Classroom Notes - IGS 2018/19

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Contents

Chapter 1

Complex manifolds

1.1 Holomorphic functions

Let $U \subset \mathbb{C}^n$ be an open subset, and $f\colon U \to \mathbb{C}$ be a function. Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, and $\mathbb{C}$ with $\mathbb{R}^2$, we may view f as a function from the open $U \subset \mathbb{R}^{2n}$ to $\mathbb{R}^2$, and hence it makes sense to state that f is, or is not, differentiable at $a \in U$.

Definition 1.1.1. Let $U \subset \mathbb{C}^n$ be an open subset. A function $f\colon U \to \mathbb{C}$ is *holomorphic* if, for each $a \in U$, it is differentiable at a, and the differential $df(a)\colon \mathbb{C}^n \to \mathbb{C}$ is *complex linear*, i.e. there exists a complex linear function $L\colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - L(h)||}{||h||} = 0.$$

Remark 1.1.2. With notation as in **Definition 1.1.1**, the linear function L is identified with the differential $df(a)$ via the standard identifications of $\mathbb{C}^n$ and $\mathbb{C}$ with $\mathbb{R}^{2n}$ and $\mathbb{R}^2$ respectively.

Let $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$ be the real vector space of $\mathbb{R}$-linear maps $\mathbb{C}^n \to \mathbb{C}$. Then $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$ contains the subspace $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ of $\mathbb{C}$-linear maps $\mathbb{C}^n \to \mathbb{C}$, and the subspace $\overline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ of $\mathbb{C}$-conjugate linear maps $\mathbb{C}^n \to \mathbb{C}$, i.e. homomorphism $f\colon \mathbb{C}^n \to \mathbb{C}$ of additive groups such that $f(\lambda v) = \overline{\lambda} f(v)$ for $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ (equivalently, such that $v \mapsto \overline{f(v)}$ is $\mathbb{C}$-linear). We have a direct sum of real vector spaces

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}) \oplus \overline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}). \tag{1.1.1}$$

Thus, a function $f\colon U \to \mathbb{C}$ is holomorphic if and only if its differential at each point of U belongs to the direct summand $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ of the above decomposition. A differentiable function $f\colon U \to \mathbb{C}$ is *antiholomorphic* if $df(a) \in \overline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ for all $a \in U$.

We rewrite the decomposition in (1.1.1) as follows. First notive that we have a natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}) \\ f \otimes \lambda & \mapsto & (v \mapsto \lambda f(v)) \end{array} \tag{1.1.2}$$

Remark 1.1.3. Let $f\colon U \to \mathbb{C}$ be differentiable, and write $f = u + iv$, where u, v are real functions. For $a \in U$, the decomposition $df(a) = du(a) + idv(a)$ illustrates (1.1.2), by rewriting it as $df(a) = du(a) + dv(a) \otimes i$.

Thus, letting

$$\Omega^{1,0}_a(U) := \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}), \quad \Omega^{0,1}_a(U) := \overline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}), \tag{1.1.3}$$

(as above, $U \subset \mathbb{C}^n$ is open) we may rewrite (1.1.1) as

$$T_a(U)^* \otimes_{\mathbb{R}} \mathbb{C} = \Omega^{1,0}_a(U) \oplus \Omega^{0,1}_a(U). \tag{1.1.4}$$

Complex bases of $\Omega^{1,0}_a(U)$ and $\Omega^{0,1}_a(U)$ are respectively

$$\{dz_1(a), \ldots, dz_n(a)\}, \quad \{d\overline{z}_1(a), \ldots, d\overline{z}_n(a)\} \tag{1.1.5}$$

Next, let

$$\partial/\partial z_1(a), \ldots, \partial/\partial z_n(a), \partial/\partial \overline{z}_1(a), \ldots, \partial/\partial \overline{z}_n(a) \in T_a(\mathbb{C}^n) \otimes \mathbb{C} \tag{1.1.6}$$

be defined by the conditions

$$\left\langle \frac{\partial}{\partial z_j}(a), dz_k(a) \right\rangle = \delta_{j,k}, \tag{1.1.7}$$

$$\left\langle \frac{\partial}{\partial z_j}(a), d\overline{z}_k(a) \right\rangle = \left\langle \frac{\partial}{\partial \overline{z}_j}(a), dz_k(a) \right\rangle = 0, \tag{1.1.8}$$

$$\left\langle \frac{\partial}{\partial \overline{z}_j}(a), d\overline{z}_k(a) \right\rangle = \delta_{j,k}. \tag{1.1.9}$$

With the above notation, a differentiable function $f\colon U \to \mathbb{C}$ is holomorphic if and only if

$$\frac{\partial f(a)}{\partial \overline{z}_k} = 0 \quad \forall k \in \{1, \ldots, n\}, \ \forall a \in U. \tag{1.1.10}$$

Informally: f is holomorphic if it depends on the z_j's, but *not* on the $\overline{z}_k$'s.

Example 1.1.4. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function of the z_j's and $\overline{z}_k$'s, i.e.

$$f(z) = \sum_{|J|+|K|\leqslant d} c_{J,K} z^J \overline{z}^K, \tag{1.1.11}$$

where $J = (j_1, \ldots, j_n)$ and $K = (k_1, \ldots, k_n)$ are multindices. Then

$$\frac{\partial f}{\partial \overline{z}_s}(z) = \sum_{\substack{J,K \\ k_s \geqslant 1}} c_{J,K} k_s z^J \overline{z}^{K-e_s}, \tag{1.1.12}$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. It follows that f is holomorphic if and only $c_{J,K} = 0$ for all $K \neq (0, \ldots, 0)$.

Remark 1.1.5. Let $f\colon U \to \mathbb{C}$ be a holomorphic function of one variable, i.e. U is an open subset of $\mathbb{C}$. For $a \in U$ we let $f'(a) := \frac{\partial f(a)}{\partial z}$.

Example 1.1.6. Let $R > 0$. Let $f\colon B(0,R) \to \mathbb{C}$ be defined by an absolutely convergent series

$$f(z) = \sum_{m=0}^{\infty} c_m z^m, \tag{1.1.13}$$

i.e. the right hand side is absolutely convergent for every $z \in B(0,R)$. We claim that f is holomorphic, and that

$$f'(z) = \sum_{m=0} (m+1) c_{m+1} z^m. \tag{1.1.14}$$

In fact, given $0 < \rho < R$, there exists $M(\rho) > 0$ such that

$$|c_m| \rho^m \leqslant M(\rho) \quad \forall m, \tag{1.1.15}$$

because the right hand side of (1.1.13) is absolute convergent for every z such that $|z| = \rho$. It follows that the right hand side of (1.1.14) is absolutely convergent for $|z| < R$. Moreover, for $|z| < R$ we have

$$\begin{aligned} f(z)-f(z_0) = \sum_{m=0}^{\infty} c_m((z_0+(z-z_0))^m - z_0^m) = \sum_{m=1}^{\infty} c_m \left(\sum_{j=1}^{m} \binom{m}{j} (z-z_0)^j z_0^{m-j} \right) = \\ = (z-z_0) \left(\sum_{m=0}^{\infty} (m+1) c_{m+1} z_0^m \right) + (z-z_0)^2 \varphi(z), \end{aligned} \tag{1.1.16}$$

where $\varphi(z)$ is uniformly bounded on $B(z_0, \epsilon)$ for $\epsilon < (R - |z_0|)$, (use (1.1.15)). Hence

$$\frac{\partial f}{\partial z}(z_0) = \sum_{m=0} (m+1) c_{m+1} z_0^m, \quad \frac{\partial f}{\partial \overline{z}}(z_0) = 0.$$

The claim follows because z_0 is an arbitrary point of $B(0,R)$.

Writing out (1.1.10) in real coordinates, one gets the Cauchy-Riemann equations. More precisely, let $z_j = x_j + iy_j$, where x_j, y_j are the real coordinate functions. Then (1.1.7), (1.1.8), and (1.1.9) are equivalent to

$$\frac{\partial}{\partial z_j}(a) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(a) - i\frac{\partial}{\partial y_j}(a)\right), \quad \frac{\partial}{\partial \bar{z}_j}(a) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(a) + i\frac{\partial}{\partial y_j}(a)\right). \tag{1.1.17}$$

In particular, we may rewrite (1.1.10) as

$$\frac{\partial f}{\partial x_j} = \frac{\partial f}{i\partial y_j} \quad \forall j \in \{1, \dots, n\}.$$

Letting $f(z) = u(z) + iv(z)$, where $u(z)$, $v(z)$ are the real and the imaginary part of $f(z)$ respectively, we get that $f\colon U \to \mathbb{C}$ is holomorphic if and only if it is differentiable and for all $j \in \{1, \dots, n\}$ the following *Cauchy-Riemann equations* hold on U:

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}, \tag{1.1.18}$$

$$\frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}. \tag{1.1.19}$$

1.2 Holomorphic maps

Definition 1.2.1. Let $U \subset \mathbb{C}^n$ be an open subset. A map $f\colon U \to \mathbb{C}^m$ is *holomorphic* if, for each $a \in U$, it is differentiable at a, and the differential $df(a)\colon \mathbb{C}^n \to \mathbb{C}^m$ is *complex linear.*

Write $f = (f_1, \dots, f_m)$; then f is holomorphic if and only if each of its component functions $f_j\colon U \to \mathbb{C}$ is holomorphic. This holds because an $\mathbb{R}$-linear map $V \to W_1 \oplus W_2$, where V, W_1, W_2 are complex vector spaces is $\mathbb{C}$-linear if and only if each of the maps $V \to W_j$ obtained by composing with the projections $(W_1 \oplus W_2) \to W_j$ is $\mathbb{C}$-linear.

Theorem 1.2.2. *Let $U \subset \mathbb{C}^n$ be a non empty open subset.*

1. *The set of holomorphic functions $f\colon U \to \mathbb{C}$ with pointwise addition and multiplication is a ring, with unit the constant function* 1*. If $f\colon U \to \mathbb{C}$ is holomorphic and nowhere zero, then $1/f(z)$ is holomorphic.*

2. *Let $U \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ be open subsets. Let $f: U \to W$ and $g: W \to \mathbb{C}^k$ be holomorphic (f holomorphic means that it is holomorphic when viewed as a map $U \to \mathbb{C}^n$). Then the composition $g \circ f: U \to \mathbb{C}^k$ is holomorphic.*

3. Holomorphic Inverse Function Theorem: *Let $U \subset \mathbb{C}^n$ be open, and let $f: U \to \mathbb{C}^n$ be holomorphic. Let $a \in U$, and assume that $df(a): \mathbb{C}^n \to \mathbb{C}^n$ is invertible. Then f is a local diffeomorophism at a, with holomorphic local inverse.*

4. Holomorphic Implicit Function Theorem: *Let $U \subset \mathbb{C}^n$ be open, and let $f: U \to \mathbb{C}^k$ be holomorphic, with components $f_1, \dots, f_k$. Write elements of $\mathbb{C}^n$ as (z, w), where $z \in \mathbb{C}^{n-k}$ $w \in \mathbb{C}^k$. Let $(a, b) \in U$. Suppose that $f(a, b) = 0$, and that the $k \times k$ matrix $\left(\frac{\partial f_l(a,b)}{\partial w_h}\right)_{1 \leqslant l, h \leqslant k}$ is non degenerate. Then there exist open (non empty) balls $B(a, R) \subset \mathbb{C}^{n-k}$, $B(b, r) \subset \mathbb{C}^k$ and a holomorhic function $\varphi: B(a, R) \to B(b, r)$ such that $B(a, R) \times B(b, r) \subset U$ and*

$$\{(z, w) \in B(a, R) \times B(b, r) \mid f(z, w) = 0\} = \{(z, \varphi(z)) \mid z \in B(a, R)\}.$$

Proof. All the statements above follow from corresponding results on differentiable maps. As an example, assume that $f: U \to \mathbb{C}$ is holomorphic and nowhere zero. Let $u, v: U \to \mathbb{C}$ be the real and imaginary parts of f. Then

$$\frac{1}{f(z)} = \frac{u}{u^2 + v^2} - i \frac{v}{u^2 + v^2}.$$

Thus $1/f$ is differentiable. Since

$$\frac{\partial}{\partial \overline{z}_j} \left(\frac{1}{f}\right) = -\frac{\frac{\partial f}{\partial \overline{z}_j}}{f^2} = 0,$$

$1/f$ is holomorphic. □

1.3 Complex valued differential forms

Definition 1.3.1. Let $U \subset \mathbb{C}^n$ be open. A *complex valued* m form ω on U is a section of $(\bigwedge^m T(U)^*) \otimes_{\mathbb{R}} \mathbb{C}$ (the complexified m-th exterior power of the cotangent bundle of U), i.e. $\omega = \alpha + i\beta$, where α, β are real m forms on U. We say that ω is continuous, differentiable or C^l if each of α, β is respectively continuous, differentiable or C^l.

We recall that the complexified cotangent space of an open $U \subset \mathbb{C}^n$ at a point a has a direct sum decomposition with addends the complex vectior subspaces $\Omega_a^{1,0}(U)$ and $\Omega_a^{0,1}(U)$. There is a similar decomposition of the fiber of $(\bigwedge^m T(U)^*) \otimes_{\mathbb{R}} \mathbb{C}$ at a.

Definition 1.3.2. Let $\Omega_a^{p,q}(U)$ be the complex subspace of $(\bigwedge^m T(U)^*) \otimes_{\mathbb{R}} \mathbb{C}$ spanned by alle elements of the form $df_1(a) \wedge \ldots \wedge df_p(a) \wedge dg_1(a) \wedge \ldots \wedge dg_q(a)$, where $f_1, \ldots, f_p$ are holomorphic defined in an open neighborhood of a, and $g_1, \ldots, g_q$ are antiholomorphic defined in an open neighborhood of a.

For multindices $J = (j_1, \ldots, j_p)$ and $K = (k_1, \ldots, k_q)$, let

$$dz_J(a) := dz_{j_1}(a) \wedge \ldots \wedge dz_{j_p}(a), \quad d\overline{z}_K(a) := d\overline{z}_{k_1}(a) \wedge \ldots \wedge d\overline{z}_{k_q}(a). \tag{1.3.1}$$

A complex basis of $\Omega_a^{p,q}(U)$ is provided by all $dz_J(a) \wedge d\overline{z}_J(a)$, where the multiindices J, K have p and q entries respectively.

We have a direct sum decomposition

$$(\bigwedge^m T_a(U)^*) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=m} \Omega_a^{p,q}(U). \tag{1.3.2}$$

Hence a complex valued m form on U can be written uniquely as

$$\omega = \sum_{|I|+|J|=m} \omega_{I,J} dz_I \wedge d\overline{z}_J, \quad \omega_{I,J} \colon U \to \mathbb{C}, \tag{1.3.3}$$

and ω is continuous, differentiable or C^k if and only if each of $\omega_{I,J}$ is respectively continuous, differentiable or C^k.

Definition 1.3.3. A differential form ω on an open $U \subset \mathbb{C}^n$ is of *type* (p, q) if $\omega(a) \in \Omega_a^{p,q}(\mathbb{C}^n)$ for all $a \in U$.

Let $U \subset \mathbb{C}^n$ be open. For a differentiable complex valued m form $\omega = u + iv$, where u, v are the real and imaginary parts of ω, we let $d\omega = du + idv$. It is convenient to split $d\omega$ according to the decomposition in (1.3.2). For a differentiable function $f \colon U \to \mathbb{C}$, we let

$$\partial f := \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j, \quad \overline{\partial} f := \sum_{k=1}^n \frac{\partial f}{\partial \overline{z}_k} d\overline{z}_k. \tag{1.3.4}$$

We extend ∂ and $\overline{\partial}$ to linear operators on differential forms by imposing Leibiniz rule. Thus

$$\partial(\sum_{|I|+|J|=m} \omega_{I,J} dz_I \wedge d\overline{z}_J) := \sum_{\substack{|I|+|J|=m\\ k\in\{1,\ldots,n\}}} \frac{\partial \omega_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J \tag{1.3.5}$$

$$\overline{\partial}(\sum_{|I|+|J|=m} \omega_{I,J} dz_I \wedge d\overline{z}_J) := \sum_{\substack{|I|+|J|=m\\ k\in\{1,\ldots,n\}}} \frac{\partial \omega_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J \tag{1.3.6}$$

A straighforward computation shows that

$$d = \partial + \overline{\partial}. \tag{1.3.7}$$

Since $d \circ d = 0$, it follows that

$$\partial \circ \partial = 0, \quad \overline{\partial} \circ \overline{\partial} = 0, \quad \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0. \tag{1.3.8}$$

In fact, it suffices to prove that the above operators are zero on a (p, q) form ω. We have

$$0 = d \circ d(\omega) = \partial \circ \partial(\omega) + (\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)(\omega) + \overline{\partial} \circ \overline{\partial}(\omega). \tag{1.3.9}$$

Since $\partial \circ \partial(\omega)$ is of type $(p + 2, q)$, $(\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)(\omega)$ is of type $(p + 1, q + 1)$, and $\overline{\partial} \circ \overline{\partial}(\omega)$ is of type $(p, q + 2)$, it follows that each vanishes.

1.4 Cauchy's integral formula

Definition 1.4.1. Let $U \subset \mathbb{C}^n$ be open, and let ω be a continuous complex valued 1 form on U. Write $\omega = \alpha + i\beta$, where α, β are (real) 1 forms on U. Given a piecewise C^1 parametrized path $\gamma \colon [a, b] \to U$, we let

$$\int_\gamma \omega := \int_\gamma \alpha + i \int_\gamma \beta = \int_a^b \gamma^*(\alpha) + i \int_a^b \gamma^*(\beta).$$

(Since γ is piecewise C^1, $\gamma^*(\alpha)$ and $\gamma^*(\beta)$ make sense over each closed interval over which γ is differentiable, and they are continuous 1 forms, hence they have finite integrals.)

Remark 1.4.2. With notation as in **Definition 1.4.1**, the integral of ω does not change if we reparametrize γ by a non decreasing differentiable function $[c, d] \to [a, b]$ (by the change of variables formula). Thus we may speak of the integral of ω over an oriented path in U.

Remark 1.4.3. Complex valued differential forms make sense on any open of $\mathbb{R}^d$ (but of course dz_i and $d\overline{z}_i$ make sense only on $\mathbb{C}^n$), and one may define differentiation and pull-back as above, by reducing to the real and imaginary parts. In **Definition 1.4.1** we could have defined $\int_\gamma \omega$ to be $\int_a^b \gamma^*(\omega)$.

Definition 1.4.4. Given $a \in \mathbb{C}$ and $R > 0$ we let $\Gamma_a(R)$ be the path

$$\begin{array}{ccc} [0, 2\pi] & \xrightarrow{\Gamma_a(R)} & \mathbb{C} \\ \theta & \mapsto & a + R\exp(i\theta) \end{array}$$

Example 1.4.5. We have

$$\int_{\Gamma_a(R)} \frac{dz}{z-a} = 2\pi i.$$

In fact

$$\Gamma_a(R)^* \left(\frac{dz}{z-a} \right) = \frac{Rie^{i\theta} d\theta}{Re^{i\theta}} = id\theta.$$

(see **Remark 1.4.3**) and the result follows.

The following integral representation is the beginning of complex analyis in one variable.

Theorem 1.4.6 (Cauchy's integral formula)**.** *Let $f: U \to \mathbb{C}$ be a holomorphic function, where $U \subset \mathbb{C}$ is open. Suppose that the closed disk $\overline{B(a,R)}$ is contained in U. Then, for all $z \in B(a,R)$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_a(R)} \frac{f(t)dt}{t-z} \tag{1.4.1}$$

We prove Cauchy's integral formula after a few preliminaries.

Key Observation 1.4.7. *Let $U \subset \mathbb{C}$ be open, and $f: U \to \mathbb{C}$ differentiable. Then*

$$d(f(z)dz) = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz = \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz.$$

In particular f is holomorphic if and only if the (differentiable) 1 form $f(z)dz$ is closed.

Theorem 1.4.8 (Cauchy-Goursat)**.** *Let $U \subset \mathbb{C}$ be open, and $f: U \to \mathbb{C}$ be holomorphic. Let $R \subset U$ be compact, with piecewise C^1 boundary ∂R, with orientation induced by the standard orientation*[1] *of $\mathbb{C}^n = \mathbb{R}^{2n}$. Then*

$$\int_{\partial R} f dz = 0. \tag{1.4.2}$$

[1]If $z_j = x_j + iy_j$, the orientation is given by $dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$.

Proof. If one assumes that f is C^1, then Stokes' Theorem applies to $f(z)dz$ and the Theorem follows from the **Key Observation 1.4.7**. For the beautiful proof (by Goursat) valid without the assumption that f is C^1, see Ahlfors [**?**]. ☐

Proof of Cauchy's integral formula. By **Example 1.4.5**, it suffices to prove that

$$\int\limits_{\Gamma_a(R)} \frac{f(t)-f(z)}{t-z}dt = 0. \tag{1.4.3}$$

Let δ be a very small (strictly) positive number. By applying **Proposition 1.4.8** to the region between the circles described by $\Gamma_z(\delta)$ and $\Gamma_a(R)$, we get that

$$\int\limits_{\Gamma_a(R)} \frac{f(t)-f(z)}{t-z}dt = \int_{\Gamma_z(\delta)} \frac{f(t)-f(z)}{t-z}dt.$$

On the other hand, $\int\limits_{\Gamma_z(\delta)} f'(a)dt = 0$ because $f'(a)dt = d(f'(a)t)$ is an exact differential, and hence

$$\int\limits_{\Gamma_a(R)} \frac{f(t)-f(z)}{t-z}dt = \int_{\Gamma_z(\delta)} \frac{f(t)-f(z)-f'(z)\cdot(t-z)}{t-z}dt. \tag{1.4.4}$$

Since f is differentiable at a, with derivative $f'(a)$, the integrand in the right hand side of (1.4.4) has absolute bounded above, say by $M > 0$. It follows that the integral in the right hand side of (1.4.4) has absolute bounded above by $2\pi\delta M$. Since δ is arbitrarily small, it follows that integral in the left hand side of (1.4.4) is zero. ☐

Corollary 1.4.9. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function, where $U \subset \mathbb{C}$ is open. Then f is C^∞, and the derivatives of any order are holomorphic. Suppose that the closed disk $\overline{B(a,R)}$ is contained in U. Then for $z \in B(a,R)$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int\limits_{\Gamma_a(R)} \frac{f(t)dt}{(t-z)^{n+1}}. \tag{1.4.5}$$

Proof. If the closed disk $\overline{B(a,R)}$ is contained in U, then (1.5.4) holds for $n = 0$ by Cauchy's integral formula. By differentiation under the integral sign (this has to be justified, we leave details to the reader) and induction on n, we get the corollary. ☐

1.5 Holomorphic functions are analytic

Definition 1.5.1. Let $U \subset \mathbb{C}^n$ be an open subset. A function $f\colon U \to \mathbb{C}$ is *analytic* if, for each $a \in U$ there exist an open ball $B(a,R) \subset U$ and an absolutely convergent power series in $B(a,R)$

$$\sum_{m\in\mathbb{N}^n} c_m(z-a)^m, \tag{1.5.1}$$

where c_m is a complex number and $(z-a)^m = (z_1-a_1)^{m_1}\cdots(z_n-a_n)^{m_1}$, whose sum is equal to $f(z)$ for all $z \in B(a,r)$.

