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Moduli of vector bundles on projective surfaces: some basic results

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In the 1970's Gieseker and Maruyama constructed moduli spaces for stable vector-bundles and for semistable torsion-free sheaves. Many detailed and interesting results have been proved regarding these moduli spaces in the case when the surface belongs to some particular class, but our knowledge decreases as the Kodaira dimension of the surface increases, and in particular very little is known if the surface is of general type. In this paper we address two basic questions:

1. When is the moduli space reduced and of the expected dimension?
2. When is the moduli space irreducible?

In order to present our results we need to introduce some notation. Let S be a smooth irreducible projective surface over $\mathbf{C}$, and let H be an ample divisor on S. To define a moduli space of sheaves on the polarized surface (S, H) we need a *set of sheaf data* ξ, i.e. a triple

$$\xi = (r_\xi, \det_\xi, c_2(\xi)),$$

where r_ξ is a positive integer, $\det_\xi$ is a line bundle on S, and $c_2(\xi) \in H^4(S;\mathbf{Z}) \cong \mathbf{Z}$. We let $\mathcal{M}_\xi$ be the moduli space of semistable (with respect to H) torsion-free sheaves, F, on S with

$$r_F = r_\xi, \qquad \det F \cong \det_\xi, \qquad c_2(F) = c_2(\xi). \tag{0.1}$$

A fundamental theorem of Gieseker and Maruyama [G1, Ma] asserts that $\mathcal{M}_\xi$ is projective. If F is a semistable sheaf satisfying (0.1), we let $[F]$ be the point in $\mathcal{M}_\xi$ corresponding to the equivalence class of F. We recall some known facts concerning the local structure of $\mathcal{M}_\xi$. First let's define the *discriminant* of a torsion-free sheaf F on S as

$$\Delta_F := c_2(F) - \frac{r_F - 1}{2r_F} c_1(F)^2$$

(warning: our normalization differs from that of [DL]). If ξ is a set of sheaf data, the discriminant Δ_ξ is defined in the obvious way: if $[F] \in \mathcal{M}_\xi$, then $\Delta_\xi = \Delta_F$. The *expected dimension* of $\mathcal{M}_\xi$ is given by

$$\exp\dim(\mathcal{M}_\xi) := 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S)\,.$$

Now assume that $[F] \in \mathcal{M}_\xi$ and that F is stable. Then deformation theory [F] gives

$$\dim_{[F]}\mathcal{M}_\xi \geqq 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S)\,, \qquad (0.2)$$

$$\dim T_{[F]}(\mathcal{M}_\xi) = 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S) + h^0(F, F\otimes K)^0\,, \qquad (0.3)$$

where, for a line bundle L on S, we set

$$h^0(F, F\otimes L)^0 := \dim\{\varphi \in \mathrm{Hom}(F, F\otimes L) |\, \mathrm{tr}\,\varphi = 0\}\,.$$

Let $[F] \in \mathcal{M}_\xi$. Following Friedman we say that $\mathcal{M}_\xi$ is *good* at $[F]$ if F is stable and $h^0(F, F\otimes K)^0$ vanishes, where K is the canonical line bundle. In this case (0.2) is an equality and the moduli space is smooth near $[F]$. We say that $\mathcal{M}_\xi$ is *good* if it is good at the generic point of every one of its irreducible components: this means that $\mathcal{M}_\xi$ is reduced and its dimension equals the expected dimension. Now we can go back to Questions (1) and (2). First of all notice that if $r_\xi = 1$ then $\mathcal{M}_\xi$ is good (at each point) for trivial reasons, and furthermore, as is well-known, it is always irreducible. Thus we will only be concerned with the case $r_\xi \geqq 2$. Our main result is that if $\Delta_\xi \gg 0$ then $\mathcal{M}_\xi$ is good and irreducible, i.e. both questions have a positive answer. The significance of the condition $\Delta_\xi \gg 0$ is the following: if $\Delta_\xi < 0$ then $\mathcal{M}_\xi$ is empty by Bogomolov's Inequality, and on the other hand $\mathcal{M}_\xi \neq \emptyset$ if $\Delta_\xi \gg 0$ (see [HL, LQ]). Actually we will prove more than the simple statement that $\mathcal{M}_\xi$ is asymptotically good. To explain this, let L be a line bundle on S and set

$$W_\xi^L = \{[F] \in \mathcal{M}_\xi |\, h^0(F, F\otimes L)^0 > 0\}\,.$$

Thus if F is stable then $\mathcal{M}_\xi$ is good at $[F]$ if and only if $[F] \notin W_\xi^K$. We prove that (if L is fixed) the growth of $\dim W_\xi^L$ (for fixed rank and increasing Δ_ξ) is smaller than that of the expected dimension of $\mathcal{M}_\xi$. The theorem about $\mathcal{M}_\xi$ being good for $\Delta_\xi \gg 0$ follows at once from this result (setting $L = K$), together with some dimension counts taking care of strictly semistable sheaves. To see that it is interesting to bound the dimension of W_ξ^L for general L consider the case $L = \mathcal{O}_S(K + C)$, where C is a smooth curve on S. In this case, if F is locally-free and stable, the geometric significance of $[F] \notin W_\xi^L$ is the following: the natural morphism from a neighborhood of $[F]$ in $\mathcal{M}_\xi$ to the deformation space of $F|_C$ surjects onto the subspace of deformations fixing the isomorphism class of $\det(F|_C)$.

The precise statements of the results we have described are given in Theorems B, C, D, E and their corollaries. One feature of these theorems is that they are for the most part effective. Thus we give an explicit upper bound for $\dim W_\xi^L$. From this one can compute an explicit lower bound for Δ_ξ guaranteeing that $\mathcal{M}_\xi$ is good. This lower bound for arbitrary rank might not be very practical, however it appears to depend on the "correct" quantities. If the rank is two our methods are somewhat stronger: in this case we have computed the lower bound explicitly, and we will show that it can not be too far off from the optimal one. Regarding irreducibility our results are less explicit, but we do give a lower bound for Δ_ξ guaranteeing irreducibility of the moduli space for rank-two bundles with trivial determinant on a complete intersection.

All of the above results spring from Propositions (2.1)–(2.2). To explain the content of these propositions let the *boundary* of $\mathcal{M}_\xi$, denoted by $\partial\mathcal{M}_\xi$, be the subset of $\mathcal{M}_\xi$ parametrizing sheaves which are singular, i.e. not locally-free. Furthermore, if $X \subset \mathcal{M}_\xi$ let the *boundary* of X be the intersection

$$\partial X := X \cap \partial\mathcal{M}_\xi \, .$$

Propositions (2.1)–(2.2) assert that if $X \subset \mathcal{M}_\xi$ is a closed subvariety whose dimension satisfies certain conditions then $\partial X \neq \emptyset$. These propositions are proved by further developing the ideas in the proof of Theorem (2.0.3) of [O]. (This theorem states that, in rank two, any irreducible component intersects the boundary, if $\Delta_\xi \gg 0$.) From Propositions (2.1)–(2.2) one obtains Theorem A, which bounds the maximum dimension of complete subvarieties of $\mathcal{M}_\xi$ not intersecting the boundary. Given Theorem A one can bound $\dim W_\xi^L$, this is the content of Theorem B, and in particular prove that $\mathcal{M}_\xi$ is asymptotically good. Theorem C gives an explicit lower bound for Δ_ξ guaranteeing that $\mathcal{M}_\xi$ is good, in rank two: this is obtained by arguments similar to those that give Theorem B. Asymptotic irreducibility (Theorem D) follows from Theorems A and B: the argument is due to Gieseker and Li. We reproduce their proof because we will then apply it to complete intersections in order to obtain an explicit result (Theorem E). Our proofs depend on certain estimates: in particular we need an upper bound for the dimension of the loci in $\mathcal{M}_\xi$ parametrizing sheaves with subsheaves with (relatively) large slope. This and other estimates are proved in the last section of the paper.

