to $P_{(a,c)}$, and such that W has the same law as a Brownian motion with diffusion coefficient a^2 and zero drift under $P^{(a,c)}$.
Then the previous properties **1.** and **2.** hold w.r.t to $P^{(a,c)}$ instead of $P_{(a,c)}$, and the equivalence between the two probability measures guarantees that **1.** and **2.** hold.

□

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References

[1] Asmussen, S., Glynn, P., and Pitman, J. Discretization error in simulation of one-dimensional reflecting Brownian motion. *Ann. Appl. Probab. 5*, 4 (1995), 875–896.

[2] Azéma, J., and Yor, M. Étude d'une martingale remarquable. In *Séminaire de Probabilités, XXIII*, Azéma, Meyer, Yor, Ed., Lecture Notes in Mathematics n. 1372. Springer, Berlin, 1989, pp. 88–130.

[3] Bramson, M., and Dai, J. G. Heavy traffic limits for some queueing networks. *Ann. Appl. Probab. 11* (2001), 49–90.

[4] Brémaud, P. *Point Process and Queues.* Springer, New York, 1981.

[5] Elliott, R. J. *Stochastic calculus and Applications.* Springer-Verlag, 1982.

[6] Harrison, J. M. *Brownian motion and stochastic flow systems.* J. Wiley & Sons, 1985.

[7] Harrison, J. M., and Williams, R. J. On the quasireversibility of a multiclass brownian service station. *Ann. Probab. 18* (1990), 1249–1268.

[8] Harrison, J. M., and Williams, R. J. Brownian models of feedforward queueing networks: Quasireversibility and product form solutions. *Ann. Appl. Probab. 2* (1992), 263–293.

[9] Joannides, M., and Le Gland, F. Nonlinear filtering with perfect discrete time observations. In *Proocedings of the 34-th conference on Decision and Control* (December 1995), IEEE-CSS, pp. 4012–4017.

[10] Joannides, M., and Le Gland, F. Small noise asymptotics of the bayesian estimator in nonidentifiable models. *Statistical Inference for Stochastic Processes 1* (2002), 95–130.

[11] Kallianpur, G. *Stochastic Filtering Theory.* Springer-Verlag, 1980.

[12] Karatzas, I., and Shreve, S. E. *Brownian Motion and Stochastic Calculus.* Springer, New York, 1988.

[13] Korezlioglu, H., and Runggaldier, W. J. Filtering for nonlinear systems driven by nonwhite noises: an approximation scheme. *Stoch. and Stoch Rep. 44* (1993), 65–102.

[14] Protter, P. *Stochastic integration and Differential Equations: A New Approach.* Springer, Berlin, 1990.