

Relations between Kendall distributions and level sets in bivariate exchangeable survival models

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Abstract

For a given bivariate survival function \bar{F} , we study the relations between the set of the level curves of \bar{F} and the *Kendall distribution*. Then we characterize the survival models simultaneously admitting a specified Kendall distribution and a specified set of level curves. Attention will be restricted to exchangeable survival models. Furthermore, we assume that the level curves of \bar{F} are regular curves within the quadrant \mathbb{R}_+^2 . For our results we combine two different methods. On the one hand we use a transformation result proven by Genest and Rivest (2001); on the other hand we resort to the semi-copula representation of the level curves of \bar{F} , introduced by Bassan and Spizzichino (2001). On the one hand we resort to the semi-copula representation of the level curves of \bar{F} , introduced by Bassan and Spizzichino (2001); on the other hand we use a semicopula version of a transformation result proven by Genest and Rivest (2001) for copulas.

Also, results by Genest and Rivest (1993), Nelsen et Al. (2003), and Nelsen et Al. (2008), concerning the equivalence class of bivariate copula sharing a same Kendall distribution, reveal to have a basic role in our analysis. In view of such result in fact it turns out that, under simple technical hypothesis, survival models with Archimedean survival copulas have a key role in the study of the general case.

Key words: Survival Copulas, Bivariate Aging Functions, Semi-copulas, Archimedean Copulas, Associative Copulas, Bivariate VaR-Curves.

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1 Introduction

Let X, Y be a pair of non-negative random variables and let $\bar{F}(x, y)$ denote their *joint survival function*, namely, for $x \geq 0, y \geq 0$,

$$\bar{F}(x, y) = P\{X > x, Y > y\}.$$

We concentrate our attention on the case when X, Y are exchangeable and assume \bar{F} to be continuous, strictly positive and strictly 1-decreasing. Denoting by $\bar{G}(\cdot)$ the common univariate marginal survival function of X and Y , i.e.

$$\bar{G}(t) = P\{X > t\} = \bar{F}(t, 0) = P\{Y > t\} = \bar{F}(0, t),$$

\bar{G} will then be continuous, strictly positive, and strictly decreasing all over the half-line $[0, \infty)$, with $\bar{G}(0) = 1$. For $v \in (0, 1)$, we consider the *level set*

$$A_v := \{(x, y) \in \mathbb{R}_+^2 \mid \bar{F}(x, y) \leq v\},$$

the set $\mathcal{L}_{\bar{F}} = \{A_v \mid v \in (0, 1)\}$, and assume that the boundary ∂A_v ,

$$\partial A_v := \{(x, y) \in \mathbb{R}_+^2 \mid \bar{F}(x, y) = v\},$$

is a continuous curve, that will be referred to as a *level curve*. In the rest of the paper all these conditions will be always assumed and will be referred to as our standing hypotheses.

We will be interested in survival models admitting a given set of level curves. We mention in this respect that, for $0 < v < 1$, ∂A_v can be interpreted as the bivariate (*upper-orthant*) *Value at Risk curve at level $1 - v$* (see [17], [8]).

We set

$$\widehat{K}(v) := P\{(X, Y) \in A_v\} = P\{\bar{F}(X, Y) \leq v\}, \quad (1)$$

and we shall write $\widehat{K}_{\bar{F}}$ when we need to stress that \widehat{K} is the upper-orthant Kendall distribution associated to \bar{F} .

We recall that, for a pair of random variables (X, Y) with joint distribution function F , the random variable $F(X, Y)$ is known as the *Bivariate Probability Integral Transformation* (BIPIT) (see e.g. [10]) and that its distribution function K is called the *Kendall distribution function* associated to F (see [9], [15], [16], and references therein for basic details about Kendall distributions). In [10] and [16] several aspects of the Kendall distribution K have been analyzed, based on the fact that it only depends on the connecting copula of X, Y . As we shall see, these papers provide an essential background for the present paper.

\widehat{K} in (1) will be termed "upper-orthant" Kendall distribution. It is easy to check (see also Section 4 below) that \widehat{K} is a Kendall distribution function associated to the bivariate distribution function \widehat{C} , the *survival copula* associated to \overline{F} , i.e.

$$\widehat{C}(u, v) := \overline{F}(\overline{G}^{-1}(u), \overline{G}^{-1}(v)).$$

Therefore, similarly to what happens for the Kendall distribution K , \widehat{K} is determined by \widehat{C} . In this respect we recall the following representation for \overline{F} in terms of \widehat{C} and \overline{G} :

$$\overline{F}(x, y) = \widehat{C}(\overline{G}(x), \overline{G}(y)), \quad (2)$$

which, under our hypotheses, uniquely characterizes the survival copula.

We shall denote by $\mathcal{L}_{\overline{F}}$ the set $\{A_v | v \in (0, 1)\}$ and write $\widehat{K}_{\overline{F}}$ when we need to stress that \widehat{K} is the upper-orthant Kendall distribution associated to \overline{F} .

There is however an infinite class of bivariate copulas that share the same Kendall distribution. There is then an infinite class of bivariate survival functions \overline{H} such that $\widehat{K}_{\overline{H}} = \widehat{K}_{\overline{F}}$; in particular this relation holds if \overline{H} and \overline{F} admit the same survival copula \widehat{C} , i.e. if it is

$$\overline{H}(x, y) = \overline{F}(\Phi(x), \Phi(y))$$

for some continuous, strictly increasing, mapping $\Phi : [0, \infty) \rightarrow [0, \infty)$.

On the other hand, it is immediate to see that any bivariate survival function \overline{M} of the form

$$\overline{M}(x, y) = \Psi[\overline{F}(x, y)],$$

with $\Psi : [0, 1] \times [0, 1] \rightarrow [0, 1]$ strictly increasing, is such that $\mathcal{L}_{\overline{M}} = \mathcal{L}_{\overline{F}}$.

For given \overline{F} , we then wonder if there exists a different bivariate survival function \overline{N} such that, simultaneously,

$$\mathcal{L}_{\overline{N}} = \mathcal{L}_{\overline{F}}, \quad \widehat{K}_{\overline{N}} = \widehat{K}_{\overline{F}},$$

and, in case, which type of relations exist between \overline{N} and \overline{F} .

More generally, in this paper we face the following *compatibility problem*: letting $K = \widehat{K}_{\overline{H}}$ be the upper-orthant Kendall distribution of a given joint survival function \overline{H} and $\mathcal{L} = \mathcal{L}_{\overline{J}}$, for a given joint survival function \overline{J} satisfying our required properties, we wonder if the system of functional equations

$$\mathcal{L}_{\overline{F}} = \mathcal{L}, \quad \widehat{K}_{\overline{F}} = K$$

admit some solution \overline{F} . If this is the case we shall say that \mathcal{L} and K are *compatible*.

Our main result concerns the compatibility problem and will be stated as Theorem 2 in the next Section 2. Before proving Theorem 2, we need some preliminary notation and results about the representation of the set \mathcal{L} , about (upper-orthant) Kendall distributions and about the relations existing between $\mathcal{L}_{\overline{F}}$ and $\widehat{K}_{\overline{F}}$, for a same \overline{F} . Essentially, our analysis will be based on three different types of arguments:

a) For a bivariate survival model satisfying our assumptions, the set \mathcal{L} can be represented in terms of the so-called *bivariate aging function* B , whose definition will be recalled in Section 3 (see (9); see also, e.g., [4]) and which is, generally, a *semi-copula*. Concerning the use of the notion of the bivariate aging function in order to describe the set \mathcal{L} , it is worthwhile mentioning that, along with the familiar representation (2), \overline{F} will also be alternatively described in terms of the pair (B, \overline{G}) (see (10), below).

b) We shall briefly recall results obtained in [9], [10], [15], and [16], concerning the equivalence class \mathcal{C}_K of all bivariate copulas admitting a same Kendall distribution K . We shall in particular use a basic result showing that, under the condition $K(t^-) > t$, the class \mathcal{C}_K contains only one Archimedean copula. The generator of such a copula will have a fundamental role in the solution of the compatibility problem.

c) We analyze some specific aspects of the relations existing between $\mathcal{L}_{\overline{F}}$ and $\widehat{K}_{\overline{F}}$, for a same \overline{F} . To this purpose we use (a slight modification of) a transformation result proven in [10].

We shall also see that the analysis of the compatibility problem is rather direct in the special case when the survival copulas \widehat{C}_H and \widehat{C}_J are Archimedean. Such a case will be analyzed in some details, since it also has a key role in the interpretation of the result for the general case.

More in details, the outline of the paper is as follows. Our main result will be stated in Section 2, and proven in Section 5, under the conditions that \overline{G}_J admits a density function g and that \widehat{K} is continuous. In Section 2, we will also recall some basic facts about the Kendall distributions. Sections 3 and 4 will be devoted to collect further developments and results that are preliminary to the proof of Theorem 2; in Section 3 we discuss in which sense the level sets of a bivariate, exchangeable, survival function can be described in terms of the semi-copula B and, then, in which sense the compatibility problem above can be formulated in terms of B . We also provide some examples and details useful for our purposes. In Section 4 we analyze a few aspects of \widehat{K} , corresponding to those obtained for K in [10]; in particular we detail the form under which \widehat{K} depends on the survival copula \widehat{C} of \overline{F} and its relations with the set of the level curves of \overline{F} . Section 5 will be devoted to present the proof of Theorem 2 and some related comments based on the analysis of special cases. In Section

6 we present a short discussion and some concluding remarks about meaning and possible applications of the obtained results.

