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HETEROGENEOUS BELIEFS, STOCHASTIC VOLATILITY AND VOLATILITY SKEW

Fabio Massimo Marchetti[a] , Giovanna Nappo[a]
and Gianluca Vagnani[b]

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The paper represents an initial effort to unfold some the determinants of the implied volatility skew in financial (derivative) markets. We propose a simple explanation of the implied volatility asymmetrical behavior, both by means of mathematical derivations and empirical analysis. In particular, traders' use of the widely accepted Black–Scholes formula with a measure of stock volatility that they calculate accordingly to their subjective and heterogeneous beliefs is a determinant of the implied volatility skew. The analysis provides also new characterizations of the behavior of the equilibrium option price that holds promise for practical modeling and forecasting.

1. Introduction

Our study deals with the long-standing problem of pricing financial options and, specifically, with the phenomenon of the asymmetry of the implied volatility curve. Using the Black–Scholes (1973) option pricing model, if we plot the implied volatility as a function of the exercise price, we should obtain a horizontal straight line. This implies that all options for buying or selling the same underlying asset with the same expiration date, but with different exercise prices, should have the same implied volatility. This is not, however, what occurs in practice in option markets worldwide. Because out-of-the-money (or in-the-money) near-maturity calls are overpriced relative to at-the-money middle-maturity calls on the same underlying stock (e.g. Rubinstein, M. (1985)) the implied volatility tends to present a strong U-shaped pattern (e.g., Black, F. (1975)). Moreover, in some markets (e.g., financial options written on index), low-strike implied volatilities are different from high-strike implied volatilities, with the resulting implied volatility curve that has a skewed U-shape (e.g., Cont, R. and da Fonseca, J. (2002)).

We offer a simple and an elementary explanation of the emergence of the volatility skew. In particular, we argue that this phenomenon might be related to the behavior of heterogenous traders who—equipped with the widely accepted formula of Black, F. and Scholes, M. (1973)—try to resolve the uncertainty about future underlying stock volatility by relying on their subjective beliefs.

[a]Department of Mathematics - Sapienza, University of Rome, 2 Aldo Moro square, 00185 Roma, Italy
marchetti@mat.uniroma1.it
nappo@mat.uniroma1.it
[b]Department of Management and Technologies - Sapienza, University of Rome, 9 Castro Laurenziano street - 00161 Roma, Italy
gianluca.vagnani@uniroma1.it

A model that incorporate heterogeneous beliefs and traders' habits in pricing options is therefore proposed. The model is a variant of the security market proposed by Hellwig, M.F. (1980) (see also Admati, A.R. (1985)). Note that on a similar version of our model is also based the Santa Fe Institute Artificial Stock Market model (e.g., Arthur, B. et. al (1997); Palmer, R. G. et. al (1994)). In the proposed model there are N traders with identical endowments and decision criterion (i.e., namely mean variance criterion with constant absolute risk aversion) but with heterogeneous beliefs. Traders' problem is to maximize their wealth by allocating their resources between a riskless and risky assets (the underlying stock or the option). To solve this problem contributes sets of traders' specific beliefs as well as practices, rules of thumb of calculating and technical systems to which traders are accustomed to and are used for pricing options.

Our model directly relates to research on the volatility smile and skew. In an attempt to incorporate stochastic volatility processes, several models have been proposed (for a review see Broadie, M. and Detemple, J. B. (2004)). Among them, we refer to the stochastic volatility models (Hull, J. and White, A. (1987)), the general equilibrium stochastic volatility models (Detemple, J. and Osakwe, C. (2000)), the pure jump-diffusion models (Merton, R. (1976)), the affine jump-diffusion models (Duffie, D. et. al (2000)). In particular, it has been demonstrated that stochastic volatility models exhibit a symmetric smile locally centered on the current forward price, as originally proposed in the seminal work of Renault, E. and Touzi, N. (1996). A simpler proof can also be found in Sircar, R. and Papanicolaou, G. (1999). The authors show the result by Renault, E. and Touzi, N. (1996) noting that if the stochastic volatility stems from a diffusion process driven by a Brownian motion uncorrelated with the Brownian motion driving the stock price, and the option price is computed by means of the so-called Hull and White formula (which is a mixture of Black–Scholes prices), then the implied volatility is locally convex and symmetric around the current forward price.

Unable to account for the case wherein the implied volatility curve presents a skewed U-shape, stochastic volatility models have been enriched by introducing a more articulated and complex form of the volatility. For example, Brigo, D. and Mercurio, F. (2000) propose a Markovian model in which the equilibrium option price is derived as convex combination of option prices, where each price is determined by a stochastic volatility option pricing model. In a prominent contribution, Fouque, J. P. et. al (2000) propose a model in which the volatility is correlated with the stock price. By doing so, the authors are able to construct a model that leads to skews in the form of the implied volatility. Moreover, Barndorff-Nielsen, O.E. and Shephard, N. (2001) introduce a Non-Gaussian Ornstein-Uhlenbeck-Based Model both to capture important distributional deviations of asset prices from Gaussianity and to incorporate the skewness shape of the implied volatility curve into stochastic volatility models.

Although we recognize the undisputed contribution of these studies, we must observe that the problem of finding a risk-neutral distribution that consistently prices all quoted options is largely undetermined (Brigo, D. and Mercurio, F.

(2000)). In addition, without a formula for pricing options under a particular stochastic volatility model, estimating the risk-neutral parameters is computationally intensive. This estimate is more complicated when correlations between the stock prices and the volatility are considered (Fouque, J. P. et. al (2000)). Therefore, the use of such additional assumptions often makes models intractable and even induces parameter estimates that are unstable and with no guarantee of uniqueness of a solution. Differently, we propose a simple model that incorporates the implied volatility skew without any explicit assumption on the structure of the volatility or on the existence of a correlation between the volatility and the stock price.

Moreover, we must observe that recent studies have pointed out the importance of traders' subjective and heterogenous beliefs for equilibrium prices and the emergence of the implied volatility skew in financial (derivative) markets. In particular, Ziegler, A. (2002) observed that under heterogeneous beliefs about the true mean of the constant instantaneous increase in expected dividends (which is another parameter of the Black–Scholes option pricing model), the state-price density function is not log-normal and the implied volatility curve becomes steeper and tends to assume a skewed U-shape. In the same perspective, Buraschi, A. and Jiltsov, A. (2006) propose a model in which traders face model uncertainty and have different dividend growth rate beliefs. Accordingly, they show that information heterogeneity can explain the smile better than can reduced-form models with stochastic volatility (see also Amin, K. and Lee C.M. (1997)). Our approach is related to theirs in that, like them, we consider heterogeneous traders that face uncertainty on some relevant parameters for option pricing. However, we differ from their approach because we do not pose any assumption on the shape of the state-price density function, or of the underlying stock-volatility density function, and we consider the underlying stock volatility as the sole source of uncertainty.

Still in an uncertain world, it is also well known that in pricing options institutions matter. On the one hand, borrowing from Callon, M. (1998) and MacKenzie, D. and Millo, Y. (2003), the Black–Scholes model (and each option pricing model, in general) does not describe an existing external financial market, but rather brings that market into being: the Black–Scholes model performs the functions of the financial market, creating an artificial state of affairs in which there is a substantive orientation, effectively enforced by the provision of an order, on the way actors behave. Therefore, the Black–Scholes model performs a sort of regulation of the market itself, with traders adapting themselves to it. As observed by Sircar, R. and Papanicolaou, G. (1999) "there is an almost perverse pressure by practitioners for real prices to conform to those of the BlackScholes world as demonstrated by prevalent use of the implied BlackScholes volatility measure on market prices" (p. 108). On the other hand, as Arthur, B. (1994) has observed, in coping with uncertainty, individuals also extensively rely on different rules of thumb for resolving their decision-making problems. The use of such rules affects not only the behavior of the individual but also fuels patterns of emergent phenomena at the market level and determines the characteristics of the equilibrium steady state (see Brock, W.A. and Hommes, C.H. (1997)). Following such suggestions, our model then directly investigates the implications for the emergence of the implied volatility skew—which represent the objective of this study—of institutionalized norms and technical devises (i.e., the Black-Scholes option pricing model) as well as rules of thumb (i.e., averaging) followed by traders in pricing options. Our attempt is to offer an alternative explanation for the emergence of the implied volatility skew by introducing a type of selective behavior and by relating it to certain concepts developed within the behavioral theory of human decision making. Note that this attempt is accomplished without rendering its subsequent analysis more difficult. Given the simplicity of the procedure to estimate its parameters (i.e., it is not needed to estimate the markets price of risk, or to use proxies to estimate traders' beliefs and their differences), our model can be expected also to have important implications for asset pricing in general (see the end of Section 4).

The paper is organized as follows. Section 2 depicts the way traders form their expectations about future options prices and, accordingly, use these expectations to buy or sell options. Section 3 identifies the determinants of the implied volatility skew and contains the mathematical results of our study. In this section we relate directly the emergence of the implied volatility skew to the way traders price options, as proposed in Section 2. Section 4 is devoted to the calibration of our model with empirical data on S&P 500 index options. Section 5 concludes. Details and proofs are in Appendix A.

2. THE MODEL

In our model, J European call options are traded with the time to maturity T and the strike price $K_j > 0$, where j belongs to a finite set J. All call options are written on only one underlying risky asset, say $S = \{S_t\}_t$, with $S_0 = s$; we denote by $\widetilde{C}(K) = \{\widetilde{C}_t(K)\}_t$ the market equilibrium price of the European call option with strike price K. Our model is completed by considering the safe asset B, paying a fixed real dividend r, the riskless rate.

At time zero every trader has to choose her strategy. Let $\tau \in (0, T]$ be the next time at which every trader may choose to change her strategy. In particular, a buyer (at time zero) may choose to sell her holdings at time τ (similarly a seller may buy options)[1]. As in Hellwig, M.F. (1980), we also introduce ω_0^i as the i trader's wealth at initial time zero, for $i = 1, ..., N$, where N is the number of traders that deal with the call option with the strike price K. Given $\lambda^i(K)$ the

[1]When τ is small the model represents traders that buy (or sell) call options in order to sell (or buy) them in short period of time. In a sense it describes the situation in which traders are speculators. Instead, the model with $\tau = T$ can be seen as a model for traders that do not intend to sell call options until their exercise time. In a sense the case $\tau = T$ describes the situation in which traders buy call options only for hedging purposes.

amount of the European call option, with strike price $K > 0$, that the agent i holds in the period $[0, \tau)$ and $\boldsymbol{\lambda} = (\lambda(K_j),\, j \in J)$, we denote the i-th trader's wealth at time τ by

$$\omega_\tau^i(\boldsymbol{\lambda}) = \sum_{j\in J} \lambda^i(K_j) \cdot \widetilde{C}_\tau(K_j) + e^{r\tau}[\omega_0^i - \sum_{j\in J} \lambda^i(K_j) \cdot \widetilde{C}_0(K_j)].$$

As in the Santa Fe Institute Artificial Stock Market model (Ehrentreich, N. (2006)), our traders aim is to maximize:

$$\widehat{\mathbb{E}}_0^i[\omega_\tau^i(\boldsymbol{\lambda})] - \gamma^i \widehat{\mathrm{Var}}_0^i(\omega_\tau^i(\boldsymbol{\lambda})), \tag{1}$$

where $\widehat{\mathbb{E}}_t^i$ denotes the expectation of trader i conditional on information up to time t, γ^i is the coefficient of absolute risk aversion, and, for a given random variable X, $\widehat{\mathrm{Var}}_t^i(X)$ is the variance of X with respect to the expectation[2] $\widehat{\mathbb{E}}_t^i$.

