Continuous time random walks and queues: explicit forms and approximations of the conditional law with respect to local times *

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Abstract

In the filtering problem here considered the state process is a continuous time random walk and the observation process is an increasing process depending deterministically on the trajectory of the state process. An explicit construction of the filter is given. This construction is then applied to a suitable approximation of a Brownian motion and to a rescaled M/M/1 queueing model. In both these cases the sequence of the observation processes converge to a local time, and a convergence result for the respective filters is given. The case of a queueing model when the observation is the idle time is also considered.

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1 Introduction

The kind of problems we are interested in arises from the following situation. Suppose that in a queue we can observe, up to time t, whether the queue is busy or idle, but we cannot observe the size of the queue, so that the observation process is the total time the queue has spent in 0, i.e. the so called *idle time* (see Prabhu [9]). Then the problem is to evaluate the size of the queue at time t, given this information, i.e. to compute the conditional law (or the filter) of the queue given the observation process up to time t. In the setup of heavy traffic limit, the rescaled queue converges to a reflected Brownian motion and the observation process converges to its local time. The limit model can be constructed as $(W_t + \Lambda_t, \Lambda_t)$, where W_t is a Brownian motion and

$$\Lambda_t = \ell_t(W), \tag{1}$$

where

$$\ell : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty),\ x \to \ell(x), \quad \text{such that} \quad \ell_t(x) = -\inf_{s\le t} x(s) \wedge 0, \tag{2}$$

is the functional involved in the solution of the Skorohod problem for $x \in D_{\mathbb{R}}[0,\infty)$, with $x(0) \geq 0$, i.e. $(z,v) = (x+\ell(x), \ell(x))$ is the unique pair of functions (z,v) satisfying $z(t) = x(t) + v(t)$, and such that $z(t) \geq 0$, for all $t \geq 0$, $v(0) = 0$, v is nondecreasing and increases only when $z(t) = 0$.

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In the limit model the corresponding filtering problem is the computation of the conditional law of a reflected Brownian motion $W_t + \Lambda_t$ when the observation process is its local time Λ_t, i.e. the computation of the filter $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]$, for g in a sufficiently large class of functions. A first problem is to find the exact expression for the filters, both for the limit model and for the rescaled queue model. A second problem concerns the convergence of the latter to $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]$.
The filter of the limit Brownian motion model is derived in G. Nappo, B. Torti [8] (Sections 4 and 6), where it is obtained by means of a suitable sequence of processes Λ^n approximating the observation process Λ. Each process Λ^n is proportional to a counting process, and therefore the nonlinear filtering techniques for counting processes are used. The filter can also be derived by means of the Azéma martingale, and this derivation is shortly discussed in [8]. For sake of completeness we recall its explicit expression.

Theorem 1.1. *Let W_t be a Brownian motion with diffusion coefficient a^2 and drift $c \in \mathbb{R}$ and let Λ_t be the local time defined in* (1)*. Let g be a bounded measurable function.*
Denote by

$$\Pi(s,l;g) = \int_0^\infty g\left(-l + y\sqrt{s}\right) y \exp\left(-\tfrac{1}{2}\,y^2\right) dy \quad s \geq 0, \tag{3}$$

$$\Pi_{a^2,c}(s,l;g) = \frac{\Pi\left(a^2 s, l; g(\cdot)\exp(\frac{c}{a^2}\,\cdot)\right)}{\Pi\left(a^2 s, l; \exp(\frac{c}{a^2}\,\cdot)\right)}, \tag{4}$$

$$\hat{\Pi}_{a^2,c}(s;g) = \Pi_{a^2,c}(s,0;g). \tag{5}$$

Then

$$\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] = \Pi_{a^2,c}(\zeta_t, \Lambda_t; g), \tag{6}$$

and

$$\hat{\pi}_t(g) = E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big] = \hat{\Pi}_{a^2,c}(\zeta_t; g) \tag{7}$$

where $\mathcal{F}_t^\Lambda$ is the history generated by Λ_u up to time t, ζ_t is the elapsed time from the last visit to 0 for the process $W_t + \Lambda_t$, i.e.

$$\zeta_t = \gamma_t^0(W + \Lambda) = \gamma_t(\Lambda), \tag{8}$$

with

$$\gamma_t^0(x) = t - \sup\{s < t : x_s = 0\}. \tag{9}$$

$$\gamma_t(x) = t - \sup\{s < t : x_s < x_t\}. \tag{10}$$

Note that $\Pi_{1,0}(s,l;g) = \Pi(s,l;g) = E\Big[g(-l + W_s^*)/\Lambda_s^* = 0\Big]$, where W^* is any standard Brownian motion and Λ^* is the local time of its Skorohod reflection.
In this paper we start with a somehow simplified version of the motivating problem: we consider a continuous time random walk Y_t and its conditional law w.r.t. $L_u = \ell_u(Y)$ up to time t, and, in analogy with the Brownian motion case, though incorrectly, in the following we refer to L_t as the local time associated to Y_t. The above problem is connected with the original one, indeed in several cases a queueing model can be represented as the reflection $Y_t + L_t$ of a continuous time random walk Y_t, and the filtration generated by L_t is strictly contained in the (right continuous) filtration generated by C_t, the time the process $Y_t + L_t$ spends in 0. It is worthwhile to observe that in contrast with L_t, the process C_t has continuous paths, and therefore it would be more natural to refer to C_t as the "local time" of $Y_t + L_t$.

It turns out (see Proposition 2.1) that the filter of Y_t w.r.t. L_u up to time t can be expressed as a probability measure depending deterministically on L_t and $\gamma_t(L)$, where γ_t is defined in (10). We derive also a more explicit expression for the filter under the assumption that the process Y_t can be decomposed as $Y_t = V_{Z_t}$, where Z_t is a renewal process, V_k is a discrete time random walk, with Z_t and V_k mutually independent. Similar results hold also for rescaled random walks.

We are interested in the situation when a sequence X_t^n of rescaled random walks converges to a Brownian motion W_t in $D_{\mathbb{R}}[0, +\infty)$. Then a continuous map argument applies to show that $(X_t^n, L_t^n) = (X_t^n, \ell_t(X^n))$ converges to $(W_t, \Lambda_t) = (W_t, \ell_t(W))$, and analogously the systems with the reflected random walk $(X_t^n + L_t^n, L_t^n)$ converge to the system with the reflected Brownian motion $(W_t + \Lambda_t, \Lambda_t)$ (see Section 5 for the details). In analogy with the second equality of (8) we denote

$$\xi_t^n = \gamma_t(L^n) = t - \sup\{u \le t \quad \text{such that} \quad L_u^n < L_t^n\}. \tag{11}$$

the elapsed time from last jump time of L^n. Then for random walk systems the corresponding filter is

$$\pi_t^n(g) = E[g(X_t^n)/\mathcal{F}_t^{L^n}] = \Sigma^n(\xi_t^n, L_t^n; g) \tag{12}$$

with $\Sigma^n(s, l)$ defined in (20), while for the reflected systems is

$$\hat{\pi}_t^n(g) = E[g(X_t^n + L_t^n)/\mathcal{F}_t^{L^n}] = \hat{\Sigma}^n(\xi_t^n; g) \tag{13}$$

with $\hat{\Sigma}^n(s) = \Sigma^n(s, 0)$.
The problem whether $\pi_t^n = \Sigma^n(\xi_t^n, L_t^n)$ converge weakly to the filter $\pi_t = \Pi_{a^2,c}(\zeta_t, \Lambda_t)$ of the limit is strictly related to the convergence of the filters $\hat{\pi}_t^n = \hat{\Sigma}^n(\xi_t^n)$ for the reflected random walks to $\hat{\pi}_t = \hat{\Pi}_{a^2,c}(\zeta_t)$. Depending on the model we privilege one or the other problem.
On the other hand, from a computational point of view, it is quite difficult to use the exact expression of $\hat{\Sigma}^n(s)$ to compute the filter $\hat{\pi}_t^n$. Then it is also interesting to find a good approximation of the filter of the discrete system, depending on the actually observed process L^n, so that it can be used in applications. For the reflected random walk system, a natural choice in order to give a manageable approximation of the filter is to use the limit functional $\hat{\Pi}_{a^2,c}(s)$, evaluated at $s = \xi_t^n$.

Approximation problems in filtering have been studied in more general situations by many authors, among which we recall in particular Bhatt et al. in [1] and Goggin ([6], [5]). Most of these results concern diffusive models and do not apply to our case. Moreover, usually the applications concern the problem to approximate a given signal/observation process with a suitably chosen sequence of signal/observation processes so that the corresponding sequence of filters converges to the the filter of the original process. We start from a different point of view: the sequence of processes is given and the problem is to show the convergence of the filters to the filter of the state/observation limit, in the sense specified above.

The problem of weak convergence of the sequence of filters is not a trivial problem as even strong convergence of random variables does not imply convergence of the conditional laws. This is clearly explained by the following simple and illuminating example (see E. Goggin [6]). Let ξ be a real random variable, and $(\xi_n, \eta_n) = (\xi, \xi/n)$. Then (ξ_n, η_n) converges strongly to (ξ, η), with $\eta = 0$. Nevertheless, for any measurable function g, $E(g(\xi_n)/\eta_n) = g(\xi)$, so that the conditional law of ξ_n given η_n is the measure concentrated in $\xi(\omega)$, while $E(g(\xi)/\eta) = E(g(\xi))$, so that the conditional law of ξ given η coincides with the (deterministic) law $P \circ \xi^{-1}$ of ξ.
In this example, although the sequence of the conditional laws of ξ_n given η_n does not converge to

the conditional law of the limit, it is a constant sequence and therefore is a converging sequence. This is not surprising indeed in the light of the next general result, which is a slight generalization of a result of E. Goggin [6] (proof of Theorem 2.1, Step 1): one has only to replace the sequence of $\sigma-$algebras used in [6] with a general sequence.

Lemma 1.2. *Let R_n be a sequence of random variables with values in a Polish space, let $\mathcal{H}^n$ be a sequence of $\sigma-$algebras, let α^n be a regular version of the conditional distribution of R_n given $\mathcal{H}^n$. If $\{R_n, n \in \mathbb{N}\}$ is tight, then $\{\alpha^n, n \in \mathbb{N}\}$ is tight.*

Then, as far as weak convergence is concerned, for the systems converging to a Brownian motion W, the main problem is to check whether the limit points of the sequence of filters are all equal to the filter of W w.r.t. the local time Λ.

The first model we consider is a non-Markovian queueing model arising when a Brownian motion W is approximated by a sequence of continuous time random walks W^n, obtained with a suitable interpolation procedure. The approximation scheme we propose for W follows some of the ideas used in [5] to study a filter approximation problem in diffusive models, and is related with the approximation scheme used in [8]. In this particular case we get a strong convergence result for the approximating filter.
The second model we consider is the case when the renewal process of the random walk is a Poisson process, so that the reflected random walk is an M/M/1 queue. The main results are weak convergence of the corresponding filters (see Theorem 5.2) and approximation in $L^p(\Omega \times [0,T])$-norm (see Theorem 5.3) and are based on the weak convergence of (ξ^n_t, L^n_t) to (ζ_t, Λ_t) (see Proposition 5.5) and on the convergence of $\hat{\Sigma}^n(\cdot)$ to $\hat{\Pi}_{2\lambda,c}(\cdot)$ (see Proposition 5.4), in the sense that

$$\lim_{n\to\infty} \hat{\Sigma}^n(s_n; g) = \hat{\Pi}_{2\lambda,c}(s; g), \tag{14}$$

for any g in a convergence determining class, whenever s_n converges to s, with $s > 0$. We prove the above key convergence result in Section 5.2, where we reformulate the problem in terms of the symmetric random walk by using a suitable change of measure and a reflection principle (see Lemma 5.7).

It is important to note that when we deal with a queue $Q^n_t = X^n_t + L^n_t$, i.e. with a reflected random walk, the previous results concern the filter w.r.t. the filtration

$$\mathcal{G}^n_t = \mathcal{F}^{L^n}_t$$

generated by the local time associated to the random walk, while the motivating problem concerns the filter of the queue w.r.t. the filtration

$$\mathcal{H}^n_t = \mathcal{F}^{C^n}_{t^+}$$

generated by the idle time C^n_t, i.e. the total time spent in 0 up to t. These two problems are strictly related: for instance (Q^n_t, C^n_t), as well as (Q^n_t, L^n_t), converges weakly to the reflected system $(W_t + \Lambda_t, \Lambda_t)$. This property, among others, allows to extend the previous convergence and approximation results to this situation (see Theorem 6.4 and Theorem 6.6 in the last section).