Example 1.1.6 shows that analytic functions of one variable are holomorphic. The same is true of analytic functions of several complex variables. What is surprising is that the converse holds, i.e. holomorphic functions are analytic that is the main result of the present subsection.

Let $a \in \mathbb{C}^n$ and let $(R_1,\ldots,R_n) \in \mathbb{R}^n_+$. Let

$$B(a_1,R_1)\times\cdots\times B(a_n,R_n) \xrightarrow{f} \mathbb{C}$$

be a continuous function. Let $0 < r_i < R_i$ for $i \in \{1,\ldots,n\}$. We let

$$\int\limits_{\Gamma_{a_1}(r_1)\times\cdots\times\Gamma_{a_n}(r_n)} \frac{f(t)dt_1\wedge\cdots\wedge dt_n}{(t_1-z_1)\cdot\cdots\cdot(t_n-z_n)} := \int\limits_{[0,2\pi]^n} \varphi^*\left(\frac{f(t)dt_1\wedge\cdots\wedge dt_n}{(t_1-z_1)\cdot\cdots\cdot(t_n-z_n)}\right), \tag{1.5.2}$$

where $\varphi := \Gamma_{a_1}(r_1)\times\ldots\times\Gamma_{a_n}(r_n)$, i.e.

$$\begin{array}{ccc} [0,2\pi]^n & \xrightarrow{\varphi} & \mathbb{C}^n \\ (\theta_1,\ldots,\theta_n) & \mapsto & (a_1+r_1\exp(i\theta_1),\ldots,a_n+r_n\exp(i\theta_1)) \end{array}$$

(Continuity of f guarantees that the integral in the right hand side of (1.5.2) is defined.)

Proposition 1.5.2. *Let $a \in \mathbb{C}^n$ and let $(R_1,\ldots,R_n) \in \mathbb{R}^n_+$. Suppose that*

$$B(a_1,R_1)\times\cdots\times B(a_n,R_n) \xrightarrow{f} \mathbb{C}$$

is a continuous function which is holomorphic in each variable separately. Let $0 < r_i < R_i$ for $i \in \{1,\ldots,n\}$. Then

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_{a_1}(r_1)\times\cdots\times\Gamma_{a_n}(r_n)} \frac{f(t_1,\ldots,t_n)\,dt_1\wedge\cdots\wedge dt_n}{(t_1-z_1)\cdot\cdots\cdot(t_n-z_n)} \tag{1.5.3}$$

for all $z \in B(a_1,R_1)\times\cdots\times B(a_n,R_n)$

Proof. By induction on n. For $n = 1$ (1.5.3) is Cauchy's formula, i.e. **Theorem 1.4.6**. Let's prove the inductive step. Let $n \geqslant 2$. By Fubini's theorem the right-hand side of (1.5.3) is equal to

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{a_n}(r_n)} \left(\frac{1}{(2\pi i)^{n-1}} \int\limits_{\Gamma_{a_1}(r_1)\times\cdots\times\Gamma_{a_{n-1}}(r_{n-1})} \frac{f(t_1, \ldots, t_n)\, dt_1 \wedge \cdots dt_{n-1}}{(t_1 - z_1) \cdot \cdots \cdot (t_{n-1} - z_{n-1})} \right) \frac{dt_n}{t_n - z_n}.$$

By the inductive hypothesis the above integral is equal to

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{a_n}(r_n)} \frac{f(z_1, \ldots, z_{n-1}, t_n)\, dt_n}{t_n - z_n},$$

and the proposition follows from Cauchy's integral formula □

Arguing as in the proof of **Corollary 1.4.9**, we get the following result.

Corollary 1.5.3. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function, where $U \subset \mathbb{C}^n$ is open. Then f is C^∞, and the partial derivatives of any order are holomorphic. Suppose that the closure of $B(a_1, R_1) \times \cdots \times B(a_n, R_n)$ is contained in U. Then for $z \in B(a_1, R_1) \times \cdots \times B(a_n, R_n)$ we have*

$$\frac{\partial^{k_1+\ldots+k_n} f(z)}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}} = \frac{k_1! \ldots k_n!}{2\pi i} \int\limits_{\Gamma_{a_1}(r_1)\times\cdots\times\Gamma_{a_n}(r_n)} \frac{f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n}{(t_1 - z_1)^{k_1+1} \cdot \cdots \cdot (t_n - z_n)^{k_n+1}}. \tag{1.5.4}$$

Theorem 1.5.4. *Let $U \subset \mathbb{C}^n$ be open and $f\colon U \to \mathbb{C}$. The following conditions on f are equivalent:*

1. *f is holomorphic.*

2. *f is a continuous function, and is holomorphic in each variable separately.*

3. *f is analytic.*

Proof. (1) $\Longrightarrow$ (2): immediate from the definitions. (2) $\Longrightarrow$ (3): by **Proposition 1.5.2** and the geometric series expansion

$$\frac{1}{t_k - z_k} = \frac{1}{t_k - a_k} \cdot \frac{1}{1 - \frac{z_k - a_k}{t_k - a_k}} = \frac{1}{t_k - a_k} + \frac{z_k - a_k}{(t_k - a_k)^2} + \frac{(z_k - a_k)^2}{(t_k - a_k)^3} + \ldots,$$

we get that for $z \in B_{a_1}(r_1) \times \ldots \times B_{a_n}(r_n)$

$$f(z) = \sum_{k \in \mathbb{N}^n} c_{k_1,\ldots,k_n}(z_1 - a_1)^{k_1} \ldots (z_n - a_n)^{k_n}, \tag{1.5.5}$$

where

$$c_{k_1,\ldots,k_n} = \frac{1}{2\pi i} \int\limits_{\Gamma_{a_1}(r_1)\times\cdots\times\Gamma_{a_n}(r_n)} \frac{f(t)dt_1 \wedge \cdots \wedge dt_n}{(t_1 - z_1)^{k_1+1} \cdot \cdots \cdot (t_n - z_n)^{k_n+1}} \tag{1.5.6}$$

(3) $\Longrightarrow$ (1): Since f is analytic, it is a continuous function, and it is analytic in each variable separately. By **Example 1.1.6**, it follows that f is holomorphic in each variable separately. By **Proposition 1.5.2** it follows that f is holomorphic (differentiation under the inegral sign). □

Corollary 1.5.5. *Let $U \subset \mathbb{C}^n$ be open and $f\colon U \to \mathbb{C}$ be holomorphic. Let $a \in U$, and let (according to* **Theorem 1.5.4***) be an expansion in power series of f around a*

$$f(z) = \sum_{m \in \mathbb{N}^n} c_m(z-a)^m, \quad z \in B(a,R) \subset U.$$

Then

$$\frac{\partial^{k_1+\ldots+k_n} f(a)}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}} = (k_1)! \ldots (k_n)! c_{k_1,\ldots,k_n}. \tag{1.5.7}$$

Proof. Follows from **Corollary 1.5.3** and (1.5.6). □

The following result is in stark contrast with what happens for C^∞ functions.

Proposition 1.5.6 (Principle of analytic continuation)**.** *Let $U \subset \mathbb{C}^n$ be open and connected. If $f, g\colon U \to \mathbb{C}$ are holomorphic, and are equal on a non empty open $V \subset U$, then they are equal on all of U.*

Proof. Since the difference of two holomorphic functions is holomorphic, it suffices to prove that if a holomorphic function is zero on a non empty open $V \subset U$, then it is zero on all of U. Let $D \subset U$ be the subset of z such that all partial derivetives of f in z vanish. Then D is closed because it is the intersection of the closed subsets of points where a specific partial derivative vanishes. In addition D is non empty because it contains Y. Since U is connected, it suffices to show that U is also open. If $a \in D$, then f vanishes in a neighborhood of a by **Corollary 1.5.5**, and hence D contains an open neighborhood of a. Thus D is open. □

1.6 Complex manifolds

Let X be a topological manifold. A *holomorphic atlas* on X is a family $\{(U_k, \varphi_k)\}_{k\in K}$, where

1. $\{U_k\}_{k\in K}$ is an open covering of X,
2. $\varphi_k \colon U_k \xrightarrow{\sim} V_k$ is a homeomorphism between U_k and an open $V_k \subset \mathbb{C}^n$,
3. and for each $k, h \in K$, the transition function $\varphi_h(U_h \cap U_k) \longrightarrow \varphi_k(U_h \cap U_k)$ is holomorphic (this makes sense because domain and codomain are open subsets of $\mathbb{C}^n$).

Remark 1.6.1. If $\{(U_k, \varphi_k)\}_{k\in K}$ is a holomorphic atlas on X, we say that (U_k, φ_k) are the charts of the atlas. Each chart determines holomorphic coordinates $(z_1 \circ \varphi_k, \dots, z_n \circ \varphi_k)$ on U_k. It is often convenient to identify U_k with its image $\varphi_k(U_k) \subset \mathbb{C}^n$, and to denote the associated coordinates by $(z_1, \dots, z_n)$.

Two holomorphic atlases on X are *compatible* if the union is a holomorphic atlas. The relation of compatibility is an equivalence relation.

Definition 1.6.2. A *complex manifold* is an equivalence class of holomorphic atlases for the relation of compatibility. If the charts take values in $\mathbb{C}^n$, the *dimension of* X is n.

Let $U \subset \mathbb{C}^n$ be open. The atlas on U defined by the identity map $U \to U$ determines an equivalence class of holomorphic atlases on U, and hence gives U the structure of a complex manifold. Below are non trivial (i.e. not zero dimensional) examples of compact complex manifolds.

Example 1.6.3. Let $\mathbb{P}^n$ be *complex* projective space, with atlas $\{(\mathbb{P}^n_{Z_i}, f_i)\}_{0\leqslant i\leqslant n}$, where $\mathbb{P}^n_{Z_i}$ is the open subset of points whose Z_i homogeneous coordinate is non zero, and

$$\begin{array}{ccc} \mathbb{P}^n_{Z_i} & \xrightarrow{f_i} & \mathbb{C}^n \\ [Z] & \mapsto & (\frac{Z_0}{Z_i}, \dots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \dots, \frac{Z_n}{Z_i}) \end{array}$$

The above atlas is holomorphic, hence it provides $\mathbb{P}^n$ a structure of complex manifold. From now on $\mathbb{P}^n$ denotes the above complex manifold. More generally the *complex* Grassmannian $\mathrm{Gr}(d, n)$ of complex vector subspaces $V \subset \mathbb{C}^n$ of dimension d has the following holomorphic atlas. First, given a multiindex $J = (j_1, \dots, j_d)$, where $1 \leqslant j_1 < \ldots < j_d \leqslant n$, let $V(Z_{j_1}, \dots, Z_{j_d})$

be the kernel of the linear map $\mathbb{C}^n \to \mathbb{C}^d$ defined by $Z \to (Z_{j_1}, \ldots, Z_{j_d})$. Let $\mathrm{Gr}(d,n)_J \subset \mathrm{Gr}(d,n)$ be the open subset defined by

$$\mathrm{Gr}(d,n)_J := \{W \in \mathrm{Gr}(d,n) \mid W \cap V(Z_{j_1}, \ldots, Z_{j_d}) = \{0\}.$$

A d dimensional subspace $W \subset \mathbb{C}^n$ belongs to $\mathrm{Gr}(d,n)_J$ if and only if it has a basis $\{v_1, \ldots, v_d\}$ given by the rows of a matrix

$$\begin{bmatrix} z_{1,1} & \ldots & z_{1,j_1-1} & 1 & z_{1,j_1+1} & \cdots & z_{1,j_2-1} & 0 & z_{1,j_2+1} & \cdots & 0 & z_{1,j_d+1} & \cdots & z_{1,n} \\ z_{2,1} & \ldots & z_{2,j_1-1} & 0 & z_{2,j_1+1} & \cdots & z_{2,j_2-1} & 1 & z_{2,j_2+1} & \cdots & 0 & z_{2,j_d+1} & \cdots & z_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{d,1} & \ldots & z_{d,j_1-1} & 0 & z_{d,j_1+1} & \cdots & z_{d,j_2-1} & 0 & z_{d,j_2+1} & \cdots & 1 & z_{d,j_d+1} & \cdots & z_{d,n} \end{bmatrix}$$

We let $f_J \colon \mathrm{Gr}(d,n)_J \to \mathbb{C}^{d(n-d)}$ be the map associating to W the entries $Z_{k,j}$ above. The atlas $\{(\mathrm{Gr}(d,n)_J, f_J)\}$ is homolorphic, and it gives $\mathrm{Gr}(d,n)$ a structure of complex manifold of dimension $d(n-d)$.

Definition 1.6.4. Let X and Y be complex manifolds. A continuous map $f \colon X \to Y$ is holomorphic if, for any atlases $\{(U_k, \varphi_k)\}_{k\in K}$ of X and $\{(W_h, \psi_h)\}_{h\in H}$ of Y, the following holds. Let $(k,h) \in K \times H$; then the map

$$\begin{array}{ccc} \varphi_k(U_k \cap f^{-1}(W_h)) & \longrightarrow & \mathbb{C}^n \\ z & \mapsto & \psi_h(f(\varphi_k^{-1}(z))) \end{array} \tag{1.6.1}$$

is holomorphic (this makes sense because the domain is an open subset of a $\mathbb{C}^n$).

If the maps in (1.6.1) are holomorphic for one choice of atlas for X, then they are holomorphic for any other choice of compatible atlas. Similarly, if the maps in (1.6.1) are holomorphic for one choice of atlas for Y, they remain holomorphic for a compatible atlas of Y. Thus, in order to check whether a given continuous function is holomorphic, it suffices to check that the maps in (1.6.1) are holomorphic for one choice of atlas for X and one choice of atlas for Y.

In particular, if $U \subset \mathbb{C}^n$ is open, the two definitions of a holomorphic map $f \colon U \to \mathbb{C}^m$, i.e. **Definition 1.2.1** and **Definition 1.6.5**, coincide. We notice that the identity map $\mathrm{Id}_X \colon X \to X$ is holomorphic, and that the composition of holomorphic maps $f \colon X \to Y$ and $g \colon Y \to Z$ is holomorphic.

Definition 1.6.5. Let X and Y be complex manifolds. A holomorphic map $f \colon X \to Y$ is an *isomorphism* if it has a holomorphic inverse, i.e. a holomorphic map $g \colon Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. An *automorphism* of X is an isomorphism between X and itself.

The set of automorphisms of X with operation given by composition is a group that we denote by $\mathrm{Aut}(X)$.

Definition 1.6.6. Let X, Y be two complex manifolds, with holomorphic atlases $\{(U_j, f_j)\}_{j\in J}$ and $\{(V_k, g_k)\}_{k\in K}$ respectively. Then $\{(U_j \times V_k, f_j \times g_k)\}_{(j,k)\in J\times K}$ is a holomorphic atlas of the topological manifold $X \times Y$. Replacing $\{(U_j, f_j)\}_{j\in J}$ and $\{(V_k, g_k)\}_{k\in K}$ by compatible holomorphic atlases, we get a holomorphic atlas compatible with $\{(U_j \times V_k, f_j \times g_k)\}_{(j,k)\in J\times K}$. Hence $X \times Y$ has a complex structure induced by those of X and Y.

From now on $X \times Y$ denotes the complex manifold defined in **Definition 1.6.6**. The projections $X \times Y \to X$ and $X \times Y \to Y$ are holomorphic, because in local coordinates they are given by the projection the first (or last) coordinates. Moreover $X \times Y$ is the product of X and Y in the category of complex manifolds, i.e. given a complex manifold W and holomorphic maps $f\colon W \to X$ and $g\colon W \to Y$, there is a unique holomorphic map $W \to X \times Y$ which composed with the two projections $X \times Y \to X$ and $X \times Y \to Y$ gives back f and g.

A complex manifold determines an *underlying C^∞ manifold*, because a holomorphic atlas is also a C^∞ atlas, and a holomorphic map between complex manifolds is a C^∞ map of the underlying C^∞ manifolds. One distinctive feature of the C^∞ manifolds underlying holomorphic manifolds (beyond having even dimension) is that they are oreintable.

Proposition 1.6.7. *The C^∞ manifold underlying a complex manifold is orientable.*

Proof. Let X be a complex manifold. Let $z = (z_1, \ldots, z_n)$ be holomorphic coordinates on a holomorphic chart (U, f) of X. Then

$$\omega_z := i^n dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n = 2^n dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$$

is a volume form on U. Let $u = (u_1, \ldots, u_n)$ be holomorphic coordinates on another holomorphic chart (V, g) of X, and let ω_u be the corresponding volume form on V. Let $z = \varphi(u)$ be the transition function, and let

$$J(\varphi) := \begin{vmatrix} \frac{\partial \varphi_1}{\partial u_1} & \ldots & \frac{\partial \varphi_1}{\partial u_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \varphi_n}{\partial u_1} & \ldots & \frac{\partial \varphi_n}{\partial u_n} \end{vmatrix}$$

be the holomorphic Jacobian determinant. Then $\omega_z = |J(\varphi)|^2 \omega_w$. Thus the holomorphic atlas of X is oriented, and hence X is orientable. $\square$

Remark 1.6.8. The proof of **Proposition 1.6.7** shows that a complex manifold has a well-defined *complex orientation*. An isomorphism $f\colon X \to Y$ of complex manifolds maps the complex oientation of Y to the complex orientation of X. Moreover the complex orientation of the product of complex manifolds X and Y is the product of the complex orientation of X and the complex orientation of Y.

One goal that we would like to reach when studying complex manifolds is to determine the isomorphism classes of complex manifolds. A necessary condition for two complex manifolds to be isomorphic is that the underlying C^∞ manifolds be diffeomorphic. The latter condition is far from being sufficient. The simplest example is provided by $\mathbb{C}$ and the unit disc $\Delta := \{z \in \mathbb{C} \mid |z| = 1\}$. A holomorphic map $\mathbb{C} \to \Delta$ is constant by Louville's Theorem (see **Exercise 1.8.1**), and hence $\mathbb{C}$ and Δ are *not* isomorphic, although they are clearly diffeomorphic. A richer family of such examples is provided by annuli in $\mathbb{C}$, see Ahlfors [**?**]. We will give plenty of compact examples later on.

Definition 1.6.9. Let X be a complex manifold. A subset $Y \subset X$ is a *complex submanifold of* X if the following holds. There exist a covering $\{U_k\}_{k\in K}$ of X by the open sets of a (holomorphic) atlas of X and, for each $k \in K$, holomorhic functions $f_k^1, \ldots, f_k^r\colon U_k \to \mathbb{C}$ (we identify U_k with an open subset of $\mathbb{C}^n$ via the local chart, see **Remark 1.6.1**) such that

1. $Y \cap U_k$ is the set of zeroes of $f_k^1, \ldots, f_k^r$:
$$Y \cap U_k = \{z \in U_k \mid f_k^1(z) = \ldots = f_k^r(z) = 0\}.$$
2. The differentials $df_k^1(z), \ldots, df_k^r(z)$ are linearly independent for each $z \in Y \cap U_k$.

Given a complex submanifold $Y \subset X$, we can define an equivalence class of holomorphic atlases on Y, by imitating the C^∞ definition - we simply replace the C^∞ Implicit Function Theorem by its holomorphic analogue, i.e. Item (4) of **Theorem 1.2.2**. Thus Y is a complex manifold, and the inclusion map $Y \hookrightarrow X$ is a holomorphic map.

1.7 Tangent space

Let M be a C^∞ manifold, and $a \in M$. The *ring of germs of smooth functions at* a, denoted $\mathscr{E}_{M,a}$, is the set of equivalence classes of couples (U, f), where

$U \subset M$ is an open subset containing a, $f \in \mathscr{C}^\infty(U)$, and couples (U, f), (V, g) are equivalent if there exists a couple (W, h) such that $W \subset U \cap V$ and $h = f_{|W} = g_{|W}$. Given $\varphi = [(U, f)] \in \mathscr{E}_{M,a}$, the evaluation $\varphi(a) := f(a)$ is well defined. Hence we may give the abelian group $\mathbb{R}$ a structure of module over $\mathscr{E}_{M,a}$ by setting $\varphi \cdot x = \varphi(a)x$, for $\varphi \in \mathscr{E}_{M,a}$ and $x \in \mathbb{R}$.

One may define the tangent space of M at a as the real vector space of $\mathbb{R}$ derivations $D\colon \mathscr{E}_{M,a} \to \mathbb{R}$, where $\mathbb{R}$ has the $\mathscr{E}_{M,a}$ module structure defined above. In local coordinates $(x_1, \ldots, x_m)$ centered at a, a basis of tangent space of M at a is given by $(\frac{\partial}{\partial x_1}|_{x=0}, \ldots, \frac{\partial}{\partial x_m}|_{x=0})$. We denote the tangent space to M at a by $T_a^{\mathbb{R}}(M)$.

Now we let $T_a^{\mathbb{C}}(M) := T_a^{\mathbb{R}}(M) \otimes_{\mathbb{R}} \mathbb{C}$ be the *complexified tangent space to M at a*. Moltiplication on the right hand side by complex numbers, gives $T_a^{\mathbb{C}}(M)$ a structure of complex vector space (of dimension $\dim M$). Let $\mathscr{E}_{M,a}^{\mathbb{C}} := \mathscr{E}_{M,a} \otimes_{\mathbb{R}} \mathbb{C}$ be the *ring of germs of complex valued smooth functions at a*. One has a canonical identification of $T_a^{\mathbb{C}}(M)$ with the complex vector space of derivations $\mathrm{Der}_{\mathbb{C}}(\mathscr{E}_{M,a}^{\mathbb{C}}, \mathbb{C})$. Concretely, an element of $\mathscr{E}_{M,a} \otimes \mathbb{C}$ is represnted by $(U, f+ig)$, where $f, g \in \mathscr{C}^\infty(U)$, and a basis (over $\mathbb{C}$) of $T_a^{\mathbb{C}}(M)$ is provided by the basis of $T_a^{\mathbb{R}}(M)$ given above.

Going from the tangent space to the complexified tangent space does not give anything new in general. On the other hand, the complexified tangent space of a complex manifold has a canonical splitting into a direct sum of complex vector spaces of equal dimensions. In order to explain this, we give a couple of definitions.

Definition 1.7.1. Let X be a complex manifold, and $x \in X$. The *ring of germs of holomorphic functions at x* is the set of $\varphi \in \mathscr{E}_{M,a}^{\mathbb{C}}$ which are represented by couples (U, f) such that f is holomorphic (clearly a subring), and is denoted $\mathscr{O}_{X,x}^{\mathrm{an}}$.

Definition 1.7.2. Let X be a complex manifold, and $x \in X$. A (complex) tangent vector $v \in T_x^{\mathbb{C}}(X)$ is *holomorphic* if $v(\overline{\varphi})$ for every $\varphi \in \mathscr{O}_{X,x}^{\mathrm{an}}$, it is *anti holomorphic* if $v(\varphi)$ for every $\varphi \in \mathscr{O}_{X,x}^{\mathrm{an}}$.