Donaldson [D, Fr, Z] first proved that $\mathcal{M}_\xi$ is asymptotically good, if the rank is two. He showed that $\dim W_\xi^L$ is bounded, up to lower order terms, by $3\Delta_\xi$. Donaldson's bound for $\dim W_\xi^L$ is asymptotically better than ours but, since the lower order terms have eluded computation, it does not give an effective result. Recently Gieseker and Li [GL2] have proved that $\mathcal{M}_\xi$ is asymptotically good in arbitrary rank. They show that the codimension of W_ξ^L in $\mathcal{M}_\xi$ goes to infinity for $\Delta_\xi \to \infty$, but the result is not effective. Asymptotic irreducibility in rank two was first obtained by Gieseker and Li [GL1].

Statement of results. Throughout the paper surface means a smooth irreducible projective surface: we will always denote if by S. We let K be its canonical divisor, and H be an ample divisor on S. We will often make the following assumption:

$$|H| \text{ is base-point-free and } \dim |H| \geqq 2 \, . \tag{0.4}$$

When considering a moduli space $\mathcal{M}_\xi$ we always tacitly assume that $r_\xi \geqq 2$.

If $r > 2$ is an integer, set

$$\rho(r) := 8(16r^3 - 39r^2 + 36r - 12)^{-1} ,$$
$$\Delta_0(r,S,H) := \rho^{-1}H^2 ,$$
$$\lambda_2(r) := 2r - \frac{r-1}{2}\rho ,$$
$$\lambda_1(r,S,H) := \sqrt{\rho}\left[\frac{r^3}{2}\frac{|K \cdot H|}{\sqrt{H^2}} + r^5\sqrt{H^2}\right] ,$$
$$\lambda_0(r,S,H) := \frac{r^7}{2}H^2 + \frac{r^4}{5}\frac{(K \cdot H)^2}{H^2} + \frac{r^2}{2}\frac{(K \cdot H + (r^2+1)H^2 + 1)^2}{H^2} + r^2|\chi(\mathcal{O}_S)| + \frac{r^3}{8}|K^2| .$$

When $r = 2$, we set

$$\Delta_0(2,S,H) := \begin{cases} 3H^2, & \text{if } K \cdot H < 0 , \\ 3H^2\left(1 + \frac{K \cdot H}{H^2}\right)^2, & \text{if } K \cdot H \geqq 0 , \end{cases}$$
$$\lambda_2(2) := \frac{23}{6} ,$$
$$\lambda_1(2,S,H) := \frac{1}{2\sqrt{3H^2}}(4H^2 + 3K \cdot H + 4) ,$$
$$\lambda_0(2,S,H) := \begin{cases} \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} \\ \quad -\frac{K^2}{4} + 4 - 3\chi(\mathcal{O}_S), & \text{if } K \cdot H < 0 , \\ \frac{3(K \cdot H)^2}{H^2} + 6K \cdot H + \frac{3H^2}{2} \\ \quad -\frac{K^2}{4} + 8 - 3\chi(\mathcal{O}_S), & \text{if } K \cdot H \geqq 0 . \end{cases}$$

Theorem A. *Let (S,H) be a polarized surface. Assume that H satisfies* (0.4). *Let ξ be a set of sheaf data such that $\Delta_\xi > \Delta_0(r_\xi,S,H)$. If $X \subset \mathcal{M}_\xi$ is a closed subvariety such that*

$$\dim X > \lambda_2(r_\xi)\Delta_\xi + \lambda_1(r_\xi,S,H)\sqrt{\Delta_\xi} + \lambda_0(r_\xi,S,H) , \tag{0.5}$$

then the boundary of X is non-empty.

Notice that since $\lambda_2(r) < 2r$ the above result is meaningful (see (0.2)). Let

$$\lambda_0'(r,S,H) := \max\{\lambda_0(r,S,H),\ \varepsilon(r,S,H) + r\} ,$$

$$\Delta_1(r,S,H) := \max\{(\lambda_2 - (2r-1))^{-1} \cdot (\Delta_0 + e_K - \lambda_0' - (r^2-1)\chi(\mathcal{O}_S)),\ \Delta_0\}$$
$$\text{if } (r,S,H) \neq (2,\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(1)) ,$$

$$\Delta_1(2,\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(1)) := 3 .$$

Here ε and e_K are given by (1.9) and (1.5) respectively.

Theorem B. *Let* (S,H) *be a polarized surface, with* H *satisfying* (0.4). *Let* L *be a line bundle on* S. *Let* ξ *be a set of sheaf data such that* $\Delta_\xi > \Delta_1(r_\xi,S,H)$. *Then*

$$\dim W_\xi^L \leqq \lambda_2(r_\xi)\Delta_\xi + \lambda_1(r_\xi,S,H)\sqrt{\Delta_\xi} + \lambda_0'(r_\xi,S,H) + e_L(r_\xi,S,H)\,, \quad (0.6)$$

where e_L *is given by* (1.5).

Applying the above result with $L = K$, one gets the following

Corollary B′. *There exists a function* $\Delta_1'(r,S,H)$ *(depending only on* r, K^2, $K \cdot H, H^2$, *and* $\chi(\mathcal{O}_S)$*) such that the following holds. Let* (S,H) *be as above, and* ξ *be a set of sheaf data such that* $\Delta_\xi > \Delta_1'(r_\xi,S,H)$. *Then* $\mathcal{M}_\xi$ *is good. Furthermore* $\mathcal{M}_\xi$ *is the closure of the open subset parametrizing* μ*-stable vector bundles.*

In the rank-two case we will carry out the computations necessary to determine explicitly the lower bound of this corollary. The result is the following.

Theorem C. *Let* (S,H) *be a polarized surface. Assume that* H *is effective and that there exists a smooth curve in* $|n_0H|$, *where* n_0 *is given by* (3.13). *Let* $\Delta_2(S,H)$ *be the function defined by* (3.12). *Let* ξ *be a set of sheaf data with* $r_\xi = 2$. *If*

$$\Delta_\xi > \Delta_2(S,H) + 2h^0(2K)\,,$$

then $\mathcal{M}_\xi$ *is good. Furthermore* $\mathcal{M}_\xi$ *is the closure of the open subset parametrizing* μ*-stable vector bundles.*

To exemplify the possible uses of this theorem we give the following two results.

Corollary C′. *Let* S *be a surface with ample canonical divisor* K. *Assume that* $p_g(S) > 0$, *and that* $K^2 \gg 0$ *(*$K^2 > 100$ *will do). Let* ξ *be a set of sheaf data with* $r_\xi = 2$, *and let* $\mathcal{M}_\xi$ *be the corresponding moduli space for semistable sheaves on the polarized surface* (S,K). *If*

$$\Delta_\xi \geqq 42K^2 + 15\chi(\mathcal{O}_S)\,,$$

then $\mathcal{M}_\xi$ *is good, and furthermore the subset parametrizing* μ*-stable vector bundles is dense in* $\mathcal{M}_\xi$.