2 The model

Fix a probability space $(\Omega, \mathcal{F}, P)$ and consider on it a sequence $\{(T_j, U_j), j \geq 1\}$, satisfying the assumption

H *The* $\mathbb{R}^+ \times \{+1, -1\}-$*valued random variables* (T_j, U_j), *for* $j \geq 1$, *are identically distributed and mutually independent.*

Put $\tau_0 = 0,\ \tau_k = \sum_{j=1}^{k} T_j$ for $k \geq 1$, consider the renewal process $Z_t = \sum_{j=1}^{\infty} \mathbb{I}(\tau_j \leq t)$ and the random walk $\{V_j, j \geq 0\}$ defined by $V_0 = 0,\ V_j = V_{j-1} + U_j, j \geq 1$.
Finally, consider the continuous time random walk

$$Y_t = V_{Z_t} = \sum_{j=1}^{\infty} U_j \mathbb{I}(\tau_j \leq t) = \sum_{j \leq Z_t} U_j. \tag{15}$$

The solution of the Skorohod problem for the process Y_t is given by the pair $(Y_t + L_t, L_t)$, with

$$L_t = \ell_t(Y) = \sum_{j=1}^{\infty} \mathbb{I}(\sigma_j \leq t), \tag{16}$$

where ℓ is defined in (2), and the sequence of its jump times $\{\sigma_j, j \geq 0\}$ is the subsequence of $\{\tau_j, j \geq 0\}$ defined by $\sigma_0 = 0$ and

$$\sigma_j = \inf \big\{ \tau_k \quad \text{s.t.} \quad Y_{\tau_k} \leq -j \big\} = \inf \big\{ t > 0 \quad \text{s.t.} \quad Y_t \leq -j \big\} \quad \text{for } j \geq 1.$$

Set

$$\mathcal{G}_t = \mathcal{F}_t^L = \sigma\{L_s, s \leq t\},$$

obviously $\{\sigma_j, j \geq 0\}$ are stopping times w.r.t. both the histories $\mathcal{G}_t$ and $\mathcal{F}_t^Y$.

Moreover condition **H** implies that the process $Y_{s+\sigma_j} - Y_{\sigma_j}$ is independent of $\mathcal{G}_{\sigma_j}$ and is equal in law to the process Y_s, and therefore the process L_t is a renewal process, with inter-arrival times $\{S_h = \sigma_h - \sigma_{h-1},\ h \geq 1\}$. The following representations for the filter of Y_t given $\mathcal{G}_t$ are then straightforward.

Proposition 2.1. *Assume* **H**, *then the conditional law of* Y_t *given* $\mathcal{G}_t$ *admits the following* P*-a.s. representations*

$$\begin{aligned} E\big[g(Y_t)/\mathcal{G}_t\big] &= \sum_{j=0}^{\infty} \left. \frac{E\Big[g(-j + Y_{s+\sigma_j} - Y_{\sigma_j})\mathbb{I}(S_{j+1} > s)\Big]}{E\Big[\mathbb{I}(S_{j+1} > s)\Big]} \right|_{s=t-\sigma_j} \mathbb{I}\big\{\sigma_j \leq t < \sigma_{j+1}\big\}, \\ &= \left. \frac{E\Big[g(-j + Y_s)\mathbb{I}(\sigma_1 > s)\Big]}{E\Big[\mathbb{I}(\sigma_1 > s)\Big]} \right|_{j=L_t, s=\gamma_t(L)}, \end{aligned}$$

where $\gamma_t(\cdot)$ *is defined in* (10).

Proof. The first representation can be obtained by standard techniques. For the second one it is enough to note that

$$E\Big[g(-j + Y_{s+\sigma_j} - Y_{\sigma_j})\mathbb{I}(S_{j+1} > s)\Big] = E\Big[g(-j + Y_s)\mathbb{I}(\sigma_1 > s)\Big],$$

and finally that if $\sigma_j \leq t < \sigma_{j+1}$, then $\sigma_j = \sup\{u \leq t \text{ s. t. } L_u < L_t\}$. □

In order to get a more explicit representation of the filter we observe that defining recursively the sequence $\{M_i, i \geq 0\}$ by $M_0 = 0$, and $M_i = \inf \big\{k \geq 0 : V_{M_0+\ldots+M_{i-1}+k} - V_{M_0+\ldots+M_{i-1}} = -1\big\}$, then

$$\sigma_0 = 0, \quad \sigma_h = \sum_{i=1}^{M_1+\ldots+M_h} T_i = \tau_{M_1+\ldots+M_h}, \qquad h \geq 1, \tag{17}$$

Under Condition **H** the sequence $\{M_i, i \geq 1\}$ is a sequence of i.i.d. random variables. Under the further assumption

K *the random variables T_1 and U_1 are mutually independent, with*

$$P(T_1 \leq t) = F(t), \qquad P(U_j = 1) = p,\ P(U_j = -1) = 1 - p = q, \quad p \in (0,1),$$

the sequences $\{T_j, j \geq 1\}$ and $\{U_j, j \geq 1\}$ are mutually independent, and clearly also $\{\tau_i, i \geq 1\}$ and $\{M_i, i \geq 1\}$ are mutually independent. Next result provides a more explicit expression for the filter.

Proposition 2.2. *Assume conditions* **H** *and* **K**. *Then*

$$E\big[g(Y_t)/\mathcal{G}_t\big] = \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\,(M_1 \geq k)\, g(-j + V_{k-1})\Big]\,(F_{k-1}(s) - F_k(s))}{\sum_{m=1}^{\infty} P\,(M_1 \geq m)\,(F_{m-1}(s) - F_m(s))}\Bigg|_{j=L_t, s=\gamma_t(L)} \tag{18}$$

where F_k is the distribution function of τ_k, i.e. $F_k = F^{*k}$, the k-fold convolution of F.

Proof. Taking into account (15) and (17), and the independence of $\{\tau_i, i \geq 1\}$ and $\{M_i, i \geq 1\}$, it is sufficient to observe that

$$\begin{aligned} E\Big[g(-j + Y_s)\mathbb{I}\,(\sigma_1 > s)\Big] &= E\left[\sum_{m=1}^{\infty} \mathbb{I}\,(M_1 = m)\, g\,(-j + V_{Z_s})\,\mathbb{I}\,(\tau_m > s)\right] \\ &= E\left[\sum_{m=1}^{\infty}\sum_{k=1}^{m} \mathbb{I}\,(M_1 = m)\, g\,(-j + V_{k-1})\,\mathbb{I}\,(\tau_{k-1} \leq s < \tau_k)\right] \end{aligned}$$

□

The state space of the process Y_t being discrete, the filter $E\big[g(Y_t)/\mathcal{G}_t\big]$ is determined by its discrete density $\nu_t(x)$, $x \in \mathbb{Z}$, i.e. $\nu_t(x) = E\big[g(Y_t)/\mathcal{G}_t\big]$ with $g(z) = \mathbb{I}_{\{x\}}(z)$, $z \in \mathbb{Z}$. Then

$$\nu_t(x) = \frac{\sum_{k=1}^{\infty} P\,(V_{k-1} = x + j, M_1 \geq k)\,\big(F_{k-1}(s) - F_k(s)\big)}{\sum_{m=1}^{\infty} P\,(M_1 \geq m)\,\big(F_{m-1}(s) - F_m(s)\big)}\Bigg|_{j=L_t, s=\gamma_t(L)},$$

where

$$P\,(V_{k-1} = a, M_1 \geq k) = \frac{a+1}{k}\binom{k}{\frac{k+a+1}{2}} p^{\frac{k+a-1}{2}} q^{\frac{k-a-1}{2}} \tag{19}$$

when $k + a - 1$ is even and $|a| \leq k - 1$, while it is 0 otherwise. Finally we note that the normalization factor in (18) can also be written as $\sum_{m=1}^{\infty} P\,(M_1 = m)\,\big(1 - F_m(s)\big)$.

3 Scaling and notations

Let $\{\tilde{X}^n,\ n \in \mathbb{N}\}$ be a sequence of continuous time random walks defined in $(\Omega^n, \mathcal{F}^n, P^n)$. We assume that $\tilde{X}^n_t = \tilde{V}^n_{\tilde{Z}^n_t}$, where $\tilde{V}^n_k$ and $\tilde{Z}^n_t$ are defined as in Section 2 starting from a sequence $\{(\tilde{T}^n_j, \tilde{U}^n_j); j \geq 1\}$. Consider the deterministic linear time-space scaling

$$X^n_t = b_n \tilde{X}^n_{a_n t},$$

where $\{a_n,\ n \in \mathbb{N}\}$ and $\{b_n,\ n \in \mathbb{N}\}$ are suitable sequences of real positive numbers. Then the process $L^n_t = \ell_t(X^n)$ can be obtained just applying the same scaling to the process $\tilde{L}^n_t = \ell_t(\tilde{X}^n)$, i.e. $L^n_t = \ell_t(X^n) = b_n \tilde{L}^n_{a_n t}$. We are interested in the conditional law of X^n_t w.r.t. $\mathcal{F}^{L^n}_t = \mathcal{F}^{\tilde{L}^n}_{a_n t}$, i.e. the filter

$$\pi^n_t(g) = E^{P^n}\big[g(X^n_t)/\mathcal{F}^{L^n}_t\big] = E^{P^n}\big[g(b_n \tilde{X}^n_{a_n t})/\mathcal{F}^{\tilde{L}^n}_{a_n t}\big],$$

where E^{P^n} denotes the expectation w.r.t. P^n. For the sake of notational convenience we will denote $\mathcal{F}^{L^n}_t$ as $\mathcal{G}^n_t$ and, when unnecessary, drop the symbol P^n in the expectation, so that the filter becomes

$$\pi^n_t(g) = E\big[g(X^n_t)/\mathcal{G}^n_t\big].$$

For each n, let σ^n_1 be the first exit time of the process X^n_t from the set $(-b_n, \infty)$, denote by $\Sigma^n(s,l)$ the probability measure such that

$$\Sigma^n(s,l;g) = \frac{E\Big[g(-l + X^n_s)\mathbb{I}\left(\sigma^n_1 > s\right)\Big]}{E\Big[\mathbb{I}(\sigma^n_1 > s)\Big]}, \tag{20}$$

and assume that the sequences $\{(\tilde{T}^n_j, \tilde{U}^n_j); j \geq 1\}$ satisfy the assumption **H** stated in Section 2. Then the filter can be shortly written as

$$\pi^n_t(g) = E\big[g(X^n_t)/\mathcal{G}^n_t\big] = \Sigma^n(\xi^n_t, L^n_t; g), \tag{21}$$

where, ξ^n_t is defined in (11).
By taking into account that $\hat{\pi}^n_t(g) = E\big[g(X^n_t + L^n_t)/\mathcal{G}^n_t\big] = E\big[g(X^n_t + m)/\mathcal{G}^n_t\big]\big|_{m=L^n_t}$ the filter can be written as

$$\hat{\pi}^n_t(g) = E\big[g(X^n_t + L^n_t)/\mathcal{G}^n_t\big] = \hat{\Sigma}^n(\xi^n_t; g), \tag{22}$$

where

$$\hat{\Sigma}^n(s;g) = \Sigma^n(s,0;g) = \frac{E\Big[g(X^n_s)\mathbb{I}\left(\sigma^n_1 > s\right)\Big]}{E\Big[\mathbb{I}(\sigma^n_1 > s)\Big]}. \tag{23}$$

Remark 3.1. *Note that $\hat{\Sigma}^n(s;g)$, as well as $\Sigma^n(s,l)$, depends also on the probability measure P^n, i.e. $\hat{\Sigma}^n(s,l) = \hat{\Sigma}^n_{P^n}(s,l)$. This dependence will be emphasized when necessary.*

When the process $\tilde{X}^n_t = \tilde{V}^n_{\tilde{Z}^n_t}$ satisfies also assumption **K** of Section 2, then Proposition 2.2 easily provides the explicit expressions of $\Sigma^n(s,l)$

$$\Sigma^n(s,l;g) = \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\left(\tilde{M}^n_1 \geq k\right) g(b_n \tilde{V}^n_{k-1} - l)\Big]\big(\tilde{F}^n_{k-1}(a_n s) - \tilde{F}^n_k(a_n s)\big)}{\sum_{m=1}^{\infty} P\big(\tilde{M}^n_1 = m\big)\big(1 - \tilde{F}^n_m(a_n s)\big)}, \tag{24}$$

where $\tilde{M}_1^n$, $\tilde{F}_k^n$ have a similar meaning as M_1, F_k in (18). More precisely $\tilde{F}_1^n$ is the distribution function of $\tilde{T}_1^n = \inf\left\{t > 0 \ \text{ s. t. } \ |\tilde{X}_t^n| \geq 1\right\}$, and if T_1^n denotes the first jump time of the process X_t^n, then $T_1^n = \tilde{T}_1^n / a_n$, and therefore $\tilde{F}_1^n(a_n s) = F_1^n(s)$ is the distribution function of T_1^n, and analogously $\tilde{F}_k^n(a_n s) = F_k^n(s)$, where F_k^n is the $k-$fold convolution of F_1^n.