In local *holomorphic* coordinates $(z_1, \ldots, z_n)$ centered at x, a basis of the complexified tangent space of X at x is given by

$$\frac{\partial}{\partial z_1}\,|_{z=0}, \ldots, \frac{\partial}{\partial z_n}\,|_{z=0}, \frac{\partial}{\partial \overline{z}_1}\,|_{z=0}, \ldots, \frac{\partial}{\partial \overline{z}_n}\,|_{z=0}\ .$$

The first n tangent vectors are holomorphic, the last n are anti holomorphic. Let $T_x(X) \subset T_x^{\mathbb{C}}(X)$ be the subspace of holomorphic tangent vectors (notice the potential for notational confusion!). Then we have a direct sum

decomposition
$$T_x^{\mathbb{C}}(X) = T_x(X) \oplus \overline{T}_x(X). \tag{1.7.1}$$

1.8 Differential forms on complex manifolds

Exercises

Exercise 1.8.1. Let f be an *entire* function, i.e. a holomorphic function $f: \mathbb{C} \to \mathbb{C}$. Suppose that there exists an integer d such that

$$\lim_{|z| \to +\infty} \frac{|f(z)|}{|z|^{d+1}} = 0. \tag{1.8.1}$$

Prove that f is a polynomial of degree at most d, i.e. there exist $a_0, \ldots a_d \in \mathbb{C}$ such that $f(z) = a_0 z^d + \ldots + a_d$. (Hint: prove that $f^{(n)}(0) = 0$ for $n > d$.) In particular one gets *Liouville's Theorem*: a bounded entire function is constant.

Exercise 1.8.2. Let $U \subset \mathbb{C}$ be open, and $a \in U$. Suppose that $f: (U \backslash \{a\}) \to \mathbb{C}$ is holomorphic, and that there exists $r > 0$ such that f is bounded on $B(a, r) \cap U$. *Riemann's extension Theorem* states that f extends to a holomorphic function $\widetilde{f}: U \to \mathbb{C}$. Prove it as follows. Let $r > 0$ be such that $\overline{B(a, r)} \subset U$. Show that the usual Cauchy integral formula holds for all $z \in (B(a, r) \backslash \{a\})$:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_a(r)} \frac{f(t)}{t - z} dt,$$

and then notice that the right hand side of the above euqation extends to a holomorphic function over a as well.

Exercise 1.8.3. Let $U \subset \mathbb{C}$ be open and connected and let $f: U \to \mathbb{C}$ be holomorphic non constant. Prove that f is *open*, i.e. it maps open sets to open sets, proceeding as follows. Let $a \in U$, and let

$$f(z) = \sum_{m=0}^{\infty} c_m (z - a)^m$$

be a power series expansion of f in a neighborhhod of a, say $B(a, r)$. Let m_0 be the minimum *strictly positive* natural number such that $c_{m_0} \neq 0$ (since f is not constant on U, such an m_0 exists by the Principle of analytic prolungation). Then, on $B(a, r)$ we have
$$f(z) = c_0 + c_{m_0} (z - a)^{m_0} g(z),$$
where g is holomorphic and $g(a) \neq 0$.

1. Prove that for a sufficiently small positive δ, there exists a homolorphic function $h: B(a, \delta)$ such that $g|_{B(a,\delta)} = h^{m_0}$ (use the Inverse function Theorem, i.e. Item (3) of **Theorem 1.2.2**).

2. Let $\varphi\colon B(a,\delta) \to \mathbb{C}$ be the holomorphic function $\varphi(z) = c_{m_0}^{1/m_0}(z-a)\cdot h(z)$. By Item (1), on $B(a,\delta)$ we have $f(z) = c_0 + \varphi(z)^{m_0}$. Check that $\varphi'(a) \neq 0$, and hence $\varphi(B(a,\delta)) \supset B(0,\delta_1)$, for some $\delta_1 > 0$ by the Inverse function Theorem.

3. Conclude that $f(B(a,\delta)) \supset B(c_0, \delta_1^{m_0})$.

Notice that the analogous statement for differentiable (or even analytic) real functions of a real variable is *false*.

Exercise 1.8.4. Prove the *Maximum modulus priciple:* Let $U \subset \mathbb{C}^n$ be open and connected, and let $f\colon U \to \mathbb{C}$ be holomorphic non constant. If $K \subset U$ is compact, any $z_0 \in K$ achieving the maximum of the absolute value function $|f(z)|$ is *not* an interior point of K, i.e $z_0 \in \partial K$. (Hint: if $n = 1$ the result follows at once from **Exercise 1.8.3**. If $n > 1$ reduce to the case $n = 1$ by restricting f to lines in $\mathbb{C}^n$.)

Exercise 1.8.5. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible 2×2 matrix. Then

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \\ [Z_0, Z_1] & \mapsto & [cZ_1 + dZ_0, aZ_1 + bZ_0] \end{array} \tag{1.8.2}$$

is an automorphism of $\mathbb{P}^1$. (The weird choice of formula in (1.8.2) is explained by the formula $f(z) = \frac{az+b}{cz+d}$ valid when using the affine coordinate $z = z_1/z_0$.) Prove that every automorphism of $\mathbb{P}^1$ (as complex manifold!) is of the above form, and hence

$$\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\mathbb{C}),$$

by arguing as follows.

1. Let $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$. Composing with a suitable automorphism in (1.8.2), we may replace φ by an automorphism ψ_0 of $\mathbb{P}^1$ such that $\psi([0,1]) = [0,1]$. The restriction of ψ to the affine line $\mathbb{P}^1 \backslash \{[0,1]\}$ defines a (holomorphic) automorphism $\psi_0 \in \mathrm{Aut}(\mathbb{C})$. It suffices to prove that there exists $(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}$ such that $\psi_0(z) = \alpha z + \beta$.

2. Prove that (1.8.1) holds for $f = \psi_0$ and $d = 1$. Conclude that ψ_0 is a polynomial function of degree 1 by **Exercise 1.8.1**.

Exercise 1.8.6. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an automorphism. Prove that the map $\widetilde{f}\colon \mathbb{P}^1 \to \mathbb{P}^1$ defined by setting

$$\widetilde{f}([Z_0, Z_1]) := \begin{cases} [1, f(\frac{Z_1}{Z_0})] & \text{if } Z_0 \neq 0, \\ [0,1] & \text{if } Z_0 = 0, \end{cases}$$

is an automorphism of $\mathbb{P}^1$. Conclude that there exists $(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}$ such that $f(z) = \alpha z + \beta$.

Exercise 1.8.7. Prove that the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

is isomorphic (as complex manifold) to the unit disc $\Delta \subset \mathbb{C}$. (Hint: find an automorphism f of $\mathbb{P}^1_{\mathbb{C}}$ which takes the closure of the real line to the boundary of the unit disc. Either f or $\frac{1}{f}$ defines an isomorphism between $\mathbb{H}$ and Δ.)

Chapter 2

Algebraic varieties

2.1 Projective varieties

Let $\mathbb{C}[Z_0,\ldots,Z_n]_d \subset \mathbb{C}[Z_0,\ldots,Z_n]$ be the degree-d subspace of the algebra of polynomials. If $F \in \mathbb{C}[Z_0,\ldots,Z_n]_d$, and $Z \in \mathbb{C}^{n+1}$, then $F(Z) = 0$ if and only if $F(\lambda Z) = 0$ for every $\lambda \in \mathbb{C}^*$, because $F(\lambda Z) = \lambda^d F(Z)$. Hence, although $F(x)$ is not defined, it makes to state $F(x) = 0$ or $F(x) \neq 0$ for a point $x \in \mathbb{P}^n$. Let $F_i \in \mathbb{C}[Z_0,\ldots,Z_n]_{d_i}$ for $i \in \{1,\ldots,r\}$; we let

$$V(F_1,\ldots,F_r) := \{x \in \mathbb{P}^n \mid F_1(x) = \ldots = F_r(x) = 0\}.$$

Definition 2.1.1. A subset $X \subset \mathbb{P}^n$ is a *projective variety* if it is equal to $V(F_1,\ldots,F_r)$ for suitable homogeneous polynomials $F_1,\ldots,F_r \in \mathbb{C}[Z_0,\ldots,Z_n]$.

Example 2.1.2. A subset $X \subset \mathbb{P}^n$ is a *hypersurface* if $X = V(F)$, where F is a non zero homogeneous polynomial of strictly positive degree. Assume that

$$V\left(\frac{\partial F}{\partial Z_0},\ldots,\frac{\partial F}{\partial Z_n}\right) = \varnothing. \tag{2.1.1}$$

Then X is a complex submanifold of $\mathbb{P}^n$, of dimension $n-1$. In fact let $(z_0,\ldots,\widehat{z_j},\ldots,z_n)$ be the (customary) holomorphic coordinates on $\mathbb{P}^n_{Z_j}$ given by $z_k := \frac{Z_k}{Z_j}$. Then

$$X \cap \mathbb{P}^n_{Z_j} = \{z \in \mathbb{C}^n \mid F(z_0,\ldots,z_{j-1},1,z_{j+1},\ldots,z_n) = 0,$$

and hence it suffices to show that for each $z \in X \cap \mathbb{P}^n_{Z_j}$, at least one of the partial derivatives $\frac{\partial F}{\partial z_k}(z_0,\ldots,z_{j-1},1,z_{j+1},\ldots,z_n)$ does not vanish. Suppose the contrary.

From Euler's relation we get that

$$0 = (\deg F)F(z_0, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n) = \\ = \sum_{\substack{0 \leqslant k \leqslant n \\ k \neq j}} z_k \frac{\partial F}{\partial z_k}(z_0, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n) + \frac{\partial F}{\partial z_j}(z_0, \ldots, z_{j-1}, 1, z_{j+1} \ldots, z_n) = \\ = \frac{\partial F}{\partial z_j}(z_0, \ldots, z_{j-1}, 1, z_{j+1} \ldots, z_n), \quad (2.1.2)$$

and hence also $\frac{\partial F}{\partial z_j}(z_0, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n)$ vanishes. This contradicts (2.1.1). Notice that $F := \sum_{j=0}^{n} Z_j^d$ provides an example satisfying (2.1.1) in an arbitrary number of variables and arbitrary degree.

Remark 2.1.3. If V is a finite dimensional complex vector space, a subset $X \subset \mathbb{P}(V)$ is a projective variety if there is a collection $F_1, \ldots, F_r$ of homogeneous elements of $\operatorname{Sym} V^\vee$ such that $X = V(F_1, \ldots, F_r)$. Everything that we do in the present section applies to this situation, but for the sake of concreteness we formulate it for $\mathbb{P}^n$.

2.2 Zariski's topology

Let $I \subset \mathbb{C}[Z_0, \ldots, Z_n]$ be a *homogeneous ideal*, i.e. such that

$$I = \bigoplus_{d=0}^{\infty} (I \cap \mathbb{C}[Z_0, \ldots, Z_n]_d). \quad (2.2.1)$$

We let

$$V(I) := \{x \in \mathbb{P}^n \mid F(x) = 0 \ \ \forall \textit{ homogeneous } F \in I\} = \{[Z] \in \mathbb{P}^n \mid F(Z) = 0 \quad \forall F\}.$$

(The second equality holds because I is homogeneous.) If I is generated by homogeneous polynomials $F_1, \ldots, F_r$, then $V(I) = V(F_1, \ldots, F_r)$, and hence $V(I)$ is a projective variety. Conversely, by Hilbert's basis Theorem a homogeneous ideal $I \subset \mathbb{C}[Z_0, \ldots, Z_n]$ is generated by homogeneous polynomials $F_1, \ldots, F_r$, and hence $V(I)$ is a projective variety.

Corollary 2.2.1. *The collection of projective varieties in $\mathbb{P}^n$ satisfies the axioms for the closed subsets of a topological space.*

Proof. We must show that the collection of subsets $V(I) \subset \mathbb{P}^n$, where $I \subset \mathbb{C}[Z_0, \ldots, Z_n]$ is a homogeneous ideal, satisfies the axioms for the closed subsets of a topological space. We have $\varnothing = V((1))$, $\mathbb{P}^n = V((0))$. If I, J are homogeneous ideal, then $I \cap J$ is a homogeneous ideal of $\mathbb{C}[Z_0, \ldots, Z_n]$, and $V(I) \cup V(J) = V(I \cap J)$. If $\{I_t\}_{t \in T}$ is a family of homogeneous ideals of $\mathbb{C}[Z_0, \ldots, Z_n]$, then

$$\bigcap_{t \in T} V(I_t) = V(\langle \{I_t\}_{t \in T} \rangle),$$

where $\langle\{I_t\}_{t\in T}\rangle$ is the (homogeneous) ideal generated by the collection of the I_t's. $\square$

Definition 2.2.2. The topology whose closed sets are projective varieties in $\mathbb{P}^n$ is the *Zariski topology.* The Zariski topology of a subset $A \subset \mathbb{P}^n$ is the topology induced by the Zariski topology of $\mathbb{P}^n$.

Notice that the Zariski topology is weaker than the classical topology of $\mathbb{P}^n$. In fact, unless $n = 0$, the Zariski is much weaker than the classical topology, in particular it is *not* Hausdorff. Given a subset $A \subset \mathbb{P}^n$, let

$$I(A) := \langle F \in \mathbb{C}[Z_0, \ldots, Z_n] \mid F \text{ is homogeneous and } F(p) = 0 \text{ for all } p \in A\rangle, \tag{2.2.2}$$

where $\langle,\rangle$ means "the ideal generated by". Clearly $I(A)$ is a homogeneous ideal of $\mathbb{C}[Z_0, \ldots, Z_n]$, and $V(I(A))$ is the closure of A in the Zariski topology.

Example 2.2.3. Identify $\mathbb{A}^n$ with the open subset $(\mathbb{P}^n\backslash V(Z_0)) \subset \mathbb{P}^n$. A subet $X \subset \mathbb{A}^n$ is closed if and only if there exist an ideal $I \subset \mathbb{C}[z_1, \ldots, z_n]$ (in general *not* homogeneous!) such that $X = V(I)$.

Definition 2.2.4. A *quasi-projective* variety is a Zariski locally closed subset of a projective space, i.e. $X \subset \mathbb{P}^n$ such that $X = U \cap Y$, where $U, Y \subset \mathbb{P}^n$ are Zariski open and Zariski closed respectively.

Definition 2.2.5. Let $X \subset \mathbb{P}^n$ be a quasi projective variety. A *principal open subset* of X is a $U \subset X$ which is equal to

$$Y_F := Y\backslash V(F),$$

where $Y \subset \mathbb{P}^n$ is closed, and $F \in \mathbb{C}[Z_0, \ldots, Z_n]$ is a homogeneous polynomial of *strictly positive degree.*

Claim 2.2.6. *Let $X \subset \mathbb{P}^n$ be locally closed. The collection of principal open subsets of X is a basis of the Zariski topology of X.*

Proof. By hypothesis there exist Zariski closed subsets $Y, W \subset \mathbb{P}^n$ such that $X = Y\backslash W$. We have $W = V(I)$, where $I \subset \mathbb{C}[Z_0, \ldots, Z_n]$ is a homogeneous ideal. Let $J \subset \mathbb{C}[Z_0, \ldots, Z_n]$ be the homogeneous ideal generated by all products $F \cdot Z_i$, where $F \in I$, and $i \in \{0, \ldots, n\}$. Then $V(J) = V(I) = W$, and J is generated by a *non empty* finite set of homogeneous polynomials $F_1, \ldots, F_r$. Then

$$X = Y\backslash V(F_1, \ldots, F_r) = Y_{F_1} \cup Y_{F_2} \cup \ldots \cup Y_{F_r}.$$

$\square$

2.3 Noetherianity and decomposition into irreducibles

A proper projective variety i $\mathbb{P}^1$ is a finite set of points. In general, a quasi projective variety is a finite union of closed subsets which are irreducible, i.e. are not the union of proper closed subsets. This will be proved in the present subsubsection.

The following is a remarkable geometric consequence of Hilbert's basis Theorem.

Proposition 2.3.1. *Let $A \subset \mathbb{P}^n$, and let $A \supset X_0 \supset X_1 \supset \ldots \supset X_m \supset \ldots$ be a descending chain of Zariski closed subsets of A, i.e X_m is defined for all $m \in \mathbb{N}$, and $X_m \supset X_{m+1}$ for all $m \in \mathbb{N}$. Then the chain is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such that $X_m = X_{m_0}$ for $m \geqslant m_0$, i.e. .*

Proof. Let $\overline{X}_i$ be the closure of X_i in $\mathbb{P}^n$. Then $X_i = A \cap \overline{X}_i$, because X_i is closed in A. Hence we may replace X_i by $\overline{X}_i$, or equivalently we may suppose that the X_i are closed in $\mathbb{P}^n$. Let $I_m = I(X_m)$. Then $I_0 \subset I_1 \subset \ldots \subset I_m \subset \ldots$ is an ascending chain of (homogeneous) ideals of $\mathbb{C}[Z_0, \ldots, Z_n]$. By Hilbert's basis Theorem and **Lemma A.1.3** the ascending chain of ideals is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such that $I_{m_0} = I_m$ for $m \geqslant m_0$. Thus $X_{m_0} = V(I_{m_0}) = V(I_m) = X_m$ for $m \geqslant m_0$. □

Corollary 2.3.2. *Let $X \subset \mathbb{P}^n$, with the Zariski topology. Every open covering of X has a finite subcover.*

Definition 2.3.3. Let X be a topological space. We say that X is *reducible* if either $X = \emptyset$ or there exist proper closed subsets $Y, W \subset Z$ such that $X = Y \cup W$. We say that X is *irreducible* if it is not reducible.

Example 2.3.4. $\mathbb{P}^n$ with the euclidean (classical) topology is reducible except if $n = 0$. $\mathbb{P}^n$ with the Zariski topology is irreducible for any n. In fact suppose that $\mathbb{P}^n = Y \cup W$ with Y and W proper closed subsets. Then there exist $F \in I(Y)$ such that $F(p) \neq 0$ for one (at least) $p \in W$ and $g \in I(W)$ such that $g(q) \neq 0$ for one (at least) $q \in Y$. Then $fg = 0$ because $\mathbb{P}^n = Y \cup W$; that is a contradiction because $\mathbb{C}[Z_0, \ldots, Z_n]$ is an integral domain.

We leave the easy proof of the following claim to the reader.

Claim 2.3.5. *Let X be a topological space. A subset of X is irreducible (with the induced topology) if and only if its closure is irreducible.*

The proof of the following result is left to the reader (see **Example 2.3.4**).

Proposition 2.3.6. *A subset $X \subset \mathbb{P}^n$ is irreducible if and only if $I(X)$ is a prime ideal.*

Remark 2.3.7. Let $I := (Z_0^2) \subset \mathbb{C}[Z_0, Z_1]$. Then $V(I) = \{[0,1]\}$ is irreducible although I is *not* prime. Of course $I(V(I))$ is prime, it equals (Z_0). In general $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime (by **Proposition 2.3.6** and the Nullstellensatz).

Definition 2.3.8. Let X be a topological space. An *irreducible decomposition of* X consists of a decomposition (possibly empty)

$$X = X_1 \cup \cdots \cup X_r \tag{2.3.1}$$

where each X_i is a closed irreducible subset of X (irreducible with respect to the induced topology) and moreover $X_i \not\subset X_j$ for all $i \neq j$.

The following result is easily proved. We leave the details to the reader.

Proposition 2.3.9. *Let X be a topological space. Suppose that an irreducible decomposition* (2.3.1) *of X exists. Then the irreducible decomposition of X is unique up to reordering the X_i's. In particular the collection of the X_i's is uniquely determined by X. The X_i's are the* irreducible components of X.

Theorem 2.3.10. *Let $A \subset \mathbb{P}^n$ with the (induced) Zariski topology. Then A admits an irreducible decomposition.*

Proof. If A is empty, then it is the empty union (of irreducibles). Assume that A is not empty. Suppose that A does not admit an irreducible decomposition; then A in reducible, i.e. $A = X_0 \cup W_0$ with $X_0, W_0 \subset A$ proper closed subsets. Suppose that both X_0 and W_0 have an irreducible decomposition; then A is the union of the irreducible components of X_0 and W_0, contradicting the assumption that A does not admit an irreducible decomposition. Hence one of X_0, W_0, does *not* have an irreducible decomposition. We may assume that X_0 does not have an irreducible decomposition. In particular X_0 is reducible, say X_0. Thus $X_0 = X_1 \cup W_1$ with $X_1, W_1 \subset X_0$ proper closed subsets. Iterating the reasoning above, we get a strictly descending chain of closed subsets

$$A \supsetneq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_m \supsetneq X_{m+1} \supsetneq \cdots$$

This contradicts **Proposition 2.3.1**. □

Example 2.3.11. Let $V(F) \subset \mathbb{P}^n$ be a hypersurface, and let $F_1, \ldots, F_r$ be the distinct prime factors of the decomposition of F into a products of primes (recall that $\mathbb{C}[Z_0, \ldots, Z_n]$ is a UFD, by **Corollary A.3.2**. The irreducible decomposition of $V(F)$ is

$$V(F) = V(F_1) \cup \ldots \cup V(F_r).$$

2.4 Regular maps

Definition 2.4.1. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be quasi-projective varieties, and let $\varphi\colon X \to Y$ be a map. Then φ is *regular at $x \in X$* if there exist an open $U \subset X$ containing x and $F_0, \ldots, F_m \in \mathbb{C}[Z_0, \ldots, Z_n]_d$ such that for all $[Z] \in U$,

1. $(F_0(Z), \ldots, F_m(Z)) \neq (0, \ldots, 0)$, and
2. $\varphi([Z]) = [F_0(Z), \ldots, F_m(Z)]$.

The map φ is *regular* if it is regular at each point of X.

The identity map of a quasi-projective variety is regular (choose $F_j(Z) = Z_j$). If $f\colon X \to Y$ and $g\colon Y \to W$ are regular maps of quasi projective varieties, the composition $g \circ f\colon X \to W$ is regular, because the composition of polynomial functions is a polynomial function. Thus we have the category of quasi projective varieties. In particular we have the notion of isomorphism between quasi-projective varieties.

Example 2.4.2. Let $\mathbb{A}^n = \mathbb{P}^n_{Z_0}$ and $\mathbb{A}^m = \mathbb{P}^m_{T_0}$, and let

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{f} & \mathbb{A}^m \\ z & \mapsto & (f_1(z), \ldots, f_m(z)) \end{array}$$

where $f_1, \ldots, f_m \in \mathbb{C}[z_1, \ldots, z_n]$. Then f is regular. In fact,

$$f([Z_0, Z_1, \ldots, Z_n]) = [Z_0^d, Z_0^d f_1\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right), \ldots, Z_0^d f_m\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right)],$$

and if d is large enough, then each of $Z_0^d, Z_0^d f_1\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right), \ldots, Z_0^d f_m\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right)$ is a homogeneous polynomial of degree d.

Example 2.4.3. Let

$$\mathcal{C}_n = \left\{[\xi_0, \ldots, \xi_n] \in \mathbb{P}^n \mid \operatorname{rk}\begin{pmatrix} \xi_0 & \xi_1 & \cdots & \xi_{n-1} \\ \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix} \leqslant 1\right\}. \tag{2.4.1}$$

Since a matrix has rank at most 1 if and only if all the determinants of its 2×2 minors vanish it follows that $\mathscr{C}_n$ is closed, and hence it is a projective variety. We have a regular map

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\varphi_n} & \mathscr{C}_n \\ [s,t] & \mapsto & [s^n, s^{n-1}t, \ldots, t^n] \end{array} \tag{2.4.2}$$

Let us prove that φ_n is an isomorphism. Let $\psi_n\colon \mathcal{C}_n \to \mathbb{P}^1$ be defined as follows:

$$\psi_n([\xi_0, \ldots, \xi_n]) = \begin{cases} [\xi_0, \xi_1] & \text{if } [\xi_0, \ldots, \xi_n] \in \mathcal{C}_n \cap \mathbb{P}^n_{\xi_0} \\ [\xi_{n-1}, \xi_n] & \text{if } [\xi_0, \ldots, \xi_n] \in \mathcal{C}_n \cap \mathbb{P}^n_{\xi_n} \end{cases}$$

Of course one has to check that the two expressions coincide for points in $\mathscr{C}_n \cap \mathbb{P}^n_{\xi_0} \cap \mathbb{P}^n_{\xi_n}$: from (2.4.1) we get that $\xi_0 \cdot \xi_n - \xi_1\xi_{n-1}$ vanishes on $\mathscr{C}_n$ and this shows the required compatibility. One checks easily that $\psi_d \circ \varphi_n = \mathrm{Id}_{\mathbb{P}^1}$ and $\varphi_n \circ \psi_n = \mathrm{Id}_{\mathscr{C}_n}$; thus φ_n defines an isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \mathscr{C}_n$.