To see the analogy with the Santa Fe Artificial market model, we recall that the mean variance criterion (1) corresponds, in the Gaussian case, to maximize a utility function like $U(x) = -e^{-2\gamma^i x}$. (see, e.g., Freund, R. (1956)).

In our simplified model, each trader invests all her limited amount of money in one option, i.e. trader i deals just with the call option with strike price K_j for a $j = j(i)$. In other words, we have that $\lambda^i(K_j) = 0$ except for $j = j(i)$. Therefore, maximizing (1) becomes equivalent to maximizing, for $K = K_{j(i)}$:

$$\lambda^i(K)\big[\widehat{C}_\tau^i(K) - e^{r\tau}\widetilde{C}_0(K)\big] - \gamma^i \cdot \big[\lambda^i(K)\big]^2 \cdot \delta_\tau^i(K) \tag{2}$$

where $\widehat{C}_\tau^i(K) := \widehat{\mathbb{E}}_0^i[\widetilde{C}_\tau(K)]$ and $\delta_\tau^i(K) := \widehat{\mathrm{Var}}_0^i(\widetilde{C}_\tau(K))$ is the variance of $\widetilde{C}_\tau(K)$, according to the distribution of the trader i.

The maximum in (2) is achieved in[3]

$$\lambda_*^i(K) = \frac{1}{2\gamma^i}\,\frac{\widehat{C}_\tau^i(K) \,-\, e^{r\tau}\widetilde{C}_0(K)}{\delta_\tau^i(K)} = \frac{e^{r\tau}}{2\gamma^i}\,\frac{\overline{C}_\tau^i(K) \,-\, \widetilde{C}_0(K)}{\delta_\tau^i(K)},$$

where

$$\overline{C}_\tau^i(K) := e^{-r\tau}\widehat{C}_\tau^i(K).$$

We now specify the heterogeneity of traders in our model. We assume the simplest case of a market populated by just two types of traders. In other words the set of traders is divided into two disjoint sets A and B, and for each $i \in \{1, \dots, N\}$, the expectation $\widehat{\mathbb{E}}_t^i$ and the risk aversion coefficient γ^i are equal either to $\mathbb{E}_{A,t}$ and γ_A, or to $\mathbb{E}_{B,t}$ and γ_B, depending on whether $i \in A$, or

[2] Note that under this assumption traders' demand for the risky asset is independent of his initial wealth (Pratt, J. (1964)).

[3] In the same context of Hellwig, M.F. (1980), where only Gaussian random variables are considered, the expression for $\lambda_*^i(K)$ is similar to the expression of $\overline{z}_i$ at page 481 (see also Grossman, S. (1976), p.575 f.).

$i \in B$. Similarly we have that either $\widehat{C}^i_\tau(K) = \widehat{C}_{A,\tau}(K)$ and $\delta^i_\tau(K) = \delta_{A,\tau}(K)$ or $\widehat{C}^i_\tau(K) = \widehat{C}_{B,\tau}(K)$ and $\delta^i_\tau(K) = \delta_{B,\tau}(K)$ where, for $I = A,\, B$

$$\widehat{C}_{I,\tau}(K) = \mathbb{E}_{I,0}[\widetilde{C}_\tau(K)] \text{ and } \delta_{I,\tau}(K) = \mathbb{E}_{I,0}\Big[\big(\widetilde{C}_\tau(K) - \mathbb{E}_{I,0}[\widetilde{C}_\tau(K)]\big)^2\Big], \tag{3}$$

and, similarly,

$$\overline{C}_{I,\tau}(K) = e^{-r\tau}\widehat{C}_{I,\tau}(K). \tag{4}$$

In this context, we give the following definition.

Definition 2.1 An equilibrium is a collection of prices $(\widetilde{C}_\tau(K))$, called market equilibrium prices, along with individual portfolio choices $(\lambda^i_*(K))$ such that:

1. Given the market equilibrium prices, portfolio choices maximize (1) subject to the budget constraints.
2. The financial market clears, that is,

$$\sum_{i=1}^N \lambda^i_*(K) = 0, \tag{5}$$

for every $K \in \{K_j,\, j \in J\}$.

In the attempt to compute the equilibrium, equation (5) simply means that for each strike price K, the number of sold call options equals the number of bought call options[4]. Indeed, if $N_A(K)$ and $N_B(K)$ denote the number of traders dealing with the call option with strike price K and belonging respectively to class A or B, then (5) can be rewritten as

$$N_A(K)\,\frac{\overline{C}_{A,\tau}(K) - \widetilde{C}_0(K)}{e^{-r\tau}\,2\gamma_A\,\delta_{A,\tau}(K)} + N_B(K)\,\frac{\overline{C}_{B,\tau}(K) - \widetilde{C}_0(K)}{e^{-r\tau}\,2\gamma_B\,\delta_{B,\tau}(K)} = 0. \tag{6}$$

Note that, unless $\overline{C}_{A,\tau}(K_j) = \overline{C}_{B,\tau}(K_j) = \widetilde{C}_0(K_j)$, the above equality implies that either $\overline{C}_{A,\tau}(K_j) > \widetilde{C}_0(K_j)$ and $\overline{C}_{B,\tau}(K_j) < \widetilde{C}_0(K_j)$, or, viceversa, $\overline{C}_{A,\tau}(K_j) < \widetilde{C}_0(K_j)$ and $\overline{C}_{B,\tau}(K_j) > \widetilde{C}_0(K_j)$. In the first case, traders of type A are on the buyer side and traders of type B are on the seller side, while the roles are inverted in the second case.

As a final condition on the market structure, we assume that $N_A(K) = N_B(K)$ for $K = K_j$. The latter condition holds if, in our market, each transaction is

[4]Note that, in Hellwig, M.F. (1980), the market clearing condition says that the sum of the optimal asset demands $\overline{z}_i$ is a random variable (see also footnote 3), and the market price depends also on the total demand, i.e. possible excesses of demand or supply. In our model equilibrium prices, and in turn the emergence of the implied volatility skew, are related directly to traders beliefs about future prices.

concluded by one trader of class A with one trader of class B. Under this final condition we obtain that, for $K = K_j$:

$$\widetilde{C}_0(K)\left(\frac{1}{\gamma_A\delta_{A,\tau}(K)} + \frac{1}{\gamma_B\delta_{B,\tau}(K)}\right) = \frac{\overline{C}_{A,\tau}(K)}{\gamma_A\delta_{A,\tau}(K)} + \frac{\overline{C}_{B,\tau}(K)}{\gamma_B\delta_{B,\tau}(K)}$$

or, equivalently, that

$$\widetilde{C}_0(K) = \eta_{A,\tau}(K)\,\overline{C}_{A,\tau}(K) + \eta_{B,\tau}(K)\,\overline{C}_{B,\tau}(K), \tag{7}$$

with

$$\eta_{A,\tau}(K) = 1 - \eta_{B,\tau}(K), \qquad \eta_{B,\tau}(K) = \frac{1}{\frac{\gamma_B\delta_{B,\tau}(K)}{\gamma_A\delta_{A,\tau}(K)} + 1}, \tag{8}$$

so that $\eta_{A,\tau}(K)$, $\eta_{B,\tau}(K) \geq 0$.

Therefore, at the initial time, the market equilibrium price $\widetilde{C}_0(K)$ of the call with strike K is a linear convex combination of the two discounted expected prices at time τ. We stress that the convex combination changes with the option strike price. The latter fact will be crucial in the analysis of the implied volatility skew.

Remark 2.2 *In the next section, aiming to show that the volatility skew is only due to different beliefs, we will also assume that $\gamma_A = \gamma_B$. It must be also observed that removing this condition does not affect formally the results of the following section (see Remark 3.2).*

Up to now we have described the structure of our market model. To complete the description we need to specify how traders compute their expected option prices (i.e., the evaluation model). First of all, we need to define how traders form their beliefs on the underlying stock volatility $\widetilde{C}_\tau(K)$, and then how they compute its expected value and its variance. In other words, we need to specify $\mathbb{E}_{I,t}$, and then compute $\widehat{C}_{I,\tau}$ and $\delta_{I,\tau}(K)$. Recognizing that there are many possible models traders can use to price options, we adopt the basic Black–Scholes option pricing model, where all agents believe that the underlying asset price follows a geometric Brownian motion, with constant stochastic volatility,

$$S_t = s\,\exp\{(r - \tfrac{\Sigma^2}{2})\,t + \Sigma\,W_t\},$$

with $\{W_t\}_t$ a Brownian motion, independent of the stochastic volatility Σ. Traders' beliefs are then summarized in the distribution of Σ. We assume[5] that, for traders in A, Σ has distribution F_A, while, for traders in B, Σ has distribution F_B.

[5]From a mathematical point of view we are assuming that the Wiener process is defined in the probability space $(\Omega', \mathcal{F}', \mathbf{P})$, that the random variable Σ is defined in the probability space $(\Omega'', \mathcal{F}'', \mathbf{Q}_A)$, and has distribution F_A, that $\Omega = \Omega' \times \Omega''$, and that $\mathbb{P}_A = \mathbf{P} \times \mathbf{Q}_A$. A similar definition holds for $\mathbb{P}_B$. To be consistent with the previous notations we will use the symbol $\mathbb{E}_{I,0}$ to denote the expectation with respect to $\mathbb{P}_I$ conditioned on the information up to time 0 (in particular the value of S_0 is assumed to be known).

The traders believe that, if Σ were known, then the equilibrium market price at time τ could be computed according to the well known Black–Scholes formula – i.e.

$$\widetilde{C}_\tau(K) = C^{BS}(S_\tau, r, T-\tau, K, \Sigma), \tag{9}$$

where

$$C^{BS}(x, r, t, K, v) = x\left[\Phi\left(\frac{l(K)}{v\sqrt{t}} + \frac{v\sqrt{t}}{2}\right) - e^{-l(K)}\Phi\left(\frac{l(K)}{v\sqrt{t}} - \frac{v\sqrt{t}}{2}\right)\right], \tag{10}$$

with Φ the standard normal distribution function, and $l(K)$ is the log-moneyness (which depends also on x, t, and r)

$$l(K) = l(x, t, r, K) = \ln\left(\frac{xe^{rt}}{K}\right).$$

In view of (9), holding the assumptions on the underlying asset evolution, we have that, for any I (either A or B)

$$\begin{aligned}
\widetilde{C}_\tau(K) &= \mathbb{E}_{I,0}\big[(S_T - K)^+ e^{-r(T-\tau)}\big|\mathcal{F}^S_\tau \vee \sigma(\Sigma)\big] \\
&= \mathbb{E}_{I,0}\Big[\big(x\, e^{(r-\frac{v^2}{2})\,(T-\tau)+v\,(W_T - W_\tau)} - K\big)^+ e^{-r(T-\tau)}\Big]\Big|_{\substack{v=\Sigma \\ x=S_\tau}} \\
&= \mathbf{E}\Big[\big(x\, e^{(r-\frac{v^2}{2})\,(T-\tau)+v\,(W_T - W_\tau)} - K\big)^+ e^{-r(T-\tau)}\Big]\Big|_{\substack{v=\Sigma \\ x=S_\tau}}
\end{aligned} \tag{11}$$

where $\mathbf{E}$ denotes the expectation with respect to the measure $\mathbf{P}$ (see footnote 5), under which $\{W_t\}_t$ is a Wiener process.

Thus, our evaluation model describes the behavior of traders who rely on the Black–Scholes formula to price options and use a simple but reasonable rule of thumb (averaging) to resolve the uncertainty around the underlying stock volatility[6]. Furthermore (see Lemma 2.3) we have also that $\overline{C}_{I,\tau}(K)$ is constant with respect to τ, with $\overline{C}_{I,\tau}(K) = \overline{C}_{I,0}(K) = \widehat{C}_{I,0}(K)$, for $\tau \in [0, T]$. In the sequel we will denote this constant value simply by $\widehat{C}_I(K)$.