We end this section by introducing the notation $\omega_g(\delta) = \sup_{|x-y|\leq\delta} \left|g(x) - g(y)\right|$ for the modulus of continuity of a uniformly continuous function g.

4 The interpolating Brownian motion model

In this Section we study the case of a continuous time random walk arising when a Brownian motion W is approximated with a sequence of processes W^n. The processes W^n are defined on the probability space of W, and are obtained pathwise by an interpolation procedure. For this model we are able to get a strong convergence result for the filter. We start by introducing the approximating models, then we show the convergence result. Successively we discuss how the approximating models W^n fall into the frame of the previous sections. In particular, when the process W is a standard Brownian motion the processes W^n corresponds to the case examined at the end of the previous section, with scaling parameters $a_n = 2^{2n}$ and $b_n = 1/2^n$, $p_n = q_n = 1/2$, and $\tilde{F}_1^n = \tilde{F}_1$, where

$$\tilde{F}_1(t) = 4\sum_{j=0}^{+\infty}(-1)^j \frac{1}{\sqrt{2\pi}} \int_{\frac{(2j+1)}{\sqrt{t}}}^{+\infty} \exp\left(-\tfrac{1}{2}\, x^2\right) dx. \tag{25}$$

The basic idea is to approximate the state W by the stepwise interpolation of the random points where W hits a uniform grid and consider as approximating observation the local time of the approximating state. This procedure is a deterministic one and therefore we describe it in the deterministic case.

Let $z \in D_{\mathbb{R}}[0, +\infty)$ and let $h \in \mathbb{R}^+$ be a fixed threshold. Consider the sequence $\{\widehat{\tau}_k^h(z), k \geq 0\}$

$$\begin{cases} \widehat{\tau}_0^h(z) = 0 \\ \widehat{\tau}_k^h(z) = \inf\left\{t > \widehat{\tau}_{k-1}^h(z) : |z(t) - z(\widehat{\tau}_{k-1}^h(z))| > h\right\} \quad , \quad k \geq 1, \end{cases} \tag{26}$$

and the function $z^h \in D_{\mathbb{R}}[0, +\infty)$

$$z^h(t) = \sum_{k=0}^{\infty} \mathbb{I}_{\left[\widehat{\tau}_k^h(z), \widehat{\tau}_{k+1}^h(z)\right)}(t) z(\widehat{\tau}_k^h(z)). \tag{27}$$

We need the following result whose proof is left to the reader.

Lemma 4.1. *Let $z \in D_{\mathbb{R}}[0, +\infty)$. Then $\sup_{t\in\mathbb{R}^+}(|z^h(t) - z(t)|) \leq h$, and $\left(z^h, \ell(z^h)\right)$ converge uniformly to $\left(z, \ell(z)\right)$, where the functional ℓ is defined by (2).*

Remark 4.2. *When z is a continuous function, the process $\ell(z^h)$ admits the representation*

$$\ell_t(z^h) = 0 \vee (-z_0) - \sum_{j=0}^{\infty} z(\sigma_j^h(z)) \mathbb{I}_{[\sigma_j^h(z), \sigma_{j+1}^h(z))}(t),$$

where

$$\sigma_j^h(z) = \inf\left\{t \quad s.t. \quad z(t) - z_0 \leq -jh\right\} = \inf\left\{\widehat{\tau}_k^h(z) \quad s.t. \quad z(\widehat{\tau}_k^h(z)) - z_0 \leq -jh\right\}. \tag{28}$$

When furthermore $z_0 = 0$, then $z(\sigma_j^h(z)) = -jh$ and

$$\ell_t(z^h) = \sum_{j=0}^{\infty} jh\mathbb{I}(\sigma_j^h(z) \le t < \sigma_{j+1}^h(z)) = \sum_{j=0}^{\infty} h\mathbb{I}(\sigma_j^h(z) \le t). \tag{29}$$

We now apply this approximating procedure to the (not necessarily standard) Brownian motion W_t. Fix now the sequence of thresholds $h_n = \frac{1}{2^n}$, and consider the stopping times $\tau_k^n := \widehat{\tau}_k^h(W)$, when using $h = h_n = \frac{1}{2^n}$ in (26). Then the approximating signal/observation process (W^n, Λ^n) is a $D_{\mathbb{R}^2}[0, +\infty)$-valued process, where

$$W_t^n = \sum_{k=0}^{\infty} W(\tau_k^n)\,\mathbb{I}_{\left[\tau_k^n, \tau_{k+1}^n\right)}(t), \quad \text{and} \quad \Lambda_t^n = \ell_t(W^n). \tag{30}$$

Note that Lemma 4.1 provides the following convergence result.

Lemma 4.3. *Let (W^n, Λ^n) be defined as in (30). Then, for each $t \in \mathbb{R}^+$*

$$|W_t^n - W_t| \le \frac{1}{2^n}, \tag{31}$$

and $(W^n, \Lambda^n) = \big(W^n, \ell(W^n)\big)$ converge to $(W, \Lambda) = \big(W, \ell(W)\big)$ a.s., w.r.t. the topology of the uniform convergence.

By (29)

$$\Lambda_t^n = \sum_{j=0}^{\infty} \frac{1}{2^n}\mathbb{I}(\sigma_j^n \le t), \tag{32}$$

where

$$\sigma_j^n = \inf\left\{t \quad \text{s.t.} \quad W_t \le -\frac{j}{2^n}\right\} = \inf\left\{t \quad \text{s.t.} \quad \Lambda_t \ge \frac{j}{2^n}\right\}. \tag{33}$$

Moreover, with the above choice of the threshold, the n-th grid is generated by considering the dyadic intervals of rank n. Then in the passage from the n-th grid to the $(n+1)$-th grid each threshold is split into two parts, and therefore $\sigma_{2j}^{n+1} = \sigma_j^n$. This property is decisive since it guarantees that, for any t, $\{\mathcal{G}_t^n = \mathcal{F}_t^{\Lambda^n}, n \in \mathbb{N}\}$ is an increasing family of σ-algebras, with $\mathcal{G}_t^n \uparrow \mathcal{F}_t^{\Lambda}$ (see Lemma 2.3 of [8], where the process Λ_t^n is defined as in (32)). The last fact allows us to show the claimed strong convergence result, which is a slight generalization of Theorem 2.4 of [8].

Theorem 4.4. *Let π_t and π_t^n be the filters defined in* (6) *and* (21). *Consider them as random variables with values in the space of probability measures on $\mathbb{R}$, endowed with the topology of weak convergence. Then the sequence π_t^n converges to π_t almost surely. As a consequence for all $g \in C_b(\mathbb{R})$*

$$\pi_t^n(g) = E\big[g(W_t^n)/\mathcal{G}_t^n\big] \to \pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^{\Lambda}\big], \quad \textit{a.s. and in } L^1. \tag{34}$$

Proof. Observe that $|\pi_t(g) - \pi_t^n(g)|$ is bounded above by

$$\Big|E\big[g(W_t)/\mathcal{F}_t^{\Lambda}\big] - E\big[g(W_t)/\mathcal{G}_t^n\big]\Big| + \Big|E\big[g(W_t)/\mathcal{G}_t^n\big] - E\big[g(W_t^n)/\mathcal{G}_t^n\big]\Big|.$$

The first term converges to zero almost surely and in L^1-sense. Indeed, as in Theorem 2.4 of [8], we apply Doob's convergence Theorem to the discrete time martingale $E\big[g(W_t)/\mathcal{G}_t^n\big]$. For all g uniformly continuous, with modulus of continuity ω_g, the second term is bounded above by

$$\Big|E\big[\big|g(W_t) - g(W_t^n)\big|/\mathcal{G}_t^n\big]\Big| \le \omega_g\left(1/2^n\right) \to 0,$$

and so we get (34) for all g in a convergence determining class. Without loss of generality we can take this class denumerable, and therefore we obtain the convergence of π_t^n to π_t. The convergence result (34) for all bounded and continuous g is then straightforward. □

One can get the analogous convergence results for the corresponding queueing model generated by reflecting W^n. In particular the conditional laws defined by $E\big[g(W_t^n+\Lambda_t^n)/\mathcal{G}_t^n\big]$ converge a.s. to the conditional law $\hat\pi_t$ defined by (7).

In addition the strong convergence of Theorem 4.4 implies the weak convergence for the filters of any rescaled model X^n sharing the same law as W^n. As an example we can take $\tilde X^n = W^0$ for all n, and $X_t^n = \frac{1}{2^n} W^0_{2^{2n}t}$.

Now we show that the approximating model falls into the frame of the previous sections. The sequence $\{(\tilde T_k^n, \tilde U_k^n),\ k\ge 1\}$ is defined as

$$\tilde T_k^n = (\tau_k^n - \tau_{k-1}^n)/2^{2n}, \qquad \tilde U_k^n = 2^n\big(W_{\tau_k^n} - W_{\tau_{k-1}^n}\big),$$

which clearly satisfies condition **H** Then

$$W_t^n = \frac{1}{2^n}\tilde V^n_{\tilde Z^n_{2^{2n}t}},$$

where $\tilde Z_t^n$ is the renewal process defined by the sequence of i.i.d. interarrival times $\{\tilde T_k^n,\ k\in\mathbb{N}\}$, and $\tilde V_k^n = 2^n W_{\tau_k^n}$. Therefore, recalling (21), $E\big[g(W_t^n)/\mathcal{G}_t^n\big] = \Sigma^n(\gamma_t(W^n),\ell_t(W^n);g)$, with $\Sigma^n(s,l;g)$ as in (23), and γ_t as in (10). Similar result holds also for $E\big[g(W_t^n+\ell_t(W^n))/\mathcal{G}_t^n\big]$.

When the drift coefficient of W is zero the random variables $\tilde T_k^n$ have common law

$$\tilde F^n(t) = P\big(\tilde T_k^n \le t\big) = \tilde F_1(2^{2n}a^2 t) \tag{35}$$

where a^2 is the diffusion coefficient and $\tilde F_1$ is defined in (25), and $\tilde U_k^n$ is symmetric and independent of $\tilde T_k^n$, for each k (see e.g. [4] page 342). Therefore $\tilde V_k^n = 2^n W_{\tau_k^n}$ is a symmetric random walk, independent of the renewal process $\tilde Z_t^n$. Then Condition **K** holds, and one can use (24) to define $\Sigma^n(s,l)$.
When the drift coefficient is $c\neq 0$ the processes $\tilde Z_t^n$ and $\tilde V_k^n$, defined as above, are not mutually independent, then one cannot use (24). Nevertheless (24), with $\tilde F^n$ as in (35) above, could be used to get the approximate expression

$$\tilde\pi_t^n(g) = \frac{E^{P_0}\big[g(W_t^n)\exp(\frac{c}{a^2}W_t^n)/\mathcal{G}_t^n\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W_t^n)/\mathcal{G}_t^n\big]} = \left.\frac{\Sigma^n\left(s,l;g(\cdot)\exp(\frac{c}{a^2}\cdot)\right)}{\Sigma^n\left(s,l;\exp(\frac{c}{a^2}\cdot)\right)}\right|_{s=\gamma_t(W^n),l=\ell_t(W^n)}.$$

Indeed by Kallianpur Striebel formula and Girsanov Theorem

$$E\big[g(W_t^n)/\mathcal{G}_t^n\big] = \frac{E^{P_0}\big[g(W_t^n)\exp(\frac{c}{a^2}W_t)/\mathcal{G}_t^n\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W_t)/\mathcal{G}_t^n\big]}$$

where P_0 is equivalent to P, and under P_0 the process W has drift coefficient zero. Then

$$E\big[g(W_t^n)/\mathcal{G}_t^n\big] = \frac{E^{P_0}\big[g(W_t^n)\exp(\frac{c}{a^2}W_t^n)\exp(\frac{c}{a^2}(W_t-W_t^n))/\mathcal{G}_t^n\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W_t^n)\exp(\frac{c}{a^2}(W_t-W_t^n))/\mathcal{G}_t^n\big]},$$

Moreover, taking into account that $|W_t - W_t^n|\le \frac{1}{2^n}$, one can get that

$$\big|E\big[g(W_t^n)/\mathcal{G}_t^n\big] - \tilde\pi_t^n(g)\big| \le 4\exp\big(2\tfrac{|c|}{a^2}\tfrac{1}{2^n}\big)\frac{|c|}{a^2}\frac{1}{2^n}\|g\|_\infty. \tag{36}$$

5 The M/M/1 queueing model

In this Section we consider a random walk with exponential interarrival times, and the M/M/1 queue, with arrival intensity λ_n and service potential μ_n, generated by reflecting the random walk. We use the techniques introduced in Section 2 to derive the filter of the M/M/1 queue (and therefore of the random walk) with respect to the local time associated to the random walk. Moreover, under a suitable set of conditions, which are related to the heavy traffic conditions, we get also the weak limit of the filter of the rescaled system (Theorem 5.2) and an approximation for the filter (Theorem 5.3).