Unless we are in the trivial case $n = 1$, it is *not* possible to define ψ_n *globally* as

$$\psi_n([\xi_0, \ldots, \xi_n]) = [P(\xi_0, \ldots, \xi_n), Q(\xi_0, \ldots, \xi_n)], \tag{2.4.3}$$

with $P, Q \in \mathbb{C}[\xi_0, \ldots, \xi_n]_e$. In fact suppose that (2.4.3) holds, and let

$$p(s,t) := P(s^n, \ldots, t^n), \quad q(s,t) := Q(s^n, \ldots, t^n).$$

Then

$$[p(s,t), q(s,t)] = [s,t] \quad \forall [s,t] \in \mathbb{P}^1. \tag{2.4.4}$$

It follows that

$$p(s,t) = as^{de}, \quad q(s,t) = bt^{de},$$

where $a, b \in \mathbb{C}^*$. That contradicts (2.4.4), unless $de = 1$.

The following lemma will be useful later on. The easy proof is left to the reader.

Lemma 2.4.4. *Let $f \colon X \to Y$ be a map between quasi projective varieties. Suppose that $Y = \bigcup_{i \in I} U_i$ is an open cover, that $f^{-1}U_i$ is open in X for each $i \in I$ and that the restriction*

$$\begin{array}{ccc} f^{-1}U_i & \longrightarrow & U_i \\ x & \mapsto & f(x) \end{array}$$

is regular for each $i \in I$. Then f is regular.

Regarding $\mathbb{A}^n$ as the open subset $\mathbb{P}^n_{Z_0}$, it makes sense to give the following.

Definition 2.4.5. An *affine variety* is a quasi projective variety isomorphic to a closed subset of $\mathbb{A}^n$.

Example 2.4.6. Let $F \in \mathbb{C}[Z_0, \ldots, Z_n]$ be a homogeneous polynomial of strictly positive degree. The principal open subset $\mathbb{P}^n_F$ (see **Definition 2.2.5**) is an affine variety. In fact, consider the *Veronese map*

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu^n_d} & \mathbb{P}^{\binom{d+n}{n}-1} \\ [Z] & \mapsto & [Z_0^d, Z_0^{d-1}Z_1, \ldots, Z_n^d] \end{array} \tag{2.4.5}$$

defined by all homogeneous monomials of degree d. The map ν^n_d is clearly regular. One checks that $\mathscr{V}^n_d := \operatorname{Im} \nu^n_d$ is a closed subset of $\mathbb{P}^{\binom{d+n}{n}-1}$ (see **Exercise 2.12.1**) - it is called a *Veronese variety.* Moreover, one shows that the map $\mathbb{P}^n \to \mathscr{V}^n_d$ defined by ν^n_d is an isomorphism. The case $n = 1$ was discussed in **Example 2.4.3**, the general case is treated similarly. From the above it follows that the restriction of ν^n_d to $\mathbb{P}^n_F$ defines an isomorphism between $\mathbb{P}^n_F$ and $\mathscr{V}^n_d \backslash H$, where $H \subset \mathbb{P}^{\binom{d+n}{n}-1}$ is a suitable hyperplane section. Equivalently, $\mathbb{P}^n_F$ is isomorphic to the intersection of the affine space $\mathbb{P}^{\binom{d+n}{n}-1} \backslash H$ and the closed set $\mathscr{V}^n_d$, which, by definition, is an affine variety.

It follows that an arbitrary principal open set Y_F, where $Y \subset \mathbb{P}^n$ is closed, and $F \in \mathbb{C}[Z_0, \ldots, Z_n]$ is homogeneous of strictly positive degree d, is an affine variety. In fact, since ν^n_d is an isomorphism $\nu^n_d(Y_F)$ is closed in the affine variety $\mathscr{V}^n_d \backslash H$, and hence is itself affine. Moreover, the restriction of ν^n_d to Y_F defines an isomorphism Y_F and the affine variety $\nu^n_d(Y_F)$.

Claim 2.2.6 and **Example 2.4.6** give the following result.

Proposition 2.4.7. *The open affine subsets of a quasi projective variety form a basis of Zariski's topology.*

In a certain sense, open affine subsets of a quasi projective variety are similar to the open subsets of a complex manifold given by charts of a holomorphic atlas.

Definition 2.4.8. A *regular function* on a quasi projective variety X is a regular map $X \to \mathbb{C}$.

Let X be a non empty quasi projective variety. The set of regular functions on X with pointwise addition and multiplication is a $\mathbb{C}$-algebra, named the *ring of regular functions* of X. We denote it by $\mathbb{C}[X]$.

Let $X \subset \mathbb{P}^n$ be a quasi projective variety which happens to be a complex submanifold, e.g. hypersurfaces satisfying (2.1.1). Then regular functions on X are holomorphic. If in addition we assume that X is closed, then it is compact (classical topology) and hence every holomorphic function on X is locally constant by the Maximum modulus principle (see **Exercise 1.8.4**). In fact, it is true in general that a regular function on a projective variety is locally constant (see **Exercise ??**). On the other hand, affine varieties have plenty of functions. In fact if $X \subset \mathbb{A}^n$ is closed we have an inclusion

$$\mathbb{C}[z_1, \ldots, z_n]/I(X) \hookrightarrow \mathbb{C}[X]. \tag{2.4.6}$$

Theorem 2.4.9. *Let $X \subset \mathbb{A}^n$ be closed. Then* (2.4.6) *is an equality, i.e. every regular function on X is the restriction of a polynomial function on $\mathbb{A}^n$.*

Before proving **Theorem 2.4.9**, we notice that, if $X \subset \mathbb{A}^n$ is closed, the Nullstellensatz for $\mathbb{C}[z_1, \ldots, z_n]$ implies a Nullstellensatz for $\mathbb{C}[z_1, \ldots, z_n]/I(X)$. First a definition: given an ideal $J \subset (\mathbb{C}[z_1, \ldots, z_n]/I(X))$ we let

$$V(J) := \{a \in X \mid f(a) = 0 \quad \forall f \in J\}.$$

The following result follows at once from the Nullstellensatz.

Proposition 2.4.10 (Nullstellensatz for a closed subset of $\mathbb{A}^n$)**.** *Let $X \subset \mathbb{A}^n$ be closed, and let $J \subset (\mathbb{C}[z_1, \ldots, z_n]/I(X))$ be an ideal. Then*

$$\left\{f \in (\mathbb{C}[z_1, \ldots, z_n]/I(X)) \mid f_{|V(J)} = 0\right\} = \sqrt{J}.$$

(The radical $\sqrt{J}$ is taken inside $\mathbb{C}[z_1, \ldots, z_n]/I(X)$.) In particular $V(J) = \emptyset$ if and only if $J = (1)$.

The following example makes it clear that **Proposition 2.4.10** must play a rôle in the proof of **Theorem 2.4.9**. Let $X \subset \mathbb{A}^n$ be closed. Suppose that $g \in \mathbb{C}[z_1, \ldots, z_n]$ and that $g(a) \neq 0$ for all $a \in Z$. Then $1/g \in \mathbb{C}[X]$ and hence **Theorem 2.4.9** predicts the existence of $f \in \mathbb{C}[z_1, \ldots, z_n]$ such that $g^{-1} = f_{|X}$. By **Proposition 2.4.10**, $(g) = (1)$ in $\mathbb{C}[z_1, \ldots, z_n]/I(X)$, because $V(\overline{g}) = \emptyset$, where $\overline{g} := g_{|X}$. hence there exists $f \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\overline{f} \cdot \overline{g} = 1$, where $\overline{f} := f_{|X}$, i.e. $g^{-1} = f_{|X}$

Proof of **Theorem 2.4.9**. Let $\varphi \in \mathbb{C}[X]$. We claim that there exist $f_i, g_i \in \mathbb{C}[z_1, \ldots, z_n]$ for $1 \leqslant i \leqslant d$ such that

1. $X = \bigcup_{1 \leqslant i \leqslant d} X_{g_i}$, i.e. $V(g_1, \ldots, g_d) \cap X = \emptyset$,
2. for all $a \in X_{g_i}$ we have $\varphi(a) = \frac{f_i(a)}{g_i(a)}$,
3. for $1 \leqslant i \leqslant j$ we have $(g_j f_i - g_i f_j)|_X = 0$.

(Notice: the last item implies that on $X_{g_i} \cap X_{g_j}$ we have $f_i/g_i = f_j/g_j$.) For $i = 1, \ldots, d$ let $\bar{g}_i := g_{i|X}$ and $\bar{f}_i := f_{i|X}$. Then

$$\overline{g}_i \varphi = \overline{f}_i. \tag{2.4.7}$$

In fact by Item (1) it suffices to check that (2.4.7) holds on X_{f_j} for $j = 1, \ldots, d$. For $j = i$ it holds by Item (2), for $j \neq i$ it holds by Item (3). (Notice: if we do not assume that Item (3) holds we only know that (2.4.7) holds on $U_j \cap U_i$.) By **Proposition 2.4.10** we have that $(\bar{g}_1, \ldots, \bar{g}_d) = (1)$, i.e. there exist $h_1, \ldots, h_d \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$1 = \bar{h}_1 \bar{g}_1 + \cdots + \bar{h}_d \bar{g}_d.$$

where $\bar{h}_i := h_{i|X}$. Multiplying by φ both sides of the above equality and remembering (2.4.7) we get that

$$\varphi = \bar{h}_1 \bar{g}_1 \varphi + \cdots + \bar{h}_d \bar{g}_d \varphi = \bar{h}_1 \overline{f}_1 + \ldots + \bar{h}_1 \overline{f}_d = (h_1 f_1 + \cdots + h_d f_d)_{|X}. \tag{2.4.8}$$

It remains to prove that there exist $f_i, g_i \in \mathbb{C}[z_1, \ldots, z_n]$ with the properties stated above. By definition of regular function there exist an open covering of X, and for each set U of the open cover a couple $\alpha, \beta \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\varphi(x) = \alpha(x)/\beta(x)$ for all $x \in U$ (it is understood that $\beta(x) \neq 0$ for all $x \in U$). By **Remark 2.4.11** we may cover U by open affine sets $X_{\gamma_1}, \ldots, X_{\gamma_r}$. Since $V(\beta) \subset \bigcap_{i=1}^{r} V(\gamma_i)$ the Nullstellensatz gives that, for each i, there exist $N_i > 0$ and $\mu_i \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\gamma_i^{N_i} = \mu_i \beta$ and hence $\varphi(x) = \mu_i(x)\alpha(x)/\gamma_i(x)^N$ for all $x \in X_{\gamma_i}$. Since $X_{\gamma_i} = X_{\gamma_i^N}$ we get that we have covered X by principal open sets $X_{g'}$ such that $\varphi = f'/g'$ for all $x \in X_{g'}$, where $f' \in \mathbb{C}[z_1, \ldots, z_n]$ (of course f' depends on g'). By **Corollary 2.3.2**, the open covering has a finite subcovering, corresponding to $f'_1, g'_1, \ldots, f'_d, g'_d$. Now let

$$f_i := f'_i g'_i, \qquad g_i := (g'_i)^2.$$

Clearly Items (1) and (2) hold. In order to check Item (3) we write

$$(g_j f_i - g_i f_j)|_X = ((g'_j)^2 f'_i g'_i - (g'_i)^2 f'_j g'_j)|_X = ((g'_i g'_j)(f'_i g'_j - f'_j g'_i))|_X.$$

Since $\varphi(z) = f'_i(z)/g'_i(z) = f'_j(z)/g'_j(z)$ for all $z \in X_{g'_i} \cap X_{g'_j}$ the last term vanishes on $X_{g'_i} \cap X_{g'_j}$, on the other hand it vanishes also on $(X \backslash X_{g'_i} \cap X_{g'_j}) = X \cap V(g'_i g'_j)$ because of the factor $(g'_i g'_j)$. □

We end the present section with a couple of consequences of **Theorem 2.4.9**.

First we give a more explicit version of **Proposition 2.4.7** in the case that the quasi projective variety itself is affine. Given a quasi projective variety X, and $f \in \mathbb{C}[X]$, let

$$X_f := X \backslash V(f), \tag{2.4.9}$$

where $V(f) := \{x \in X \mid f(x) = 0\}$. The following remark is easily verified.

Remark 2.4.11. Let $X \subset \mathbb{A}^n$ be closed (and hence an affine variety). Let $f \in \mathbb{C}[X]$, and hence by **Theorem 2.4.9** there exists $\widetilde{f} \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\widetilde{f}_{|X} = f$. Let $Y \subset \mathbb{A}^{n+1}$ be the subset of solutions of $g(z_1, \ldots, z_n) = 0$ for all $g \in I(X)$, and the extra equation $f(z_1, \ldots, z_n) \cdot z_{n+1} - 1 = 0$. Then the map

$$\begin{array}{ccc} X_f & \longrightarrow & Y \\ (z_1, \ldots, z_n) & \mapsto & (z_1, \ldots, z_n, \frac{1}{f(z_1, \ldots, z_n)}) \end{array}$$

is an isomorphism. In particular X_f is an open affine subset of X. Moreover, the open affine subset X_f, for $f \in \mathbb{C}[X]$ form a basis for the Zariski topology of X.

Notice that, by **Theorem 2.4.9** and the above isomorphism, every regular function on X_f is given by the restriction to X_f of $\frac{g}{f^m}$, where $g \in \mathbb{C}[X]$ and $m \in \mathbb{N}$.

Next, we give a few remarkable consequences of **Theorem 2.4.9**.

Proposition 2.4.12. *Let R be a finitely generated $\mathbb{C}$ algebra without nilpotents. There exists an affine variety X such that $\mathbb{C}[X] \cong R$ (as $\mathbb{C}$ algebras).*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be generators (over $\mathbb{C}$) of R, and let $\varphi\colon \mathbb{C}[z_1, \ldots, z_n] \to R$ be the surjection of algebras mapping z_i to α_i. The kernel of φ is an ideal $I \subset \mathbb{C}[z_1, \ldots, z_n]$, which is radical because R has no nilpotents. Let $X := V(I) \subset \mathbb{A}^n$. Then $\mathbb{C}[X] \cong R$ by **Theorem 2.4.9**. □

In order to introduce the next result, consider a regular map $f\colon X \to Y$ of (non empty) quasi projective varieties. The pull-back $f^*\colon \mathbb{C}[Y] \to \mathbb{C}[X]$ is the homomorphism of $\mathbb{C}$-algebras defined by $f^*(\varphi) := \varphi \circ f$.

Proposition 2.4.13. *Let Y be an affine variety, and let X be a quasi projective variety. The map*

$$\begin{array}{ccc} \{f\colon X \to Y \mid f \text{ regular}\} & \longrightarrow & \{\varphi\colon \mathbb{C}[Y] \to \mathbb{C}[X] \mid \varphi \text{ homomorphism of } \mathbb{C}\text{-algebras}\} \\ f & \mapsto & f^* \end{array} \tag{2.4.10}$$

is a bijection.

Proof. We may assume that $Y \subset \mathbb{A}^n$ is closed; let $\iota\colon Y \hookrightarrow \mathbb{A}^n$ be the inclusion map. Suppose that $f, g\colon X \to Y$ are regular maps, and that $f^* = g^*$. Then $f^*(\iota^*(z_i)) = g^*(\iota^*(z_i))$ for $i \in \{1, \ldots, n\}$, and hence $f = g$. This proves injectivity of the map in (2.4.10). In order to prove surjectivity, let $\varphi\colon \mathbb{C}[Y] \to \mathbb{C}[X]$ be a homomorphism of $\mathbb{C}$ algebras. Let $f_i := \varphi(\iota^*(z_i))$, and let $f\colon X \to \mathbb{A}^n$ be the regular map defined by $f(x) := (f_1(x), \ldots, f_n(x))$ for $x \in X$. Then $f(x) \in Y$ for all $x \in X$. In fact, since Y is closed, it suffices to show that $g(f(x)) = 0$ for all $g \in I(X)$. Now

$$g(f_1(x), \ldots, f_n(x)) = g(\varphi(\iota^*(z_1)), \ldots, \varphi(\iota^*(z_n)) = \varphi(g(\iota^*(z_1)), \ldots, \iota^*(z_n)) = \varphi(0) = 0.$$

(The second and last equality hold because φ is a homomorphism of $\mathbb{C}$-algebras.) Thus f is a regular map $f\colon X \to Y$ such that $f^*(\iota^*(z_i)) = \varphi(\iota^*(z_i))$ for $i \in \{1, \ldots, n\}$. By **Theorem 2.4.9** the $\mathbb{C}$-algebra $\mathbb{C}[Y]$ is generated by $\iota^*(z_1), \ldots, \iota^*(z_n)$; it follows that $f^* = \varphi$. □

Corollary 2.4.14. *In* **Proposition 2.4.12***, the affine variety X such that $\mathbb{C}[X] \cong R$ is unique up to isomorphism.*

2.5 Products

The category of quasi projective sets has products. If $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are quasi projective sets which happen to be complex submanifolds (or just locally complex submanifolds), then the product $X \times Y$ in the category of quasi projective sets is a complex submanifold of $\mathbb{P}^{m \cdot n+m+n}$ isomorphic to the product of X and Y in the category of complex manifolds. We go thorugh the construction of products in the category of quasi projective sets. Proofs are absent or sketched.

First let X, Y be affine varieties. Thus, we may assume that $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are closed subsets. Then $X \times Y \subset \mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$ is a closed subset, and the maps $X \times Y \to X$ and $X \times Y \to Y$ given by the two projections are regular. One checks easily that $X \times Y$ with the two projection maps is the product of X and Y in the category of quasi projective varieties (use **Proposition 2.4.13**). The ring of regular functions of $X \times Y$ is constructed from $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ as follows. Let $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ be the projections. The $\mathbb{C}$-bilinear map

$$\begin{array}{ccc} \mathbb{C}[X] \times \mathbb{C}[Y] & \longrightarrow & \mathbb{C}[X \times Y] \\ (f,g) & \mapsto & \pi_X^*(f) \cdot \pi_Y^*(g) \end{array} \tag{2.5.1}$$

induces a linear map

$$\mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y] \longrightarrow \mathbb{C}[X \times Y]. \tag{2.5.2}$$

Proposition 2.5.1. *The map in* (2.5.2) *is an isomorphism.*

Proof. We may assume that $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are closed subsets. Then $X \times Y \subset \mathbb{A}^{m+n}$ is closed subset, and hence the map in (2.5.2) is surjective by **Theorem 2.4.9**. It remains to prove injectivity, i.e. the following: if $A \subset \mathbb{C}[X]$ and $B \subset \mathbb{C}[Y]$ are finite-dimensional complex vector subspaces, then the map $A \otimes B \to \mathbb{C}[X \times Y]$ obtained by restriction of (2.5.2) is injective. Let $\{f_1, \ldots, f_a\}$, $\{g_1, \ldots, g_b\}$ be bases of A and B. By considering the maps

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{C}^a \\ z & \mapsto & (f_1(z), \ldots, f_a(z)) \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & \mathbb{C}^b \\ z & \mapsto & (g_1(z), \ldots, g_b(z)) \end{array} \tag{2.5.3}$$

we get that there exist $p_1, \ldots, p_a \in X$ and $q_1, \ldots, q_b \in Y$ such that the square matrices $(f_i(p_j))$ and $(g_i(q_j))$ are non-singular. By change of bases, we may assume that $f_i(p_j) = \delta_{ij}$ and $g_k(q_h) = \delta_{kh}$. Computing the values of $\pi_X^*(f_i) \cdot \pi_Y^*(g_j)$ on (p_s, q_t) for $1 \leqslant i, s \leqslant a$ and $1 \leqslant j, t \leqslant b$ we get that the functions $\ldots, \pi_X^*(f_i) \cdot \pi_Y^*(g_j), \ldots$ are linearly independent. Thus $A \otimes B \to \mathbb{C}[W \times Z]$ is injective. □

Since every quasi projective variety has an open cover by affine varieties, one could try to define the product of quasi projective varieties X and Y by gluing together the products of the affine varieties in open coverings of X and Y. This is done in scheme theory, where schemes are algebriac varieties defined by atlases with charts given by affine schemes. However, one wants to show more, for example that the product of projective varieties is a projective varietry. This is why we need the more elaborate construction presented below.

Let $\mathscr{M}_{m+1,n+1}$ be the vector space of complex $(m+1)\times(n+1)$ matrices. Let

$$\Sigma_{m,n} := \{[A] \in \mathbb{P}(\mathscr{M}_{m+1,n+1}) \mid \operatorname{rk} A = 1\}.$$

Then $\Sigma_{m,n}$ is a projective variety in $\mathbb{P}(\mathscr{M}_{m+1,n+1}) = \mathbb{P}^{mn+m+n}$. In fact the entries of a non zero matrix $A \in \mathscr{M}_{m+1,n+1}$ define homogegeous coordinates on $\mathbb{P}(\mathscr{M}_{m+1,n+1})$, and $\Sigma_{m,n}$ is the set of zeroes of determinants of all 2×2 minors of A. Let $[W] \in \mathbb{P}^m$ and $[Z] \in \mathbb{P}^n$; then $W^t \cdot Z$ is a complex $(m+1)\times(n+1)$ matrix of rank 1, determined up to recsaling. Thus we have the *Segre map*

$$\begin{array}{ccc} \mathbb{P}^m \times \mathbb{P}^n & \xrightarrow{\sigma_{m,n}} & \Sigma_{m,n} \\ ([W],[Z]) & \mapsto & [W^t \cdot Z] \end{array} \tag{2.5.4}$$

Proposition 2.5.2. *The map in* (2.5.4) *is a bijection.*

From now on, we identify $\mathbb{P}^m \times \mathbb{P}^n$ with the projective variety $\Sigma_{m,n}$. In particular $\mathbb{P}^m \times \mathbb{P}^n$ has a Zariski topology.

Claim 2.5.3. *A subset $X \subset \mathbb{P}^m \times \mathbb{P}^n$ is closed if and only if there exist bihomogeneous polynomials* [1]

$$F_1, \ldots, F_r \in \mathbb{C}[W_0, \ldots, W_m, Z_0, \ldots, Z_n]$$

such that

$$X = V(F_1, \ldots, F_r) := \{([W],[Z]) \in \mathbb{P}^n \times \mathbb{P}^m \mid 0 = F_1(W;Z) = \cdots = F_r(W;Z)\}. \tag{2.5.5}$$

Remark 2.5.4. If $m \neq 0$ and $n \neq 0$, then the Zariski topology on the product $\mathbb{P}^m \times \mathbb{P}^n$ is *not* the product topology. In fact it is finer than the product topology

Example 2.5.5. The diagonal $\Delta_{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^n$ is closed. In fact, Δ is the set of couples $([W],[Z])$ such that the matrix with rows W and Z has rank less than 2, and hence it is the zero locus of the bihomogeneous polynomials $W_iZ_j - W_jZ_i$ for $(i,j) \in \{0,\ldots,n\}$. Notice that this is not in contrast with the fact that, if $n \neq 0$, the Zariski topology on $\mathbb{P}^n$ is *not* Hausdorff, because of **Remark 2.5.4**.