2 Strict Archimedean Kendall distributions and solution to the compatibility problem.

In order to state our main result we recall here some relevant facts about the Kendall distributions and we introduce some terminology and notation, convenient for our purposes. In particular we focus on the concept of *Strict Archimedean Kendall distribution*.

Let X, Y be two random variables, with joint distribution function $F(x, y) = P\{X \leq x, Y \leq y\}$, and with continuous univariate marginal distributions F_X and F_Y , respectively. As mentioned in the Introduction, the probability distribution function of the random variable $F(X, Y)$ has been called Kendall distribution and several basic aspects of it have been studied in [9], [10], [15], and in the recent paper [16]. For $0 \leq u \leq 1$, set then

$$K(u) = P\{F(X, Y) \leq u\}.$$

K satisfies the conditions

$$K(0^-) = 0, \quad K(u) \geq u, \quad u \in [0, 1]. \quad (3)$$

Furthermore, in view of Sklar's Theorem (see e.g. [14]), it is also easily seen that K only depends on the *connecting copula* of F

$$C(u, v) = F(F_X^{-1}(u), F_Y^{-1}(v)), \quad u, v \in [0, 1].$$

We remind that a bivariate copula is *Archimedean* if it is of the form

$$C^\phi(u, v) = \phi^{-1}[\phi(u) + \phi(v)] \quad (4)$$

with $\phi : (0, 1] \rightarrow [0, +\infty)$ a convex, continuous, decreasing function. ϕ is called an (*additive*) *generator* of C^ϕ and it is obviously determined up to a positive multiplicative constant. ϕ is said to be *strict* when $\phi(0) = \phi(0^+) = +\infty$.

Consider the family of bivariate copulas C , such that $C_v(\cdot) := C(\cdot, v)$ is a strictly increasing and continuous function for every v and, on this family, define the operator \mathcal{K} as follows:

$$\mathcal{K}C(t) := t + \int_t^1 \frac{\partial C}{\partial u}(u, v)|_{v=C_{u,t}} du, \quad (5)$$

with $v_{u,t}^C = C_u^{-1}(t)$. Set also $\lambda_C(t) := - \int_t^1 \frac{\partial C}{\partial u}(u, v)|_{v=v_{u,t}^C} du$ so that

$$\mathcal{K}C(t) = t - \lambda_C(t).$$

Then the Kendall distribution associated to C is given by $\mathcal{K}C(t)$ (see e.g. [10]).

For any distribution function K satisfying (3), it is possible to find a bivariate copula such that its Kendall distribution coincides with K . Generally, there is an equivalence class \mathcal{C}_K of different bivariate copulas admitting the same Kendall distribution K . If, moreover, it is

$$K(t^-) > t, \forall t \in (0, 1), \quad (6)$$

then there exists a unique Archimedean copula C^ϕ belonging to \mathcal{C}_K .

Letting t_0 be an arbitrary constant, arbitrarily chosen in $(0, 1)$, any possible generator of C^ϕ is a decreasing function $\phi : (0, 1] \rightarrow [0, +\infty)$ of the form

$$\phi(t) = \theta \cdot \exp\left\{\int_{t_0}^t \frac{1}{s - K(s)} ds\right\}, \quad (7)$$

with $\theta > 0$. This result had been proven by Genest and Rivest in [9] and has been exploited by Nelsen et al. in [16], to prove a result valid in the case when (3) holds but without assuming the condition (6). Such a result says that, under the conditions (3), there exists a unique *associative* copula C belonging to \mathcal{C}_K . As well known, a copula C is said to be associative, when

$$C(C(u, v), w) = C(u, C(v, w))$$

holds for every u, v and w in $[0, 1]$. Since an Archimedean copula is associative, C coincides with C^ϕ above, in the case when $K(t^-) > t$ for all $t \in (0, 1)$.

In order to state our main result, we introduce the following

Definition 1 *A distribution function K is an Archimedean Kendall distribution when it satisfies the conditions in (6). We say that K is a strict Archimedean Kendall distribution when the generators of the associated Archimedean copula, that are then given by (7), are strict. For an Archimedean Kendall distribution K , we will use the notation*

$$\Upsilon_{t_0, K}(t) = \exp\left\{\int_{t_0}^t \frac{1}{s - K(s)} ds\right\}.$$

Concerning the upper-orthant Kendall distributions, it is important to point out that also \widehat{K} is a Kendall distribution, as it is easy to check (see also Section 4 below) and as we mentioned in the Introduction.

Now we consider a pair (K, \bar{J}) where K is a Kendall distribution and \bar{J} is an exchangeable bivariate survival function, jointly continuous and strictly one-decreasing, with marginal survival function $\bar{G}_{\bar{J}}$. We look for bivariate survival functions \bar{F} such that

$$\widehat{K}_{\bar{F}} = K, \mathcal{L}_{\bar{F}} = \mathcal{L}_{\bar{J}}. \quad (8)$$

Our result is the following

Theorem 2 *Let \bar{J} be as above, and let K be a strict Archimedean Kendall distribution with generator $\hat{\phi}$. Assume furthermore that $\bar{G}_{\bar{J}}$ is differentiable and that $\widehat{K}_{\bar{J}}$ is also a strict Archimedean Kendall distribution, with generator $\Upsilon_{t_0, \widehat{K}_{\bar{J}}}(t)$. Let*

$$\begin{aligned} \varphi(t) &:= \Upsilon_{t_0, \widehat{K}_{\bar{J}}}(\bar{G}_{\bar{J}}(-\log t)), \\ \bar{G}_{\theta}(x) &:= \hat{\phi}^{-1}(\theta \varphi(e^{-x})) = \hat{\phi}^{-1}\left(\theta \Upsilon_{t_0, \widehat{K}_{\bar{J}}}(\bar{G}_{\bar{J}}(x))\right). \end{aligned}$$

Then

(a) $\bar{G}_{\theta}(x)$ is a survival function for any $\theta > 0$

(b) A bivariate survival function \bar{F} , satisfying our standing hypotheses, satisfies (8) if and only if it is of the form

$$\bar{F}_{\theta}(x, y) = \bar{G}_{\theta}(\bar{G}_{\bar{J}}^{-1}(\bar{J}(x, y)))$$

(c) \bar{G}_{θ} is the marginal survival function of \bar{F}_{θ}

In order to better focus our developments, a few remarks and comments are now in order.

First we notice that the bivariate function $\bar{G}_{\theta}(\bar{G}_{\bar{J}}^{-1}(\bar{J}(x, y)))$ is not necessarily a joint survival function. Even though sufficient conditions under which $\bar{F}_{\theta}(x, y)$ is a joint survival function could be given, we do not detail them here for brevity's reasons.

Remark 3 *The solution of the compatibility problem rather depends on the set $\mathcal{L}_{\bar{J}}$ and not on the specific survival function \bar{J} . In fact the solution depends on $\bar{G}_{\bar{J}}^{-1}(\bar{J}(x, y))$ and all the different survival functions sharing the same \mathcal{L} give rise to the same solution (see also the arguments in the next Section). Furthermore, it cannot come as a surprise that a solution \bar{F} is of the form $\bar{F}(x, y) = \bar{G}_{\bar{F}}(\bar{G}_{\bar{J}}^{-1}(\bar{J}(x, y)))$ (in this respect see, more specifically, Lemma 8 below). The actual problem is rather to understand whether we can establish compatibility for given $(K, \mathcal{L}_{\bar{J}})$ and to identify the possible acceptable marginals $\bar{G}_{\bar{F}}$.*

As a consequence of our analysis we will also single out some conditions under which a pair (K, \mathcal{L}) is not compatible.

In order to prove Theorem 2 we need some preliminary notation and results that will be provided in the next two Sections.

3 On the semi-copula representation of $\mathcal{L}_{\bar{F}}$

Let \bar{F} be an exchangeable bivariate survival function satisfying our standing hypotheses. To start this Section, we first consider the function $h : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ defined by

$$h(x, y) = h_{\bar{F}}(x, y) = \bar{G}^{-1} [\bar{F}(x, y)].$$

In view of our assumptions on \bar{F} , h is continuous and strictly 1-increasing; furthermore, ∂A_v is the image of a function $\gamma_v : [0, 1] \rightarrow [0, \infty) \times [0, \infty)$, connecting the Cartesian axes, i.e. such that $\gamma_v(0) = (x_v, 0)$ and $\gamma_v(1) = (0, y_v)$, for some strictly positive x_v and y_v . Concerning x_v , y_v , and the curve γ_v , the following properties can be easily checked:

$$x_v = y_v = \bar{G}^{-1}(v), \quad x_{\bar{F}(x, y)} = h(x, y).$$

Moreover, the curve γ_v can be parametrized as a function $y(v, x)$ of the variable x , with $x \in [0, x_v]$,

$$y(v, 0) = x_v, y(v, x_v) = 0,$$

and, for $x \in [0, x_v]$, $y(v, x) = h_x^{-1}(\bar{G}^{-1}(1 - v))$, where $h_x(\cdot) = h(x, \cdot)$.

