Some remarks on the implied volatility smile and on a related identification problem *

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Abstract

A simple and elementary explanation of the volatility smile is provided by using some properties of the associative means. Some remarks on the identification problem for the volatility distribution are posed, along with a characterization problem for the implied volatility as a function of log-moneyness squared.

Key Words: volatility smile, option price, associative means

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1 Introduction

It is well known that in the Black and Scholes ([4]) option price model if we plot implied volatility as a function of the exercise price, we should obtain a horizontal straight line. This implies that all options with the same expiration date on the same underlying asset but with different exercise prices should have the same implied volatility. Contrary to this, the implied volatilities differ across strike prices for the same maturity and across diverse expiration periods. Early examples include: Black ([3]), MacBeth and Merville ([20]), Galai ([14]) and Rubinstein ([24]). More recently, the presence of the volatility smile has been empirically observed that in US financial markets options, where far out of the money or deep into the money

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are traded at higher implied volatility than options at the money (e.g. Heynen [17], Derman and Kani [9]). Similar patterns have also been documented in the Germany, Japanese and U.K. index option markets (Gemmill and Kamiyama [15]). As a consequence of that, the graph of the observed implied volatility function often looks like the smile of the "Cheshire cat" (e.g. Cont and Fonseca [6]).

There is an abundance of work that deals directly with the volatility smile or volatility skew as well as several approaches have been proposed for explaining it. For example, Rubinstein ([25]) related the smile effect to the presence (i) of jumps in the price of the underlying asset between successive opportunities to trade, (ii) of market imperfections and frictions, such as transaction costs, illiquidity, and other trading restrictions, that imply that a single arbitrage-free option price no longer exists and (iii) of disturbances in the price process of the underlying assets that does not follow a geometric Brownian motion with constant volatility.

Along this line, many models have been proposed by a number of authors, including, among others, the stochastic volatility models, (Hull and White [18]; Wiggins [28]; Stein and Stein [27]), the general equilibrium stochastic volatility model (Detemple and Osakwe [10]), the pure jump diffusion models (Merton [21], Bates [2]), the affine jump-diffusion model (Heston [16]), the double exponential jump-diffusion (Kou [19]).

Stochastic volatility models exhibit a smile locally centered around the current forward price, as it was originally shown by Renault and Touzi ([23]) in 1996, when the stochastic volatility is a diffusion process driven by a Brownian motion uncorrelated with the Brownian motion driving the stock price, and the market price of risk depends only on the volatility process. A simpler proof can also be found in the paper by Sircar and Papanicolau ([26]). In the latter paper the authors show the result by Renault and Touzi, by noticing that in the setting of [23], the price is computed by means of the so-called Hull and White formula, which is an expectation of the Black and Scholes price, where the volatility is considered as a random variable Σ. In the Renault and Touzi setting the probability belongs to a suitable class of equivalent martingale measures and the random variable is suitably obtained starting by the stochastic volatility process. Furthermore Sircar and Papanicolau show that whenever the price is computed by means of the Hull and White formula, then the implied volatility is locally convex around the current forward price.

Recognizing the contribution of these studies, this paper offers a simple and elementary explanation of this phenomena: whenever the price is computed via an expectation of the Black and Scholes price, then the implied volatility is an associative mean. By using some properties of the associative means, and in particular an observation due to de Finetti ([7]), we show that the implied volatility is an increasing function f of the log-moneyness squared (Proposition 2.1). Then the local convexity property is an easy exercise (see Remark 2.2).

Furthermore we face two related problems (i) an identification problem: is the distribution function of the random variable Σ uniquely determined by the implied volatility function? (ii) a characterization problem: is there a characterization of the increasing function f? The identification problem is faced using some properties of the completely monotone functions, under reasonable hypotheses on the volatility distribution (see Theorem 3.2), and, though we cannot give an exhaustive answer to the characterization problem, we show that the increasing function f solves an ordinary differential equation (see Theorem 3.3). The above results have mainly a theoretical interest and are valid under some regularity conditions on the the option price function, which are satisfied when the random variable $1/\Sigma$ has finite moments. The latter condition holds when the volatility squared has a Generalized Inverse Gaussian (GIG) distribution, and in this parametric case one can compute explicitly the second derivative of the option price with respect to the strike price.

In the parametric case of GIG distributions, and when call option prices are available only for a finite number of strike prices, as it happens in real market data, one can use a least square method in order to compute the implied volatility for "unobserved" strike prices. The explicit form of the second derivative of the option price has inspired a modification of the least square criterion. We have briefly compared numerically the two methods with some artificially generated data. The numerical results show that our modified least square method is a slight improvement of the usual method.

2 The model and the smile effect

Let the dynamics of price follow the model of Black and Scholes, with current stock price $S_0 = s$, and volatility v. If the riskless interest rate is constant and the stock pays no dividend during the option life, then the price of an European call with exercise time T and strike price K is evaluated by traders according to the following well known formula

$$\begin{aligned} C^{BS}(s, r, T, K, v) &= e^{-rT} E[(S_T - K)^+] \\ &= s\,\Phi\left(\frac{\log(\frac{s}{K}) + (r + \frac{v^2}{2})T}{v\sqrt{T}}\right) - e^{-rT} K \Phi\left(\frac{\log(\frac{s}{K}) + (r + \frac{v^2}{2})T}{v\sqrt{T}} - v\sqrt{T}\right) \end{aligned} \tag{1}$$

where $\Phi(x)$ is the standard normal distribution function.