Claim 2.5.6. *The projections of $\mathbb{P}^m \times \mathbb{P}^n$ on its two factors are regular maps.*

Proof. Let a_{ij}, where $(i,j) \in \{0,\ldots,m\}\times\{0,\ldots,n\}$, be the homogeneous coordinates on $\mathbb{P}(\mathscr{M}_{m+1,n+1})$ given by the entries of a matrix $A \in \mathscr{M}_{m+1,n+1}$. Then

$$\mathbb{P}^m \times \mathbb{P}^n = \bigcup_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} (\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}. \tag{2.5.6}$$

[1] A polynomial $F \in \mathbb{C}[W;Z]$ is *bihomogeneous of degree* (d,e) if $F = \sum_{\substack{\deg I = d\\ \deg J = e}} a_{I,J} W^I Z^J$.

On the open subset $(\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}$, the projections $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$, $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ are given by

$$\begin{array}{ccc} \mathbb{P}^m \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^m \\ [A] & \mapsto & [a_{0j}, \ldots, a_{mj}] \end{array} \qquad \begin{array}{ccc} \mathbb{P}^m \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ [A] & \mapsto & [a_{i0}, \ldots, a_{in}] \end{array}$$

respectively. $\square$

Proposition 2.5.7. *Let X be a quasi projective variety, and let $f\colon X \to \mathbb{P}^m$ and $g\colon X \to \mathbb{P}^n$ be regular maps. Then*

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^m \times \mathbb{P}^n \\ x & \mapsto & (f(x), g(x)) \end{array} \tag{2.5.7}$$

is a regular map.

Proof. We have the open cover of $\mathbb{P}^m \times \mathbb{P}^n$ given by (2.5.6), with open sets indicized by $\{0, \ldots, m\} \times \{0, \ldots, n\}$. By **Lemma 2.4.4**, it suffices to prove that, for each $(i, j) \in \{0, \ldots, m\} \times \{0, \ldots, n\}$, the following hold:

1. $(f \times g)^{-1}(\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}})$ is open in X.
2. The restriction

$$\begin{array}{ccc} (f \times g)^{-1}(\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}) & \longrightarrow & (\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}} \\ x & \mapsto & (f(x), g(x)) \end{array} \tag{2.5.8}$$

   is regular.

We have

$$(f \times g)^{-1}((\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}) = X \backslash (f^{-1}V(W_i) \cup g^{-1}V(Z_j)).$$

Both f and g are continuous, because they are regular, and hence $f^{-1}V(X_i)$ and $g^{-1}V(Y_j)$ are closed. It follows that Item (1) holds. The map

$$\begin{array}{ccc} \mathbb{A}^m \times \mathbb{A}^n & \longrightarrow & (\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}} \\ ((w_0,\ldots,\widehat{w}_i,\ldots,w_m),(z_0,\ldots,\widehat{z}_j,\ldots,z_n)) & \mapsto & ([w_0,\ldots,w_{i-1},1,w_{i+1}\ldots,w_m],[z_0,\ldots,z_{j-1},1,z_{j+1},\ldots,z_n]) \end{array}$$

is an isomorphism commuting with the projections. Item (2) follows. $\square$

It follows that $\mathbb{P}^m \times \mathbb{P}^n$ with the two projections is the product of $\mathbb{P}^m$ and $\mathbb{P}^n$ in the category of quasi projective varieties.

Now suppose that $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are locally closed sets. It follows from **Claim 2.5.3** that $Y \times Y \subset \mathbb{P}^m \times \mathbb{P}^n$ is locally closed, i.e. we have identified $W \times Z$ with a quasi-projective set. Moreover, the projections of $X \times Y$ to X and Y are regular, because they are the restrictions of the projections of $\mathbb{P}^m \times \mathbb{P}^n$ to $X \times Y$.

The proof of the following result is easy; we leave details to the reader.

Proposition 2.5.8. *Keep notation as above. The quasi projective variety $X \times Y$, with the projections to the two factors, is the product of X and Y in the category of quasi projective sets.*

Notice that if $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are closed then $X \times Y$ is closed in $\mathbb{P}^m \times \mathbb{P}^n$. Hence the product of projective varieties is a projective variety. On the othar hand, we have already observed that the product of affine varieties is an affine varietry.

Remark 2.5.9. Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ be locally closed sets. Let $\varphi\colon X \xrightarrow{\sim} X'$, $\psi\colon Y \xrightarrow{\sim} Y'$ be isomorphisms, where $X' \subset \mathbb{P}^a$ and $Y' \subset \mathbb{P}^b$ are locally closed sets. Then

$$\begin{array}{ccc} X \times Y & \longrightarrow & X' \times Y' \\ (p, q) & \mapsto & (\varphi(p), \psi(q)) \end{array} \tag{2.5.9}$$

is an isomorphism. This follows from the formal property of a categorical product. Thus the isomorphism class of $X \times Y$ is independent of the embeddings $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$. This is why we say that $X \times Y$ is *the* product of X and Y.

Since the product of two quasi projective varieties exists, also the product $X_1 \times \ldots \times X_r$ of a finite collection $X_1, \ldots, X_r$ of quasi-projective varieties exists; it is given by $(X_1 \times (X_2 \times (X_3 \ldots \times X_r) \ldots)$ (we may rearrange the parenthesis arbitrarily, and we will get an isomorphic variety).

Let X be a quasi projective variety, and let $\Delta_X \subset X \times X$ be the diagonal. It follows from **Example 2.5.5** that Δ_X is closed in $X \times X$ (this is not in contradiction with the fact that, if X is not finite, then it is not Hausdorff, see **Remark 2.5.4**). This property of quasi projective varieties goes under the name of *properness*. The following is a consequence of properness.

Proposition 2.5.10. *Let X, Y be quasi projective varieties, and let f, g be regular maps $X \to Y$. If $f(x) = g(x)$ for x in a dense subset of X, then $f = g$.*

Proof. Let $\varphi\colon X \to Y \times Y$ be the map defined by $\varphi(x) := (f(x), g(x))$. Then φ is regular, because $Y \times Y$ is the categorical square of Y. Since Δ_Y is closed, $\varphi^{-1}(\Delta_Y)$ is closed. By hypothesis $\varphi^{-1}(\Delta_Y)$ contains a dense subset of X, hence it is equal to X, i.e. $f(x) = g(x)$ for all $x \in X$. □

2.6 Elimination theory

Let M be a topological space. Then M is quasi compact, i.e. every open covering has a finite subcovering, if and only if M is universally closed, i.e. for any topological space T, the projection map $T \times M \to T$ is closed, i.e. it maps closed sets to closed sets. (See tag/005M in [TSPR].)

A quasi projective variety X is quasi compact, but it is not generally true that, for a variety T, the projection $T \times X \to T$ is closed. In fact, let $X \subset \mathbb{P}^n$ be locally closed; then Δ_X, the diagonal of X, is closed in $X \times \mathbb{P}^n$, because it is the intersection of $X \times X \subset \mathbb{P}^n \times \mathbb{P}^n$ with the diagonal $\Delta_{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^n$, which is closed. The projection $X \times \mathbb{P}^n \to \mathbb{P}^n$ maps X to X, hence if X is not closed in $\mathbb{P}^n$, then X is not universally closed. This does not contradict the result in topology quoted above, because the Zariski topology of the product of quasi projective varieties is not the product topology.

The following key result states that projective varieties are the equivalent of compact topological spaces in the category of quasi projective varieties.

Theorem 2.6.1 (Main Theorem of elimination theory)**.** *Let T be a quasi-projective variety and X be a projective variety. Then the projection*

$$\pi\colon T \times X \to T$$

is closed.

Proof. By hypothesis we may assume that $X \subset \mathbb{P}^n$ is closed. It follows that $T \times X \subset T \times \mathbb{P}^n$ is closed. Thus it suffices to prove the result for $X = \mathbb{P}^n$. Since T is covered by open affine subsets, we may assume that T is affine, i.e. T is (isomorphic to) a closed subset of $\mathbb{A}^m$ for some m. It follows that it suffices to prove the proposition for $T = \mathbb{A}^m$. To sum up: it suffices to prove that if $X \subset \mathbb{A}^m \times \mathbb{P}^n$ is closed, then $\pi(X)$ is closed in $\mathbb{A}^m$, where $\pi\colon \mathbb{A}^m \times \mathbb{P}^n \to \mathbb{A}^m$ is the projection. We will show that $(\mathbb{A}^m \backslash \pi(X))$ is open. By **Claim 2.5.3** there exist $F_i \in \mathbb{C}[t_1, \ldots, t_m, Z_0, \ldots, Z_n]$ for $i = 1, \ldots, r$, *homogeneous* as polynomial in $X_0, \ldots, X_n$ such that

$$X = \{(t, [Z]) \mid 0 = F_1(t, Z) = \ldots = F_r(t, Z)\}.$$

Suppose that $F_i \in \mathbb{C}[t_1, \ldots, t_m][Z_0, \ldots, Z_n]_{d_i}$ i.e. F_i is homogeneous of degree d_i in $Z_0, \ldots, Z_n$. Let $\bar{t} \in (T \backslash \pi(X))$. By Hilbert's Nullstellensatz, there exists $N \geqslant 0$ such that

$$(F_1(\bar{t}, Z), \ldots, F_r(\bar{t}, Z)) \supset \mathbb{C}[Z_0, \ldots, Z_n]_N. \tag{2.6.10}$$

We may assume that $N \geqslant d_i$ for $1 \leqslant i \leqslant r$. For $t \in \mathbb{A}^m$ let

$$\begin{array}{ccc} \mathbb{C}[Z_0, \ldots, Z_n]_{N-d_1} \times \ldots \times [Z_0, \ldots, Z_n]_{N-d_r} & \xrightarrow{\Phi(t)} & \mathbb{C}[Z_0, \ldots, Z_n]_N \\ (G_1, \ldots, G_r) & \mapsto & \sum_{i=1}^r G_i \cdot F_i \end{array}$$

Thus $\Phi(t)$ is a linear map: choose bases of domain and codomain and let $M(t)$ be the matrix associated to $\Phi(t)$. Clearly the entries of $M(t)$ are elements of $\mathbb{C}[t_1, \ldots, t_m]$. By hypothesis $\Phi(\bar{t})$ is surjective and hence there exists a maximal minor of $M(t)$, say $M_{I,J}(t)$, such that $\det M_{I,J}(\bar{t}) \neq 0$. The open $(\mathbb{A}^m_{\mathbb{C}} \backslash V(\det M_{I,J}))$ is contained in $(T \backslash \pi(X))$. This finishes the proof of **Theorem 2.6.1**. $\square$

We will give a few corollaries of **Theorem 2.6.1**. First, we prove an elemntary auxiliary result.

Lemma 2.6.2. *Let $f\colon X \to Y$ be a regular map between quasi-projective varieties. The graph of f*

$$\Gamma_f := \{(x, f(x)) \mid \ p \in X\}$$

is closed in $X \times Y$.

Proof. The map

$$f \times \mathrm{Id}_Y\colon X \times Y \to Y \times Y$$

is regular, and $\Gamma_f = (f \times \mathrm{Id}_X)^{-1}(\Delta_Y)$. Hence Γ_f is closed because Δ_Y is closed in $Y \times Y$. $\square$

Proposition 2.6.3. *Let X be a projective variety and Y be a quasi-projective set. A regular map $f: X \to Y$ is closed.*

Proof. Since closed subsets of X are projective it suffices to prove that $f(X)$ is closed in Y. Let $\pi: X \times Y \to Y$ be the projection map. Then $f(X) = \pi(\Gamma_f)$. By **Lemma 2.6.2** and the Main Theorem of elimination theory we get that $f(X)$ is closed. □

Corollary 2.6.4. *A locally-closed subset of $\mathbb{P}^n_{\mathbb{C}}$ is projective if and only if it is closed.*

Corollary 2.6.5. *Let X be a projective set. A regular map $f: X \to \mathbb{C}$ is locally constant.*

Proof. Composing f with the inclusion $j: \mathbb{C} \hookrightarrow \mathbb{P}^1$ we get a regular map $\bar{f}: X \to \mathbb{P}^1$. By **Proposition 2.6.3** $\bar{f}(X)$ is closed. Since $\bar{f}(X) \not\ni [0,1]$ it follows that $\bar{f}(X) = f(X)$ is a finite set. □

2.7 Rational maps

Let X and Y be quasi projective varieties. We define a relation on the set of couples (U, φ) where $U \subset X$ is open dense and $\varphi: U \to Y$ is a regular map, as follows: $(U, \varphi) \sim (V, \psi)$ if the restrictions of φ and ψ to $U \cap V$ are equal. One checks easily that $\sim$ is an equivalence relation.

Definition 2.7.1. A *rational map* $f: X \dashrightarrow T$ is a $\sim$-equivalence class of couples (U, φ) where $U \subset X$ is open dense and $\varphi: U \to Y$ is a regular map. Let $f: X \dashrightarrow Y$ be a rational map.

1. The map f is *regular* at $x \in X$ (equivalently x is a *regular point* of f), if there exists (U, φ) in the equivalence class of f such that $x \in U$. We let $\mathrm{Reg}(f) \subset X$ be the set of regular points of f.
2. The point $x \in X$ is a *point of indeterminancy* if it is in $X \backslash \mathrm{Reg}(f)$.

From now on we will consider only rational maps between *irreducible* quasi projective varieties. Let $f: X \dashrightarrow Y$ and $g: Y \dashrightarrow W$ be rational maps between (irreducible) quasi projective varieties. It might happen that for all $x \in \mathrm{Reg}(f)$ the image $f(x)$ does not belong to $\mathrm{Reg}(g)$, and then the composition $g \circ f$ makes no sense. In order to deal with compositions of reational maps, we give the following definition.

Definition 2.7.2. A rational map $f: X \dashrightarrow Y$ between irreducible quasi projective varieties is *dominant* if it is represented by a couple (U, φ) such that $\varphi(U)$ is dense in Y.

Notice that if $f: X \dashrightarrow Y$ is dominant and (V, ψ) is an arbitrary representative of f then $\psi(V)$ is dense in Y.

Definition 2.7.3. Let $f\colon X \dashrightarrow Y$ be a dominant rational map, and let $g\colon Y \dashrightarrow W$ be a rational map (X, Y, W are irreducible). Let (U, φ) and (V, ψ) be representatives of f and g respectively. Then $\varphi^{-1}V$ is open dense in X. We let $g \circ f\colon X \dashrightarrow W$ be the rational map represented by $(\varphi^{-1}V, \psi \circ \varphi)$. (The equivalence class of $(\varphi^{-1}V, \psi \circ \varphi)$ is independent of the representatives (U, φ) and (V, ψ).)

Definition 2.7.4. A dominant rational map $f\colon X \dashrightarrow Y$ between irreducible quasi projective varieties is *birational* if there exists a dominant rational map $g\colon Y \dashrightarrow X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. An irreducible quasi projective variety X is *rational* if it is birational to $\mathbb{P}^n$ for some n, it is *unirational* if there exists a dominant rational map $f\colon \mathbb{P}^n \dashrightarrow X$.

Example 2.7.5.

1. Of course isomorphic irreducible quasi projective varieties are birational. On the other a quasi projective (irreducible) variety is birational to any of its dense open subsets. In particular $\mathbb{P}^n$ is birational to $\mathbb{A}^n$, although they are not isomorphic if $n > 0$ (if they were isomorphic, they would be diffeomorphic as C^∞ manifolds, but $\mathbb{P}^n$ is compact, $\mathbb{A}^n$ is not).

2. Let $0 \neq F \in \mathbb{C}[Z_0, \ldots, Z_n]_2$, and let $Q^{n-1} := V(F) \subset \mathbb{P}^n$. Suppose that F is prime, i.e that $\operatorname{rk} F \geqslant 3$, and hence Q^{n-1} is irreducible. We claim that Q^{n-1} is rational. In fact, after a suitable change of coordinates, we may assume that $F = Z_0 Z_n - G$, where $0 \neq G \in \mathbb{C}[Z_1, \ldots, Z_{n-1}]_2$. The rational maps
$$\begin{array}{ccc} Q^{n-1} & \overset{f}{\dashrightarrow} & \mathbb{P}^{n-1} \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_{n-1}] \end{array}$$
and
$$\begin{array}{ccc} \mathbb{P}^{n-1} & \overset{g}{\dashrightarrow} & Q^{n-1} \\ [T_0, \ldots, T_{n-1}] & \mapsto & [T_0^2, T_0T_1, \ldots, T_0T_{n-1}, G(T_1, \ldots, T_{n-1})] \end{array}$$
are dominant, and they are inverses of each other. Notice that if $n = 2$, then f and g are regular (see **Example 2.4.3**), while for $n \geqslant 3$, the quadric Q^{n-1} is not isomorphic to $\mathbb{P}^{n-1}$, because the underlying C^∞ manifolds are not homeomorphic.

Proposition 2.7.6. *Irreducible quasi varieties X, Y are birational if and only if there exist open dense subsets $U \subset X$ and $V \subset Y$ that are isomorphic.*

Proof. An isomorphism $\varphi\colon U \xrightarrow{\sim} V$ clearly defines a birational map $f\colon X \dashrightarrow Y$. Conversely, suppose that $f\colon X \dashrightarrow Y$ is birational with inverse $g\colon Y \dashrightarrow X$. Let (U, φ) represent f and (V, ψ) represent g. Then $\varphi^{-1}V \subset U$ and $\psi^{-1}U \subset V$ are open dense. By hypothesis the composition $\psi \circ \left(\varphi_{|\varphi^{-1}V}\right) : \varphi^{-1}V \to U$ is equal to the identity on an open non-empty subset of $\varphi^{-1}V$. By **Proposition 2.5.10**, we get that $\psi \circ \left(\varphi_{|\varphi^{-1}V}\right) = \mathrm{Id}_{\varphi^{-1}V}$. In particular $\psi \circ \varphi\left(\varphi^{-1}V\right) \subset U$ i.e. $\varphi\left(\varphi^{-1}V\right) \subset \psi^{-1}U$, and similarly $\varphi \circ \left(\psi_{|\psi^{-1}U}\right) = \mathrm{Id}_{\psi^{-1}U}$ and $\psi\left(\psi^{-1}U\right) \subset \varphi^{-1}V$. Thus we have isomorphisms $\varphi^{-1}V \xrightarrow{\sim} \psi^{-1}U$ and $\psi^{-1}U \xrightarrow{\sim} \varphi^{-1}V$. $\square$

Many natural invariants of projective varieties do not separate between (projective) birational varieties. This fact gives practical criteria that allow to establish that certain projective varieties are not birational. On the other hand, it leads us to approach the classification of isomorphism classes of projective varieties in two steps: first we classify equivalence classes for birational equivalence, then we distinguish isomorphim classes within each birational equivalence class.

2.8 The field of rational functions

If we consider the category whose objects are irreducible quasi projective varieties, and morphisms are dominant rational maps, we get a familiar algebraic category. In order to explain this, we introduce a key definition. Let X be an irreducible quasi projective variety. The *field of rational functions on X* is

$$\mathbb{C}(X) := \{f\colon X \dashrightarrow \mathbb{C} \mid f \text{ is a rational map}\}. \tag{2.8.1}$$

Addition and multiplication are defined on representatives. Let $f, g \in \mathbb{C}(X)$ be represented by (U, φ) and (V, ψ) respectively. Then

$$\begin{aligned} f+g &:= [\big(U \cap V, \varphi_{|U\cap V} + \psi_{|U\cap V}\big)], \\ f\cdot g &:= [\big(U \cap V, \varphi_{|U\cap V} \cdot \psi_{|U\cap V}\big)]. \end{aligned}$$

Example 2.8.1.

- $\mathbb{C}(\mathbb{P}^n) \cong \mathbb{C}(z_1, \ldots, z_n)$ is the purely transcendental extension of $\mathbb{C}$ of transcendence degree n.
- Let $p \in \mathbb{C}[z]$ be free of square factors (and $\deg p \geqslant 1$). Then $t^2 - p(z)$ is prime and hence $X := V\left(t^2 - p(z)\right) \subset \mathbb{A}^2$ is irreducible. Then $\mathbb{C}(z) \subset \mathbb{C}(X)$ is an extension of degree 2. We may ask whether $\mathbb{C}(X)$ is a purely trascendental extension of $\mathbb{C}$. The answer is *yes* if $\deg p = 1, 2$ (see **Example 2.4.3**), *no* if $\deg p \geqslant 3$ (this requires new ideas).

Let $f\colon X \dashrightarrow Y$ be a dominant rational map of irreducible quasi projective varieties. We have a well-defined *pull-back*

$$\begin{array}{ccc} \mathbb{C}(Y) & \xrightarrow{\varphi^*} & \mathbb{C}(X) \\ \varphi & \mapsto & \varphi \circ f \end{array}$$

(The composition is well defined because by hypothesis f is dominant.) The map f^* is an inclusion of extensions of $\mathbb{C}$. Suppose that $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow W$ are dominant rational maps of irreducible quasi projective varieties. Then $g \circ f\colon X \dashrightarrow W$ is dominant and

$$f^* \circ g^* = (g \circ f)^*. \tag{2.8.2}$$

Of course $\mathrm{Id}_X^*\colon \mathbb{C}(X) \to \mathbb{C}(X)$ is the identity map. We will prove the following result.

Theorem 2.8.2. *By associating to each quasi projective variety its field of fractions, and to each dominant rational map $f\colon X \dashrightarrow Y$ of irreducible quasi projective varieties the pull back, we get an equivalence between the category of irreducible quasi projective varieties with homomorphisms dominant rational maps, and the category of finitely generated field extensions of* $\mathbb{C}$.

What must be proved are the following two statements:

1. An extension of fields $\mathbb{C} \subset E$ is isomorphic to the filed of rational functions $\mathbb{C}(X)$ of a quasi projective variety X if and only it it is finitely generated over $\mathbb{C}$.

2. Let E, F be finitely generated field extensions of $\mathbb{C}$, and let $\alpha\colon E \to F$ be a homomorphism of $\mathbb{C}$ extensions (i.e. an inclusion $E \hookrightarrow F$ which is the identity on $\mathbb{C}$). Let Y, X be irreducible quasi projective varieties such that $\mathbb{C}(Y), \mathbb{C}(X)$ are isomorphic to E and F respectively as extensions of $\mathbb{C}$ (they exist by Item (1)). Then there exists a unique dominant rational map $f\colon X \dashrightarrow Y$ such that $f^* = \alpha$.

Item (1) is proved in **Proposition 2.8.4**. Item (2) is proved in **Proposition 2.8.5**.

