Irreducible symplectic 4-folds numerically equivalent to $(K3)^{[2]}$

Kieran G. O'Grady*
Università di Roma "La Sapienza"

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Abstract

First steps towards a classification of irreducible symplectic 4-folds whose integral 2-cohomology with 4-tuple cup product is isomorphic to that of $(K3)^{[2]}$. We prove that any such 4-fold deforms to an irreducible symplectic 4-fold of Type A or Type B. A 4-fold of Type A is a double cover of a (singular) sextic hypersurface and a 4-fold of Type B is birational to a hypersurface of degree at most 12. We conjecture that Type B 4-folds do not exist.

1 Introduction

Kodaira [16] proved that any two $K3$ surfaces are deformation equivalent. A $K3$ surface is the same as an irreducible symplectic 2-fold - recall that a compact Kähler manifold is irreducible symplectic if it is simply connected and it carries a holomorphic symplectic form spanning $H^{2,0}$ (see [1, 13]). A classification of higher-dimensional irreducible symplectic manifolds up to deformation equivalence appears to be out of reach at the moment (see [1, 13]). We will take the first steps towards a solution of the classification problem for numerical $(K3)^{[2]}$'s. We explain our terminology: two irreducible symplectic manifolds M_1, M_2 of dimension $2n$ are *numerically equivalent* if there exists an isomorphism of abelian groups $\psi\colon H^2(M_1;\mathbb{Z}) \xrightarrow{\sim} H^2(M_2;\mathbb{Z})$ such that $\int_{M_1}\alpha^{2n} = \int_{M_2}\psi(\alpha)^{2n}$ for all $\alpha \in H^2(M_1;\mathbb{Z})$. Recall [1] that if S is a $K3$ then $S^{[n]}$ - the Douady space parametrizing length-n analytic subsets of S - is an irreducible symplectic manifold of dimension $2n$. A *numerical* $(K3)^{[2]}$ is an irreducible symplectic 4-fold numerically equivalent to $S^{[2]}$ where S is a $K3$.

Theorem 1.1. *Let M be a numerical $(K3)^{[2]}$. Then M is deformation equivalent to one of the following:*

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(1) An irreducible symplectic 4*-fold* X *carrying an anti-symplectic involution* $\phi\colon X \to X$ *such that the quotient* $X/\langle\phi\rangle$ *is isomorphic to a sextic hypersurface* $Y \subset \mathbb{P}^5$*. Let* $f\colon X \to Y$ *be the quotient map and* $H := f^*\mathcal{O}_Y(1)$*; the fixed locus of* ϕ *is a smooth irreducible Lagrangian surface* F *such that*

$$c_2(F) = 192, \quad \mathcal{O}_F(2K_F) \cong \mathcal{O}_F(6H), \quad c_1(F)^2 = 360. \tag{1.0.1}$$

(2) An irreducible symplectic 4*-fold* X *admitting a rational map* $f\colon X \dashrightarrow \mathbb{P}^5$ *which is birational onto its image* Y*, with* $6 \le \deg Y \le 12$*.*

We give a brief outline of the proof of the theorem. By applying surjectivity of the period map and Huybrechts' projectivity criterion [13, 14] we will be able to deform M to an irreducible symplectic 4-fold X such that Items (1) through (6) of Proposition (3.2) hold. The first item gives (via Hirzebruch-Riemann-Roch and Kodaira Vanishing) that there is an ample divisor H on X such that

$$\int_X c_1(H)^4 = 12, \quad h^0(\mathcal{O}_X(H)) = 6. \tag{1.0.2}$$

Let $h := c_1(H)$; Items (2), (3) and (4) state that h generates $H^{1,1}_{\mathbb{Z}}(X)$ and that $H^4(X)$ has no rational Hodge substructures other than those forced by the existence of h and the Beauville-Bogomolov bilinear symmetric form. Items (5)-(6) imply, via Proposition (4.1), the following *Irreducibility property of* $|H|$: if $D_1, D_2 \in |H|$ are distinct then $D_1 \cap D_2$ is a reduced and irreducible surface in X. Next we will study the rational map $f\colon X \dashrightarrow |H|^\vee \cong \mathbb{P}^5$. Let $U \subset X$ be the open set where f is regular and $Y := \overline{f(U)} \subset \mathbb{P}^5$. A straightforward argument based on ampleness of H and the Irreducibility property of $|H|$ will show that one of the following holds: there exists an involution ϕ on X such that Item (1) of Theorem (1.1) holds and f equals the quotient map $X \to X/\langle\phi\rangle$ followed by the inclusion $X/\langle\phi\rangle \hookrightarrow \mathbb{P}^5$, or f is as in Item (2) of Theorem (1.1), or else Y is one of the following:

(a) a 3-fold of degree at most 6,

(b) a 4-fold of degree at most 4.

Thus in order to complete the proof of Theorem (1.1) we will need to show that (a) nor (b) can hold. We will argue by contradiction: assuming that (a) or (b) holds we will get that either $H^4(X)$ has a non-existent Hodge substructure or the Irreducibilty property of $|H|$ does not hold (the case of Y a normal quartic 4-fold is exceptional, it will require an ad hoc argument). Thus we will need to analyze 3-folds and 4-folds in $\mathbb{P}^5$ of low degree. In particular we will prove some results on cubic 4-folds $Y \subset \mathbb{P}^5$ which might be of independent interest. First we will show that if $\dim(sing Y) \ge 1$ then Y contains a plane. Secondly we will prove that if Y is singular with isolated singularities and it does not contain planes then $Gr^W_4 H^4(Y)$ contains a Hodge substructure isomorphic to the transcendental part of the H^2 of a $K3$ surface (shifted by $(1,1)$), namely the minimal desingularization of the set of lines in Y through any of its singular points. This result should be equivalent to a statement about degenerations of the variety $F(Y)$ parametrizing lines on a cubic 4-fold $Y \subset \mathbb{P}^5$ - recall that if Y is smooth then $F(Y)$ is a deformation of $(K3)^{[2]}$ (see [2]) and if Y is singular then $F(Y)$ is singular [11]. The relevant statement should be the following: Let

$\mathcal{U}$ be the parameter space for cubic 4-folds $Y \subset \mathbb{P}^5$ not containing a plane; then there exists a finite cover $\widetilde{\mathcal{U}} \to \mathcal{U}$ such that the pull-back to $\widetilde{\mathcal{U}}$ of the family over $\mathcal{U}$ with fiber $F(Y)$ at $[Y]$ has a simultaneous resolution of singularities. Going back to the proof of Theorem (1.1) we notice that in its barest outline it resembles the proof given in [25] of Kodaira's theorem on deformation equivalence of $K3$ surfaces. In [21] we described explicitly the X satisfying Item (1) of Theorem (1.1): they are double covers of certain special singular sextic hypersurfaces in $\mathbb{P}^5$ that where constructed by Eisenbud-Popescu-Walter [7]. We also proved that the X satisfying Item (1) of Theorem (1.1) belong to a single locally complete irreducible family of projective irreducible symplectic varieties.

Conjecture 1.2. *Suppose that X is a numerical $(K3)^{[2]}$ and that Items (1) through (6) of Proposition (3.2) hold. Then Item (1) of Theorem (1.1) holds.*

Assume that Conjecture (1.2) is true: then any numerical $(K3)^{[2]}$ is deformation equivalent to an X satisfying Item (1) of Theorem (1.1) and by the results of [21] it follows that X is a deformation of $(K3)^{[2]}$.

Notation: If X is a topological space then $H^*(X)$ denotes cohomology with complex coefficients.

Topology of algebraic varieties (or analytic spaces) will be either the classical topology or the Zariski topology: in general it will be clear from the context in which topology we are working.

Let X be a smooth projective variety. If W is a closed subscheme of X of pure dimension d we let

$$[W] \in Z_d(X) \tag{1.0.3}$$

be the fundamental cycle associated to W as in [9], p. 15.

Let $\mathbb{P}(V)$ be a projective space. If $A \subset \mathbb{P}(V)$ we let $span(A) \subset \mathbb{P}(V)$ be the span of A, i.e. the intersection of all linear subspaces containing A. If $A, B \subset \mathbb{P}(V)$ we let

$$J(A,B) := \bigcup_{p \in A, q \in B} span(p,q). \tag{1.0.4}$$

If A, B are closed and $A \cap B = \emptyset$ then $J(A,B)$ is closed - in general $J(A,B)$ is not closed.

Let X be a scheme and $x \in X$ a (closed) point; we let $\Theta_x X$ be the Zariski tangent space to X at x. Now assume that X is a subscheme of a projective space $\mathbb{P}(V)$. Then $\Theta_x X \subset \Theta_x \mathbb{P}(V)$: the *projective tangent space to X at x* is the unique linear subspace

$$T_x X \subset \mathbb{P}(V) \tag{1.0.5}$$

containing x whose Zariski tangent space at x is equal to $\Theta_x X$.

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2 Preliminaries

2.1 The Beauville-Bogomolov form and Fujiki's constant

Let M be an irreducible symplectic manifold of (complex) dimension $2n$. By Beauville and Fujiki (see [1] and Thm. (4.7) of [8]) there exist a rational positive number c_M and an integral indivisible symmetric bilinear form $(,)_M$ on $H^2(M)$ characterized by the following two properties. First if $\omega \in H^{1,1}(M;\mathbb{R})$ is a Kähler class then the restriction of $(,)_M$ to the span of ω and $\{\sigma+\overline{\sigma}\}_{\sigma\in H^{2,0}(M)}$ is positive definite. Secondly we have the equality

$$\int_M \alpha^{2n} = c_M(\alpha,\alpha)_M^n, \qquad \alpha \in H^2(M). \tag{2.1.1}$$

The above two properties of $(,)_M$ imply that

$$H^{p,q}(X) \bot H^{p',q'}(X), \quad \text{if } p+p' \neq 2, \tag{2.1.2}$$

and that the signature of $(,)_M$ is $(3, b_2(M)-3)$, see [1]. In particular $(,)_M$ is non-degenerate and hence $H^2(M;\mathbb{Z})$ has a canonical structure of lattice. Let M_1, M_2 be irreducible symplectic manifolds of dimension $2n$. If the lattices $H^2(M_1;\mathbb{Z})$ and $H^2(M_2;\mathbb{Z})$ are isometric and the Fujiki constants of M_1, M_2 are equal then M_1, M_2 are numerically equivalent by (2.1.1). The converse is true unless n is even and $b_2(M_1) = b_2(M_2) = 6$: in this case numerical equivalence implies that $c_{M_1} = c_{M_2}$ and that $H^2(M_1;\mathbb{Z})$ is isometric either to $H^2(M_2;\mathbb{Z})$ or to the lattice $H^2(M_2;\mathbb{Z})(-1)$ i.e. $H^2(M_2;\mathbb{Z})$ equipped with the symmetric bilinear form defined by $(\alpha,\beta) := -(\alpha,\beta)_{M_2}$. Let Λ be the lattice given by

$$\Lambda := U^{\oplus 3} \widehat{\oplus} (-E_8)^{\oplus 2} \widehat{\oplus} (-2), \tag{2.1.3}$$

where U is the standard hyperbolic plane. Let S be a $K3$ surface; the Beauville-Bogomolov form and Fujiki constant of $S^{[2]}$ are given (see [1]) by

$$H^2(S^{[2]};\mathbb{Z}) \cong \Lambda, \qquad c_{S^{[2]}} = 3. \tag{2.1.4}$$

In particular $b_2(S^{[2]}) = 23 \neq 6$ and hence an irreducible symplectic 4-fold M is a numerical $(K3)^{[2]}$ if and only if

$$H^2(M;\mathbb{Z}) \cong \Lambda, \tag{2.1.5}$$

$$c_M = 3. \tag{2.1.6}$$

Let M be a numerical $(K3)^{[2]}$; since $b_2(M) = 23$ we have [10, 23]

$$H^3(M;\mathbb{Q}) = 0, \tag{2.1.7}$$

$$Sym^2 H^2(M;\mathbb{Q}) \xrightarrow{\sim} H^4(M;\mathbb{Q}), \tag{2.1.8}$$

where the second isomorphism is given by cup-product. In particular the Hodge numbers of M are uniquely determined.

2.2 Quadratic forms on V and $Sym^2 V$

Let V be a finite-dimensional complex vector space. Let $(V\otimes V)^+, (V\otimes V)^- \subset V\otimes V$ be the subspaces of tensors which are invariant, respectively anti-invariant, for the involution of $V\otimes V$ interchanging the factors. We let $Sym_2 V := (V\otimes V)^+$ and $Sym^2 V := V\otimes V/(V\otimes V)^-$. Assume that $(,)$ is a symmetric bilinear form on V; we let $\langle,\rangle$ be the unique symmetric bilinear form on $Sym^2 V$ such that

$$\langle\alpha_1\alpha_2,\alpha_3\alpha_4\rangle=(\alpha_1,\alpha_2)(\alpha_3,\alpha_4)+(\alpha_1,\alpha_3)(\alpha_2,\alpha_4)+(\alpha_1,\alpha_4)(\alpha_2,\alpha_3) \quad (2.2.1)$$

for $\alpha_1,\ldots,\alpha_4\in V$. Using (2.1.1) and (2.1.6) one gets the following result.

Remark 2.1. *Let M be a numerical $(K3)^{[2]}$. The intersection form on*

$$Sym^2 H^2(M)\cong H^4(M) \quad (2.2.2)$$

is the bilinear form given by (2.2.1) for $V := H^2(M)$ and $(,) := (,)_M$.

Now assume that the symmetric bilinear form $(,)$ on V is non-degenerate. Let $q\in Sym_2 V^\vee$ be the symmetric tensor associated to $(,)$; there is a dual $q^\vee\in Sym^2 V$ defined as follows. Since q is non-degenerate it defines an isomorphism $L_q\colon V^\vee\xrightarrow{\sim} V$ and hence we get $Sym_2(L_q)(q)\in Sym_2 V$. Let $\Pi_2\colon Sym_2 V\to Sym^2 V$ be the composition of the inclusion $Sym_2 V\hookrightarrow V\otimes V$ and the projection map $V\otimes V\to Sym^2 V$; we let

$$q^\vee := \Pi_2\circ Sym_2(L_q)(q)\in Sym^2 V. \quad (2.2.3)$$

Explicitly: let $r:=\dim V$ and let $\{e_1,\ldots,e_r\}$ be a basis of V. Let $\{e_1^\vee,\ldots,e_r^\vee\}$ be the dual basis of $V^\vee$. Then

$$q=\sum_{ij} g_{ij} e_i^\vee\otimes e_j^\vee \quad (2.2.4)$$

where (g_{ij}) is a symmetric matrix and

$$q^\vee=\sum_{ij} m_{ij}e_ie_j, \qquad (m_{ij})=(g_{ij})^{-1}. \quad (2.2.5)$$

Proposition 2.2. *Suppose that V is a complex vector space of dimension r equipped with a non-degenerate bilinear symmetric form $(,)$. Let $\langle,\rangle$ be the bilinear symmetric form on $Sym^2 V$ defined by (2.2.1). Then $\langle,\rangle$ is non-degenerate and furthermore*

$$\langle q^\vee,\alpha\beta\rangle=(r+2)(\alpha,\beta), \quad \alpha,\beta\in V \quad (2.2.6)$$
$$\langle q^\vee,q^\vee\rangle=r(r+2). \quad (2.2.7)$$

Proof. Since $(,)$ is a symmetric bilinear non-degenerate form on a complex r-dimensional vector space there exists a basis $\{e_1,\ldots,e_r\}$ of V such that $q=\sum_{i=1}^r e_i^\vee\otimes e_i^\vee$. A straightforward computation gives that the discriminant of $\langle,\rangle$ with respect to the basis $\{e_1^2,\ldots,e_r^2,e_1e_2,\ldots,e_{r-1}e_r\}$ is equal to $(r+2)2^{r-1}$; thus $\langle,\rangle$ is non-degenerate. Equations (2.2.6)-(2.2.7) are obtained by a straightforward computation. □

3 The deformation

Let M be a numerical $(K3)^{[2]}$. We will show that M can be deformed to a projective irreducible symplectic 4-fold X such that $H^*(X)$ has few integral Hodge substructure. First we introduce the tautological rational Hodge substructures of $H^*(X)$ for X a numerical $(K3)^{[2]}$ with an $h \in H^{1,1}_{\mathbb{Q}}(X)$ such that

$$\int_X h^4 \neq 0. \tag{3.0.1}$$

The above inequality is equivalent to $(h,h)_X \neq 0$ by (2.1.1). We have an orthogonal direct sum decomposition

$$H^2(X) = \mathbb{C}h \widehat{\oplus} h^\perp \tag{3.0.2}$$

(orthogonality is with respect to $(,)_X$) into Hodge substructures of levels 0 and 2 respectively. We will systematically identify $H^4(X)$ with $Sym^2 H^2(X)$ because of (2.1.8); thus (3.0.2) gives a direct sum decomposition

$$H^4(X) = \mathbb{C}h^2 \oplus \left(\mathbb{C}h \otimes h^\perp\right) \oplus Sym^2(h^\perp) \tag{3.0.3}$$

into Hodge substructures of levels 0, 2 and 4 respectively. There is a refinement of Decomposition (3.0.3) obtained as follows. Let $q_X \in Sym_2 H^2(X)^\vee$ be the symmetric tensor associated to the Beauville-Bogomolov form; let

$$q_X^\vee \in Sym^2 H^2(X) \cong H^4(X) \tag{3.0.4}$$

be the dual of q_X - see Subsection (2.2). By (2.2.6)-(2.2.7) we have

$$\langle q_X^\vee, \alpha\beta\rangle = 23(\alpha,\beta)_X, \quad \alpha,\beta \in H^2(X) \tag{3.0.5}$$

$$\langle q_X^\vee, q_X^\vee\rangle = 575. \tag{3.0.6}$$

Equation (2.1.2) gives that $q_X^\vee \in H^{2,2}(X)$ and since q_X is integral we have $q_X^\vee \in H^4_{\mathbb{Q}}(X)$; thus $q_X^\vee \in H^{2,2}_{\mathbb{Q}}(X)$. In terms of Decomposition (3.0.3) we have $q_X^\vee \in \mathbb{C}h^2 \oplus Sym^2(h^\perp)$. More precisely let $q_h \in Sym_2(h^\perp)^\vee$ be the symmetric tensor associated to the restriction of $(,)_X$ to $h^\perp$. Since $(h,h) \neq 0$ the restriction of $(,)_X$ to $h^\perp$ is non-degenerate and hence we have the dual $q_h^\vee \in Sym^2(h^\perp)$; as is easily checked

$$q_X^\vee = (h,h)^{-1}h^2 + q_h^\vee. \tag{3.0.7}$$

Let $\langle,\rangle_X$ be the intersection form on $H^4(X)$: identifying $H^4(X)$ with $Sym^2 H^2(X)$ the intersection form $\langle,\rangle_X$ gets identified with the symmetric bilinear form constructed from $(,)_X$ as in Subsection (2.2) - see Remark (2.1). Let

$$W(h) := (q_X^\vee)^\perp \cap Sym^2(h^\perp), \tag{3.0.8}$$

where the first orthogonality is with respect to $\langle,\rangle_X$.

Claim 3.1. *Keep notation as above and assume that (3.0.1) holds. Then $W(h)$ is a codimension-1 rational sub-Hodge-structure of $Sym^2(h^\perp)$, and we have a direct sum decomposition*

$$\mathbb{C}h^2 \oplus Sym^2(h^\perp) = \mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee \oplus W(h). \tag{3.0.9}$$

Proof. Since $q_X^\vee$ is a Hodge class $W(h)$ is a rational sub-Hodge-structure. It follows immediatly from (3.0.5) that $Sym^2(h^\perp) \not\subset (q_X^\vee)^\perp$ and thus $W(h)$ has codimension 1. Now let's prove that we have (3.0.9). First we prove that $h^2, q_X^\vee$ are linearly independent. Assume that

$$\lambda h^2 + \mu q_X^\vee = 0, \quad \lambda, \mu \in \mathbb{C}. \tag{3.0.10}$$

Since $(,)_X$ is non-degenerate there exists $\alpha \in H^2(X)$ such that $(\alpha, \alpha)_X = 0$ and $(h, \alpha)_X \neq 0$. Then from (3.0.10) and (3.0.5) we get that

$$0 = \langle \lambda h^2 + \mu q_X^\vee, \alpha \rangle_X = 2\lambda (h, \alpha)_X^2, \tag{3.0.11}$$

and hence $\lambda = 0$. Since $q_X^\vee \neq 0$ we get that $\mu = 0$. This shows that $h^2, q_X^\vee$ are linearly independent. To finish the proof of the claim it suffices to show that

$$\left(\mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee\right) \cap W(h) = \{0\}. \tag{3.0.12}$$

It follows from (3.0.5) that

$$\left(\mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee\right) \cap (q_X^\vee)^\perp = \mathbb{C}(23h^2 - (h,h)q_X^\vee). \tag{3.0.13}$$

On the other hand by (3.0.7) we have

$$\left(\mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee\right) \cap Sym^2(h^\perp) = \mathbb{C}(h^2 - (h,h)q_X^\vee). \tag{3.0.14}$$

Equation (3.0.12) follows immediately from (3.0.13)-(3.0.14). □

Keep notation as above and assume that (3.0.1) holds; by (3.0.3) and Claim (3.1) we have a decomposition

$$H^4(X;\mathbb{C}) = \left(\mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee\right) \oplus \left(\mathbb{C}h \otimes h^\perp\right) \oplus W(h) \tag{3.0.15}$$

into sub-H.S.'s of levels 0, 2 and 4 respectively. The following is the main result of this section.

Proposition 3.2. *Keep notation as above. Let M be a numerical $(K3)^{[2]}$. There exists an irreducible symplectic manifold X deformation equivalent to M such that:*

(1) X has an ample divisor H with $(h,h)_X = 2$, where $h := c_1(H)$,

(2) $H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}h$,

(3) Let $\Sigma \in Z_1(X)$ be an integral algebraic 1-cycle on X and $cl(\Sigma) \in H^{3,3}_{\mathbb{Q}}(X)$ be its Poincaré dual. Then $cl(\Sigma) = mh^3/6$ for some $m \in \mathbb{Z}$.

(4) if $V \subset H^4(X)$ is a rational sub Hodge structure then $V = V_1 \oplus V_2 \oplus V_3$ where $V_1 \subset \left(\mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee\right)$, V_2 is either 0 or equal to $\mathbb{C}h \otimes h^\perp$ and V_3 is either 0 or equal to $W(h)$.

(5) the image of h^2 in $H^4(X;\mathbb{Z})/Tors$ is indivisible, (we denote by $Tors$ the torsion subgroup of $H^4(X;\mathbb{Z})$)

(6) $H^{2,2}_{\mathbb{Z}}(X)/Tors \subset \mathbb{Z}(h^2/2) \oplus \mathbb{Z}(q_X^\vee/5)$.