For simplicity we assume that the current stock price s, the riskless interest rate r and the time until option expiration T are fixed. Thus we can consider the price of the European option as a function of the volatility and the strike price:

$$\varphi(v,K) := C^{BS}(s,r,T,K,v).$$

The main problem in using the latter formula is that the value of v is unknown to traders. On the one hand, it has been observed that "the future volatility of the stock must be estimated. The past volatility [...] is not an infallible guide" (Black [3]: p.36). Moreover, it has been pointed out that "volatility forecasts are sensitive to the specification of the volatility model [...], correctly estimating the parameters of a volatility model can be difficult, because volatility is not observable [...]. Volatility forecasts are anchored at noisy proxies or estimates of the current level of volatility. Even with a perfectly specified and estimated volatility model, forecasts of future volatility inherit and potentially even amplify the uncertainty about the current level of volatility" (Brandt, Jones [5]: p.470).

On the other hand, it is well known that the function

$$h_K : v \to h_K(v) := \varphi(v,K) \tag{2}$$

is a strictly increasing continuous function and therefore invertible. Thus, another approach to infer the value of the volatility is to observe the market price of a call option $\widehat{C}(K)$ with strike price K. Then, by applying a suitable option pricing formula – in a sense backwards – the trader can calculate the volatility that would have to be input into the option pricing formula to obtain that price for the option. In this manner, the trader obtains the volatility implied by the option price.

Nevertheless the implicit stock volatility is not a constant, but varies as K varies, i.e. the implied volatility is a function of K:

$$\widehat{\sigma}(K) = h_K^{-1}\big(\widehat{C}(K)\big). \tag{3}$$

As already recalled in the Introduction, this phenomenon is called the smile effect and shows that the market price does not follow the Black and Scholes model.

This fact can be explained in the following simple way: Since the stock volatility appears to be frequently ambiguous, it is natural to believe that the option price is evaluated by traders following some rules of the thumb like the following:

$$\widehat{C}(K) = \int_0^\infty \varphi(v,K) F_\Sigma\{dv\} \tag{4}$$

with F_Σ is the distribution function representing the set of beliefs of the representative trader over the possible values of the volatility, i.e., in a Bayesian framework, the volatility is considered as a random variable Σ. Indeed, as mentioned in the Introduction, when the option price is given by (4), the implied volatility function $K \mapsto \widehat{\sigma}(K)$ is an increasing function of $l^2(K)$, where

$$l(K) := \log\left(\frac{se^{rT}}{K}\right)$$

is the so called log-moneyness.

Note that the dependence of the implied volatility on $l(K)$ is obvious: when (4) holds, then clearly $\widehat{\sigma}(K)$ has to be a function of $l(K)$, as it is readily seen by observing that

$$\varphi(v, K) = s\left[\Phi\left(\frac{l(K)}{v\sqrt{T}} + \frac{v\sqrt{T}}{2}\right) - e^{-l(K)}\Phi\left(\frac{l(K)}{v\sqrt{T}} - \frac{v\sqrt{T}}{2}\right)\right].$$

Proposition 2.1. *Assume that (3) and (4) hold, i.e. that the implied volatility is defined as*

$$\widehat{\sigma}(K) = h_K^{-1}(E[h_K(\Sigma)]),$$

with Σ a random variable with distribution function F_Σ. Then there exists an increasing function $f = f_{F_\Sigma}$ such that

$$\widehat{\sigma}(K) = f_{F_\Sigma}\big(l^2(K)\big).$$

Furthermore if Σ is a non degenerate random variable, then the function f is strictly increasing.

Proof. First of all, note that the thesis is equivalent to

$$\widehat{\sigma}(K_1) \leq \widehat{\sigma}(K_2) \Leftrightarrow l^2(K_1) \leq l^2(K_2).$$

According to Nagumo-Kolmogoroff-de Finetti theorem, given a stochastic variable X, and a strictly increasing and continuous function h

$$M_h(X) := h^{-1}(E[h(X)]) \tag{5}$$

is an associative mean. Then the implied volatility $\widehat{\sigma}(K)$ is the associative mean of Σ, with $h = h_K$. Furthermore, as observed in de Finetti's paper ([7]), given a random variable X whose distribution function is defined in $D := \mathrm{supp}(X)$ and two invertible and strictly increasing regular functions h and g in D, we have:

$$M_h(X) := h^{-1}(E[h(X)]) \leq M_g(X) := g^{-1}(E[g(X)]) \tag{6}$$

if and only if

$$\frac{h''(x)}{h'(x)} \leq \frac{g''(x)}{g'(x)} \tag{7}$$

for all x that belongs to D. The above property is equivalent to the convexity of the function $g(h^{-1}(\cdot))$. The convexity is strict when the inequality in (7) is strict; if this is the case then also the inequality in (6) is strict, unless in the degenerate case. To obtain the thesis it is enough to show that

$$\frac{h''_{K_1}(v)}{h'_{K_1}(v)} < \frac{h''_{K_2}(v)}{h'_{K_2}(v)} \quad \forall\, v > 0, \qquad \text{if and only if} \qquad l^2(K_1) < l^2(K_2). \quad (8)$$