We can also write

$$A_v = \{(x, y) \in \mathbb{R}_+^2 | h(x, y) \geq \bar{G}^{-1}(v)\}$$

and

$$\partial A_v = \{(x, y) \in \mathbb{R}_+^2 | h(x, y) = \bar{G}^{-1}(v)\}.$$

The function h then determines $\mathcal{L}_{\bar{F}}$ or, equivalently, the set $\mathcal{D}_{\bar{F}}$ of the level curves ∂A_v of \bar{F} , while the marginal \bar{G} determines the level on each level curve.

It is also immediate to see that the following Lemma holds.

Lemma 4 *Let \bar{F} and \bar{J} be two different survival functions satisfying our hypotheses, then $\mathcal{L}_{\bar{F}} = \mathcal{L}_{\bar{J}}$ if and only if $h_{\bar{F}} = h_{\bar{J}}$.*

For our purposes we follow however the approach introduced in [2], [4], and replace the function h by the function $B : [0, 1] \times [0, 1] \rightarrow [0, 1]$ defined by:

$$B(u, v) = \exp\{-h(-\log u, -\log v)\}, \quad u, v \in [0, 1],$$

i.e.

$$B(u, v) = \exp\{-\bar{G}^{-1}(\bar{F}(-\log u, -\log v))\}. \quad (9)$$

We can then write

$$\bar{F}(x, y) = \bar{G}(-\log B(e^{-x}, e^{-y})). \quad (10)$$

Notice that Eq. (10) provides the already mentioned representation of $\bar{F}(x, y)$ in terms of the pair (B, \bar{G}) ; this is in a sense analogous to (but different from) the representation (2).

In order to fix ideas it is useful to consider the set of the level curves, and the related functions B , corresponding to three very special cases that will also have a role in the subsequent analysis.

Example 5 (Perfect dependence) *Let $P\{X = Y\} = 1$. Then $\bar{F}(x, y) = \bar{G}(x \vee y)$ and, for $0 \leq v \leq 1$,*

$$A_v = \{(x, y) : x \vee y \geq \bar{G}^{-1}(v)\}. \quad (11)$$

In this case, B is the maximal copula

$$B(u, v) = u \wedge v \quad (12)$$

Example 6 ("Schur-constant" \bar{F}) *Here we consider the case*

$$\bar{F}(x, y) = \bar{G}(x + y) \quad (13)$$

where \bar{G} is a univariate continuous, convex, strictly positive and strictly decreasing survival function on $[0, +\infty)$. It is immediate to check that \bar{G} has also the role of univariate marginal and that

$$A_v = \{(x, y) : x + y \geq \bar{G}^{-1}(v)\}.$$

The condition (13) holds if and only if B is the product copula:

$$B(u, v) = u \cdot v.$$

Example 7 (i.i.d. variables) *When $\bar{F}(x, y) = \bar{G}(x) \cdot \bar{G}(y)$, we can write*

$$A_v = \{(x, y) : \Lambda(x) + \Lambda(y) \geq -\log v\},$$

by setting $\Lambda(x) := -\log \bar{G}(x)$. In terms of the function B , we can say that the above condition holds if and only if

$$B(u, v) = Q^{-1}[Q(u) + Q(v)], \quad (14)$$

where we set

$$Q(u) := \Lambda(-\log u) = -\log \bar{G}(-\log u). \quad (15)$$

The function B turns out to be a convenient tool in the study of certain notions of multivariate ageing (see in particular [2], [4], [7]) and has also been termed *bivariate ageing function*.

In view of what has been discussed so far, B can be seen as a tool for representing $\mathcal{L}_{\bar{F}}$ or $\mathcal{D}_{\bar{F}}$. Taking into account the definition (9), we can in fact replace Lemma 2 above by the following Lemma, whose proof is immediate (see also [3]).

Lemma 8 *Let \bar{F} and \bar{J} be two different survival functions satisfying our hypotheses. Then the following conditions are equivalent*

(a) $\mathcal{L}_{\bar{F}} = \mathcal{L}_{\bar{J}}$

(b) $B_{\bar{F}} = B_{\bar{J}}$.

(c) *there exists a continuous, strictly increasing, function $\psi : [0, 1] \rightarrow [0, 1]$, such that*

$$\bar{F}(x, y) = \psi(\bar{J}(x, y))$$

(d) *it is*

$$\bar{F}(x, y) = \bar{G}_{\bar{F}} \left[\bar{G}_{\bar{J}}^{-1}(\bar{J}(x, y)) \right]. \quad (16)$$

Notice that point (b) of Theorem 2 provides more detailed information than Eq. (16); the latter in fact has been obtained by imposing the condition (a) (or (b)) above and we do not take into account here possible information about $\widehat{K}_{\bar{F}}$ (see also Remark 3 above).

Let us now look at analytical properties of the function B , as defined by (9).

Since X, Y are exchangeable it is

$$B(u, v) = B(v, u).$$

It is immediate to check that $B : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is strictly increasing in each variable and also shares with a copula the property to be *grounded*, i.e.

$$B(u, 0) = 0, \quad B(u, 1) = u, \quad \forall 0 \leq u \leq 1.$$

However it is not necessarily a copula: in some cases it can happen that is not *two-increasing*, as for example in (14) whenever Q is not convex.

The term *semi-copula* has been introduced to describe functions $B : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that are grounded and strictly increasing in each variable. See [4], [6]; see in particular [6] for the analysis of some technical aspects of the notion of semi-copula and also [7] for the relations with the concept of *Choquet capacity*. This notion can have however other interesting applications in problems of rather different type (see e.g. [5], [12]).

In the special case (14), B is in any case an *Archimedean t-norm* (see [13]); in some of the cited past papers the term *Archimedean semi-copula* had also been used.

Generally, for an Archimedean t-norm

$$S^\varphi(u, v) = \varphi^{-1}[\varphi(u) + \varphi(v)],$$

the *generator* φ is a decreasing function, with $\varphi(1) = 0$. When $\varphi(0+) = \infty$, then we say that φ is a *strict generator*. It is clear that if φ is a generator, then $\theta \varphi$ is also a generator, for any $\theta > 0$.

As mentioned just above, when the generator is a convex function ϕ , then S^ϕ is a copula and it coincides with the Archimedean copula C^ϕ , where we used the notation (4).

We now come back to the compatibility problem stated in the Introduction. In view of Lemma 8, we can in fact restate it as follows:

Let K be a given Kendall distribution and B a (semi-)copula such that $B = B_{\bar{J}}$, for a given joint survival function \bar{J} satisfying our required properties, we wonder if B and K are *compatible*, i.e., if we can find some solution \bar{F} to the system of equations

$$B_{\bar{F}} = B, \quad \widehat{K}_{\bar{F}} = K.$$

Since $\widehat{K}_{\bar{F}}$ is a characteristic of $\widehat{C}_{\bar{F}}$, for the study of this problem it is necessary to analyze the relations existing between $B_{\bar{F}}$ and $\widehat{C}_{\bar{F}}$.

Regardless of B being or not a copula, the relations between B and \widehat{C} involve the marginal survival function \bar{G} ; in fact we can write, from (10) and (2),

$$B(u, v) = \exp \left\{ -\bar{G}^{-1} \left(\widehat{C} \left(-\log \bar{G}(u), -\log \bar{G}(v) \right) \right) \right\},$$

$$\widehat{C}(u, v) = \bar{G} \left(-\log B \left(e^{-\bar{G}^{-1}(u)}, e^{-\bar{G}^{-1}(v)} \right) \right),$$

or

$$B(u, v) = \gamma \left(\widehat{C} \left(\gamma^{-1}(u), \gamma^{-1}(v) \right) \right), \quad \widehat{C}(u, v) = \gamma^{-1} \left(B(\gamma(u), \gamma(v)) \right) \quad (17)$$

where $\gamma, \gamma^{-1} : [0, 1] \rightarrow [0, 1]$ are the increasing functions defined by

$$\gamma(w) = \exp\{-\overline{G}^{-1}(w)\}, \quad \gamma^{-1}(u) = \overline{G}(-\log u). \quad (18)$$

Example 9 *In the perfect dependence case it is, trivially,*

$$\widehat{C}(u, v) = B(u, v) = u \wedge v.$$

Notice that the maximal copula $u \wedge v$ is a fixed point of both the transformations in (17), regardless of the marginal \overline{G} .

As to the Schur-constant case, with given marginal \overline{G} , we observe that $B(u, v) = u \cdot v$ holds if and only if it is

$$\widehat{C}(u, v) = \overline{G} \left(\overline{G}^{-1}(u) + \overline{G}^{-1}(v) \right),$$

i.e., $\widehat{C} = C^{\overline{G}^{-1}}$.

Finally, we consider the case of independence, with given marginal \overline{G} . In this case, the necessary condition (14) for B , can be rewritten as $B = S^Q$, with Q given by (15).