Corollary C″. *Let* S *be a surface with ample canonical divisor* K. *Assume that* H *is a divisor satisfying* (0.4), *and that* $c_1(K) = kc_1(H)$ *for* $k \gg 0$ *(say* $k > 100$*). Let* ξ *be a set of sheaf data with* $r_\xi = 2$, *and let* $\mathcal{M}_\xi$ *be the corresponding moduli space for semistable sheaves on* (S,K). *If*

$$\Delta_\xi \geqq 17K^2 + 10\chi(\mathcal{O}_S)\,,$$

then the conclusions of the previous corollary hold.

These two corollaries only apply to minimal surfaces of general type with no (-2)-curves. Of course Theorem C applies to any surface, in particular we can apply it to surfaces of general type with (-2)-curves, but the lower bound one gets is not as nice as that of Corollaries C′–C″. However we believe that bounds similar to those of these corollaries hold for any surface of general type, if K^2 is replaced by $\omega^2_{S_{\rm can}}$, where $S_{\rm can}$ is the canonical model of S, and the polarization is close enough to $\omega_{S_{\rm can}}$ (how "close" will depend on Δ_ξ). When $S_{\rm can}$ is smooth this should follow from the results in Sect. 2 of [MO]; the case when S contains (-2)-curves should be analyzable by similar methods.

Finally we come to irreducibility.

Theorem D. *There exists a function $\Delta_3(r,S,H)$ such that the following holds. If (S,H) is a polarized surface, and if ξ is set of sheaf data such that $\Delta_\xi > \Delta_3(r_\xi,S,H)$, then $\mathscr{M}_\xi$ is irreducible (and the open subset parametrizing μ-stable vector bundles is dense).*

We give an explicit value for Δ_3 valid for complete intersections of large degree, if the rank is two and the determinant is trivial.

Theorem E. *Let S be a complete intersection in a projective space, and let $H = \mathscr{O}_S(1)$. Suppose also that the integer k such that $K \sim kH$ is very large ($k > 100$ suffices). Let*

$$\xi = (2, \mathscr{O}_S, c_2(\xi)) \, .$$

If

$$\Delta_\xi > 95K^2 + 11\chi(\mathscr{O}_S) + 1 \, ,$$

then $\mathscr{M}_\xi$ is irreducible.

Notational warning. A reference, in the course of a proof, to a formula labeled by a symbol such as $*$, †, etc. always refers to the (unique!) formula with that label appearing in that same proof.

1. Preliminaries

We introduce notation that will be used throughout the paper. We also state some technical results whose proof is deferred to the last section.

We work throughout over the complex numbers. We let $\pi_X : X \times Y \to X$ be the projection. Sheaves are always coherent. If F, G are sheaves on the same scheme we set

$$h^i(F,G) := \dim \operatorname{Ext}^i(F,G), \qquad \chi(F,G) := \sum_i (-1)^i h^i(F,G) \, .$$

A family of sheaves on X parametrized by B consists of a sheaf $\mathscr{F}$ on $X \times B$, flat over B. If $b \in B$ we set $\mathscr{F}_b := \mathscr{F}|_{X\times\{b\}}$. Let ξ be a set of sheaf data for S, and let $U \subset \mathscr{M}_\xi$ be an algebraic subset all of whose points parametrize stable sheaves. A *tautological family* parametrized by U consists of a family of stable sheaves $\mathscr{F}$ on S, parametrized by U, such that for all $b \in U$ the isomorphism class of $\mathscr{F}_b$ is represented by b. If F is a sheaf on a smooth curve or surface, we let $\operatorname{Def}(F)$ be the versal deformation space of F, and $\operatorname{Def}^0(F)$ be the subscheme parametrizing deformations which fix the isomorphism class of $\det F$. (See [Fr].)

If F is a semistable torsion-free sheaf on S, we denote by $\mathrm{Gr}(F)$ the direct sum of the successive quotients of any Jordan–Hölder filtration of F. Recall that the (closed) points of $\mathcal{M}_\xi$ are in one-to-one correspondence with equivalence classes of torsion-free semistable sheaves F satisfying (0.1): two semistable sheaves F_1, F_2 are *equivalent* if and only if $\mathrm{Gr}(F_1) \cong \mathrm{Gr}(F_2)$.

Let X be a projective irreducible variety, and D be an ample divisor on X. The *slope* of a torsion-free sheaf F on X with respect to D is given by

$$\mu_F := \frac{1}{r_F} c_1(F) \cdot D^{n-1} ,$$

where $n := \dim X$. We recall that F is *μ-semistable* (equivalently *slope-semistable*) if, for all subsheaves $E \subset F$ we have

$$\mu_E \leqq \mu_F .$$

If the above inequality is strict whenever $r_E < r_F$, then F is *μ-stable* (*slope-stable*). If F is semistable then it is also μ-semistable, if it is μ-stable then it is also stable.

α-stability

Let $\alpha \in \mathbf{R}$. A torsion-free sheaf F on S is *α-stable* if, for every subsheaf $E \subset F$ with $0 < r_E < r_F$, one has

$$\mu_E < \mu_F - \frac{\alpha}{r_E}\sqrt{H^2} .$$

(Thus μ-stability is equivalent to 0-stability.) As is immediately verified F is α-stable if and only if, for every non-trivial torsion-free quotient $F \to Q$, one has

$$\mu_Q > \mu_F + \frac{\alpha}{r_Q}\sqrt{H^2} .$$

Furthermore the notion of α-stability only depends on the ray spanned by $c_1(H)$, and F is α-stable if and only if so is F^*. We let

$$\mathcal{M}_\xi(\alpha) := \{[F] \in \mathcal{M}_\xi | F \text{ is } \textit{not } \alpha\text{-stable}\} .$$

The estimates of the dimension of $\mathcal{M}_\xi(\alpha)$ given below will be an essential technical ingredient in the proof of Theorem A. Let

$$s(r) := ((r-1)^2 + 1)H^2 + 1 ,$$

$$T(r,S,H) := \begin{cases} -\frac{r}{8}K^2 - \frac{1}{2}r(r+1)\chi(\mathcal{O}_S) & \text{if } \chi(\mathcal{O}_S) \geqq 0 \text{ and } K^2 \geqq 0 , \\ \frac{r^2}{2}|\chi(\mathcal{O}_S)| + \frac{r^3}{8}|K^2| & \text{otherwise} . \end{cases} \tag{1.1}$$

(1.2) Proposition. *Assume that H satisfies* (0.4). *Let ξ be a set of sheaf data, and $\alpha \in \mathbf{R}$. Then $\mathcal{M}_\xi(\alpha)$ is a constructible subset of $\mathcal{M}_\xi$. If $K \cdot H \geqq 0$,*

$$\begin{aligned}\dim \mathcal{M}_\xi(\alpha) \leqq\ & (2r_\xi - 1)\Delta_\xi + (2r_\xi - 1)\alpha^2 + (r_\xi^2 - 2r_\xi + 2)\frac{K \cdot H + 2s(r_\xi)}{2\sqrt{H^2}}\alpha \\ & + (r_\xi - 1)(r_\xi^3 - 4r_\xi^2 + 6r_\xi - 2)\frac{(K \cdot H)^2}{8H^2} \\ & + [(r_\xi - 1)^3 + r_\xi]\frac{s(r_\xi)^2}{2H^2} + (r_\xi^2 - r_\xi + 1)\frac{(K \cdot H + s(r_\xi))^2}{2H^2} \\ & - q_S + T(r_\xi, S, H)\,.\end{aligned}$$

If $K \cdot H < 0$,

$$\begin{aligned}\dim \mathcal{M}_\xi(\alpha) \leqq\ & (2r_\xi - 1)\Delta_\xi + (2r_\xi - 1)\alpha^2 \\ & + \left[(r_\xi^2 - 2r_\xi + 2)\frac{s(r_\xi)}{\sqrt{H^2}} - (r_\xi^2 - 2r_\xi - 2)\frac{K \cdot H}{2\sqrt{H^2}}\right]\alpha \\ & + \frac{r_\xi^3}{16}(17r_\xi + 4)\frac{(K \cdot H)^2}{8H^2} + \left[(r_\xi - 1)^3 + r_\xi\right]\frac{s(r_\xi)^2}{2H^2} \\ & + (r_\xi^2 - r_\xi + 1)\frac{(K \cdot H + s(r_\xi))^2}{2H^2} \\ & + \frac{r_\xi^2}{2}(1 - 2\chi(\mathcal{O}_S)) + \frac{r_\xi^3}{8}|K^2| - q_S\,.\end{aligned}$$

The next proposition provides a better bound in the case when the rank is two.