The latter statement is easily seen from the equality

$$\frac{h''_K(v)}{h'_K(v)} = \frac{l^2(K) - (\frac{v^2 T}{2})^2}{v^2 T},$$

which is a straightforward consequence of the fact that h'_K coincides with the Vega (which measures the sensitivity of option price to its implied volatility and is positive for each value of K)

$$h'_K(v) = \frac{\partial}{\partial v}\varphi(v, K) = s\,\sqrt{T}\,\frac{1}{\sqrt{2\,\pi}}\exp\big\{-\frac{1}{2}\big(\tfrac{l(K)}{v\,\sqrt{T}} + \tfrac{v\sqrt{T}}{2}\big)^2\big\}.$$

$\square$

Proposition 2.1 implies that the minimum point of the implied volatility $\widehat{\sigma}(K)$ is the value $K_{at} := se^{rT}$. This fact can already be considered as a simple theoretical explanation of the emergence of the volatility smile effect. Under some further mild conditions one can get a new proof of the announced local convexity property for the implied volatility, as it is explained in the following remark.

Remark 2.2. *Let $f = f_{F_\Sigma}$, the increasing function of the Proposition 2.1, be a $C^2\big([0,\infty)\big)$ function, with $f'(0) > 0$. Then the function $\widehat{\sigma}$* is convex on a neighborhood around the future price K_{at}. *This result was established, with different techniques, by Sircar and Papanicolau (see Theorem 1 in [26]). In fact, since $\widehat{\sigma}(x) = f(l^2(x))$, it follows that*

$$\widehat{\sigma}''(x) = 2l(x)\Big(2l(x)[l'(x)]^2 f''(l^2(x)) + l''(x) f'(l^2(x))\Big) + 2[l'(x)]^2 f'(l^2(x)).$$

As x tends to K_{at} the log-moneyness $l(x)$ tends to 0, while $l'(x)$ and $l''(x)$ remain bounded, thus we have that

$$\lim_{x \to K_{at}} \widehat{\sigma}''(x) = \frac{2}{s^2 e^{2rT}} f'(0) > 0.$$

Finally we note that the regularity assumption on $f = f_{F_\Sigma}$ translates into a condition on the distribution of Σ, and it is satisfied when Σ takes values in a compact set not containing 0 (for a proof, under milder assumptions on Σ, see Theorem 3.3 and Proposition 3.1).

3 The identification problem

Suppose, as in the previous section and in Proposition 2.1, that the market call price $\widehat{C}(K)$ is given by (4) and therefore

$$\widehat{C}(K) = \varphi(\widehat{\sigma}(K), K) = \varphi\big(f(l^2(K)), K\big). \tag{9}$$

In this section we treat two problems:

Given the implied volatility function $K \mapsto \widehat{\sigma}(K)$, or equivalently the call price function $K \mapsto \widehat{C}(K)$,
• is it possible to identify the distribution F_Σ in (4)?
• is it possible to characterize the function f?

We need the following preliminary result

Proposition 3.1. *Suppose that (4) holds together with*

$$\int_0^\infty \frac{1}{v} F_\Sigma\{dv\} < \infty \quad \big(i.e.\ 1/\Sigma \textit{ is a integrable random variable}\big). \tag{10}$$

Then $\widehat{C}$ is a $C^2\big((0,\infty)\big)$ function, with

$$\widehat{C}''(K) = \frac{e^{-2rT} e^{\frac{3}{2}l(K)}}{s\sqrt{2\pi T}} \int_0^\infty e^{-\frac{v^2T}{8}} \frac{e^{-\frac{l^2(K)}{2v^2T}}}{v} F_\Sigma\{dv\} \tag{11}$$

for all $K > 0$.

Proof. To start, the first derivative of $\varphi(v, K)$ with respect to K is

$$\frac{\partial}{\partial K}\varphi(v, K) = -e^{-rT}\Phi\left(\frac{l(K)}{v\sqrt{T}} - \frac{v\sqrt{T}}{2}\right), \tag{12}$$

hence

$$\left|\frac{\partial}{\partial K}\varphi(v, K)\right| \leq e^{-rT}$$

uniformly in K. Hence $\widehat{C}$ is a derivable function, with

$$\widehat{C}'(K) = \int_0^\infty \frac{\partial}{\partial K}\varphi(v, K) F_\Sigma\{dv\}. \tag{13}$$

The second derivative of $\varphi(v, K)$ with respect to K is

$$\frac{\partial^2}{\partial K^2}\varphi(v, K) = \frac{e^{-2rT} e^{\frac{3}{2}l(K)}}{s\sqrt{2\pi T}} e^{-\frac{v^2T}{8}} \frac{e^{-\frac{l^2(K)}{2v^2T}}}{v}, \tag{14}$$

as follows immediately by deriving (12). For the sake of simplicity, we denote the first derivative of $\varphi(v,K)$ with respect to K by $\varphi'(v,K)$, and we get

$$\begin{aligned}\frac{\varphi'(v,K+\triangle K)-\varphi'(v,K)}{\triangle K} &= \frac{1}{\triangle K}\int_K^{K+\triangle K}\frac{\partial^2}{\partial H^2}\varphi(v,H)dH\\ \leq \sqrt{\frac{s}{2\pi Te^{rT}}}\frac{1}{v}\frac{1}{\triangle K}\int_K^{K+\triangle K}\frac{dH}{H^{\frac{3}{2}}} &\leq \sqrt{\frac{s}{2\pi Te^{rT}}}\frac{1}{v}\frac{2^{\frac{3}{2}}}{K^{\frac{3}{2}}},\end{aligned} \tag{15}$$