Example 10 (Exponential marginals) *When $\overline{G}(x) = e^{-\beta x}$, then γ and γ^{-1} in (18) are given respectively by $\gamma(w) = w^{\frac{1}{\beta}}$ and $\gamma^{-1}(u) = u^\beta$, and (17) becomes*

$$B(u, v) = \left(\widehat{C} \left(u^\beta, v^\beta \right) \right)^{\frac{1}{\beta}}, \quad \text{and} \quad \widehat{C}(u, v) = \left(B \left(u^{\frac{1}{\beta}}, v^{\frac{1}{\beta}} \right) \right)^\beta.$$

Note that if we take the copula $\widehat{C}(u, v) = (u + v - 1)^+$, i.e. the lower Fréchet-Hoeffding bound, then B coincides with the Clayton semi-copula, given by

$$B(u, v) = \left[\left(u^{-\beta} + v^{-\beta} - 1 \right)^+ \right]^{\frac{1}{\beta}}.$$

The condition $\overline{G}(x) = e^{-x}$ implies $\gamma(w) = w$, and therefore

$$B(u, v) = \widehat{C}(u, v).$$

Thus we see that if B is compatible with the standard exponential marginal distribution, i.e., if there exists a survival function \overline{F} with a standard exponential marginal and such that $B_{\overline{F}} = B$, then necessarily B is a copula. However the condition $\overline{G}(x) = e^{-x}$ is not necessary for $B = \widehat{C}$ as we see, for instance, by considering the perfect dependence case. Another instance will be met in

Section 5, where we will consider the case of Marshall-Olkin models, whose marginal distributions are exponential, but non-necessarily standard exponential.

We conclude this Section with the following remarks and preliminary results, the role of which will emerge in the rest of the paper.

Remark 11 *The property of associativity can immediately be extended to semi-copulas: we will say that a semi-copula S is associative when*

$$S(S(u, v), w) = S(u, S(v, w))$$

holds for every u, v and w in $[0, 1]$. It can be derived from (17) that B is associative if and only if \widehat{C} is such. In the latter case the aging function is then a t -norm (see [13]).

Remark 12 *If \widehat{C} is Archimedean then it is associative; whence B is associative as well. In such a case we can however say more. In view of (17), we also see that B is Archimedean, with an invertible generator, if and only if \widehat{C} is such; if $\widehat{\phi}$ is a convex generator of \widehat{C} then*

$$\widehat{\phi}(\gamma^{-1}(u)) = \widehat{\phi}(\overline{G}(-\log u)) \quad (19)$$

is a generator of B . Furthermore, since generators of Archimedean semi-copulas are determined up to a constant, we can conclude as follows: if $\widehat{\phi}$ is an invertible, convex, generator of \widehat{C} and φ is a generator of B , then

$$\theta \varphi(u) = \widehat{\phi}(\gamma^{-1}(u)) = \widehat{\phi}(\overline{G}(-\log u)), \quad (20)$$

for some constant $\theta > 0$. From the above relation we also see that, in our framework, the generator φ is continuous.

We now state a useful result that is suggested by the Eq. (20).

Lemma 13 (a) *Assume that $\theta > 0$, $\widehat{\phi}$, and \overline{G} are given, with $\widehat{\phi}$ a strict and convex generator, and \overline{G} a strictly positive survival function with $\overline{G}(0) = 1$. Then φ defined by (20) is a strict generator of a semi-copula. Furthermore if $\widehat{\phi}$ and \overline{G} are strictly decreasing, then the same holds for φ .*

(b) *Assume that $\theta > 0$, $\widehat{\phi}$, and φ are given, with $\widehat{\phi}$ a strict and convex generator and φ a strict generator. Assume furthermore that $\widehat{\phi}$ and φ are strictly decreasing, and that \overline{G}_θ is defined by means of (20), i.e.*

$$\overline{G}_\theta(x) := \widehat{\phi}^{-1}(\theta \varphi(e^{-x})). \quad (21)$$

Then \overline{G}_θ is a strictly decreasing survival function, with $\overline{G}_\theta(0) = 1$.

(c) Assume that $\hat{\phi}$ and φ satisfy the same properties of item (b), except the strictness property, i.e. $\hat{\phi}(0) = \hat{\phi}(0^+)$ and $\varphi(0) = \varphi(0^+)$ are finite. Then (21) defines a survival function \overline{G}_θ if and only if $\theta \geq \theta_0$, where $\theta_0 := \varphi(0)/\hat{\phi}(0)$. Furthermore \overline{G}_θ is strictly decreasing on $[0, \infty)$ if and only if $\theta = \theta_0$.

PROOF. The proof of part (a) is a simple verification: since $\theta > 0$, $\hat{\phi}$, \overline{G} and $-\log$ are (strictly) decreasing, then φ is (strictly) decreasing. Since $\lim_{u \rightarrow 0^+} -\log(u) = +\infty$, $\lim_{x \rightarrow +\infty} \overline{G}(x) = 0$ and $\lim_{u \rightarrow 0^+} \hat{\phi}(u) = +\infty$, then $\lim_{u \rightarrow 0^+} \varphi(u) = +\infty$.

Finally it is $\varphi(1) = 0$, since $-\log(1) = 0$, $\overline{G}(0) = 1$, and $\hat{\phi}(1) = 0$.

The proof of part (b) is similar: since $\hat{\phi}$, φ and e^{-x} are strictly decreasing, then $\hat{\phi}^{-1}$ and therefore \overline{G}_θ are also strictly decreasing. Since $\varphi(1) = 0$, $\hat{\phi}^{-1}(0) = 1$, then $\overline{G}_\theta(0) = \hat{\phi}^{-1}(\theta\varphi(1)) = 1$. Finally, since $\hat{\phi}$ and φ are strict generators, then

$$\lim_{x \rightarrow \infty} \overline{G}_\theta(x) = \lim_{y \rightarrow 0^+} \hat{\phi}^{-1}(\theta\varphi(y)) = 0.$$

The proof of part (c) is readily obtaining by observing that $\hat{\phi}^{-1}(\theta x) = 0$ if and only if $\theta x \geq \hat{\phi}(0)$.

Note that in the proof of Lemma 13 we have not actually used the fact that $\hat{\phi}$ is convex, while the convexity property has a key role in the following Corollary; the latter is a simple consequence of the previous Lemma 13, and its proof is left to the reader.

Corollary 14 *Assume the same hypotheses of Lemma 13 part (b). Assume furthermore that φ is continuous and that \overline{G}_θ is defined by (21). Then, for every $\theta > 0$, the bivariate function*

$$\hat{\phi}^{-1}\left(\theta\left(\varphi(e^{-x}) + \varphi(e^{-y})\right)\right) \quad (22)$$

coincides with the bivariate survival function defined by

$$\overline{F}_\theta(x, y) := \hat{\phi}^{-1}\left(\hat{\phi}\left(\overline{G}_\theta(x)\right) + \hat{\phi}\left(\overline{G}_\theta(y)\right)\right).$$

Furthermore $\overline{F}_\theta(x, y)$ satisfies our standing hypotheses, its marginal survival function is \overline{G}_θ and has Archimedean survival function and ageing function with generators $\hat{\phi}$ and φ respectively, i.e.

$$\widehat{C}_{\overline{F}_\theta} = C^{\hat{\phi}}, \quad B_{\overline{F}_\theta} = S^\varphi$$

Viceversa, if \overline{F} is a bivariate survival function with $\widehat{C}_{\overline{F}} = C^{\hat{\phi}}$ and $B_{\overline{F}} = S^\varphi$ then $\overline{F} = \overline{F}_\theta$ and $\overline{G}_{\overline{F}} = \overline{G}_\theta$, for some $\theta > 0$.

This result is the starting point for the proof of the compatibility problem, at least in the cases when given \widehat{K} and B are both Archimedean. In the next Section we shall analyze this point more in details and in a larger generality.

4 Survival copulas and upper-orthant Kendall distributions

In this Section we concentrate our attention on the upper-orthant Kendall distribution \widehat{K} defined in (1). We start by introducing the bivariate function

$$\overline{\Phi}(x, y) := \widehat{K}(\overline{F}(x, y)), \quad (23)$$

which has the interesting meaning:

$$\overline{\Phi}(x, y) = P\{(X, Y) \in A_{\overline{F}(x, y)}\}. \quad (24)$$

Since $\overline{G}(0) = 1$ and $\widehat{K}(0) = 0$, $\overline{\Phi}(x, 0) = \widehat{K}(\overline{G}(x))$ is a univariate survival function, with $\overline{\Phi}(0, 0) = 1$; moreover, since \widehat{K} is strictly increasing, the function $\overline{\Phi}(x, 0)$ is strictly decreasing. When $\overline{\Phi}(x, y)$ turns out to be a bivariate survival function, then we can write

$$\overline{G}_{\overline{\Phi}}(x) = \widehat{K}(\overline{G}(x)), B_{\overline{\Phi}}(u, v) = B_{\overline{F}}(u, v).$$

We notice that, as it can also be shown by taking into account Eq. (24), it is

$$\overline{F}(x, y) \leq \widehat{K}(\overline{F}(x, y)) = \overline{\Phi}(x, y). \quad (25)$$

As an immediate consequence we get that for any bivariate survival function \overline{F} it holds

$$\widehat{K}(u) \geq u, \quad \forall u \in [0, 1]. \quad (26)$$

The previous inequality is also a consequence of the fact that \widehat{K} is a Kendall distribution (that depends only on the survival copula \widehat{C}). The latter property can be easily seen by taking into account the relation between the survival copula and the connecting copula $C(u, v)$, which is the joint distribution function of the pair (U, V) , where $U = G(X), V = G(Y)$: similarly, $\widehat{C}(u, v)$ is the joint distribution function of the pair $(\widehat{U}, \widehat{V})$, where $\widehat{U} := \overline{G}(X), \widehat{V} := \overline{G}(Y)$. Then

$$\widehat{K}(t) = P\{\overline{F}(X, Y) \leq t\} = P\{\widehat{C}(\widehat{U}, \widehat{V}) \leq t\}. \quad (27)$$

The following Lemma points out further properties of \widehat{K} , that are important in our frame.