We start by observing that we may restrict our attention to affine (irreducible) varieties. In fact, let X be an irreducible quasi projective variety, and let $Y \subset X$ be an open dense affine subset (e.g. a prinipal open subset). We have a well-defined restriction map

$$\mathbb{C}(X) \dashrightarrow \mathbb{C}(Y). \tag{2.8.3}$$

In fact, let $f \in \mathbb{C}(X)$, and let (U, φ) be a couple representing an element. Then $U \cap Y$ is an open dense subset of Y, and the couple $(U \cap Y, \varphi_{|U\cap Y})$ represents an element $\overline{f} \in \mathbb{C}(Y)$, which is independnet of the representative of f. The restriction map in (2.8.3) is an isomorphism of $\mathbb{C}$ extensions. Hence, when dealing with the field of fractions of a quasi projective variety, we may assume that the variety is affine.

Let X be an irreducible quasi projective variety. We have an inclusion of $\mathbb{C}$ extensions:

$$\begin{array}{ccc} (\text{field of fractions of } \mathbb{C}[X]) & \hookrightarrow & \mathbb{C}(X) \\ \frac{\alpha}{\beta} & \mapsto & [(X\backslash V(\beta), \frac{\alpha}{\beta})] \end{array} \tag{2.8.4}$$

Claim 2.8.3. *Let X be an affine irreducible variety. Then* (2.8.4) *is an isomorphism.*

Proof. We must prove that the map in (2.8.4) is surjective. Let $f \in \mathbb{C}(X)$, and let (U, φ) represent f. By **Remark 2.4.11**, there exists $0 \neq \gamma \in \mathbb{C}[X]$ such that the dense principal open subset X_γ is contained in U. Moreover, by **Remark 2.4.11** and **Theorem 2.4.9**, $\mathbb{C}[X_f]$ is generated as $\mathbb{C}$-algebra by $\mathbb{C}[X]$ and γ^{-1}, hence ϕ is represented by $(X_\gamma, \frac{\alpha}{\gamma^m})$ where $\alpha \in \mathbb{C}[X]$. Let $\beta := \gamma$. Since $X_\gamma = X_\beta$, we have proved that f belongs to the image of (2.8.4). □

Proposition 2.8.4. *A field extension of $\mathbb{C}$ is isomorphic to the field of fractions of an irreducible quasi projective variety if and only if it is finitely generated over $\mathbb{C}$.*

Proof. Let X be a quasi projective variety. The field $\mathbb{C}(X)$ is isomorphic to the field of fractions of an open dense affine subset of X. Thus we may assume that $X \subset \mathbb{A}^n$ is closed. By **Claim 2.8.3** $\mathbb{C}(X)$ is the field of quotients of $\mathbb{C}[X]$, and by **Theorem 2.4.9** $\mathbb{C}[X]$ is generated (over $\mathbb{C}$) by the restrictions of the coordinate functions $z_1, \ldots, z_n$. Hence the restrictions of the coordinate functions $z_1, \ldots, z_n$ to X generate $\mathbb{C}(X)$ over $\mathbb{C}$.

Now assume that E is a finitely generated field extension of $\mathbb{C}$. In particular the transcendenece degree of E over $\mathbb{C}$ is finite, say d. Let $f_1, \ldots, f_m \in \mathbb{C}(X)$ be a transcendence basis of $\mathbb{C}(X)$ over $\mathbb{C}$. Then $\mathbb{C}(X)$ is a finitely generated algebraic extension of $\mathbb{C}(f_1, \ldots, f_m)$. By the Theorem on the primitive element, i.e. **Theorem A.4.1**, there exists $g \in \mathbb{C}(X)$ algebraic of degree d over $\mathbb{C}(f_1, \ldots, f_m)$ and such that $\mathbb{C}(X)$ is generated over $\mathbb{C}$ by $f_1, \ldots, f_m, g$. Let $P \in \mathbb{C}(f_1, \ldots, f_n)[y]$ be the minimal polynomial of g over $\mathbb{C}(f_1, \ldots, f_n)$. Thus

$$P(y) = y^d + c_1 y^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{C}(f_1, \ldots, f_m).$$

Write $c_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{C}[f_1, \ldots, f_m]$. Let $\widetilde{Q} \in \mathbb{C}[f_1, \ldots, f_m][y]$ be obtained from P by clearing denominators, i.e. $\widetilde{Q} = (b_1 \cdot \ldots \cdot b_d)P$. Let $Q \in \mathbb{C}[f_1, \ldots, f_m][y]$ be obtained from $\widetilde{Q}$ by factoring out the maximum common divisor of the coefficients (recall that $\mathbb{C}[f_1, \ldots, f_m]$ is a UFD). Notice that Q is irreducible and hence prime. Write

$$Q = e_0 y^d + e_1 y^{d-1} + \cdots + e_d, \qquad e_i \in \mathbb{C}[f_1, \ldots, f_m], \quad e_0 \neq 0.$$

Let $\theta \colon \mathbb{C}[f_1, \ldots, f_m, y] \xrightarrow{\sim} \mathbb{C}[z_1, \ldots, z_m, y]$ be the isomorphism of $\mathbb{C}$-algebrae mapping f_i to z_i and y to itself. Let $\Phi := \theta(Q)$. Then $X := V(\Phi) \subset \mathbb{A}^{n+1}$ is an irreducible hypersurface because Φ is prime. Let $\overline{z}_i := z_{i|X}$. We claim that the rational functions on X represented by $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{C}$. In fact suppose that $R \in \mathbb{C}[t_1, \ldots, t_m]$ and $R(\overline{z}_1, \ldots, \overline{z}_n) = 0$. By the fundamental Theorem of Algebra, for any $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^m \backslash V(e_0))$ there exists $\xi_{m+1} \in \mathbb{C}$ such that $(\xi_1, \ldots, \xi_m, \xi_{m+1}) \in X$. It follows that $R(\xi_1, \ldots, \xi_m) = 0$ for all $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^n \backslash V(e_0))$, and hence $R \cdot e_0$ vanishes identically on $\mathbb{A}^m$. Thus $R \cdot e_0 = 0$, and since $e_0 \neq 0$ it follows that $R = 0$. This proves that $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{C}$. On the other hand $\overline{y} := y|_X$ is algebraic over $\mathbb{C}(\overline{z}_1, \ldots, \overline{z}_m)$ and its minimal polynomial equals Φ. Since the field of fractions of X is the field of quotients of $\mathbb{C}[X] = \mathbb{C}[z_1, \ldots, z_{m+1}]/(\Phi)$, we get that

$$E \cong \mathbb{C}(f_1, \ldots, f_m)[y]/(Q(y)) \cong \mathbb{C}(\overline{z}_1, \ldots, \overline{z}_m)[y]/(\Phi) \cong \mathbb{C}(X).$$

□

Proposition 2.8.5. *Let X and Y be irreducible quasi projective varieties. Suppose that $\alpha \colon \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ is an inclusion of extensions of $\mathbb{C}$. There exists a unique dominant rational map $f \colon X \dashrightarrow Y$ such that $f^* = \alpha$.*

Proof. We may assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are closed. By **Claim 2.8.3** $\mathbb{C}(X)$, $\mathbb{C}(Y)$ are the fields of fractions of $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ respectively, and by **Theorem 2.4.9**, $\mathbb{C}[X] = \mathbb{C}[z_1, \ldots, z_n]/I(X)$ and $\mathbb{C}[Y] = \mathbb{C}[w_1, \ldots, w_m]/I(Y)$. Given $p \in \mathbb{C}[z_1, \ldots, z_n]$ and $q \in \mathbb{C}[w_1, \ldots, w_m]$ we let $\overline{p} := p|_X$ and $\overline{q} := q|_Y$. We have

$$\alpha(\overline{w}_i) = \frac{\overline{f}_i}{\overline{g}_i}, \quad f_i, g_i \in \mathbb{C}[z_1, \ldots, z_n], \quad \overline{g}_i \neq 0.$$

Let $U := X \backslash (V(g_1) \cup \ldots \cup V(g_m))$. Then U is open and dense in X. Let

$$\begin{array}{ccc} U & \stackrel{\widetilde{\phi}}{\longrightarrow} & \mathbb{A}^m \\ a & \mapsto & \left(\frac{f_1(a)}{g_1(a)}, \ldots, \frac{f_m(a)}{g_m(a)}\right) \end{array}$$

We claim that $\widetilde{\phi}(U) \subset Y$. In fact let $h \in I(Y)$. Since α is an inclusion of extensions of $\mathbb{C}$,

$$h(\overline{f}_1/\overline{g}_1, \ldots, \overline{f}_m/\overline{g}_m) = h(\alpha(\overline{w}_1), \ldots, \alpha(\overline{w}_m)) = \alpha(h(\overline{w}_1, \ldots, \overline{w}_m) = \alpha(0) = 0.$$

This proves that if $h \in I(Y)$ then h vanishes on $\widetilde{\phi}(U)$, i.e. $\widetilde{\phi}(U) \subset Y$. Thus $\widetilde{\phi}$ induces a regular map $\phi\colon U \to Y$. Let $f\colon X \dashrightarrow Y$ be the equivalence class of (U, ϕ). Then $f^* = \alpha$.

It is clear by the above construction that f is the unique rational (dominant) map such that $f^* = \alpha$. □

The result below follows at once from what has been proved above.

Corollary 2.8.6. *Irreducible quasi projective varieties are birational if and only if their fields of rational functions are isomorphic as extensions of* $\mathbb{C}$*.*

The result below follows from the above corollary and the proof of **Proposition 2.8.4**.

Proposition 2.8.7. *Let X be an irreducible quasi projective variety and let* $m := \mathrm{Tr.deg}_{\mathbb{C}}\, \mathbb{C}(X)$*. Then X is birational to an irreducible hypersurface in* $\mathbb{A}^{m+1}$*.*

2.9 Dimension

Let X be an irreducible quasi projective variety. The *dimension of* X is defined to be the transcendence degree of $\mathbb{C}(X)$ over $\mathbb{C}$. Next, let X be an arbitrary quasi projective variety, and let $X = X_1 \cup \cdots \cup X_r$ be its irreducible decomposition.

1. The *dimension of* X is the maximum of the dimensions of its irreducible components.

2. Let $p \in X$. The *dimension of X at p* is the maximum of the dimensions of the irreducible components of X containing p.

Notice that the dimension of X is equal to the dimension of any open dense subset $U \subset X$.

Example 2.9.1. 1. The dimension of $\mathbb{A}^n$ is equal to n because $\{z_1, \ldots, z_n\}$ is a transcendence basis of $\mathbb{C}(z_1, \ldots, z_n)$ over $\mathbb{C}$.

2. Let $X \subset \mathbb{A}^{n+1}_{\mathbb{C}}$ be an irreducible hypersurface. Let $I(X) = (f)$. Reordering the coordinates $(z_1, \ldots, z_n, z_{n+1})$ we may assume that

$$f = c_0 z_{n+1}^d + c_1 z_{n+1}^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{C}[z_1, \ldots, z_n], \quad c_0 \neq 0, \quad d > 0.$$

In proving **Proposition 2.8.7** we showed that the restrictions to X of the z_i's, for $i = 1, \ldots, d$ give a transcendence basis of $\mathbb{C}(X)$. Thus $\dim X = n$. It follows that the dimension of any hypersurface $X \subset \mathbb{P}^{n+1}$ is also n.

Proposition 2.9.2. *Let X be an irreducible quasi projective variety and $Y \subset X$ be a proper closed subset. Then* $\dim Y < \dim X$.

Proof. We may assume that Y is irreducible. Since X is covered by open affine varieties, we may assume that X is affine. Thus $X \subset \mathbb{A}^n$ is a closed (irreducible) subset, and so is Y. We may choose a transcendence basis $\{f_1, \ldots, f_d\}$ of $\mathbb{C}(Y)$, where each f_i is a regular function on Y (for example a coordinate function).

Let $\tilde{f}_1, \ldots, \tilde{f}_d \in \mathbb{C}[X]$ such that $\tilde{f}_{i|}W = f_i$. Since Y is a proper closed subset of X, there exists a non zero $g \in \mathbb{C}[X]$ such that $g_{|Y} = 0$. It suffices to prove that $\tilde{f}_1, \ldots, \tilde{f}_d, g$ are algebraically independent over. We argue by contradiction. Suppose that there exists $0 \neq P \in \mathbb{C}[S_1, \ldots, S_d, T]$ such that $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$. Since X is irreducible we may assume that P is irreducible. Restricting to Y the equality $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$, we get that $P(f_1, \ldots, f_d, 0) = 0$. Thus $P(S_1, \ldots, S_d, 0) = 0$, because $f_1, \ldots, f_d$ are algebraically independent. This means that T divides P. Since P is irreducible $P = cT$, $c \in \mathbb{C}^*$. Thus $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$ reads $g = 0$, and that is a contradiction. □

Proposition 2.9.3. *Let X and Y be quasi projective varieties. Then* $\dim(X \times Y) = \dim X + \dim Y$.

Proof. We may assume that X and Y are irreducible affine varieties. There exist transcendence bases $\{f_1, \ldots, f_d\}$, $\{g_1, \ldots, g_e\}$ of $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ respectively given by regular functions. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projections. We claim that $\{\pi_X^*(f_1), \ldots, \pi_X^*(f_d), \pi_Y^*(g_1), \ldots, \pi_Y^*(g_e)\}$ is a transcendence basis of $\mathbb{C}(X \times Y)$.

First, by **Proposition 2.5.1** $\mathbb{C}[X \times Y]$ is algebraic over the subring generated (over $\mathbb{C}$) by $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$.

Secondly, let us show that $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$ are algebraically independent. Suppose that there is a polynomial relation

$$\sum_{0 \leqslant m_1, \ldots, m_e \leqslant N} P_{m_1, \ldots, m_e}(\pi_X^*(f_1), \ldots, \pi_X^*(f_d)) \cdot \pi_Y^*(g_1)^{m_1} \cdot \ldots \cdot \pi_Y^*(g_e)^{m_e} = 0,$$

where each $P_{m_1,\ldots,m_e}$ is a polynomial. Since $g_1,\ldots,g_e$ are algebraically independent we get that $P_{m_1,\ldots,m_e}(f_1(a),\ldots,f_d(a)) = 0$ for every $a \in X$. Since $f_1,\ldots,f_d$ are algebraically independent, it follows that $P_{m_1,\ldots,m_e} = 0$ for every $0 \leqslant m_1,\ldots,m_e \leqslant N$, and hence $P = 0$. This proves that $\pi_X^*(f_1),\ldots,\pi_Y^*(g_e)$ are algebraically independent. □

2.10 Tangent space

One definition of tangent space of a C^∞ manifold M at a point $x \in M$ is as the real vector space of derivations of the space $\mathscr{E}_{M,x}$ of germs of C^∞ functions at x. Similarly, the holomorphic tangent space of a complex manifold X at a point $x \in X$ is as the complex vector space of derivations of the space $\mathscr{O}_{X,x}$ of germs of holomorphic functions at x. We will give an analogous definition of the tangent space of a quasi projective variety. A fundamental difference between quasi projective varieties and the previous examples is that the dimension of the tangent space at a point might depend on the point. Intuitively, the reason is that a quasi projective variety can have non smooth points, meaning that in a neighborhood of such a point the variety is not a complex submanifold of the ambient projective space.

Let X be a quasi projective variety. We start by defining the ring of germs of regular functions at $x \in X$.

Definition 2.10.1. Let X be a quasi projective variety, and let $x \in X$. Let (U,ϕ) and (V,ψ) be couples where U,V are open subsets of X containing x, and $\phi \in \mathbb{C}[U]$, $\psi \in \mathbb{C}[V]$. Then $(U,\phi) \sim (V,\psi)$ if there exists an open subset $W \subset X$ containing x such that $W \subset U \cap V$ and $\phi_{|W} = \psi_{|W}$.

One checks easily that $\sim$ is an equivalence relation: an equivalence class for the realtion $\sim$ is a *germ of regular function of X at x*. We may define a sum and a product on the set of germs of regular functions of X at x by setting

$$[(U,\phi)] + [(V,\psi)] := [\big(U \cap V, \phi_{|U\cap V} + \psi_{|U\cap V}\big)], \tag{2.10.1}$$

and

$$[(U,\phi)] \cdot [(V,\psi)] := [\big(U \cap V, \phi_{|U\cap V} \cdot \psi_{|U\cap V}\big)]. \tag{2.10.2}$$

Of course one has to check that the equivalence class of the sum and product is independent of the choice of representatives: this is easy, we leave details to the reader. With these operations, the set of germs of regular functions of X at x is a ring.

Definition 2.10.2. Let X be a quasi projective variety, and let $x \in X$. The *local ring of X at x* is the ring of germs of regular functions of X at x, and is denoted $\mathscr{O}_{X,x}$.

We have a natural homomorphism of rings

$$\begin{array}{ccc} \mathbb{C}[X] & \xrightarrow{\rho} & \mathscr{O}_{X,x} \\ f & \mapsto & [(X,f)] \end{array} \tag{2.10.3}$$

Lemma 2.10.3. *Suppose that X is an affine variety, and let $x \in X$. If $\varphi \in \mathscr{O}_{X,x}$ then there exist $f, g \in \mathbb{C}[X]$, with $g(x) \neq 0$, such that $\varphi = \frac{\rho(f)}{\rho(g)}$.*

Proof. Let φ be represented by (U, h), where $U \subset X$ is open, and $x \in U$. Since the principal open affine subsets of X form a basis of the Zariski topology, there exists $h \in \mathbb{C}[X]$ such that $X_h \subset U$ and $x \in X_h$ (see **Remark 2.4.11**). Then $\varphi = [(X_h, h_{|X_h})]$. By **Remark 2.4.11**, there exist $f \in \mathbb{C}[X]$ and $m \in \mathbb{N}$ such that h is the restriction to X_h of $\frac{f}{h^m}$. Then $\varphi = \frac{\rho(f)}{\rho(h^m)}$. □

There is a well-defined surjective homomorphism

$$\begin{array}{ccc} \mathscr{O}_{X,x} & \longrightarrow & \mathbb{C} \\ [(U, \phi)] & \mapsto & \phi(a) \end{array} \tag{2.10.4}$$

The kernel

$$\mathfrak{m}_x := \{[(U, \phi)] \mid \phi(x) = 0\}$$

of (2.10.4) is a maximal ideal, because (2.10.4) is a surjection to a field.

Proposition 2.10.4. *With notationas above, $\mathfrak{m}_x$ is the unique maximal ideal of $\mathscr{O}_{X,x}$, and hence $\mathscr{O}_{X,x}$ is a local ring. Moreover, $\mathscr{O}_{X,x}$ is Noetherian.*

Proof. Let $f = [(U, \phi)] \in (\mathscr{O}_{X,x} \backslash \mathfrak{m}_x)$. Then $W := (U \backslash V(\phi))$ is an open subset of X containing x and hence $g := [(W, (\phi|_W)^{-1}]$ belongs to $\mathscr{O}_{X,x}$. Since $gf = 1$ we get that f is invertible. It follows that $\mathfrak{m}_x$ contains any proper ideal of $\mathscr{O}_{X,x}$ and hence is the unique maximal ideal of $\mathscr{O}_{X,x}$.

In order to prove that $\mathscr{O}_{X,x}$ is Noetherian, we notice that if $U \subset X$ is Zariski open and contains x, then the natural homomorphism $\mathscr{O}_{U,x} \to \mathscr{O}_{X,x}$ is an isomorphism. Since X is covered by open affine ssubsets, it follows that we may assume that X is affine. Let $I \subset \mathscr{O}_{X,x}$ be an ideal. Then $\rho^{-1}(I)$ is a finitely generated ideal, because $\mathbb{C}[X]$ is Noetherian. Let $f_1, \ldots, f_r$ be generators of $\rho^{-1}(I)$. Then $\rho(f_1), \ldots, \rho(f_r)$ generate I. In fact let $\varphi \in I$. By **Lemma 2.10.3**, there exist $f, g \in \mathbb{C}[X]$, with $g(x) \neq 0$, such that $\varphi = \frac{\rho(f)}{\rho(g)}$. We have $f = \sum_{i=1}^r a_i f_i$, and hence $\varphi = \sum_{i=1}^r \frac{\rho(a_i)}{\rho(g)} \rho(f_i)$. □

The homomorphism (2.10.4) equips $\mathbb{C}$ with a structure of $\mathscr{O}_{X,x}$-module. Moreover $\mathscr{O}_{X,x}$ is a $\mathbb{C}$-algebra. Thus it makes sense to speak of $\mathbb{C}$-derivations of $\mathscr{O}_{X,x}$ to $\mathbb{C}$.

Definition 2.10.5. Let X be a quasi projective variety, and let $x \in X$. The *Zariski tangent space to X at x* is $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}_{X,x}, \mathbb{C})$, and will be denoted by $\Theta_x X$. Thus $\Theta_x X$ is an $\mathscr{O}_{X,x}$-module (see **Section A.5**), and since $\mathfrak{m}_x$ annihilates every derivation $\mathscr{O}_{X,x} \to \mathbb{C}$, it is a complex vector space.

The result below shows that the Zariski tangent space at a point of $\mathbb{A}^n$ agrees with the holomorphic tangent space.

Lemma 2.10.6. *Let $a \in \mathbb{A}^n$. The complex linear map*

$$\begin{array}{ccc} \Theta_a \mathbb{A}^n & \longrightarrow & \mathbb{C}^n \\ D & \mapsto & (D(z_1), \ldots, D(z_n)) \end{array} \tag{2.10.5}$$

is an isomorphism.

Proof. The formal partial derivative $\frac{\partial}{\partial z_m}$ defined by (A.5.1) defines an element of $\Theta_a \mathbb{A}^n$ by the familiar formula

$$\frac{\partial}{\partial z_m}\left(\frac{f}{g}\right)(a) := \frac{\frac{\partial f}{\partial z_m}(a) \cdot g(a) - f(a) \cdot \frac{\partial g}{\partial z_m}(a)}{g(a)^2}.$$

(See **Example A.5.3**.) Since $\frac{\partial}{\partial z_m}(z_j) = \delta_{mj}$, the map in (2.10.5) is surjective.

Let's prove that the map in (2.10.5) is injective. Assume that $D \in \Theta_{X,x}$ is mapped to 0 by the map in (2.10.5), i.e. $D(x_j) = 0$ for $j \in \{1, \ldots, n\}$. Let $f, g \in \mathbb{C}[z_1, \ldots, z_n]$, with $g(a) \neq 0$. Then

$$D\left(\frac{f}{g}\right) = \frac{D(f) \cdot g(a) - f(a) \cdot D(g)}{g(a)^2}.$$

(See **Example A.5.3**.) Hence it suffices to show that $D(f) = 0$ for every $f \in \mathbb{C}[z_1, \ldots, z_n]$. Consider the first-order expansion of f around a i.e. write

$$f = f(a) + \sum_{i=1}^{n} c_i(z_i - a) + R, \qquad R \in \mathfrak{m}_a^2. \tag{2.10.6}$$

Since D is zero on constants (because D is a $\mathbb{C}$-derivation) and $D(z_j) = 0$ for all j it follows that $D(f) = D(R)$, and the latter vanishes by Leibniz' rule and the hypothesis $D(z_j) = 0$ for all j. □

The differential of a regular map at a point of the domain is defined by the usual procedure. Explicitly, let $f \colon X \to Y$ be a regular map of quasi projective varieties, let $x \in X$ and $y := f(x)$. There is a well-defined pull-back homomorphism

$$\begin{array}{ccc} \mathscr{O}_{Y,y} & \xrightarrow{f^*} & \mathscr{O}_{X,x} \\ [(U, \phi)] & \mapsto & [(f^{-1}U, \phi \circ (f_{|f^{-1}U}))] \end{array} \tag{2.10.7}$$

The *differential of f at x* is the linear map of complex vector spaces

$$\begin{array}{ccc} T_x X & \xrightarrow{df(x)} & T_y Y \\ D & \mapsto & (\phi \mapsto D(f^* \phi)) \end{array} \tag{2.10.8}$$

The differential has the customary functorial properties. Explicitly, suppose that we have

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3\,, \quad x_1 \in X_1, \quad x_2 = f_1(x_1)\,.$$

Since $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$ we have

$$d(f_2 \circ f_1)(x_1) = df_2(x_2) \circ df_1(x_1)\,. \tag{2.10.9}$$

Moreover $d\,\mathrm{Id}_X(x) = \mathrm{Id}_{T_x X}$ for $x \in X$.