Lemma 2.3 *Under the above assumptions on the evaluation model for traders in class A and B, the value of $\overline{C}_{A,\tau}(K)$ is constant with respect to τ and its value is*

$$\overline{C}_{A,\tau}(K) = \widehat{C}_A(K) := \int_0^\infty C^{BS}(s, r, T, K, v) F_A\{dv\}. \tag{12}$$

Analogous relations hold by replacing A with B.

[6]Note that our model allows different specifications for the equilibrium price, as expected by traders. Using such different specifications for the equilibrium price and exploring its impact on the emergence of the implied volatility skew represents an important issue and offers avenue for future enquiries.

Lemma 2.3 states that traders believe that the expectation of the discounted equilibrium price is a mixture of Black–Scholes prices.

The value of $\delta_{I,\tau}(K)$ can be computed with similar techniques, but, contrarily to $\overline{C}_{I,\tau}(K)$, its value depends on τ (see Lemma 2.4). Then, through (8), the same holds for the equilibrium price $\widetilde{C}_0(K)$ and for the implied volatility. Furthermore, the presence of the skew appears for every value of τ, although the degree of the skew may depend on the value of τ (see Theorem 3.4). We close this section with the announced result on $\delta_{I,\tau}(K)$:

Lemma 2.4 *Under the above assumptions on the evaluation model for traders in class A and B, the value of $\delta_{A,\tau}(K)$ is given by*

$$\delta_{A,\tau}(K) = \mathbb{E}_{A,0}\left[\left(C^{BS}(S_\tau, r, T-\tau, K, \Sigma)\right)^2\right] - e^{2r\tau}\left(\widehat{C}_A(K)\right)^2. \tag{13}$$

When $\tau = T$, it turns out that

$$\delta_{A,T}(K) = Var_{A,0}\big((S_T - K)^+\big). \tag{14}$$

Furthermore, setting

$$C_2^{BS}(x, r, t, K, v) := \mathbf{E}\Big[\big(\big(x\,\exp\{(r - \tfrac{v^2}{2})\,t + v\,W_t\} - K\big)^+\big)^2 e^{-2rt}\Big], \tag{15}$$

the following relation holds:

$$\begin{aligned}\delta_{A,\tau}(K) &= e^{-2r(T-\tau)}\,\delta_{A,T}(K)\\ &\quad - \mathbb{E}_{A,0}\left[C_2^{BS}(S_\tau, r, T-\tau, K, \Sigma) - \left(C^{BS}(S_\tau, r, T-\tau, K, \Sigma)\right)^2\right].\end{aligned} \tag{16}$$

Similar relations hold for A replaced by B.

For the proof see Appendix A.

Remark 2.5 *The function $C_2^{BS}(x, r, t, K, v)$, given by (15), can be expressed analytically by*

$$\begin{aligned}C_2^{BS}(x, r, t, K, v) = x^2 \Bigg[&e^{v^2\,t}\,\Phi\Big(\tfrac{l(K)}{v\,\sqrt{t}} + \tfrac{3\,v\sqrt{t}}{2}\Big) - 2\,e^{-l(K)}\,\Phi\Big(\tfrac{l(K)}{v\,\sqrt{t}} + \tfrac{v\sqrt{t}}{2}\Big)\\ &+ e^{-2l(K)}\Phi\Big(\tfrac{l(K)}{v\,\sqrt{t}} - \tfrac{v\sqrt{t}}{2}\Big)\Bigg].\end{aligned} \tag{17}$$

Details can be found in Appendix A (Remark A.1).

3. THE VOLATILITY SKEW

Given our model, we relate the emergence of the implied volatility skew to the ways traders are accustomed to price options. To this aim, we study the behavior of the market price $\widetilde{C}_0(K)$ and of its implied volatility near to $K_{at} := se^{rT}$. Therefore, we consider K as a continuous variable in a neighborhood of K_{at} and, taking into account (12), express (7) as function of K,

$$\widetilde{C}_0(K) = \eta_{A,\tau}(K)\,\widehat{C}_A(K) + \eta_{B,\tau}(K)\,\widehat{C}_B(K), \qquad K > 0. \tag{18}$$

To simplify the notations, from now on we will drop the dependence on τ, unless noted otherwise, and furthermore we assume that $\gamma_A = \gamma_B$.

Then we will write (18) and (8) as follows:

$$\widetilde{C}_0(K) = \eta_A(K)\,\widehat{C}_A(K) + \eta_B(K)\,\widehat{C}_B(K), \qquad K > 0, \tag{19}$$

with

$$\eta_A(K) = 1 - \eta_B(K), \qquad \eta_B(K) = \frac{1}{\frac{\delta_B(K)}{\delta_A(K)} + 1}. \tag{20}$$

Furthermore, to better analyze the emergence of the implied volatility skew it is more convenient to use the log-moneyness instead of the strike price, i.e.

$$l = l(K) = \ln\left(\frac{K_{at}}{K}\right),$$

and to consider each of the above functions evaluated in the inverse log-moneyness function, i.e., evaluated in

$$K = K(l) = se^{rT}e^{-l}. \tag{21}$$

We will denote by $\widetilde{\sigma}(K(l))$ the implied volatility of $\widetilde{C}_0(K(l))$. In order to further simplify the notations, we will drop the symbol $K(l)$ and use mainly the shorthand notations $\widetilde{C}_0(l)$, $\widehat{C}_I(l)$, $\delta_I(l)$, $\eta_I(l)$ instead of $\widetilde{C}_0(K(l))$, $\widehat{C}_I(K(l))$, $\delta_I(K(l))$, $\eta_I(K(l))$, $I = A, B$, respectively, though is an abuse of notation. The reader should take care of this fact.

We are now ready to prove how heterogeneity of trader preferences and the way traders are accustomed to price options, given our market structure, may explain the emergence of the implied volatility skew. Namely, we show that the implied volatility, as a function of the log-moneyness, can be written as follows

$$\widetilde{\sigma}\big(K(l)\big) = f(l^2) + \alpha\, l + o(l),$$

where f is an opportune increasing function of the log-moneyness squared and α is a constant that can be computed whenever the distribution functions F_A, F_B are known (see Theorem 3.4).

PROPOSITION 3.1 *In our market model, with the above short notations, let* $\widetilde{C}_0(l)$ *be the market price of a call option with log-moneyness* l*, i.e. with strike price* $K = K(l)$*. Assume that the functions* $\delta_I(l)$ *admit first continuous derivatives* $\delta_I'(l) = \frac{d}{dl}\delta_I(l)$*. Then, in a neighborhood of* $l = 0$*, we have that*

$$\widetilde{C}_0(l) = \widehat{C}(l) + l\psi(A, B) + o(l), \tag{22}$$

where

$$\widehat{C}(l) := \eta_A(0)\widehat{C}_A(l) + \eta_B(0)\widehat{C}_B(l), \tag{23}$$

and

$$\psi(A, B) := \left[\frac{\delta_B'(0)}{\delta_B(0)} - \frac{\delta_A'(0)}{\delta_A(0)}\right] \eta_A(0)\, \eta_B(0)\big(\widehat{C}_A(0) - \widehat{C}_B(0)\big). \tag{24}$$

For the proof see Appendix A.

REMARK 3.2 *When* $\gamma_A \neq \gamma_B$*, then we have to substitute* $\delta_{\tau,I}$ *with* $\gamma_I\, \delta_{\tau,I}$*. This would not affect formally expression* (24)*. Indeed, since*

$$\frac{(\gamma_I \delta_I)'(0)}{\gamma_I \delta_I(0)} = \frac{\delta_I'(0)}{\delta_I(0)}, \qquad \textit{for } I = A,\, B,$$

then (24) *still holds, but of course* $\eta_B(0)$ *(and therefore* $\eta_A(0)$*) depends on the ratio* γ_A/γ_B*:* $\eta_B(l) = 1/\big(\frac{\gamma_B}{\gamma_A}\frac{\delta_B(l)}{\delta_A(l)} + 1\big)$ *(see* (8)*).*

Proposition 3.1 states that in a neighborhood of $l = 0$, the first order approximation of the equilibrium market option price $\widetilde{C}(l)$ is given by a mixture of the traders' expected option prices plus a linear component. Note, that this result is valid regardless of the evaluation model used. When the functions $\delta_I(l)$ admit also second derivatives, then it is possible to get the second order approximation of $\widetilde{C}(l)$ (see Remark A.2 in Appendix A).

When traders use the Black–Scholes option pricing model with constant stochastic volatility, the expression (23) can be interpreted as a mixture of Black–Scholes prices. Indeed, by (12) and its analogous with B, we can state that

$$\widehat{C}(l) = \mathbb{E}[C^{BS}(s, r, T, K(l), \Sigma)] = \int_0^\infty C^{BS}(s, r, T, K(l), v)dF_\Sigma(v), \tag{25}$$

where the expectation is taken with respect to the probability measure

$$\mathbb{P}_0 = \eta_A(0)\mathbb{P}_{A,0} + \eta_B(0)\mathbb{P}_{B,0},$$

which is a mixture of the probability measures $\mathbb{P}_{A,0}$ and $\mathbb{P}_{B,0}$, and similarly F_Σ is a mixture of F_A and F_B, i.e.

$$F_\Sigma(v) = \eta_A(0)\, F_A(v) + \eta_B(0)\, F_B(v). \tag{26}$$

We can say that the corresponding call price is evaluated through (12), with F_Σ instead of F_A.

We recall that the implied volatility function is the function $\widetilde{\sigma}(K)$ such that

$$\widetilde{C}_0(K) = C^{BS}(s, r, T, K, \widetilde{\sigma}(K)).$$

It is well known that for s, r, T fixed, and for every strike price K, the function

$$h_K : v \mapsto h_K(v) := C^{BS}(s, r, T, K, v) \tag{27}$$

is a strictly increasing continuous function and therefore invertible, so the implied volatility is equivalently defined by $\widetilde{\sigma}(K) = h_K^{-1}(\widetilde{C}_0(K))$.

To the price $\widehat{C}(l(K))$ is attached an implied volatility that can be expressed as $\widehat{\sigma}(K) = h_K^{-1}\big(\widehat{C}(l(K))\big)$. By the result of the seminal work of Renault, E. and Touzi, N. (1996), in the stochastic volatility model it is well known that $\widehat{\sigma}(K)$ is a function of $|l(K)|$, and is increasing (decreasing) for $K > K_{at}$ ($K < K_{at}$), so that the minimum point of the implied volatility $\widehat{\sigma}(K)$ is the value $K = K_{at} := se^{rT}$. We recast their results in our context and in the following Theorem 3.3, we characterize $\widehat{\sigma}(K)$ as a strictly increasing continuous function of $l^2(K)$. In addition we also give some further properties that we will use in the sequel.

THEOREM 3.3 *Assume that (25) holds, with Σ a random variable with distribution function F_Σ. Then the implied volatility is a C^1-function and there exists an increasing function $f = f_{F_\Sigma}$ such that*

$$\widehat{\sigma}(K) = f\big(l^2(K)\big). \tag{28}$$

Furthermore if Σ is a non degenerate random variable, then the function f is strictly increasing. Finally, if all the moments of $1/\Sigma$ are finite, then f is C^∞-function.

A proof of Theorem 3.3 can be obtained by using the properties of associative means as proposed in Marchetti et al. (2007) (see also Vagnani, G. (2009) for a sketch of the proof), and the result in Renault, E. and Touzi, N. (1996) can be recovered by taking Σ^2 as the time average of the stochastic volatility process. In the latter paper, in order to obtain the equivalent martingale measure, it is necessary to assume that Σ has finite second moment. On the contrary, we do not need this assumption since the measure $\mathbb{P}_{I,0}$, and therefore $\mathbb{P}_0$, are already martingale measures[7].