for all $\triangle K$ such that $|\triangle K|\leq K/2$.
Let $\{(\triangle K)_n\}_{n\in\mathbb{N}}$ be a sequence of real nonzero numbers such that $(\triangle K)_n$ tends to 0 as $n\to\infty$, with $|(\triangle K)_n|\leq K/2$; let $\{X_n\}_n$ be the sequence of random variables given by

$$X_n := \frac{\varphi'\big(\Sigma,K+(\triangle K)_n\big)-\varphi'\big(\Sigma,K\big)}{(\triangle K)_n},$$

and set $Y := 2\sqrt{\frac{s}{\pi Te^{rT}}}K^{-\frac{3}{2}}\Sigma^{-1}$. By (10) and (15), Y is a integrable random variable such that $|X_n|\leq Y$ for all n. Since $X_n\to X:=\varphi''\big(\Sigma,K\big)$ a.s. as $n\to\infty$, by the dominated convergence theorem applied to the sequence $\{X_n\}_n$ we obtain the result. $\square$

Note that (11) can be achieved for all positive $K\neq se^{rT}$, without assuming condition (10). Indeed, when $l(K)\neq 0$, the second partial derivative of $\varphi(v,K)$ is bounded uniformly in v by a suitable constant depending on K, as can be readily seen by (14). Therefore, in this case, the proof can be repeated using a positive constant in place of the random variable Y. In other words condition (10) is essential only in the case $K=se^{rT}$: in the latter case, condition (10) is not only sufficient, but also necessary in order to obtain (11). To this aim we observe that, for every constant $a\geq 0$, the following inequality holds

$$\int_0^\infty \frac{1}{v}F_\Sigma\{dv\}\leq e^a\int_0^\infty\frac{e^{-av^2}}{v}F_\Sigma\{dv\}+1,$$

and that (11), with $K=se^{rT}$, implies that, when $a=\frac{T}{8}$, the above upper bound is finite. With similar techniques it is possible to show that $\widehat{C}$ is C^∞, with derivatives of all orders equal to the expectations of the corresponding partial derivatives of φ evaluated in Σ, if and only if all the moments of $1/\Sigma$ are finite.

The following theorem provides an answer about the first question of this section.

Theorem 3.2. *Suppose that:*

(i) $\widehat{C}$, given by (4), is a $C^\infty\big((0,\infty)\big)$ function, and set

$$\eta(x) := se^{2rT}\sqrt{2\pi T}e^{-\frac{3}{2}\sqrt{x}}\widehat{C}''\big(l^{-1}(\sqrt{x})\big), \quad x \in [0,\infty); \tag{16}$$

(ii) the volatility Σ is an absolutely continuous integrable random variable with continuous density $g(v)$;

(iii) $1/\Sigma$ is an integrable random variable;

Then for every v

$$g(v) = \frac{e^{\frac{v^2T}{8}}}{v^2T}\frac{d}{dz}\beta(z)\big|_{z=\frac{1}{2v^2T}}, \tag{17}$$

where

$$\beta(z) = \lim_{a\to\infty}\sum_{n\le az}\frac{(-a)^n}{n!}\eta^{(n)}(a), \quad z \in [0,\infty). \tag{18}$$

Before giving the proof, we point out that the main aim is to consider the integral at the right hand side of (11) as a Laplace transform, up to a suitable changes of variable; in other words the aim is to prove that η is a Laplace transform. The latter property is equivalent to the complete monotonicity of the function η, i.e. η possesses derivatives $\eta^{(n)}$ of all orders and

$$(-1)^n\eta^{(n)}(x) \ge 0, \quad x > 0.$$

Proof. By (i), (ii), (iii), the assumptions of Proposition 3.1 hold. Therefore, multiplying both the members of (11) by $s\,e^{2rT}e^{-\frac{3}{2}l(K)}\sqrt{2\pi T}$, we obtain

$$se^{2rT}\sqrt{2\pi T}e^{-\frac{3}{2}l(K)}\widehat{C}''\big(l^{-1}(l(K))\big) = \int_0^\infty e^{-\frac{v^2T}{8}}\frac{e^{-\frac{l^2(K)}{2v^2T}}}{v}F_\Sigma\{dv\}, \tag{19}$$

where $l^{-1}(y) = se^{rT}e^{-y}$ is the inverse of $l(K)$. Therefore the left hand side of (19) is a symmetric function with respect to $l(K)$. Taking into account the latter remark, we set $x := l^2(K)$; furthermore we change variable in the integral at the right hand side of (19), by setting $z = \frac{1}{2v^2T}$:

$$se^{2rT}\sqrt{2\pi T}e^{-\frac{3}{2}\sqrt{x}}\widehat{C}''\big(l^{-1}(\sqrt{x})\big) = \int_0^\infty e^{-xz}g\Big(\frac{1}{\sqrt{2Tz}}\Big)\frac{e^{-\frac{1}{16z}}}{2z}dz \tag{20}$$