5.1 Description of the model and main results

The sequence of random walks we consider is defined by means of the same rule as in (15), namely for each $n \in \mathbb{N}$

$$\tilde{X}_t^n = \tilde{V}^n_{\tilde{Z}_t^n} = \sum_{j=1}^{\tilde{Z}_t^n} \tilde{U}_j^n,$$

where

A1 $\tilde{Z}_t^n$ is a Poisson process with intensity $(\lambda_n + \mu_n)$

A2 $\tilde{V}_j^n$ is defined by $\tilde{V}_j^n = \tilde{V}_{j-1}^n + \tilde{U}_j^n$, where $\{\tilde{U}_j^n, j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with $P^n(\tilde{U}_k^n = +1) = \frac{\lambda_n}{\lambda_n+\mu_n}$ and $P^n(\tilde{U}_k^n = -1) = \frac{\mu_n}{\lambda_n+\mu_n}$.

A3 $\{\tilde{U}_k^n, k \in \mathbb{N}\}$ and $\tilde{Z}_t^n$ are mutually independent.

In this case the interarrival times $\tilde{T}_k^n$ of the renewal process $\tilde{Z}_t^n$ are exponential random variables with expectation $1/(\lambda_n + \mu_n)$. Therefore we are in the situation discussed at the end of Section 3, with $p_n = \lambda_n/(\lambda_n + \mu_n)$, $\tilde{F}_1^n$ the distribution function of an exponential random variable of parameter $\lambda_n + \mu_n$, moreover the scaling parameters are $a_n = n$ and $b_n = \sqrt{n}$, and then F_k^n is the distribution function Gamma of parameter $(k, n(\lambda_n + \mu_n))$.

The conditions **C1**, **C2**, and **C3** are defined as follows.

C1 $\lambda_n, \mu_n > 0$

C2 $(\lambda_n, \mu_n) \underset{n\to+\infty}{\to} (\lambda, \lambda)$

C3 $\sqrt{n}(\lambda_n - \lambda) \underset{n\to+\infty}{\to} c(1) \qquad \sqrt{n}(\mu_n - \lambda) \underset{n\to+\infty}{\to} c(2)$

Condition **C1** avoids to consider pure birth or pure death processes, and **C3** clearly implies condition **C2** and condition

C3* $\sqrt{n}(\lambda_n - \mu_n) \underset{n\to+\infty}{\to} c = c(1) - c(2)$.

We recall that, when $\lambda_n < \mu_n$, the set of conditions **C1**, **C2**, **C3*** are known in literature as the *heavy traffic conditions*, and in this case $c \leq 0$. These conditions guarantee the existence of the diffusive limit of the rescaled system, more precisely, the sequence of processes $X_t^n = \tilde{X}_{nt}^n/\sqrt{n}$ converges weakly in $D_{\mathbb{R}}[0, +\infty)$ to a Brownian motion W_t, with diffusion coefficient 2λ and drift coefficient c.

Remark 5.1. *It is interesting to note that conditions* **C1**, **C2**, **C3** *are equivalent to the weak convergence in* $D_{\mathbb{R}^2}[0,+\infty)$ *of the processes* (X_t^n, Z_t^n), *where* $Z_t^n = (\tilde{Z}_{nt}^n - 2\lambda nt)/\sqrt{n}$, *to a pair of independent Brownian motions* (W_t, B_t) *with drift* $c = c(1) - c(2)$ *and* $d = c(1) + c(2)$ *respectively, and both with variance* 2λ. *Indeed the processes* $\tilde{X}_t^n$ *and* $\tilde{Z}_t^n$ *can be represented as*

$$\tilde{X}_t^n = \tilde{A}_t^n - \tilde{N}_t^n, \quad \tilde{Z}_t^n = \tilde{A}_t^n + \tilde{N}_t^n \tag{37}$$

where, if $\tilde{\tau}_k^n$ *are the jump times of* $\tilde{Z}^n$,

$$\tilde{A}_t^n = \sum_{k=0}^{\infty} \mathbb{I}(\tilde{U}_k^n = 1)\mathbb{I}(\tilde{\tau}_k^n \leq t) \tag{38}$$

$$\tilde{N}_t^n = \sum_{k=0}^{\infty} \mathbb{I}(\tilde{U}_k^n = -1)\mathbb{I}(\tilde{\tau}_k^n \leq t), \tag{39}$$

So that

$$X_t^n = \frac{\tilde{A}_{nt}^n - \tilde{N}_{nt}^n}{\sqrt{n}} = \frac{\tilde{A}_{nt}^n - n\lambda_n t}{\sqrt{n}} - \frac{\tilde{N}_{nt}^n - n\mu_n t}{\sqrt{n}} + \sqrt{n}(\lambda_n - \mu_n)t,$$

$$Z_t^n = \frac{\tilde{A}_{nt}^n - n\lambda_n t}{\sqrt{n}} + \frac{\tilde{N}_{nt}^n - n\mu_n t}{\sqrt{n}} + \sqrt{n}(\lambda_n + \mu_n - 2\lambda)t.$$

By Watanabe's Theorem (see, for instance, [2]) the processes $\tilde{A}^n$ *and* $\tilde{N}^n$ *are mutually independent Poisson processes with intensities* λ_n *and* μ_n *respectively, as a consequence the processes* $\left(\frac{\tilde{A}_{nt} - n\lambda_n t}{\sqrt{n}}, \frac{\tilde{N}_{nt} - n\mu_n t}{\sqrt{n}}\right)$ *converge weakly to a pair of independent Brownian motions with zero drift and diffusion coefficient* 2λ.

The solution of the Skorohod problem for the process $\tilde{X}_t^n$ is the pair $(\tilde{Q}_t^n, \tilde{L}_t^n)$, where

$$\tilde{Q}_t^n = \tilde{X}_t^n + \tilde{L}_t^n$$

is a M/M/1 queue (see, for instance, [2]), and $\tilde{L}_t^n$ is the local time associated to the process $\tilde{X}_t^n$. Then, thanks to the weak convergence of X^n, a continuous map argument applies to show that

$$(X_t^n, Q_t^n, L_t^n) = \left(\frac{\tilde{X}_{nt}^n}{\sqrt{n}}, \frac{\tilde{Q}_{nt}^n}{\sqrt{n}}, \frac{\tilde{L}_{nt}^n}{\sqrt{n}}\right) \Rightarrow (W_t, W_t + \Lambda_t, \Lambda_t), \tag{40}$$

where $\Lambda_t = \ell_t(W)$ is the local time of the Skorohod reflection of W_t. Indeed the functional ℓ_t is continuous with respect to the topology of uniform convergence on bounded intervals of time, and X_t^n converges to W_t with respect to this topology, since W_t has continuous trajectories.

The main results are stated in the following theorems which are proven at the end of this subsection. We recall (see (12) and (13)) that π_t^n and $\hat{\pi}_t^n$ denote the filters of X_t^n and of Q_t^n given the filtration $\mathcal{G}_t^n$, respectively, where, as in Section 3, $\mathcal{G}_t^n$ denotes the filtration generated by L_t^n.

Theorem 5.2. *Assume* **A1**, **A2**, **A3** *and* **C1**, **C2**, **C3**, *then, for any* $t \geq 0$, π_t^n *converge weakly to* π_t, *and* $\hat{\pi}_t^n$ *converge weakly to* $\hat{\pi}_t$, *as random variables with values in the space of probability measures endowed with the topology of weak convergence.*
In particular, for any $t \geq 0$, *and for any bounded continuous function* g

$$\pi_t^n(g) = E\big[g(X_t^n)/\mathcal{G}_t^n\big] \Rightarrow \pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] \tag{41}$$

and

$$\hat{\pi}_t^n(g) = E\big[g(Q_t^n)/\mathcal{G}_t^n\big] \Rightarrow \hat{\pi}_t(g) = E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]. \tag{42}$$

As explained in the introduction it is interesting to find a good approximation for the filter $\hat{\pi}_t^n = \hat{\Sigma}^n(\xi_t^n)$ which is, at the same time, simpler to handle, and depends on the actually observed trajectory, and a natural candidate is $\hat{\Pi}_{2\lambda,c}(\xi_t^n)$, where $\hat{\Pi}_{2\lambda,c}(s)$ is defined in (5). We prove that this natural candidate is an $L_p\,(\Omega \times [0,T])$-norm approximation of the filter $\hat{\pi}_t^n$.

Theorem 5.3. *Under the same assumptions of Theorem 5.2, for all g bounded and continuous and for each $T > 0$, $p > 0$*

$$\int_0^T E\left|\hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\xi_t^n; g)\right|^p dt \underset{n\to\infty}{\longrightarrow} 0.$$

The proofs of the previous theorems are based on the representations for $\pi_t^n = \Sigma^n(\xi_t^n, L_t^n)$ and $\hat{\pi}_t^n = \hat{\Sigma}^n(\xi_t^n)$, respectively, and the results of Propositions 5.4 and 5.5 below. The first result is the the key convergence result (14) announced in the Introduction.

Proposition 5.4. *Under the same assumptions of Theorem 5.2, $\hat{\Sigma}^n(s; g)$ converge pointwise to $\hat{\Pi}_{2\lambda,c}(s; g)$, for every bounded continuous function $g : \mathbb{R}^+ \to \mathbb{R}$, and $s \geq 0$. Moreover the convergence is uniform on bounded intervals contained in $(0,\infty)$, i.e. whenever $s_n \to s$, with $s > 0$*

$$\hat{\Sigma}^n(s_n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(s; g). \tag{43}$$

Proof. The proof of this key result is postponed to the next subsection. □

The second results concerns the weak convergence of $\xi_t^n = \gamma_t(L^n)$ to $\zeta_t = \gamma_t^0(W + \Lambda) = \gamma_t(\Lambda)$, with γ_t^0 and γ_t defined in (9) and (10) respectively. However we show a slightly stronger result concerning the weak convergence of $\gamma_t^0(X^n + L^n) = \gamma_t^0(Q^n)$ to ζ_t. This stronger result is used later in Section 6.

Proposition 5.5. *Assume* **A1**, **A2**, **A3** *and* **C1**, **C2**, **C3***, *then for each $t > 0$*

$$(\gamma_t^0(Q^n), \gamma_t(L^n), L_t^n) \Rightarrow (\zeta_t, \zeta_t, \Lambda_t).$$

Proof. Define

$$\begin{aligned} \eta_t^n &= \sup\{s < t : L_s^n < L_t^n\}, \quad \eta_t = \sup\{s < t : \Lambda_s < \Lambda_t\}, \\ \beta_t^n &= \sup\{s < t : \;\; Q_s^n = 0\}, \quad \beta_t = \sup\{s < t : W_s + \Lambda_s = 0\}, \end{aligned}$$

with $\eta_t^n = t$, $\eta_t = t$, $\beta_t^n = t$ and $\beta_t = t$ when the corresponding sets are empty. Note that

$$\begin{aligned} \eta_t^n &= \sup\{s < t : X_t^n - X_s^n < Q_t^n - Q_s^n\}, \\ \eta_t &= \sup\{s < t : W_t - W_s < W_t + \Lambda_t - W_s - \Lambda_s\}, \end{aligned}$$

Applying the Skorohod representation theorem, we can assume that all the processes live on the same probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, and that

$$\sup_{s\leq t}(|X_s^n - W_s| + |Q_s^n - W_s - \Lambda_s|) \to 0 \quad \bar{P}\text{-a.s.} \tag{44}$$

This implies that $L^n \to \Lambda$ uniformly in $[0,t]$, $\bar{P}$-a.s., and

$$\liminf_{n\to\infty} \; \eta_t^n \geq \eta_t.$$

Then, since $\gamma_t^0(Q^n) = t - \beta_t^n$, and $\gamma_t(L^n) = t - \eta_t^n$, the result is achieved once we prove that the sequence (β_t^n, η_t^n) converges $\bar{P}$-a.s. to (η_t, η_t) and $\zeta_t = t - \eta_t$. Let $\beta_t^\infty = \limsup_{n\to\infty} \beta_t^n$ and note that $Q^n(\beta_t^n)$ assumes only the values 0 or $\frac{1}{\sqrt{n}}$. Then, by (44), $W_{\beta_t^\infty} + \Lambda_{\beta_t^\infty} = 0$. It follows that

$$\limsup_{n\to\infty} \beta_t^n \leq \beta_t.$$

Moreover, if $\eta_t^n < t$, then $Q^n(\eta_t^n) = 0$, if $\eta_t^n = t$, then $\beta_t^n = t$, and it follows that

$$\eta_t^n \leq \beta_t^n \qquad\qquad \text{for all } t,\ \bar{P}\text{-a.s.}$$

and then

$$\eta_t \leq \liminf_{n\to\infty}\ \eta_t^n \leq \limsup_{n\to\infty} \beta_t^n \leq \beta_t, \qquad\qquad \text{for all } t,\ \bar{P}\text{-a.s.}$$

The proof is achieved since $\bar{P}(\eta_t = \beta_t) = 1$, and then $\zeta_t = \gamma_t^0(W + \Lambda) = t - \beta_t = t - \eta_t = \gamma_t(\Lambda)$.