The proof of the proposition will be given after some preliminary results. We recall Huybrechts' Theorem on surjectivity of the global period map [13, 14] - in the context of numerical $(K3)^{[2]}$'s. Let M be a numerical $(K3)^{[2]}$ and $\mathcal{M}$ be the moduli space of marked irreducible symplectic 4-folds deformation equivalent to M; thus a point of $\mathcal{M}$ is an equivalence class of couples (X, ψ) where X is an irreducible symplectic 4-fold deformation equivalent to M and $\psi\colon \Lambda \xrightarrow{\sim} H^2(X;\mathbb{Z})$ is an isometry of lattices where Λ is the lattice given by (2.1.3). Couples (X, ψ) and (X', ψ') are equivalent if there exists an isomorphism $f\colon X \to X'$ such that $H^2(f) \circ \psi' = \pm\psi$. If $t \in \mathcal{M}$ we let (X_t, ψ_t) be a representative of t. It is known that $\mathcal{M}$ is a non-separated complex analytic space, see Thm.(2.4) of [18]. The period domain $Q \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ is given by

$$Q := \{[\sigma] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) |\ (\sigma, \sigma)_\Lambda = 0, \quad (\sigma, \overline{\sigma})_\Lambda > 0\} \tag{3.0.16}$$

where $(,)_\Lambda$ is the symmetric bilinear form on $\Lambda \otimes \mathbb{C}$ obtained by extending linearly the integral symmetric bilinear form on Λ. The period map is given by

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{P} & Q \\ (X, \psi) & \mapsto & \psi^{-1} H^{2,0}(X). \end{array} \tag{3.0.17}$$

Here and in the following ψ denotes both the isometry $\Lambda \xrightarrow{\sim} H^2(X;\mathbb{Z})$ and its linear extension $\Lambda \otimes \mathbb{C} \to H^2(X;\mathbb{C})$. The map P is locally an isomorphism, see [1]. Let $\mathcal{M}^0$ be a connected component of $\mathcal{M}$. Huybrechts' Theorem on surjectivity of the global period map (Thm. (8.1) of [13]) states that the restriction of P to $\mathcal{M}^0$ is surjective. Given $\alpha \in \Lambda$ we let

$$\mathcal{M}^0_\alpha := \{t \in \mathcal{M}^0 |\ \psi_t(\alpha) \in H^{1,1}_{\mathbb{Z}}(X_t)\}. \tag{3.0.18}$$

Of course $\mathcal{M}^0_0 = \mathcal{M}^0$. Assume that $\alpha \neq 0$ and let

$$Q_\alpha := \alpha^\perp \cap Q = \{[\sigma] \in Q |\ (\sigma, \alpha)_\Lambda = 0\}. \tag{3.0.19}$$

Then Q_α is a non-empty codimension 1 analytic subset of Q (a submanifold if $(\alpha, \alpha)_\Lambda \neq 0$) and furthermore

$$\mathcal{M}^0_\alpha = P^{-1}(Q_\alpha). \tag{3.0.20}$$

By surjectivity of the period map we get that $\mathcal{M}^0_\alpha$ is non-empty of dimension 20.

Lemma 3.3. *Let M be a numerical $(K3)^{[2]}$ and $\mathcal{M}^0$ be a connected component of the moduli space of marked irreducible symplectic 4-folds deformation equivalent to M. Let $\alpha \in \Lambda$ such that $(\alpha, \alpha)_\Lambda \neq 0$. For $t \in \mathcal{M}^0_\alpha$ outside of a countable union of proper analytic subsets we have:*

(1) $H^{1,1}_{\mathbb{Q}}(X_t) = \mathbb{Q}\psi_t(\alpha)$,

(2) any rational sub-Hodge-structure of $W(\psi_t(\alpha))$ is trivial.

Proof. It is well-known that Item (1) holds for $t \in \mathcal{M}^0_\alpha$ outside of a countable union of proper analytic subsets; we recall the proof for the reader's convenience. Let $\mathcal{N}^0_\alpha \subset \mathcal{M}^0_\alpha$ be the subset of t such that Item (1) does not hold. Let $\overline{\mathbb{Z}\alpha} \subset \Lambda$

be the saturation of $\mathbb{Z}\alpha$ i.e. the set of $\beta \in \Lambda$ such that β is proportional to α. Then

$$\mathcal{N}^0_\alpha = \bigcup_{\beta \in \Lambda \setminus \overline{\mathbb{Z}\alpha}} P^{-1}(Q_\alpha \cap Q_\beta). \tag{3.0.21}$$

Let $\beta \in (\Lambda \setminus \overline{\mathbb{Z}\alpha})$: then $(Q_\alpha \cap Q_\beta)$ is a proper analytic subset of Q_α because $(,)_\Lambda$ is non-degenerate. Since P is a (holomorphic) local isomorphism we get that $P^{-1}(Q_\alpha \cap Q_\beta)$ is a proper analytic subset of $\mathcal{M}^0_\alpha$; Item (1) follows because $(\Lambda \setminus \overline{\mathbb{Z}\alpha})$ is countable. Now let's prove that the set of $t \in \mathcal{M}^0_\alpha$ for which (2) does not hold is a countable union of proper analytic subsets of $\mathcal{M}^0_\alpha$. Let $\langle,\rangle_\Lambda$ be the symmetric bilinear form on $Sym^2(\Lambda \otimes \mathbb{C})$ constructed from $(,)_\Lambda$ as in Subsection (2.2) - see (2.2.1). Let $q_\Lambda \in Sym_2(\Lambda \otimes \mathbb{C})^\vee$ be the symmetric tensor associated to $(,)_\Lambda$ and $q^\vee_\Lambda \in Sym^2(\Lambda \otimes \mathbb{C})$ be its dual. Let

$$W(\alpha) := Sym^2(\alpha^\perp) \cap (q^\vee_\Lambda)^\perp \subset Sym^2(\Lambda \otimes \mathbb{C}) \tag{3.0.22}$$

where the first orthogonality is with respect to $(,)_\Lambda$ and the second orthogonality is with respect to $\langle,\rangle_\Lambda$. For a linear subspace $V \subset W(\alpha)$ defined over $\mathbb{Q}$ let

$$\mathcal{M}^0_\alpha(V) := \{t \in \mathcal{M}^0_\alpha |\ Sym^2(\psi_t)(V) \text{ is a sub-H.S. of } H^4(X_t)\}. \tag{3.0.23}$$

Let $t \in \mathcal{M}^0_\alpha$; then (2) does not hold if and only if $t \in \mathcal{M}^0_\alpha(V)$ for some linear subspace $0 \neq V \underset{\neq}{\subset} W(\alpha)$ defined over $\mathbb{Q}$. Since the set of subspaces $V \subset W(\alpha)$ defined over $\mathbb{Q}$ is countable we get that it suffices to prove that $\mathcal{M}^0_\alpha(V)$ is a proper analytic subset of $\mathcal{M}^0_\alpha$ whenever $V \neq 0$ or $V \neq W(\alpha)$. First of all $\mathcal{M}^0_\alpha(V)$ is an analytic subset of $\mathcal{M}^0_\alpha$ because the period map is holomorphic. Assume that $\mathcal{M}^0_\alpha(V)$ contains a non-empty open subset

$$U \subset \mathcal{M}^0_\alpha; \tag{3.0.24}$$

we will show that either $V = W(\alpha)$ or $V = 0$. We have

(a) $Sym^2(\psi_t)(V) \cap H^{4,0}(X_t) \neq \{0\}$ for all $t \in U$, or

(b) $Sym^2(\psi_t)(V) \cap H^{4,0}(X_t) = \{0\}$ for all $t \in U$.

Assume that (a) holds. Then

$$Sym^2(\psi_t)(V) \supset H^{4,0}(X_t) = H^{2,0}(X_t) \wedge H^{2,0}(X_t) \tag{3.0.25}$$

for all $t \in U$ and hence

$$V \supset \{\sigma^2 |\ [\sigma] \in P(U)\}, \tag{3.0.26}$$

where P is the period map. We claim that this implies that $V = W(\alpha)$. Let

$$\begin{array}{ccc} \mathbb{P}(\alpha^\perp) & \xrightarrow{\nu_\alpha} & \mathbb{P}(Sym^2(\alpha^\perp)) \\ [\sigma] & \mapsto & [\sigma^2] \end{array} \tag{3.0.27}$$

be the Veronese map associated to the complete linear system of quadrics on $\mathbb{P}(\alpha^\perp)$. Let $\mathcal{V}_\alpha := Im(\nu_\alpha)$. Let $Z_\alpha \subset \mathbb{P}(\alpha^\perp)$ be the quadric of one-dimensional subspaces isotropic for the restriction of $(,)_\Lambda$ to $\alpha^\perp$. Since $(\alpha,\alpha)_\Lambda \neq 0$ the quadric Z_α is smooth. Furthermore by (2.2.6) we get that

$$\nu_\alpha(Z_\alpha) = \mathcal{V}_\alpha \cap \mathbb{P}(W(\alpha)). \tag{3.0.28}$$

Since Z_α is an irreducible (actually smooth) quadric we get that $\nu_\alpha(Z_\alpha)$ spans $\mathbb{P}(W(\alpha))$. In fact the following stronger statement holds: if $\mathcal{U} \subset Z_\alpha$ is a non-empty subset which is open in the classical topology then $\nu_\alpha(\mathcal{U})$ spans $\mathbb{P}(W(\alpha))$. Now notice that Q_α is a non-empty subset of Z_α which is open in the classical topology. Let $U \subset \mathcal{M}^0_\alpha$ be the open non-empty subset of (3.0.24). Then $P(U)$ is open non-empty in Q_α and hence it is open non-empty in Z_α; thus

$$span\{\sigma^2 |\ [\sigma] \in P(U)\} = W(\alpha). \tag{3.0.29}$$

By (3.0.26) we get that $V = W(\alpha)$. Now assume that (b) holds. Then

$$\langle Sym^2(\psi_t)(V), H^{0,4}(X_t)\rangle_X = 0 \tag{3.0.30}$$

for all $t \in U$ and hence $V \otimes \mathbb{C} \bot \{\overline{\sigma}^2 |\ [\sigma] \in P(U)\}$, where orthogonality is with respect to $\langle , \rangle_\Lambda$. Since $V \otimes \mathbb{C}$ is real, that is invariant under conjugation, we get that $V \otimes \mathbb{C} \bot \{\sigma^2 |\ [\sigma] \in P(U)\}$. By (3.0.29) we get that $V \bot W(\alpha)$. Thus in order to finish the proof it suffices to show that the restriction of $\langle , \rangle_\Lambda$ to $W(\alpha)$ is non-degenerate. First we claim that the restriction of $\langle , \rangle_\Lambda$ to $Sym^2(\alpha^\perp)$ is non-degenerate: in fact since $(\alpha, \alpha)_\Lambda \neq 0$ the restriction of $(,)_\Lambda$ to $\alpha^\perp$ is non-degenerate and the claim follows from Proposition (2.2). Let $q_\alpha \in Sym_2(\alpha^\perp)^\vee$ be the symmetric tensor associated to the restriction of $(,)_\Lambda$ to $\alpha^\perp$; we claim that

$$W(\alpha) = \{\zeta \in Sym^2(\alpha^\perp) |\ \langle q^\vee_\alpha, \zeta\rangle_\Lambda = 0\}; \tag{3.0.31}$$

In fact by (2.2.6) we have

$$\langle q^\vee_\Lambda, \gamma\delta\rangle = 25(\gamma, \delta)_\Lambda, \tag{3.0.32}$$
$$\langle q^\vee_h, \gamma\delta\rangle = 24(\gamma, \delta)_\Lambda \tag{3.0.33}$$

for all $\gamma, \delta \in \alpha^\perp$; this implies (3.0.31). By (2.2.7) we have $\langle q^\vee_h, q^\vee_h\rangle_\Lambda = 22 \cdot 24$; by (3.0.31) we get that $W(h)$ is the orthogonal (in $Sym^2(\alpha^\perp)$) of a non-isotropic vector and since the restriction of $\langle , \rangle_\Lambda$ to $Sym^2(\alpha^\perp)$ is non-degenerate it follows that the restriction of $\langle , \rangle_\Lambda$ to $W(h)$ is non-degenerate. □

We will apply Lemma (3.3) with a particular choice of α. We will need the following two results.

Lemma 3.4. *The vectors $\alpha \in \Lambda$ with*

$$(\alpha, \alpha)_\Lambda = 2 \tag{3.0.34}$$

belong to a single $O(\Lambda)$-orbit.

Proof. Let $\alpha, \beta \in \Lambda$ be indivisible: by Proposition (2.3) of [12] the elements α, β belong to the same $O(\Lambda)$-orbit if and only if $(\alpha, \alpha)_\Lambda = (\beta, \beta)_\Lambda$ and

$$(\alpha, \Lambda)_\Lambda := \{(\alpha, \gamma)_\Lambda |\ \gamma \in \Lambda\} = \{(\beta, \delta)_\Lambda |\ \delta \in \Lambda\} =: (\beta, \Lambda)_\Lambda. \tag{3.0.35}$$

Let $\alpha \in \Lambda$ such that $(\alpha, \alpha)_\Lambda = 2$. Then α is indivisible because $(\alpha, \alpha)_\Lambda$ has no square factors. Thus to finish the proof of the lemma it suffices to show that

$$(\alpha, \Lambda)_\Lambda = \mathbb{Z}. \tag{3.0.36}$$

Assume that (3.0.36) does not hold. Since α is indivisible and the discriminant of $(,)_\Lambda$ is 2 it follows that

$$(\alpha, \Lambda)_\Lambda = 2\mathbb{Z}. \tag{3.0.37}$$

Let $\xi \in \Lambda$ be a generator of the summand (-2) appearing in (2.1.3). Then $\alpha = v + x\xi$ where $v \in \xi^\perp$ and $x \in \mathbb{Z}$. Since the restriction of $(,)_\Lambda$ to $\xi^\perp$ is unimodular we get from (3.0.37) that $v = 2w$. Thus

$$1 = \frac{(\alpha,\alpha)_\Lambda}{2} = 4\frac{(w,w)_\Lambda}{2} - x^2. \tag{3.0.38}$$

Since $(,)_\Lambda$ is even we get that $x^2 \equiv -1 \pmod 4$: that is absurd and hence (3.0.37) does not hold. This proves (3.0.36). □

Lemma 3.5. *Let M be a numerical $(K3)^{[2]}$. Let $\mathcal{M}^0$ be a connected component of the moduli space of marked irreducible symplectic 4-folds deformation equivalent to M. Suppose that $\alpha_1, \alpha_2 \in \Lambda$ satisfy*

$$(\alpha_1,\alpha_1)_\Lambda = (\alpha_2,\alpha_2)_\Lambda = 2, \quad (\alpha_1,\alpha_2)_\Lambda \equiv 1 \mod 2. \tag{3.0.39}$$

There exists $1 \le i \le 2$ such that for every $t \in \mathcal{M}^0$ the class of $\psi_t(\alpha_i)^2$ in $H^4(X_t;\mathbb{Z})/Tors$ is indivisible.

Proof. By Remark (2.1) we have

$$\langle \psi_t(\alpha_1)^2, \psi_t(\alpha_2)^2\rangle_{X_t} = (\alpha_1,\alpha_1)_\Lambda \cdot (\alpha_2,\alpha_2)_\Lambda + 2(\alpha_1,\alpha_2)^2_\Lambda \equiv 2 \pmod 4. \tag{3.0.40}$$

Next we show that the class of $\psi_t(\alpha_i)^2$ in $H^4(X_t;\mathbb{Z})/Tors$ is divisible at most by 2. We claim that there exists $\beta_i \in \Lambda$ with

$$(\alpha_i,\beta_i)_\Lambda = 1, \quad (\beta_i,\beta_i)_\Lambda = 0. \tag{3.0.41}$$

In fact by Lemma (3.4) it suffices to exhibit $\alpha', \beta' \in \Lambda$ such that

$$(\alpha',\alpha')_\Lambda = 2, \quad (\alpha',\beta')_\Lambda = 1, \quad (\beta',\beta')_\Lambda = 0, \tag{3.0.42}$$

and this is a trivial exercise. Now let β_i be as above; by Remark (2.1) we have

$$\langle \psi_t(\alpha_i)^2, \psi_t(\beta_i)^2\rangle_{X_t} = 2(\alpha_i,\beta_i)_\Lambda = 2 \tag{3.0.43}$$

and thus the class of $\psi_t(\alpha_i)^2$ in $H^4(X_t;\mathbb{Z})/Tors$ is divisible at most by 2. Now we prove the lemma arguing by contradiction. Assume that the class of $\psi_t(\alpha_i)^2$ in $H^4(X_t;\mathbb{Z})/Tors$ is divisible for $i = 1$ and $i = 2$; then it is divisible by 2 for $i = 1$ and $i = 2$. Thus $\langle \psi_t(\alpha_1)^2, \psi_t(\alpha_2)^2\rangle_{X_t} \equiv 0 \pmod 4$ and this contradicts (3.0.40). □

Proof of Proposition (3.2). Let $\mathcal{M}^0$ be a connected component of the moduli space of marked irreducible symplectic 4-folds deformation equivalent to M. By Lemma (3.5) there exists $\alpha \in \Lambda$ such that $(\alpha,\alpha) = 2$ and the class of $\psi_t(\alpha)^2$ in $H^4(X_t;\mathbb{Z})/Tors$ is indivisible for every $t \in \mathcal{M}^0$. Let $t \in \mathcal{M}^0_\alpha$ be such that both Item (1) and Item (2) of Lemma (3.3) are satisfied. Set $X := X_t$. Since $\psi_t(\alpha) \in H^{1,1}_{\mathbb{Z}}(X)$ and $(\psi_t(\alpha),\psi_t(\alpha)) = 2$ we know that X is projective by Huybrechts' projectivity criterion [13]: since $H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}\psi_t(\alpha)$ either $\psi_t(\alpha)$ or $-\psi_t(\alpha)$ is the class of an ample divisor. Let $h := \psi_t(\alpha)$ in the former case and

$h := -\psi_t(\alpha)$ in the latter case. We let H be a divisor with $c_1(H) = h$. Let's prove that Proposition (3.2) holds for (X, H). First X is a deformation of M by definition of $\mathcal{M}^0$. Next Items (1)-(2) hold by construction. Let's prove Item (3): By Item (2) and Hard Lefschetz we have $H^{3,3}_{\mathbb{Q}}(X) = \mathbb{Q}h^3$ and hence $cl(\Gamma) = xh^3$ for some $x \in \mathbb{Q}$. There exists $e \in H^2(X;\mathbb{Z})$ with $(e,h)_X = 1$, see (3.0.41), and hence by Remark (2.1) we have

$$\mathbb{Z} \ni \int_\Gamma e = \langle xh^3, e\rangle_X = 3x(h,h)_X(h,e)_X = 6x. \tag{3.0.44}$$

This proves Item (3). Item (4) follows from Item (2) of Lemma (3.3), from the fact that $\mathbb{C}h \otimes h^\perp$ has no non-trivial sub-H.S.'s and an easy argument based on the observation that the three summands of Decomposition (3.0.15) have pairwise distinct levels. Item (5) holds by our choice of α, thanks to Lemma (3.5). Finally we prove Item (6). First we show that

$$c_2(X) = 6q_X^\vee/5 \quad \text{in } H^4(X;\mathbb{Q}). \tag{3.0.45}$$

Let Z be an irreducible symplectic manifold: it is known that if

$$\theta \in Im(Sym^2 H^2(Z) \to H^4(Z)) \tag{3.0.46}$$

is a $(2,2)$ class which remains of type $(2,2)$ for all small deformations of Z then θ is a multiple of $q_Z^\vee$. Clearly $c_2(X)$ remains of type $(2,2)$ for all small deformations of X, and since X is a numerical $(K3)^{[2]}$ we have $H^4(X) \cong Sym^2 H^2(X)$; thus $c_2(X) = aq_X^\vee$ for some a. We claim that $a \geq 0$; in fact Theorem (1.1) of [19] together with (3.0.5) gives that

$$0 \leq \langle c_2(x), h^2\rangle_X = \langle aq_X^\vee, h^2\rangle_X = 50a. \tag{3.0.47}$$

On the other hand applying Hirzebruch-Riemann-Roch and keeping in mind that all odd Chern classes of X vanish we get that

$$3 = \chi(\mathcal{O}_X) = \frac{1}{240}\left(c_2(X)^2 - \frac{1}{3}c_4(X)\right). \tag{3.0.48}$$

By (2.1.7)-(2.1.8) we have that

$$c_4(X) = 324 \tag{3.0.49}$$

and hence it follows that $c_2(X)^2 = 828$. Applying Formula (3.0.6) we get that

$$828 = c_2(X)^2 = \langle aq_X^\vee, aq_X^\vee\rangle_X = a^2 23 \cdot 25. \tag{3.0.50}$$

Since $a \geq 0$ we get that $a = 6/5$; this proves (3.0.45). Now notice that by (2.2.5) we have $2q_X^\vee \in Sym^2 H^2(X;\mathbb{Z})/Tors$. Thus Formula (3.0.45) gives that

$$H^4(X;\mathbb{Z})/Tors \ni (2c_2(X) - 2q_X^\vee) = 2q_X^\vee/5. \tag{3.0.51}$$

In particular

$$\Omega(h) := \mathbb{Z}h^2 \oplus \mathbb{Z}(2q_X^\vee/5) \subset (H^{2,2}_{\mathbb{Z}}(X)/Tors) \tag{3.0.52}$$

By Item (4) of Proposition (3.2) we know that $h^{2,2}_{\mathbb{Q}}(X) = 2$; since h^2 and $q_X^\vee$ are linearly independent we get that $\Omega(h)$ is of finite index in $H^{2,2}_{\mathbb{Z}}(X)/Tors$. A straightforward computation (use (3.0.6) and (3.0.5)) shows that

$$\text{discr}\left(\langle,\rangle_X|_{\Omega(h)}\right) = 2^6 \cdot 11, \tag{3.0.53}$$

and hence

$$[H^{2,2}_{\mathbb{Z}}(X)/Tors : \Omega(h)] \le 8. \tag{3.0.54}$$

Now let $xh^2 + y(2q^{\vee}_X/5) \in H^{2,2}_{\mathbb{Z}}(X)/Tors$: in order to prove Item (6) we must show that $2x \in \mathbb{Z}$ and $2y \in \mathbb{Z}$. Let $\beta \in H^2(X;\mathbb{Z})$ with $(h,\beta)_X = 1$ and $(\beta,\beta)_X = 0$: such a β exists, see (3.0.41). Using (3.0.5) we get that

$$\mathbb{Z} \ni \langle xh^2 + y(2q^{\vee}_X/5), \beta^2\rangle_X = 2x. \tag{3.0.55}$$

Next let $\gamma,\delta \in H^2(X;\mathbb{Z})$ with $(\gamma,\delta)_X = 1$. By Remark (2.1) we have

$$\mathbb{Z} \ni \langle xh^2 + y(2q^{\vee}_X/5), \gamma\delta\rangle_X = 2x(1 + (h,\gamma)_X(h,\delta)_X) + 10y. \tag{3.0.56}$$

By (3.0.55) we have $2x \in \mathbb{Z}$ and hence we get that $10y \in \mathbb{Z}$. By (3.0.54) we know that $8y \in \mathbb{Z}$ and hence $2y \in \mathbb{Z}$. This finishes the proof of Item (6) and of Proposition (3.2). □

Let M be a numerical $(K3)^{[2]}$: by Proposition (3.5) there exists $\gamma \in H^2(M;\mathbb{Z})$ such that $(\gamma,\gamma)_M = 2$ and the class of γ^2 in $H^4(M;\mathbb{Z})/Tors$ is indivisible, however we cannot exclude a priori the existence of some $\gamma \in H^2(M;\mathbb{Z})$ such that $(\gamma,\gamma)_M = 2$ and the image of γ^2 in $H^4(M;\mathbb{Z})/Tors$ is divisible by 2. If M is a deformation of $(K3)^{[2]}$ the picture is simpler.