(1.3) Proposition. *Assume that H is effective. Let ξ be a set of sheaf data with $r_\xi = 2$, and let $\alpha \in \mathbf{R}$. Then*

$$\begin{aligned}\dim \mathcal{M}_\xi(\alpha) \leqq\ & 3\Delta_\xi + 3\alpha^2 + \frac{K \cdot H + 2H^2 + 2}{\sqrt{H^2}}\alpha \\ & + \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(\mathcal{O}_S) - q_S\,.\end{aligned}$$

When $r_\xi = 2$ we will also consider a subset of $\mathcal{M}_\xi(\alpha)$, defined as follows. Let $C \subset S$ be a smooth irreducible curve. For $\alpha \in \mathbf{R}$ we let $\mathcal{M}_\xi^C(\alpha) \subset \mathcal{M}_\xi$ be the subset of points $[F]$ such that $F|_C$ is locally-free, and such that there exists a rank-one subsheaf $A \subset F$ with:

1. $\mu_A \geqq \mu_F - \alpha\sqrt{H^2}$ (so A is α-destabilizing).
2. The restriction $A|_C$ spans a destabilizing subline bundle of $F|_C$.

The precise meaning of Item (2) is that $F|_C$ is not stable, and that there exists a destabilizing sub-line-bundle $L \subset F|_C$ containing $A|_C$.

(1.4) Proposition. *Assume H is effective. Let $C \in |nH|$ be a smooth curve, and let ξ be a set of sheaf data with $r_\xi = 2$. Let $\varphi(\xi, \alpha)$ be the right-hand side of the inequality in the previous proposition. Then $\mathcal{M}_\xi^C(\alpha_0)$ is a constructible* *subset of* $\mathcal{M}_\xi$, *and*

$$\dim \mathcal{M}_\xi^C(\alpha_0) \leqq \max\{\varphi(\xi,\alpha) - n\alpha\sqrt{H^2}\}_{0\leqslant\alpha\leqslant\alpha_0} .$$

Twisted endomorphisms

Let L be a line bundle. The number of linearly independent "twisted endomorphisms" $F \to F \otimes L$, for F a semistable torsion-free sheaf, is bounded by a quantity depending on the rank of F, the line bundle L, and the polarized surface. The remarkable fact is that the bound is independent of the discriminant Δ_F. Let

$$e_L(r,S,H) := \begin{cases} \frac{r^2}{2H^2}(\mu_L + (r^2+1)H^2 + 1)^2 - h^0(L), & \text{if } L \cdot H \geqq 0\,, \\ 0, & \text{if } L \cdot H < 0\,. \end{cases} \tag{1.5}$$

(1.6) *Assume that* H *satisfies* (0.4). *Let* F *be a* μ*-semistable rank-*r *torsion-free sheaf on* S. *Then*

$$\dim \operatorname{Hom}(F, F\otimes L)^0 \leqq e_L(r,S,H)\,.$$

We will also use the following two results.

(1.7) Lemma. *Let* A,B *be torsion-free sheaves on a projective irreducible variety. Suppose that* A,B *are both* μ*-semistable, and that* $\mu_A = \mu_B$. *Then*

$$h^0(A,B) \leqq r_A r_B\,.$$

(1.8) *Let* ξ *be a set of sheaf data for* S, *with* $r_\xi = 2$. *Let* $\Sigma_\xi \subset \mathcal{M}_\xi$ *be the subset parametrizing sheaves* F *such that*

$$h^0(F, F\otimes K)^0 > h^0(2K)\,.$$

Then $\Sigma_\xi \subset \mathcal{M}_\xi((K\cdot H)/2\sqrt{H^2})$.

Strictly semistable sheaves

Let F be a torsion-free sheaf on S which is strictly μ-semistable, i.e. μ-semistable but not μ-stable. We will need a bound for the dimension of the locus $V^0(F) \subset \operatorname{Def}^0(F)$ parametrizing strictly μ-semistable sheaves.

(1.9) *For* $r \geqq 2$ *an integer, set*

$$\varepsilon(r,S,H) := \begin{cases} \frac{r^2}{2H^2}[K\cdot H + s(r)]^2 + \frac{r^2}{16}\left[\frac{(K\cdot H)^2}{H^2} - K^2\right] & \\ \quad + r^2 + pr^2|\chi(\mathcal{O}_S)| - q_S & \text{if } r > 2\,, \\ \frac{3}{2H^2}(K\cdot H + H^2 + 1)^2 + \frac{1}{8}\left[\frac{(K\cdot H)^2}{H^2} - K^2\right] & \\ \quad + 4 - 3\chi(\mathcal{O}_S) - q_S & \text{if } r = 2\,, \end{cases}$$

where $p := -3/4$ *if* $\chi(\mathcal{O}_S) \geqq 0$, *and* $p := 1$ *if* $\chi(\mathcal{O}_S) < 0$.

(1.10) Proposition. *Let* (S,H) *be a polarized surface. Let* F *be a torsion-free sheaf on* S *of rank* $r_F \geqq 2$, *which is* μ*-semistable but not* μ*-stable* (*i.e. strictly* μ*-semistable*). *If* $r_F > 2$ *assume that* H *satisfies* (0.4), *if* $r_F = 2$ *assume only that* H *is effective. Then, letting* $V^0(F) \subset \mathrm{Def}^0(F)$ *be as above, we have*

$$\dim V^0(F) \leqq (2r_F - 1)\Delta_F + \varepsilon(r_F, S, H)\,.$$

(1.11) Corollary. *Let* (S,H) *be a polarized surface, and* ξ *be set of sheaf data for* S. *If* $r_\xi > 2$ *we assume that* H *satisfies* (0.4), *if* $r_\xi = 2$ *we only assume that* H *is effective. Let* $X \subset \mathscr{M}_\xi$ *be a subset such that*

$$\dim X > (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H)\,.$$

Then there exists a point of X *parametrizing a* μ*-stable sheaf.*

(1.12) Corollary. *Let* (S,H) *and* ξ *be as in the previous corollary. If*

$$2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathscr{O}_S) > (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H)\,,$$

then the generic point of any irreducible component of $\mathscr{M}_\xi$ *parametrizes a* μ*-stable sheaf.*

Non-stable vector-bundles on a curve

Let C be a smooth irreducible curve of genus g. Let $\mathscr{F}$ be a family of rank-r vector bundles on C, parametrized by an equidimensional variety B. Let $B^{\mathrm{ns}} \subset B$ be the subset parametrizing bundles which are not stable (i.e. either unstable or strictly semistable). The following result gives an upper bound for the codimension of B^{ns}.