Lemma 15

$$\widehat{K}(t-) = \widehat{K}(t) \quad \text{if and only if} \quad P(\overline{F}(X, Y) = t) = 0, \quad (28)$$

i.e. if and only if $P((X, Y) \in \partial A_t) = 0$.

Under our standing assumptions on \overline{F} , the inequality in (26) is strict for all $u \in (0, 1)$, namely

$$\widehat{K}(u) > u, \quad \forall u \in (0, 1). \quad (29)$$

PROOF. The proof of (28) is readily obtained by observing that $\widehat{K}(t-) = P(\overline{F}(X, Y) < t)$.

We now prove that the strict inequality (29) holds; actually we want to show that the condition $\widehat{K}(t) = t$ for some $t \in (0, 1)$ is incompatible with our assumptions. To this purpose we notice that, for all $u \in (0, 1)$,

$$P(\overline{G}(X) \leq u) = u$$

and that the event $\{\overline{F}(X, Y) \leq u\}$ contains both the events

$$\{\overline{G}(X) \leq u\} \text{ and } \{\overline{G}(Y) \leq u\},$$

whence

$$P(\overline{F}(X, Y) \leq u) \geq P(\overline{G}(X) \leq u).$$

Suppose that the condition $\widehat{K}(t) = t$ holds for some $t \in (0, 1)$. We may write then

$$t = P(\overline{F}(X, Y) \leq t) \geq P(\overline{G}(X) \leq t) = t$$

which entails

$$P(\overline{F}(X, Y) \leq t) = P(\overline{G}(X) \leq t).$$

In its turn, the latter equality implies

$$P(X > \overline{G}^{-1}(t), Y \leq \overline{G}^{-1}(t)) = P(X \leq \overline{G}^{-1}(t), Y > \overline{G}^{-1}(t)) = 0.$$

This condition implies that $\overline{F}(x, \overline{G}^{-1}(t)) = t$ for every $x \in [0, \overline{G}^{-1}(t)]$ and this is impossible since we assumed that \overline{F} is strictly 1-increasing.

Remark 16 *It is interesting to note that if the condition $\widehat{K}(t) = t$ holds for every $t \in (t_1, t_2)$, then $P(X = Y | X \in (t_1, t_2)) = 1$. In this respect see also Theorem 5.2 in [1]. Moreover we note that the continuity condition $\widehat{K}(t-) = \widehat{K}(t)$ is not automatically fulfilled, under our assumptions; as a counter-example we can take $(X, Y) = (X_1, Y_1)$ with probability $\lambda \in (0, 1)$, and $(X, Y) =$*

$(U, 1 - U)$ with probability $1 - \lambda$, where (X_1, Y_1) are independent and standard exponential, and U is uniformly distributed in $(0, 1)$, then

$$\begin{aligned}\bar{F}(x, y) &= P(X > x, Y > y) \\ &= \lambda P(X_1 > x, Y_1 > y) + (1 - \lambda) P(U > x, 1 - U > y) \\ &= \lambda e^{-(x+y)} + (1 - \lambda) (1 - x - y)^+ = \bar{G}(x + y).\end{aligned}$$

Then \bar{F} is a Schur-constant survival function, satisfying all our standing assumptions, nonetheless

$$P(\bar{F}(X, Y) = \lambda e^{-1}) = 1 - \lambda > 0.$$

It was recalled in Section 2 that, under natural regularity conditions, the Kendall distribution K , associated to a bivariate distribution $F(x, y)$ with marginal distributions G_X and G_Y , can be obtained analytically from the connecting copula $C_F(u, v) = F(G_X^{-1}(u), G_Y^{-1}(v))$: whenever $C = C_F$ is strictly 1-increasing and with $u \rightarrow C(u, v)$ continuous, $K_F(t) = \mathcal{K}C_F(t)$, where \mathcal{K} is defined in (5). The latter is a consequence on the fact that

$$P(V \leq v | U = u) = \frac{\partial C}{\partial u}(u, v).$$

By using a similar argument, and the representation (27), it can be checked that an analogous representation connects the upper-orthant Kendall distribution function \widehat{K} to the survival copula $\widehat{C} = \widehat{C}_{\bar{F}}$, when \widehat{C} satisfies the above conditions; more precisely,

$$\widehat{K}_{\bar{F}}(t) = \mathcal{K}\widehat{C}_{\bar{F}}(t). \quad (30)$$

For a given bivariate \bar{F} , then, the negative function $t - \widehat{K}_{\bar{F}}(t)$ depends only on the survival copula $\widehat{C}_{\bar{F}}$.

Consider now the family of bivariate semi-copulas S , such that $S_v(\cdot) := S(\cdot, v)$ is a strictly increasing and continuous function for every v . Extending to this family the definition given in (5), we put:

$$\mathcal{K}S(t) := t + \int_t^1 \frac{\partial S}{\partial u}(u, v)|_{v=v_{u,t}^S} du, \quad (31)$$

with $v_{u,t}^S = S_u^{-1}(t)$, i.e. $S_{\bar{F}}(u, v_{u,t}^S) = t$.

Furthermore, also for a semi-copula S , we put

$$\lambda_S(t) := t - \mathcal{K}S(t).$$

Notice that, while for the copulas C_F and $\widehat{C}_{\bar{F}}$, the functions $\mathcal{K}C_F$ and $\mathcal{K}\widehat{C}_{\bar{F}}$ are distribution functions, $\mathcal{K}S$ is not generally a distribution function, and

therefore we can refer to $\mathcal{K}S$ by using the term *Kendall pseudo-distribution function*.

When $B = B_F$ is a strictly 1-increasing semi-copula, with B_v continuous, we can therefore define $\mathcal{K}B$ and λ_B .

Now we can turn to studying the relations existing between $\mathcal{K}B$, $\mathcal{K}\widehat{C}$ and \overline{G} for $(B, \widehat{C}, \overline{G}) = (B_{\overline{F}}, \widehat{C}_{\overline{F}}, \overline{G}_{\overline{F}})$. Such relations constitute in fact an essential ingredient in the proof of results about the compatibility problem. As first, we direct the reader's attention to the following known result.

Theorem 17 (Genest and Rivest, [10]) *If C and C^* are copulas that are related via the relation*

$$C^*(u, v) = \gamma^{-1}(C(\gamma(u), \gamma(v)))$$

by a strictly increasing, differentiable bijection γ , then

$$\lambda_{C^*}(v) = \frac{\lambda_C(\gamma(v))}{\gamma'(v)}, 0 < v < 1.$$

Now we point out that this result can be applied in a rather straightforward way to the case when C^* is not a copula but a semi-copula. Taking into account the Eqs. (17) and (18), for a joint survival \overline{F} satisfying our current assumptions, we can then get the following

Proposition 18 *Assume the marginal survival function $\overline{G}_{\overline{F}}$ admit a strict positive density $g_{\overline{F}}$, so that $\gamma^{-1}(u) = \overline{G}_{\overline{F}}(-\log t)$ is differentiable and bijective from $[0, 1]$ to $[0, 1]$. Then*

$$\mathcal{K}B_{\overline{F}}(t) = t + \frac{1}{(\gamma^{-1})'(t)} (\lambda_{\widehat{C}}\gamma^{-1}(t)). \quad (32)$$

Corollary 19 *Let us maintain the assumption of the Proposition above and assume furthermore that $\widehat{K} = \mathcal{K}\widehat{C}$ is Archimedean, then $\Upsilon_{t_0, \mathcal{K}B}$ is well defined, with*

$$\Upsilon_{t_0, \mathcal{K}B}(t) = \exp \left\{ \int_{t_0}^t \frac{1}{s - \mathcal{K}B(s)} ds \right\} = \exp \left\{ \int_{\gamma^{-1}(t_0)}^{\gamma^{-1}(t)} \frac{1}{s - \mathcal{K}\widehat{C}(s)} ds \right\}, \quad (33)$$

and there exists a positive constant θ such that the following relation holds

$$\theta \Upsilon_{t_0, \mathcal{K}B}(t) = \Upsilon_{t_0, \widehat{K}}(\overline{G}(-\log(t))). \quad (34)$$

PROOF.

