RELATIONS between KENDALL DISTRIBUTIONS and FAMILIES of BIVARIATE VALUES at RISK in EXCHANGEABLE SURVIVAL MODELS

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Outline of talk

The family $\mathcal{D}$ of Bivariate VaR Curves

(upper-orthant) Kendall distribution $\mathcal{K}$

The problem of compatibility between $\mathcal{D}$ and $\mathcal{K}$

* Semicopulas & representation of $\mathcal{D}$

* The relation between $\mathcal{D}$ and the survival copula $\hat{C}$

A transformation result by Genest & Rivest (2001)

The Archimedean case

Necessary conditions for the general case

A non Archimedean Example

*based on previous papers of Bruno Bassan and Fabio Spizzichino
\( X, Y \) non-negative exchangeable random quantities

with \( \overline{F}(x, y) \) joint survival function

\[
\overline{F}(x, y) := P\{X > x, Y > y\}, \ x > 0, y > 0.
\]

with \( G(x) \) marginal survival function

\[
\overline{G}(x) := \overline{F}(x, 0) = P\{X > x\} = P\{Y > x\}
\]

**Definition:** (Bivariate (upper-orthant) Value at Risk at a probability level \( v \), [Embrechts & Puccetti (2006)]

\[
\text{VaR}_v(\overline{F}) := \partial A_v, \quad v \in [0, 1]
\]

where

\[
A_v := \{(x, y) \in \mathbb{R}_+^2 : \overline{F}(x, y) \leq 1 - v\}.
\]

(for \( A \subset \mathbb{R}_+^2, \ \partial A \) boundary of \( A \))

**Assumptions:** \( \overline{G} \) continuous, strictly decreasing, strictly positive on \([0, \infty), \overline{G}(0) = 1\)

\[
\partial A_v \text{ regular curve, } \forall v \in [0, 1]
\]

**Notation:** \( \mathcal{D}_\overline{F} := \text{family of bivariate upper-orthant Value at Risk curves:} \)

\[
\mathcal{D}_\overline{F} := \{\partial A_v; v \in [0, 1]\}\]
(upper-orthant) **Bivariate Probability Integral Transformation:**

\[ Z := \overline{F}(X, Y) \quad \text{(BIPIT)} \]

(upper-orthant) **Kendall distribution:**

\[ \hat{K}_{\overline{F}}(v) := P\{Z \leq v\} = P\{\overline{F}(X, Y) \leq v\} = 1 - P\{(X, Y) \in A_{1-v}\}, \; v \in [0, 1] \]

("upper-orthant" used in order to distinguish from the objects respectively considered in [Genest, Rivest (2001)], [Nelsen et al. (2003)] and ref. therein)

**Facts:**

* \( \hat{K}_{\overline{F}}(= \hat{K}_{\hat{C}_{\overline{F}}}) \) only determined by the **survival copula** \( \hat{C}_{\overline{F}} \), where

\[ \hat{C}_{\overline{F}}(u, v) = \overline{F}\left(\overline{G}^{-1}(u), \overline{G}^{-1}(v)\right), \]

* \( \hat{K}_{\overline{F}} \) is a Kendall distribution, i.e. \( K(t) \geq t, \; \forall t \in [0, 1] \)

Kendall distributions are characterized as distribution functions on \([0,1]\), with the above property
(upper-orthant) **Kendall distribution:**

\[ \hat{K}_F(v) := P\{Z \leq v\} = P\{F(X, Y) \leq v\} \]

\[ = P\left( (X, Y) \in \{(x, y) : F(x, y) \leq 1 - (1 - v)\} \right) \]

\[ = P\{(X, Y) \in A_{1-v}\}, \quad v \in [0, 1] \]
Let $\hat{K}$ and $\mathcal{D}$ be given, with

$\hat{K}$ a Kendall distribution ($\hat{K}(t) \geq t$)

$\mathcal{D}$ a family of biv. (upper-orthant) Value at Risk curves

**Compatibility Problem:** Can we find $F$ such that

$$\hat{K}_F = \hat{K}, \quad \mathcal{D}_F = \mathcal{D}$$

Responses will be given under the further assumptions

* $F$ strictly one-decreasing
* $G$ abs. continuous $\Rightarrow \hat{K}_F(t^-) > t, \ t \in (0, 1)$
* $P(F(X, Y) = t) = 0, \ \forall t \in [0, 1]$