(1.13) Proposition. *Keeping notation as above, assume that* B^{ns} *is not empty. Then*

$$\mathrm{cod}(B^{\mathrm{ns}}, B) \leqq \frac{r^2}{4}g\,.$$

2. A criterion for non-emptiness of the boundary

The goal of this section is to prove the following two propositions.

(2.1) Proposition. *Let* (S,H) *be a polarized surface, with* H *satisfying* (0.4). *Let* ξ *be a set of sheaf data, and let* $X \subset \mathscr{M}_\xi$ *be a closed subvariety. Assume that there exists a positive integer* n *such that*:

1. $\dim X > \frac{1}{2}(r_\xi^2 - 1)(H^2n^2 + K \cdot Hn)$,
2. $\dim X > \frac{1}{8}r_\xi^2(H^2n^2 + K \cdot Hn) + \frac{1}{4}r_\xi^2 + (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H)$,
3. $\dim X > \frac{1}{8}r_\xi^2(H^2n^2 + K \cdot Hn) + \frac{1}{4}r_\xi^2 + \dim \mathscr{M}_\xi((r_\xi - 1)\sqrt{H^2}n)$,
4. $\dim X > 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathscr{O}_S) + e_K(r_\xi, S, H)$
$+\frac{1}{4}r_\xi^2 - \frac{1}{2}(r_\xi - 1)(H^2n^2 - K \cdot Hn)$,

where $e_K(r_\xi, S, H), \varepsilon(r_\xi, S, H)$ *are as in* (1.5) *and* (1.9) *respectively. Then* $\partial X \neq \emptyset$.

(2.2) Proposition. *Let* (S, H) *be a polarized surface. Assume that* H *is effective. Let* ξ *be a set of sheaf data with* $r_\xi = 2$. *Let* $X \subset \mathcal{M}_\xi$ *be a closed subvariety. Suppose that there exist a positive integer n and a smooth curve* $C \in |nH|$ *such that Items* (1) *and* (2) *of Proposition* (2.1) *are satisfied, and furthermore*

$$\dim X > \tfrac{1}{2}(H^2 n^2 + K \cdot Hn) + 1 + \dim \mathcal{M}_\xi^C(\sqrt{H^2}n)\,, \qquad (2.3)$$

$$\dim X > \tfrac{1}{2}(H^2 n^2 + K \cdot Hn) + 1 + \dim \mathcal{M}_\xi\left(\frac{K \cdot H}{2\sqrt{H^2}}\right), \qquad (2.4)$$

$$\dim X > 4\Delta_\xi - 3\chi(\mathcal{O}_S) + h^0(2K) + 1 - \tfrac{1}{2}(H^2 n^2 - K \cdot Hn)\,. \qquad (2.5)$$

Then $\partial X \neq \emptyset$.

An examination of the inequalities contained in Propositions (2.1) and (2.2) will convince the reader that a statement similar to that of Theorem A must follow from these propositions. In the next section we will prove that in fact Theorem A follows from (2.1)–(2.2).

Before getting into the details we sketch the proof of Propositions (2.1)–(2.2). First we give a sufficient criterion for non-emptiness of the boundary of a closed subset $X \subset \mathcal{M}_\xi$, this is Proposition (2.6). More or less the hypotheses one needs are: first the existence of a smooth curve $C \subset S$ and a point $[F] \in X$ such that $F|_C$ is locally-free and not stable, secondly the codimension of X must be relatively small. To prove that $\partial X \neq \emptyset$ we construct a family of torsion-free sheaves on S, with parameter space denoted by Y_F, all of whose members are obtained from a fixed sheaf by an elementary modification along C. One of the sheaves of this family is isomorphic to F. Furthermore under suitable hypotheses all these sheaves are μ-stable, and hence we get a morphism $\varphi : Y_F \to \mathcal{M}_\xi$, whose image contains $[F] \in X$. One shows that any closed subset of Y_F which has dimension larger than $r_\xi^2/4$ does intersect the boundary $\varphi^{-1}(\partial \mathcal{M}_\xi)$. Since under our hypotheses $\dim Y_F$ is large, we see that $\dim \varphi^{-1}X$ is larger than $r_\xi^2/4$, and hence $\varphi^{-1}X$ intersects $\dim \varphi^{-1}(\partial \mathcal{M}_\xi)$, i.e. $\partial X \neq \emptyset$. Propositions (2.1)–(2.2) are proved by applying the above criterion for non-emptiness of the boundary. We do not proceed directly, rather by reductio ad absurdum. Assume X is closed, of "small" codimension, and ∂X is empty. First we show that for a carefully chosen n, and for any smooth curve $C \in |nH|$, there exists $[F] \in X$ such that $F|_C$ is not stable. The proof of this is again by contradiction. Suppose $F|_C$ is stable for all $[F] \in X$; then there exists a smooth $D \in |kH|$, with k arbitrary, such that the restriction $F|_D$ is stable for all $[F] \in X$. Taking k very large we conclude that

$$\langle c_1(\mathcal{L}_H)^d, [X] \rangle > 0\,, \qquad (*)$$

where $\mathcal{L}_H$ is the determinant line bundle on $\mathcal{M}_\xi$ associated to H, and d is the dimension of X. On the other hand if $F|_C$ is stable for all $[F] \in X$ we get that the left-hand side of $(*)$ is zero, the reason being that, by our careful choice of n, the dimension of the moduli space of semistable bundles on C is smaller than $\dim X$. This contradicts the hypothesis that $F|_C$ is stable for all $[F] \in X$. Hence there exists $[F] \in X$ such that $F|_C$ is not stable: applying Proposition (2.6) we conclude that $\partial X \neq \emptyset$, which contradicts the initial assumption. This proves that $\partial X \neq \emptyset$.

Certain families of elementary modifications

In this subsection we will prove:

(2.6) Proposition. *Let (S,H) be a polarized surface. Let ξ be a set of sheaf data. Let $[F] \in \mathcal{M}_\xi$, and assume that F is locally-free and μ-stable. Let $C \subset S$ be a smooth irreducible curve. Set $\alpha_C := (r_\xi - 1)(C \cdot H)/\sqrt{H^2}$. Assume that:*
1. *$F|_C$ is not stable,*
2. *$[F] \notin \mathcal{M}_\xi(\alpha_C)$.*

Let $X \subset \mathcal{M}_\xi$ be a closed subvariety containing $[F]$, and such that

$$\dim X > 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S) + h^0(F, F \otimes K_S)^0 + \frac{r_\xi^2}{4} - \frac{1}{2}(r_\xi - 1)C^2 + \frac{1}{2}(r_\xi - 1)C \cdot K \,. \tag{2.7}$$

Then the boundary of X is non-empty. If $r_\xi = 2$, the same conclusion holds if Item (2) is replaced by the condition

$$[F] \notin \mathcal{M}_\xi^C(\alpha_C)\,.$$

The proof of this proposition will be given at the end of the subsection. The key ingredient is provided by a certain family of elementary modifications which we now introduce. Let $C \subset S$ be a smooth irreducible curve. Let $[F] \in \mathcal{M}_\xi$, and assume that $F|_C$ is locally-free and not stable. Choose a destabilizing sequence

$$0 \to \mathcal{L}_0 \to F|_C \to \mathcal{Q}_0 \to 0\,. \tag{2.8}$$

By definition we have:

$$\mathcal{L}_0 \text{ and } \mathcal{Q}_0 \text{ are locally-free}\,, \tag{2.9}$$

$$\mu_{\mathcal{L}_0} - \mu_{\mathcal{Q}_0} \geqq 0\,. \tag{2.10}$$

Let E be the sheaf on S fitting into the exact sequence

$$0 \to E \to F \xrightarrow{g} \iota_*\mathcal{Q}_0 \to 0\,, \tag{2.11}$$

where $\iota : C \hookrightarrow S$ is the inclusion. In other words E is the elementary modification of F associated to the destabilizing quotient of (2.8). Restricting the above sequence to C, one gets an exact sequence

$$0 \to \mathcal{Q}_0 \otimes \mathcal{O}_C(-C) \to E|_C \xrightarrow{f_0} \mathcal{L}_0 \to 0\,. \tag{2.12}$$