□

Remark 5.6. *In the Skorohod space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ used in the proof of Proposition 5.5, choose a jointly measurable version of $\xi_t^n = \gamma_t(L^n) = t - \eta_t^n$. Then $M = \big\{(\omega, t) \in \Omega \times [0, T] \ \ s.t.\ \xi_t^n(\omega) \not\to \zeta_t(\omega)\big\}$ is a zero $d\bar{P} \times dt$-measure set. Moreover a similar result holds for $\gamma_t^0(Q^n) = t - \beta_t^n$, namely $M_0 = \big\{(\omega, t) \in \Omega \times [0, T] \ \ s.t.\ \gamma_t^0(Q^n)(\omega) \not\to \zeta_t(\omega)\big\}$ is a zero $d\bar{P} \times dt$-measure set.*

We are now ready to prove Theorems 5.2 and 5.3

Proof of Theorem 5.2
The weak convergence for filters of the reflected random walk follows since we can use the Skorohod representation probability space as in Proposition 5.5, and in this space, for each $t > 0$, $\bar{P}(\xi_t^n = \gamma_t(L^n) \to \zeta_t) = 1$, and, on the other hand $\bar{P}\{\omega :\ \zeta_t(\omega) > 0\} = 1$, as observed in Remark 5.6. As a consequence, taking into account the key convergence result of Proposition 5.4,

$$\bar{P}\big(\hat{\Sigma}^n(\xi_t^n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(\zeta_t; g), \quad \text{ for every } g : \mathbb{R}^+ \to \mathbb{R} \text{ bounded and continuous}\big) = 1,$$

and the above property is equivalent to show that $\hat{\pi}_t^n$ converges weakly to $\hat{\pi}_t$.
The proof of the weak convergence for the filter of the random walk is similar, since the convergence of $\Sigma^n(s_n, l_n; g)$ to $\Pi_{2\lambda,c}(s, l; g)$ whenever (s_n, l_n) converges to (s, l), with $s > 0$ is just a slight extension of Proposition 5.4, that is for any $g : \mathbb{R}^+ \to \mathbb{R}$ bounded and uniformly continuous,

$$\Sigma^n(s_n, l_n; g) \underset{n\to\infty}{\longrightarrow} \Pi_{2\lambda,c}(s, l; g). \tag{45}$$

Indeed on the one hand $|\Sigma^n(s_n, l_n; g) - \Sigma^n(s_n, l; g)| \leq \omega_g(|l_n - l|)$, and therefore converge to zero, and on the other hand, $\Sigma^n(s_n, l; g) = \hat{\Sigma}^n(s_n; g_l)$ converge to $\Pi(s, l; g) = \hat{\Pi}(s; g_l)$, where $g_l(x) = g(-l + x)$. The set of bounded and uniformly continuous functions is a convergence determining class and then (45) follows for all bounded continuous functions g. Then, using again the Skorohod representation space, we get that $\bar{P}((\xi_t^n, L_t^n) \to (\zeta_t, \Lambda_t), \zeta_t > 0) = 1$, and

$$\bar{P}\big(\Sigma^n(\xi_t^n, L_t^n; g) \underset{n\to\infty}{\longrightarrow} \Pi_{2\lambda,c}(\zeta_t, \Lambda_t; g), \quad \text{ for every } g : \mathbb{R}^+ \to \mathbb{R} \text{ bounded and continuous}\big) = 1.$$

Therefore π_t^n converge weakly to π_t, and Theorem 5.2 is completely achieved.

□

Proof of Theorem 5.3
The limit we are looking for depends only on the distribution of ξ_t^n, therefore using the Skorohod representation space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ as in the proof of Proposition 5.5, the thesis is equivalent to

$$\int_0^T E^{\bar{P}} \left| \hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\xi_t^n; g) \right|^p dt \underset{n\to\infty}{\longrightarrow} 0.$$

As observed in Remark 5.6, we can assume that $\xi_t^n(\omega)$ converge to $\zeta_t(\omega)$ $d\bar{P} \times dt-$a.e., and then by Proposition 5.4 we get

$$\hat{\Sigma}^n(\xi_t^n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(\zeta_t; g) \qquad \text{and} \qquad \hat{\Pi}_{2\lambda,c}(\xi_t^n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(\zeta_t; g) \tag{46}$$

for each (ω, t) such that $\zeta_t(\omega) > 0$.

The observation that $\{(\omega, t) \in \Omega \times [0, T]$ such that $\zeta_t(\omega) = 0\}$ is a zero measure set with respect to $d\bar{P} \times dt$, and an easy application of the dominated convergence theorem imply that

$$\int_0^T E^{\bar{P}} \left[\left| \hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\zeta_t; g) \right|^p \right] \to 0, \qquad \text{for any } p > 0$$

and

$$\int_0^T E^{\bar{P}} \left[\left| \hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\xi_t^n; g) \right|^p \right] \to 0, \qquad \text{for any } p > 0. \tag{47}$$

□

5.2 The key result

In this subsection our aim is to prove Proposition 5.4, i.e.the key result (14) under conditions **A1**, **A2**, **A3** and **C1**, **C2**, **C3**.
Without loss of generality, we can assume that all the processes involved are defined on the same measurable space $(\Omega, \mathcal{F})$, but with different probability measures P^n. Moreover we can assume that ***(i)*** the processes defined in (37) are the same for all n, namely we can take $\tilde{X}_t^n = \tilde{X}_t = \tilde{V}_{\tilde{Z}_t}$ and $\tilde{Z}_t^n = \tilde{Z}_t$, with $\tilde{V}_k = \sum_{j=1}^k \tilde{U}_j$, ***(ii)*** the measures P^n are all absolutely continuous with respect to a given measure P (see (48) below), and finally ***(iii)*** under the measure P, the process $\tilde{Z}_t$ is a Poisson process $\tilde{Z}_t$ of intensity 2λ and $\tilde{V}_k$ is a symmetric random walk.
Starting from the processes $\tilde{X}_t$ and $\tilde{Z}_t$, and in analogy with (38) and (39) of Remark 5.1, we can define the process $\tilde{A}_t$ as the process counting the positive jumps of $\tilde{X}_t$, and the process $\tilde{N}_t$ as the process counting the negative jumps of $\tilde{X}_t$.

On $(\Omega, \mathcal{F})$ we consider the filtration $\{\mathcal{F}_t^n,\ t \in [0, T]\}$ generated by the time-rescaled processes $(\hat{A}_t^n, \hat{N}_t^n) = (\tilde{A}_{nt}, \tilde{N}_{nt})$, and the probability measure P^n, absolutely continuous with respect to P, such that

$$\frac{dP^n}{dP}\Big|_{\mathcal{F}_t^n} = \mathcal{L}_t^n = \left(\frac{\lambda_n}{\lambda}\right)^{\hat{A}_t^n} \exp\left\{-n\left(\lambda_n - \lambda\right)t\right\} \left(\frac{\lambda_n}{\lambda}\right)^{\hat{N}_t^n} \exp\left\{-n\left(\mu_n - \mu\right)t\right\} \tag{48}$$

Under the measure P, the processes $\hat{A}_t^n$, $\hat{N}_t^n$ are mutually independent Poisson processes with intensities $n\lambda$, $n\lambda$, while (see [2], Chapter VIII) under P^n the processes $\hat{A}_t^n$, $\hat{N}_t^n$ are mutually independent Poisson processes with intensities $n\lambda_n$, $n\mu_n$. Finally note that, in the probability space $(\Omega, \mathcal{F}, P^n)$, the conditions **A1**, **A2**, **A3** are satisfied with $\tilde{Z}_t^n = \tilde{Z}_t = \tilde{A}_t + \tilde{N}_t$, $\tilde{U}_j^n = \tilde{U}_j$.

From now to the end of this Section, we denote by E^P and E^{P^n} the expectations with respect to the probability measures P and P^n, respectively. Then, by Kallianpur Striebel formula, we get

$$E^{P^n}\big[g(X_t^n)/\mathcal{G}_t^n\big] = \frac{E^P\big[g(X_t^n)\mathcal{L}_t^n/\mathcal{G}_t^n\big]}{E^P\big[\mathcal{L}_t^n/\mathcal{G}_t^n\big]}. \tag{49}$$

Moreover, setting $\hat{X}_t^n = \hat{A}_t^n - \hat{N}_t^n$, and $\hat{Z}_t^n = \hat{A}_t^n + \hat{N}_t^n$,the rescaled processes are $X_t^n = \frac{\hat{X}_t^n}{\sqrt{n}}$ and $Z_t^n = \frac{\hat{Z}_t^n - 2n\lambda t}{\sqrt{n}}$, and under the measure P the sequence of processes (X_t^n, Z_t^n) converge weakly in $D_{\mathbb{R}^2}([0,\infty))$ to two independent Brownian motions $(W_t, B_t) := (W_t^A - W_t^N, W_t^A + W_t^N)$, indeed

$$X_t^n = \frac{\hat{X}_t^n}{\sqrt{n}} = \frac{\tilde{A}_{nt} - \tilde{N}_{nt}}{\sqrt{n}} = \frac{\tilde{A}_{nt} - n\lambda t}{\sqrt{n}} - \frac{\tilde{N}_{nt} - n\lambda t}{\sqrt{n}} \Rightarrow W_t^A - W_t^N,$$

$$Z_t^n = \frac{\hat{Z}_t^n - 2n\lambda t}{\sqrt{n}} = \frac{\tilde{A}_{nt} - n\lambda t}{\sqrt{n}} + \frac{\tilde{N}_{nt} - n\lambda t}{\sqrt{n}} \Rightarrow W_t^A + W_t^N,$$

where W_t^A and W_t^N are clearly independent Brownian motion (the last property implies the independence of W and B). Therefore it is natural to get an alternative expression of $\mathcal{L}_t^n$ in terms of the processes X_t^n and Z_t^n. Taking into account that $\hat{A}_t^n = (\hat{Z}_t^n + \hat{X}_t^n)/2$, $\hat{N}_t^n = (\hat{Z}_t^n - \hat{X}_t^n)/2$, and that

$$log(\mathcal{L}_t^n) = \log\left(\frac{\lambda_n}{\lambda}\right)\hat{A}_t^n - n\left(\lambda_n - \lambda\right)t + \log\left(\frac{\mu_n}{\lambda}\right)\hat{N}_t^n - n\left(\mu_n - \lambda\right)t,$$

we get immediately that

$$log(\mathcal{L}_t^n) = c_n X_t^n + d_n Z_t^n + e_n t, \tag{50}$$

where

$$c_n = \tfrac{1}{2}\sqrt{n}\left[\log\left(\tfrac{\lambda_n}{\lambda}\right) - \log\left(\tfrac{\mu_n}{\lambda}\right)\right] \tag{51}$$

$$d_n = \frac{1}{2}\sqrt{n}\left[\log\left(\tfrac{\lambda_n}{\lambda}\right) + \log\left(\tfrac{\mu_n}{\lambda}\right)\right] \tag{52}$$

$$e_n = n\left[\log\left(\tfrac{\lambda_n}{\lambda}\right) + \log\left(\tfrac{\mu_n}{\lambda}\right)\right]\lambda - n\left(\lambda_n + \mu_n - 2\lambda\right). \tag{53}$$

Therefore (49) can be rewritten as

$$E^{P^n}\big[g(X_t^n)/\mathcal{G}_t^n\big] = \frac{E^P\big[g(X_t^n)\exp(c_n X_t^n)\exp(d_n Z_t^n)/\mathcal{G}_t^n\big]}{E^P\big[\exp(c_n X_t^n)\exp(d_n Z_t^n)/\mathcal{G}_t^n\big]}. \tag{54}$$

Under conditions **C1**, **C2** and **C3**, the sequence (c_n, d_n) converges to $(\bar{c}, \bar{d})$, where $\bar{c} = c/(2\lambda)$ (see Lemma 5.9 at the end of the section). If we substitute in the right hand side of the above expression the formal limits we get

$$\frac{E\big[g(W_t)\exp(\bar{c}W_t)\exp(\bar{d}B_t)/\mathcal{F}_t^\Lambda\big]}{E\big[\exp(\bar{c}W_t)\exp(\bar{d}B_t)/\mathcal{F}_t^\Lambda\big]} = \frac{E\big[g(W_t)\exp(\bar{c}W_t)/\mathcal{F}_t^\Lambda\big]}{E\big[\exp(\bar{c}W_t)/\mathcal{F}_t^\Lambda\big]} = \hat{\Pi}_{2\lambda,c}(\zeta_t; g), \tag{55}$$

where the first equality holds since $\mathcal{F}_t^\Lambda \subset \mathcal{F}_t^W$ and the processes W and B are independent, while the second equality follows by the fact that W_t has drift zero and diffusion coefficient 2λ, by the value of the limit $\bar{c}$ and by using Kallianpur-Striebel formula again.