Remark 2.10.7. It follows from the above that if f is an isomorphism, then $df(x)\colon T_xX \to T_{f(x)}Y$ is an isomorphism, in particular $\dim T_xX = \dim T_yY$.

The next result shows how to compute the Zariski tangent space of a closed subset of $\mathbb{A}^n$. Since every point x of a quasi projective variety X is contained in an open affine subset U, and $\Theta_xX = \Theta_xU$ (because restriction defines an identification $\mathscr{O}_{X,x} = \mathscr{O}_{U,x}$), the result will allow to compute the Zariski tangent space in general.

Proposition 2.10.8. *Let $\iota\colon X \hookrightarrow \mathbb{A}^n$ be the inclusion of a closed subset and $a \in X$. The differential $d\iota(a)\colon \Theta_aX \to \Theta_a\mathbb{A}^n$ is injective and, identifying $\Theta_a\mathbb{A}^n$ with $\mathbb{C}^n$ via* (2.10.5)*, we have*

$$\operatorname{Im} dj(a) = \left\{ v = (v_1, \dots, v_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) \cdot v_i = 0 \quad \forall f \in I(X) \right\}. \tag{2.10.10}$$

Proof. The differential $d\iota(a)$ is injective because the pull-back $\iota^*\colon \mathscr{O}_{\mathbb{A}^n_{\mathbb{C}},a} \to \mathscr{O}_{X,a}$ is surjective. Let $D \in \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_{X,a}, \mathbb{C})$. If $f \in I(X) \subset \mathbb{C}[z_1, \dots, z_n]$, then $d\iota(D)(f) = D(\iota^* f) = D(0) = 0$. Hence $\operatorname{Im} d\iota(a)$ is contained in the right-hand side of (2.10.10). Let's prove that $\operatorname{Im} d\iota(a)$ contains the right-hand side of (2.10.10). Let $\widetilde{D} \in \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_{\mathbb{A}^n,a}, \mathbb{C})$ belong to the right hand side of (2.10.10), i.e. $\widetilde{D}(f) = 0$ for all $f \in I(X)$. By Item (3) of **Example A.5.3** it follows that $\widetilde{D}(\frac{f}{g}) = 0$ whenever $f, g \in \mathbb{C}[z_1, \dots, z_n]$ and $f \in I(X)$ (of course we assume that $g(a) \neq 0$). Thus $\widetilde{D}$ descends to a $\mathbb{C}$-derivation $D \in \operatorname{Der}(\mathscr{O}_{X,a}, \mathbb{C})$, and $\widetilde{D} = d\iota_*(a)(D)$. □

Remark 2.10.9. With the hypotheses of **Proposition 2.10.8**, suppose that $I(X)$ is generated by $f_1, \dots, f_r$. Then

$$\operatorname{Im} dj(a) = \left\{ v = (v_1, \dots, v_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \frac{\partial f_k}{\partial z_i}(a) \cdot v_i = 0 \quad k \in \{1, \dots, r\} \right\}.$$

In fact, the right hand side of the above equation is equal to the right hand side of (2.10.10), because if $f = \sum_{j=1}^{r} g_j f_j$, then $\frac{\partial f}{\partial z_i}(a) = \sum_{j=1}^{r} g_j(a) \frac{\partial f_j(a)}{\partial z_i}$.

Example 2.10.10. Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial without multiple factors, i.e. such that $\sqrt{(f)} = (f)$, and let $X = V(f)$. Let $a \in X$; by **Remark 2.10.9** Zariski's tangent space to X is the subspace of $\mathbb{C}^n$ defined by

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) \cdot v_i = 0.$$

Hence

$$\dim \Theta_a X = \begin{cases} n-1 & \text{if } (\frac{\partial f}{\partial z_1}(a), \dots, \frac{\partial f}{\partial z_n}(a)) \neq 0, \\ n & \text{if } (\frac{\partial f}{\partial z_1}(a), \dots, \frac{\partial f}{\partial z_n}(a)) = 0. \end{cases}$$

Let us show that

$$X \backslash V\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right) \tag{2.10.11}$$

is an open *dense* subset of X (it is obviously open, the point is that it is dense), i.e. $\dim \Theta_a X = n-1$ for a in an open dense subset of X.

First assume that f is irreducible. Reordering the coordinates if necessary, we may assume that

$$f = a_0 z_n^d + a_1 z_n^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{C}[z_1, \ldots, z_{n-1}], \quad a_0 \neq 0, \quad d > 0.$$

Thus

$$\frac{\partial f}{z_n} = da_0 z_n^{d-1} + (d-1)a_1 z_n^{d-2} + \cdots + a_{d-1} \neq 0.$$

The degree in z_n of f is d (i.e. f has degree d as element of $\mathbb{C}[z_1, \ldots, z_{n-1}][z_n]$) while the degree in z_n of $\frac{\partial f}{z_n}$ is $(d-1)$ and hence $f \nmid \frac{\partial f}{z_n}$. This shows that the set in (2.10.11) is dense in X if f is irreducible.

In general, let $f = f_1 \cdot \cdots \cdot f_r$ be the decomposition of f as product of prime factors. Let $X_i = V(f_i)$. Then

$$X = X_1 \cup \cdots \cup X_r$$

is the irreducible decomposition of X. As shown above, for each $i \in \{1, \ldots, r\}$

$$X_j \backslash V\left(\frac{\partial f_j}{z_1}, \ldots, \frac{\partial f_j}{z_n}\right) \neq \varnothing.$$

Hence there exists $a \in X_j$ such that $\frac{\partial f_j}{z_h}(a) \neq 0$ for a certain $1 \leqslant h \leqslant n$, and in addition a does not belong to any other irreducible component of X. It follows that

$$\frac{\partial f}{z_h}(a) = \frac{\partial f_j}{z_h}(a) \cdot \prod_{k \neq j} f_k(a) \neq 0.$$

This proves that the open set in(2.10.11) has non empty intersection with every irreducible component of X, and hence is dense in X.

The result below shows that the behaviour of the tangent space examined in the above example is typical of what happens in general.

Proposition 2.10.11. *Let X be a quasi projective variety. The function*

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{N} \\ x & \mapsto & \dim \Theta_x X \end{array} \tag{2.10.12}$$

is Zariski upper-semicontinuous, i.e. for every $k \in \mathbb{N}$

$$X_k := \{x \in X \mid \dim \Theta_x X \geqslant k\}$$

is closed in X.

Proof. Since X has an open affine covering, we may suppose that $X \subset \mathbb{A}^n$ is closed. Let $I(X) = (f_1, \ldots, f_r)$. For $x \in \mathbb{A}^n_{\mathbb{C}}$ let

$$J(f_1, \ldots, f_s)(x) := \begin{pmatrix} \frac{\partial f_1}{z_1}(x) & \cdots & \frac{\partial f_1}{z_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{z_1}(x) & \cdots & \frac{\partial f_r}{z_n}(x) \end{pmatrix}$$

be the Jacobian matrix of $(f_1, \ldots, f_s)$ at x. By **Proposition 2.10.8** we have that

$$X_k = \{x \in X \mid \operatorname{rk} J(f_1, \ldots, f_r)(x) \leqslant n - k\}\,. \tag{2.10.13}$$

Given multi-indices $I = \{1 \leqslant i_1 < \ldots < i_m \leqslant s\}$ and $J = \{1 \leqslant j_1 < \ldots < j_m \leqslant n\}$ let $J(f_1, \ldots, f_s)(x)_{I,J}$ be the $m \times m$ minor of $J(f_1, \ldots, f_r)(x)$ with rows corresponding to I and columns corresponding to J (if $m > \min\{r, n\}$ we set $J(f_1, \ldots, f_s)(x)_{I,J} = 0$). We may rewrite (2.10.13) as

$$X_k = X \cap V\left(\ldots, \det J(f_1, \ldots, f_r)(x)_{I,J}, \ldots\right)_{|I|=|J|=n-k+1}.$$

It follows that X_k is closed. $\square$

2.11 Cotangent space

Let X be a quasi projective variety, and let $x \in X$. The *cotangent space to X at x* is the dual complex vector space of the tangent space $\Theta_x X$, and is denoted $\Omega_X(x)$:

$$\Omega_X(x) := (\Theta_x X)^\vee\,. \tag{2.11.1}$$

We define a map

$$\mathscr{O}_{X,x} \xrightarrow{d} \Omega_X(x) \tag{2.11.2}$$

as follows. Let $f \in \mathscr{O}_{X,x}$ be represented by (U, ϕ). The codomain of the differential $d\phi(x)\colon \Theta_x U \to \Theta_{\phi(x)}\mathbb{C}$ is identified with with $\mathbb{C}$, because of the isomorphism in (2.10.5), and hence $d\phi(x) \in (\Theta_x U)^\vee$. Since $U \subset Z$ is an open subset containing x, the differential at x of the inclusion map defines an identification $\Theta_x U \xrightarrow{\sim} \Theta_x X$. Thus $d\phi(x) \in (\Theta_x X)^\vee = \Omega_X(x)$. One checkes immediately that if (V, ψ) is another representative of f then $d\psi(x) = d\phi(x)$. We let

$$df(x) := d\phi(x), \qquad (U, \phi) \text{ any representative of } f.$$

Remark 2.11.1. We equip $\Omega_X(x)$ with a structure of $\mathscr{O}_{X,x}$-module by composing the evaluation map $\mathscr{O}_{X,x} \to \mathbb{C}$ given by (2.10.4) and scalar multiplication of the complex vector-space $\Omega_Z(a)$. With this structure (2.11.2) is a derivation over $\mathbb{C}$.

Remark 2.11.2. Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ and $a \in \mathbb{A}^n$. Then the familiar formula

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) dz_i(a)$$

holds. In fact this follows from the first-order Taylor expansion of f at a:

$$f = f(a)+\sum_{i=1}^{n}\frac{\partial f}{\partial z_i}(a)(z_i-a_i)+\sum_{1\leqslant i,j\leqslant n} m_{ij}(z_i-a_i)(z_j-a_j), \qquad m_{ij}\in\mathbb{C}[z_1,\ldots,z_n]. \tag{2.11.3}$$

Remark 2.11.3. Let $X\subset\mathbb{A}^n$ be closed, and let $a\in X$. Identify $\Theta_a\mathbb{A}^n$ with $\mathbb{C}^n$ via **Lemma 2.10.6**. By **Remark 2.11.2** we have the identification

$$T_aX = \operatorname{Ann}\{df(a)\mid f\in I(X)\}.$$

Let X be a quasi projective variety, and let $x\in X$. Let $\mathfrak{m}_x\subset\mathscr{O}_{X,x}$ be the maximal ideal. By Leibiniz' rule $d\phi(x)=0$ if $\phi\in\mathfrak{m}_x^2$ (recall that $d\colon\mathscr{O}_{X,x}\to\Omega_X(x)$ is a derivation over $\mathbb{C}$). Thus we have an induced $\mathbb{C}$-linear map

$$\begin{array}{ccc} \mathfrak{m}_x/\mathfrak{m}_x^2 & \xrightarrow{\delta(x)} & \Omega_X(x)\\ [\phi] & \mapsto & d\phi(a)\end{array} \tag{2.11.4}$$

Proposition 2.11.4. *Keep notation as above. Then $\delta(x)$ is an isomorphism of complex vector spaces.*

Proof. Since Θ_xX is a finite dimensional complex vector space, it is the dual of its dual, i.e. the dual of $\Omega_X(x)$. Thus, in order to prove that $\delta(x)$ is surjective it suffices to show that no non zero $D\in\Theta_xX$ annihilates the image of $\delta(x)$. Suppose that $d\phi(x)(D)=0$ for all $[\phi]\in\mathfrak{m}_x/\mathfrak{m}_x^2$. Since the differential of a constant is zero we get that $d\phi(x)(D)=0$ for all $\phi\in\mathscr{O}_{X,x}$, and hence $D=0$. This proves surjectivity of $\delta(x)$.

In order to prove injectivity of $\delta(x)$, we must show that if $\phi\in\mathfrak{m}_x$ is such that $d\phi(x)(D)=0$ for all $D\in\Theta_xX$, then $\phi\in\mathfrak{m}_x^2$. We may suppose that X is a closed subset of $\mathbb{A}^n$. In order to avoid confusion, we let $x=a=(a_1,\ldots,a_n)$. Let $(U,f/g)$ be a representative of ϕ, where $f,g\in\mathbb{C}[X]$, and $f(a)=0$, $g(a)\neq 0$. It will suffice to prove that $f\in\mathfrak{m}_a^2$. Since $0=d\phi(a)=g(a)^{-1}df(a)$ we have $df(a)=0$. By **Theorem 2.4.9** there exists $\tilde{f}\in\mathbb{C}[z_1,\ldots,z_n]$ such that $\tilde{f}_{|X}=f$. By **Proposition 2.10.8** we may identify Θ_aX with the subspace of $T_a\mathbb{C}^n=\mathbb{C}^n$ given by (2.10.10). By hypothesis $d\tilde{f}(a)(D)=0$ for all $D\in\Theta_aX$, i.e.

$$d\tilde{f}(a)\in\operatorname{Ann}(\Theta_aX)\subset\Omega_{\mathbb{A}^n}(x).$$

By (2.10.10) there exists $h\in I(X)$ such that $d\tilde{f}(a)=dh(a)$. Then $(\tilde{f}-h)_{|X}=f$ and $d(\tilde{f}-h)(a)=0$. Thus $(\tilde{f}-h)\in\mathbb{C}[z_1,\ldots,z_n]$ has vanishing value and differential at a. It follows (first-order Taylor expansion of $\tilde{f}-h$ at a) that

$$(\tilde{f}-h)\in(z_1-a_1,\ldots,z_n-a_n)^2.$$

Since $h\in I(X)$ we get that $f\in\mathfrak{m}_a^2$. $\square$

2.12 Smooth points of quasi projective varieties

Definition 2.12.1. Let X be a quasi projective variety, and let $x \in X$. Then X is *smooth at* x if $\dim \Theta_x X = \dim_x X$, it is *singular* at x otherwise. The set of smooth points of X is denoted by X^{sm}. The set of singular points of X is denoted by $\operatorname{sing} X$.

Example 2.12.2. Let $Y \subset \mathbb{A}^{m+1}$ be a hypersurface. By **Example 2.9.1**, the dimension of Y is equal to m, and hence the set of smooth points of Y is an open dense subset of Y by **Example 2.10.10**.

The main result of the present section extends the picture for hypersurfaces to the general case.

Theorem 2.12.3. *Let X be a quasi projective variety. Then the following hold:*

1. *The set X^{sm} of smooth points of X is an open dense subset of X.*
2. *For $x \in X$ we have $\dim \Theta_x X \geqslant \dim_x X$.*
3. *If $X \subset \mathbb{P}^n$ is locally closed, then X^{sm} is a complex submanifold of $\mathbb{P}^n \backslash \overline{\operatorname{sing} X}$, and for $x \in X^{\mathrm{sm}}$ the dimension of X^{sm} as complex manifold equals its dimension as quasi projective variety.*

We will prove **Theorem 2.12.3** at the end of the section. First we go through some preliminary results.

Our first result proves a weaker version of Item (1) of **Theorem 2.12.3**, and proves Item (2) of the same theorem.

Proposition 2.12.4. *Let X be a quasi projective variety. Then the following hold:*

1. *The set X^{sm} of smooth points of X contains an open dense subset of X.*
2. *For $x \in X$ we have $\dim \Theta_x X \geqslant \dim_x X$.*

Proof. Suppose that X is irreducible of dimension d. By **Proposition 2.8.7** there is a birational map $g \colon X \dashrightarrow Y$, where $Y \subset \mathbb{A}^{d+1}$ is a hypersurface. By **Proposition 2.7.6** there exist open dense subsets $U \subset X$ and $V \subset Y$ such that g is regular on U, and it defines an isomorphism $f \colon U \xrightarrow{\sim} V$. By **Example 2.12.2**, the set of smooth points Y^{sm} of Y is open and dense in Y. Since V is open and dense in Y the intersection $Y^{\mathrm{sm}} \cap V$ is open and dense dense in Y and hence $f^{-1}(Y^{\mathrm{sm}} \cap V)$ is an open dense subset of X. Since $f^{-1}(Y^{\mathrm{sm}} \cap V)$ is contained in U^{sm}, we have proved that the set of smooth points of X contains an open dense subset of X. We have proved that Item (1) holds if X is irreducible. In general, let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. Let

$$X_j^0 := (X \backslash \bigcup_{i \neq j} X_i) = (X_j \backslash \bigcup_{i \neq j} X_i)$$

By the result that was just proved, $(X_j^0)^{\mathrm{sm}}$ contains an open dense subset of smooth points. Every smooth point of X_j^0 is a smooth point of X, because X_j^0 is open in

X. Thus $\bigcup_i (X_i^0)^{\mathrm{sm}}$ is an open dense subset of X, containing an open dense subset of X. This proves Item (1).

Let us prove Item (2). Let $x_0 \in X$, and let X_0 be an irreducible component of X containing x_0 such that $\dim X_0 = \dim_{x_0} X$. By Item (1) X_0^{sm} contains an open dense subset of points x such that $\dim \Theta_x X_0 = \dim_x X_0$, and hence by **Proposition 2.10.11** we have $\dim \Theta_x X_0 \geqslant \dim_x X_0$ for *all* $x \in X$. In particular $\dim \Theta_{x_0} X_0 \geqslant \dim_{x_0} X_0 = \dim_{x_0} X$. Since $\Theta_{x_0} X_0 \subset \Theta_{x_0} X$, it follows that $\dim \Theta_{x_0} X \geqslant \dim_{x_0} X$. □

The next result involves more machinery. We will give an algebraic version of the (analytic) Implicit Function Theorem. The algebraic replacement for the ring of analytic functions defined in a neighborhood of $0 \in \mathbb{A}^n$ is the ring $\mathbb{C}[[z_1, \ldots, z_n]]$ of formal power series in $z_1, \ldots, z_n$ with complex coefficients. We have inclusions

$$\mathbb{C}[z_1, \ldots, z_n] \subset \mathscr{O}_{\mathbb{A}^n,0} \subset \mathbb{C}[[z_1, \ldots, z_n]]. \tag{2.12.1}$$

(The second inclusion is obtained by developing $\frac{f}{g}$ as convergent power series centered at 0, where $f, g \in \mathbb{C}[z_1, \ldots, z_n]$ and $g(0) \neq 0$.) We will need the following elementary results.

Lemma 2.12.5. *Let $\mathfrak{m} \subset \mathbb{C}[z_1, \ldots, z_n]$, $\mathfrak{m}' \subset \mathscr{O}_{\mathbb{A}^n,0}$ and $\mathfrak{m}'' \subset \mathbb{C}[[z_1, \ldots, z_n]]$ be the ideals generated by $z_1, \ldots, z_n$ in the corresponding ring. Then for every $i \geqslant 0$ we have $(\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^i$, and $(\mathfrak{m}')^i \cap \mathbb{C}[z_1, \ldots, z_n] = \mathfrak{m}^i$.*

Proof. By induction on i. For $i = 0$ the statement is trivially true. The proof of the inductive step is the same in both cases. For definiteness let us show that $(\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^{i+1}$, assuming that $(\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^i$. The non trivial inclusion is $(\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0} \subset (\mathfrak{m}')^{i+1}$. Assume that $f \in (\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0}$. Then $f \in (\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0}$, and hence $f \in (\mathfrak{m}')^i$ by the inductive hypothesis. Thus we may write

$$f = \sum_{|I|} \alpha_J z^J,$$

where the sum is over all multiindices $J = (j_1, \ldots, j_n)$ of weight $|J| = \sum_{s=1}^n j_s = i$, and $\alpha_J \in \mathscr{O}_{\mathbb{A}^n,0}$ for all J. Since $f \in (\mathfrak{m}'')^{i+1}$, we have $\alpha_J(0) = 0$ for all J. It follows that $\alpha_J \in \mathfrak{m}'$ for all J, and hence $f \in (\mathfrak{m}')^{i+1}$. □

Proposition 2.12.6 (Formal Implicit Function Theorem)**.** *Let $\varphi \in \mathbb{C}[[z_1, \ldots, z_n]]$, and suppose that $\varphi = z_1 + \varphi_2 + \ldots + \varphi_d + \ldots$, where $\varphi_d \in \mathbb{C}[z_1, \ldots, z_n]_d$. Given $\alpha \in \mathbb{C}[[z_1, \ldots, z_n]]$, there exists a unique $p \in \mathbb{C}[[z_1, \ldots, z_n]]$ such that*

$$(\alpha - p \cdot \varphi) \in \mathbb{C}[[z_2, \ldots, z_n]]. \tag{2.12.2}$$

Hence the natural map $\mathbb{C}[[z_2, \ldots, z_n]] \to \mathbb{C}[[z_1, \ldots, z_n]]/(\varphi)$ is an isomorphism.

Proof. Write $p = p_0 + p_1 + \ldots + p_d + \ldots$, where $p_d \in \mathbb{C}[z_1, \ldots, z_n]_d$. Expand the product $(\alpha - p \cdot \varphi)$, and solve for p_0 (we get $p_0 = 1$), then for p_1, etc. At each stage there is one and only one solution. □

Proposition 2.12.7. *Let $f_1, \ldots, f_k \in \mathbb{C}[z_1, \ldots, z_n]$ and $a \in \mathbb{A}^n$. Suppose that*

(i) *each f_i vanishes at a, and*

(ii) *the differentials $df_1(a), \ldots, df_k(a)$ are linearly independent.*

Then $V(f_1, \ldots, f_k) = X \cup Y$, where

1. *X, Y are closed in $\mathbb{A}^n$, $a \in X$, while Y does not contain a;*
2. *X is irreducible of dimension $n-k$, it is smooth at a, and $\Theta_a(X) = \operatorname{Ann}(\langle df_1(a), \ldots, df_k(a)\rangle)$ (as subspace of $\Theta_a\mathbb{A}^n$).*

Moreover, there exists a principal open affine set $\mathbb{A}^n_g$ containing a such that $f_{1|\mathbb{A}^n_g}, \ldots, f_{k|\mathbb{A}^n_g}$ generate the ideal of $X \cap \mathbb{A}^n_g$.