Hence by (2.9) $E|_C$ is locally-free. Since E and F are isomorphic outside of C we conclude that E is locally-free. Let

$$Y_F := \mathrm{Quot}(E|_C; \mathscr{L}_0)$$

be the Grothendieck Quot-scheme parametrizing quotients of $E|_C$ which have the same Hilbert polynomial as $\mathscr{L}_0$. Notice that the notation is slightly imprecise, since a destabilizing sequence for $F|_C$ is not necessarily unique. However this will not create confusion because we will always fix a sequence (2.8) once and for all. We denote by 0 the point of Y_F corresponding to f_0 (see (2.12)). We are now ready to define a family of elementary modifications of E parametrized by Y_F. Let

$$\pi_C^*(E|_C) \xrightarrow{f} \mathscr{L}$$

be the tautological quotient sheaf on $C \times Y_F$, and let $\mathscr{G}$ be the sheaf on $S \times Y_F$ fitting into the exact sequence

$$0 \to \mathscr{G} \to \pi_S^* E \xrightarrow{\phi} (\imath \times \mathrm{id}_{Y_F})_* \mathscr{L} \to 0\,,$$

where ϕ is the composition of restriction to $C \times Y_F$ and f.

(2.13) Lemma. *The sheaf $\mathscr{G}$ is flat over Y_F, and thus we can regard it as a family of sheaves on S parametrized by Y_F. Let $y \in Y_F$. Then $\mathscr{G}_y$ fits into the exact sequence*

$$0 \to \mathscr{G}_y \to E \xrightarrow{\phi_y} \imath_* \mathscr{L}_y \to 0\,, \tag{2.14}$$

where $\phi_y := \phi|_{S\times\{y\}}$. In particular $\mathscr{G}_y$ is torsion-free. Finally, $\mathscr{G}_y$ is singular if and only if so is $\mathscr{L}_y$.

Proof. The sheaf $\mathscr{G}$ is Y_F-flat because so are $\pi_S^* E$ (obvious) and $(\imath \times \mathrm{id}_{Y_F})_* \mathscr{L}$ (by definition of the Quot-scheme). Flatness of the latter implies that (2.14) is exact, because of the long exact Tor sequence. For $y \in Y_F$ let $f_y := f|_{C\times\{y\}}$ and set $\mathscr{R}_y := \ker f_y$; since $E|_C$ is locally-free, so is $\mathscr{R}_y$. The exact sequence

$$0 \to \mathscr{L}_y(-C) \to \mathscr{G}_y|_C \to \mathscr{R}_y \to 0$$

shows that $\mathscr{G}_y$ is singular at a point $P \in C$ if and only if $\mathscr{L}_y$ is singular at P. Since

$$\mathscr{G}_y|_{(S-C)} \cong E|_{(S-C)}\,,$$

and E is locally-free, we conclude that $\mathscr{G}_y$ is singular if and only if so is $\mathscr{L}_y$.
QED

Let $\mathscr{F} := \mathscr{G} \otimes \pi_S^* \mathcal{O}_S(C)$. By the above lemma, we can regard $\mathscr{F}$ as a family of torsion-free sheaves on S parametrized by Y_F. Let $\partial Y_F \subset Y_F$ be the subset parametrizing singular sheaves.

(2.15) Lemma. *Let notation be as above. Then*:

1. $\mathscr{F}_0 \cong F$.
2. *For $y \in Y_F$ we have* $\det \mathscr{F}_y \cong \det_\xi$ *and* $c_2(\mathscr{F}_y) = c_2(\xi)$.
3. *Let $\Sigma \subset Y_F$ be a closed subvariety. If* $\dim \Sigma > r_\xi^2/4$, *then* $\Sigma \cap \partial Y_F \neq \emptyset$.

Proof. Clearly the subsheaf $F(-C) \hookrightarrow F$ is in the kernel of the map g of (2.11). Hence $F(-C)$ is actually a subsheaf of E; let $\lambda\colon F(-C) \to E$ be the inclusion map. As is easily checked

$$\mathrm{Im}(\lambda|_C) = \ker f_0 \,.$$

Since λ is an isomorphism outside of C, we conclude that $F(-C)$ fits into the exact sequence

$$0 \to F(-C) \to E \xrightarrow{\phi_0} \iota_* \mathscr{L}_0 \to 0 \,.$$

By Lemma (2.13) the sheaf $\mathscr{G}_0$ fits into the same exact sequence, and thus $\mathscr{G}_0 \cong F(-C)$. This proves Item (1). Let's consider the second Item. It follows from Exact sequences (2.11) and (2.14) that $\det E \cong \det F(-r_{\mathscr{Q}_0} C)$ and that $\det \mathscr{G}_y \cong \det E(-r_{\mathscr{L}_y} C)$. This gives $\det \mathscr{F}_y \cong \det F \cong \det_\xi$. Since $c(\iota_* \mathscr{L}_y)$ is independent of $y \in Y_F$, so is $c(\mathscr{G}_y)$, and hence also $c(\mathscr{F}_y)$. Since $c_2(\mathscr{F}_0) = c_2(F) = c_2(\xi)$, we conclude that $c_2(\mathscr{F}_y) = c_2(\xi)$ for all $y \in Y_F$. Now let's prove Item (3). Assume that $\Sigma \cap \partial Y_F = \emptyset$; we will arrive at a contradiction. First we define a morphism from Σ to the Hilbert scheme parametrizing subvarieties of $\mathbf{P}(E|_C)$. For $y \in \Sigma$ let

$$0 \to \mathscr{R}_y \xrightarrow{h_y} E|_C \xrightarrow{f_y} \mathscr{L}_y \to 0$$

be the exact sequence determined by f_y. Since $\Sigma \cap \partial Y_F = \emptyset$, we know by Lemma (2.13) that $\mathscr{L}_y$ is locally-free and hence h_y is injective on the fiber of $\mathscr{R}_y$ at any point of C. Thus h_y induces a morphism

$$\mathbf{P}(\mathscr{R}_y) \xrightarrow{\overline{h}_y} \mathbf{P}(E|_C)$$

whose image is a closed subvariety intersecting each fiber of the ruling $\mathbf{P}(E|_C) \mapsto C$ in a $\mathbf{P}^{r_0 - 1}$, where $r_0 := r_{\mathscr{Q}_0}$. Mapping

$$\Sigma \ni y \mapsto \overline{h}_y(\mathbf{P}(\mathscr{R}_y))$$

we get a morphism θ from Σ to the Hilbert scheme parametrizing subvarieties of $\mathbf{P}(E|_C)$. Clearly θ is injective, and hence

$$\dim \theta(\Sigma) = \dim \Sigma \,. \qquad (*)$$

Now choose $P \in C$, and let E_P be the fiber of E at P. Let

$$\rho\colon \Sigma \to \mathbf{Gr} := \mathbf{Gr}(r_0 - 1, \mathbf{P}(E_P))$$

be the morphism associating to $y \in \Sigma$ the subspace $\rho(y) := \mathbf{P}(\mathscr{R}_{y,P}) \subset \mathbf{P}(E_P)$. Notice that ρ factors through θ. Since $\dim \Sigma > r_\xi^2/4 \geqq \dim \mathbf{Gr}$,

$$\dim \rho^{-1}([V]) \geqq 1 \qquad (**)$$

for any $[V] \in \mathbf{Gr}$. Choose a $[V] \in \mathrm{Im}\,\rho$ (clearly ρ has non-empty image) and let $\Omega \subset \mathbf{P}(E|_C)$ be defined by

$$\Omega := \bigcup_{y \in \rho^{-1}([V])} \overline{h}_y(\mathbf{P}(\mathscr{R}_y)) \,.$$