The above considerations lead to a heuristic proof of our main result. However we do not formalize the above heuristic reasoning to get the proof, but we use the following expression for the filter of the queue

$$E^{P^n}\big[g(Q_t^n)/\mathcal{G}_t^n\big] = \frac{E^P\big[\, g(X_s^n)\, \exp(c_n X_s^n)\, \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}{E^P\big[\exp(c_n X_s^n) \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}\Bigg|_{s=\xi_t^n}. \tag{56}$$

The above expression can be easily obtained taking into account (22), (23), the definition (48) of P^n by means of (50), namely, using the notations of Remark 3.1, $E^{P^n}\big[g(Q_t^n)/\mathcal{G}_t^n\big] = \hat{\Sigma}_{P^n}^n(\xi_t^n; g)$, and

$$\begin{aligned}\hat{\Sigma}_{P^n}^n(s;g) &= \frac{E^{P^n}\Big[g(X_s^n)\mathbb{I}\,(\sigma_1^n > s)\Big]}{E^{P^n}\Big[\mathbb{I}(\sigma_1^n > s)\Big]} = \frac{E^P\Big[\mathcal{L}_s^n g(X_s^n)\mathbb{I}\,(\sigma_1^n > s)\Big]}{E^P\Big[\mathcal{L}_s^n\mathbb{I}(\sigma_1^n > s)\Big]} \\ &= \frac{E^P\big[\, g(X_s^n)\, \exp(c_n X_s^n)\, \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}{E^P\big[\exp(c_n X_s^n) \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}.\end{aligned} \tag{57}$$

In order to get the limit of previous filter the idea is to show the convergence of the function $\hat{\Sigma}_{P^n}^n(s;g)$, and then a first essential step consists in evaluating

$$E^P\big[\, f(X_s^n)\, \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]$$

either for $f(x) = g_n(x) = g(x)\exp(c_n x)$ or for $f(x) = \exp(c_n x)$, and this can be found in the following lemma, which is based on the reflection principle.

Lemma 5.7. *Let f be a function with continuous derivative f'. Then*

$$\begin{aligned}E^P\big[\, f(X_s^n)\, \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big] =& E^P\Big[\frac{2}{\sqrt{n}}\mathbb{I}(X_s^n \geq 2/\sqrt{n}) f'(X_s^n - 2\theta_f^n/\sqrt{n}) \exp(d_n Z_s^n)\Big] \\ &+ E^P\Big[\mathbb{I}(0 \leq X_s^n < 2/\sqrt{n}) f(X_s^n)\, \exp(d_n Z_s^n)\Big],\end{aligned}$$

where θ_f^n is a random variable with values in $(0,1)$.

Proof. It is sufficient to prove that

$$\begin{aligned}&E^P\big[\, f(X_s^n)\, \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big] = E^P\big[\, f(X_s^n)\mathbb{I}(X_s^n \geq 0)\, \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big] \\ &= E^P\big[\, \tilde{f}(X_s^n)\, \exp(d_n Z_s^n)\big] - E^P\big[\, \tilde{f}(X_s^n)\, \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n \leq s)\big] \\ &= E^P\big[\, \{\tilde{f}(X_s^n) - \tilde{f}(X_s^n - \frac{2}{\sqrt{n}})\}\, \exp(d_n Z_s^n)\big],\end{aligned} \tag{58}$$

where $\tilde{f}(x) = f(x)\mathbb{I}(x \geq 0)$, and where we apply the reflection principle in order to get the last equality. Indeed, if $\bar{X}_s^n$ is the process obtained by reflecting X_s^n at time σ_1^n, i.e. if $\bar{X}_s^n =$

$(\bar{A}^n_s - \bar{N}^n_s)/\sqrt{n}$, where $\big(\bar{A}^n_s, \bar{N}^n_s\big)$ is defined as $\big(\hat{A}^n_s, \hat{N}^n_s\big)$ for $s < \sigma^n_1$, and as $\big(\hat{A}^n_{\sigma^n_1} + \big(\hat{N}^n_s - \hat{N}^n_{\sigma^n_1}\big), \hat{N}^n_{\sigma^n_1} + \big(\hat{A}^n_s - \hat{A}^n_{\sigma^n_1}\big)\big)$ for $s \geq \sigma^n_1$, then, on the one hand,

$$\begin{aligned} & f(X^n_s)\mathbb{I}(X^n_s \geq 0)\,\exp(d_n Z^n_s)\mathbb{I}(\sigma^n_1 \leq s) \\ &= f(-\bar{X}^n_s - 2/\sqrt{n})\mathbb{I}(\bar{X}^n_s \leq -2/\sqrt{n})\,\exp(d_n Z^n_s)\,\mathbb{I}(\bar{\sigma}^n_1 \leq s) \\ &= f(-\bar{X}^n_s - 2/\sqrt{n})\mathbb{I}(\bar{X}^n_s \leq -2/\sqrt{n})\,\exp(d_n Z^n_s), \end{aligned} \tag{59}$$

since
(i) $\sigma^n_1 \leq s$ if and only if $\bar{\sigma}^n_1 \leq s$, where $\bar{\sigma}^n_1 = \inf\{u \text{ such that } \bar{X}^n_u \leq -1/\sqrt{n}\}$,
(ii) if $\bar{\sigma}^n_1 \leq s$ then $X^n_s + 1/\sqrt{n} = -1/\sqrt{n} - \bar{X}^n_s$,
and therefore, when $\bar{\sigma}^n_1 \leq s$,
(iii) $X^n_s \geq 0$ if and only if $-2/\sqrt{n} \geq \bar{X}^n_s$, which implies $\bar{\sigma}^n_1 \leq s$,
and, on the one hand, $(-\bar{X}^n_s, Z^n_s)$ has the same law as (X^n_s, Z^n_s) under P, so that (58) follows by (59). □

We are now ready to prove the main result of this subsection.

Proof of Proposition 5.4
We start with the case $s_n = s > 0$. The idea is to prove the following chain of equalities

$$\lim_{n\to\infty} \Sigma^n_{P^n}(s; g) = \lim_{n\to\infty} \frac{\sqrt{n}\, E^P\big[\, g(X^n_s)\exp(c_n X^n_s)\,\exp(d_n Z^n_s)\mathbb{I}(\sigma^n_1 > s)\big]}{\sqrt{n}\, E^P\big[\exp(c_n X^n_s)\exp(d_n Z^n_s)\mathbb{I}(\sigma^n_1 > s)\big]}$$

$$= \frac{\int_0^\infty g(x)\exp(\bar{c}x)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx}{\int_0^\infty \exp(\bar{c}x)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx}$$

$$= \frac{\Pi\big(2\lambda s, 0; g(\cdot)\exp(\frac{c}{2\lambda}\cdot)\big)}{\Pi\big(2\lambda s, 0; \exp(\frac{c}{2\lambda}\cdot)\big)} = \hat{\Pi}_{2\lambda,c}(s; g),$$

where $\bar{c} = \lim_n c_n = c/(2\lambda)$. The first equality is immediately obtained by multiplying the numerator and the denominator of (57) by $\sqrt{n}$. So we need only to prove the second equality since the others are obvious.
Without loss of generality, we can assume that g has continuous bounded derivative. Then, by Lemma 5.7, we need to evaluate the limit of

$$\sqrt{n}E^P\Big[\frac{2}{\sqrt{n}}\mathbb{I}(X^n_s \geq 2/\sqrt{n})g_n'(X^n_s - 2\theta_g/\sqrt{n})\exp(d_n Z^n_s)\Big] \tag{60}$$

$$+\sqrt{n}E^P\Big[\mathbb{I}(0 \leq X^n_s < 2/\sqrt{n})g_n(X^n_s)\,\exp(d_n Z^n_s)\Big], \tag{61}$$

when $g_n(x) = g(x)\exp(c_n x)$, for the numerator, and then the limit of the denominator follows taking $g(x) = 1$. Recalling that (under P) (X^n_t, Z^n_t) converge weakly in $D_{\mathbb{R}^2}([0,\infty))$ to two independent Brownian motions (W_t, B_t), the limit of (60) is

$$\begin{aligned} & 2E\Big[\mathbb{I}(0 < W_s < \infty)f'(W_s)\,\exp\big(\bar{d}B_s\big)\Big] \\ &= 2E\Big[\mathbb{I}(0 < W_s < \infty)f'(W_s)\Big]\,E\Big[\exp\big(\bar{d}B_s\big)\Big], \end{aligned}$$

with $f(x) = g(x)\exp(\bar{c}x)$. By standard computations, using the integration by part formula,

$$E\Big[\mathbb{I}(0 < W_s < \infty)f'(W_s)\Big] = \frac{1}{\sqrt{2\pi}\sqrt{2\lambda s}}\Big(-g(0) + \int_0^\infty g(x)\exp(\tfrac{c}{2\lambda}x)\frac{x}{2\lambda s}\exp\{-\tfrac{1}{2}\tfrac{x^2}{2\lambda s}\}dx\Big).$$

Furthermore, by Lemma 5.8 below, the addend (61) converge to

$$\frac{2}{\sqrt{2\pi}\sqrt{2\lambda s}}\, g(0)\, E\Big[\exp\big(\bar{d}B_s\big)\Big].$$

Then the result is achieved for any constant sequence $s_n = s$. The case of s_n converging to $s > 0$, is achieved in a similar way, by the weak convergence of (X^n_s, Z^n_s) to (W_s, B_s) in the uniform norm on bounded intervals, and again by Lemma 5.8. □

Lemma 5.8. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a bounded continuous function, and let*

$$q_n(s) := E^P\Big[\mathbb{I}(0 \le X^n_s < 2/\sqrt{n})\, g(X^n_s)\, \exp(c_n X^n_s) \exp(d_n Z^n_s)\Big],$$

then

$$\lim_{n\to\infty} \sqrt{n}\, q_n(s_n) = \frac{2}{\sqrt{2\pi}\sqrt{2\lambda s}}\, g(0)\, E^P\Big[\exp\big(\bar{d}B_s\big)\Big], \tag{62}$$

whenever $s_n \to s$, with $s > 0$.

Proof. We start with the symmetric case, i.e. when $\lambda_n = \mu_n = \lambda$, since the proof is technically simpler. Indeed in this case $c_n = d_n = 0$ and therefore (62) is achieved by proving that

$$\lim_{n\to\infty} \sqrt{n}\, P(X^n_{s_n} \in [0, 2/\sqrt{n})) = \frac{2}{\sqrt{2\pi}\sqrt{2\lambda s}}.$$

Without loss of generality we can assume $\lambda = \frac{1}{2}$. Otherwise we can use the deterministic change of time $t/(2\lambda)$ instead of t and consider the sequence of processes $X^n_{t/(2\lambda)}$, which converges to a standard Brownian motion. Let F^n_s denote the distribution function of X^n_s. Then as an easy consequence of Berry-Esseen theorem we get

$$\sup_{x\in\Gamma_n} \big|F^n_s(x) - \Phi\big(\tfrac{x}{\sqrt{s}}\big)\big| = o\left(\tfrac{1}{\sqrt{ns}}\right), \tag{63}$$

where $\Gamma_n = \left\{\frac{1}{\sqrt{n}}\left(z + \frac{1}{2}\right), z \in \mathbb{Z}\right\}$, and $\Phi(x)$ is the distribution function of a standard normal random variable. Clearly $P\big(X^n_{s_n} \in [0, 2/\sqrt{n})\big)$ is equal to $P\big(X^n_{s_n} \in (-\frac{1}{\sqrt{n}}, \frac{3}{2}\frac{1}{\sqrt{n}}]\big)$, and therefore to

$$\Phi\big(\tfrac{3}{2}\tfrac{1}{\sqrt{ns_n}}\big) - \Phi\big(-\tfrac{1}{2}\tfrac{1}{\sqrt{ns_n}}\big) + o\big(\tfrac{1}{\sqrt{ns_n}}\big) \simeq \frac{2}{\sqrt{ns_n}}\Phi'(\gamma_n) + o\big(\tfrac{1}{\sqrt{ns_n}}\big),$$

where $\gamma_n \in \big(-\frac{1}{2}\frac{1}{\sqrt{ns_n}}, \frac{3}{2}\frac{1}{\sqrt{ns_n}}\big)$. Moreover, as $s_n \to s$ and $s > 0$, there exists $\bar{n}$ such that $s_n \ge \frac{1}{2}s$, for any $n > \bar{n}$, and then $o\big(\frac{1}{\sqrt{ns_n}}\big) = o\big(\frac{1}{\sqrt{n}}\big)$ and we obtain the limit (62) in the symmetric case.