The relation (33) is a direct consequence of Eq. (32), while the relation (34) follows by observing that

$$\Upsilon_{t_0, \mathcal{K}B}(t) = \Upsilon_{\gamma^{-1}(t_0), \mathcal{K}\hat{C}}(\gamma^{-1}(t)) = \hat{\theta} \Upsilon_{t_0, \hat{K}}(\gamma^{-1}(t)),$$

where

$$\hat{\theta} = \exp\left\{\int_{\gamma^{-1}(t_0)}^{t_0} \frac{1}{s - \mathcal{K}\hat{C}(s)} ds\right\}.$$

In the next Example 20 we compute the upper-orthant Kendall distribution functions $\mathcal{K}\hat{C}$ and the functions $\lambda_{\hat{C}}$, together with $\mathcal{K}B$ and λ_B , for the basic cases considered in Examples 5, 6, and 7, replacing the Schur-constant case with the more general Archimedean case.

Example 20 *In the perfect dependence case, in agreement with Remark 16, it holds*

$$\hat{K}_{\bar{F}}(t) = \mathcal{K}\hat{C}(t) = t, \quad \lambda_{\hat{C}}(t) = t - t = 0,$$

since $\bar{F}(x, y) = \bar{G}(x \vee y)$, and $\hat{U} = \bar{G}(X)$ is a uniform random variable on $(0, 1)$. This corresponds to the fact that $\bar{F}(x, y) = \bar{\Phi}(x, y)$, where $\bar{\Phi}$ is the function defined in (23). Furthermore $\mathcal{K}B(t) = t$ and $\lambda_B(t) = 0$, since $B = \hat{C}$.

In the case of an Archimedean survival copula with convex generator $\hat{\phi}$, i.e. (using the notation (4))

$$\hat{C}(u, v) = C^{\hat{\phi}}(u, v) = \hat{\phi}^{-1}(\hat{\phi}(u) + \hat{\phi}(v)), \quad (35)$$

it is (see e.g. [14] p.102)

$$\mathcal{K}\hat{C}(t) = \mathcal{K}C^{\hat{\phi}}(t) = t - \frac{\hat{\phi}(t)}{\hat{\phi}'(t+)}, \quad \lambda_{C^{\hat{\phi}}}(t) = \frac{\hat{\phi}(t)}{\hat{\phi}'(t+)}. \quad (36)$$

As we already know, in this case the aging function is an Archimedean semi-copula, i.e. (by extending to Archimedean t -norms the notation in (4)), $B = S^\varphi = \varphi^{-1}(\varphi(u) + \varphi(v))$, with φ satisfying (20). As a consequence it is not difficult to get that

$$\mathcal{K}B(t) = \mathcal{K}S^\varphi(t) = t - \frac{\varphi(t)}{\varphi'(t+)}, \quad \lambda_B(t) = \lambda_{S^\varphi}(t) = \frac{\varphi(t)}{\varphi'(t+)}. \quad (37)$$

Finally, for the product copula $\hat{C}(u, v) = u \cdot v$, it holds $\mathcal{K}\hat{C}(t) = t - t \log t$ (see e.g. [10], p. 394); so that it is

$$\lambda_{\hat{C}}(t) = t \log t.$$

The above relations can also be obtained by the previous Archimedean case, observing that $\widehat{C} = C^{\widehat{\phi}_0}$, with $\widehat{\phi}_0 = -\log t$. This observation allows also to compute immediately \mathcal{KB} and λ_B , by means of (37), with $\varphi = Q$, and Q defined as in (15).

In view of the arguments in the next Section, we add a couple of useful comments concerning Eq. (34). To this end we recall that the equivalence class \mathcal{C}_K contains one and only one associative copula. The latter is Archimedean (and its generators are given by (7)) if and only if the condition (6) holds. In this respect, we note that, under the assumptions considered here, the condition (6) for \widehat{K} is granted if the equivalent conditions in (28) hold.

Note that the unique Archimedean copula C belonging to the class \mathcal{C}_K (when (6) holds) does not depend on the choice of the constant θ appearing in (7). Nevertheless this constant will play a role in the compatibility conditions in the next section.

Consider now a joint survival function \overline{F} such that $\widehat{K}_{\overline{F}}$ is an Archimedean Kendall distribution, so that the unique copula in $\mathcal{C}_{\widehat{K}_{\overline{F}}}$ is the copula $C^{\widehat{\phi}}$ that admits, as a generator, the convex function

$$\widehat{\phi}(t) := \Upsilon_{t_0, \widehat{K}_{\overline{F}}}(t).$$

Then by letting

$$\varphi(t) := \Upsilon_{t_0, \mathcal{KB}_{\overline{F}}}(t),$$

the relation (34) becomes exactly the relation (19).

If, furthermore, the convex generator $\widehat{\phi}$ is strict then (34) necessarily implies that

$$\overline{G}_{\overline{F}}(x) = \widehat{\phi}(\theta\varphi(e^{-x})). \quad (38)$$

5 Conditions for compatibility between Kendall distributions and level sets

Consider the following objects that appear in the statement of Theorem 2: K is a strict Archimedean Kendall distribution and $\widehat{\phi}$ is its strict generator. Furthermore, \overline{J} is a jointly continuous and strictly one-decreasing bivariate survival function, with differentiable marginal survival function $\overline{G}_{\overline{J}}$, and such that $\widehat{K}_{\overline{J}}$ is also a strict Archimedean Kendall distribution, with generator $\Upsilon_{t_0, \widehat{K}_{\overline{J}}}(t)$.

Basic aspects of the arguments developed in the Sections 3 and 4 will be

collected and used in this Section to provide the proof of Theorem 2 and to present examples and related comments.

Before proceeding with the announced proof it is convenient however to focus the reader's attention on the following remarks.

First we recall from Section 3 that, by letting $B = B_{\bar{J}}$, any solution \bar{F} of the compatibility problem (besides satisfying $\widehat{K}_{\bar{F}} = K$) must be such that $B_{\bar{F}} = B$. In view of the latter condition and of Eq. (10), \bar{F} must be of the form

$$\bar{F}(x, y) = \bar{G}_{\bar{F}} \left(-\log B \left(e^{-x}, e^{-y} \right) \right),$$

and then, in order to identify it, we only need to determine the marginal survival function $\bar{G}_{\bar{F}}$.

The latter, if it exists, must (and can) be obtained then by taking into account the simultaneous conditions $\widehat{K}_{\bar{F}} = K$ and $B_{\bar{F}} = B$.

In this respect, Corollary 19, and in particular the Eq. (34), and (33), help us in giving a necessary condition concerning the possible candidates for the marginal survival function $\bar{G}_{\bar{F}}$ when

$$\widehat{\phi}(t) := \Upsilon_{t_0, K}(t), \varphi_B(t) := \Upsilon_{t_0, \mathcal{K}B}(t)$$

are strict generators: by (38) the possible candidates for $\bar{G}_{\bar{F}}$ are necessarily of the form

$$\bar{G}_{\theta}(x) = \widehat{\phi}^{-1} \left(\theta \varphi_B(e^{-x}) \right) \quad \text{for some } \theta > 0. \quad (39)$$

On the other hand φ_B can be obtained by applying once again Corollary 19 and (38) to the survival function \bar{J} , in view of the conditions that $B_{\bar{J}} = B$ and that $\widehat{K}_{\bar{J}}$ is also a strict Archimedean Kendall distribution, with generator $\Upsilon_{t_0, \widehat{K}_{\bar{J}}}(t)$.

We now proceed more formally with the proof of our result.

PROOF. (of Theorem 2) (a) First of all note that φ_B is well-defined since, for some $\theta_B > 0$,

$$\theta_B \Upsilon_{t_0, \mathcal{K}B}(t) = \Upsilon_{t_0, \widehat{K}_{\bar{J}}} \left(\bar{G}_{\bar{J}}(-\log(t)) \right),$$

as immediately follows from the assumption that $\widehat{K}_{\bar{J}}$ is a strict Archimedean Kendall distribution, with generator $\Upsilon_{t_0, \widehat{K}_{\bar{J}}}(t)$ and from Eq. (34), applied to \bar{J} .

Taking again into account that $\widehat{K}_{\bar{J}}$ is a strict Archimedean Kendall distribution, we have that $\widehat{\phi}_{\bar{J}} := \Upsilon_{t_0, \widehat{K}_{\bar{J}}}(t)$ is a strict and convex generator. It also

follows from our assumptions on \bar{J} that $\bar{G}_{\bar{J}}$ is strictly positive and strictly decreasing. Then φ_B is a strict generator and (a) follows by part (b) of Lemma 13.

(b) In view of (a), $\bar{G}_\theta(x)$ is a bona-fide one-dimensional survival function for any $\theta > 0$. We can combine (39) and (10) in order to immediately obtain the equation $\bar{F}_\theta(x, y) = \hat{\phi}^{-1}(\theta\varphi_B(B(e^{-x}, e^{-y})))$. Then

$$\bar{F}_\theta(x, y) = \hat{\phi}^{-1}\left(\theta\Upsilon_{t_0, \hat{K}_{\bar{J}}}\left(\bar{J}(x, y)\right)\right),$$

and the result is obtained by taking into account the definition of \bar{G}_θ .

(c) is obvious since $\bar{F}_\theta(x, 0) = \bar{G}_\theta(\bar{G}_{\bar{J}}^{-1}(\bar{J}(x, 0))) = \bar{G}_\theta(\bar{G}_{\bar{J}}^{-1}(\bar{G}_{\bar{J}}(x))) = \bar{G}_\theta(x)$.