Since θ is injective Inequality $(**)$ gives

$$\dim \Omega \geqq (1 + r_0) . \qquad (\dagger)$$

Since Y_F is complete so is Σ, and hence Ω is a closed subvariety of $\mathbf{P}(E|_C)$. Hence by $(\dagger)$

$$\dim \Omega \cap \mathbf{P}(E_P) \geqq r_0 .$$

This is absurd because by definition the above intersection is equal to $\mathbf{P}(V)$, and $\dim \mathbf{P}(V) = (r_0 - 1)$. QED

In order to use the above lemma, we need to ensure that the dimension of Y_F is large, and that $\mathscr{F}$ is a family of semistable sheaves. (By Item (1) the sheaves $\mathscr{F}_y$ are stable for y varying in an open non-empty subset of Y_F; however this will not suffice.)

(2.16) Lemma. *Keep notation as above. Then*

$$\dim Y_F \geqq \tfrac{1}{2}(r_\xi - 1)C^2 - \tfrac{1}{2}(r_\xi - 1)C \cdot K .$$

Proof. By (2.12) and by the deformation theory of the Quot-scheme, we have the lower bound

$$\dim Y_F \geqq \chi(\mathscr{Q}_0^* \otimes \mathcal{O}_C(C) \otimes \mathscr{L}_0) .$$

Let g be the genus of C. Riemann–Roch gives

$$\chi(\mathscr{Q}_0^* \otimes \mathcal{O}_C(C) \otimes \mathscr{L}_0) = r_{\mathscr{L}_0} r_{\mathscr{Q}_0} [\mu_{\mathcal{O}_C(C)} + \mu_{\mathscr{L}_0} - \mu_{\mathscr{Q}_0} + 1 - g] ,$$

where the slopes are as *bundles on C*. By Inequality (2.10) we conclude that

$$\dim Y_F \geqq r_{\mathscr{L}_0} r_{\mathscr{Q}_0} (C^2 + 1 - g) .$$

Using adjunction one gets the lemma. (Notice that if $(C^2 - C \cdot K) < 0$ then the lemma is trivially verified.) QED

Regarding stability we have the following

(2.17) Lemma. *Keep notation as above, and let α_C be as in the statement of Proposition* (2.6). *If $[F] \notin \mathscr{M}_\xi(\alpha_C)$, then $\mathscr{F}$ is a family of stable sheaves. In the case $r_\xi = 2$ the same conclusion holds if*:

1. *F is μ-stable, and*
2. *$[F] \notin \mathscr{M}_\xi^C(\alpha_C)$.*

Proof. Let $y \in Y_F$. We will show that $\mathscr{G}_y$ is μ-stable; this will prove the lemma. Item (2) of Lemma (2.15) gives

$$\mu_{\mathscr{G}_y} = \mu_F - C \cdot H . \qquad (*)$$

Now let $A \hookrightarrow \mathscr{G}_y$ be a subsheaf with $0 < r_A < r_{\mathscr{G}_y}$. Let $\lambda \colon A \to F$ be the composition (see (2.14) and (2.11))

$$A \hookrightarrow \mathscr{G}_y \to E \to F .$$

Since λ is injective outside of C, and since A is torsion-free, we conclude that λ is an injection. If $[F] \notin \mathcal{M}_\xi(\alpha_C)$ then

$$\mu_A < \mu_F - \frac{\alpha_C}{r_A}\sqrt{H^2} \leqq \mu_F - C \cdot H = \mu_{\mathcal{G}_y} , \qquad (\dagger)$$

and hence $\mathcal{G}_y$ is μ-stable. Now assume $r_\xi = 2$ and $[F] \notin \mathcal{M}_\xi^C(\alpha_C)$. If λ is zero at the generic point of C, then we get an injection $A(C) \hookrightarrow F$. By hypothesis F is μ-stable, and hence

$$\mu_A + C \cdot H < \mu_F .$$

By $(*)$ we get $\mu_A < \mu_{\mathcal{G}_y}$. Now assume λ is not zero at the generic point of C. Then

$$\mathrm{Im}\,(\lambda|_C) = \mathrm{Im}(E|_C \to F|_C) \quad \text{at the generic point of } C .$$

Since the right-hand side is a destabilizing subline bundle of $F|_C$, and since $[F] \notin \mathcal{M}_\xi^C(\alpha_C)$ we conclude that $(\dagger)$ holds. Thus $\mathcal{G}_y$ is μ-stable. QED

Proof of Proposition (2.6). Since F satisfies Item (1), we can construct Y_F and the family $\mathcal{F}$ of sheaves on S parametrized by Y_F. By Lemma (2.17) this is a family of stable sheaves. Hence, by Item (2) of Lemma (2.15), $\mathcal{F}$ induces a classifying morphism

$$\varphi \colon Y_F \to \mathcal{M}_\xi .$$

By Item (1) of (2.15), we have $\mathcal{F}_0 \cong F$. Hence $[F] \in X$ and the inverse image $\varphi^{-1}X$ is a closed subvariety of Y_F containing the point 0. We have

$$\dim \varphi^{-1}X \geqq \dim Y_F - (\dim T_{[F]}\mathcal{M}_\xi - \dim X) .$$

To be precise: the right-hand side is a lower bound for the dimension of any irreducible component of $\varphi^{-1}X$ containing 0. By (0.3), (2.7) and Lemma (2.16) we conclude that $\dim \varphi^{-1}X > r_\xi^2/4$. By Item (3) of Lemma (2.15) there exists $y \in \varphi^{-1}X$ such that $\mathcal{F}_y$ is singular. Then $\varphi(y) \in \partial X$, and hence $\partial X \neq 0$.
QED

Determinant bundles

We let C be a smooth irreducible curve in the linear system $|nH|$. Let $\mathcal{M}(C;\xi)$ be the moduli space of rank-r_ξ semistable bundles on C with determinant isomorphic to $\det_\xi|_C$. Let $X \subset \mathcal{M}_\xi$ be a subvariety such that, for all $[F] \in X$, the restriction $F|_C$ is a stable locally-free bundle. Since $C \in |nH|$ this implies that F is μ-stable for all $[F] \in X$. Then, as is easily verified there exists a morphism

$$\rho \colon X \to \mathcal{M}(C;\xi)$$

given by restriction, i.e. $\rho([F]) = [F|_C]$. Our goal will be to prove the following

(2.18) Proposition. *Let X, C, ρ be as above. Assume also that X is closed and irreducible, and that all sheaves parametrized by points of X are locally-free.*

Let Θ be the theta-divisor on $\mathcal{M}(C;\xi)$ (see [DN]). *Then*

$$(\rho^*\Theta)^{\dim X} > 0\,.$$

We will first prove a series of preliminary results. We start with a weak version of closedness of non-stability for a family of vector bundles on a variable degenerating curve. More precisely: Let B be a smooth curve, $0 \in B$ be a base point, and $B^0 := (B-0)$. Let $\pi\colon \mathscr{C} \to B$ be a family of curves; for $b \in B$ set $C_b := \pi^{-1}(b)$. We assume that:

1. $\mathscr{C}$ is smooth outside a finite set of points in $\mathscr{C}_0$,
2. all fibers C_b are reduced and connected, and for $b \neq 0$ they are smooth.