We switch now to the general case. First of all we observe that

$$\begin{aligned} q_n(s) = &\sum_{h=0}^{\infty} P(\tilde{A}_{ns} = h) P(\tilde{N}_{ns} = h) g(0) \mathrm{e}^{c_n 0} \mathrm{e}^{d_n \frac{2h - 2n\lambda s}{\sqrt{n}}} \\ &+ \sum_{h=0}^{\infty} P(\tilde{A}_{ns} = h+1) P(\tilde{N}_{ns} = h) g(1/\sqrt{n}) \mathrm{e}^{c_n \frac{1}{\sqrt{n}}} \mathrm{e}^{d_n \frac{2h+1-2n\lambda s}{\sqrt{n}}}, \end{aligned}$$

and that $\sqrt{n}\; q_n(s_n)$ has the same behaviour as $g(0)\,\sqrt{n}\bar{q}_n(s_n)$, where

$$\bar{q}_n(s) := \sum_{h=0}^{\infty} P(\tilde{A}_{ns}=h)P(\tilde{N}_{ns}=h)\mathrm{e}^{d_n \frac{2h-2n\lambda s}{\sqrt{n}}} + \sum_{h=0}^{\infty} P(\tilde{A}_{ns}=h+1)P(\tilde{N}_{ns}=h)\mathrm{e}^{d_n \frac{2h+1-2n\lambda s}{\sqrt{n}}},$$

as can be immediately seen by

$$\min\big(g(0), g(1/\sqrt{n})\big)\sqrt{n}\bar{q}_n(s) \le \sqrt{n}q_n(s) \le \mathrm{e}^{c_n\frac{1}{\sqrt{n}}} \max\big(g(0), g(1/\sqrt{n})\big)\sqrt{n}\bar{q}_n(s).$$

Taking into account that $\tilde{Z}_{ns} = \tilde{A}_{ns} + \tilde{N}_{ns}$ is a Poisson random variable of parameter $2\lambda ns$, one can see that

$$P(\tilde{A}_{ns}=h)P(\tilde{N}_{ns}=h) = P(\tilde{Z}_{ns}=2h)\frac{(2h)!}{h!\;h!}\frac{1}{2^{2h}}$$

and that

$$P(\tilde{A}_{ns}=h+1)P(\tilde{N}_{ns}=h) = P(\tilde{Z}_{ns}=2h+1)\frac{(2h+1)!}{(h+1)!\;h!}\frac{1}{2^{2h+1}}.$$

Then, setting

$$r(k) = \frac{k!}{(k-[k/2])!\;[k/2]!}\frac{1}{2^k},$$

we can rewrite

$$\begin{aligned}\bar{q}_n(s) &= \sum_{k=0}^{\infty} P(\tilde{Z}_{ns}=k)\mathrm{e}^{d_n \frac{k-2n\lambda s}{\sqrt{n}}}\, r(k)\\ &= E^P\Big[r(\tilde{Z}_{ns})\;\exp(d_n\frac{\tilde{Z}_{ns}-2n\lambda s}{\sqrt{n}})\Big] = E^P\Big[r(\tilde{Z}_{ns})\;\exp(d_n Z^n_s)\Big]\end{aligned}$$

Now by Stirling formula $\hat{r}(m) = \sqrt{m}\,r(m)$ converge to $2/\sqrt{2\pi}$ as m increases to infinity, and therefore, we rewrite

$$\sqrt{n}\;\bar{q}_n(s) = \sqrt{n}\;P(\tilde{Z}_{ns}=0) + E^P\left[\mathbb{I}(\tilde{Z}_{ns}>0)\;\hat{r}(\tilde{Z}_{ns})\frac{1}{\sqrt{\tilde{Z}_{ns_n}/n}}\;\exp\big(d_n Z^n_s\big)\right].$$

We are interested in the asymptotic behaviour of $\sqrt{n}\;\bar{q}_n(s_n)$, and first of all we note that the sequence $nP(Z^n_{s_n}=0)$ converges to zero, as can be seen by direct calculations. Then we observe that for each $T>0$, by Kolmogorov inequality

$$P\big(\sup_{s\le T}\big|\tfrac{1}{n}\,\tilde{Z}_{ns} - 2\lambda s\big| \ge \varepsilon\big) \le \frac{Var(\tilde{Z}_{nT})}{n^2\varepsilon^2} = \frac{2n\lambda T}{n^2\varepsilon^2}.$$

Furthermore Z^n_s converge weakly to B_s in $D_R([0,\infty))$, w.r.t. the topology of uniform convergence on bounded intervals, the limit process having continuous paths. Therefore the pair $(\tilde{Z}_{ns}/n, Z^n_s)$ converges in $D_R([0,\infty))\times D_R([0,\infty))$, each component endowed with the topology of uniform convergence, and then $(\tilde{Z}_{ns}/n, Z^n_s)$ converge in distribution to $(2\lambda s, B_s)$, in the space $D_{R^2}([0,\infty))$ endowed with the topology of uniform convergence on bounded intervals.
By Skorohod theorem we can assume w.l.o.g. that the above pair converges $P-$a.s., and uniformly on bounded intervals. Then,

$$\hat{r}(\tilde{Z}_{ns_n})\,\mathbb{I}(\tilde{Z}_{ns_n}>0)\;\frac{1}{\sqrt{\tilde{Z}_{ns_n}/n}}\;\exp\big(d_n Z^n_{s_n}\big)\;\to \frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{2\lambda s}}\exp\big(\bar{d}B_s\big) \qquad P-a.s. \qquad (64)$$

whenever $s_n \to s$, with $s > 0$. The above convergence is equivalent to the uniform convergence on bounded and compact intervals of $(0, \infty)$, and its proof is straightforward. We only observe that, if we set $h_n(s) = \sqrt{\tilde{Z}_{ns}/n}$, then, for any $T > 0$, $h_n(s)$ converge to $h(s) = \sqrt{2\lambda s}$ uniformly in $[0, T]$, and therefore, for any $\underline{h} > 0$, $1/h_n(s)$ converge to $1/h(s)$ uniformly in the set $\{s, \text{such that } h(s) \geq \underline{h}\}$. The thesis follows, as the sequence at the l.h.s. of (64) is uniformly integrable. Indeed

$$\sup_n E^P \left[\left(\sqrt{n}\; r(\tilde{Z}_{ns_n}) \; \exp\left(d_n Z^n_{s_n}\right) \right)^2 \right] \leq L < \infty,$$

since $\sup_m \hat{r}(m) = \sqrt{m}\, r(m) \leq L' < \infty$, and

$$\begin{aligned}
&E^P \left[\mathbb{I}(\tilde{Z}_{ns_n} > 0) \; \frac{n}{\tilde{Z}_{ns_n}} \; \exp\left(2 \; d_n Z^n_{s_n}\right) \right] \\
&= \exp(-2d_n/\sqrt{n}) \frac{1}{2\lambda s_n} \sum_{k=1}^{\infty} \frac{k+1}{k} \exp\left(2d_n \frac{k+1-2\lambda ns}{\sqrt{n}}\right) \frac{(2\lambda n s_n)^{k+1}}{(k+1)!} \mathrm{e}^{-2\lambda n s_n} \\
&\leq \exp(-2d_n/\sqrt{n}) \frac{1}{\lambda s_n} E^P[\exp(2d_n Z^n_{s_n})]
\end{aligned}$$

□

We end this section with the statement of the elementary technical lemma, which has been used in the proof of the previous results.

Lemma 5.9. *If condition* **C3** *holds, then*

$$\lim_{n\to\infty} (c_n, d_n, e_n) = (\bar{c}, \bar{d}, \bar{e}), \tag{65}$$

where c_n, d_n and e_n are defined in (51),(52) and (53), and where

$$\bar{c} = \frac{c}{2\lambda} = \frac{c(1) - c(2)}{2\lambda}, \quad \bar{d} = \frac{d}{2\lambda} = \frac{c(1) + c(2)}{2\lambda}, \quad \bar{e} = -(c^2(1) + c^2(2))/2\lambda.$$

6 The M/M/1 queueing model: observing the idle time process

In this section we are interested in the conditional law of the M/M/1 queue $\tilde{Q}^n_s$, when the observation process is the *idle time* process, i.e.

$$\tilde{C}^n_t = \int_0^t \mathbb{I}(\tilde{Q}^n_s = 0)\, ds,$$

the cumulative time the queue has spent in 0, up to t.

Equivalently one can consider as observation process the bivariate point process $(\tilde{I}^n_t, \tilde{B}^n_t)$, where $\tilde{I}^n_t$ is the process that counts the times when the system starts an idle period and $\tilde{B}^n_t$ is the process that counts the times when the system starts a busy period, that is

$$\tilde{I}^n_t = \int_0^t \mathbb{I}(\tilde{Q}^n_{s-} = 1)\, d\tilde{N}^n_s \tag{66}$$

$$\tilde{B}^n_t = \int_0^t \mathbb{I}(\tilde{Q}^n_{s-} = 0)\, d\tilde{A}^n_s. \tag{67}$$

Indeed the filtration generated by the idle time process $\tilde{C}^n_t$ and the filtration generated by the observation process $(\tilde{I}^n_t, \tilde{B}^n_t)$ coincide, or more precisely $\mathcal{F}^{\tilde{C}^n}_{t^+} = \mathcal{F}^{\tilde{I}^n,\tilde{B}^n}_t$.

Our first aim is to study the conditional law

$$E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big], \tag{68}$$

where for the notational convenience we denote

$$\tilde{\mathcal{H}}^n_t = \mathcal{F}^{\tilde{I}^n,\tilde{B}^n}_t = \mathcal{F}^{\tilde{C}^n}_{t^+},$$

and the explicit expression for the filter (68) in terms of $\gamma^0_t(\tilde{Q}^n)$, is given in (75).
Then we consider the rescaled processes

$$Q^n_t := \frac{\tilde{Q}^n_{nt}}{\sqrt{n}}, \qquad I^n_t := \frac{\tilde{I}^n_{nt}}{\sqrt{n}}, \qquad B^n_t := \frac{\tilde{B}^n_{nt}}{\sqrt{n}}, \qquad C^n_t := \sqrt{n}\mu_n \tilde{C}^n_{nt}, \tag{69}$$

and the conditional law of the rescaled queue

$$E\big[g(Q^n_t)/\mathcal{H}^n_t\big], \tag{70}$$

where $\mathcal{H}^n_t$ is the filtration generated by the rescaled observation

$$\mathcal{H}^n_t = \tilde{\mathcal{H}}^n_{nt} = \mathcal{F}^{I^n,B^n}_t = \mathcal{F}^{C^n}_{t^+}. \tag{71}$$

We are interested in the limit behaviour of the filter (70) under the same assumption **A1**, **A2**, **A3** and conditions **C1**, **C2**, **C3** of Section 5. Under these assumptions we already know that Q^n_t converge weakly to a Brownian motion W_t with diffusion coefficient 2λ and drift coefficient c.
If one defines $\bar{X}^n_t := Q^n_t - C^n_t$, then clearly $Q^n_t = \bar{X}^n_t + C^n_t$, and therefore, since by definition C^n_t increases only when $Q^n_t = 0$, the pair (Q^n_t, C^n_t) is the solution of the Skorohod problem corresponding to $\bar{X}^n_t$. Moreover

$$\big(\bar{X}^n_t, Q^n_t, C^n_t\big) \Rightarrow (W_t, W_t + \Lambda_t, \Lambda_t)\,,$$

where, as usual, Λ_t is defined as in (1) (for a deeper investigation of these results, we refer to Kurtz [7]). It is therefore natural to expect that $E\big[g(Q^n_t)/\mathcal{H}^n_t\big]$ converges weakly to $E\big[g(W_t + \Lambda_t)/\mathcal{F}^{\Lambda}_t\big] = \hat{\Pi}_{2\lambda,c}(\zeta_t; g)$. This result is proven in Theorem 6.4. Moreover $\hat{\Pi}_{2\lambda,c}(\gamma^0_t(Q^n); g)$ is a good approximation of the filter for the rescaled model (see Theorem 6.6).

Let $\{\sigma^{Bn}_k,\ k \in \mathbb{N}\}$ and $\{\sigma^{In}_k,\ k \in \mathbb{N}\}$ be the jump times of the process $\tilde{I}^n_t$ and the process $\tilde{B}^n_t$, respectively. Under the assumption $\tilde{Q}^n_0 = 0$, it easy to verify that

$$\sigma^{Bn}_k < \sigma^{In}_k < \sigma^{Bn}_{k+1} < \sigma^{In}_{k+1}, \qquad \text{for each } k \geq 1,$$

and that $Q^n_t = 0$ when $\sigma^{In}_k \leq t < \sigma^{Bn}_{k+1}$, for $k \geq 0$, while $Q^n_t > 0$ otherwise.
We start by observing some regenerative properties of the above jump times, which are fundamental in the sequel. The first one is due to the strong Markov property for the process $\tilde{Q}^n_t$, and is given the following lemma.