for all $x \ge 0$. Setting

$$h(z) := \begin{cases} g\big(\frac{1}{\sqrt{2Tz}}\big)\frac{e^{-\frac{1}{16z}}}{2z}, & z \in (0,\infty)\\ 0 = \lim_{z\to 0^+} g\big(\frac{1}{\sqrt{2Tz}}\big)\frac{e^{-\frac{1}{16z}}}{2z}, & z = 0 \end{cases} \tag{21}$$

and taking into account (16), we now can rewrite (20) as

$$\eta(x) = \int_0^\infty e^{-xz} h(z) dz, \quad x \in [0, \infty). \tag{22}$$

Observe that, since condition *(iii)* holds, the function h is in $\mathrm{L}^1([0,\infty))$, and it is not necessary in (22) to rule out the value 0 from the set in which x runs through. By [13](Theorem 1a, XIII.4, p. 439) η is completely monotone (in particular $\widehat{C}''(K) \geq 0$ for all $K > 0$) and, moreover – since

$$\beta(z) := \int_0^z h(y) dy$$

is absolutely continous – by [13] (Theorem 2, XIII.4, p. 440) we can invert (22), obtaining that

$$\lim_{a\to\infty} \sum_{n \leq az} \frac{(-a)^n}{n!} \eta^{(n)}(a) = \beta(z), \quad z \in [0, \infty). \tag{23}$$

Since $\big|\frac{e^{-\frac{1}{16z}}}{2z}\big| \leq 8e^{-1}$ for $z \in (0, \infty)$, we get

$$g\left(\frac{1}{\sqrt{2Tz}}\right) \frac{e^{-\frac{1}{16z}}}{2z} = \frac{d}{dz}\beta(z), \tag{24}$$

at all points of continuity of $g\big(\frac{1}{\sqrt{2Tz}}\big)$. $\square$

From the above proof it is straightforward to note that if g is not continuous, then (17) only holds at the continuity points of g.

We now face the characterization problem of the increasing function of Proposition 2.1: when the option price is evaluated as in (4), we give a necessary condition for a function $f : [0, \infty) \to [0, \infty)$ in order that the implied volatility is $K \mapsto f(l^2(K))$. Therefore, we have the following:

Theorem 3.3. *Let $\widehat{C} : (0,\infty) \to (0,\infty)$ be a known call price C^2-function given by (4), and*

$$f \,:\, [0, \infty) \to (0, \infty), \quad x \mapsto f(x)$$

be the increasing function (see Proposition 2.1) such that $K \mapsto f(l^2(K))$ is the implied volatility.

Then $f \in C^1\big([0,\infty)\big) \cap C^2\big((0,\infty)\big)$ and solves the following differential equations for $x \in (0, \infty)$

$$y'(x) = -\frac{e^{rT}\widehat{C}'\big(K_{at}\, e^{-\sqrt{x}}\big) + \Phi\big(\frac{\sqrt{x}}{y\sqrt{T}} - \frac{y\sqrt{T}}{2}\big)}{\sqrt{\frac{2T}{\pi}}\,\sqrt{x}\, e^{\frac{\sqrt{x}}{2}}\, e^{-\frac{y^4T^2+4x}{8y^2T}}}. \tag{25}$$

$$e^{-\frac{y^4T^2+4x}{8y^2T}} \big\{\frac{1}{y}\big[\frac{2xy'}{y} - 1\big]^2 - T^2xy(y')^2 + 4Txy'' + 2Ty'\big\} = \eta(x), \tag{26}$$

where $\eta : [0,\infty) \to \mathbb{R}$ is the continuous function given by (16). Moreover f satisfies the following initial conditions

$$f(0) = \frac{2}{\sqrt{T}}\,\phi_{\frac{s+\widehat{C}(K_{at})}{2s}}$$
$$f'(0) = \frac{1}{2T}\{se^{2rT}\sqrt{2\pi T}\widehat{C}''(K_{at})\,e^{\frac{f^2(0)T}{8}} - \frac{1}{f(0)}\}$$

where ϕ_α is the α-quantile of a $\mathcal{N}(0,1)$ random variable.

Before giving the proof we recall that the regularity assumption on $\widehat{C}$ corresponds to an assumption on the moments of $1/\Sigma$ (see Proposition 3.1)

Proof. By assumption the function $\widehat{C}(K)$ has continuous first and second derivatives, and by the first equality in (9) the same holds for the implied volatility $\widehat{\sigma}(K)$. By Proposition 2.1 $\widehat{\sigma}(K) = f(l^2(K))$, where f is an increasing function. Therefore $\widehat{\sigma}(K)$ has a minimum in $K_{at} = s\,e^{rT}$ and $\widehat{\sigma}'(K_{at}) = 0$. Furthermore the following identities hold

$$\begin{aligned}
f(x) &= \widehat{\sigma}(l^{-1}(\sqrt{x})) = \widehat{\sigma}\big(K_{at}\,e^{-\sqrt{x}}\big), \quad \text{for } x \geq 0,\\
f'(x) &= -\widehat{\sigma}'\big(K_{at}\,e^{-\sqrt{x}}\big)K_{at}\,e^{-\sqrt{x}}\frac{1}{2\sqrt{x}}, \quad \text{for } x > 0,\\
f''(x) &= \widehat{\sigma}''\big(K_{at}\,e^{-\sqrt{x}}\big)\ \frac{K_{at}^2 e^{-2\sqrt{x}}}{4x} + \widehat{\sigma}'\big(K_{at}\,e^{-\sqrt{x}}\big)K_{at}\,e^{-\sqrt{x}}\frac{1}{2\sqrt{x}}\frac{\sqrt{x}+1}{2x},\\
&= \frac{1}{2x}\left[\widehat{\sigma}''\big(K_{at}\,e^{-\sqrt{x}}\big)\ \frac{K_{at}^2}{2}\,e^{-2\sqrt{x}} - f'(x)\big(\sqrt{x}+1\big)\right], \quad \text{for } x > 0.
\end{aligned}$$