We notice that in the proof of part (b), we have obtained that \bar{F}_θ can also be written in the form

$$\bar{F}_\theta(x, y) = \hat{\phi}^{-1}\left(\theta\varphi_B(B(e^{-x}, e^{-y}))\right). \quad (40)$$

Theorem 2 provides conditions on the form of \bar{F} that are necessary for \bar{F} to be a solution of the compatibility problem, but it does not ensure that the bivariate functions \bar{F}_θ are actually survival functions, for a given choice of $B = B_{\bar{J}}$ and \hat{K} .

The following two remarks, that point out some further aspects about the function B , can be of help in the discussion about sufficient conditions that guarantee that the functions \bar{F}_θ are survival functions.

Remark 21 Let $(\bar{G}, B) = (\bar{G}_F, B_F)$ be the marginal survival function and the semi-copula respectively obtained, from a given joint survival function \bar{F} , by means of the Eq. (9). Assume that F has a jointly continuous and strictly positive density f , so that $G = G_F$ has a strictly positive, differentiable, density and $B = B_F$ has continuous second-order derivatives. Then necessarily

$$1 + \frac{g'(x)}{g(x)} \Big|_{x=-\log B(u,v)} \leq \frac{B(u,v) \frac{\partial^2}{\partial u \partial v} B(u,v)}{\frac{\partial}{\partial u} B(u,v) \frac{\partial}{\partial v} B(u,v)} \quad (41)$$

or, equivalently,

$$1 + \frac{d}{dx} \log g(x) \Big|_{x=-\log B(u,v)} \leq \frac{\frac{\partial}{\partial u} \log \frac{\partial}{\partial v} B(u,v)}{\frac{\partial}{\partial u} \log B(u,v)}$$

Note that when $\bar{G}(x) = e^{-x}$, then the compatibility condition (41) may be satisfied if and only if $\frac{\partial^2}{\partial u \partial v} B(u, v) \geq 0$, i.e. if and only if B is a copula. In fact (see Example (10)) we know that B has to be a copula, in order to be compatible with a standard exponential marginal. Similar considerations hold when $\bar{G}(x) = e^{-\theta x}$, with $\theta \geq 1$.

Remark 22 Let \bar{M} be an arbitrary continuous, strictly decreasing and strictly positive survival function over $[0, +\infty)$, and S be a semi-copula. The bivariate function defined by

$$\bar{M} \left(-\log S \left(e^{-x}, e^{-y} \right) \right)$$

is a bivariate survival function (with aging function S and marginal survival function \bar{M}) provided appropriate compatibility conditions hold. For instance, if we assume that M has a strictly positive density $m \in C^1$ and that $S = B_{\bar{F}}$, with F as in Remark 21, i.e., admitting a jointly continuous and strictly positive density f (and therefore $B \in C^2$), then the necessary and sufficient compatibility conditions are given by (41) with $g = m$. As a consequence, in this case, a sufficient condition is given by

$$\frac{m'(x)}{m(x)} \leq \frac{g'_F(x)}{g_F(x)} \quad \Leftrightarrow \quad \frac{d}{dx} \log m(x) \leq \frac{d}{dx} \log g_F(x), \quad (42)$$

which implies that

$$1 + \frac{m'(x)}{m(x)} \Big|_{x=-\log B(u,v)} \leq 1 + \frac{g'_F(x)}{g_F(x)} \Big|_{x=-\log B(u,v)} \leq \frac{B(u,v) \frac{\partial^2}{\partial u \partial v} B(u,v)}{\frac{\partial}{\partial u} B(u,v) \frac{\partial}{\partial v} B(u,v)}.$$

For instance, this happens when $g(x) = k \exp\{-A_F(x)\}$, with A_F positive and increasing, and $m(x) := k_\theta \exp\{-\theta A_F(x)\}$, with $\theta \geq 1$.

For fixed \widehat{K} and \bar{J} , and under suitable regularity assumptions on \widehat{K} and on \bar{J} (and therefore on $B = B_{\bar{J}}$), we may assume that $g_\theta = -\bar{G}'_\theta$ is a C^1 probability density, and we may use the conditions of Remark 22 to find sufficient conditions for compatibility (take also into account the above Remark 21): When B admits continuous second order derivatives, a necessary and sufficient condition is given by (41) with g replaced by g_θ , and a sufficient condition is given by

$$\frac{d}{dx} \log g_\theta(x) \leq \frac{d}{dx} \log g_J(x),$$

as immediately follows by (42).

Actually the above conditions are not often satisfied. In a number of cases, however, it can be checked directly that the bivariate functions \bar{F}_θ in (41) are bivariate survival functions and therefore they are solutions of our compatibility problem.

A relevant special case of this kind arises when, besides the assumptions of

Theorem 2, we add the condition that B is Archimedean. Even though this remark is nothing but an application of Corollary 14, we prefer in the following example to summarize some details about the Archimedean cases and to briefly discuss a few related aspects.

Example 23 Consider the special case of exchangeable survival models characterized by the condition that the survival copula $\widehat{C}_{\overline{F}}$ is Archimedean with a (convex) invertible generator $\widehat{\phi}$. In this case $B_{\overline{F}}$ is Archimedean, with a continuous and invertible generator φ , as well.

Concerning the role of $\widehat{\phi}$, we first recall Eq. (35) and also write

$$\overline{F}(x, y) = \widehat{\phi}^{-1} \left(\widehat{\phi} \left(\overline{G}_{\overline{F}}(x) \right) + \widehat{\phi} \left(\overline{G}_{\overline{F}}(y) \right) \right).$$

As to φ , it is easy to check, by (9), that it is

$$B_{\overline{F}} = S^{\varphi},$$

for

$$\varphi(u) = \widehat{\phi} \left(\overline{G}_{\overline{F}}(-\log u) \right).$$

Thus, in terms of $\widehat{\phi}$ and φ , $\overline{G}_{\overline{F}}$ is given by Eq. (21), and it must be

$$\overline{F}(x, y) = \widehat{\phi}^{-1} \left(\varphi(e^{-x}) + \varphi(e^{-y}) \right).$$

In the latter formulas, $\overline{G}_{\overline{F}}(x)$ and $\overline{F}(x, y)$ respectively are a one-dimensional and a two-dimensional survival function, for any choice of the generators $\widehat{\phi}$ and φ , under our assumptions. Now, let \widehat{K} and \widehat{J} be given, with \widehat{K} Archimedean copula having an invertible generator $\widehat{\phi}$. From above we see that the compatibility problem is determined by the pair $\widehat{\phi}$ and φ , where φ is the generator of $B_{\widehat{J}}$ (and then of $B_{\overline{F}}$). Furthermore we see that, for any choice of such $\widehat{\phi}$ and φ , the compatibility problem admits solutions.

Of course, for $\theta > 0$, $\theta \varphi$ is also a generator of $B_{\overline{F}}(u, v)$ and, correspondingly, any bivariate survival function having the form (22), i.e.

$$\widehat{\phi}^{-1} \left(\theta \left(\varphi(e^{-x}) + \varphi(e^{-y}) \right) \right),$$

is still a solution of the same compatibility problem.

Two comments are here in order.

First, Eq. (40) can be seen as the natural generalization of Eq. (22). However Eq. (22) can be derived directly, just by taking into account (17) and (21); the derivation of Eq. (40) on the contrary stands on the appropriate extension to semi-copulas of Theorem 17.

Second, the condition that $B_{\overline{F}}$ is Archimedean with strict generator φ characterizes the bivariate models such that the set of the level curves is the same as in the case of stochastic independence; in fact (see also [2] and (14) in Example 6) a pair of independent, identically distributed, non-negative variables with marginal survival function $e^{-\theta\varphi(e^{-x})}$ admits S^φ as its aging function. We can then conclude by saying that any strict Archimedean Kendall distribution is compatible with the set of the level curves associated to any independent model, provided that the marginal of the latter is strictly positive, strictly decreasing, and continuous all over $[0, +\infty)$.

Theorem 2 can also be used in cases where B and \widehat{K} are not Archimedean, nor admit second derivatives. The following example is related to the celebrated *Marshall-Olkin models*, where the corresponding \overline{F} is not absolutely continuous, as in the basic perfect dependence case.

Example 24 (Marshall-Olkin models and Cuadras-Augé copulas)

Consider independent, exponentially distributed, non-negative variables A, B, τ , where $A, B \sim \text{Exp}(\lambda), \tau \sim \text{Exp}(\mu)$, and set $X = \min(A, \tau), Y = \min(B, \tau)$. The joint survival function of X, Y is

$$\overline{F}(x, y) = \exp\{-\lambda(x + y) - \mu(x \vee y)\}. \quad (43)$$

In this case it is

$$\overline{G}(x) = e^{-(\lambda+\mu)x}, \quad \overline{G}^{-1}(u) = -\frac{\log u}{\lambda + \mu},$$

By (2), and by (10) in Example 10, with $\beta = \lambda + \mu$, we find

$$\widehat{C}(u, v) = B(u, v) = (uv)^\alpha (u \wedge v)^{1-\alpha}, \quad (44)$$

with $\alpha = \frac{\lambda}{\lambda+\mu}$, i.e. $\widehat{C} = B$ is a Cuadras-Augé copula. Notice that this is not an Archimedean copula.