Proposition 3.6. *Let M be a deformation of $(K3)^{[2]}$ and $\gamma \in H^2(M;\mathbb{Z})$ such that $(\gamma,\gamma)_M = 2$. The image of γ^2 in $H^4(M;\mathbb{Z})/Tors$ is indivisible.*

Proof. Let S be a $K3$ surface. We may assume that $\gamma \in H^2(S^{[2]};\mathbb{Z})$. Let $\Delta \subset S^{[2]}$ be the codimension-1 locus parametrizing non-reduced subschemes of S. There exists $\xi \in H^2(S^{[2]};\mathbb{Z})$ such that $2\xi = c_1(\Delta)$. Furthermore there is an orthogonal direct sum decomposition (see Prop. 6, p. 768 and pp. 777-778 of [1])

$$H^2(S^{[2]};\mathbb{Z}) = \mu(H^2(S;\mathbb{Z}))\widehat{\oplus}\mathbb{Z}\xi \tag{3.0.57}$$

where $\mu\colon H^2(S;\mathbb{Z}) \to H^2(S^{[2]};\mathbb{Z})$ is the symmetrization map (Donaldson map). If $\Gamma \subset S$ is an oriented closed C^∞ surface representing a class $[\Gamma] \in H^2(S;\mathbb{Z})$ then a representative of $\mu([\Gamma])$ is given by

$$\{[Z] \in S^{[2]} \mid Z \cap \Gamma \neq \emptyset\}. \tag{3.0.58}$$

By (3.0.57) we have $\gamma = x\mu(\alpha) + y\xi$ where $\alpha \in H^2(S;\mathbb{Z})$ is indivisible (in particular non-zero) and $x,y \in \mathbb{Z}$ are coprime. Let $(,)_S$ be the intersection form on $H^2(S)$. Since α is indivisible and the group of isometries of the $K3$-lattice acts transitively on indivisible vectors of a given length there exists $\beta \in H^2(S;\mathbb{Z})$ such that

$$(\beta,\beta)_S = -2, \tag{3.0.59}$$

$$x(\alpha,\beta)_S + y \equiv 1 \pmod 2. \tag{3.0.60}$$

By deforming the complex structure of S we may assume that $H^{1,1}_{\mathbb{Z}}(S) = \mathbb{Z}\beta$. From (3.0.59) one gets that $\pm\beta$ is represented by a smooth irreducible rational curve $C \subset S$. Let $\Sigma := C^{(2)} \subset S^{[2]}$; thus $\Sigma \cong \mathbb{P}^2$. Then

$$\xi|_\Sigma = c_1(\mathcal{O}_\Sigma(1)), \tag{3.0.61}$$

$$\mu(\alpha)|_\Sigma = c_1(\mathcal{O}_\Sigma(\pm(\alpha,\beta)_S)). \tag{3.0.62}$$

Thus (3.0.60) gives that

$$\gamma|_{\Sigma} = c_1(\mathcal{O}_{\Sigma}(d)), \quad d \equiv 1 \pmod 2. \tag{3.0.63}$$

Thus $\gamma^2|_{\Sigma}$ is not divisible by 2. On the other hand we know that the image of γ^2 in $H^4(M;\mathbb{Z})/Tors$ is divisible at most by 2, see the proof of Lemma (3.5); thus the image of γ^2 in $H^4(S^{[2]};\mathbb{Z})/Tors$ is not divisible. □

4 The linear system $|H|$

In this section we let X, H be as in Proposition (3.2); we will prove some basic properties of the complete linear system $|H|$. We let $h := c_1(H)$. First we claim that

$$h^0(\mathcal{O}_X(nH)) = \frac{1}{2}n^4 + \frac{5}{2}n^2 + 3, \quad n \in \mathbb{N}_+. \tag{4.0.1}$$

In fact applying Hirzebruch-Riemann-Roch and keeping in mind that all odd Chern classes of X vanish we get that

$$\chi(\mathcal{O}_X(nH)) = \frac{1}{24}\left(\int_X h^4\right)n^4 + \frac{1}{24}\left(\int_X c_2(X)h^2\right)n^2 + \chi(\mathcal{O}_X). \tag{4.0.2}$$

Using (3.0.45) and (3.0.5) we get that

$$\chi(\mathcal{O}_X(nH)) = \frac{1}{2}n^4 + \frac{5}{2}n^2 + 3, \quad n \in \mathbb{Z}. \tag{4.0.3}$$

Since $K_X \cong \mathcal{O}_X$ Kodaira vanishing gives that for $n > 0$ we have $h^0(\mathcal{O}_X(nH)) = \chi(\mathcal{O}_X(nH))$. Thus (4.0.1) follows from (4.0.3). In particular we have $h^0(\mathcal{O}_X(H)) = 6$. We choose once and for all an isomorphism

$$|H|^{\vee} \xrightarrow{\sim} \mathbb{P}^5 \tag{4.0.4}$$

and we let

$$f\colon X \dashrightarrow \mathbb{P}^5 \tag{4.0.5}$$

be the composition $X \dashrightarrow |H|^{\vee} \xrightarrow{\sim} \mathbb{P}^5$. Let B be the base-scheme of $|H|$, i.e.

$$B := \bigcap_{D \in |H|} D, \tag{4.0.6}$$

and $\pi\colon \widetilde{X} \to X$ be the blow-up of B. Let

$$\widetilde{f}\colon \widetilde{X} \to \mathbb{P}^5 \tag{4.0.7}$$

be the regular map which resolves the indeterminacies of f. Let $Y := Im(\widetilde{f})$; thus $Y \subset \mathbb{P}^5$ is closed and we have (abusing notation) a dominant map

$$f\colon X \dashrightarrow Y. \tag{4.0.8}$$

We let $\deg f$ be the degree of the map above. Let $X_0 := (X \setminus B)$; thus $X_0 \subset X$ is open and dense. The restriction of f to X_0 is regular; we let $Y_0 := f(X_0)$. Thus $Y_0 \subset Y$ is a constructible dense subset of Y, in particular Y_0 contains an open dense subset of Y. Let

$$f_0\colon X_0 \to Y_0 \tag{4.0.9}$$

be the restriction of f to X_0. The next proposition is the key technical result of this section.

Proposition 4.1. *If $D_1, D_2 \in |H|$ are distinct then $D_1 \cap D_2$ is a reduced irreducible surface.*

Proof. Assume that $\Gamma \in Z^2(X)$ is an effective non-zero algebraic cycle of pure codimension 2. Assume that

$$cl(\Gamma) = (sh^2 + t(2q^\vee/5)) \in H^4(X;\mathbb{Z})/Tors, \tag{4.0.10}$$

where $cl(\Gamma)$ is the image of the Poicaré dual of the homology class represented by Γ. Let $\sigma \in \Gamma(\Omega^2_X)$ be a symplectic form. Then by Remark (2.1) and (3.0.5)-(2.1.2) we have

$$0 < \langle cl(\Gamma), h^2\rangle_X = 12s + 20t, \tag{4.0.11}$$

$$0 \leq \langle cl(\Gamma), (\sigma+\overline{\sigma})^2\rangle_X = (2s+10t)(\sigma+\overline{\sigma}, \sigma+\overline{\sigma})_X. \tag{4.0.12}$$

Since $(\sigma+\overline{\sigma}, \sigma+\overline{\sigma})_X > 0$ we get that

$$3s + 5t > 0, \qquad s + 5t \geq 0. \tag{4.0.13}$$

Now let $D_1, D_2 \in |H|$ be distinct. By Item (2) of Proposition (3.2) we know that $D_1 \cap D_2$ is a subscheme of X of pure codimension 2 representing h^2. Assume that $D_1 \cap D_2$ is not reduced and irreducible: then we have an equality of cycles $[D_1 \cap D_2] = A + B$ with A, B effective non-zero. By Item (6) of Proposition (3.2) we have

$$cl(A) = xh^2 + y(2q^\vee/5), \qquad cl(B) = (1-x)h^2 - y(2q^\vee/5) \tag{4.0.14}$$

with

$$2x, 2y \in \mathbb{Z}. \tag{4.0.15}$$

Applying (4.0.13) to A and B we get that

$$0 < 3x + 5y < 3, \qquad 0 \leq x + 5y \leq 1. \tag{4.0.16}$$

"Eliminating x"we get that

$$-3/5 < 2y < 3/5. \tag{4.0.17}$$

Thus by (4.0.15) we get that $y = 0$ and hence $cl(A) = xh^2$ with $0 < x < 1$. That contradicts Item (5) of Proposition (3.2). □

Corollary 4.2. *Keep notation as above.*

(1) If $L \subset \mathbb{P}^5$ is a linear subspace of codimension at most 2 then $L \cap Y_0$ is reduced and irreducible and, if non-empty, it has pure codimension equal to $cod(L, \mathbb{P}^5)$.

(2) The base-scheme B of $|H|$ has dimension at most 1. Let B_{red} be the reduced scheme associated to B i.e. the scheme defined by the radical of the ideal sheaf of B; let B^1_{red} be the union of 1-dimensional irreducible components of B_{red}. If D_1, D_2, D_3 are linearly independent then $D_1 \cap D_2 \cap D_3$ is purely 1-dimensional and there is a unique decomposition

$$[D_1 \cap D_2 \cap D_3] = \Gamma + \Sigma \tag{4.0.18}$$

where Γ, Σ are effective 1-cycles such that:

(2a) supp$(\Gamma)\cap B_{red}$ *is zero-dimensional or empty,*

(2b) supp$\Sigma = B^1_{red}$.

Proof. Let's prove Item (1). If $L=\mathbb{P}^5$ there is nothing to prove. Assume that $\mathrm{cod}(L,\mathbb{P}^5)=1$. Let $D\in|H|$ be the divisor corresponding to L via (4.0.4). Then $D\cap X_0=f_0^*L$; since X_0 is open dense in X and f_0 is surjective Item (1) follows from Item (2) of Proposition (3.2). Assume that $\mathrm{cod}(L,\mathbb{P}^5)=2$ and write $L=L_1\cap L_2$ where $L_1,L_2\subset\mathbb{P}^5$ are hyperplanes. Let $D_1,D_2\in|H|$ be the divisors corresponding to L_1,L_2 via (4.0.4). Then $D_1\cap D_2\cap X_0=f_0^*L$; since X_0 is open dense in X and f_0 is surjective Item (1) follows from Proposition (4.1). This finishes the proof of Item (1). Let's prove Item (2). By Proposition (4.1) the intersection $D_1\cap D_2\cap D_3$ is purely 1-dimensional and hence in particular the dimension of the base-scheme B of $|H|$ is at most 1. It remains to prove that there is a unique decomposition (4.0.18) with the stated properties. Let $\Gamma_0\in Z_1(X_0)$ be the fundamental cycle of $((D_1\cap D_2\cap D_3)\setminus B)$ and $\Gamma\in Z_1(X)$ be its closure. Since $D_1\cap D_1\cap D_3\supset B$ and $\dim B\le 1$ we get that Item (1) holds with this choice of Γ. On the other hand it is clear that if we have a decomposition (4.0.18) such that (2a) and (2b) hold then necessarily Γ is the closure of Γ_0 and hence decomposition (4.0.18) is unique. □

Corollary 4.3. *Keeping notation as above, we have* $\dim Y\ge 3$.

Proof. Suppose that $\dim Y=1$. Since Y is an irreducible non-degenerate curve in $\mathbb{P}^5$ we have $\deg Y\ge 5$. Let $L\subset\mathbb{P}^5$ be a generic hyperplane; since Y_0 contains an open dense subset of Y the intersection $Y_0\cap L$ consists of $\deg Y$ points, contradicting Item (1) of Corollary (4.2). Now suppose that $\dim Y=2$; since Y is an irreducible non-degenerate surface in $\mathbb{P}^5$ we have $\deg Y\ge 4$. Let $L\subset\mathbb{P}^5$ be a generic linear subspace of codimension 2; since Y_0 contains an open dense subset of Y the intersection $Y_0\cap L$ consists of $\deg Y$ points, contradicting Item (1) of Corollary (4.2). □

The following result is the first step towards the proof that the variety X satisfies (1) or (2) of Theorem (1.1).

Proposition 4.4. *Let* (X,H) *be as in Proposition (3.2) and* $Y\subset|H|^\vee\cong\mathbb{P}^5$ *be the image of* $f\colon X\dashrightarrow|H|^\vee$ *- see (4.0.8). One of the following holds:*

(1) $\dim Y=3$ *and* $3\le\deg Y\le 6$. *Furthermore if* $\dim Y=3$ *and* $\deg Y=6$ *then* B *is* 0*-dimensional.*

(2) $\dim Y=4$, $\deg Y=2$.

(3) $\dim Y=4$, $\deg Y=3$ *and* $\deg f=3$.

(4) $\dim Y=4$, $\deg Y=3$, $\deg f=4$ *and* $B=\emptyset$.

(5) $\dim Y=4$, $\deg Y=4$, $\deg f=3$ *and* $B=\emptyset$.

(6) There exists a regular anti-symplectic involution $\phi\colon X\to X$ *such that* $Y\cong X/\langle\phi\rangle$ *and the quotient map* $X\to X/\langle\phi\rangle$ *is identified with* $f\colon X\to Y$ *- in particular* f *is regular. The* (±1)*-eigenspaces of* $H^2(\phi)$ *are* $\mathbb{C}h$ *and* $h^\perp$ *respectively. The fixed locus of* ϕ *is a smooth irreducible Lagrangian surface* F *such that*

$$c_2(F)=192,\quad \mathcal{O}_F(2K_F)\cong\mathcal{O}_F(6H),\quad c_1(F)^2=360. \tag{4.0.19}$$

(7) $\dim Y = 4$, *$f: X \dashrightarrow Y$ is birational and* $6 \leq \deg Y \leq 12$.

The proof of the above proposition will be given after a series of preliminary results.

Proposition 4.5. *Keep notation as above and assume that* $\dim Y = 3$. *Then* $3 \leq \deg Y \leq 6$. *If* $\deg Y = 6$ *then the base-scheme B is* 0*-dimensional.*

Proof. Let $d := \deg Y$. Since Y is an irreducible non-degenerate 3-fold in $\mathbb{P}^5$ we have $3 \leq d$. Let's prove that $d \leq 6$. Let $L_1, L_2, L_3 \subset \mathbb{P}^5$ be generic linearly independent hyperplanes. Then the intersection $Y \cap L_1 \cap L_2 \cap L_3$ is transverse and it consists of d points $p_1, \ldots, p_d \in Y_0$. Let $\Gamma_{0,i} := f_0^{-1}(p_i)$ and Γ_i be its closure in X. Let $D_1, D_2, D_3 \in |H|$ correspond to L_1, L_2, L_3 via (4.0.4). Since L_1, L_2, L_3 are generic we may assume that D_1, D_2, D_3 are linearly independent and hence by Corollary (4.2) we have Decomposition (4.0.18). By our assumptions on $Y \cap L_1 \cap L_2 \cap L_3$ we have

$$\Gamma = \sum_{i=1}^{d} \Gamma_i. \tag{4.0.20}$$

By Item (3) of Proposition (3.2)

$$cl(\Gamma_i) = m_i h^3/6, \quad m_i \in \mathbb{N}_+. \tag{4.0.21}$$

Since the 1-cycle $[D_1 \cap D_2 \cap D_3]$ represents h^3 Equations (4.0.20) and (4.0.21) give that

$$12 = \int_X h^4 = \langle h, \sum_{i=1}^{d} \Gamma_i + \Sigma \rangle = 2 \sum_{i=1}^{d} m_i + \langle h, \Sigma \rangle \geq 2d + \langle h, \Sigma \rangle. \tag{4.0.22}$$

Since h is ample and Σ is effective we get that $d \leq 6$. Now assume that $d = 6$. Then $\langle h, \Sigma \rangle = 0$ and hence $\Sigma = 0$; by Item (2b) of Proposition (4.2) we get that $\dim B = 0$. □

Proposition 4.6. *Keep notation as above and assume that* $\dim Y = 4$. *Let* $D_1, D_2, D_3, D_4 \in |H|$ *be generic - in particular we may assume that* D_1, D_2, D_3 *are linearly independent. Let* Γ, Σ *be as in (4.0.18). Then*

$$\deg Y \cdot \deg f + \sum_{p \in B_{red}} mult_p(\Gamma \cdot D_4) + \int_\Sigma h = 12. \tag{4.0.23}$$

(Notice that the summation appearing in (4.0.23) is finite by Item (2a) of Corollary (4.2).)

Proof. Let $L_1, L_2, L_3, L_4 \subset \mathbb{P}^5$ be the hyperplanes corresponding respectively to $D_1, D_2, D_3, D_4 \in |H|$ - see (4.0.4). Then $L_1, \cdots, L_4$ are generic because $D_1, \ldots, D_4$ are generic. Let $Z \subset Y$ be the subset of points p such that $\dim \widetilde{f}^{-1}(p) > 0$; then Z is closed and $\dim Z \leq 2$. Of course $\dim \widetilde{f}(suppE) \leq \dim(suppE) = 3$. Furthermore $\dim Y = 4$ by hypothesis. Since $L_1, \ldots, L_4$ are generic we get that

$$\emptyset = L_1 \cap \cdots \cap L_4 \cap Z = L_1 \cap \cdots \cap L_4 \cap \widetilde{f}(E), \quad |L_1 \cap \cdots \cap L_4 \cap Y| < \infty. \tag{4.0.24}$$

Let $\pi\colon \widetilde{X} \to X$ be the blow-up of the base-schem B of $|H|$; it follows from (4.0.24) that the effective divisors $\widetilde{f}^*L_1, \ldots, \widetilde{f}^*L_4$ on $\widetilde{X}$ intersect properly and that the intersection is contained in the open subset X_0. Thus

$$\deg Y \cdot \deg f = \widetilde{f}^*L_1 \cdot \widetilde{f}^*L_2 \cdot \widetilde{f}^*L_3 \cdot \widetilde{f}^*L_4 = \sum_{p \in X_0} mult_p(D_1 \cdot \cdots \cdot D_4). \quad (4.0.25)$$

On the other hand by (4.0.18) we have

$$12 = \int_X h^4 = (\Gamma + \Sigma) \cdot H = \Gamma \cdot D_4 + \int_\Sigma h. \quad (4.0.26)$$

Furthermore the restrictions of Γ and $[D_1 \cap D_2 \cap D_3]$ to X_0 are equal. Thus

$$\Gamma \cdot D_4 = \sum_{p \in X_0} mult_p(D_1 \cdot D_2 \cdot D_3 \cdot D_4) + \sum_{p \in B_{red}} mult_p(\Gamma \cdot D_4). \quad (4.0.27)$$

Equation (4.0.23) follows from (4.0.25), (4.0.26) and (4.0.27). □

Corollary 4.7. *Assume that* $\dim Y = 4$. *Then*

$$\deg Y \cdot \deg f \leq 12 \quad (4.0.28)$$

with equality if and only if $B = \emptyset$.

Proof. Let $D_1, \ldots, D_4 \in |H|$ be generic. By Proposition (4.6) we have (4.0.23). Since H is ample $\int_\Sigma h \geq 0$ and hence (4.0.28) follows from (4.0.23). If $B = \emptyset$ then $B_{red} = \emptyset$ and furthermore $\Sigma = 0$ by Item (2b) of Corollary (4.2); thus (4.0.28) is an equality by (4.0.23). Assume that (4.0.28) is an equality. By (4.0.23) we have $\Sigma = 0$ and hence Item (2b) of Proposition (4.6) gives that $\dim B = 0$. From (4.0.18) we get that $supp\Gamma \supset B_{red}$. Since $supp D_4 \supset B_{red}$ it follows that every $p \in B_{red}$ is contained in $D_4 \cap \Gamma$. By (4.0.23) we get that $B = \emptyset$. □

Proposition 4.8. *Assume that* $f\colon X \dashrightarrow Y$ *is birational. Then* $6 \leq \deg Y \leq 12$.

Proof. From Corollary (4.7) we get that $\deg Y \leq 12$. One gets the lower bound $6 \leq \deg Y$ by adjunction. Explicitly, let $\widetilde{Y} \subset \widetilde{\mathbb{P}}^5$ be an embedded resolution of $Y \subset \mathbb{P}^5$: then

$$h^0(K_{\widetilde{Y}}) = 1 \quad (4.0.29)$$

because $\widetilde{Y}$ is birational to X. On the other hand by adjunction and vanishing of the Hodge numbers $h^{5,1}(\mathbb{P}^5), h^{5,0}(\mathbb{P}^5), h^{4,0}(\mathbb{P}^5)$ we get an isomorphism

$$H^0(K_{\widetilde{Y}}) = H^0(I_Z(\deg Y - 6)), \quad (4.0.30)$$

where $Z \subset \mathbb{P}^5$ is a subscheme supported on $singY$. From (4.0.29) we get that $6 \leq \deg Y$. □

Proposition 4.9. *Assume that* $\dim Y = 4$ *and that* $\deg f = 2$. *Then there exists a regular anti-symplectic involution* $\phi\colon X \to X$ *such that* $Y \cong X/\langle\phi\rangle$ *and the quotient map* $\rho\colon X \to X/\langle\phi\rangle$ *is identified with* $f\colon X \to Y$ *- in particular* f *is regular. The* (± 1)*-eigenspaces of* $H^2(\phi)$ *are* $\mathbb{C}h$ *and* $h^\perp$ *respectively. The fixed locus of* ϕ *is a smooth irreducible Lagrangian surface* F *such that*

$$c_2(F) = 192, \quad (4.0.31)$$
$$\mathcal{O}_F(2K_F) \cong \mathcal{O}_F(6H), \quad (4.0.32)$$
$$c_1(F)^2 = 360. \quad (4.0.33)$$

Proof. Since $f\colon X \dashrightarrow Y$ is generically a double cover it defines a birational involution $\phi\colon X \dashrightarrow X$. We claim that ϕ is regular: since $K_X \sim 0$ there exist closed subsets $I_1, I_2 \subset X$ of codimension at least 2 such that ϕ restricts to a regular map $(X \setminus I_1) \to (X \setminus I_2)$. Since $H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}h$ we have $\phi^* H \sim H$; it follows by a well-known argument (see [13]) that ϕ is regular. The map $f\colon X \dashrightarrow Y$ factors as

$$X \xrightarrow{\rho} X/\langle\phi\rangle \overset{\overline{f}}{\dashrightarrow} Y \qquad (4.0.34)$$

where ρ is the quotient map. Since $\deg f = 2$ we have $\deg \overline{f} = 1$, i.e. $\overline{f}$ is birational. We claim that

$$\deg Y = 6, \quad \overline{f} \text{ is regular}, \quad \dim(singY) \le 2. \qquad (4.0.35)$$

Let σ be a symplectic form on X: since $H^0(\Omega^2_X) = \mathbb{C}\sigma$ and since ϕ is an involution we have $\phi^*\sigma = \pm\sigma$ and hence $\phi^*(\sigma \wedge \sigma) = \sigma \wedge \sigma$. Thus if W is any desingularization of $X/\langle\phi\rangle$ we have $H^0(K_W) \ne 0$. Since $\overline{f}$ is birational we get that $H^0(K_{\widetilde{Y}}) \ne 0$ for any desingularization $\widetilde{Y} \to Y$. By (4.0.30) we get that $\deg Y \ge 6$, and hence Corollary (4.7) gives that $\deg Y = 6$ and that $B = \emptyset$. Since $B = \emptyset$ the map $\overline{f}$ is regular. Since $\deg Y = 6$ we get that $\dim(singY) \le 2$ - if $\dim(singY) = 3$ then $singY$ certainly "imposes conditions on adjoints". We have proved (4.0.35). Let's show that $\overline{f}$ is an isomorphism. The fibers of $\overline{f}$ are finite because $\overline{f}^*\mathcal{O}_Y(1)$ is ample, furthermore Y is normal because it is a hypersurface smooth in codimension 1: this implies that the regular birational map $\overline{f}$ is an isomorphism. Let $H^2_{\pm}(X) \subset H^2(X)$ be the (± 1)-eigenspace of $H^2(\phi)$ respectively. Then $\dim H^2_+(X)$ is equal to $h^2(Y)$, which is 1 by Lefschetz' Hyperplane Section Theorem: since h belongs to $H^2(\phi)_+$ we get that

$$H^2(\phi)_+ = \mathbb{C}h. \qquad (4.0.36)$$

Since ϕ preserves the Beauville-Bogomolov form $(,)_X$ we get that

$$H^2(\phi)_- = h^{\perp}. \qquad (4.0.37)$$

In particular ϕ is anti-symplectic. Let's prove that the fixed locus F has the stated properties. Since F is the fixed locus of an involution on a smooth manifold it is smooth. Since ϕ is anti-symplectic F has pure dimension equal to $\dim X/2 = 2$, and F is Lagrangian. Let's prove that F is irreducible. Let $F = \bigcup_{i \in I} F_i$ be the decomposition into irreducible components. For $i \in I$ let $cl(F_i) \in H^{2,2}_{\mathbb{Q}}(X)$ be the Poincaré dual of F_i; we claim that

$$cl(F_i) = k_i(15h^2 - c_2(X)), \quad k_i \in \mathbb{Q}_+. \qquad (4.0.38)$$

In fact since F_i is effective and Lagrangian we have

$$\int_X cl(F_i) \wedge h^2 > 0, \quad \int_X cl(F_i) \wedge \sigma \wedge \overline{\sigma} = 0. \qquad (4.0.39)$$