**Methods**

- Describe $\mathcal{D}_F$ by means of a *semicopula* $B_F$
- Find the relation between $\mathcal{D}_F$ and $\hat{C}_F$ (i.e. between $B_F$ and $\hat{C}_F$)
- Exploit (and extend the use of) a transformation result by Genest & Rivest (2001)
- Exploit the properties of the class $\mathcal{C}_\hat{K}$ of survival copulas $\hat{C}$ such that $\hat{K}_{\hat{C}} = \hat{K}$
For $x \geq 0, y \geq 0$, let

$$h(x, y) := \overline{G}^{-1} \left[ \overline{F} (x, y) \right]$$

Properties of $h$

* $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is one-increasing, $h(x, 0) = x, \forall x \geq 0$

* $h$ describes $\mathcal{D}_\overline{F}$:

$$\mathcal{V}aR_v (\overline{F}) = \{ (x, y) : h(x, y) = \overline{G}^{-1}(1 - v) \}$$

* $\overline{F}$ is determined from the knowledge of $h$ and $\overline{G}$: in fact, trivially,

$$\overline{F} (x, y) = \overline{G} [h(x, y)]$$

Remark: In the above representation of $\overline{F}$ the information provided by the marginal is separated from the information provided by $\mathcal{D}$

**Question:** In the representation of $\mathcal{D}$, can one replace $h$ by a copula $B_{\overline{F}}$?

The idea is to transform $h$ into a function $B$ such that

$$B_{\overline{F}} : [0, 1] \times [0, 1] \to [0, 1]$$

Define:

$$B_{\overline{F}} (u, v) = \exp \{-h(- \log u, - \log v)\}$$
\[ B_F(u, v) = \exp\{-G^{-1}(F(-\log u, -\log v))\} \]

whence

\[ \overline{F}(x, y) = \overline{G}(-\log B_F(e^{-x}, e^{-y})) \]

\(B_F\) is called \textit{aging function}, and it turns out that [Bassan, Spizzichino (2001), (2003), (2005)]:

* \(B_F\) can be used to describe \(\mathcal{D}_F\)

* \(B_F\) is a \textit{semicopula} (\(B_F\) has all the properties of a copula but ... it may be not 2-increasing)

If \(\overline{F}(x, y) = (e^{-\beta x} + e^{-\beta y} - 1)^+\), with \(\beta > 0\), then \(B(u, v) = [(u^\beta + v^\beta - 1)^+]^{1/\beta}\). When \(\beta > 1\), \(\exists a u_\beta\) such that \(\frac{1}{2} < u_\beta \leq (\frac{1}{2})^{1/\beta}\), and therefore \(B(1, 1) - B(u_\beta, 1) - B(1, u_\beta) + B(u_\beta, u_\beta) = 1 - 2u_\beta + (\left(2(u_\beta)^\beta - 1\right)^+)^{1/\beta} = 1 - 2u_\beta < 0\).

\textbf{Equivalent Compatibility Problem:}

Given a Kendall distribution \(\hat{K}\) and an aging function \(B\), can we find a bivariate survival function \(\overline{F}\) such that

\[ \hat{K}_F = \hat{K}, \quad B_F = B ? \]
Example 1. (perfect dependence): When
\[ \overline{F}(x, y) = \overline{G}(x \lor y) \]
\( B_\overline{F} \) is the maximal copula
\[ B_\overline{F}(u, v) = u \land v \]

Example 2. ("Schur-constant" case): The condition
\[ \overline{F}(x, y) = \overline{G}(x + y) \]
holds if and only if \( B_\overline{F} \) is the product copula:
\[ B_\overline{F}(u, v) = u \cdot v. \]

Example 3. (i.i.d. variables):
\[ \overline{F}(x, y) = \overline{G}(x) \cdot \overline{G}(y) \]
if and only if
\[ B_\overline{F}(u, v) = Q^{-1}[Q(u) + Q(v)], \]
with \( Q(u) := - \log \overline{G}(- \log u) \),
\[ Q(u) := - \log \overline{G}(- \log u). \]
The relation between $B_F$ and $\hat{C}_F$ is

$$B_F(u, v) = \gamma \left( \hat{C}_F (\gamma^{-1}(u), \gamma^{-1}(v)) \right),$$

$$\hat{C}_F(u, v) = \gamma^{-1} \left( B_F(\gamma(u), \gamma(v)) \right)$$

where $\gamma : [0, 1] \rightarrow [0, 1]$ is the increasing function defined by

$$\gamma(w) = \exp\{-\overline{G}^{-1}(w)\},$$

and such that, then,

$$\gamma^{-1}(w) = \overline{G}(-\log w).$$

Indeed, starting from

$$B_F(u, v) = \exp\{-h(-\log u, -\log v)\}$$

$$\overline{F}(x, y) = \overline{G}(-\log B_F(e^{-x}, e^{-y}))$$

$$\hat{C}_F(u, v) = \overline{F} \left( \overline{G}^{-1}(u), \overline{G}^{-1}(v) \right),$$

one can easily obtain

$$B_F(u, v) = \exp\{-\overline{G}^{-1} \left( \hat{C}_F(-\log \overline{G}(u), -\log \overline{G}(v)) \right) \},$$

$$\hat{C}_F(u, v) = \overline{G} \left( -\log B_F \left( e^{-\overline{G}^{-1}(u)}, e^{-\overline{G}^{-1}(v)} \right) \right),$$

i.e.
Solutions to the *compatibility* problem have been inspired by the following results (see [Genest & Rivest (2001)] and [Nelsen et al. (2003)])

\[ \hat{K}_F = \hat{K}, \quad B_F = B \]