It must be observed that, in the absence of any source of heterogeneity in traders' beliefs around the underlying stock volatility, $\widetilde{C}_0(l) = \widehat{C}(l)$ and, according to Theorem 3.3, the implied volatility presents a symmetric U-shape, as a

[7]For the regularity of f see Marchetti et al. (2007) and Marchetti, F.M. (2009), in particular Proposition 1.3 and Theorem 3.3.

function of the log-moneyness. It follows that the smile appears as an inevitable equilibrium outcome as long as a sufficient number of traders tend to form the same beliefs around the stock volatility and use the Black–Scholes option pricing model to price options. Note, that this can already be considered as a simple theoretical and alternative explanation of the emergence of the implied volatility smile.

On the contrary, when traders' beliefs around the stock volatility tend to be different, the implied volatility looses its symmetry and takes a skewed U-shape. Indeed, thanks to Theorem 3.3, we are now in a position to state and prove the announced result about the relation between traders' heterogeneous beliefs and the emergence of the implied volatility skew.

THEOREM 3.4 *Assume that, for all $k \in \mathbb{N}$,*

$$\mathbb{E}_{I,0}\big[\tfrac{1}{\Sigma^k}\big] = \int_0^\infty \frac{1}{v^k}\, F_I\{dv\} < \infty, \qquad I = A,\, B. \tag{29}$$

Let f be the increasing function of Theorem 3.3 associated to the distribution function F_Σ, defined in (26). Let $\widetilde{\sigma}(K)$ be the implied volatility of $\widetilde{C}_0(K)$. Then $\widetilde{\sigma}$ is a C^∞-function and satisfies the following relation

$$\widetilde{\sigma}(K(l)) = f(l^2) + \alpha\, l + o(l), \tag{30}$$

with

$$\alpha = \psi(A,B)\sqrt{\frac{2\,\pi}{s^2\,T}}\,\exp\big\{\tfrac{\widetilde{\sigma}^2(K_{at})T}{8}\big\}, \tag{31}$$

where $\psi(A,B)$ is defined as in (24).

For the proof see Appendix A.

Theorem 3.4 shows that, for a fixed maturity, the implied volatility not only has a minimum around the underlying forward price but also that low-strike implied volatilities are different from high-strike implied volatilities[8].

Under suitable hypotheses on the integrability of $1/\Sigma$, the functions $\delta_I(l)$ admit second derivatives and thus a second order approximation for (30) (see Remark A.3 in the Appendix). However, in a neighborhood of 0, the main contribution to the asymmetry is again due to the presence of the term $\alpha\, l$, indeed there exists a $\beta \in \mathbb{R}$ such that $\widetilde{\sigma}(K(l)) = \alpha\, l + \beta\, l^2 + f(l^2) + o(l^2)$.

Under some mild conditions, our model allows us also to demonstrate that, as observed in financial markets, low-strike implied volatilities are higher than high-strike implied volatilities. To this aim, we recast (30) as

$$\widetilde{\sigma}(K) = \widehat{\sigma}(K) - \alpha\big(\log(K) - \log(K_{at})\big) + o\big(\log(\tfrac{K_{at}}{K})\big).$$

[8] We have also computed an explicit expression of $\psi(A,B)$ for $\tau = T$ (see Lemma A.4 in Appendix A), under the condition (29).

Accordingly, in our model low strikes implied volatilities are higher than high strikes implied volatilities if the sign of α is positive. This is the case when F_A and F_B are concentrated on v_A and v_B respectively, i.e. all the traders (either of type A or of type B) use the Black–Scholes model in order to forecast future call prices, but use different (deterministic) volatility parameters v_A and v_B.

Lemma 3.5 *Assume that $F_A(v) = \mathbf{1}_{[v_A,\infty)}(v)$ and $F_B(v) = \mathbf{1}_{[v_B,\infty)}(v)$, i.e. F_A and F_B are concentrated on v_A and v_B respectively, for v_A, $v_B > 0$. Then condition (29) holds and, for every choice of $v_A \neq v_B$, the slope α is positive.*

For the proof see Appendix A.

The above Lemma shows that, due to continuity of the $\psi(A, B)$ with respect to F_A and F_B, the slope α remains positive if F_A and F_B are concentrated in a suitable neighborhood of v_A and v_B, with $v_A \neq v_B$. Therefore, the heterogeneity of traders' beliefs combined with traders' reliance on the Black–Scholes option pricing model not only accounts for the emergence of an implied volatility that is an asymmetric function of the log-moneyness (see Theorem 3.4) but also—as observed in financial (derivative) markets worldwide—imply that low strikes implied volatilities are typically higher than high strikes.

4. ESTIMATION ISSUES

In this section we evaluate how the proposed model is able to capture the reality of financial (derivative) markets. This is a tedious point, involving some complex analysis that can be carried out in different ways (e.g., see Nelson, B. (2004)). Thus, we compare the model outputs (i.e., option prices and the implied volatility curve) with the empirical data, assuming that inputs of the model are representative of inputs used by traders to price options in financial markets.

We consider the market model proposed in Section 2, assuming that, for traders in A, Σ has a distribution F_A, while, for traders in B, has a distribution F_B. Both distributions have a probability density function of the following kind:

$$g(v) = g_\Sigma(v) = \frac{\big(a/b\big)^{\frac{q+1}{4}}}{\mathcal{K}_{\frac{q+1}{2}}\big(\sqrt{ab}\big)}\, v^q e^{-(av^2+b/v^2)/2}, \tag{32}$$

with parameters $a > 0$, $b > 0$, $q \in \mathbb{R}$, and where $\mathcal{K}_\nu(z)$ is the modified Bessel function of the second kind. The latter assumption is equivalent to assume that Σ^2 has a generalized inverse Gaussian distribution (GIG) with probability density function

$$g_{\Sigma^2}(v) = \frac{1}{2}\frac{\big(a/b\big)^{\frac{q+1}{4}}}{\mathcal{K}_{\frac{q+1}{2}}\big(\sqrt{ab}\big)}\, v^{(\frac{q+1}{2}-1)} e^{-\frac{a}{2}v} e^{-\frac{b}{2v}}. \tag{33}$$

This assumption on the distribution implies that traders assume a generalized hyperbolic distribution (which is a mixture of normals) for daily returns, which seems to be appropriate (see e.g. the pioneering paper of Eberlein, E., Keller, U. (1995) for the case $q = 1$; Rydberg, T.H (1999)), instead of a normal distribution (as in the standard Black–Scholes model). Note that by using such a hyperbolic distribution, a modified version of Black–Scholes can be obtained to price options (see Barndorff-Nielsen, O.E. and Shephard, N. (2001)).

Finally, for the sake of simplicity, we propose that all traders share the same coefficient of absolute risk aversion ($\gamma_A = \gamma_B = \gamma$), so that our model depends (remembering Remark 3.2 and that $\frac{\gamma_A}{\gamma_B} = \frac{\gamma}{\gamma} = 1$) only on the set of parameters $\phi = (a_A, b_A, q_A, a_B, b_B, q_B)$.

This version of our model was calibrated to S&P 500 call option data, which Bakshi, Cao and Chen kindly provided to us. The sample period extends from June 1, 1989, through May 31, 1991. Using such data allows us: to refer to a period in which the Black–Scholes model and its generalizations (see in particular Hull, J. and White, A. (1987)) were widely used; to test our model in periods of financial stability and instability (in early 90's a crash occurred); to compare our results with those obtained for other option pricing models in Bakshi et al. (1997). In addition note that other authors have tested their models on similar time intervals, see e.g. Jackwerth, J.C. and Rubinstein, M. (1996).

The data have the following characteristics: i) options prices are the last bid-ask quote (prior to 3:00 PM Central Standard Time); ii) the free interest rate is extracted by the daily Treasury-bill bid and ask discounts with maturity up to one year; iii) the spot stock price is adjusted for discrete dividends; iv) price quotes that are time-stamped later than 3:00, are lower than \$3/8, or do not satisfying the arbitrage restriction are not included in the sample; v) options with less than six days to expiration are also excluded.

Options included in the sample are divided into two categories, according to moneyness or term to expiration. In order to make our results comparable with those of Bakshi et al. (1997), a call option is then said to be at-the-money if the ratio between the future price of the underlying S&P 500 asset S and the strike price K is between 0.97 and 1.03; out-of-the-money if the ratio is less than or equal to 0.97; and in-the-money if the ratio is greater than or equal to 1.03. By the term to expiration, options are classified as short-term (< 60 days), medium-term ($60 - 180$ days) and long-term(> 180 days). The reported moneyness and maturity classification produce different options categories for which the empirical results will be reported.

In Table I, we report the average bid-ask mid-point price, the average implied volatility which is shown in parentheses, and the total number of observations (in braces) for each moneyness-maturity category.

From our data set, we extract M option prices on the S&P 500 index, taken from the same day, for any M greater than or equal to 1, plus number of parameters to be estimated. For each $m = 1, ..., M$, let $\widetilde{C}_m(t, T_m, K_m)$ be the observed

TABLE I

SAMPLE PROPERTIES OF S&P 500 INDEX OPTIONS.

	Moneyness S/K	Days-to-Expiration < 60	$60 - 180$	≥ 180	Subtotal
OTM	<0.94	$1.68	$4.38	$8.58	
		(0.06)	(0.16)	(0.26)	
		{542}	{2330}	{1847}	{4719}
	0.94-0.97	$2.35	$8.02	$15.12	
		(0.09)	(0.23)	(0.35)	
		{1943}	{1966}	{965}	{4874}
ATM	0.97-1.00	$4.83	$12.79	$20.17	
		(0.15)	(0.29)	(0.40)	
		{2703}	{1919}	{1003}	{5625}
	1.00-1.03	$10.42	$18.72	$26.44	
		(0.23)	(0.35)	(0.45)	
		{2543}	{1793}	{931}	{5267}
ITM	1.03-1.06	$17.77	$25.52	$33.00	
		(0.30)	(0.41)	(0.50)	
		{2255}	{1566}	{738}	{4559}
	≥1.06	$39.40	$48.06	$58.12	
		(0.37)	(0.46)	(0.50)	
		{5137}	{5269}	{3299}	{13705}
Subtotal		{15123}	{14843}	{8783}	{38749}

price and $\widetilde{C}^{\phi}_m(t, T_m, K_m)$ be the estimated price as determined by our model, with S and r taken directly from the financial market, and where ϕ is the set of parameters. Therefore, the difference between $\widetilde{C}_m(t, T_m, K_m)$ and $\widetilde{C}^{\phi}_m(t, T_m, K_m)$ is assumed to be a function of the value taken by $\phi = (a_A, b_A, q_A, a_B, b_B, q_B)$. Thus, for each m we define

$$g_m[\phi] = \widetilde{C}_m(t, T_m, K_m) - \widetilde{C}^{\phi}_m(t, T_m, K_m) \tag{34}$$

and find the parameter vector ϕ to solve

$$SSE(t) = \inf_{\phi} \sum_{m \in M} |g_m[\phi]|^2 \tag{35}$$

Implementing the above procedure, we first use all call options available at each given day as inputs to estimate the relevant parameters. The results under "All-Options-Based" are then contrasted with those under "Maturity-Based" (i.e., using all call options of a given maturity—short-,medium-,or long-term—at each given day to estimate relevant parameters) and under "Moneyness-Based" (i.e., using all call options of a given moneyness—Out-, At-, or In-the-money—to estimate relevant parameters).