It was shown in [11] that

$$\widehat{K}_{\overline{F}}(t) = t - \frac{1 + \alpha^2}{1 + \alpha} t \log t,$$

i.e.

$$\lambda_{\widehat{C}}(t) = \frac{1 + \alpha^2}{1 + \alpha} t \log t.$$

Let us now consider the compatibility problem with $B = B_{\overline{F}}$ the Cuadras-Augé (44) copula and $\widehat{K}(t) = \mathcal{K}B(t)$. We then write $\widehat{K}(t) = t - \widehat{\rho} t \log t$, with $\widehat{\rho} = \frac{2\alpha}{1+\alpha}$ and notice that \widehat{K} is a strict Archimedean Kendall distribution, in fact

$$\varphi_B(t) = \widehat{\phi} = \exp\left\{\int_{t_0}^t \frac{1}{\widehat{\rho}} \frac{1}{\log u} d \log u\right\} = \text{const} |\log t|^{1/\widehat{\rho}}.$$

Therefore (39) shows that the compatible marginal survival functions have to be exponential. These are exactly the marginal survival functions of exchangeable Marshall-Olkin models in (43).

Summarizing, if B is the Cuadras-Augé copula and $\widehat{K} = \mathcal{K}B$, then \widehat{K} and B are compatible. The choice of such a pair (\widehat{K}, B) characterizes the exchangeable Marshall-Olkin models in (43).

In the conclusion of this Section we present a remark about $\overline{F}_\theta(x, y)$ and about their survival copulas. \overline{F}_θ depends on $\theta, \widehat{\phi}$ and B . The marginal \overline{G}_θ , more specifically, depends on $\theta, \widehat{\phi}$ and φ_B (see the formula (39)). But, after computing the marginal, we generally need the entire knowledge of B in order to recover both \overline{F}_θ (see the formula (40)) and its survival copulas, say \widehat{C}_θ . The latter is Archimedean if and only if B (as a semi-copula) is such, i.e. if and only if the survival function \overline{J} , such that $B = B_{\overline{J}}$, is of the form

$$\overline{J}(x, y) = \psi(\overline{G}(x)\overline{G}(y))$$

for \overline{G} a one-dimensional survival function respecting our conditions, and for $\psi : [0, 1] \rightarrow [0, 1]$ a strictly increasing function.

6 Summary and concluding remarks

Our attention has been focused on exchangeable bivariate survival models characterized by joint survival functions \overline{F} , that satisfy a convenient set of regularity conditions.

For any such model, we considered the (upper-orthant) Kendall distribution $\widehat{K}_{\overline{F}}$ and the set $\mathcal{D}_{\overline{F}}$ of level curves. From a *geometric* viewpoint, the relation between these two objects becomes immediately clear when considering the level sets A_v ($0 < v < 1$) of \overline{F} .

For a given Kendall distribution \widehat{K} and a given set \mathcal{D} of level curves, it is then natural to wonder whether they are *compatible*, i.e. whether it there exists \overline{F} such that $\widehat{K}_{\overline{F}} = \widehat{K}$, $\mathcal{D}_{\overline{F}} = \mathcal{D}$. In the paper we developed a method that allows us to find sufficient or necessary conditions for compatibility.

We described the family $\mathcal{D}_{\overline{F}}$ in terms of the “aging function” $B_{\overline{F}}$ and this allowed us to rephrase that problem in an *analytical* form, as follows: for a given \widehat{K} and a given aging function B does there exist \overline{F} such that $\widehat{K}_{\overline{F}} = \widehat{K}$, $B_{\overline{F}} = B$?

The advantage offered by this formulation is based on the possibility to use (a slight extension of) the transformation result presented in [10] and here

recalled as Theorem 17. In the present context this result plays a key role in describing the relations between the (semi-)copula B and the copula \widehat{C} (see (17)). Furthermore we also relied on (a slight extension of) the representation (7), which provides the generators of the unique Archimedean (semi-)copula having a given strict Archimedean (pseudo-) Kendall distribution. The above-mentioned extension was summarized in Proposition 19.

As an application of such results, we could determine the form that possible solutions to the system of equations $\widehat{K}_{\overline{F}} = \widehat{K}$, $B_{\overline{F}} = B$ must have. To this purpose, the strategy followed is simply summarized as follows: since, by Eq.(10), \overline{F} is determined from the knowledge of both $B_{\overline{F}}$ and the marginal $\overline{G}_{\overline{F}}$, then, in view of the condition $B_{\overline{F}} = B$, the condition $\widehat{K}_{\overline{F}} = \widehat{K}$ is to be used to obtain $\overline{G}_{\overline{F}}$. For this reason we first found (see Eq. (21)) that the possible marginals are of the form displayed in (39), i.e.

$$\overline{G}_{\theta}(x) = \widehat{\phi}^{-1} \left(\theta \cdot \varphi_B \left(e^{-x} \right) \right),$$

where $\widehat{\phi}$ is the generator of the unique Archimedean copula with Kendall distribution \widehat{K} , and, analogously, φ_B is the generator of the Archimedean semi-copula $B_{\overline{J}}$ associated to the Archimedean model \overline{J} , and such that $\mathcal{K}B_{\overline{J}} = \mathcal{K}B$. Then, by applying Eq. (10), in Theorem 2 we obtained the family of “candidate” joint survival functions, given by (40), i.e.

$$\overline{F}_{\theta}(x, y) = \widehat{\phi}^{-1} \left(\theta \cdot \varphi_B \left(B \left(e^{-x}, e^{-y} \right) \right) \right).$$

Whenever \overline{F}_{θ} turns out to be a joint survival function, then it actually is a solution to the compatibility problem. As we discussed, this can happen when some suitable further conditions are satisfied. We can easily exhibit however simple cases where this does not happen; for instance, if $\widehat{\phi}$ and φ_B are not strict, we can find values of θ for which even $\overline{G}_{\theta}(x) = \widehat{\phi}^{-1} \left(\theta \cdot \varphi_B \left(e^{-x} \right) \right)$ is not a one-dimensional survival function (see point (c) of Lemma 13).

However, under the assumptions of Theorem 2, we proved that $\overline{G}_{\theta}(x)$ is a one-dimensional survival function.

One basic assumption that we used along the paper is that \widehat{K} is a strict Archimedean Kendall distribution so that there exists a unique Archimedean copula in the equivalence class determined by \widehat{K} . This yields that $\overline{G}_{\theta}(x)$ coincides with the marginal survival function of an Archimedean model.

Furthermore we note that $\varphi = \Upsilon_{t_0, \mathcal{K}B_{\overline{J}}}$, the generator of the Archimedean semi-copula $B_{\overline{J}}$, is strict if and only if the convex generator $\widehat{\phi} = \Upsilon_{t_0, \widehat{K}}$ is such. This observation leads us to another source of incompatibility: B and \widehat{K} are certainly not compatible whenever \widehat{K} is a strict Archimedean Kendall distribution, and the generator φ_B is not strict.

The above arguments point out the interest of the fact that any equivalence class associated to a strict Archimedean Kendall distribution contains one and only one Archimedean copula.

The assumption that \widehat{K} is a strict Archimedean Kendall distribution actually allows for the possibility that $\overline{F}_\theta(x, y)$ in (40) are solutions to the compatibility problem for infinite different values of the parameter θ .

An example of this situation is provided by the family of Marshall-Olkin models, that is obtained by taking B as a Cuadras-Augé copula (see 44) and $\widehat{K} = \mathcal{K}B = t - \widehat{\rho}t \log t$, with $\widehat{\rho} = \frac{2\alpha}{1+\alpha}$. In fact, in this case we have that all the exponential marginals are compatible.

For the sake of simplicity, we assumed that \overline{F} is strictly 1-decreasing all over \mathbb{R}_+^2 , and this was actually used in Theorem 2. Actually, in Proposition 19, we strictly used the assumption that γ^{-1} is a differentiable bijection, i.e. that \overline{G} admits a strictly positive density. However we one can find examples of solutions to the compatibility problem where this assumption does not hold.

As already remarked, in the case of models with strict Archimedean \widehat{K} a central role for the analysis of the compatibility problem is played by the unique Archimedean copula with the given strict Archimedean Kendall distribution. We can conjecture that, in the more general case when $\widehat{K}(t^-) = t$ for some t , a similar role would be played by the unique associative copula in the class of \widehat{K} .

Some applications of our method can be found in the field of risk and (bivariate) models of interacting defaults. In fact, as mentioned in the Introduction, the set $\mathcal{D}_{\overline{F}}$ of level curves of \overline{F} can be interpreted as the set of the (upper-orthant) bivariate VaR curves.

A different type of application is, instead, the characterization of special (one-parameter) families of bivariate models. A particular case has been described in the above example about Marshall-Olkin models. A more complete treatment may be the object of a future paper.

Finally we mention an open problem that, in our setting, arises as a natural one: to characterize the semi-copulas B that are “aging functions”, i.e. that can be obtained by applying a continuous transformation as in the first line of (17) on a bivariate copula \widehat{C} .

For the Archimedean case Corollary 14 gives at least a sufficient condition: any Archimedean semi-copula with continuous, strict, and strictly decreasing generator φ is an aging function. More in general, we expect that our arguments can be used to study this open problem.

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