✓ The *Kendall distribution* \( K_F(v) := P\{F(X, Y) \leq v\} \) of a bivariate distribution \( F \) only depends on its copula \( C = C_F \):

\[
K_F(t) = K_C(t) = t - \int_t^1 \frac{\partial}{\partial u} C(u, v_{u,t}) du \lambda_C(t)
\]

where \( v_{u,t} = C_u^{-1}(t) \), with \( C_u(v) := C(u, v) \).

✓ If \( C \) and \( C^* \) are copulas that are related via relation

\[ BC^*(u, v) = \gamma^{-1} (C(\gamma(u), \gamma(v))) \]

by a strictly increasing, differentiable bijection \( \gamma \), then

\[
\lambda_{C^*}(t) = \frac{\lambda_C(\gamma(v))}{\gamma'(v)}, \quad 0 < v < 1.
\]

✓ For any Kendall distribution \( K \), there is a unique *associative* copula \( C \) such that \( K_C = K \) (i.e. the Kendall distribution of \( C \) coincides with \( K \)).

\( C \) is *associative* \( \iff C(C(u, v), w) = C(u, C(u, w)) \) holds \( \forall u, v \) and \( w \) in \([0, 1]\).
Remark A  If \( \hat{K}(t^-) > t \) (necessarily true under our conditions) the unique associative copula \( \hat{C} \) in the equivalence class of the (upper orthant) Kendall distribution \( \hat{K} \) is actually \textit{ARCHIMEDEAN}, with generators (determined up to a constant)

\[
\hat{\theta} \hat{\phi}(t), \text{ where } \hat{\phi}(t) = \hat{\phi}_{\hat{K}}(t) = \exp \left\{ - \int_{t_0}^{t} \frac{1}{\hat{\lambda}(s)} ds \right\},
\]

and where \( \hat{\lambda}(s) = s - \hat{K}(s) \), and \( \hat{\theta} > 0 \).

A copula \( C \) is \textit{Archimedean} \iff \( C(u, v) = \phi^{-1}[\phi(u) + \phi(v)] \ \forall \ u, v \in [0, 1] \). The function \( \phi \) (decreasing, convex, \( \phi(1) = 0 \)) is called generator of \( C \).

Remark B  The survival copula \( \hat{C}_F \) is Archimedean iff the function \( B_F \) is such:

\[
\hat{C}_F(u, v) = \phi^{-1}[\hat{\phi}(u) + \hat{\phi}(v)] \quad \iff \quad B_F(u, v) = \gamma \left( \hat{C}_F(\gamma^{-1}(u), \gamma^{-1}(v)) \right) = \gamma \left( \phi^{-1}[\hat{\phi}(\gamma^{-1}(u)) + \hat{\phi}(\gamma^{-1}(v))] \right)
\]

\[
B_F(u, v) = \varphi^{-1}[\varphi(u) + \varphi(v)] \quad \text{with} \quad \varphi(t) = \hat{\phi}(\gamma^{-1}(t))
\]

Replace \( \hat{\phi} \rightsquigarrow \hat{\theta} \hat{\phi} \), then, for \( \theta = 1/\hat{\theta} \)

\[
\theta \varphi(t) = \hat{\phi}(\gamma^{-1}(t)) \quad \iff \quad G(x) = G_\theta(x) := \hat{\phi}^{-1}(\theta \varphi(e^{-x}))
\]