By Item (6) of Proposition (3.2) and by (3.0.45) we have

$$cl(F_i) = x_i h^2 + y_i c_2(X), \quad x_i, y_i \in \mathbb{Q}. \qquad (4.0.40)$$

Substituting the above expression for $cl(F_i)$ in (4.0.39) and applying (2.1)-(2.2.1) and (3.0.5) we get (4.0.38). We will be using the formula

$$\int_X (15h^2 - c_2(X))^2 = 1728 \tag{4.0.41}$$

which follows from (2.1)-(2.2.1) and (3.0.5). Now suppose that there exist two distinct irreducible components F_i, F_j of F. Then $F_i \cap F_j = \emptyset$ because F is smooth and hence by (4.0.38) we get that

$$0 = \int_X cl(F_i) \wedge cl(F_i) = k_i k_j \int_X (15h^2 - c_2(X))^2. \tag{4.0.42}$$

Thus $\int_X (15h^2 - c_2(X))^2 = 0$, and this contradicts (4.0.41); this proves that F is irreducible. Now let's prove (4.0.31). First we compute the Euler characteristic of Y. We have $b_i(Y) = \dim H^i(\phi)_+$. Thus $b_i(Y) = 0$ for odd i and $b_2(Y) = 1$ by (4.0.36). By (2.1.8) and (4.0.36)-(4.0.37) we get that $H^4(\phi)_+ = \mathbb{C}h^2 \oplus Sym^2(h^\perp)$ and hence $b_4(Y) = 254$. Thus

$$\chi(Y) = 258. \tag{4.0.43}$$

On the other hand the decompositions $X = (X \setminus F) \coprod F$ and $Y = (Y \setminus \rho(F)) \coprod \rho(F)$ give that

$$324 = \chi(X) = 2\chi(Y \setminus \rho(F)) + \chi(F) = 2\chi(Y) - \chi(F). \tag{4.0.44}$$

By (4.0.43) we get that $\chi(F) = 192$, and this proves (4.0.31). Before proving (4.0.32) we show that

$$cl(F) = 5h^2 - \frac{1}{3}c_2(X). \tag{4.0.45}$$

We have

$$\int cl(F) \wedge cl(F) = \int_F c_2(N_{F/X}) = \int_F c_2(\Omega_F^1) = 192, \tag{4.0.46}$$

where the second equality holds because F is Lagrangian and the third equality follows from (4.0.31); replacing $cl(F)$ by the right-hand side of (4.0.38) and using (4.0.41) one gets (4.0.45). Now let's prove (4.0.32). Let $F' := \rho(F)$; thus $\rho \colon F \to F'$ is an isomorphism. The embedding of $Y \cong (X/\langle\phi\rangle)$ into $\mathbb{P}^5$ defines by pull-back an isomorphism

$$\rho^* N^\vee_{F'/\mathbb{P}^5} \cong Sym^2(N^\vee_{F/X}). \tag{4.0.47}$$

Since F is Lagrangian in X we have $N^\vee_{F/X} \cong \Theta_F$; substituting in (4.0.47) and taking determinants we get an isomorphism

$$\rho^* \det(N_{F'/\mathbb{P}^5}) \cong \mathcal{O}_F(3K_F). \tag{4.0.48}$$

On the other hand the normal sequence for the embedding $F' \hookrightarrow \mathbb{P}^5$ gives

$$\det(N_{F'/\mathbb{P}^5}) \cong \mathcal{O}_{F'}(6) \otimes \mathcal{O}_{F'}(K_{F'}). \tag{4.0.49}$$

The restriction of ρ to F defines an isomorphism $F \xrightarrow{\sim} F'$ and furthermore $(\rho|_F)^*\mathcal{O}_{F'}(1) \cong \mathcal{O}_F(H)$; thus

$$(\rho|_F)^* \det(N_{F'/\mathbb{P}^5}) \cong \mathcal{O}_F(6H) \otimes \mathcal{O}_F(K_F). \tag{4.0.50}$$

The above isomorphism together with (4.0.48) gives (4.0.32). Finally to get (4.0.33) use (4.0.32) and (4.0.45) together with (2.1)-(2.2.1) and (3.0.5). □

Proof of Proposition (4.4). By Corollary (4.3) we have $\dim Y \geq 3$ and of course $\dim Y \leq 4$. If $\dim Y = 3$ then Item (1) holds by Proposition (4.5). Now assume that $\dim Y = 4$. If $\deg f = 1$ then Item (7) holds by Proposition (4.8). If $\deg f = 2$ then Item (6) holds by Proposition (4.9). Thus we may assume that $\deg f \geq 3$. By Corollary (4.7) we get that $\deg Y \leq 4$, and of course $\deg Y \geq 2$ because Y is a non-degenerate hypersurface. If $\deg Y = 2$ then Item (2) holds. If $\deg Y = 3$ then by Corollary (4.7) either Item (3) or Item (4) holds. If $\deg Y = 4$ then Item (5) holds by Corollary (4.7). □

We remark that the (X, H) satisfying Item (6) of Proposition (4.4) are stable under small deformations. More precisely let X be a numerical $(K3)^{[2]}$ and suppose that there exist an anti-symplectic involution $\phi\colon X \to X$ with quotient map $\rho\colon X \to X/\langle\phi\rangle =: Y$ and an embedding $j\colon Y \hookrightarrow \mathbb{P}^5$ with $j(Y)$ a sextic hypersurface. Let $f := j \circ \rho$ and $H = c_1(f^*\mathcal{O}_Y(1)) \in Pic(X)$.

Proposition 4.10. *Let (X', H') be a small deformation of (X, H). Then ϕ deforms to an involution $\phi'\colon X' \to X'$. Let $\rho'\colon X' \to X'/\langle\phi'\rangle =: Y'$ be the quotient map; there is an embedding $j'\colon Y' \hookrightarrow \mathbb{P}^5$ which deforms $j\colon Y \hookrightarrow \mathbb{P}^5$. Furthermore $H' = c_1((f')^*\mathcal{O}_{Y'}(1))$.*

Proof. Let $h := c_1(H)$. Since $j(Y)$ is a sextic and $\deg f = 2$ we have $\int_X h^4 = 12$. By Remark (2.1) and Equation (2.2.1) we get that $(h, h) = 2$. The invariant subspace $H^2(X)_+ \subset H^2(X)$ for the action of $H^2(\phi)$ contains h and has rank 1 because $H^2(Y)$ has rank 1; thus $H^2(X)_+ = \mathbb{C}h$. It follows that $H^2(\phi)$ is the reflection in the span of h. The result then follows from Proposition (3.3) of [20]. (Notice that in that proposition we assume that X is a deformation of $(K3)^{[n]}$ however the proof is valid for any irreducible symplectic manifold satsifying the hypothesis of Proposition (3.3) of [20] because all we need is Corollary (3.2) of [20] which is valid for an arbitrary irreducible symplectic manifold.) □

Of course also the (X, H) satisfying Item (7) of Proposition (4.4) are stable under small deformations.

5 Proof of Theorem (1.1)

We will prove that Items (1) through (5) of Proposition (4.4) do not hold. Subsections (5.1), (5.2), (5.3), (5.5), (5.6) are devoted to the proof that (1), (2), (3), (4), (5) of Proposition (4.4) respectively do not hold. Subsection (5.4) is preliminary to Subsection (5.5) and contains results on cubic hypersurfaces in $\mathbb{P}^5$.

5.1 (1) of Proposition (4.4) does not hold

We will prove the following result.

Proposition 5.1. *Let $Y \subset \mathbb{P}^5$ be an irreducible non-degenerate linearly normal 3-dimensional subvariety of degree at most 6.*

(1) If $\deg Y \leq 5$ then given an arbitrary non-empty subset $U \subset Y$ there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap U$ is reducible.

(2) If $\deg Y = 6$ then there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y$ is not reduced or not irreducible.

Let's grant the above proposition for the moment and prove that (1) of Proposition (4.4) does not hold. The proof is by contradiction. First assume that (1) of Proposition (4.4) holds with $\deg Y \leq 5$. Clearly Y is irreducible non-degenerate and linearly normal. Let $U \subset Y$ be the interior of Y_0; then U is non-empty and hence by Proposition (5.1) there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap U$ is reducible. Since U is open in Y_0 we get that $L \cap Y_0$ is reducible; that contradicts Item (1) of Corollary (4.2). This proves that (1) of Proposition (4.4) with $\deg Y \leq 5$ does not hold. In order to show that the remaining case, i.e. $\deg Y = 6$, does not hold we first prove the following result.

Claim 5.2. *Suppose that (1) of Proposition (4.4) holds with $\deg Y = 6$. Then $Y_0 = Y$.*

Proof. By Item (1) of Proposition (4.4) we know that $\dim B = 0$. Let n be such that nH is very ample and let $D \in |nH|$ be generic; in particular since $\dim B = 0$ we have $D \subset (X \setminus B) = X_0$. Let f_0 be the map of (4.0.9); it suffices to show that

$$f_0(D) = Y. \tag{5.1.1}$$

Since $\dim Y_0 = 3$ the generic fiber of f_0 is 1-dimensional and hence its intersection with D consists of a finite set of points. Thus $f_0(D)$ is 3-dimensional. Since $f_0(D)$ is closed in Y and Y is irreducible of dimension 3 we get (5.1.1). □

Now assume that (1) of Proposition (4.4) holds with $\deg Y = 6$; we will arrive at a contradiction. By Claim (5.2) we have $Y_0 = Y$. Since Y is irreducible non-degenerate and linearly normal Proposition (5.1) applies and we get that there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y_0$ is not reduced or not irreducible, contradicting Item (1) of Corollary (4.2).

Proof of Proposition (5.1). The proof consists in a case-by-case analysis. We classify the 3-fold Y according to $\deg Y$ and the nature of $singY$.

Assume that Y is a cone: We have $Y = J(p, \overline{Y})$ where $\overline{Y}$ is a surface with $\dim(span\overline{Y}) = 4$. (See (1.0.4) for the notation $J(\cdot, \cdot)$.) Let $\overline{L} \subset (span\overline{Y})$ be a linear subspace of dimension 2. Then $L := J(p, \overline{L})$ is a 3-dimensional linear subspace of $\mathbb{P}^5$ and

$$L \cap Y = J(p, \overline{L} \cap \overline{Y}). \tag{5.1.2}$$

If $\overline{L}$ is generic then $L \cap U$ has $\deg \overline{Y}$ irreducible components - they are open dense subsets of lines through p. Since $\deg \overline{Y} = \deg Y \geq 3$ we get that $L \cap Y$ is reducible. This proves the proposition for Y a cone.

Assume that $\deg Y \leq 5$, Y *is singular and not a cone:* Let $p \in sing(Y)$ and let m be its multiplicity. Let $A \subset \mathbb{P}^5$ be a hyperplane not containing p and let

$$\rho\colon (Y \setminus \{p\}) \to A \tag{5.1.3}$$

be projection from p. Let $Z := Im(\rho)$ and let $\overline{Z}$ be its closure. Since Y is not a cone $\overline{Z}$ is a hypersurface with $\deg \overline{Z} = (\deg Y - m)$. Thus $\overline{Z}$ is a hypersurface in $A \cong \mathbb{P}^4$ of degree at most 3 and hence it is covered by lines. The image $\rho(U \setminus \{p\}) \subset \overline{Z}$ contains an open dense $V \subset Z$. Let $\ell \subset \overline{Z}$ be a generic line: then $\ell \cap V$ is dense in ℓ. Let $q \in (V \setminus \ell)$ be generic and let $\overline{L} := J(q, \ell)$. Thus $\overline{L} \subset A$ is a plane and

$$\overline{L} \cap V = (\ell \cap V) \cup C \tag{5.1.4}$$

where C is an open dense subset of a line or of a conic. (Notice that $\overline{L} \not\subset \overline{Z}$ because ℓ and q are generic in $\overline{Z}$.) Let $L := J(p, \overline{L})$; this is a 3-dimensional linear subspace of $\mathbb{P}^5$. We have

$$L \cap (\rho^{-1} V) = \rho^{-1}(\overline{L} \cap V) \tag{5.1.5}$$

and hence $L \cap (\rho^{-1} V)$ is reducible because of (5.1.4). Since $\rho^{-1} V$ is an open subset of U we get that $L \cap U$ is reducible.

Assume that $\deg Y \leq 5$ *and* Y *is smooth:* All smooth non-degenerate linearly normal 3-folds $Y \subset \mathbb{P}^5$ of degree at most 5 have been classified, see [15]: Y is the Segre 3-fold i.e. $\mathbb{P}^1 \times \mathbb{P}^2$ embedded by $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$, or a complete intersection of two quadric hypersurfaces, or a quadric fibration, i.e. it fibers over $\mathbb{P}^1$ with fibers which are embedded quadric surfaces. In each case Y is covered by lines; it follows immediately that Item (1) of Proposition (5.1) holds for Y.

It remains to prove that Proposition (5.1) holds for Y of degree 6. Before going into the case-by-case analysis we state the following elementary result.

Claim 5.3. *Suppose that* $Y \subset \mathbb{P}^5$ *is an irreducible* 3*-dimensional non-degenerate subvariety containing a plane curve. Then there exists a* 3*-dimensional linear subspace* $L \subset \mathbb{P}^5$ *such that* $L \cap Y$ *is reducible.*

Proof. Let $C \subset Y$ be a plane curve and $\Lambda := span(C)$. Thus $\dim \Lambda = 1$ if C is a line and $\dim \Lambda = 2$ otherwise. Let $L \subset \mathbb{P}^5$ be a generic 3-dimensional linear subspace containing Λ. Then $L \cap Y \not\subset \Lambda$ and every irreducible component of $L \cap Y$ has dimension at least 1. Thus $L \cap Y$ is reducible unless possibly if each of its irreducible components not contained in Λ has dimension at least 2: thus we may assume that

$$\dim((L \cap Y) \setminus \Lambda) \geq 2 \text{ for all } \Lambda \subset L \subset \mathbb{P}^5,\ L \text{ linear},\ \dim L = 3. \tag{5.1.6}$$

Let $\Omega \subset \mathbb{P}^5$ be a linear subspace complementary to Λ, i.e. such that $\Omega \cap \Lambda = \emptyset$ and $\dim \Omega + \dim \Lambda = 4$. Let

$$\rho\colon (Y \setminus \Lambda) \to \Omega \tag{5.1.7}$$

be the projection from $Y \cap \Lambda$. Let $Z := Im(\rho)$ and $\overline{Z}$ be its closure. Since Y is irreducible $\overline{Z}$ is irreducible and since Y is non-degenerate $\overline{Z}$ is non-degenerate. By (5.1.6) we have $\dim Z \leq 1$ and hence $\overline{Z}$ is an irreducible non-degenerate curve in Ω. Thus $\deg \overline{Z} \geq 2$ and since Z is open dense in $\overline{Z}$ there exists a hyperplane $D \subset \Omega$ such that $D \cap Z$ is finite with $|D \cap Z| = \deg \overline{Z} \geq 2$. The hyperplane $J(\Lambda, D) \subset \mathbb{P}^5$ intersects Y in a reducible surface - in fact $J(\Lambda, D) \cap Y$ has at least $\deg \overline{Z}$ irreducible components. Let $L_1 \subset \mathbb{P}^5$ be a generic hyperplane and $L := L_1 \cap J(\Lambda, D)$; then $L \cap Y$ is reducible. $\square$

Assume that $\deg Y = 6$ *and* $\dim(singY) = 2$*:* Let $V \subset singY$ be a 2-dimensional component. We claim that

$$\deg V \leq 4. \tag{5.1.8}$$

In fact let $\Sigma \subset \mathbb{P}^5$ be a generic 3-dimensional linear subspace: then

$$sing(\Sigma \cap Y) = \Sigma \cap singY \supset \Sigma \cap V \tag{5.1.9}$$

and $|\Sigma \cap V| = \deg V$. Now $\Sigma \cap Y$ is an irreducible non-degenerate degree-6 curve in $\Sigma \cong \mathbb{P}^3$ and hence it has at most 4 singular points. Thus (5.1.8) follows from (5.1.9). A straightforward argument shows that any surface V of degree at most 4 contains a plane curve. Explicitely: If $\dim(spanV) = 2$ there is nothing to prove. If $\dim(spanV) \leq 3$ intersect V with a plane contained in $span(V)$. If $\dim(spanV) \geq 4$ and V is singular the projection of V from $q \in (singV)$ is a quadric surface Q; if $\ell \subset Q$ is a line the intersection $J(q, \ell) \cap V$ has dimension 1. If $\dim(spanV) \geq 4$ and V is smooth then (see [15]) V is a rational scroll, a complete intersection of quadric hypersurfaces in a hyperplane of $\mathbb{P}^5$ or the Veronese surface. In the first two cases V contains lines, in the third case it contains conics. Thus we proved that V contains a plane curve. Since $V \subset Y$ we get that Y contains a plane curve and hence we are done by Claim (5.3).

Assume that $\deg Y = 6$ *and* $\dim(singY) = 1$*:* Let $W \subset (singY)$ be a 1-dimensional component. If $\dim(spanW) \leq 2$ then Y contains a plane curve and we are done by Claim (5.3). Now assume that $\dim(spanW) = 3$. If $\dim((spanW) \cap Y) = 2$ then Y contains plane curves and we are done by Claim (5.3). If $\dim((spanW) \cap Y) = 1$ let $L := spanW$; since Y is singular along W the intersection $L \cap Y$ is not reduced along W. Finally assume that $\dim(spanW) \geq 4$. Then $\dim((spanW) \cap Y) \geq 2$ and hence there exists $p \in ((spanW) \cap (Y \setminus W))$. Since curves are never defective (see [4]) there exists a 3-secant plane of W containing p, call it Ω. We claim that $\dim(\Omega \cap Y) \geq 1$. In fact assume that this is not the case; then $\dim(\Omega \cap Y) = 0$ and hence by Bezout's Theorem

$$\sum_x mult_x(\Omega \cdot Y) = 6. \tag{5.1.10}$$

If $x \in \Omega \cap W$ then $mult_x(\Omega \cdot Y) \geq 2$ because $W \subset (singY)$. Since Ω is 3-secant to W we get that the points in $\Omega \cap W$ give a contribution of at least 6 to the left-hand side of (5.1.10). On the other hand we have an extra contribution of at least 1 from p, and hence we get that the left-hand side of (5.1.10) is at least 7; that contradicts (5.1.10) and hence we get that $\dim(\Omega \cap Y) \geq 1$. Since Ω is a plane we get that Y contains a plane curve and thus we are done by Claim (5.3).

Assume that $\deg Y$ *and* $\dim(singY) \leq 0$*:* Let $\Lambda \subset \mathbb{P}^5$ be a generic hyperplane; thus $S := \Lambda \cap Y$ is an irreducible smooth non-degenerate (in Λ!) surface of degree 6. Since $\deg(S) \neq 4$ we know that S is linearly normal (Severi) and hence we may invoke the classification of irreducible smooth non-degenerate degree-6 surfaces in $\mathbb{P}^4$ (see [15]): S is either the complete intersection of a quadric and a cubic or a Bordiga surface i.e. the blow up of $\mathbb{P}^2$ at 10 points embedded by the linear system of plane quartics through the 10 points. Assume that S is the complete intersection of a quadric and a cubic. Then since Y is linearly normal the quadric hypersurface in Λ containing S lifts to a quadric hypersurface $Q \subset \mathbb{P}^5$ containing Y. There exist 3-dimensional linear spaces $L \subset \mathbb{P}^5$ such that $L \cap Q$ is the union of 2 planes; if L is a generic such space then $L \cap Y$ is reducible. Now assume that S is a Bordiga surface. Then S contains lines, namely the image of the 10 exceptional lines; thus Proposition (5.1) holds for Y by Claim (5.3). $\square$

5.1.1 Comment

One may ask the following: does there exist a numerical $(K3)^{[2]}$ with an ample H with $(c_1(H), c_1(H)) = 2$ and $Y := Im(f\colon X \dashrightarrow |H|^\vee)$ of dimension strictly smaller than 4? We do not know of any such example however we do have examples with H big and nef such that $\dim Y < \dim X$. (Couples (X, H) with H a big and nef divisors will be needed in order to construct complete moduli spaces.) An explicit example is the following. Let $\pi\colon S \to \mathbb{P}^2$ be a double cover ramified over a smooth sextic; thus S is a $K3$ surface. Let $H_S := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and let $X := M(0, H_S, 0)$ be the Moduli space of H_S-semistable rank-0 pure sheaves G on S with $c_1(G) = c_1(H_S)$ and $\chi(G) = 0$: a typical G is given by $\iota_*\xi$ where $\iota\colon C \hookrightarrow S$ is the inclusion of a curve $C \in |H_S|$ and ξ is a degree-1 line-bundle on C. It is known that X is a deformation of $(K3)^{[2]}$ - see [24]. There is a Lagrangian fibration $\rho\colon X \to |H_S|$ mapping $[G] \in M(0, H_S, 0)$ to its support; the fiber over $C \in |H_S|$ is $Jac^1(C)$ (suitably defined if C is singular). Thus on X we have the divisor class $F := \rho^*\mathcal{O}_{|H_S|}(1)$. We also have a unique effective divisor A on X whose restriction to any Lagrangian fiber $\rho^{-1}([C]) \cong Jac^1(C)$ is the canonical Θ-divisor. Let $H := A + 2F$; a straightforward argument shows that $(c_1(H), c_1(H)) = 2$. One can also show that H is nef; since $\int_X c_1(H)^4 = 12$ we get that H is big. The image $Y = Im(f\colon X \dashrightarrow |H|)$ is the Veronese surface in $\mathbb{P}^5$.

5.2 (2) of Proposition (4.4) does not hold

Let's assume that $Y \subset \mathbb{P}^5$ is a quadric hypersurface. Of course Y is irreducible. Since Y_0 contais an open dense subset of the irreducible quadric 4-fold Y there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y_0$ is reducible; that contradicts Item (1) of Corollary (4.2) and hence we get that Y can not be a quadric hypersurface.

5.2.1 Comment

There exist examples (X, H) with X a deformation of $(K3)^{[2]}$ and H an ample divisors with $(c_1(H), c_1(H)) = 2$ such that Y is a quadric hypersurface - see (4.1) of [20].

5.3 (3) of Proposition (4.4) does not hold

First we will analyze the base-scheme B of $|H|$ under the hypothesis that $\dim Y = 4$ and $(\deg f \cdot \deg Y) \geq 9$. As a consequence we will get that under these hypotheses the hypersurface $Y \subset \mathbb{P}^5$ contains a 3-dimensional linear subspace. Then we will concentrate on the case when $\deg f = \deg Y = 3$. First we will prove an elementary result on 4-dimensional cubic hypersurfaces containing a 3-dimensional linear subspace, and then we will prove that (3) of Proposition (4.4) does not hold.

Proposition 5.4. *Let $f\colon X \dashrightarrow Y$ be as in Proposition (4.4). Assume that*

$$\dim Y = 4, \tag{5.3.1}$$

$$\deg Y \cdot \deg f \geq 9. \tag{5.3.2}$$

Then the following hold:

(1) Let B_i be a 0-dimensional connected component of the base-scheme B of $|H|$; then B_i is curvilinear, i.e. it is contained in a curve which is smooth at the point $(B_i)_{red}$.