Table II reports the average and the standard error (in parentheses) for each parameter implied by All-, Short-Term-, and At-the-Money-Options under, respectively, All-Options-Based, Maturity-Based, and Moneyness-Based approaches. The daily average sum of squared errors (SSE) is also reported.

TABLE II

Implied parameters and in-sample fit.

Parameters	All Options	Short-Term Options	At-The-Money Options
a_A	0.14	0.14	0.14
	(0.00)	(0.00)	(0.00)
b_A	192.61	214.17	349.22
	(3.25)	(7.02)	(11.25)
q_A	1.15	1.13	1.13
	(0.00)	(0.01)	(0.01)
a_B	0.15	0.15	0.14
	(0.00)	(0.00)	(0.00)
b_B	398.98	423.87	351.41
	(6.39)	(8.41)	(10.73)
q_B	1.17	1.17	1.14
	(0.02)	(0.01)	(0.01)
SSE	12.84	5.81	8.51

Under the "All-options-Based", the average stock volatility—evaluated as the mean of the average daily traders' expectations about the underlying stock volatility, as estimated by our model—is equal to 22.30 for traders A and to 17.27 for traders B. Considering that $\mathbb{P}_0 = \eta_A(0)\mathbb{P}_{A,0} + \eta_B(0)\mathbb{P}_{B,0}$, the global average stock volatility is equal to 18.37 and its standard deviation is equal to 0.023. These values resemble those presented by the study of Bakshi et al. (1997) which, for the implied volatility, reports values that range from 18.23 to 19.38. Moreover, the in-sample fit (SSE) ranges between 5 and 12. These values are fairly acceptable, in particular considering that, for stochastic volatility model, Bakshi et al. (1997) reported SSE values that range from 5.45 (for short-term options under the Maturity-Based approach) to 10.63 (under the All-Options-Based approach). Note that our model performs always better (in terms of SSE) than the Black–Scholes model (i.e., the SSE values range between 25.34 and 68.40) but always worst than the stochastic-volatility random-jump model (i.e., the SSE values range between 2.63 and 6.46).

Since option pricing models that perform well in terms of in-sample pricing errors might display large out-of-sample pricing errors (see Dumas, B. et. al (1998)), we contrast the out-of-sample results of Bakshi et. al (1997) for the Black–Scholes (BS), stochastic-volatility (SV), and stochastic-volatility random-jump (SVJ) models with the results of our model (NMV). To this aim, we use previous day's option prices to estimate the parameters of our model. Then, parameters are updated using the following properties.

In a Bayesian statistics setting, assuming that the prices are observed in discrete time, if the prior distribution of Σ^2 is a GIG distribution with parameters (a, b, q), then the posterior distribution is still a GIG distribution, with parameters (a', b', q'). Thus, we compute $\mathcal{L}(\Sigma|S_{t_1}, \cdots, S_{t_n})$, the conditional law of Σ, given the observations S_u (i.e., the S&P500 spot price at time u), for u that ranges in the set $\{t_1, t_2, \ldots, t_n\}$, where $t_0 = 0 < t_1 < t_2 < \cdots < t_i < t_{i+1} <$

$\cdots < t_n$. Setting

$$Y(t) = \log(S(t)/S_0),$$

then we have $\mathcal{L}(\Sigma|S_{t_1}, \cdots, S_{t_n}) = \mathcal{L}(\Sigma|Y_{t_1}, \cdots, Y_{t_n})$. Furthermore, since $Y(t) = (r - \frac{1}{2}\Sigma^2)\, t + \Sigma\, W_t$ and, conditionally on Σ, the increments $Y(t_i) - Y(t_{i-1})$ are independent, then, conditionally on Σ, the law of $Y_{t_i} - Y_{t_{i-1}}$ is Gaussian, namely

$$\mathcal{L}(Y_{t_i} - Y_{t_{i-1}}|\Sigma = v) = N\big((r - \frac{1}{2}\, v^2)(t_i - t_{i-1}), v^2\,(t_i - t_{i-1})\big),$$

and therefore, if the initial distribution of Σ has a density $g_\Sigma(v)$, then, conditionally on $Y(t_1), Y(t_2), \cdots, Y(t_n)$, the distribution of Σ has density

$$g_\Sigma(v|Y_{t_1}, Y_{t_2}, \cdots, Y_{t_n}) \propto g_\Sigma(v)\, \prod_{i=1}^{n} \frac{e^{-\frac{(y_i - y_{i-1} - (r - \frac{1}{2} v^2)(t_i - t_{i-1}))^2}{2\, v^2\, (t_i - t_{i-1})}}}{\sqrt{2\pi\, v^2\, (t_i - t_{i-1})}}\Bigg|_{y_i = Y(t_i)}$$

where, as usual, $\propto$ is the symbol for proportional to.

After some manipulation, for the conditional density, we get that

$$\begin{aligned} g_\Sigma^{(n)}(v) &:= g_\Sigma(v|Y_{t_1}, Y_{t_2}, \cdots, Y_{t_n}) \\ &= g_{a_n, b_n, q_n}(v) \left(= \frac{(a_n/b_n)^{\frac{q_n+1}{4}}}{\mathcal{K}_{\frac{q_n+1}{2}}\big(\sqrt{a_n b_n}\big)}\, v^{q_n}\, e^{-(a_n v^2 + b_n/v^2)/2} \right), \quad (v > 0) \end{aligned}$$

where the parameters (a_n, b_n, q_n) can be obtained by an updating formula:

$$\begin{aligned} a_n &= a_{n-1} + \frac{t_n - t_{n-1}}{4}, \\ b_n &= b_{n-1} + \frac{[Y(t_n) - Y(t_{n-1})]^2}{t_n - t_{n-1}} + 2\, [Y(t_n) - Y(t_{n-1})]\, r + r^2 (t_n - t_{n-1}) \\ &= b_{n-1} + \left(\frac{[Y(t_n) - Y(t_{n-1})]}{\sqrt{t_n - t_{n-1}}} + r\sqrt{t_n - t_{n-1}} \right)^2, \\ q_n &= q_{n-1} - 1. \end{aligned}$$

Note that from the recursive expression for b_n, it occurs that the parameter b_n is positive for all n.

Once updated[9] the parameters a_I, b_I, and q_I, we use them as input to compute current day's model-based option prices. Next, we derive the absolute difference between the option price and the model price for each call in a given moneyness-maturity category. Note that we repeat this procedure for each option and each day in the sample. Results are summarized in Table III.

[9]We have considered $t_{i+1} - t_i$ equal to one day.

TABLE III

Out-of-Sample Absolute Pricing Errors.

		All-Options-Based Days-to-Expiration			Maturity-Based Days-to-Expiration			Moneyness-Based Days-to-Expiration		
Moneyness	Model	< 60	$60-180$	≥ 180	< 60	$60-180$	≥ 180	< 60	$60-180$	≥ 180
< 0.94	BS	\$0.78	\$1.39	\$1.89	\$1.02	\$1.48	\$1.78	\$0.41	\$0.63	\$0.78
	SV	0.42	0.43	0.61	0.38	0.42	0.58	0.32	0.36	0.53
	SVJ	0.37	0.40	0.59	0.27	0.40	0.58	0.33	0.36	0.54
	NMV	0.31	0.42	0.53	0.29	0.37	0.54	0.39	0.47	0.61
$0.94-0.97$	BS	0.76	1.02	1.16	0.73	1.07	1.15	0.45	0.53	0.69
	SV	0.46	0.41	0.54	0.33	0.41	0.54	0.34	0.38	0.53
	SVJ	0.38	0.38	0.53	0.25	0.39	0.53	0.33	0.38	0.51
	NMV	0.43	0.57	0.65	0.31	0.42	0.69	0.42	0.57	0.56
$0.97-1.00$	BS	0.61	0.62	0.66	0.51	0.64	0.66	0.70	0.74	0.94
	SV	0.48	0.41	0.53	0.39	0.41	0.52	0.40	0.43	0.60
	SVJ	0.42	0.40	0.52	0.31	0.40	0.51	0.36	0.41	0.63
	NMV	0.47	0.76	0.98	0.42	0.56	0.97	0.34	0.52	0.76
$1.00-1.03$	BS	0.52	0.69	0.81	0.45	0.65	0.84	0.47	0.50	0.69
	SV	0.41	0.43	0.53	0.40	0.41	0.51	0.38	0.43	0.54
	SVJ	0.40	0.42	0.51	0.37	0.41	0.50	0.37	0.41	0.51
	NMV	0.28	0.51	0.72	0.33	0.38	0.65	0.32	0.42	0.70
$1.03-1.06$	BS	0.76	1.21	1.30	0.77	1.14	1.37	0.51	0.85	1.76
	SV	0.45	0.47	0.55	0.41	0.41	0.51	0.48	0.48	0.67
	SVJ	0.39	0.44	0.53	0.39	0.41	0.51	0.39	0.42	0.53
	NMV	0.33	0.35	0.63	0.32	0.37	0.66	0.29	0.43	0.63
≥ 1.06	BS	0.82	1.39	1.57	0.79	1.35	1.64	0.56	0.62	0.72
	SV	0.54	0.49	0.65	0.47	0.40	0.51	0.44	0.41	0.54
	SVJ	0.43	0.43	0.56	0.36	0.39	0.50	0.40	0.42	0.54
	NMV	0.35	0.40	0.85	0.34	0.42	0.89	0.30	0.31	0.53

Although the stochastic-volatility random-jump model accounts for the lowest SSE, we observe that our model appears to be the most suitable model for predicting pricing of far-from-money options that are close to maturity—which have been considered in the literature as the most challenging (e.g., Brigo, D. et al. (2003))—and less for options that are still far from the maturity.

Since our model incorporate the skew in a stochastic volatility framework, it is able to better predict those options that, being far from the money, are more likely to be over-priced than options that are at-the-money. In addition, concerning the effect of time to maturity on absolute pricing errors, our model performs better for options that are close to maturity than for those that are still far from expiration. A possible explanation is that our model assumes that traders' beliefs holding constant over time, and this assumption is fairly reasonable for options that are close to maturity, but not for those that are still far from maturity. This observation asks for future enquires aimed at extending our model by considering ways traders can update their beliefs over time and accordingly find suitable classes of models that are able to grasp these changing expectations over time. This is an important extension for our model that needs to be addressed in future studies.

5. CONCLUSIONS

In this paper, we developed a parsimonious model of a financial (derivative) market that pays particular attention to issues of heterogeneity of traders' beliefs and institutionalized technical devices for pricing options, and inspects their implications for the emergence of the volatility skew. In our model traders believe that the equilibrium price is the Black–Scholes price.

However, the uncertainty around the underlying stock volatility complicates the problem of correctly estimate this equilibrium price. Traders tend to form different expectations on the underlying stock volatility and, equipped with the well-know Black–Scholes option pricing model, use these expectations to forecast the "true" equilibrium price. In accordance with these expectations they price and trade options and, as we formally demonstrate, they favor the emergence of the implied volatility skew. Moreover, using a simplified version of our model, we also reproduce the structure of the skew (i.e., with low-strike implied volatilities are higher than high-strike implied volatilities) that is observed in "real" financial markets. Additionally, we provide evidence of the ability of our model to grasp the reality of financial (derivative) markets. Thus, it is confirmed that shared sets of beliefs, practices, ways of calculating, and technical systems (as the Black–Scholes model), which MacKenzie, D. (2009) has termed as "evaluation cultures" are determinants of the emergence of the implied volatility skew.

While this paper has made a contribution by offering a simple but novel approach to modeling the processes of option pricing, the goal is far from accomplished: it remains to develop a model in which traders deal with portfolios of options and may change asynchronously their habits, explore new models for predicting the equilibrium price and, accordingly, use them to price options. New sources of heterogeneity are then introduced in the model that can help to better explain the different shapes the implied volatility skew has assumed over time. These tasks await further research.