since \( \gamma^{-1}(t) = \G(- \log t) \), and \( t \rightsquigarrow e^{-x} \iff \gamma^{-1}(e^{-x}) = G(x) \).
Therefore, if the aging function
\[ B(u, v) = \varphi^{-1}[\varphi(u) + \varphi(v)], \text{ with } \varphi = \varphi_B \]
is an Archimedean semicopula, with \( \varphi(0^+) = \infty \), and
\[ \hat{K}(t^-) > t, \]
then the compatibility problem
\[ \hat{K}_F = \hat{K}, \quad B_F = B \]
has infinitely many solutions:
Indeed, necessarily \( \hat{C} \) is Archimedean
\[ \hat{C}(u, v) = \hat{C}^\phi(u, v) := \hat{\phi}^{-1}[\hat{\phi}(u) + \hat{\phi}(v)], \text{ with } \hat{\phi} = \hat{\phi}_{\hat{K}}. \]
the possible marginal distributions are
\[ \bar{G}_\theta(x) := \hat{\phi}^{-1}(\theta \varphi(e^{-x})), \quad \theta > 0. \]
and the solutions are \( F = F_\theta \)
\[ \bar{F}_\theta(x, y) = \hat{C}^\phi \left( \bar{G}_\theta(x), \bar{G}_\theta(y) \right), \]
or equivalently
\[ \bar{F}_\theta(x, y) = \bar{G}_\theta \left( -\log B(e^{-x}, e^{-y}) \right). \]
(If \( \varphi(0^+) < \infty \), then \( \bar{G}_\theta \) are distribution functions only when \( \hat{\phi}(0^+) < \infty \)
and \( \theta \geq \hat{\phi}(0^+)/\varphi(0^+) \); furthermore \( \bar{G}_\theta(x) > 0 \ \forall \ x > 0 \iff \theta \geq \hat{\phi}(0^+)/\varphi(0^+) \))
Let $B$ be an aging function (not necessarily Archimedean). It is possible to define:

✓ A pseudo Kendall distribution $KB$

\[
KB(t) = t - \int_t^1 \frac{\partial}{\partial u} B(u, v_{u,t}^B) du
\]

where $v_{u,t}^B = B_u^{-1}(t)$, with $B_u(v) := B(u, v)$.

✓ A generator $\varphi_B$ of a semicopula

\[
\varphi_B(t) = \exp \left\{ - \int_0^t \frac{1}{s - KB(s)}(s) ds \right\}
\]

Therefore, under the assumptions that $\varphi_B(0^+) = \infty$ and $\hat{K}(t^-) > 0$, the possible solutions to compatibility problem $\hat{K}_F = \hat{K}$, $B_F = B$ are given by

\[
F_\theta(x, y) = G_\theta(-\log B(e^{-x}, e^{-y})).
\]

where, recalling that $\hat{\phi}_\hat{K}$ is a generator of the unique Archimedean copula, with Kendall distribution $\hat{K}$,

\[
G_\theta(x) := \hat{\phi}_\hat{K}^{-1}(\theta \varphi_B(e^{-x})) \quad \theta > 0.
\]

(The proof is based on an extension of the transformation result by Genest & Rivest (2001))
A non Archimedean Example

\[ B(u, v) = (uv)^\alpha (u \land v)^{1-\alpha}, \]

with \( \alpha \in (0, 1) \), i.e. \( B \) is a Cuadras-Augé copula.

(Note that the Cuadras-Augé copula is not associative, and therefore is not Archimedean.)

\[ \hat{K}(t) = t - H t \log t, \text{ with } H = \frac{2\alpha}{1+\alpha}. \]

Then

\[ \phi_B(t) = \phi_{\hat{K}}(t) = \text{const} \mid \log t\mid^{1/H}, \]

and therefore necessarily the marginal distributions are exponential \( Exp(\theta) \), and the compatible models are the \textit{exchangeable Marshal-Olkin models}:

Consider three independent, exponentially distributed, non-negative variables \( A, B, \tau \), where \( A, B \sim Exp(\lambda), \tau \sim Exp(\mu) \) and set

\[ X = \min(A, \tau), \quad Y = \min(B, \tau). \]

with \( \alpha = \frac{\lambda}{\lambda+\mu} \) and \( \theta = \lambda + \mu. \)
OPEN PROBLEM:

WHAT HAPPENS IF WE DROP THE ASSUMPTION THAT 
$
\hat{K}(t^-) > t
$?

CONJECTURE:

THE ASSOCIATIVE COPULAS TAKE THE ROLE OF ARCHIMEDEAN COPULAS (?)

IF WE DROP THE ASSUMPTION THAT $\varphi_B(0^+) = \infty$, THEN THERE IS ONLY ONE MARGINAL DISTRIBUTION, WHICH IS STRICTLY POSITIVE ON $[0, \infty)$.

EXAMPLE???(still Archimedean) Take $B(u, v)$ a Clayton semicopula, but not a copula:

$$B(u, v) = [(u^\beta + v^\beta - 1)^+]^{\frac{1}{\beta}} = [(u^{-b} + v^{-b} - 1)^+]^{-\frac{1}{b}}.$$ 

When $\beta > 1$, i.e. $b \leq -1$ it is not a copula, but it is a copula for $\beta \in (0, 1]$. The generator is $\varphi(u) = 1 - u^\beta = 1 - u^{-b}$. 