(2) Assume that $\dim B > 0$. Then

$$\deg Y \cdot \deg f = 9 \tag{5.3.3}$$

and B is an irreducible and reduced l.c.i. curve. If $D_1, D_2, D_3 \in |H|$ are generic - in particular linearly independent - and Γ, Σ are as in (4.0.18) then $\Sigma = [B]$ and

$$|supp\Gamma \cap B| = 1. \tag{5.3.4}$$

Letting $\{p\} = supp\Gamma \cap B$ there is a unique irreducible component of $supp\Gamma$ containing p, call it Γ_p, and it appears with multiplicity 1 in Γ. Furthermore both Γ_p and B are smooth at p, and they have distinct tangent spaces at p.

Proof. Throughout the proof we let $D_1, \ldots, D_4 \in |H|$ be generic and Γ, Σ be as in (4.0.18). By (4.0.23) and (5.3.2) we have

$$\sum_{p \in B_{red}} mult_p(D_4 \cdot \Gamma) + \int_{[\Sigma]} h = 12 - \deg Y \cdot \deg f \leq 3. \tag{5.3.5}$$

Let's prove Item (1). Let $p_i := (B_i)_{red}$. Of course $p_i \in (supp\Gamma \cup supp\Sigma)$, and since B_i is a 0-dimensional connected component of B we get by Item (2b) of Corollary (4.2) that

$$p_i \in (supp\Gamma \setminus supp\Sigma). \tag{5.3.6}$$

Since $[D_1 \cap D_2 \cap D_3] = \Gamma$ outside Σ we get from (5.3.6) and (5.3.5) that $D_1, \ldots, D_4$ intersect properly at p_i and that

$$mult_{p_i}(D_1 \cdot D_2 \cdot D_3 \cdot D_4) \leq 3. \tag{5.3.7}$$

We claim that the connected component of $D_1 \cap \cdots \cap D_4$ supported at p_i is curvilinear, i.e. that

$$c := \dim \left(\bigcap_{j=1}^{4} \Theta_{p_i} D_j \right) \leq 1. \tag{5.3.8}$$

There exist $D'_1, \ldots, D'_4 \in |H|$ which span the same linear system as $D_1, \ldots, D_4$ and such that $D'_1, \ldots, D'_c$ are singular at p_i. By (5.3.7) we get that

$$3 \geq mult_{p_i}(D_1 \cdot D_2 \cdot D_3 \cdot D_4) = mult_{p_i}(D'_1 \cdot D'_2 \cdot D'_3 \cdot D'_4) \geq 2^c. \tag{5.3.9}$$

Thus $c \leq 1$ and this proves (5.3.8). Since B_i is a subscheme of the connected component of $D_1 \cap \cdots \cap D_4$ supported at p_i and the latter is curvilinear we get that B_i is curvilinear. This proves Item (1). Now we prove Item (2). Assume that $\dim B > 0$. By Item (2) of Corollary (4.2) we have $\dim B = 1$ and

$$supp\Sigma = B^1_{red}. \tag{5.3.10}$$

Thus $\Sigma \neq 0$ and since H is ample we get that $\int_\Sigma h > 0$. By Item (3) of Proposition (3.2) we have $cl(\Sigma) = mh^3/6$ for some $m \in \mathbb{Z}$ and hence $m \in \mathbb{N}_+$. Thus $\int_\Sigma h = 2m > 0$ and by (5.3.5) we get that

$$\int_\Sigma h = 2. \tag{5.3.11}$$

Hence by Item (3) of Proposition (3.2) we get that

$$cl(\Sigma) = h^3/6. \tag{5.3.12}$$

Let's prove that (5.3.3) holds. We notice that $\Sigma \neq 0$ by (5.3.11) and that $\Gamma \neq 0$ because D_1, D_2, D_3 are generic. Since H is ample the 1-cycle $\Gamma + \Sigma$ is connected and hence there exists $p \in (supp\Gamma \cap supp\Sigma)$. By (5.3.10) we have $p \in B^1_{red}$. Since B is the base-scheme of $|H|$ we get that $p \in D_4$. Thus $p \in (B^1_{red} \cap \Gamma \cap D_4)$ and hence

$$\sum_{p \in B^1_{red}} mult_p(D_4 \cdot \Gamma) \geq 1. \tag{5.3.13}$$

This together with (5.3.11) and (5.3.5) gives that $\deg Y \cdot \deg f \leq 9$; by hypothesis $\deg Y \cdot \deg f \geq 9$ and hence (5.3.3) holds. From (5.3.5)-(5.3.11) we get that

$$\sum_{p \in B_{red}} mult_p(D_4 \cdot \Gamma) = 1, \tag{5.3.14}$$

and hence (5.3.13) gives that $B^1_{red} = B_{red}$ and that (5.3.4) holds. In particular we have proved that B is purely 1-dimensional. On the other hand (5.3.11) together with Item (3) of Proposition (3.2) gives that

$$\Sigma \text{ is reduced and irreducible,} \tag{5.3.15}$$

that is Σ is the fundamental cycle of an irreducible curve in X; by (5.3.10) we get that B is irreducible. Now choose $p_0 \in B_{red}$; let's prove that there exists a generic $(D_1, \ldots, D_4) \in |H|^4$ such that if Γ is as in (4.0.18) then

$$p_0 \notin supp\Gamma. \tag{5.3.16}$$

First we may assume that the open dense $\mathcal{U}_{gen} \subset |H|^4$ parametrizing generic $(D_1, \ldots, D_4)$ is invariant under permutation of the factors. Let $(D_1, \ldots, D_4) \in \mathcal{U}_{gen}$; since $B \subset (D_1 \cap \cdots \cap D_4)$ and B is purely 1-dimensional we have

$$\dim V_{p_0}(D_1 \cap \cdots \cap D_4) := \dim(\bigcap_{i=1}^4 \Theta_{p_0} D_i) \geq 1. \tag{5.3.17}$$

Thus there exist 3 of the D_i's such that the intersection of their tangent spaces at p_0 equals $V_{p_0}(D_1 \cap \cdots \cap D_4)$. By invariance of $\mathcal{U}_{gen}$ under permutation of the factors we may assume that

$$\bigcap_{i=1}^{3} \Theta_{p_0} D_i = V_{p_0}(D_1 \cap \cdots \cap D_4). \tag{5.3.18}$$

Now assume that $p_0 \in supp\Gamma$; since $supp\Gamma \subset (D_1 \cap \cdots \cap D_4)$ we get that

$$\Theta_{p_0} supp\Gamma \subset V_{p_0}(D_1 \cap \cdots \cap D_4) \subset \Theta_{p_0} D_4. \tag{5.3.19}$$

Thus $mult_{p_0}(D_4 \cdot \Gamma) \geq 2$; that contradicts (5.3.14) and hence proves that there exists $(D_1, \ldots, D_4) \in \mathcal{U}_{gen}$ such that (5.3.16) holds. Let's prove that B is a reduced l.c.i. curve. Let $p_0 \in B_{red}$ and let $(D_1, \ldots, D_4) \in \mathcal{U}_{gen}$ be such that (5.3.16) holds. Then in a neighborhood of p_0 we have $\Sigma = [D_1 \cap D_2 \cap D_3]$ and the multiplicity of intersection of D_1, D_2, D_3 along Σ is 1 by (5.3.15). Thus in a neighborhhod of p_0 the scheme $D_1 \cap D_2 \cap D_3$ is a l.c.i. which is generically reduced; since it is a l.c.i. it has no embedded components and hence we get that it is reduced. Of course $B \subset (D_1 \cap D_2 \cap D_3)$ and by (5.3.10) we get that B is also locally around p_0 the complete intersection of D_1, D_2, D_3 and that it is reduced. This proves that B is a reduced l.c.i. curve. Now let $D_1, D_2, D_3 \in |H|$ be generic. We have proved that (5.3.4) holds; let $supp\Gamma \cap B = \{p\}$. It remains to prove that there is a unique irreducible component Γ_p of $supp\Gamma$ containing p, that it appears with multiplicity 1 in Γ and that the tangent spaces to Γ_p and B at p are distinct. Since D_1, D_2, D_3 are generic the point p is a generic point of B by (5.3.16) and hence B is smooth at p. Since D_1, D_2, D_3 are generic there exists D_4 such that (D_1, D_2, D_3, D_4) is generic. All the statements that reamain to be proved follow from (5.3.14). □

Corollary 5.5. *Let $f\colon X \dashrightarrow Y$ be as in Proposition (4.4). Assume that (5.3.1) and (5.3.2) hold. Then $Y \subset \mathbb{P}^5$ contains a 3-dimensional linear subspace.*

Proof. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of B and $\widetilde{f}\colon \widetilde{X} \to \mathbb{P}^5$ be the resolution of indeterminacise of f. First assume that B contains a 0-dimensional connected component B_i. Then B_i is curvilinear by Item (1) of Proposition (5.4), in particular B_i is a l.c.i. and hence $E_i := \pi^{-1}(B_i) \cong \mathbb{P}^3$ and $\widetilde{f}^*\mathcal{O}_Y(1) \cong \mathcal{O}_{E_i}(1)$. Thus $\widetilde{f}(E_i) \subset Y$ is a 3-dimensional linear space. Now assume that B does not contain 0-dimensional connected components. By Item (2) of Proposition (5.4) the base scheme B is an irreducible, reduced 1-dimensional l.c.i. Thus the exceptional divisor of π is an irreducible divisor E. Let $L_1, L_2, L_3 \subset \mathbb{P}^5$ be generic hyperplanes; it follows from Item (2) of Proposition (5.4) that

$$|\widetilde{f}(E) \cap L_1 \cap L_2 \cap L_3| = 1. \tag{5.3.20}$$

Thus $\widetilde{f}(E) \subset Y$ is a 3-dimensional linear space. □

Before applying the above corollary to our case we prove an elementary result on 4-dimensional cubic hypersurfaces.

Proposition 5.6. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface containing a 3-dimensional linear space Ω. There exists a hyperplane $Z \subset \mathbb{P}^5$ containing Ω such that $Z \cap Y$ is swept out by planes, i.e. either $Z \subset Y$ or $Z \cdot Y = \Omega + Q$ where $Q \subset Z$ is a singular quadric hypersurface.*

Proof. Let $I \subset \mathbb{G}r(3,\mathbb{P}^5) \times |\mathcal{O}_{\mathbb{P}^5}(3)| \times (\mathbb{P}^5)^\vee$ be the set of triples (Ω, Y, Z) where $\Omega \subset Y$ and $\Omega \subset Z$. Let $J \subset \mathbb{G}r(3,\mathbb{P}^5) \times |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the set of couples (Ω, Y) where $\Omega \subset Y$ and let

$$\begin{array}{ccc} I & \xrightarrow{\rho} & J \\ (\Omega, Y, Z) & \mapsto & (\Omega, Y) \end{array} \tag{5.3.21}$$

be the forgetful map. Let $I^0 \subset I$ be the (open) subset of triples (Ω, Y, Z) such that $Z \cdot Y = \Omega + Q$ with $Q \subset Z$ a smooth quadric hypersurface. We must show that $\rho(I \setminus I^0) = J$. The map ρ is proper and surjective with 1-dimensional fibers, J is irreducible and $(I \setminus I^0)$ is closed of pure codimension 1 in I; thus it suffices to exhibit one couple $(\Omega, Y) \in J$ such that

$$\rho^{-1}(\Omega, Y) \cap I^0 \neq \emptyset, \quad \rho^{-1}(\Omega, Y) \cap (I \setminus I^0) \neq \emptyset. \tag{5.3.22}$$

Let $[X_0, \ldots, X_5]$ be homogeneous coordinates on $\mathbb{P}^5$. Let $\Omega = V(X_4, X_5)$ and $Y = V(F \cdot X_4 + G \cdot X_5)$ where $F, G \in \mathbb{C}[X_0, \ldots, X_5]_2$ are such that $F(X_0, \ldots, X_4, 0)$ and $G(X_0, \ldots, X_3, 0, X_5)$ are quadratic forms of rank 4 and 5 respectively. Then $(\Omega, Y) \in J$ and (5.3.22) holds: in fact $(\Omega, Y, V(X_4)) \in (\rho^{-1}(\Omega, Y) \cap I^0)$ and $(\Omega, Y, V(X_5)) \in (\rho^{-1}(\Omega, Y) \cap (I \setminus I^0))$. □

Proof that Item (3) of Proposition (4.4) does not hold. By contradiction. Assume that Item (3) of Proposition (4.4) holds. By Corollary (5.5) we get that Y contains a 3-dimensional linear space Ω. By Proposition (5.6) there exists a hyperplane $Z \subset \mathbb{P}^5$ containing Ω such that $Z \cap Y$ is swept out by planes. We claim that $Z \cap Y_0 \neq \emptyset$. In fact let E be the exceptional divisor of the blow-up $\pi \colon \widetilde{X} \to X$ of B and $\widetilde{f} \colon \widetilde{X} \to \mathbb{P}^5$ be the resolution of indeterminacies of f. If $Z \cap Y_0 = \emptyset$ then $supp(\widetilde{f}^* Z) \subset supp(E)$ and that is absurd - an ample divisor cannot be linearly equivalent to a divisor supported on components of an exceptional divisor. Let $y \in Z \cap Y_0$; by Proposition (5.6) there exists a plane $\Lambda \subset (Z \cap Y)$ with $y \in \Lambda$. Now let $y' \in (Y_0 \setminus Z)$ and let $L \subset \mathbb{P}^5$ be the 3-dimensional linear space $L := J(y', \Lambda)$. Then

(a) either $L \subset Y$, or

(b) $L \cap Y = \Lambda \cup \Gamma$ where $\dim \Gamma = 2$ and $\Gamma \ni y'$.

If Item (a) holds then $L \cap Y_0$ is non-empty 3-dimensional (notice: we do not know whether our "original"3-dimensional linear space $\Omega \subset Y$ intersects Y_0) and if Item (b) holds then $L \cap Y_0$ is reducible. In either case we contradict Item (1) of Corollary (4.2). □

5.4 Cubic 4-folds that do not contain planes

We will prove two propositions on cubic hypersurfaces in $\mathbb{P}^5$. These results will be the key ingredients in the proof that (4) of Proposition (4.4) does not hold - see Subsection (5.5). In order to state our first result we introduce some notation.

Definition 5.7. *Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface and $p \in sing(Y)$. We let $S_p \subset \mathbb{G}r(1,\mathbb{P}^5)$ be the subscheme parametrizing lines contained in Y and containing p.*

Thus the associated reduced variety $(S_p)_{red}$ is given by

$$(S_p)_{red} := \{\ell \in \mathbb{G}r(1,\mathbb{P}^5) |\ p \in \ell \subset Y\}. \tag{5.4.1}$$

The scheme structure on S_p is defined as follows. Let $\mathcal{F}$ be the tautological globally generated rank-2 vector-bundle on $\mathbb{G}r(1,\mathbb{P}^5)$. The Fano scheme of lines on Y is the zero-scheme $F(Y)$ of the section of $Sym^3\mathcal{F}$ defined by Y; we let S_p be the scheme-theoretic intersection of $F(Y)$ and the Schubert variety of lines containing p. The first result on 4-dimensional cubic hypersurfaces that we will prove in this subsection is the following.

Proposition 5.8. *Suppose that $Y \subset \mathbb{P}^5$ is a cubic hypersurface that contains no planes. Then either Y is smooth or the following holds.*

(a) The cubic Y has isolated quadratic singularities.

(b) If $p \in singY$ then S_p is a reduced and irreducible normal surface with du Val singularities[1] *and the minimal desingularization of S_p is a $K3$ surface $\widetilde{S}_p$.*

(c) If $p, q \in singY$ the surfaces $\widetilde{S}_p$ and $\widetilde{S}_q$ are isomorphic.

The second result that we will prove is on cubic hypersurfaces $Y \subset \mathbb{P}^5$ for which Items (a), (b) and (c) of Proposition (5.8) hold. Before stating the result we recall that $H^4(Y)$ has a (mixed) Hodge structure with weight filtration [6]

$$\ldots \subset W_3H^4(Y) \subset W_4H^4(Y) = H^4(Y). \tag{5.4.2}$$

If $\zeta\colon \widetilde{Y} \to Y$ is an arbitrary desingularization then (see Proposition (8.5.2) of [6]) we have

$$W_3H^4(Y) = \ker(H^4(Y) \overset{H^4(\zeta)}{\longrightarrow} H^4(\widetilde{Y})). \tag{5.4.3}$$

In particular $W_3H^4(Y)$ is contained in the kernel of the intersection form on $H^4(Y)$ and hence the intersection form is well-defined on

$$Gr_4^W H^4(Y) := H^4(Y)/W_3H^4(Y). \tag{5.4.4}$$

Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface such that (a), (b) and (c) of Proposition (5.8) hold. Let $p \in singY$. Then $\widetilde{S}_p$ is a $K3$ surface: let $T(\widetilde{S}_p) \subset H^2(\widetilde{S}_p;\mathbb{Z})$ be the transcendental lattice of $\widetilde{S}_p$ i.e.

$$T(\widetilde{S}_p) := \{\alpha \in H^2(\widetilde{S}_p;\mathbb{Z}) |\ \alpha \bot H^{1,1}_{\mathbb{Z}}(\widetilde{S}_p)\}. \tag{5.4.5}$$

Then

$$T(\widetilde{S}_p)_{\mathbb{C}} := T(\widetilde{S}_p) \otimes_{\mathbb{Z}} \mathbb{C} \subset H^2(\widetilde{S}_p) \tag{5.4.6}$$

is a sub-Hodge structure of level 2 with

$$h^{2,0}(T(\widetilde{S}_p)_{\mathbb{C}}) = h^{0,2}(T(\widetilde{S}_p)_{\mathbb{C}}) = 1, \quad 1 \le h^{1,1}(T(\widetilde{S}_p)_{\mathbb{C}}) \le 19. \tag{5.4.7}$$

The second main result of this subsection is the following.

[1]See Ch.4 of [17] for definition and properties of du Val singularities.

Proposition 5.9. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface such that Items (a), (b) and (c) of Proposition (5.8) hold. Let $p \in singY$. There is a morphism of type $(1,1)$ of Hodge structures*

$$\gamma \colon T(\widetilde{S}_p)_{\mathbb{C}} \longrightarrow Gr_4^W H^4(Y) \tag{5.4.8}$$

such that

$$\int_Y \gamma(\eta) \wedge \gamma(\theta) = -\int_{\widetilde{S}_p} \eta \wedge \theta, \quad \eta, \theta \in T(\widetilde{S}_p)_{\mathbb{C}}. \tag{5.4.9}$$

Propositions (5.8) and (5.9) will be proved at the end of the present subsection.

5.4.1 Singular cubic 4-folds: preliminary considerations

Both hyperplanes and quadric hypersurfaces in $\mathbb{P}^5$ contain planes and hence we have the following.

Claim 5.10. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface which is not reduced or not irreducible. Then Y contains a plane.*

Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface. Suppose that $p, q \in singY$ are distinct points: then by Bézout's Theorem we get that $span(p,q) \subset Y$. It follows that if $W \subset singY$ is a closed subset then

$$chord(W) \subset Y, \tag{5.4.10}$$

where $chord(W) \subset \mathbb{P}^5$ is the subvariety swept out by the chords of W i.e.

$$chord(W) := \text{closure of } \{span(p,q) | \ p, q \in W, \quad p \neq q\}. \tag{5.4.11}$$

Let $p \in singY$. Choose homogeneous coordinates $[X_0, \ldots, X_4, Z]$ on $\mathbb{P}^5$ such that $p = [0, \ldots, 0, 1]$. We have

$$Y = V(F(X_0, \ldots, X_4)Z + G(X_0, \ldots, X_4)) \tag{5.4.12}$$

where F, G are homogeneous of degrees 2 and 3 respectively. We may and will view

$$[X] = [X_0, \ldots, X_4] \text{ as homogeneous coordinates on } \mathbb{P}(\Theta_p Y). \tag{5.4.13}$$

Let $S_p \subset \mathbb{G}r(1, \mathbb{P}^5)$ be the subscheme parametrizing lines in Y that contain p - see Definition (5.7); the natural inclusion $S_p \subset \mathbb{P}(\Theta_p Y)$ is given by

$$S_p = V(F, G) \subset \mathbb{P}^4_{[X]} = \mathbb{P}(\Theta_p Y). \tag{5.4.14}$$

The following remark follows immediatly from (5.4.12) and (5.4.14).

Remark 5.11. *Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface which is reduced and irreducible. Let $p \in singY$. Suppose that Y is not a cone with vertex p. Then referring to (5.4.12), (5.4.13) and (5.4.14) we have:*

(1) $F \neq 0$ and hence $\mathbb{P}(C_p Y) \subset \mathbb{P}(\Theta_p Y)$ is identified with $V(F)$,

(2) the scheme S_p is the complete intersection of $V(F)$ and $V(G)$ - in particular it is a l.c.i. surface.

Claim 5.12. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface which is a cone. Then Y contains a plane.*

Proof. Suppose that Y is a cone with vertex p. Let $F, G \in \mathbb{C}[X_0, \ldots, X_4]$ be as in (5.4.12). Then $F = 0$. Since $V(G) \subset \mathbb{P}^4_{[X]}$ is a cubic hypersurface it contains a line L. The lines in Y parametrized by L sweep out a plane. $\square$

Claim 5.13. *Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface which is reduced and irreducible. Let $p \in singY$ and suppose that Y is not a cone with vertex p - thus by Item (2) of Remark (5.11) we know that S_p is a complete intersection in $\mathbb{P}^4$. Suppose that S_p is not reduced or not irreducible. Then Y contains a plane.*

Proof. By (5.4.14) and Item (2) of Remark (5.11) we get that there exists a surface $T \subset S_p$ of degree at most 3. It follows that T contains a line ℓ. The lines in $\mathbb{P}^5$ parametrized by points of ℓ sweep out a plane contained in Y. $\square$

Now assume that Y is irreducible and reduced and furthermore that it is not a cone with vertex p. Then the rational map

$$\psi_p \colon Y \dashrightarrow \mathbb{P}(\Theta_p Y) \tag{5.4.15}$$

given by projection from p is birational. The inverse of ψ_p is given by

$$\begin{array}{ccc} \mathbb{P}(\Theta_p Y) & \overset{\psi_p^{-1}}{\dashrightarrow} & Y \\ {[X]} & \mapsto & [F(X)X_0, \ldots, F(X)X_4, -G(X)] \end{array} \tag{5.4.16}$$

Proposition 5.14. *Let $Y \subset \mathbb{P}^5$ be an irreducible and reduced singular cubic hypersurface. Let $p \in singY$ and suppose that Y is not a cone with vertex p. The resolution of indeterminacies of ψ_p defines an isomorphism*

$$\widetilde{\psi}_p \colon Bl_p Y \xrightarrow{\sim} Bl_{S_p} \mathbb{P}(\Theta_p Y). \tag{5.4.17}$$

Proof. Let $F, G \in \mathbb{C}[X_0, \ldots, X_4]$ be as in (5.4.12). By Remark (5.11) we get that S_p is a (possibly non-reduced, non-irreducible) surface, complete intersection of $V(F)$ and $V(G)$. The indeterminacy locus of ψ_p^{-1} is clearly identified with S_p and ψ_p^{-1} is defined by the linear system $|I_{S_p}(3)|$ on $\mathbb{P}(\Theta_p Y)$. Since $I_{S_p}(3)$ is globally generated the proposition follows. $\square$

We will need to relate properties of Y and of S_p. A first observation: if $y \in sing(Y \setminus \{p\})$ then $span(p, y) \subset Y$ by (5.4.10) and hence

$$\psi_p(singY \setminus \{p\}) \subset S_p \subset \mathbb{P}(\Theta_p Y). \tag{5.4.18}$$

Proposition 5.15. *Suppose that $Y \subset \mathbb{P}^5$ is a singular reduced and irreducible cubic hypersurface, that $p \in singY$ and that Y is not a cone with vertex p.*

(1) If $y \in sing(Y \setminus \{p\})$ then $s := \psi_p(y) \in sing(S_p)$. If $span(p, y) \subset sing(Y)$ then $\dim \Theta_s(S_p) = 4$, in particular $\mathbb{P}(C_p Y)$ is singular at s. If $span(p, y) \not\subset sing(Y)$ then $\mathbb{P}(C_p Y)$ is smooth at s.