APPENDIX A: PROOFS

Proof of Lemma 2.3: Recalling (4), (3) and (11) we have that

$$\begin{aligned}\overline{C}_{I,\tau}(K) &= e^{-r\tau}\widehat{C}_{I,\tau}(K) = e^{-r\tau}\mathbb{E}_{I,0}\left[\widetilde{C}_\tau(K)\right]\\ &= e^{-r\tau}\mathbb{E}_{I,0}\left[\mathbb{E}_{I,0}\left[(S_T-K)^+e^{-r(T-\tau)}\middle|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\right]\right]\end{aligned}$$

and, therefore, on the one hand

$$\overline{C}_{I,\tau}(K) = e^{-r\tau}\mathbb{E}_{I,0}\left[C^{BS}(S_\tau,r,T-\tau,K,\Sigma)\right],$$

while, on the other hand

$$\overline{C}_{I,\tau}(K) = \mathbb{E}_{I,0}\left[(S_T-K)^+e^{-rT}\right],$$

independently of τ. □

Proof of Lemma 2.4: First of all note that

$$\begin{aligned}\delta_{A,\tau}(K) &= \mathbb{E}_{A,0}\left[(\widetilde{C}_\tau(K))^2\right] - \left(\mathbb{E}_{A,0}\left[\widetilde{C}_\tau(K)\right]\right)^2\\ &= e^{2r\tau}\left\{\mathbb{E}_{A,0}\left[\left(\mathbb{E}_{A,0}\left[(S_T-K)^+\,e^{-rT}\middle|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\right]\right)^2\right] - \left(\widehat{C}_A(K)\right)^2\right\}\\ &= \mathbb{E}_{A,0}\left[\left(C^{BS}(S_\tau,r,T-\tau,K,\Sigma)\right)^2\right] - e^{2r\tau}\left(\widehat{C}_A(K)\right)^2.\end{aligned}$$

In the particular case when $\tau=T$, $(S_T-K)^+$ is measurable with respect to $\mathcal{F}^S_\tau\vee\sigma(\Sigma)$, and therefore

$$\delta_{A,T}(K) = \mathbb{E}_A\left[\left((S_T-K)^+\right)^2\right] - e^{2rT}\left(\widehat{C}_A(K)\right)^2.$$

Setting

$$\xi := (S_T-K)^+e^{-r(T-\tau)},\qquad \mathcal{G} := \mathcal{F}^S_\tau\vee\sigma(\Sigma),\qquad\text{and}\qquad \widetilde{\xi} := \mathbb{E}_{A,0}\left[\xi\middle|\mathcal{G}\right],$$

then $\widetilde{C}_\tau(K)=\widetilde{\xi}$ and $\delta_{A,\tau}(K)$ is the variance of $\widetilde{\xi}$, under $\mathbb{P}_{0,A}$. By the well known formula of Pythagoras (also known as the law of total variance)

$$Var_{A,0}\left(\xi\right) = Var_{A,0}\left(\widetilde{\xi}\right) + \mathbb{E}_{A,0}\left[Var_{A,0}\left(\xi\,|\mathcal{G}\right)\right],$$

we get

$$\begin{aligned}\delta_{A,\tau}(K) &= \mathbb{E}_{A,0}\left[\left(\xi-\mathbb{E}_{A,0}\left[\xi\right]\right)^2\right] - \mathbb{E}_{A,0}\left[\mathbb{E}_{A,0}\left[\xi^2\middle|\mathcal{G}\right] - \left(\mathbb{E}_{A,0}\left[\xi|\mathcal{G}\right]\right)^2\right]\\ &= e^{2r\tau}\mathbb{E}_{A,0}\left[\left((S_T-K)^+e^{-rT} - \mathbb{E}_{A,0}[(S_T-K)^+e^{-rT}]\right)^2\right]\\ &\quad - \Bigg(\mathbb{E}_{A,0}\Bigg[\mathbb{E}_{A,0}\left[\left((S_T-K)^+\right)^2e^{-2r(T-\tau)}\middle|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\right]\\ &\qquad\qquad - \left(\mathbb{E}_{A,0}\left[(S_T-K)^+e^{-r(T-\tau)}\middle|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\right]\right)^2\Bigg]\Bigg).\end{aligned}$$

For the general case, we can rewrite

$$\begin{aligned}\delta_{A,\tau}(K) &= e^{-2r(T-\tau)}\,\delta_{A,T}(K)\\ &\quad - \Big(\mathbb{E}_{A,0}\Big[\mathbb{E}_{A,0}\big[\big((S_T-K)^+\big)^2 e^{-2r(T-\tau)}\big|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\big]\\ &\qquad\qquad - \Big(\mathbb{E}_{A,0}\big[(S_T-K)^+ e^{-r(T-\tau)}\big|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\big]\Big)^2\Big]\Big).\end{aligned}$$

Then (16) is obtained by observing that

$$\mathbb{E}_{A,0}\big[\big((S_T-K)^+\big)^2 e^{-2r(T-\tau)}\big|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\big] = C_2^{BS}(S_\tau,r,T-\tau,K,\Sigma)$$

and

$$\mathbb{E}_{A,0}\big[(S_T-K)^+ e^{-r(T-\tau)}\big|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\big] = C^{BS}(S_\tau,r,T-\tau,K,\Sigma).$$

□

Remark A.1 *Considering (16), an other expression of $\delta_{A,\tau}(K)$ is given by:*

$$\begin{aligned}\delta_{A,\tau}(K) + e^{2r\tau}\big(\widehat{C}_A(K)\big)^2 &= \mathbb{E}_A\Big[\Big(\mathbb{E}_A\big[(S_T-K)^+\,e^{-r(T-\tau)}\big|\mathcal{F}^S_\tau\vee\sigma(\Sigma)\big]\Big)^2\Big]\\ &= \mathbb{E}_A\Big[S_\tau^2\Big[\Phi\Big(\tfrac{l(S_\tau,T-\tau,r,K)}{\Sigma\sqrt{T-\tau}}+\tfrac{\Sigma\sqrt{T-\tau}}{2}\Big)\\ &\quad - e^{-l(S_\tau,T-\tau,r,K)}\Phi\Big(\tfrac{l(S_\tau,T-\tau,r,K)}{\Sigma\sqrt{T-\tau}}-\tfrac{\Sigma\sqrt{T-\tau}}{2}\Big)\Big]^2\Big]\\ &= \int_0^\infty \mathbf{E}\Big[S_\tau^2(W_\tau,v)\Big[\Phi\Big(\tfrac{l(S_\tau(W_\tau,v),T-\tau,r,K)}{v\sqrt{T-\tau}}+\tfrac{v\sqrt{T-\tau}}{2}\Big)\\ &\quad - e^{-l(S_\tau(W_\tau,v),T-\tau,r,K)}\Phi\Big(\tfrac{l(S_\tau(W_\tau,v),T-\tau,r,K)}{v\sqrt{T-\tau}}-\tfrac{v\sqrt{T-\tau}}{2}\Big)\Big]^2\Big]F_A\{dv\}.\end{aligned} \tag{36}$$

Furthermore, as already stated in Remark 2.5, C_2^{BS} is given by (17). In order to prove it one can use the following property:

$$\int_a^\infty \exp\{\alpha+\beta z\}\,\tfrac{1}{\sqrt{2\,\pi}}\,e^{-\frac{z^2}{2}}\,dz = \exp\{\alpha+\tfrac{1}{2}\beta^2\}\,\Phi(\beta-a) \tag{37}$$

for every $a, \alpha, \beta \in \mathbb{R}$. Indeed, we have that

$$
\begin{aligned}
& e^{2rT}\, C_2^{BS}(s,r,T,K,v) = \\
&= \int_{-\frac{l(K)}{v\sqrt{T}}+\frac{v\sqrt{T}}{2}}^{\infty} \left(s\, e^{(r-\frac{v^2}{2})\,T+v\,\sqrt{T}\,z} - K\right)^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\,\pi}}\, dz \\
&= s^2 \int_{-\frac{l(K)}{v\sqrt{T}}+\frac{v\sqrt{T}}{2}}^{\infty} e^{(2r-v^2)\,T+2v\,\sqrt{T}\,z}\, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\,\pi}}\, dz \\
&\quad - 2K\, s \int_{-\frac{l(K)}{v\sqrt{T}}+\frac{v\sqrt{T}}{2}}^{\infty} e^{(r-\frac{v^2}{2})\,T+v\,\sqrt{T}\,z}\, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\,\pi}}\, dz + K^2 \int_{-\frac{l(K)}{v\sqrt{T}}+\frac{v\sqrt{T}}{2}}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\,\pi}}\, dz \\
&= s^2\, e^{(2r-v^2)\,T+\frac{(2v\,\sqrt{T})^2}{2}}\, \Phi\Big(2v\,\sqrt{T} + \frac{l(K)}{v\,\sqrt{T}} - \frac{v\sqrt{T}}{2}\Big) \\
&\quad - 2K\, s\, e^{(r-\frac{v^2}{2})\,T+\frac{(v\,\sqrt{T})^2}{2}}\, \Phi\Big(v\sqrt{T} + \frac{l(K)}{v\,\sqrt{T}} - \frac{v\sqrt{T}}{2}\Big) + K^2 \Phi\Big(\frac{l(K)}{v\,\sqrt{T}} - \frac{v\sqrt{T}}{2}\Big) \\
&= s^2\, e^{2r\,T} e^{v^2\,T}\, \Phi\Big(\frac{l(K)}{v\,\sqrt{T}} + \frac{3\,v\sqrt{T}}{2}\Big) \\
&\qquad\qquad - 2K\, s\, e^{r\,T}\, \Phi\Big(\frac{l(K)}{v\,\sqrt{T}} + \frac{v\sqrt{T}}{2}\Big) + K^2\, \Phi\Big(\frac{l(K)}{v\,\sqrt{T}} - \frac{v\sqrt{T}}{2}\Big),
\end{aligned}
$$

and (17) immediately follows.