Proof. By changing affine coordinates, if necessary, we may assume that $a = 0$, and that $df_i(0) = z_i$ for $i \in \{1, \ldots, k\}$. Let $J' \subset \mathscr{O}_{\mathbb{A}^n,0}$ be the ideal generated by $f_1, \ldots, f_k$ (to be consistent with our notation, we should write $J' = (\varphi(f_1), \ldots, \varphi(f_k)))$, let $J := J' \cap \mathbb{C}[z_1, \ldots, z_n]$, and let $J'' \subset \mathbb{C}[[z_1, \ldots, z_n]]$ be the ideal generated by $f_1, \ldots, f_k$. Lastly, let $I \subset \mathbb{C}[z_1, \ldots, z_n]$ be the ideal generated by $f_1, \ldots, f_k$. We claim that

$$J \cdot g \subset I \subset J. \tag{2.12.3}$$

for a suitable $g \in \mathbb{C}[z_1, \ldots, z_n]$ with $g(0) \neq 0$. In fact, the second inclusion is trivially true. In order to prove the first inclusion, let $h_1, \ldots, h_r$ be generators of the ideal $J \subset \mathbb{C}[z_1, \ldots, z_n]$. By definition of J, there exist $a_i, g_i \in \mathbb{C}[z_1, \ldots, z_n]$, for $i \in \{1, \ldots, r\}$, such that $a_i \in I$, $g_i(0) \neq 0$, and $h_i = \frac{a_i}{g_i}$. Hence the second inclusion in (2.12.3) holds with $g = g_1 \cdot \ldots \cdot g_r$. This proves (2.12.3), and hence we have $V(J) \subset V(I) \subset (V(J) \cup V(g))$. It follows that, letting $X := V(J)$, there exists a closed $Y \subset V(g)$ such that

$$V(f_1, \ldots, f_k) = X \cup Y, \quad 0 \notin Y. \tag{2.12.4}$$

Let us prove that J is a prime ideal, so that in particular X is irreducible. First, we claim that

$$J'' \cap \mathscr{O}_{\mathbb{A}^n,0} = J'. \tag{2.12.5}$$

The non trivial inclusion to be proved is $J'' \cap \mathscr{O}_{\mathbb{A}^n,0} \subset J'$. Let $f \in J'' \cap \mathscr{O}_{\mathbb{A}^n,0}$. Then there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{C}[[z_1, \ldots, z_n]]$ such that $f = \sum_{j=1}^k \alpha_j f_j$. Given $s \in \mathbb{N}$, let α_j^s be the MacLaurin polynomial of α_j of degree s, i.e. such that $(\alpha_j - \alpha_j^s) \in (\mathfrak{m}'')^{s+1}$, where $\mathfrak{m}''$ is as in **Lemma 2.12.5**. Then

$$f = \sum_{j=1}^k \alpha_j^{(s)} f_j + \sum_{j=1}^k (\alpha_j - \alpha_j^s) f_j.$$

Both addends are in $\mathscr{O}_{\mathbb{A}^n,0}$. In addition, the first addend belongs to J', and the second one belongs to $(\mathfrak{m}'')^{s+1}$. By **Lemma 2.12.5**, it follows that the second one belongs to $(\mathfrak{m}')^{s+1}$. Hence $f \in \bigcap_{s=0}^{\infty}(I' + (\mathfrak{m}')^{s+1})$. By **Corollary A.6.2**, it follows that $f \in I'$. This proves (2.12.5). By (2.12.5) and the definition of J, we have an inclusion

$$\mathbb{C}[z_1, \dots, z_n]/J \subset \mathbb{C}[[z_1, \dots, z_n]]/J''.$$

Hence, in order to prove that J is prime, it suffices to show that $\mathbb{C}[[z_1, \dots, z_n]]/J''$ is an integral domain. In fact we will see that the natural map

$$\mathbb{C}[z_{k+1}, \dots, z_n] \longrightarrow \mathbb{C}[[z_1, \dots, z_n]]/J'' \tag{2.12.6}$$

is an isomorphism of rings. This follows from the algebraic version of the Implicit Function Theorem, i.e. **Proposition 2.12.6**. In fact, by **Proposition 2.12.6**, the natural map $\mathbb{C}[[z_2, \dots, z_n]] \to \mathbb{C}[[z_1, \dots, z_n]]/(f_1)$ is an isomorphism. Let $i \in \{2, \dots, k\}$. Given the identification $\mathbb{C}[[z_1, \dots, z_n]]/(f_1) = \mathbb{C}[[z_2, \dots, z_n]]$, the image of f_i under the quotient map $\mathbb{C}[[z_1, \dots, z_n]] \to \mathbb{C}[[z_1, \dots, z_n]]/(f_1)$ is an element $z_i + f_i'$, where $f_i' \in (\mathfrak{m}'')^2$ (notation as in **Lemma 2.12.5**). Iterating, we get that the map in (2.12.6) is an isomorphism of rings. As explained above, this proves that J is a prime ideal. In particular X is irreducible. Moreover, since $z_{k+1}, \dots, z_n \in \mathbb{C}[X]$, the isomorphism in (2.12.6) shows that $\mathbb{C}(X)$ has transcendence degree $n-k$, i.e. X has dimension $n-k$. Since $f_1, \dots, f_k$ vanish on X, and their differentials are linearly independent, it follows that $\dim \Theta_0(X) \leqslant (n-k) = \dim_0 X$. Hence $\dim \Theta_0(X) = (n-k) = \dim_0 X$, by Item (2) of **Proposition 2.12.4**, i.e. X is smooth at 0, and $\Theta_0(X) \subset \Theta_0\mathbb{A}^n$ is the annihilator of $df_1(0), \dots, df_k(0)$. This proves Items (1) and (2). The last statement in the proposition holds with the polynomial g appearing in (2.12.3). □

Corollary 2.12.8. *Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Let a be a smooth point of X, and let $k = n - \dim_a X$. Then following hold:*

1. *there exist $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_n]$ with linerly independent differentials $df_1(a), \dots, df_k(a)$, and a Zariski open affine subset $U \subset \mathbb{A}^n$ containing a, such that $I(X \cap U) = (f_{1|U}, \dots, f_{k|U})$;*
2. *there is a unique irreducible component of X containing a;*
3. *there exists an open (classical topology) $\mathscr{U} \subset \mathbb{A}^n$ containing a such that $X \cap \mathscr{U}$ is a complex submanifold of $\mathscr{U}$, of dimension $n-k$;*
4. *the natural map $T_a(X \cap \mathscr{U}) \to \Theta_a X$, where $T_a(X \cap \mathscr{U})$ is the holomorphic tangent space, induced from the injection of rings of germs $\mathscr{O}_{X,a} \hookrightarrow \mathscr{O}^{\mathrm{hol}}_{X \cap \mathscr{U},a}$ is an isomorphism.*

Proof. Since X is smooth at a, and $\dim_a X = n-k$, there exist $f_1, \dots, f_k \in I(X)$ such that $df_1(a), \dots, df_k(a)$ are linearly independent. Of course $X \subset V(f_1, \dots, f_k)$. By **Proposition 2.12.7** there is a unique irreducible component of $V(f_1, \dots, f_k)$ containing a, call it Y, and $\dim Y = n-k$. Every irreducible component of X containing a is contained in Y. Since $\dim_a X = n-k$, there exists (at least) one irreducible component of X containing a of dimension $n-k$. Let X' be such an irreducible component; by **Proposition 2.9.2**, $X' = Y$. It follows that there is a single component of X containing a, and it is equal to the unique irreducible component of $V(f_1, \ldots, f_k)$ containing a. Hence the corollary follows from **Proposition 2.12.7**. $\square$

Proof of **Theorem 2.12.3**. Let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. Since X is covered by open affine subset, **Corollary 2.12.8** gives that

$$X^{\mathrm{sm}} \subset X \backslash \bigcup_{\substack{i,j \in I \\ i \neq j}} \cap X_i \cap X_j. \tag{2.12.7}$$

The right hand side of (2.12.7) is an open dense subset of X. Let X_i^0 be an irreducible component of the right hand side of (2.12.7). Thus $X_i^0 \subset X_i$ is the complement of the intersection of X_i with the other irreducible componets of X. The set of smooth points of X_i^0 is non empty by **Proposition 2.12.4**, and it is open by upper semiconinuity of the dimension of $\Theta_x X$ (**Proposition 2.10.11**), because $\dim_x X$ is independent of $x \in X_i^0$. Hence X^{sm} is an open dense subset of the open dense subset of X given by the right hand side of (2.12.7), and hence is open and dense in X. This proves Item (1) of **Theorem 2.12.3**. Item (2) has been proved in **Proposition 2.12.4**. Item (3) follows at once from **Corollary 2.12.8**, because X is covered by open affine subset. $\square$

Exercises

Exercise 2.12.1. Let $V \subset \mathbb{P}(\mathbb{C}[Z_0, \ldots, Z_n]_d)$ be defined by

$$V := \{[L^d] \mid 0 \neq L \in \mathbb{C}[Z_0, \ldots, Z_n]_d\}.$$

1. Prove that $[F] \in V$ if and only if

$$\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n} \text{ span a 1-dimensional subspace of } \mathbb{C}[Z_0, \ldots, Z_n].$$

[*Hint.* By induction on $\deg F$. Moreover use *Euler's identity*

$$\sum_{j=0}^{n} Z_j \frac{\partial F}{\partial Z_j} = (\deg F) \cdot F$$

for F *homogeneous.*]

2. Deduce from (1) that V is closed in $\mathbb{P}(\mathbb{C}[Z_0, \ldots, Z_n]_d)$.

3. Identify up to projectivities V with the Veronese variety $\mathscr{V}_d^n$.

Appendix A

Commutative algebra à la carte

A.1 Noetherian rings

In what follows, rings are always commutative with 1. The proofs of the results below are contained in most Algebra textbooks (e.g. Lang [**?**]).

Definition A.1.1. A (commutative unitary) ring R is *Noetherian* if every ideal of R is finitely generated.

Example A.1.2. A field K is Noetherian, because the only ideals are $\{0\} = (0)$ and $K = (1)$. The ring $\mathbb{Z}$ is Noetherian, because every ideal has a single generator.

Lemma A.1.3. *A (commutative unitary) ring R is Noetherian if and only if every ascending chain*

$$I_0 \subset I_1 \subset \ldots \subset I_m \subset \ldots$$

of ideals of R (here I_m is defiend for all $m \in \mathbb{N}$, and $I_m \subset I_{m+1}$ for all $m \in \mathbb{N}$) is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such hat $I_m = I_{m_0}$ for $m \geqslant m_0$.

Proof. Suppose that R is Noetherian. The union $I := \bigcup_{m \in \mathbb{N}} I_m$ is an ideal because the $\{I_m\}$ form a chain. By Noetherianity I is finitely generated, say $I = (a_1, \ldots, a_r)$. There exists m_0 such that $a_j \in I_{m_0}$ for $j \in \{1, \ldots, r\}$, and hence $I = I_{m_0}$. Let $m \geqslant m_0$; then $I_m \subset I$ and $I \subset I_m$, hence $I = I_m$. Thus $I_{m_0} = I_m$ for $m \geqslant m_0$.

Now suppose that every ascending chain of ideals of R is stationary. Let $I \subset R$ be an ideal. Suppose that I is *not* finitely generated. Let $a_1 \in I$. Then $(a_1) \subsetneq I$ because I is not finitely generated; let $a_2 \in (I \backslash (a_1))$. Then $(a_1, a_2) \subsetneq I$ because I is not finitely generated. Iterating, we get a non stationary chain of ideals (contained in I)

$$(a_1) \subsetneq (a_1, a_2) \subsetneq \ldots \subsetneq (a_1, \ldots, a_m) \subsetneq$$

This is a contradiction. □

Example A.1.4. The ring Hol($\mathbb{C}$) of entire functions of one variable is *not* Noetherian. In fact let $f_m \in \mathrm{Hol}(\mathbb{C})$ be defined by

$$f_m(z) := \prod_{n=m}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \qquad m \geqslant 1.$$

Then $(f_m) \subsetneq (f_{m+1})$. Thus $(f_1) \subset (f_2) \subset \ldots \subset (f_m) \subset \ldots$ is a non-stationary ascending chain of ideals, and hence Hol($\mathbb{C}$) is not Noetherian by **Lemma A.1.3**.

Theorem A.1.5. *Let R be a Noetherian commutative ring. Then $R[t]$ is Noetherian.*

Theorem A.1.6 (Hilbert's basis Theorem)**.** *Every ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is finitely generated.*

Proof. By induction on n. If $n = 0$, the ring is a field, and hence is Noetherian. The inductive step follows from **Theorem A.1.5**, because $\mathbb{C}[x_1, \ldots, x_n] \cong \mathbb{C}[x_1, \ldots, x_{n-1}][t]$. □

A.2 The Nullstellensatz

We will denote $\mathbb{C}^n$ by $\mathbb{A}^n$ when we will view it as an n dimensional complex affine space. If $I \subset \mathbb{C}[z_1, \ldots, z_n]$ is an ideal, we let

$$V(I) := \{z \in \mathbb{A}^n \mid f(z) = 0 \quad \forall f \in I\}.$$

(The above notation is the same that is used for closed subsets of $\mathbb{P}^n$, and hence there is potential for confusion. Which of the two definitions of $V(I)$ applies in each instance will be clear from the context.)

If $Y \subset \mathbb{A}^n$ is a subset, we let $I(Y) := \{f \in \mathbb{C}[z_1, \ldots, z_n] \mid f|_Y = 0\}$. We recall that the *radical* of an ideal I ina ring R, is the set of elements $a \in R$ such that $a^m \in I$ for some $m \in \mathbb{N}$. As is easily checked, the radical is an ideal; it is denoted by $\sqrt{I}$,

Theorem A.2.1 (Hilbert's Nullstellensatz)**.** *Let $I \subset \mathbb{C}[z_1, \ldots, z_n]$ be an ideal. Then $I(V(I)) = \sqrt{I}$.*

Corollary A.2.2 (Weak Nullstellensatz)**.** *Let $I \subset \mathbb{C}[z_1, \ldots, z_n]$ be an ideal. Then $V(I) = \varnothing$ if and only if $I = (1)$.*

Proof. If $I = (1)$, then $V(I) = \varnothing$.Assume that $V(I) = \varnothing$. By the Nullstellensatz, $\sqrt{I} = I(V(I)) = I(\varnothing) = (1)$. Thus $1^m \in I$ for some $m \in \mathbb{N}$, and hence $1 \in I$. □

A.3 Unique factorization

Theorem A.3.1. *Let R be a UFD. Then $R[t]$ is a UFD. Moreover a polynomial $p = a_0t^d + a_1t^{d-1} + \ldots + a_d$ is prime if and only if*

1. *p is prime when viewed as element of $K[t]$, where K is the field of fractions of R,*
2. *and the greatest common divisor of $a_0, a_1, \ldots, a_d$ is 1.*

Corollary A.3.2. *The ring $\mathbb{C}[x_1, \ldots, x_n]$ is a unique factorization domain.*

Proof. By induction on n. If $n = 0$, the ring is a field, and hence it is trivially a UFD. The inductive step follows from **Theorem A.3.1**, because $\mathbb{C}[x_1, \ldots, x_n] \cong \mathbb{C}[x_1, \ldots, x_{n-1}][t]$. □

A.4 Extensions of fields

Let $F \subset E$ be an extension of fields. Elements $\alpha_1, \ldots, \alpha_n \in E$ are *algebraically dependent* over F is there exists a non zero polynomial $f \in F[z_1, \ldots, z_n]$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$ (strictly speaking, we should say that the set $\{\alpha_1, \ldots, \alpha_n\}$ is algebraically dependent over F). A collection $\{\alpha_i\}_{i \in I}$ of elements of E is *algebraically independent* over F if there does *not* exist a non empty finite $\{i_1, \ldots, i_n\} \subset I$ such that $\alpha_{i_1}, \ldots, \alpha_{i_n}$ are algebraically dependent (with the usual abuse of language, we also say that the α_i's are algebarically independent). A *transcendence basis* of E over F is a maximal set of algebraically independent elements of E over F. There always exists a transcendence basis, by Zorn's Lemma. One proves that any two transcendence bases have the same cardinality, which is by definition the *transcendence degree* of E over F; we denote it by $\mathrm{Tr.deg}_F(E)$. An extension $F \subset E$ is *algebraic* if the transcendence degree is 0. Every finitely generated extension $F \subset E$ can be obtained as a composition of extensions $F \subset K$ and $K \subset E$, where $F \subset K$ is a *purely transcendental extension*, i.e. there exists a transcendence basis $\{\alpha_1, \ldots, \alpha_n\}$ of K over F such that $K = F(\alpha_1, \ldots, \alpha_n)$ (thus $F(\alpha_1, \ldots, \alpha_n)$ is isomorphic to the filed of rational functions in n indeterminates with coefficients in F), and $F \subset K$ is a finitely generated algebraic extension.

Theorem A.4.1. *Let $F \subset E$ be a finite extension of fields, i.e. the dimension of E as F-vector space is finite. Suppose that F is of characteristic 0. Then there exists a* primitive *element of E over F, i.e. $\alpha \in E$ such that $E = F(\alpha)$.*

A.5 Derivations

Let R be a ring (commutative with unit), and let M be an R-module.

Definition A.5.1. A *derivation from R to M* is a map $D \colon R \to M$ such that additivitity and Leibinitz' rule hold, i.e. for all $a, b \in R$,

$$D(a + b) = D(a) + D(b), \quad D(ab) = bD(a) + aD(b).$$

If k is a field and R is a k-algebra a *k-derivation* (or *derivation over k*) $D\colon R \to M$ is a derivation such that $D(c) = 0$ for all $c \in k$. We let $\mathrm{Der}(R, M)$ be the set of derivations from R to M. If R is a k-algebra we let $\mathrm{Der}_k(R, M) \subset \mathrm{Der}(R, M)$ be the subset of k-derivations.

Example A.5.2. Let k be a field, and let $f = \sum_I a_I z^I$ be a polynomial in $k[z_1, \ldots, z_n]$, where the summation is over multiindices I, $a_I \in \mathbb{C}$ for every I, and a_I is almost always zero. The formal derivative of f with respect to z_m is defined by the familar formula

$$\frac{\partial f}{\partial z_m} = \sum_{I \text{ s.t. } i_m > 0} i_h a_I z_1^{i_1} \cdot \ldots \cdot z_{m-1}^{i_{m-1}} \cdot z_m^{i_m - 1} \cdot z_{m+1}^{i_{m+1}} \cdot \ldots z_n^{i_n}. \tag{A.5.1}$$

The map

$$\begin{array}{ccc} k[z_1, \ldots, z_n] & \xrightarrow{\frac{\partial}{\partial z_m}} & k[z_1, \ldots, z_n] \\ f & \mapsto & \frac{\partial f}{\partial z_m} \end{array} \tag{A.5.2}$$

is a k-derivation of the k algebra to istelf. We claim that $\mathrm{Der}_k(k[z_1, \ldots, z_n], k[z_1, \ldots, z_n])$ is freely generated (as $k[z_1, \ldots, z_n]$ module) by $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$. In fact there is no relation between $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ because $\frac{\partial z_j}{\partial z_m} = \delta_{jm}$, and moreover, given a k derivation

$$D\colon k[z_1, \ldots, z_n] \to k[z_1, \ldots, z_n]$$

we have $D = \sum_{m=1}^n \alpha_m \frac{\partial}{\partial z_m}$, where $\alpha_m := D(z_m)$.

Example A.5.3. Let $D\colon R \to M$ be a derivation.

1. By Leibniz we have $D(1) = D(1 \cdot 1) = D(1) + D(1)$ and hence $D(1) = 0$.
2. Suppose that $g \in R$ is invertible. Then
$$0 = D(1) = D(g \cdot g^{-1}) = g^{-1} Dg + f D(g^{-1}) \tag{A.5.3}$$
and hence $D(g^{-1}) = -g^{-2} D(f)$.
3. Suppose that $f, g \in R$ and that g is invertible. By Item (2) we get that the following familiar formula holds:
$$D(f \cdot g^{-1}) = g^{-2}(D(f) \cdot g - f \cdot D(g)). \tag{A.5.4}$$

Let $D, D' \in \mathrm{Der}(R, M)$ and $z \in R$ we let

$$\begin{array}{ccc} R & \xrightarrow{D+D'} & M \\ a & \mapsto & D(a) + D'(a) \end{array} \tag{A.5.5}$$

and

$$\begin{array}{ccc} R & \xrightarrow{zD} & M \\ a & \mapsto & zD(a) \end{array} \tag{A.5.6}$$

Both $D + D'$ and zD are derivations and with these operations $\mathrm{Der}(R, M)$ is an R-module. If R is a k-algebra then $\mathrm{Der}_k(R, M)$ is an R-submodule of $\mathrm{Der}(R, M)$.

A.6 Order of vanishing

The prototype of a Noetherian local ring $(R, \mathfrak{m})$ is the ring $\mathscr{O}_{X,x}$ of germs of regular functions of a quasi projective variety X at a point $x \in X$, with maximal ideal $\mathfrak{m}_x$, see **Proposition 2.10.4**. The following result of Krull can be interpreted as stating that a non zero element of $\mathscr{O}_{X,x}$ can not vanish to arbitrary high order at x. In other words, elements of $\mathscr{O}_{X,x}$ behave like analytic functions (as opposed to C^∞ functions).

Theorem A.6.1 (Krull)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$\bigcap_{i \geqslant 0} \mathfrak{m}^i = \{0\}.$$

Proof. Since R is Noetherian the ideal $\mathfrak{m}$ is finitely generated; say $\mathfrak{m} = (a_1, \ldots, a_n)$. Let $b \in \bigcap_{i \geqslant 0} \mathfrak{m}^i$. Let $i \geqslant 0$; since $b \in \mathfrak{m}^i$ there exists $P_i \in R[X_1, \ldots, X_n]_i$ such that $P_i(a_1, \ldots, a_n) = b$. Let $J \subset R[X_1, \ldots, X_n]$ be the ideal generated by the P_i's. Since R is Noetherian so is $R[X_1, \ldots, X_n]$. Thus J is finitely generated and hence there exists $N > 0$ such that $J = (P_0, \ldots, P_N)$. Thus there exists $Q_{N+1-i} \in R[X_1, \ldots, X_n]_{N+1-i}$ for $i = 0, \ldots, N$ such that $P_{N+1} = \sum_{i=0}^N Q_{N+1-i} P_i$. It follows that

$$b = P_{N+1}(a_1, \ldots, a_n) = \sum_{i=0}^N Q_{N+1-i}(a_1, \ldots, a_n) P_i(a_1, \ldots, a_n) = b \sum_{i=0}^N Q_{N+1-i}(a_1, \ldots, a_n). \tag{A.6.7}$$

Now $Q_{N+1-i}(a_1, \ldots, a_n) \in \mathfrak{m}$ for $i = 0, \ldots, N$ and hence $\epsilon := \sum_{i=0}^N Q_{N+1-i}(a_1, \ldots, a_n) \in \mathfrak{m}$. Equality (A.6.7) gives that $(1 - \epsilon) b = 0$: since $\epsilon \in \mathfrak{m}$ the element $(1 - \epsilon)$ is invertible and hence $b = 0$. □

Corollary A.6.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\mathfrak{I} \subset R$ be an ideal. Then*

$$\bigcap_{i \geqslant 0} (\mathfrak{I} + \mathfrak{m}^i) = \{0\}.$$

Proof. Let $S := R/\mathfrak{I}$. Then S is a Noetherian local ring, with maximal ideal $\mathfrak{m}_S := \mathfrak{I} + \mathfrak{m}$. The corollary follows by applying **Theorem A.6.1** to $(S, \mathfrak{m}_S)$. □

Bibliography

[TSPR] A. J. de Jong & c. *The Stacks Project,* https://stacks.math.columbia.edu