(2) Let $s \in sing(S_p)$ and assume that $\dim \Theta_s(S_p) = 4$. Then Y is singular at all points of the line corresponding to s.

(3) *Let* $s \in sing(S_p)$ *and assume that* $\dim \Theta_s(S_p) = 3$. *If* $\mathbb{P}(C_pY)$ *is smooth at* s *there exists a unique* $y \in sing(Y \setminus \{p\})$ *such that* $\psi_p(y) = s$. *If* $\mathbb{P}(C_pY)$ *is singular at* s *there is no* $y \in sing(Y \setminus \{p\})$ *such that* $\psi_p(y) = s$.

(4) Y *contains a plane if and only if* S_p *contains a line or a conic.*

Proof. Let $[X_0, \ldots, X_4, Z]$ be homogeneous coordinates on $\mathbb{P}^5$ with $p = [0, \ldots, 0, 1]$; thus we have (5.4.12)-(5.4.14). Let $y = [a_0, \ldots, a_4, b] \in \mathbb{P}^5 \setminus \{p\}$: thus

$$\psi_p(y) = [a_0, \ldots, a_4] = [a]. \tag{5.4.19}$$

Differentiating the defining equation of Y we get that $y \in sing(Y \setminus \{p\})$ if and only if

$$b \cdot \frac{\partial F}{\partial X_i}(a) + \frac{\partial G}{\partial X_i}(a) = 0 \quad i = 0, \ldots, 4, \quad \text{and} \quad F(a) = 0. \tag{5.4.20}$$

Let's prove Item (1). From (5.4.20) we get that $G(a) = 0$ and hence $[a] \in S_p$ (we already noticed this), and the first equation shows that $s \in sing(S_p)$. Assume that for a fixed $a \neq (0, \ldots, 0)$ the first equation holds with an arbitrary choice of b: then both $V(F)$ and $V(G)$ are singular at s and this proves the second statement (recall that $\mathbb{P}(C_pY) = V(F)$ by Item (1) of Remark (5.11).). Assume that for a fixed $a \neq (0, \ldots, 0)$ the first equation holds for some but not for all choices of b: then $V(F)$ is smooth at s and this proves the third statement. Items (2)-(3) are proved by similar elementary considerations. Now let's prove Item (4). Assume that Y contains a plane L. If $p \in L$ then $\psi_p(L \setminus \{p\})$ is a line contained in S_p. If $p \notin L$ then $\Lambda := \psi_p(L)$ is a plane in $\mathbb{P}^4_{[X]}$. The restriction of ψ_p^{-1} to Λ is the linear system $|I_{\Lambda \cap S_p}(3)|$. Since $\psi_p^{-1}(\Lambda) = L$ is a plane we get that necessarily $\Lambda \cap S_p$ is a conic in Λ; thus S_p contains a conic. The proof of the converse is similar. □

5.4.2 Cubic 4-folds with positive-dimensional singular set

We will prove the following result.

Proposition 5.16. *Let* $Y \subset \mathbb{P}^5$ *be a cubic hypersurface such that* $singY$ *has strictly positive dimension. Then* Y *contains a plane.*

The proof of the proposition will be given at the end of this subsubsection.

Lemma 5.17. *Let* $Y \subset \mathbb{P}^5$ *be a cubic hypersurface such that* $\dim(singY) = 3$. *Then* Y *contains a plane.*

Proof. By Claim (5.10) we may assume that Y is reduced and irreducible. The intersection of Y and a generic plane is a singular reduced and irreducible cubic curve and hence it has exactly one singular point. Thus $singY$ has exactly one 3-dimensional irreducible component, call it V, and V is a linear space. Thus Y contains (many) planes. □

Lemma 5.18. *Let* $Y \subset \mathbb{P}^5$ *be a cubic hypersurface such that* $\dim(singY) = 2$. *Then* Y *contains a plane.*

Proof. Assume that there exists a 2-dimensional irreducible component V of $singY$ with $\dim(span(V)) \leq 4$. Then $chord(V) = span(V)$ and hence by (5.4.10) we get that Y contains a linear subspace of dimension at least 2. Thus we may assume that every 2-dimensional irreducible component V of $singY$ is non-degenerate. By (5.4.10) we have $chord(V) \subset Y$ and hence $\dim(chord(V)) \leq 4$, i.e. the non-degenerate surface $V \subset \mathbb{P}^5$ is defective. By a classical result of Severi (see [4]) we get that V is either a cone over a degree-4 rational normal curve or the Veronese surface. One verifies easily that in both cases $chord(V)$ is a cubic hypersurface in $\mathbb{P}^5$ and hence

$$Y = chord(V). \tag{5.4.21}$$

If V is a cone over a degree-4 rational normal curve then $chord(V)$ is itself a cone, and hence by (5.4.21) we get that Y is cone; thus Y contains a plane by Claim (5.12). If V is a Veronese surface let $\psi\colon \mathbb{P}^2 \xrightarrow{\cong} V$ be an isomorphism with $\psi^*\mathcal{O}_V(1) \cong \mathcal{O}_{\mathbb{P}^2}(2)$; if $\ell \subset \mathbb{P}^2$ is a line then $\psi(\ell)$ is a conic spanning a plane contained in $chord(V)$. By (5.4.21) we get that Y contains a plane. □

Proposition 5.19. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface such that*

$$\dim(singY) = 1. \tag{5.4.22}$$

Let $(singY)^1$ be the union of 1-dimensional irreducible components of $singY$. Suppose that there exists $p \in (singY)^1$ such that

$$(singY)^1 \textit{ is smooth at } p, \tag{5.4.23}$$

and

$$S_p \textit{ is a reduced and irreducible surface.} \tag{5.4.24}$$

Then

$$\deg(singY)^1 \leq 5. \tag{5.4.25}$$

Proof. If $\deg(singY)^1 = 1$ then (5.4.25) holds, thus we may assume that

$$\deg(singY)^1 \geq 2. \tag{5.4.26}$$

Let $L_p \subset singY$ be defined by

$$L_p := \{q \in singY \mid span(p,q) \subset singY\}. \tag{5.4.27}$$

By (5.4.23) either $L_p = \{p\}$ or L_p is a line. By (5.4.22) the cubic Y is reduced and irreducible and furthermore by (5.4.24) it is not a cone with vertex p; thus we have the birational map $\psi_p\colon Y \dashrightarrow \mathbb{P}(\Theta_p Y)$ - see (5.4.15). Let

$$W_p := \overline{\psi_p((singY)^1 \setminus L_p))}. \tag{5.4.28}$$

By (5.4.26) we know that $W_p \neq \emptyset$; in fact by (5.4.22) and Proposition (5.15) we get that $W_p \subset sing(S_p)$ and that W_p is of pure dimension 1. Furthermore by (5.4.23) we have

$$\deg W_p = \deg(singY)^1 - 1. \tag{5.4.29}$$

By hypothesis S_p is a reduced surface and hence $\dim(sing(S_p)) \leq 1$. Let $(singS_p)^1$ be the union of 1-dimensional irreducible components of $sing(S_p)$.

Thus W_p is a union of irreducible components of $sing(S_p)^1$, and by (5.4.29) it follows that it suffices to prove that

$$\deg(singS_p)^1 \leq 4. \tag{5.4.30}$$

Let $\Lambda \subset \mathbb{P}(\Theta_p Y)$ be a generic 3-dimensional linear space; then $S_p \cap \Lambda$ is irreducible by (5.4.24). By (5.4.14) $S_p \cap \Lambda$ is a complete intersection of a quadric and a cubic in $\Lambda \cong \mathbb{P}^3$ and hence it has arithmetic genus 4; since it is irreducible we get that it has at most 4 singular points. Inequality (5.4.30) follows because $sing(S_p \cap \Lambda) = sing(S_p) \cap \Lambda$. □

Corollary 5.20. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface satisfying the hypotheses of Proposition (5.19). Then one of the following holds:*

- *(I) $(singY)^1$ contains a line.*
- *(II) There is an irreducible component Γ of $(singY)^1$ with $2 \leq \dim(span(\Gamma)) \leq 3$.*
- *(III) There is an irreducible component Γ of $(singY)^1$ with $\dim(span(\Gamma)) = 4$ and $4 \leq \deg(\Gamma) \leq 5$.*
- *(IV) $(singY)^1$ is the rational normal curve of degree 5 in $\mathbb{P}^5$.*

Proof. By Proposition (5.19) there exists an irreducible component Γ of $(singY)^1$ which has degree at most 5. If $\dim(span(\Gamma)) = 1$ then (I) holds, if $2 \leq \dim(span(\Gamma)) \leq 3$ then (II) holds, if $\dim(span(\Gamma)) = 4$ then (III) holds and if $\dim(span(\Gamma)) = 5$ then (IV) holds. □

Lemma 5.21. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface such that singY contains a line ℓ. There exists a plane $\Lambda \subset Y$ containing ℓ.*

Proof. Let $[X_0, \ldots, X_5]$ be homogeneous coordinates on $\mathbb{P}^5$ such that $\ell = V(X_0, \ldots, X_3)$. Let $Y = V(P)$ where $P \in \mathbb{C}[X_0, \ldots, X_3]_3$. Since Y is singular along ℓ we have $P \in (X_0, \ldots, X_3)^2$ and hence we may write

$$P = A \cdot X_4 + B \cdot X_5 + C \tag{5.4.31}$$

where $A, B, C \in \mathbb{C}[X_0, \ldots, X_3]$ are homogeneous with $\deg A = \deg B = 2$ and $\deg C = 3$. There exists a point

$$[a_0, \ldots, a_3] \in V(A, B, C) \subset \mathbb{P}^3_{[X_0, \ldots, X_3]}. \tag{5.4.32}$$

The plane

$$\Lambda := \{[\lambda a_0, \ldots, \lambda a_3, \mu, \theta] | \ [\lambda, \mu, \theta] \in \mathbb{P}^2\} \tag{5.4.33}$$

is contained in Y. □

Lemma 5.22. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface such that singY contains an irreducible curve Γ such that $2 \leq \dim(span(\Gamma)) \leq 3$. Then Y contains a plane.*

Proof. Since $\dim(span(\Gamma)) \leq 3$ we have $chord(\Gamma) = span(\Gamma)$. By (5.4.10) we know that $Y \supset span(\Gamma)$. Since by hypothesis $\dim(span(\Gamma)) \geq 2$ we get that Y contains a plane. □

Next we examine cubic hypersurfaces Y for which Proposition (5.19) and Item (III) of Corollary (5.20) hold.

Lemma 5.23. *Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface which satsifies the hypotheses of Proposition (5.19) and such that Item (III) of Corollary (5.20) holds. Then Γ is a degree-4 rational normal curve and $Y \cap (span(\Gamma))$ is the cubic 3-fold $chord(\Gamma)$.*

Proof. By (5.4.22) the hypersurface Y is reduced and irreducible and hence $Y \cap (span(\Gamma))$ is a hypersurface. By (5.4.10) $chord(\Gamma) \subset Y$. Since $chord(\Gamma)$ is a hypersurface in $span(\Gamma)$ we get that

$$3 = \deg(Y \cap span(\Gamma)) \geq \deg(chord(\Gamma)), \tag{5.4.34}$$

with equality only if $(Y \cap span(\Gamma)) = (chord(\Gamma))$. From our hypotheses we get that either Γ is a degree-4 rational normal curve in $span(\Gamma)$ or it has degree 5 and arithmetic genus at most 1. A straightforward computation shows that

$$\deg(chord\Gamma) = \begin{cases} 3 & \text{if } \deg \Gamma = 4, \\ 6 & \text{if } \deg \Gamma = 5 \text{ and } p_a(\Gamma) = 0, \\ 5 & \text{if } \deg \Gamma = 5 \text{ and } p_a(\Gamma) = 1. \end{cases} \tag{5.4.35}$$

The lemma follows from the above formulae and (5.4.34). □

Now fix a degree-4 rational normal curve $\Gamma \subset \mathbb{P}^5$ (normal in its span, of course). Let $\mathcal{I}_\Gamma \subset \mathcal{O}_{\mathbb{P}^5}$ be the ideal sheaf of Γ; then $|\mathcal{I}_\Gamma^2(3)|$ is the linear system of cubic hypersurfaces $Y \subset \mathbb{P}^5$ such that $\Gamma \subset sing Y$. We will show that if $Y \in |\mathcal{I}_\Gamma^2(3)|$ then Y contains a plane. If it were true that $chord(\Gamma)$ contains a plane then the result would follow immediatly from Lemma (5.23); unfortunately $chord(\Gamma)$ does not contain planes.

Proposition 5.24. *Let $\Gamma \subset \mathbb{P}^5$ be a degree-4 rational normal curve. Let $Y \in |\mathcal{I}_\Gamma^2(3)|$. Then Y contains a 1-dimensional family of planes Λ such that $\Lambda \cap span(\Gamma)$ is a chord of Γ.*

Proof. Let $Z \subset \Gamma^{(2)} \times \mathbb{G}r(2, \mathbb{P}^5)$ be the subset defined by

$$Z := \{(p+q, \Lambda) | \ \Lambda \supset \overline{p,q}\}, \tag{5.4.36}$$

where $\overline{p,q} = span(p,q)$ if $p \neq q$ and $\overline{p,p} = T_p\Gamma$. Projecting Z to the first factor we get that Z is smooth irreducible and

$$\dim Z = 5. \tag{5.4.37}$$

Let $W \subset Z \times |\mathcal{I}_\Gamma^2(3)|$ be defined by

$$W := \{(p+q, \Lambda, Y) | \ \Lambda \subset Y\}. \tag{5.4.38}$$

One verifies easily that $\mathrm{cod}(W, Z \times |\mathcal{I}_\Gamma^2(3)|) \leq 4$; thus (5.4.37) gives that

$$\dim W \geq \dim |\mathcal{I}_\Gamma^2(3)| + 1. \tag{5.4.39}$$

Let $\rho \colon W \to |\mathcal{I}_\Gamma^2(3)|$ be the restriction of the projection map $Z \times |\mathcal{I}_\Gamma^2(3)| \to |\mathcal{I}_\Gamma^2(3)|$.

Claim 5.25. *Keep notation as above. There exist* $(z_0, Y_0) \in W$ *and an open* $U \subset W$ *containing* (z_0, Y_0) *such that* $U \cap \rho^{-1}(Y_0)$ *is purely* 1*-dimensional.*

Proof. As is easily checked there exists a smooth $Q \in |\mathcal{I}_\Gamma(2)|$. Since Γ is cut out by quadrics we may assume that

$$Q \not\supset chord(\Gamma). \tag{5.4.40}$$

let $Y_0 := Q + span(\Gamma)$; clearly $Y_0 \in |\mathcal{I}^2_\Gamma(3)|$. Before choosing z_0 we notice that

$$\Sigma_Q := \{p + q \in \Gamma^{(2)} | \ \overline{p, q} \subset Q\} \tag{5.4.41}$$

is 1-dimensional because of (5.4.40). Let $p_0 + q_0 \in \Sigma_Q$. There exist two planes $\Lambda \subset Q$ which contain $\overline{p_0, q_0}$, let Λ_0 be one of them: we set $z_0 := (p_0 + q_0, \Lambda_0)$. We let $U \subset W$ be the open subset given by

$$U := \{(p + q, \Lambda, Y) \in W | \ \Lambda \not\subset span(\Gamma)\}. \tag{5.4.42}$$

One easily checks that with these choices the claim holds. □

Let's finish the proof of the proposition. By Claim (5.25) the fibers of ρ restricted to U have dimension at most 1 in a neighborhood of $\rho^{-1}(Y)_0$; by (5.4.39) we get that

$$\dim \rho(W) = \dim |\mathcal{I}^2_\Gamma(3)|. \tag{5.4.43}$$

Since ρ is proper and $|\mathcal{I}^2_\Gamma(3)|$ is irreducible we get that $\rho(W) = |\mathcal{I}^2_\Gamma(3)|$, i.e. every $Y \in |\mathcal{I}^2_\Gamma(3)|$ contains a plane intersecting $span(\Gamma)$ in a chord of Γ. Furthermore the set of such planes has dimension at least 1 because every fiber of $\rho|_W$ has dimension at least 1 by (5.4.39) and because every plane in $\mathbb{P}^5$ intersects Γ in a finite set of points. □

Now let $\Gamma \subset \mathbb{P}^5$ be a degree-5 rational normal curve. First we will give an explicit construction of cubic hypersurfaces $Y \in |\mathcal{I}^2_\Gamma(3)|$ and then we will prove that every $Y \in |\mathcal{I}^2_\Gamma(3)|$ is realized by that procedure. Let $L \to \Gamma$ be "the"degree-1 line-bundle. Given a degree-3 linear system G of dimension 2 on Γ i.e. $G \in |L^{\otimes 3}|^\vee$, we let

$$Y_G := \bigcup_{p_1+p_2+p_3 \in G} \overline{p_1, p_2, p_3} \tag{5.4.44}$$

be the variety swept out by the planes spanned by divisors parametrized by G - of course if $p_1 = p_2 = p$ and $p_3 \neq p$ then $\overline{p_1, p_2, p_3} := J(T_p\Gamma, p_3)$ and if $p_1 = p_2 = p_3 = p$ then $\overline{p_1, p_2, p_3}$ is the the projective osculating plane to Γ at p. One easily checks that Y_G is a hypersurface and that $sing(Y_G) = \Gamma$. Furthermore Y_G is a cone with vertex p if and only if $p \in \Gamma$ and $G = p + |L^{\otimes 2}|$; if this is the case then $Y_G = \langle p, chord(\Gamma_p)\rangle$ where $\Gamma_p \subset \mathbb{P}(\Theta_p(\mathbb{P}^5))$ is the projection of Γ from p. Since Γ_p is a degree-4 rational normal curve $chord(\Gamma_p)$ is a cubic 3-fold and hence we get that $\deg(Y_G) = 3$ whenever G has a base point. Since $\deg(Y_G)$ is independent of G we get that Y_G is a cubic hypersurface for all $G \in |L^{\otimes 3}|^\vee$. Thus we have defined an injection

$$\begin{array}{ccc} |L^{\otimes 3}|^\vee & \hookrightarrow & |\mathcal{I}^2_\Gamma(3)| \\ G & \mapsto & Y_G \end{array} \tag{5.4.45}$$

Proposition 5.26. *Keep notation as above. The map (5.4.45) is an isomorphism.*

Proof. Let $p \in \Gamma$ and let $\Sigma_p \subset |\mathcal{I}_\Gamma^2(3)|$ be the linear subspace of cubics which are cones with vertex p. Let $G_p := (p + |L^{\otimes 2}|) \in |L^{\otimes 3}|^\vee$; a straightforward argument shows that

$$\Sigma_p = \{Y_{G_p}\}. \tag{5.4.46}$$

Now let's prove that

$$\mathrm{cod}(\Sigma_p, |\mathcal{I}_\Gamma^2(3)|) \leq 3. \tag{5.4.47}$$

Let $U \ni p$ be an open affine space containing p; associating to $Y \in |\mathcal{I}_\Gamma^2(3)|$ an affine cubic equation of $Y \cap U$ we may identify $H^0(\mathcal{I}_\Gamma^2(3))$ with a sub-vector-space $A \subset \mathbb{C}[U]$. If $Y \in |\mathcal{I}_\Gamma^2(3)|$ then Y is singular at p; thus p is a critical point of ϕ for all $\phi \in A$. Associating to $\phi \in A$ its Hessian at p we get a linear map

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{H}} & Sym_2(\Omega_p^1(\mathbb{P}^5)) \\ \phi & \mapsto & \text{Hessian of } \phi \text{ at } p. \end{array} \tag{5.4.48}$$

Since $\Sigma_p = \mathbb{P}(\ker \mathcal{H})$ it suffices to prove that

$$\dim(Im\mathcal{H}) \leq 3. \tag{5.4.49}$$

Let $Q \in \mathbb{P}(Im\mathcal{H})$; we may view Q as a quadric hypersurface in $\mathbb{P}^5$ with vertex at p. Since cubics in $|\mathcal{I}_\Gamma^2(3)|$ are singular at all points of Γ the quadric Q is singular at all points of $T_p\Gamma$. Moreover Q contains all the lines $\overline{p,q}$ for $q \in \Gamma$ because such lines are contained in any $Y \in |\mathcal{I}_\Gamma^2(3)|$. Hence projecting Q from the line $T_p\Gamma$ we get a quadric $\overline{Q} \subset \mathbb{P}(N_{T_p\Gamma,\mathbb{P}^5})$ containing the degree-3 rational normal curve $\overline{\Gamma}$ obtained projecting Γ from $T_p\Gamma$. The linear system of quadrics in $\mathbb{P}(N_{T_p\Gamma,\mathbb{P}^5}) \cong \mathbb{P}^3$ containing $\overline{\Gamma}$ has (projective) dimension 2 and hence we get (5.4.49). This proves (5.4.47). By (5.4.46) we get that $\dim |\mathcal{I}_\Gamma^2(3)| \leq 3$. Since the map of (5.4.45) is injective and since $\dim |L^{\otimes 3}|^\vee = 3$ we get the proposition. □

Proof of Proposition (5.16) If $\dim(singY) = 4$ then Y is not reduced and hence it contains a plane by by Claim (5.10). If $\dim(singY) = 3$ or $\dim(singY) = 2$ then Y contains a plane by Lemma (5.17) and Lemma (5.18) respectively. Now assume that $\dim(singY) = 1$. Of course this implies that Y is reduced and irrreducible. If Y is a cone then Y contains a plane by Claim (5.12), thus we may suppose that Y is not a cone. Let $(singY)^1$ be the union of 1-dimensional irreducible components of $singY$. Let $p \in (singY)^1$ be such that $(singY)^1$ is smooth at p. By Item (2) of Remark (5.11) we get that S_p is a l.c.i. intersection surface. First suppose that S_p is not reduced or not irreducible. Then Y contains a plane by Claim (5.13). Thus we may suppose that S_p is a reduced and irreducible surface. It follows that Y satisfies the hypotheses of Proposition (5.19) and hence one of Items (I) - (IV) of Corollary (5.20) holds. If (I) holds then Y contains a plane by Lemma (5.21). If (II) holds then Y contains a plane by Lemma (5.22). If (III) holds then Y contains a plane by Lemma (5.23) and Proposition (5.24). If (IV) holds then by Proposition (5.26) we have $Y = Y_G$ for some $G \in |L^{\otimes 3}|^\vee$ where Y_G is defined by (5.4.44). Since Y_G is sept out by planes we get that Y contains a plane. □

5.4.3 Proof of Proposition (5.8)

Let $Y \subset \mathbb{P}^5$ be a cubic hypersurface which contains no planes. We suppose that Y is singular: we must prove that Items (a), (b) and (c) of Proposition (5.8) hold. Let's prove that (a) holds. The hypersurface Y is reduced and irreducible by Claim (5.10), and it is not a cone by Claim (5.12). Thus Y has quadratic singularities. Furthermore $singY$ is finite by Proposition (5.16). This proves that Item (a) of Proposition (5.8) holds. Next we address Item (b).

Proposition 5.27. *Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface which does not contain any plane. Let $p \in sing(Y)$. Then:*

(1) The projectivizied tangent cone $\mathbb{P}(C_pY)$ is a quadric hypersurface with $\dim(sing\mathbb{P}(C_pY)) \leq 1$.

(2) $S_p \subset \mathbb{P}(\Theta_pY)$ is a reduced and irreducible complete intersection of $\mathbb{P}(C_pY)$ and a cubic hypersurface.

(3) S_p has isolated hypersurface singularities (embedding dimension 3) - in particular S_p is normal.