Proof of Proposition 3.1: Under the market model's hypotheses, due to (21), equation (19) becomes

$$\widetilde{C}_0(l) = \eta_A(l)\widehat{C}_A(l) + \eta_B(l)\widehat{C}_B(l), \tag{38}$$

where

$$\eta_B(l) = \frac{1}{\frac{\delta_B(l)}{\delta_A(l)} + 1}, \quad \eta_A(l) = 1 - \eta_B(l).$$

Consider the Taylor expansion of $\eta_I(l)$, $I = A, B$, in a neighborhood of $l = 0$: $\eta_I(l) = \eta_I(0) + l\eta_I'(0) + o(l)$. Substituting in (38) we get

$$
\begin{aligned}
\widetilde{C}_0(l) &= \eta_A(0)\widehat{C}_A(l) + \eta_B(0)\widehat{C}_B(l) + l\left[\eta_A'(0)\widehat{C}_A(l) + \eta_B'(0)\widehat{C}_B(l)\right] + o(l) \\
&= \widehat{C}(l) + l\left[\eta_A'(0)\widehat{C}_A(l) + \eta_B'(0)\widehat{C}_B(l)\right] + o(l),
\end{aligned} \tag{39}
$$

expanding at the first order in (39) $\widehat{C}_I(l) = \widehat{C}_I(0) + o(1)$, we get

$$\widetilde{C}_0(l) = \widehat{C}(l) + l\left[\eta_A'(0)\widehat{C}_A(0) + \eta_B'(0)\widehat{C}_B(0)\right] + o(l)$$

i.e., since $\eta_A'(l) = \big(1 - \eta_B(l)\big)'$,

$$\widetilde{C}_0(l) = \widehat{C}(l) + l\,\eta_B'(0)\left[\widehat{C}_B(0) - \widehat{C}_A(0)\right] + o(l). \tag{40}$$

Thus

$$\psi(A,B) = \eta_A'(0)\widehat{C}_A(0) + \eta_B'(0)\widehat{C}_B(0) = \eta_B'(0)\left[\widehat{C}_B(0) - \widehat{C}_A(0)\right].$$

Clearly $\psi(A,B) = \psi(B,A)$(as was expected by the symmetry of the roles of A and B), moreover

$$\eta_B'(l) = -\Big(\frac{\delta_B'(l)}{\delta_B(l)} - \frac{\delta_A'(l)}{\delta_A(l)}\Big)\eta_A(l)\eta_B(l), \tag{41}$$

and hence we have

$$\psi(A,B) = \left[\frac{\delta'_B(0)}{\delta_B(0)} - \frac{\delta'_A(0)}{\delta_A(0)}\right]\eta_A(0)\eta_B(0)\big(\widehat{C}_A(0) - \widehat{C}_B(0)\big). \tag{42}$$

□

REMARK A.2 *When δ_I has second derivatives, then*

$$\widetilde{C}_0(l) = \widehat{C}(l) + l\,\eta'_B(0)\big[\widehat{C}_B(0) - \widehat{C}_A(0)\big] + \Big(\eta''_B(0)\big[\widehat{C}_B(0) - \widehat{C}_A(0)\big] + 2\,\eta'_B(0)\big[\widehat{C}'_B(0) - \widehat{C}'_A(0)\big]\Big)\; l^2 + o(l^2),$$

where $\eta'(0)$ and $\eta''(0)$ can be computed by (41) and

$$\eta''_B(l) = -\left[\left(\frac{\delta'_B(l)}{\delta_B(l)} - \frac{\delta'_A(l)}{\delta_A(l)}\right)' + \left(\frac{\delta'_B(l)}{\delta_B(l)} - \frac{\delta'_A(l)}{\delta_A(l)}\right)^2 [\eta_A(l) - \eta_B(l)]\right]\eta_A(l)\eta_B(l),$$

respectively.

PROOF OF THEOREM 3.4: As far as the regularity of the implied volatility is concerned, note that condition (29) implies that $\widehat{C}_I$ and δ_I, are C^∞-functions. Then $\widetilde{C}_0$ and, by the implicit function theorem, also the implied volatility are C^∞-functions. The same holds for the function f of Theorem 3.3 associated to F_Σ given by (26).

As far as the expansion of $\widetilde{C}_0(l)$ is concerned, first of all note that, by (38) and (23), the functions $\widetilde{C}_0(l)$ and $\widehat{C}(l)$ coincide in $l = 0$. For simplicity of notation we set $g(l) := \widetilde{\sigma}(K(l))$. In our model, g and f are defined by

$$\widetilde{C}_0(l) = \varphi\big(g(l), K(l)\big) \qquad \text{and} \qquad \widehat{C}(l) = \varphi\big(f(l^2), K(l)\big),$$

where we have used the notation

$$\varphi(v,K) = C^{BS}(s,r,T,K,v), \quad (s,\ r \text{ and } T, \text{ being fixed}) \tag{43}$$

Therefore

$$g(0) = f(0) = \widetilde{\sigma}(K_{at}) \tag{44}$$

(recall that $K(0) = K_{at}$). Furthermore, we can recast (22) as

$$\varphi(g(l), K(l)) = \varphi(f(l^2), K(l)) + l\psi(A,B) + o(l), \tag{45}$$

looking for to get g as $g(l) = f(l^2) + h(l)$, where h is a function we need to determine at the first order expansion. By definition, and thanks to (44), we have $h(0) = 0$. To recover $h'(0)$, its first derivative in 0, we observe that $h'(0) = g'(0)$, since $g'(l) = f'(l^2)2\,l + h'(l)$. Then we derive both members of (45) with respect to l and compute them at $l = 0$, and obtain

$$g'(0)\,\frac{\partial\varphi}{\partial v}(g(0), K_{at}) = \psi(A,B), \tag{46}$$

and obtain

$$h'(0) = g'(0) = \frac{\psi(A,B)}{\dfrac{\partial\varphi}{\partial v}\big(g(0), K_{at}\big)}.$$

Then α is immediately computed by recalling that

$$\frac{\partial\varphi}{\partial v}\big(v,K\big) = s\sqrt{T}\,\frac{1}{\sqrt{2\,\pi}}\exp\Big\{-\frac{1}{2}\Big(\tfrac{l(K)}{v\sqrt{T}} + \tfrac{v\sqrt{T}}{2}\Big)^2\Big\},$$

$l(K_{at}) = 0$, and using again (44). □

Remark A.3 *Using the same notations of the previous proof, one can get the second order expansion* $\widetilde{\sigma}(K(l)) = f(l^2) + \alpha\, l + \beta\, l^2 + o(l^2)$*, where* $\alpha = h'(0)$ *and* $\beta = h''(0)$*. The latter value can be recovered with techniques similar to those used in the previous proof and get that*

$$\begin{aligned} h''(0)\, \frac{\partial \varphi}{\partial v}(g(0), K_{at}) &\\ &= \eta''_B(0)\Big[\widehat{C}_B(0) - \widehat{C}_A(0)\Big] + 2\,\eta'_B(0)\Big[\widehat{C}'_B(0) - \widehat{C}'_A(0)\Big] \\ &\quad - 2\,\frac{\partial^2 \varphi}{\partial v \partial K}(g(0), K_{at})\eta'(0)\, K'(0) - 2\,\frac{\partial^2 \varphi}{\partial v^2 \partial K}(g(0), K_{at})\big(\eta'(0)\big)^2. \end{aligned}$$

The exact values of $\beta = h''(0)$*, can then be obtained in terms of the first and second derivatives of* δ_I*, by using the computations in the previous proof of Theorem 3.4, those of Remark A.2, and observing that* $K'(0) = -K(0) = -K_{at}$*.*

In the following lemma we compute the value of $\psi(A, B)$ for the case $\tau = T$, under the further condition that $1/\Sigma$ is an integrable random variable, under $\mathbb{P}_{0,I}$, for $I = A,\ B$, i.e. (29).

Lemma A.4 *Assume condition (29), and set, for* $I = A,\ B$*,*

$$q_I := \int_0^\infty F_I\{dv\}\, e^{v^2 T}\, \Phi\Big(\tfrac{3v\sqrt{T}}{2}\Big), \quad p_I := \int_0^\infty F_I\{dv\}\, \Phi\Big(\tfrac{v\sqrt{T}}{2}\Big).$$

then, when $\tau = T$*, the following equality holds*

$$\begin{aligned} \psi(A, B) = 4s\, &\frac{\frac{\gamma_B}{\gamma_A}(q_A + p_A - 4p_A^2)(q_B + p_B - 4p_B^2)}{\Big((q_A + p_A - 4p_A^2) + \frac{\gamma_B}{\gamma_A}(q_B + p_B - 4p_B^2)\Big)^2}\cdot \\ &\cdot \left(\frac{(2\,p_B - 1)\,p_B}{q_B + p_B - 4p_B^2} - \frac{(2\,p_A - 1)\,p_A}{q_A + p_A - 4p_A^2}\right)\big(p_A - p_B\big). \end{aligned} \tag{47}$$

Proof of Lemma A.4: First of all, taking into account the general expression (24) for $\psi(A, B)$ and the expressions (8) for η_I, we need to compute

$$(i)\ \widehat{C}_I(0); \qquad (ii)\ \delta_{I,T}(0); \qquad (iii)\ \delta'_{I,T}(0); \quad \text{for } I = A,\ B.$$

As point (i) is concerned, we start by setting, for any distribution function F on $[0, \infty)$:

$$p_{\pm}(F, l) := \int_0^\infty F\{dv\}\Phi\Big(\frac{l}{v\sqrt{T}} \pm \frac{v\sqrt{T}}{2}\Big), \tag{48}$$

$$q(F, l) := \int_0^\infty F\{dv\}\, e^{v^2 T}\, \Phi\Big(\frac{l}{v\sqrt{T}} + \frac{3v\sqrt{T}}{2}\Big), \tag{49}$$

and

$$\widehat{C}_F(0) := \int_0^\infty C^{BS}(s, r, T, K_{at}, v)\, F\{dv\}. \tag{50}$$

From (10) and by observing that $l(K_{at}) = 0$, we have that

$$C^{BS}(s, r, T, K_{at}, v) = s\Big[\Phi\Big(\frac{v\sqrt{T}}{2}\Big) - \Phi\Big(\frac{-v\sqrt{T}}{2}\Big)\Big] = s\Big[2\Phi\Big(\frac{v\sqrt{T}}{2}\Big) - 1\Big],$$

and, therefore, by considering (48), we get immediately that

$$\widehat{C}_F(0) = s\Big[p_+(F,0) - p_-(F,0)\Big] = s\Big[2p_+(F,0) - 1\Big]. \tag{51}$$

In particular, we get the explicit expression of $\widehat{C}_I(0)$, $I = A,\, B$.

As point (ii) is concerned (we recall that $\delta_{I,T}(l)$ is a shorthand for $\delta_{I,T}(K(l))$), we are to compute

$$\delta_{I,T}(K) = \mathbb{E}_{I,0}\Big[\big((S_T-K)^+\big)^2\Big] - \Big(\mathbb{E}_{I,0}\big[(S_T-K)^+\big]\Big)^2.$$

For the ease of the reader, we start with the second addend, by showing directly that

$$e^{-rT}\mathbb{E}_{I,0}[(S_T-K)^+] = \int_0^\infty C^{BS}(s,r,T,K,v)F_I\{dv\},$$

without making use of conditional expectations. Indeed, by (37), we have

$$\begin{aligned}
&\mathbb{E}_{I,0}[(S_T-K)^+]\\
&= \int_0^\infty F_I\{dv\}\int_{-\infty}^\infty \Big(s\,\exp\{(r-\tfrac{v^2}{2})\,T + v\,\sqrt{T}\,z\} - K\Big)^+ \,\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\,\pi}}\,dz\\
&= \int_0^\infty F_I\{dv\}\int_{\frac{-l(K)}{v\,\sqrt{T}}+\frac{v\sqrt{T}}{2}}^\infty \Big(s\,\exp\{(r-\tfrac{v^2}{2})\,T + v\,\sqrt{T}\,z\} - K\Big)\,\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\,\pi}}\,dz\\
&= \int_0^\infty F_I\{dv\}\Big[s\,\exp\{(r-\tfrac{v^2}{2})\,T + \tfrac{v^2\,T}{2}\}\,\Phi(v\,\sqrt{T} + \tfrac{l(K)}{v\,\sqrt{T}} - \tfrac{1}{2}v\sqrt{T})\\
&\qquad\qquad\qquad\qquad\qquad\qquad -K\,\Phi(\tfrac{l(K)}{v\,\sqrt{T}} - \tfrac{1}{2}v\sqrt{T})\Big]\\
&= \int_0^\infty F_I\{dv\}\Big[s\,e^{rT}\,\Phi(\tfrac{l(K)}{v\,\sqrt{T}} + \tfrac{1}{2}v\sqrt{T}) - K\,\Phi(\tfrac{l(K)}{v\,\sqrt{T}} - \tfrac{1}{2}v\sqrt{T})\Big]\\
&= e^{rT}\int_0^\infty C^{BS}(s,r,T,K,v)F_I\{dv\}.
\end{aligned}$$

Using the inverse of the log-moneyness, i.e. the function $K(l) = K_{at}e^{-l}$ defined in (21), and the notations $p_\pm(F,l)$ we can rewrite

$$\mathbb{E}_{I,0}[(S_T-K(l))^+] = se^{rT}\Big\{p_+(F_I,l) - e^{-l}p_-(F_I,l)\Big\}, \tag{52}$$