As a consequence we have that

$$f'(0) = \lim_{x\to 0^+} f'(x) = -\frac{K_{at}}{2}\lim_{x\to 0^+}\frac{\widehat{\sigma}'(K_{at}e^{-\sqrt{x}})}{\sqrt{x}} = \frac{K_{at}^2}{2}\,\widehat{\sigma}''(K_{at})$$

and $f \in C^1\big([0,\infty)\big)$, $f \in C^2\big((0,\infty)\big)$, and $\lim_{x\to 0^+} x\,f''(x) = 0$.

By using the second equality in (9), the first derivative of $\widehat{C}$ with respect to K is

$$\begin{aligned}
\widehat{C}'(K) &= \widehat{C}'\big(l^{-1}(l(K))\big)\\
&= -e^{-rT}\sqrt{\frac{2T}{\pi}}\,l(K)\,e^{\frac{l(K)}{2}}\,f'(l^2(K))e^{-\frac{f^4(l^2(K))T^2+4l^2(K)}{8f^2(l^2(K))T}}\\
&\quad -e^{-rT}\Phi\big(\tfrac{l(K)}{f(l^2(K))\sqrt{T}} - \tfrac{f(l^2(K))\sqrt{T}}{2}\big). \qquad (27)
\end{aligned}$$

Therefore, after some manipulation, and using $K = K_{at}e^{-l(K)}$, we have

$$\begin{aligned}\widehat{C}''(K) &= \frac{e^{-2rT}}{s\sqrt{2\pi T}} e^{\frac{3}{2}l(K)} e^{-\frac{f^2(l^2(K))T}{8}} e^{-\frac{l^2(K)}{2f^2(l^2(K))T}} \\ &\left\{\frac{1}{f(l^2(K))}\left[\frac{2l^2(K)}{f(l^2(K))}f'(l^2(K)) - 1\right]^2 - T^2 l^2(K) f(l^2(K))\big[f'(l^2(K))\big]^2 \right. \\ &\left. + 4Tl^2(K)f''(l^2(K)) + 2Tf'(l^2(K))\right\}. \end{aligned} \tag{28}$$

We obtain (25) and (26), setting $x := l^2(K)$ in (27) and (28):

$$\begin{aligned}\widehat{C}'\big(l^{-1}(\sqrt{x})\big) = -e^{-rT}\sqrt{\frac{2T}{\pi}}\,\sqrt{x}\,e^{\frac{\sqrt{x}}{2}} f'(x) e^{-\frac{f^4(x)T^2+4x}{8f^2(x)T}} \\ - e^{-rT}\Phi\big(\tfrac{\sqrt{x}}{f(x)\sqrt{T}} - \tfrac{f(x)\sqrt{T}}{2}\big).\end{aligned}$$

$$\begin{aligned}\eta(x) = e^{-\frac{f^4(x)T^2+4x}{8f^2(x)T}} &\left\{\frac{1}{f(x)}\left[\frac{2xf'(x)}{f(x)} - 1\right]^2\right. \\ &\left. - T^2 x f(x)\big[f'(x)\big]^2 + 4Txf''(x) + 2Tf'(x)\right\}. \end{aligned} \tag{29}$$

Since (27) and (28) hold for all $K > 0$, the function f is a general solution of (26). To obtain the boundary values of f and f' respectively we observe that

$$\widehat{C}(K_{at}) = s\left[\Phi\big(\tfrac{f(0)\sqrt{T}}{2}\big) - \Phi\big(-\tfrac{f(0)\sqrt{T}}{2}\big)\right] = s\left[2\,\Phi\big(\tfrac{f(0)\sqrt{T}}{2}\big) - 1\right], \tag{30}$$

that $\eta(0) = se^{2rT}\sqrt{2\pi T}\widehat{C}''\big(K_{at}\big)$, and take the limit for x going to 0 in (29). $\square$

Remark 3.4. *We would like to point out here that, if $\widehat{C}(K)$ is a C^2-function, then, by Theorem 3.3, there are some necessary conditions in order that (9) holds: there exists a general solution $y(x) > 0$ of (25) and (26), with $y'(x) \geq 0$, $x \in [0, \infty)$, and with $\frac{e^{-\frac{y^2(0)T}{8}}}{y(0)} \leq se^{2rT}\sqrt{2\pi T}\widehat{C}''\big(K_{at}\big)$.*

3.1 An application: Σ^2 has a GIG distribution

Let us now turn to a particular case: we assume that the density of Σ is

$$g(v) = g_\Sigma(v) = \frac{\big(a/b\big)^{\frac{q+1}{4}}}{\mathcal{K}_{\frac{q+1}{2}}\big(\sqrt{ab}\big)}\, v^q e^{-(av^2+b/v^2)/2}, \tag{31}$$

with parameters $a > 0$, $b > 0$, $q \in \mathbb{R}$, and where $\mathcal{K}_\nu(z)$ is the modified Bessel function of the second kind. It goes without saying that Σ^2 has a GIG distribution

$$g_{\Sigma^2}(v) = \frac{1}{2}\frac{\big(a/b\big)^{\frac{q+1}{4}}}{\mathcal{K}_{\frac{q+1}{2}}\big(\sqrt{ab}\big)}\, v^{(\frac{q+1}{2}-1)} e^{-\frac{a}{2}v} e^{-\frac{b}{2v}}. \tag{32}$$