Proof. Let F, G be as in (5.4.12). We know that Y is reduced, irreducible and it has quadratic singularities; thus by Item (2) of Remark (5.11) we have

$$S_p = V(F) \cap V(G). \tag{5.4.50}$$

Furthermore by Item (1) of Remark (5.11) we have $\mathbb{P}(C_pY) = V(F)$ - here we refer to (5.4.13). Let's prove Item (1). Since $\deg F = 2$ we get that $\mathbb{P}(C_pY)$ is a quadric hypersurface. Suppose that $\dim(sing\mathbb{P}(C_pY)) \geq 2$. Then $\mathbb{P}(C_pY))$ is the union of two hyperplanes in $\mathbb{P}(\Theta_pY) \cong \mathbb{P}^4_{[X]}$ or a double hyperplane. Thus by (5.4.50) we get that S_p is the union of two cubic surfaces or a double cubic surface. In either case S_p contains a line, contradicting Item (4) of Proposition (5.15). Let's prove Item (2): S_p is reduced and irreducible by Claim (5.13), and it is a complete intersection of $\mathbb{P}(C_pY)$ and a cubic hypersurface by (5.4.50). Let's prove Item (3). First we show that S_p has hypersurface singularities. Let $s \in sing(S_p)$ and suppose that $\dim(\Theta_sS_p) \geq 4$; since $S_p \subset \mathbb{P}(C_pY) \cong \mathbb{P}^4$ we get that $\dim(\Theta_sS_p) = 4$ and hence by Item (2) of Proposition (5.15) the line corresponding to s is contained in $singY$. By Lemma (5.21) we get that Y contains a plane, that is a contradiction. It remains to show that $sing(S_p)$ is finite. By Item (3) of Proposition (5.15) we have a one-to-one correspondence

$$\begin{array}{ccc} sing(Y \setminus \{p\}) & \longrightarrow & sing(S_p) \setminus sing\mathbb{P}(C_pY) \\ y & \mapsto & \psi_p(y). \end{array} \tag{5.4.51}$$

By Item (a) of Proposition (5.8) (which we have already proved) $singY$ is finite and hence we get that $|sing(S_p) \setminus sing\mathbb{P}(C_pY)| < \infty$. Thus it remains to prove that

$$|sing(S_p) \cap sing\mathbb{P}(C_pY)| < \infty. \tag{5.4.52}$$

By Item (1) we now that $sing\mathbb{P}(C_pY)$ is either empty or a single point or a line. If $sing\mathbb{P}(C_pY)$ is empty or a single point then (5.4.52) holds, so let's assume that $sing\mathbb{P}(C_pY)$ is a line. By Item (4) of Proposition (5.15) the surface S_p does not contain $sing\mathbb{P}(C_pY)$ and hence we get that (5.4.52) holds. □

Proposition 5.28. *Let $Y \subset \mathbb{P}^5$ be singular cubic hypersurface which does not contain planes. Let $p \in singY$. Then S_p has du Val singularities. (This makes sense because S_p is a reduced normal surface by Proposition (5.27).)*

The above proposition will be proved at the end of the present subsubsection. For the moment we grant the above proposition and we finish the proof of Proposition (5.8). Let's prove that Item (b) holds. That S_p is reduced, irreducible and normal follows from Items (2) and (3) of Proposition (5.27). Furthermore S_p has duVal singularities by Proposition (5.28). By Items (1), (2) of Proposition (5.27) we know that S_p is an intersection of a quadric and a cubic in $\mathbb{P}^4$ and hence by simultaneous resolution of du Val singularities it follows that the minimal desingularization $\widetilde{S}_p$ is a deformation of a smooth intersection of a quadric and a cubic in $\mathbb{P}^4$. Since a smooth intersection of a quadric and a cubic in $\mathbb{P}^4$ is a $K3$ surface we get that $\widetilde{S}_p$ is a $K3$. This proves that Item (b) of Proposition (5.8) holds. In order to prove that Item (c) holds we examine the relation between S_p and S_q for $p, q \in singY$. By Item (a) of Proposition (5.8) we know that $singY$ is a finite set: let

$$k := |singY|, \qquad sing(Y) = \{p_1, \ldots, p_k\}. \tag{5.4.53}$$

Suppose that $k > 1$ and let $i \neq j \in \{1, \ldots, k\}$. Let $r_{ij} := span(p_i, p_j)$; thus $r_{ij} \subset Y$. Let $\Sigma_{ij} \subset \mathbb{G}r(2, \mathbb{P}^5)$ be defined by

$$\Sigma_{ij} := \{\Lambda \in \mathbb{G}r(2, \mathbb{P}^5) | \ \Lambda \supset r_{ij}\}. \tag{5.4.54}$$

If $\Lambda \in \Sigma_{ij}$ then $Y|_\Lambda$ is an effective divisor because Y does not contain planes and we have

$$Y|_\Lambda = r_{ij} + c_\Lambda, \qquad c_\Lambda \in |\mathcal{O}_\Lambda(2)|. \tag{5.4.55}$$

Let $\Gamma^0_{ij} \subset \Sigma_{ij}$ be the subset parametrizing planes Λ such that the conic c_Λ is reducible and $r_{ij} \not\subset supp(c_\Lambda)$. Let

$$\Gamma_{ij} \subset \mathbb{G}r(2, \mathbb{P}^5) \text{ be the closure of } \Gamma^0_{ij}. \tag{5.4.56}$$

If $\Lambda \in \Gamma^0_{ij}$ then $p_i, p_j \in supp(c_\Lambda)$ because Y is singular at p_i and p_j; hence there is a unique decomposition $c_\Lambda = \ell_i + \ell_j$ with $p_i \in \ell_i$ and $p_j \in \ell_j$. Thus we have regular maps

$$\begin{array}{ccc} \Gamma^0_{ij} & \overset{\pi^0_{ij}}{\longrightarrow} & S_{p_i} \\ \Lambda & \mapsto & \ell_i \end{array} \qquad \begin{array}{ccc} \Gamma^0_{ij} & \overset{\tau^0_{ij}}{\longrightarrow} & S_{p_j} \\ \Lambda & \mapsto & \ell_j \end{array} \tag{5.4.57}$$

As is easily verified the above maps extend to regular maps

$$\pi_{ij} \colon \Gamma_{ij} \to S_{p_i}, \qquad \tau_{ij} \colon \Gamma_{ij} \to S_{p_j}. \tag{5.4.58}$$

The fiber of π_{ij} over a point of $S_{p_i} \setminus \{r_{ij}\}$ consists of a single point, and the same holds for the fiber of τ_{ij} over a point of $S_{p_j} \setminus \{r_{ij}\}$. By Item (3) of Proposition (5.27) both S_{p_i} and S_{p_j} are normal and hence π_{ij} and τ_{ij} define isomorphisms

$$(\Gamma_{ij} \setminus \pi^{-1}_{ij}(r_{ij})) \overset{\sim}{\longrightarrow} S_{p_i} \setminus \{r_{ij}\}, \qquad (\Gamma_{ij} \setminus \tau^{-1}_{ij}(r_{ij})) \overset{\sim}{\longrightarrow} S_{p_j} \setminus \{r_{ij}\}. \tag{5.4.59}$$

In particular π_{ij} and τ_{ij} are birational maps and hence S_{p_i} is birational to S_{p_j}. Thus the minimal desingularizations $\widetilde{S}_{p_i}$ and $\widetilde{S}_{p_j}$ are birational: by Item (b) of Proposition (5.8) (already proved modulo Proposition (5.28))both $\widetilde{S}_{p_i}$ and $\widetilde{S}_{p_j}$ are $K3$ surfaces and hence we get that they are isomorphic. This proves that Item (c) holds. We are left with the task of proving Proposition (5.28). The proof will be given after some preliminary results. Let V be a smooth surface, W a normal surface and $\varphi\colon W \to V$ be a double cover branched over the effective reduced divisor $D \in Div(V)$. One can get a desingularization $\widehat{W}$ of W by constructing an embedded resolution $\widehat{D}$ of D in a suitable blow-up $\widehat{V}$ of V and taking a double cover $\widehat{W} \to \widehat{V}$ branched over $\widehat{D}$ and a suitable sum of components of the exceptional divisors: from this construction one easily gets the following criterion.

Condition 5.29. *Keep notation as above. Let $w \in W$ and $v := \varphi(w)$. Suppose that $mult_v(D) \leq 3$ and moreover that if $mult_v(D) = 3$ the strict transform of D in $Bl_v(V)$ intersects the exceptional divisor in at least two distinct points. Then W has a du Val singularity at w.*

Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface which does not contain any plane. Let k be as in (5.4.53); for $1 \leq i \leq k$ let

$$U_{p_i} := S_{p_i} \setminus \{r_{i1}, \ldots, r_{i,i-1}, r_{i,i+1}, \ldots, r_{ik}\}. \tag{5.4.60}$$

Proposition 5.30. *Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface which does not contain planes. Let $p_i \in sing(Y)$. Then U_{p_i} is a du Val surface.*

Proof. By Proposition (5.15) we know that U_{p_i} is smooth away from $sing(\mathbb{P}(C_{p_i}Y)) \cap S_{p_i}$. Thus we must prove that U_{p_i} has a du Val singularity at all $s \in sing(\mathbb{P}(C_{p_i}Y)) \cap S_{p_i}$. Choose such an s. Let $[X_0, \ldots, X_4, Z]$ be homogeneous coordinates on $\mathbb{P}^5$ such that $p_i = [0, \ldots, 0, 1]$. Let $F, G \in \mathbb{C}[X_0, \ldots, X_4]$ be as in (5.4.12); by Proposition (5.27) S_{p_i} is the complete intersection

$$S_{p_i} = V(F, G) \subset \mathbb{P}^4_{[X]} = \mathbb{P}(\Theta_{p_i}Y) \tag{5.4.61}$$

and the cubic $V(G)$ is smooth at s. By Proposition (5.27) the quadric $\mathbb{P}(C_{p_i}Y)$ has rank at least 3; since $V(G)$ is smooth at s it follows that $mult_s(S_{p_i}) = 2$. Let $\varphi\colon \widetilde{S}_{p_i} \to S_{p_i}$ be the blow-up of s. Let $\omega_{S_{p_i}}$ and $\omega_{\widetilde{S}_{p_i}}$ be the dualizing sheaves of S_{p_i} and $\widetilde{S}_{p_i}$ respectively; since S_{p_i} is a surface with a hypersurface singularity of multiplicity 2 at s we have $\varphi^*\omega_{S_{p_i}} \cong \omega_{\widetilde{S}_{p_i}}$. Thus it suffices to prove that

$$\widetilde{S}_{p_i} \text{ has du Val singularities along } \varphi^{-1}(s). \tag{5.4.62}$$

Since $\mathbb{P}(C_{p_i}Y)$ is singular at s we have $\mathbb{P}(C_{p_i}Y) = J(s, Q)$ where $Q \subset \mathbb{P}^4_{[X]}$ is a quadric surface not containing s. By Item (1) of Proposition (5.27) we know that Q is either smooth or the cone over a smooth conic. By Item (4) of Proposition (5.15) we know that S_{p_i} contains no lines and hence projection from s defines a regular finite map $\psi\colon \widetilde{S}_{p_i} \to Q$ of degree 2. We will write out explicit formulae for ψ. We may assume that $s = [0, 0, 0, 0, 1]$ and $span(Q) = V(X_4)$. Thus $[X_0, \ldots, X_3]$ are projective coordinates on $span(Q)$; we let $\mathbb{P}^3_{[X]} = span(Q)$. We recall that $\mathbb{P}(C_{p_i}Y) = V(F)$; since $s = [0, 0, 0, 0, 1] \in sing V(F)$ we have $F \in \mathbb{C}[X_0, \ldots, X_3]_2$. Since $s = [0, 0, 0, 0, 1] \in V(G)$ we have

$$G = AX_4^2 + BX_4 + C \tag{5.4.63}$$

where $A, B, C \in \mathbb{C}[X_0, \ldots, X_3]$ are homogeneous of degrees $1, 2, 3$ respectively. We notice that since S_{p_i} contains no lines we have

$$\mathbb{P}^3_{[\underline{X}]} \supset V(F, A, B, C) = \emptyset. \tag{5.4.64}$$

Since $\widetilde{S}_{p_i}$ is normal the branch divisor of $\psi\colon \widetilde{S}_{p_i} \to Q$ is the reduced effective divisor $D(\psi) \in Div(Q)$ defined by

$$D(\psi) = V(F, B^2 - 4A \cdot C) \subset Q = V(F) \subset \mathbb{P}^3_{[X]}. \tag{5.4.65}$$

Let $t \in \varphi^{-1}(s)$ and let $[e] = \psi(t)$. We have

$$[e] \in \psi(\pi^{-1}(s)) = V(F, A) \subset Q \subset \mathbb{P}^3_{[X]}. \tag{5.4.66}$$

If $B(e) \neq 0$ then by (5.4.66) and (5.4.65) we get that $[e] \notin D(\psi)$. Thus a neighborhood of t in $\widetilde{S}_q$ is isomorphic to a neighborhood of $[e]$ in Q. Since Q has du Val singularities we get that $\widetilde{S}_q$ is du Val at t. Thus we may assume from now on that

$$B(e) = 0. \tag{5.4.67}$$

By (5.4.64) we have

$$C(e) \neq 0. \tag{5.4.68}$$

We treat separately the two cases:

(α) Q is smooth at $[e]$.

(β) Q is singular at $[e]$.

Assume that Item (α) holds. If $V(A)$ is transverse to $Q = V(F)$ at $[e]$ then by (5.4.68) we get that $D(\psi)$ is smooth at $[e]$ and hence $\widetilde{S}_q$ is du Val at t - actually smooth. If $V(A)$ is tangent to Q at $[e]$ we distinguish two cases: Q smooth and Q singular - of course if Q is singular then it is singular at a point different from $[e]$ because we are assuming that Item (α) holds. If Q is smooth then $V(A, F)$ is the union of two distinct lines through $[e]$ and we get from (5.4.68) and (5.4.65) that $D(\psi)$ has a quadratic singularity at $[e]$: thus $\widetilde{S}_q$ is du Val at t by Criterion (5.29). If Q is singular then $V(A, F)$ is a "double line"supported on $\ell := span([e], sing Q)$. If $V(B)$ is singular at $[e]$ or if it is smooth at $[e]$ and transverse to ℓ then $D(\psi)$ has a quadratic singularity at $[e]$; thus $\widetilde{S}_q$ is du Val at t by Criterion (5.29). Finally assume that $V(B)$ is smooth at $[e]$ and that ℓ is tangent to $V(B)$ at $[e]$. We notice that

$$(\ell \cdot V(B))_{[e]} = 2. \tag{5.4.69}$$

In fact if this does not hold then $\ell \subset V(B)$ because $V(B)$ is a quadric and hence $\ell \cap V(C) \subset V(F, A, B, C)$; this contradicts (5.4.64). Let $[e] = [e_0, \ldots, e_3]$ and choose $0 \leq h \leq 3$ such that $e_h \neq 0$. Let $a, b, c \in \mathbb{C}[\mathbb{P}^3 \setminus V(X_h)]$ be the regular functions $a := A/X_h$, $b := B/X_h^2$, $c := C/X_h^3$. From (5.4.69) we get that there exists an open (in the classical topology) $U \subset Q$ containing $[e]$ and analytic coordinates (x, y) on U centered at $[e]$ such that

$$I(\ell \cap U) = (y), \qquad b|_U = y + x^2. \tag{5.4.70}$$

Then $a|_U = \lambda y^2$ and $c|_U = \mu$ with $\lambda, \mu \in \mathbb{C}\{x, y\}$ units. Let $\lambda \cdot \mu = \sum_{i,j} f_{i,j} x^i y^j$, where $f_{i,j} \in \mathbb{C}$. Then

$$(b^2 - 4a \cdot c)|_U \equiv (1 - 4f_{0,0}) y^2 + 2y(x^2 - 2f_{1,0} xy - 2f_{0,1} y^2) \mod (x, y)^4. \quad (5.4.71)$$

If $4f_{0,0} \neq 1$ then $D(\psi)$ has a quadratic singularity at $[e]$ and hence $\widetilde{S}_{p_i}$ is du Val at t by Criterion (5.29). On the other hand if $4f_{0,0} = 1$ then the multiplicity of $D(\psi)$ at $[e]$ is 3 and the strict transform of $D(\psi)$ under the blow-up of Q at $[e]$ intersects the exceptional divisor in at least 2 distinct points; thus Criterion (5.29) applies again and we get that $\widetilde{S}_{p_i}$ is du Val at t. This finishes the proof of (5.4.62) under the assumption that Item (α) above holds. Now assume that Item (β) holds, i.e. that Q is a cone with vertex $[e]$ over a smooth conic. Let $\rho \colon \widehat{Q} \to Q$ be the blow-up of $[e]$ and R be the exceptional divisor of ρ. Let $\widehat{D}(\psi) \subset \widehat{Q}$ be the strict transform of $D(\psi)$. Since $0 = A([e]) = B([e])$ and $C([e]) \neq 0$ we get that

$$\rho^* D(\psi) = \widehat{D}(\psi) + R. \quad (5.4.72)$$

Thus $\rho^* D(\psi)$ is reduced, and there is a unique square-root of $\mathcal{O}_{\widehat{Q}}(\rho^* D(\psi))$, namely $\rho^* \mathcal{O}_Q(2)$; let $\nu \colon W \to \widehat{Q}$ be the corresponding normal double cover with branch divisor $\rho^* D(\psi)$. We have a natural map $\zeta \colon W \to \widetilde{S}_q$ which is an isomorphism outside t and such that $\zeta^{-1}(t) = \nu^{-1}(R)$. Furthermore the dualizing sheaf ω_W is locally-free because W has hypersurface singularities and we have

$$\omega_W \cong \zeta^* \omega_{\widetilde{S}_q}. \quad (5.4.73)$$

Thus it suffices to prove that W has du Val singularities at all points of $\nu^{-1}(R)$. Since $\rho^* D(\psi)$ is smooth at all points of $R \setminus (supp \widehat{D}(\psi))$ we get that W is smooth at points of $\left(\varphi^{-1}(R) \setminus \varphi^{-1}(supp \widehat{D}(\psi))\right)$. Let $\widehat{V}(A, F) \subset \widehat{Q}$ be the strict transform of $V(A, F) \subset Q$; we have

$$R \cap (supp \widehat{D}(\psi)) = R \cap \widehat{V}(A, F). \quad (5.4.74)$$

Either $V(A, F)$ consists of two lines ℓ_1, ℓ_2 or it is a "double line"supported on a single line ℓ. In the first case $R \cap \widehat{V}(A, F)$ consists of two points r_1, r_2. One easily checks that $\rho^* D(\psi)$ has a quadratic singularity at r_1 and at r_2; thus W is du Val at $\nu^{-1}(r_1), \nu^{-1}(r_2)$ by Criterion (5.29). In the second case $R \cap \widehat{V}(A, F)$ consists of a single point r: one easily checks that the multiplicity of $\rho^* D(\psi)$ at r is at most 3 and that if it is equal to 3 then the strict transform of $\rho^* D(\psi)$ under the blow-up of r intersects the exceptional divisor in 2 distinct points; thus W is du Val at $\nu^{-1}(r)$ by Criterion (5.29). □

Now let Σ_{ij} and Γ_{ij} be as in (5.4.54) and (5.4.56) respectively. Let $\pi_{ij} : \Gamma_{ij} \to S_{p_i}$ be as in (5.4.58.

Proposition 5.31. *Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface which does not contain any plane. Suppose that $k := |singY| > 1$ and let $1 \leq i, j \leq k$ with $i \neq j$. The embedding $\Gamma_{ij} \hookrightarrow \Sigma_{ij} \cong \mathbb{P}^3$ realizes Γ_{ij} as a quartic surface. Furthermore $\pi_{ij}^{-1}(U_{p_i} \cup \{r_{ij}\})$ is an open subset of Γ_{ij} with du Val singularities.*

Proof. Over Σ_{ij} we have a tautological family of conics: the conic over Λ is given by the divisor c_Λ appearing in (5.4.55). Thus we have a discriminant divisor $\Delta_{ij} \subset \Sigma_{ij}$ locally defined by the determinant of a symmetric matrix defining the family of conics; of course $\Gamma_{ij} \subset supp(\Delta_{ij})$. Let's show that $\Gamma_{ij} \neq supp(\Delta_{ij})$. Let

$$\Omega_{ij} := \{\Lambda \in \Sigma_{ij} | \quad Y|_\Lambda = 2r_{ij} + \ell_\Lambda, \quad \ell_\Lambda \in |\mathcal{O}_\Lambda(1)|\}. \tag{5.4.75}$$

Clearly $\Omega_{ij} \subset supp(\Delta_{ij})$ and

$$supp(\Delta_{ij}) = \Gamma_{ij} \cup \Omega_{ij}. \tag{5.4.76}$$

A plane $\Lambda \in \Sigma_{ij}$ is parametrized by a point of Ω_{ij} if and only if it is tangent to Y at each point of r_{ij}, i.e. if Λ is contained in

$$L_{ij} := \bigcap_{y \in r_{ij}} \Theta_y Y. \tag{5.4.77}$$

By hypothesis Y is singular at p_i and p_j but $r_{ij} \not\subset sing(Y)$ by Lemma (5.21). It follows that L_{ij} is a hyperplane and hence Ω_{ij} is a plane. Thus Ω_{ij} is an irreducible component of $supp(\Delta_{ij})$. We will write out explicit equations for Ω_{ij} and Γ_{ij}. Let $[X_0, X_1, X_2, X_3, Z_0, Z_1]$ be projective coordinates on $\mathbb{P}^5$ such that

$$p_i = [0, \ldots, 0, 1, 0], \qquad p_j = [0, \ldots, 0, 1]. \tag{5.4.78}$$

Thus $r_{ij} = V(X_0, X_1, X_2, X_3)$ and we have an obvious identification $\Sigma_{ij} \cong \mathbb{P}^3_{[X]}$. Since $r_{ij} \subset Y$ we have $Y = V(\sum_t A_t X_t)$ where $A_t \in \mathbb{C}[X, Z]$ is homogeneous of degree 2. Since Y is singular at p_i and p_j we have $0 = A_t(0, \ldots, 0, 1, 0) = A_t(0, \ldots, 0, 1)$. Thus

$$A_t = B_t + C_t Z_0 + D_t Z_1 + F_t Z_0 Z_1 \tag{5.4.79}$$

where $B_t, C_t, D_t, F_t \in \mathbb{C}[X]$ are homogeneous of degrees 2, 1, 1 and 0 respectively. By hypothesis Y does not contain any plane and hence by Lemma (5.21) the line r_{ij} is not contained in $singY$; thus

$$(F_0, F_1, F_2, F_3) \neq (0, 0, 0, 0). \tag{5.4.80}$$

An easy computation gives that

$$\Omega_{ij} = V(\sum_t F_t X_t). \tag{5.4.81}$$

Let $[X]$ correspond to the plane $\Lambda \in \Sigma_{ij}$; a straightforward computation gives that the conic c_Λ appearing in (5.4.55) is defined by the 3×3 symmetric matrix

$$M_{ij}(X) := \begin{pmatrix} \sum_t B_t X_t & \sum_t C_t X_t & \sum_t D_t X_t \\ \sum_t C_t X_t & 0 & \sum_t F_t X_t \\ \sum_t D_t X_t & \sum_t F_t X_t & 0 \end{pmatrix}. \tag{5.4.82}$$

In particular since Y does not contain any plane we get that

$$V\left(\sum_t B_t X_t, \sum_t C_t X_t, \sum_t D_t X_t, \sum_t F_t X_t\right) = \emptyset. \tag{5.4.83}$$

The divisor Δ_{ij} is defined by

$$\det M_{ij} = \left(\sum_t F_t X_t\right) \cdot \left(\sum_{t,h} (2C_t X_t D_h X_h - B_t X_t F_h X_h)\right). \qquad (5.4.84)$$

Let $P_{ij} \in \mathbb{C}[X]$ be the second factor appearing in the right-hand side of (5.4.84). It follows from (5.4.81) and (5.4.83) that P_{ij} does not vanish identically on Ω_{ij}; thus by (5.4.76) the zero-set of P_{ij} is equal to Γ_{ij}. By Item (2) of Proposition (5.27) we know that Γ_{ij} is irreducibile and hence we get that

$$div(P_{ij}) = m_{ij}\Gamma_{ij} \qquad (5.4.85)$$

for some positive integer m_{ij}. Let

$$[e] \in V(\sum_t F_t X_t, \sum_t C_t X_t, \sum_t D_t X_t). \qquad (5.4.86)$$

Then

$$P_{ij}(e) = 0, \qquad \frac{\partial P_{ij}}{\partial X_s}(e) = -F_s \sum_t B_t(e) e_t. \qquad (5.4.87)$$

By (5.4.80) we have $F_s \neq 0$ for some $0 \leq s \leq 3$ and by (5.4.83) we have $\sum_t B_t(e) e_t \neq 0$; thus

$$\text{if (5.4.86) holds then } P_{ij}(e) = 0 \text{ and } \nabla P_{ij}(e) \neq 0\ . \qquad (5.4.88)$$

This proves that the m_{ij} appearing in (5.4.85) is equal to 1; since $\deg P_{ij} = 4$ we get that Γ_{ij} is a quartic, defined by the vanishing of P_{ij}. Let's show that $\pi_{ij}^{-1}(U_{p_i} \cup \{r_{ij}\})$ is an open subset of Γ_{ij} with du Val singularities. The subset $(U_{p_i} \cup \{r_{ij}\}) \subset S_{p_i}$ is open, see (5.4.60), and hence $\pi_{ij}^{-1}(U_{p_i} \cup \{r_{ij}\})$ is open. Next we notice that if $\Lambda \in \Gamma_{ij}$ and $\pi_{ij}(\Lambda) = r_{iu}$ with $u \neq j$ then $\tau_{ij}(\Lambda) = r_{ju}$ and hence

$$\tau_{ij}\left(\pi_{ij}^{-1}(U_{p_i} \cup \{r_{ij}\})\right) = U_{p_j} \cup \{r_{ij}\}. \qquad (5.4.89)$$

Let $\Lambda \in \pi_{ij}^{-1}(U_{p_i} \cup \{r_{ij}\})$. By (5.4.89) one of the following holds:

(1) $\pi_{ij}(\Lambda) \in U_{p_i}$.