Lemma 6.1. *For each $k \in \mathbb{N}$ the processes $\tilde{Q}^{In}_{k,t} = \tilde{Q}^n_{t+\sigma^{In}_k} - \tilde{Q}^n_{\sigma^{In}_k}$ and $\tilde{Q}^{Bn}_{k,t} = \tilde{Q}^n_{t+\sigma^{Bn}_k} - \tilde{Q}^n_{\sigma^{Bn}_k}$ are independent of $\mathcal{F}^{\tilde{Q}^n}_{\sigma^{In}_k}$ and $\mathcal{F}^{\tilde{Q}^n}_{\sigma^{Bn}_k}$ respectively. Moreover, the process $\tilde{Q}^{In}_{k,t}$ has the same law as the process $\tilde{Q}^n_t$.*

The process $\tilde{I}_t^n$ is a renewal process, and $\tilde{B}_t^n$ is a delayed renewal process, i.e. the random variables $\sigma_{k+1}^{Bn} - \sigma_k^{Bn}$ are mutually independent for $k \geq 0$ and identically distributed for $k \geq 1$. Also $\{\sigma_k^{In} - \sigma_k^{Bn}\}_{k\geq 1}$ is a sequence of mutually independent random variables.
In the setting of this section, the above considerations and Lemma 6.1 guarantee that the filter of $\tilde{Q}_t^n$ given $\tilde{\mathcal{H}}_t^n$ admits a representation similar to that given in Proposition 2.1. The proof of the following proposition is left to the reader, however we point out that since the processes involved are all Markovian, the proof could be given by the techniques used in [3].

Proposition 6.2. *The conditional law of $\tilde{Q}_t^n$ given $\tilde{\mathcal{H}}_t^n$ admits the following representation*

$$E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big] = \mathbb{I}(\tilde{Q}_t^n = 0)g(0) + \tag{72}$$
$$+\mathbb{I}(\tilde{Q}_t^n > 0)\sum_{j=1}^{\infty} \frac{E\Big[g(\tilde{Q}^n_{s+\sigma_j^{Bn}} - \tilde{Q}^n_{\sigma_j^{Bn}} + 1)\,\mathbb{I}(\sigma_j^{In} - \sigma_j^{Bn} > s)\Big]_{s=t-\sigma_j^{Bn}}}{E\Big[\mathbb{I}(\sigma_j^{In} - \sigma_j^{Bn} > s)\Big]_{s=t-\sigma_j^{Bn}}}\mathbb{I}\{\sigma_j^{Bn} \leq t < \sigma_j^{In}\}.$$

It is important to note that

$$\sigma_j^{In} - \sigma_j^{Bn} = \inf\Big\{u \geq 0: \ \tilde{Q}_{j,u}^{Bn} + 1 = 0\Big\}, \tag{73}$$

and that the process $\tilde{Q}_{j,s}^{Bn} + 1$ for $s < \sigma_j^{In} - \sigma_j^{Bn}$ behaves like the continuous time random walk $\tilde{X}_s^n + 1$ for $s < \tilde{\sigma}_1^n = \inf\Big\{u \geq 0: \ \tilde{X}_u^n = -1\Big\}$, and hence

$$\frac{E\Big[g(\tilde{Q}_{j,s}^{Bn} + 1)\,\mathbb{I}(\sigma_j^{In} - \sigma_j^{Bn} > s)\Big]}{E\Big[\mathbb{I}(\sigma_j^{In} - \sigma_j^{Bn} > s)\Big]} = \frac{E\Big[g(\tilde{X}_s^n + 1)\,\mathbb{I}(\tilde{\sigma}_1^n > s)\Big]}{E\Big[\mathbb{I}(\tilde{\sigma}_1^n > s)\Big]}. \tag{74}$$

As a consequence and observing that, by definition (9),

$$\gamma_t^0(\tilde{Q}^n) = t - \sup\{s < t \text{ such that } \tilde{Q}_s^n = 0\} = \sum_{j=1}^{\infty}(t - \sigma_j^{Bn})\,\mathbb{I}\{\sigma_j^{Bn} \leq t < \sigma_j^{In}\}$$

we can rewrite (72) as

$$E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big] = \mathbb{I}(\tilde{Q}_t^n = 0)g(0) + \mathbb{I}(\tilde{Q}_t^n > 0)\left.\frac{E\Big[g(\tilde{X}_s^n + 1)\,\mathbb{I}(\tilde{\sigma}_1^n > s)\Big]}{E\Big[\mathbb{I}(\tilde{\sigma}_1^n > s)\Big]}\right|_{s=\gamma_t^0(\tilde{Q}^n)}. \tag{75}$$

The above considerations leads us to state the following result

Theorem 6.3. *Consider the rescaled process Q_t^n, the rescaled observation processes I_t^n and B_t^n, defined in* (69)*, and the history generated by (I_u^n, B_u^n) for $u \leq t$, i.e. $\mathcal{H}_t^n$ defined in* (71)*. Then*

$$E\big[g(Q_t^n)/\mathcal{H}_t^n\big] = \mathbb{I}\,(Q_t^n = 0)\,g(0) + \mathbb{I}\,(Q_t^n > 0)\,\bar{\Sigma}^n(\gamma_t^0(Q^n); g), \tag{76}$$

where $\bar{\Sigma}^n(s; g) = \hat{\Sigma}^n(s; \overline{g}_n)$, with $\hat{\Sigma}^n(s)$ the probability defined in (23), and $\overline{g}_n(x) = g(x + \frac{1}{\sqrt{n}})$.

Proof. Equality (75) implies

$$E\big[g(Q_t^n)/\mathcal{H}_t^n\big] = \mathbb{I}\,(Q_t^n = 0)\, g(0) + \mathbb{I}\,(Q_t^n > 0)\; \frac{E\Big[g\left(X_s^n + \frac{1}{\sqrt{n}}\right) \mathbb{I}\,(\sigma_1^n > s)\Big]}{E\Big[\mathbb{I}\,(\sigma_1^n > s)\Big]}\Bigg|_{s=\gamma_t^0(Q^n)}$$

and clearly

$$\frac{E\Big[g\left(X_s^n + \frac{1}{\sqrt{n}}\right) \mathbb{I}\,(\sigma_1^n > s)\Big]}{E\Big[\mathbb{I}\,(\sigma_1^n > s)\Big]} = \hat{\Sigma}^n(s; \overline{g}_n).$$

□

As a consequence of the above Theorem, the conditional law of Q_t^n given $\mathcal{H}_t^n$ can be written as

$$\mathbb{I}\,(Q_t^n = 0)\left[\delta_{\{0\}} - \bar{\Sigma}^n(\gamma_t^0(Q^n))\right] + \bar{\Sigma}^n(\gamma_t^0(Q^n)).$$

Moreover, for any g uniformly continuous

$$\bar{\Sigma}^n(\gamma_t^0(Q^n); g) = \hat{\Sigma}^n(\gamma_t^0(Q^n); g) + \varepsilon(n, g) \tag{77}$$

with $|\varepsilon(n, g)| \leq \omega_g(1/\sqrt{n})$. We are now ready to prove the main result of this section.

Theorem 6.4. *Assume conditions* **C1**, **C2**, **C3** *and* **A1**, **A2**, **A3**. *Then, for any $t \geq 0$, the sequence of measure-valued random variables defined by (70) converge weakly to $\hat{\pi}_t$, on the space of probability measures endowed with the topology of weak convergence. In particular, for any bounded and continuous function g*

$$E\big[g(Q_t^n)/\mathcal{H}_t^n\big] \Rightarrow E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] = \hat{\Pi}_{2\lambda,c}(\zeta_t; g), \qquad \textit{for any } t \geq 0.$$

Proof. As in the proof of Theorem 5.2, using (77), it is possible to show that in the Skorohod space of Proposition 5.5

$$\bar{P}\big(\bar{\Sigma}^n(\gamma_t^0(Q^n); g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(\zeta_t; g), \quad \text{for every } g: \mathbb{R}^+ \to \mathbb{R} \text{ bounded and continuous}\big) = 1,$$

since in that space $\gamma_t^0(Q^n)$ converges to ζ_t almost surely. On the other hand, the total variation of the measure $\delta_{\{0\}} - \bar{\Sigma}^n(\gamma_t^0(Q^n))$ is at most 2, so that the result is achieved once we prove that $\mathbb{I}\,(Q_t^n = 0)$ converges to zero in probability. Indeed, as recalled in (40), the sequence Q_t^n converges weakly to a reflected Brownian motion $W_t + \Lambda_t$. Then, the above convergence can be obtained by noting that the function $\mathbb{I}\,(x = 0)$ has a discontinuity point at $x = 0$, $P(W_t + \Lambda_t = 0) = 0$, and that $\mathbb{I}\,(Q_s^n = 0)$ converges to zero, by the continuous mapping theorem. □

Remark 6.5. *As already observed at the beginning of this section,*

$$\left(\bar{X}_t^n, Q_t^n, C_t^n\right) \Rightarrow (W_t, W_t + \Lambda_t, \Lambda_t)$$

where $\bar{X}_t^n := Q_t^n - C_t^n$. Thanks to the continuity of the limit processes, the convergence can be considered in the space $D_{\mathbb{R}^3[0,\infty)}$ endowed with the topology of the uniform convergence on compact sets. Moreover it is interesting to note that $\gamma_t^0(Q^n) = \gamma_t(C^n) \Rightarrow \gamma_t(\Lambda) = \zeta_t$, Then, similarly to Proposition 5.5, it is possible to prove that

$$\left(\gamma_t^0(Q^n), \gamma_t(C^n), C_t^n\right) \Rightarrow \left(\gamma_t^0(W_t + \Lambda_t), \gamma_t^0(\Lambda_t), \Lambda_t\right) = (\zeta_t, \zeta_t, \Lambda_t)\,,$$

and therefore an alternative proof of the previous Theorem can be achieved, by using these properties.

We end this section by noting that, even in this new situation, it is possible to give the same approximation for the filter as in Theorem 5.3, namely for $E\big[g(Q^n_t)/\mathcal{H}^n_t\big]$ the following result holds.

Theorem 6.6. *For all g bounded and continuous and for each $T > 0$, $p > 0$*

$$\int_0^T E\Big|E\big[g(Q^n_t)/\mathcal{H}^n_t\big] - \hat{\Pi}_{2\lambda,c}(\gamma^0_t(Q^n); g)\Big|^p dt \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. Note that, by (77),

$$\begin{aligned}
&\Big|E\big[g(Q^n_t)/\mathcal{H}^n_t\big] - \hat{\Pi}_{2\lambda,c}(\gamma^0_t(Q^n); g)\Big|^p = \\
&= \Big|\mathbb{I}\,(Q^n_t = 0)\, g(0) + \mathbb{I}\,(Q^n_t > 0)\, \hat{\Sigma}^n(\gamma^0_t(Q^n); g) + \varepsilon(n, g) - \hat{\Pi}_{2\lambda,c}(\gamma^0_t(Q^n); g)\Big|^p \\
&\quad \le C(p)\Big|\mathbb{I}\,(Q^n_t = 0)\,\Big(g(0) + \hat{\Sigma}^n(\gamma^0_t(Q^n); g)\Big) + \varepsilon(n, g)\Big|^p \\
&\quad + C(p)\Big|\hat{\Sigma}^n(\gamma^0_t(Q^n); g) - \hat{\Pi}_{2\lambda,c}(\gamma^0_t(Q^n); g)\Big|^p,
\end{aligned}$$

where $C(p)$ is a suitable constant. Then

$$\begin{aligned}
&E\left[\Big|E\big[g(Q^n_t)/\mathcal{H}^n_t\big] - \hat{\Pi}_{2\lambda,c}(\gamma^0_t(Q^n); g)\Big|^p\right] \le \\
&\le C(p)E\left[\Big|\mathbb{I}\,(Q^n_t = 0)\, 2 \parallel g \parallel_\infty + \varepsilon(n, g)\Big|^p\right] + C(p)E\left[\Big|\hat{\Sigma}^n(\gamma^0_t(Q^n); g) - \hat{\Pi}_{2\lambda,c}(\gamma^0_t(Q^n); g)\Big|^p\right].
\end{aligned}$$

The thesis follows since both the addends at the right hand side of the previous inequality converge to zero. The first addend converges to zero by the bounded convergence theorem. To prove that the the second addend converges to zero one has just to substitute ξ^n_t with $\gamma^0_t(Q^n)$ in the proof of Theorem 5.3. □

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