This assumption on the distribution implies that the traders assume a generalized hyperbolic distribution for daily returns, which seems to be appropriate (see e.g. the pioneering paper by Eberlein and Keller ([11]) for the case $q = 1$). Moreover, under the above assumption $E\big[(1/\Sigma)^k\big]$ is finite for all integers k, and therefore $\widehat{C} = \widehat{C}_{a,b,q}$ is a C^∞ function and all the results in the previous section hold. In this setting, the regularity of $\widehat{C}_{a,b,q}$, and of η as well, can be obtained also directly: by Proposition 3.1 the second derivative of the option price is

$$\widehat{C}''_{a,b,q}(K) = \frac{e^{-2rT}}{s\sqrt{2\pi T}}\,\frac{\big(a/b\big)^{\frac{q+1}{4}}}{\mathcal{K}_{\frac{q+1}{2}}\big(\sqrt{ab}\big)}\,\frac{e^{\frac{3}{2}l(K)}\,\mathcal{K}_{\frac{q}{2}}\Big(\sqrt{\big(a+\frac{T}{4}\big)\big(b+\frac{l^2(K)}{T}\big)}\Big)}{\Big[\big(a+\frac{T}{4}\big)/\big(b+\frac{l^2(K)}{T}\big)\Big]^{\frac{q}{4}}}, \tag{33}$$

then the regularity of $\widehat{C}_{a,b,q}$ follows from the regularity of the modified Bessel functions. The regularity of

$$\eta(x) = \eta_{a,b,q}(x) = \frac{\big(a/b\big)^{\frac{q+1}{4}}}{\mathcal{K}_{\frac{q+1}{2}}\big(\sqrt{ab}\big)}\cdot\frac{\mathcal{K}_{\frac{q}{2}}\Big(\sqrt{\big(a+\frac{T}{4}\big)\big(b+\frac{x}{T}\big)}\Big)}{\Big[\big(a+\frac{T}{4}\big)/\big(b+\frac{x}{T}\big)\Big]^{\frac{q}{4}}}$$

follows similarly.

In conclusion we illustrate numerically the behaviour of the smile when Σ^2 has a GIG distribution. Figure 1 shows the results of the numerical experiment for a particular choice of the (true) parameters, together with a numerical estimation of the parameters.

Assuming, as "in practice", that the price for call options is observed only for a few values of the strike price K_j, and taking the current stock price $s = 10$, the strike prices $K_j = 10 + j\,\delta$, with $j \in \{-4, -3, ..., +3, +4\}$ and $\delta = 0.5$, the time to maturity $T = 1$, and the riskless interest rate $r = 0$, we have computed the data prices $\widehat{C}(K_j) = \widehat{C}_{a,b,q}(K_j)$, for fixed parameters a, b, q (we have used a Montecarlo method, with 10.000 samples). We have estimated the parameters by two slightly different least square methods:

$$\inf_{\alpha,\beta,\gamma} G_i(\alpha,\,\beta,\,\gamma), \qquad i = 1,2$$

with

$$G_1(\alpha,\,\beta,\,\gamma) = \sum_{j=-4}^{4} \left(\widehat{C}_{\alpha,\beta,\gamma}(K_j) - \widehat{C}(K_j)\right)^2,$$

and

$$G_2(\alpha,\,\beta,\,\gamma) = \sum_{j=-4}^{4} \left(\widehat{C}_{\alpha,\beta,\gamma}(K_j) - \widehat{C}(K_j)\right)^2 + \sum_{j=-3}^{3} \left(\widehat{C}''_{\alpha,\beta,\gamma}(K_j) - \triangle_2\widehat{C}(K_j)\right)^2,$$

where

$$\triangle_2\widehat{C}(K) = \frac{\widehat{C}(K+\delta) - 2\widehat{C}(K) + \widehat{C}(K-\delta)}{\delta^2}.$$

The idea behind the choice of G_2 is that $\triangle_2\widehat{C}(K_j)$ can be computed only by means of the observed data $\widehat{C}(K_j)$ and $\triangle_2\widehat{C}(K) \approx \widehat{C}''(K)$, when δ is small with respect to K.

We have repeated the numerical experiment with all the parameters a, b, $q \in \{1,2,3,4\}$. The results about the implied volatility are shown in Figure 1, for a particular choice of the parameters. Concerning the overall results about the implied volatility we notice that the second method seems to give a slight improvement with respect to the first one, since average relative errors are $1,6\%$ and $1,5\%$ for the first and second method, respectively.