(2) $\tau_{ij}(\Lambda) \in U_{p_j}$.

(3) $\Lambda \in \pi_{ij}^{-1}(r_{ij}) \cap \tau_{ij}^{-1}(r_{ij})$.

Suppose that (1) holds. By (5.4.59) the map π_{ij} is a local isomorphism onto S_{p_i} in a neighborhood of Λ. By Proposition (5.30) we get that Γ_{ij} is du Val at Λ. If (2) holds a similar proof gives that Γ_{ij} is du Val at Λ. Finally suppose that (3) holds. We claim that

$$\pi_{ij}^{-1}(r_{ij}) = V(\textstyle\sum_t F_t X_t, \sum_t D_t X_t), \qquad (5.4.90)$$

$$\tau_{ij}^{-1}(r_{ij}) = V(\textstyle\sum_t F_t X_t, \sum_t C_t X_t). \qquad (5.4.91)$$

In fact let $\Lambda \in \pi_{ij}^{-1}(r_{ij})$ and let $[X]$ be its projective coordinates. Since $\Lambda \cap Y = 2r_{ij} + \ell_\Lambda$ where $p_j \in \ell_\Lambda$ we have $\Lambda \in \Omega_{ij}$ and $span(p_j, [X, 0, 0]) \subset \mathbb{P}(C_{p_j} Y)$. This gives that $\pi_{ij}^{-1}(r_{ij})$ consists of those points of the right-hand side of (5.4.90) which are contained in Γ_{ij}. Since Γ_{ij} is the zero-locus of P_{ij} we get that the right-hand side of (5.4.90) is contained in Γ_{ij}; this proves (5.4.90). Exchanging the rôles of p_i and p_j we get Equation (5.4.91). From (5.4.90)-(5.4.91) we get that

$$\pi_{ij}^{-1}(r_{ij}) \cap \tau_{ij}^{-1}(r_{ij}) = V(\sum_t F_t X_t, \sum_t C_t X_t, \sum_t D_t X_t). \tag{5.4.92}$$

By (5.4.88) we get that Γ_{ij} is smooth at every point of $\pi_{ij}^{-1}(r_{ij}) \cap \tau_{ij}^{-1}(r_{ij})$. □

Proof of Proposition (5.28). By Item (a) of Proposition (5.8) (already proved) $singY$ is finite; let $k := |singY|$. We let $singY = \{p_1, \ldots, p_k\}$. If $k = 1$ then $U_{p_1} = S_{p_1}$ and hence S_{p_1} has du Val singularities by Proposition (5.30). Now assume that $k > 1$ and let $1 \leq i \leq k$. Then U_{p_i} has du Val singularities by Proposition (5.30). It remains to show that S_{p_i} has a du Val singularity at each r_{ij}, where $1 \leq j \leq k$ and $j \neq i$. Let $[X_0, X_1, X_2, X_3, Z_0, Z_1]$ be homogeneous coordinates on $\mathbb{P}^5$ such that (5.4.78) holds. Projection of S_{p_i} from r_{ij} defines an embedding $Bl_{r_{ij}}(S_{p_i}) \hookrightarrow \mathbb{P}^3_{[X]}$; the image of this embedding is Γ_{ij} and it gives an identification of $\pi_{ij} \colon \Gamma_{ij} \to S_{p_i}$ with the blow-up of r_{ij}. In particular since $\deg S_{p_i} = 6$ and $\deg \Gamma_{ij} = 4$ we get that $mult_{r_{ij}} S_{p_i} = 2$. On the other hand S_{p_i} has embedding dimension 3 at r_{ij} by Item (3) of Proposition (5.27) and hence we get that

$$\omega_{\Gamma_{ij}} = \pi_{ij}^*(\omega_{S_{p_i}}). \tag{5.4.93}$$

Let $\rho_{ij} \colon \widetilde{\Gamma}_{ij} \to \Gamma_{ij}$ be the minimal desingularization of the singularities belonging to $\pi_{ij}^{-1}(r_{ij})$. By Proposition (5.31) we get that Γ_{ij} has du Val singularities along $\pi_{ij}^{-1}(r_{ij})$ and hence

$$\omega_{\widetilde{\Gamma}_{ij}} = \rho_{ij}^*(\omega_{\Gamma_{ij}}). \tag{5.4.94}$$

The regular map $\pi_{ij} \circ \rho_{ij} \colon \widetilde{\Gamma}_{ij} \to S_{p_i}$ gives a desingularization of the singular point r_{ij} and by (5.4.93)-(5.4.94) we have

$$\omega_{\widetilde{\Gamma}_{ij}} = (\pi_{ij} \circ \rho_{ij})^*(\omega_{S_{p_i}}). \tag{5.4.95}$$

This proves that S_{p_i} has a du Val singularity at r_{ij}. □

5.4.4 Proof of Proposition (5.9)

Let $S_p^{sm} \subset S_p$ be the smooth locus of S_p. We have a cylinder map

$$cyl \colon H_2(S_p^{sm}; \mathbb{Z}) \to H^4(Bl_{S_p}\mathbb{P}(\Theta_p Y); \mathbb{Z}) \tag{5.4.96}$$

defined as follows. Let

$$\pi \colon Bl_{S_p}\mathbb{P}(\Theta_p Y) \to \mathbb{P}(\Theta_p Y) \tag{5.4.97}$$

be the blow-down map. Given a homology class $\alpha \in H_2(S_p^{sm}; \mathbb{Z})$ represented by an oriented closed smooth real surface $\Sigma \subset S_p^{sm}$ the oriented smooth real 4-fold $\pi^{-1}\Sigma$ is in the smooth locus of $Bl_{S_p}\mathbb{P}(\Theta_p Y)$, hence $\pi^{-1}\Sigma$ has a well-defined Poincaré dual class $PD(\pi^{-1}\Sigma) \in H^4(Bl_{S_p}\mathbb{P}(\Theta_p Y); \mathbb{Z})$ independent of the choice of representative Σ: we set $cyl(\alpha) := PD(\pi^{-1}\Sigma)$. Now let $\ldots, R_i, \ldots$ be the irreducible components of the desingularization map $\widetilde{S}_p \to S_p$; thus we have

$$j\colon S_p^{sm} \hookrightarrow \widetilde{S}_p, \qquad j(S_p^{sm}) = \left(\widetilde{S}_p \setminus \bigcup_i R_i\right). \tag{5.4.98}$$

Since S_p has du Val singularities the map $H_2(j)$ is injective and it gives an identification

$$H_2(S_p^{sm}) = \{\alpha \in H_2(\widetilde{S}_p; \mathbb{Z}) | \ \langle \alpha, R_i \rangle = 0 \quad \forall R_i\}, \tag{5.4.99}$$

where $\langle \cdot, \cdot \rangle$ is the intersection pairing on $H_2(\widetilde{S}_p; \mathbb{Z})$. If $\alpha \in H_2(\widetilde{S}_p; \mathbb{Z})$ is Poincaré dual to a class in $T(\widetilde{S}_p)$ then α belongs to the right-hand side of (5.4.99). Thus via Poincaré duality we get an injection

$$T(\widetilde{S}_p) \hookrightarrow H_2(S_p^{sm}; \mathbb{Z}). \tag{5.4.100}$$

Composing the above inclusion with the cylinder map (5.4.96) and tensoring with $\mathbb{C}$ we get a map

$$\widetilde{\gamma}\colon T(\widetilde{S}_p)_{\mathbb{C}} \longrightarrow H^4(Bl_{S_p}\mathbb{P}(\Theta_p Y)). \tag{5.4.101}$$

A moment's thought will convince the reader that the map above is a morphism of type $(1,1)$ of Hodge structures. Furthermore for $\alpha, \beta \in T(\widetilde{S}_p)_{\mathbb{C}}$ we have

$$\int_{Bl_{S_p}\mathbb{P}(\Theta_p Y)} \widetilde{\gamma}(\alpha) \wedge \widetilde{\gamma}(\beta) = -\int_{\widetilde{S}_p} \alpha \wedge \beta. \tag{5.4.102}$$

In fact this follows from a standard computation based on the fact that the normal bundle of the exceptional divisor of (5.4.97) has degree -1 on a fiber of the $\mathbb{P}^1$-bundle $\pi^{-1}(S_p) \to S_p$. By Isomorphism (5.4.17) we may replace the right-hand side of (5.4.101) by $H^4(Bl_p Y)$; thus $\widetilde{\gamma}$ defines a morphism (we do not change its name) of type $(1,1)$

$$\widetilde{\gamma}\colon T(\widetilde{S}_p)_{\mathbb{C}} \longrightarrow H^4(Bl_p Y). \tag{5.4.103}$$

Let

$$\rho\colon Bl_p Y \to Y \tag{5.4.104}$$

be the blow-down map. The exceptional divisor of ρ is the projectivized normal cone $\mathbb{P}(C_p Y)$. Composing the map of (5.4.103) with the restriction map $H^4(Bl_p Y) \to H^4(\mathbb{P}(C_p Y))$ we get

$$T(\widetilde{S}_p)_{\mathbb{C}} \to H^4(\mathbb{P}(C_p Y)). \tag{5.4.105}$$

We claim that the above map is zero. It suffices to prove triviality of the map

$$T(\widetilde{S}_p)_{\mathbb{C}} \to H^4(\mathbb{P}(C_p Y))/W_3 H^4(\mathbb{P}(C_p Y)) \tag{5.4.106}$$

obtained by composing (5.4.105) with the quotient map. The right-hand side of (5.4.106) is a sub Hodge structure of H^4 of any desing ularization of $\mathbb{P}(C_p Y)$; since $\mathbb{P}(C_p Y)$ is a quadric we get that the right-hand side of (5.4.106) is of pure type $(2,2)$. By (5.4.7) we get that (5.4.106) has a non-zero kernel, and since $T(\widetilde{S}_p)_{\mathbb{C}}$ has no non-trivial rational sub-Hodge structure we get that the kernel of (5.4.106) is all of $T(\widetilde{S}_p)_{\mathbb{C}}$. Thus (5.4.105) is zero and $Im(\widetilde{\gamma}) \subset ImH^4(\rho)$ where ρ is the blow-down map (5.4.104). Hence there exists a morphism of type $(1,1)$ of Hodge structures

$$\widehat{\gamma}\colon T(\widetilde{S}_p)_{\mathbb{C}} \longrightarrow H^4(Y)/\ker(\rho^*). \tag{5.4.107}$$

such that $\widetilde{\gamma} = H^4(\rho) \circ \widehat{\gamma}$. Clearly $\ker(\rho^*) \subset W_3H^4(Y)$; we let γ be the composition of $\widehat{\gamma}$ with the quotient map $H^4(Y)/\ker(\rho^*) \to Gr_4^W H^4(Y)$. This defines the morphism of Hodge structures (5.4.8). Equation (5.4.9) follows from (5.4.102). □

5.5 (4) of Proposition (4.4) does not hold

The proof is by contradiction. We assume that we have $f\colon X \to Y$ a finite regular map of degree 4 onto a cubic 4-fold $Y \subset \mathbb{P}^5$ and we reach a contradiction. Since f is regular $Y = Y_0$ and hence Y does not contain planes by Item (1) of Corollary (4.2). By Propositions (5.8) either Y is smooth or else it is singular and Items (a), (b) and (c) of Proposition (5.8) hold. Suppose first that the latter holds. By Proposition (5.9) we have the morphism of type $(1,1)$ of Hodge structures γ of (5.4.8). Composing γ with f^* we get a morphism of type $(1,1)$ of Hodge structures

$$T(\widetilde{S}_p)_{\mathbb{C}} \overset{f^*\circ\gamma}{\longrightarrow} H^4(X). \tag{5.5.1}$$

Let $\eta, \theta \in T(\widetilde{S}_p)_{\mathbb{C}}$; by (5.4.9) we have

$$\int_X f^*\gamma(\eta) \wedge f^*\gamma(\theta) = -4 \int_{\widetilde{S}_p} \eta \wedge \theta. \tag{5.5.2}$$

Since the restriction to $T(\widetilde{S}_p)_{\mathbb{C}}$ of the intersection form on $H^2(\widetilde{S}_p)$ is non-degenerate we get that $f^*\circ\gamma$ is injective. Thus $Im(f^*\circ\gamma)$ is a rational Hodge substructure of $H^4(X)$ with Hodge numbers $h^{p,q} = h^{p-1,q-1}(T(\widetilde{S}_p)_{\mathbb{C}})$. By (5.4.7) this contradicts Item (4) of Proposition (3.2). Now suppose that Y is smooth. Since $\deg f = 4$ we have

$$\langle f^*\alpha, f^*\beta\rangle_X = 4\langle \alpha, \beta\rangle_Y, \quad \alpha, \beta \in H^4(Y) \tag{5.5.3}$$

where $\langle,\rangle_X$ and $\langle,\rangle_Y$ are the intersection forms on $H^4(X)$ and $H^4(Y)$ respectively. Thus $f^*\colon H^4(Y) \to H^4(X)$ is an injection of rational Hodge structures. Let

$$H^4(Y)_{prim} := \{\alpha \in H^4(Y) \mid \alpha \wedge c_1(\mathcal{O}_Y(1)) = 0\} \tag{5.5.4}$$

be the primitive cohomology of Y: this a rational sub Hodge structure of $H^4(Y)$. Since $\dim H^4(Y)_{prim} = 22$ Item (4) of Proposition (3.2) gives that $f^*H^4(Y)_{prim} = \mathbb{C}h \otimes h^\perp$. Thus

$$f^*H^4(Y;\mathbb{Q})_{prim} = \mathbb{Q}h \otimes h^\perp_{\mathbb{Q}} \tag{5.5.5}$$

where $h^\perp_{\mathbb{Q}} := h^\perp \cap H^2(X;\mathbb{Q})$. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_{22}\}$ be a $\mathbb{Z}$-basis of $H^4(Y;\mathbb{Z})_{prim}$. Let $\mathcal{Q}_{\mathcal{B}}$ be the matrix of the restriction of $\langle,\rangle_Y$ to $H^4(Y;\mathbb{Z})_{prim}$ in the basis $\mathcal{B}$. Since $\langle,\rangle_Y$ is unimodular and $\deg Y = 3$ we have

$$|\det(\mathcal{Q}_{\mathcal{B}})| = 3. \tag{5.5.6}$$

Let $\mathcal{B}' := \{f^*\alpha_1, \ldots, f^*\alpha_{22}\}$; by (5.5.5) we know that $\mathcal{B}'$ is a $\mathbb{Q}$-basis of $\mathbb{Q}h \otimes h^{\perp}_{\mathbb{Q}}$. Let $\mathcal{Q}_{\mathcal{B}'}$ be the matrix of the restriction of $\langle,\rangle_X$ to $\mathbb{Q}h \otimes h^{\perp}_{\mathbb{Q}}$ in the basis $\mathcal{B}'$; by (5.5.6)-(5.5.3) we have

$$|\det(\mathcal{Q}_{\mathcal{B}'})| = 3 \cdot 2^{44}. \tag{5.5.7}$$

Now let $\{\beta_1, \ldots, \beta_{22}\}$ be a $\mathbb{Z}$-basis of $h^{\perp}_{\mathbb{Z}} := H^2(X;\mathbb{Z}) \cap h^{\perp}$; then $\mathcal{B}'' := \{h\beta_1, \ldots, h\beta_{22}\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}h \otimes h^{\perp}_{\mathbb{Q}}$. Let $\mathcal{Q}_{\mathcal{B}''}$ be the matrix of the restriction of $\langle,\rangle_X$ to $\mathbb{Q}h \otimes h^{\perp}_{\mathbb{Q}}$ in the basis $\mathcal{B}''$. By Remark (2.1) one gets (use also Lemma (3.4)) that

$$|\det(\mathcal{Q}_{\mathcal{B}''})| = 2^{24}. \tag{5.5.8}$$

Since both $\mathcal{B}'$ and $\mathcal{B}''$ are $\mathbb{Q}$-bases of $\mathbb{Q}h \otimes h^{\perp}_{\mathbb{Q}}$ the determinants appearing in Equations (5.5.7)-(5.5.8) must represent the same class in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$. This is visibly false, contradiction. □

5.5.1 Comment

The following is an example of X a numerical $(K3)^{[2]}$ and H a big and nef divisor on X with $(c_1(H), c_1(H)) = 2$ such that $f\colon X \to |H|^\vee$ is a regular double covering of a cubic hypersurface - we do not know of any such example with H ample. Let V be a 3-dimensional complex vector space and $\pi\colon S \to \mathbb{P}(V)$ be a double covering ramified over a smooth sextic curve; thus S is a $K3$ surface. Let $X := S^{[2]}$ and let f be the composition

$$S^{[2]} \to S^{(2)} \to \mathbb{P}(V)^{(2)} \hookrightarrow \mathbb{P}(Sym^2 V) \cong \mathbb{P}^5. \tag{5.5.9}$$

The image of $\mathbb{P}(V)^{(2)} \hookrightarrow \mathbb{P}(Sym^2 V)$ is the discriminant cubic hypersurface; since f has degree 4 onto its image we get that $\int_X c_1(H)^4 = 12$ and hence $(c_1(H), c_1(H)) = 2$ by (2.1.4). The divisor H is big and nef and f can be identified with the natural map $f\colon X \to |H|^\vee$: thus f has the stated properties.

5.6 (5) of Proposition (4.4) does not hold

In Subsubsection (5.6.1) we will prove the following result.

Proposition 5.32. *Let $Y \subset \mathbb{P}^5$ be a quartic hypersurface such that* $\dim(singY) \geq 3$. *Then Y contains a plane.*

Granting the above proposition let's prove that Item (5) of Proposition (4.4) does not hold. We argue by contradiction. Assume that we have $f\colon X \to Y$ regular of degree 3 onto a quartic hypersurface $Y \subset \mathbb{P}^5$. By Item (1) of Corollary (4.2) and Proposition (5.32) we get that $\dim(singY) \leq 2$. Let $R \in Div(X)$ be the ramification divisor of f. Applying the adjunction formula to $Y^{sm} := (Y \setminus singY)$ and Hurwitz' formula to $f^{-1}(Y^{sm}) \xrightarrow{f} Y^{sm}$ we get that

$$R \in |\mathcal{O}_X(2H)|. \tag{5.6.1}$$

By applying (4.0.1) we get that

$$h^0(\mathcal{O}_X(2H)) = 21 = h^0(\mathcal{O}_Y(2)). \tag{5.6.2}$$

Thus the pull-back map $f^*\colon H^0(\mathcal{O}_Y(2)) \to H^0(\mathcal{O}_X(2H))$ is an isomorphism and from (5.6.1) we get that there exists an effective Cartier divisor $D \in Div(Y)$ such that $f^*D = R$. Comparing the orders of vanishing of f^*D and R at a prime component of R we get a contradiction.

5.6.1 Proof of Proposition (5.32)

If Y is not reduced or not irreducible then there is an irreducible component of Y of degree at most 2 and the result follows immediately. Thus we may assume that Y is irreducible and reduced. Let V be an irreducible component of $singY$; intersecting Y with a generic plane we get that $\deg V \le 3$. If $\deg V = 1$ there is nothing to prove. Assume that $\deg V = 2$. If V is singular then V contains planes and we are done. Thus we may assume that V is smooth. Let $L := span(V)$. Then $L \cong \mathbb{P}^4$ and V is a quadric hypersurface in L. Since Y is irreducible of degree 4 we have the cycle-theoretic intersection

$$Y \cdot L = 2V. \tag{5.6.3}$$

We claim that there exists a complete intersection of two quadrics

$$\widetilde{Y} = Q_1 \cap Q_2 \subset \mathbb{P}^6 \tag{5.6.4}$$

such that Y is isomorphic to the projection of $\widetilde{Y}$ from a point outside $\widetilde{Y}$. In fact let $\mathcal{I}_V \subset \mathcal{O}_{\mathbb{P}^5}$ be the ideal sheaf of (the reduced) V. The linear system $|\mathcal{I}_V(2)|$ has dimension 6. The rational map

$$\varphi\colon \mathbb{P}^5 \dashrightarrow |\mathcal{I}_V(2)|^\vee \cong \mathbb{P}^6 \tag{5.6.5}$$

is the composition of the blow-up of V and contraction of the strict transform of L to a point, call it p. The image of φ is a smooth quadric $Q_1 \subset \mathbb{P}^6$. The inverse of $\mathbb{P}^5 \dashrightarrow Q_1$ is projection from p. The image (strict transform) of Y under φ is a codimension-1 subset $\widetilde{Y} \subset Q_1$ which does not intersect p - use (5.6.3) to get this last statement. Thus $\deg \widetilde{Y} = \deg Y = 4$ and hence there exists a quadric $Q_2 \subset \mathbb{P}^6$ such that (5.6.4) holds. By a theorem of Debarre-Manivel [5] we get that $\widetilde{Y}$ contains a plane Λ. Since projection from p will map Λ to a plane in Y we are done. Finally assume that $\deg V = 3$. The variety is non-degenerate: in fact if $\dim(span(V)) = 4$ then $span(V) \subset Y$ contradiction. Since V is non-degenerate of degree 3 we get that V is smooth and linearly normal; as is well-known [15] it follows that V is the Segre 3-fold i.e. $\mathbb{P}^1 \times \mathbb{P}^2$ embedded by $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$. Since the Segre 3-fold contains planes we are done. □

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Kieran G. O'Grady
Università di Roma "La Sapienza",
Dipartimento di Matematica "Guido Castelnuovo",
Piazzale Aldo Moro n. 5, 00185 Rome, Italy,
e-mail: ogrady@mat.uniroma1.it.