The first addend in the definition of $\delta_{I,T}(K)$ is

$$\begin{aligned}
&\mathbb{E}_{I,0}\Big[\big((S_T-K)^+\big)^2\Big]\\
&= \int_0^\infty F_I\{dv\}\int_{-\infty}^\infty \Big[\Big(s\,\exp\{(r-\tfrac{v^2}{2})\,T + v\,\sqrt{T}\,z\} - K\Big)^+\Big]^2\,\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\,\pi}}\,dz\\
&= \int_0^\infty F_I\{dv\}e^{2rT}C_2^{BS}(s,r,T,K,v) \qquad (53)
\end{aligned}$$

where we have used the notation $C_2^{BS}(s,r,T,K,v)$ introduced in (15). Using its analytical expression (17) and the notations $p_\pm(F,l)$ and $q(F,l)$, we can rewrite (53) as

$$\mathbb{E}_{I,0}\Big[\big((S_T-K(l))^+\big)^2\Big] = s^2e^{2rT}\Big\{q(F,l) - 2e^{-l}p_+(F_I,l) + e^{-2l}p_-(F_I,l)\Big\}. \tag{54}$$

The expression of $\delta_{I,T}(l)$ is then obtained considering the difference between (54) and (52) squared:

$$\begin{aligned}\delta_I(l) = \delta_{I,T}(l) = s^2 e^{2rT}\Big\{ & q(F_I,l) - 2e^{-l} p_+(F_I,l) + e^{-2l} p_-(F_I,l) \\ & - \Big(p_+(F_I,l) - e^{-l} p_-(F_I,l)\Big)^2\Big\},\end{aligned} \tag{55}$$

and in particular

$$\begin{aligned}\delta_I(0) &= \delta_{I,T}(0) \\ &= s^2 e^{2rT}\Big\{q(F_I,0) - 2p_+(F_I,0) + p_-(F_I,0) - \Big(p_+(F_I,0) - p_-(F_I,0)\Big)^2\Big\} \\ &= s^2 e^{2rT}\Big\{q(F_I,0) - 2p_+(F_I,0) + \Big(1 - p_+(F_I,0)\Big) - \Big(2p_+(F_I,0) - 1\Big)^2\Big\} \\ &= s^2 e^{2rT}\Big\{q(F_I,0) + 1 - 3p_+(F_I,0) - 4p_+^2(F_I,0) - 1 + 4p_+(F_I,0)\Big\} \\ &= s^2 e^{2rT}\Big\{q(F_I,0) + p_+(F_I,0) - 4p_+^2(F_I,0)\Big\}.\end{aligned} \tag{56}$$

As far as (iii) is concerned, from (48) and (49) it is immediate to get their derivatives for $l \neq 0$:

$$p'_\pm(F,l) := \frac{d}{dl} p_\pm(F,l) = \int_0^\infty F\{dv\} \frac{1}{v\sqrt{T}} \Phi'\Big(\frac{l}{v\sqrt{T}} \pm \frac{v\sqrt{T}}{2}\Big), \quad l \neq 0,$$

and

$$q'(F,l) := \frac{d}{dl} q(F,l) = \int_0^\infty F\{dv\} \frac{e^{v^2 T}}{v\sqrt{T}} \Phi'\Big(\frac{l}{v\sqrt{T}} + \frac{3v\sqrt{T}}{2}\Big), \quad l \neq 0.$$

Furthermore, since the integrability condition

$$\int_0^\infty \frac{1}{v} F\{dv\} < \infty$$

holds, then the derivatives in $l = 0$ of $p_\pm(F,l)$ and $q(F,l)$ are finite and equal: in fact, once we observe that

$$p'_\pm(F,0) = \int_0^\infty F\{dv\} \frac{1}{v\sqrt{T}} \Phi'\Big(\frac{v\sqrt{T}}{2}\Big),$$
$$q'(F,0) = \int_0^\infty F\{dv\} \frac{e^{v^2 T}}{v\sqrt{T}} \Phi'\Big(\frac{3v\sqrt{T}}{2}\Big),$$

and

$$e^{v^2 T}\, \Phi'\Big(\frac{3v\sqrt{T}}{2}\Big) = \frac{1}{\sqrt{2\pi}} e^{v^2 T - \frac{9v^2 T}{8}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2 T}{8}} = \Phi'\Big(\frac{v\sqrt{T}}{2}\Big),$$

then we get

$$p'_+(F,0) = p'_-(F,0) = q'(F,0) = \int_0^\infty F\{dv\} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{v^2 T}{8}}}{v\sqrt{T}}. \tag{57}$$

By (55), it follows immediately that

$$\begin{aligned}\delta'_{I,T}(l) = s^2 e^{2rT}\Big\{ & q'(F_I,l) + 2e^{-l} p_+(F_I,l) - 2e^{-l} p'_+(F_I,l) \\ & - 2e^{-2l} p_-(F_I,l) + e^{-2l} p'_-(F_I,l) \\ & - 2\Big(p_+(F_I,l) - e^{-l} p_-(F_I,l)\Big)\Big(p'_+(F_I,l) + e^{-l} p_-(F_I,l) - e^{-l} p'_-(F_I,l)\Big)\Big\},\end{aligned} \tag{58}$$

and, in particular, for $l = 0$, we have that

$$\delta'_{I,T}(0) = s^2 e^{2rT}\Big\{ q'(F_I,0) + 2p_+(F_I,0) - 2p'_+(F_I,0) - 2p_-(F_I,0) + p'_-(F_I,0) \\ - 2\Big(p_+(F_I,0) - p_-(F_I,0)\Big)\Big(p'_+(F_I,0) + p_-(F_I,0) - p'_-(F_I,0)\Big)\Big\}$$

taking into account that $p'_\pm(F_I,0) = q'(F_I,0)$, all the derivatives cancel,

$$= s^2 e^{2rT}\Big\{2p_+(F_I,0) - 2p_-(F_I,0) - 2\Big(p_+(F_I,0) - p_-(F_I,0)\Big)p_-(F_I,0)\Big\}$$
$$= s^2 e^{2rT} 2\Big\{\Big(p_+(F_I,0) - p_-(F_I,0)\Big)\Big(1 - p_-(F_I,0)\Big)\Big\}$$

taking into account that $p_+(F_I,0) = 1 - p_-(F_I,0)$,

$$= s^2 e^{2rT} 2\Big(2\,p_+(F_I,0) - 1\Big)\,p_+(F_I,0). \tag{59}$$

We are now able to compute the explicit value of $\psi(A,B)$ in the special case $\tau = T$. Indeed, recalling that $p_+(F_I,0) = p_I$ and $q(F_I,0) = q_I$, taking into account the definition (24) of $\psi(A,B)$, and (51), (56), (59), we get (47).

□

Proof of Lemma 3.5: The constant α being $\psi(A,B)$ times a positive constant (see (31)), and the constants $q_I + p_I - 4p_I^2$, for $I = A$, B, being strictly positive (see the expression (56) for the variance $\delta_I(0) > 0$, in the proof of Lemma A.4, where we compute $\psi(A,B)$ for the case $\tau = T$), from (47) we immediately get that the sign of α is positive if and only if

$$\left(\frac{(2\,p_B - 1)\,p_B}{q_B + p_B - 4p_B^2} - \frac{(2\,p_A - 1)\,p_A}{q_A + p_A - 4p_A^2}\right)\Big(p_A - p_B\Big) > 0. \tag{60}$$

Taking into account that $q_I = e^{v_I^2 T}\,\Phi(\frac{3v_I\sqrt{T}}{2})$ and $p_I = \Phi(\frac{v_I\sqrt{T}}{2})$, and that

$$p_A > p_B, \quad\Leftrightarrow\quad \Phi(\tfrac{v_A\sqrt{T}}{2}) > \Phi(\tfrac{v_B\sqrt{T}}{2}) \quad\Leftrightarrow\quad v_A > v_B,$$

the above condition is a consequence of the observation that the function $x \mapsto \Psi(x) := \frac{2\,\Phi^2(x) - \Phi(x)}{e^{(2x)^2}\Phi(3x) + \Phi(x) - 4\,\Phi^2(x)}$ is strictly decreasing in $(0,\infty)$. The rest of the proof is devoted to get this monotonicity property.

Setting $A(x) := 2\,\Phi^2(x) - \Phi(x)$ and $B(x) := e^{(2x)^2}\Phi(3x) - \Phi(x)$, we can write $\Psi(x) = 1/(\frac{B(x)}{A(x)} - 2)$. Thus the function Ψ is strictly decreasing in $(0,\infty)$ if and only if the function

$$x \mapsto \frac{B(x)}{A(x)} = \frac{e^{(2x)^2}\Phi(3x) - \Phi(x)}{2\,\Phi^2(x) - \Phi(x)}$$

is strictly increasing. Using the relation $e^{(2x)^2}\Phi'(3x) = \Phi'(x)$, and $\Phi''(x) = -x\Phi'(x)$ one can easily check that

$$A'(x) = [4\,\Phi(x) - 1]\Phi'(x),$$
$$A''(x) = 4\Big(\Phi'(x)\Big)^2 - x\,A'(x),$$
$$B'(x) = 8x\Big(B(x) + \Phi(x)\Big) + 2\,\Phi'(x),$$
$$B''(x) = \Big(8 + (8x)^2\Big)\Big(B(x) + \Phi(x)\Big) + 22\,x\,\Phi'(x),$$

and that $\lim_{x\to 0^+} B(x)/A(x) = 2$. Taking into account that $A(0) = B(0) = 0$, and therefore $A(x) = \int_0^x A'(y)\,dy$, and $B(x) = \int_0^x B'(y)\,dy$, the function $B(x)/A(x)$ is strictly increasing if and only if

$$\int_0^x [B'(x)\,A'(y) - A'(x)\,B'(y)]\,dy > 0, \qquad x > 0. \tag{61}$$

Taking into account that A' is a strictly positive function in $(0,\infty)$, the condition $B'(x)/A'(x) \geq B'(y)/A'(y)$, for $0 < y \leq x$, implies inequality (61). In other words it is sufficient to prove that the function $\frac{B'(x)}{A'(x)}$ is strictly increasing, i.e.,

$$\begin{aligned} &B''(x)A'(x) - B'(x)A''(x) \\ &= \Big(B(x) + \Phi(x)\Big)\left[\left[\Big(8 + (8x)^2\Big) + 8x^2\right] A'(x) - 8\,x\,\Big(\Phi'(x)\Big)^2\right] \\ &+ \Big(22 + 2\Big)\,x\,\Phi'(x)\,A'(x) - 8\,\Big(\Phi'(x)\Big)^3 > 0, \quad \text{for } x > 0. \end{aligned} \tag{62}$$

Considering that $B(x) + \Phi(x) \geq \Phi(x) > \frac{1}{2}$, and that $x\,\Phi'(x)\,A'(x) > 0$, for $x > 0$, by using the above expression of $A'(x)$, it follows that (62) is greater or equal to

$$\begin{aligned} &4\,\Phi'(x)\left[\Big(1 + 9x^2\Big)\Big(4\Phi(x) - 1\Big) - x\,\Phi'(x) - 2\Big(\Phi'(x)\Big)^2\right] \\ &\geq 4\,\Phi'(x)\left(9x^2 - \frac{1}{\sqrt{2\pi}}\,x + 1 - \frac{1}{\pi}\right) > 0, \end{aligned}$$

where the first inequality is a consequence of $4\Phi(x) - 1 \geq 1$ and $\Phi'(x) \leq \Phi'(0) = \frac{1}{\sqrt{2\pi}}$. $\square$

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