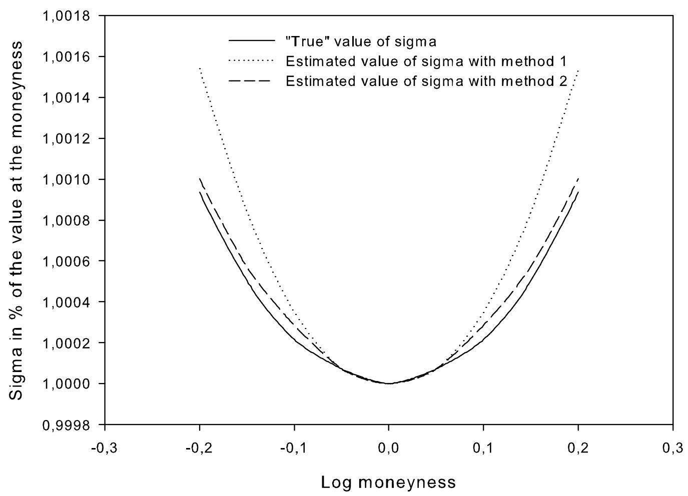

Figure 1: Volatility smile $a = b = q = 2$, $l = l(K) \in [-0.2, 0.2]$

References

[1] Arthur, B., J. Holland, B. LeBaron, R. Palmer, P. Taylor: *Asset pricing under endogenous expectations in an artificial stock market.* In Arthur B., S.N. Durlauf, D.A. Lane (Eds). The economy as an evolving complex system, Addison-Wesley, Reading, MA, (1997).

[2] Bates, D.: *The crash of 87: Was it expected. The evidence from options markets.* Journal of Finance, 42, 301-320, (1991).

[3] Black F.: *Fact and fantasy in the use of options.* Financial Analysis Journal, 31, 36-41, (1975).

[4] Black F., Scholes M.: *The pricing of options and corporate liabilities.* Journal of Political Economy, 81, 659-683, (1973).

[5] Brandt M.W., C.S. Jones: *Volatility forecasting with range-based EGARCH models.* Journal of Business & Economic Statistics, 24, 470-486, (2006).

[6] Cont R., J. da Fonseca: *The dynamics of implied volatility surfaces* Quantitative Finance, 2, 45-60, (2002).

[7] de Finetti, B.: *Sul concetto di media.* Giornale dell'Istituto Italiano degli Attuari, 2, 369-396, (1931).

[8] DeLong, J.B., A. Shleifer, L.H. Summers, R.J. Waldmann: *Noise trader risk in financial markets.* Journal of Political Economy, 98, 703-738, (1990).

[9] Derman E., Kani I.: *Riding on a smile.* Risk, 7, 32-39, (1994).

[10] Detemple, J.B., C. Osakwe: *The valuation of volatility options.* European Finance Review, 4, 21-50, (2000).

[11] Eberlein, E., U. Keller: *Hyperbolic Distributions in Finance.* Bernoulli, 1, 281-299, (1995).

[12] Ehrentreich N.: *Technical Trading in the Santa Fe Institute Artificial Stock Market.* Revisited Journal of Economic Behavior and Organization,(2006).

[13] Feller W.: *An Introduction to Probability Theory and Its Applications, Volume II.* (Second Edition), Wiley, New York, (1971).

[14] Galai D.: *A survey of empirical tests of option pricing models. In Brenner, M. (Ed). Option pricing: Theory and applications.* Lexington Books, Lexington, (1983).

[15] Gemmill G., Kamiyama N.: *International transmissions of option volatility and skewness: when you are smiling, does the whole world smile?* Working paper, City University of London, (1997).

[16] Heston, S.L.: *A closed-form solution for options with stochastic volatility with applications to bond and currency options.* Review of Financial Studies, 6, 327-343, (1993).

[17] Heynen R.: *An empirical investigation of observed smile patterns.* Review of Future Markets, 13, 317-353, (1993).

[18] Hull J., White A.: *The pricing of options on assets with stochastic volatilities.* Journal of Finance, 42, 281-300, (1987).

[19] Kou, S.G.: *A jump-diffusion model for option pricing.* Management Science, 48, 1086-1101, (2002).

[20] Macbeth J., Merville L.J.: *An empirical examination of the Black-Scholes call option pricing model.* Journal of Finance, 34, 1173-1186, (1979).

[21] Merton, R.: *Option pricing when the underlying stock returns are discontinuous.* Journal fo Financial Economics, 4, 125-144, (1976).

[22] Palmer, R.G., Arthur, B., Holland, J., LeBaron, B., P. Taylor: *Artificial economic life: A simple model of stockmarket.* Physica D, 75, 264-274, (1994).

[23] Renault, E., N. Touzi: *Option hedging and implied volatilities in a stochastic volatility model.* Mathematical Finance, 9, 279-302, (1996).

[24] Rubinstein M.: *Non-parametric tests of alternative option pricing models.* Journal of Finance, 40, 455-480, (1985).

[25] Rubinstein M.: *Implied binomial trees.* Journal of Finance, 49, 771–818, (1994).

[26] Sircar, R., G. Papanicolaou: *Stochastic volatility, smile and asymptotics.* Applied Mathematical Finance, 6, 107-145, (1999).

[27] Stein, E., J.Stein: *Stock price distributions with stochastic volatility: An analytical approach.* Review of Financial Studies, 4, 727-752, (1991).

[28] Wiggins, J.B.: *Option values under stochastic volatility - theory and empirical estimates.* Journal of Financial Economics, 19, 351-372, (1987).

[29] Ziegler, A.: *State-price densities under heterogeneous beliefs, the smile effect, and implied risk aversion.* European Economic Review, 46, 1539-1557, (2002).