Let $D_1, \ldots, D_s$ be the irreducible components of the central fiber C_0.

(2.19) Lemma. *Keep notation as above. Let $\mathscr{F}$ be a vector bundle on $\mathscr{C}$, and set $\mathscr{F}_b := \mathscr{F}|_{C_b}$. Assume that for all $b \neq 0$ the bundle $\mathscr{F}_b$ is non-stable. There exists i with $1 \leqq i \leqq s$ such that, letting $\lambda_i\colon \tilde{D}_i \to D_i$ be the normalization, the bundle $\lambda_i^*(\mathscr{F}|_{D_i})$ is not stable.*

Proof. For generic $b \in B^0$ the Harder–Narasimhan filtration of $\mathscr{F}_b$ has constant type (i.e. length, and rank and slope of the successive quotients). Thus, shrinking B^0 if necessary, we can assume that there exists a vector bundle $\mathscr{Q}^0$ on $\mathscr{C}^0 := \pi^{-1}(B^0)$, and an exact sequence

$$\mathscr{F}|_{\mathscr{C}^0} \xrightarrow{\alpha} \mathscr{Q}^0 \to 0\,, \tag{$*$}$$

whose restriction to C_b is a destabilizing sequence, for all $b \in B^0$. By properness of the relative Quot-scheme parametrizing quotients of $\mathscr{F}_b$, there is a B-flat sheaf $\mathscr{Q}$ on $\mathscr{C}$ extending $\mathscr{Q}^0$, and an exact sequence

$$\mathscr{F} \to \mathscr{Q} \to 0\,. \tag{$**$}$$

By flatness $\mathscr{Q}$ is torsion-free. In particular it is locally-free outside a finite set of points in C_0. Let $f : \tilde{\mathscr{C}} \to \mathscr{C}$ be a desingularization such that, for all $1 \leqq i \leqq s$, the proper transform of D_i is smooth. We denote this proper transform by $\tilde{D}_i$. Let $\mathrm{Tor}(f^*\mathscr{Q})$ be the torsion subsheaf of $f^*\mathscr{Q}$: since $\mathscr{Q}$ is torsion-free $\mathrm{Tor}(f^*\mathscr{Q})$ is supported on the exceptional divisors of f. Pulling back ($**$) we get an exact sequence

$$0 \to \mathscr{K} \to f^*\mathscr{F} \to f^*\mathscr{Q}/\mathrm{Tor}(f^*\mathscr{Q}) \to 0\,. \tag{$\dagger$}$$

Since $\tilde{\mathscr{C}}$ is a smooth surface, and since $f^*\mathscr{F}$ is locally-free, $\mathscr{K}$ is also locally-free. Let $\varphi := \pi \circ f$, and set $\tilde{\mathscr{C}}_b := \varphi^{-1}(b)$. If $b \neq 0$, then the restriction of ($\dagger$) to $\tilde{\mathscr{C}}_b = \mathscr{C}_b$ is the destabilizing sequence associated to α_b (see ($*$)), and hence

$$\frac{1}{r_{\mathscr{K}}}\, c_1(\mathscr{K}) \cdot \tilde{\mathscr{C}}_b \geqq \frac{1}{r_{\mathscr{F}}}\, c_1(f^*\mathscr{F}) \cdot \tilde{\mathscr{C}}_b\,. \tag{$\sharp$}$$

The same inequality holds also when $b = 0$. We have

$$\tilde{\mathscr{C}}_0 = \sum_{i=1}^{s} \tilde{D}_i + \sum_{j=1}^{p} n_j E_j\,, \tag{$\flat$}$$

where $E_1,\ldots,E_p$ are the exceptional divisors of f, and $n_j > 0$ for all j. The map $\mathscr{K} \to f^*\mathscr{F}$ has isolated zeros, and hence the restriction of (†) to $\tilde{D}_i$ and E_j is exact. Since $f^*\mathscr{F}|_{E_j}$ is trivial,

$$\frac{1}{r_{\mathscr{K}}} c_1(\mathscr{K}) \cdot E_j \leqq \frac{1}{r_{\mathscr{F}}} c_1(f^*\mathscr{F}) \cdot E_j \,.$$

By (♯) and (♭) we conclude that there exists $1 \leqq i \leqq s$ such that

$$\frac{1}{r_{\mathscr{K}}} c_1(\mathscr{K}) \cdot \tilde{D}_i \geqq \frac{1}{r_{\mathscr{F}}} c_1(f^*\mathscr{F}) \cdot \tilde{D}_i \,.$$

Since $0 < r_{\mathscr{K}} < r_{f^*\mathscr{F}}$, we conclude that $f^*\mathscr{F}|_{\tilde{D}_i}$ is not stable. QED

(2.20) Proposition. *Let $C \in |nH|$ be a smooth curve. Let $X \subset \mathscr{M}_\xi$ be a closed subset such that $F|_C$ is locally-free and stable for all $[F] \in X$. Let k be a positive integer such that there exist smooth curves in $|knH|$ (e.g. $k \gg 0$). Then there exists a smooth $D_k \in |knH|$ such that $F|_{D_k}$ is stable for all $[F] \in X$.*

Proof. Since $C \in |nH|$ it follows from our hypothesis that F is μ-stable for all $[F] \in X$. Thus we can cover X by subsets X_i, open in the analytic topology, so that there exists a tautological sheaf on each $S \times X_i$. For convenience of exposition we will assume that these local tautological sheaves fit together to give a tautological sheaf on $S \times X$; however, as the reader will readily check, the proof works in general. Now let $U_k \subset |knH|$ be the open dense subset parametrizing smooth curves. Let

$$\tilde{Z}_k := \{([D],[F]) \in U_k \times X |\ F|_D \text{ is not stable}\} \,.$$

By openness of stability, $\tilde{Z}_k$ is closed in $U_k \times X$. Let Z_k be the image of $\tilde{Z}_k$ under the projection $U_k \times X \to U_k$. Since X is closed in $\mathscr{M}_\xi$, it is proper, and hence Z_k is closed in U_k. We must show that $Z_k \neq U_k$. (Of course, if $k = 1$ this is true by hypothesis.) The proof is by contradiction, so we assume $Z_k = U_k$. Let

$$[R_1],\ldots,[R_k] \in U_1 - Z_1$$

be distinct. Set

$$C_0 := R_1 + \cdots + R_k \,.$$

Since $|knH| \times X \to |knH|$ is proper, and since $Z_k = U_k$, there exists a smooth connected curve B, a point $0 \in B$ and a map

$$g = (g_1, g_2)\colon B \to |knH| \times X \,,$$

with the following properties:

1. $g_1(B^0) \subset U_k$, where $B^0 := (B - 0)$.
2. $g_1^{-1}([C_0]) = 0$.
3. The family $\pi : \mathscr{C} \to B$, obtained pulling back by g_1 the tautological family of curves parametrized by $|knH|$, satisfies Items (1) and (2) preceding Lemma (2.19).

4. Let $\psi : \mathscr{C} \to S \times X$ be given by (ψ_1, ψ_2), where $\psi_1 : \mathscr{C} \to S$ is the natural map, and $\psi_2 := g_2 \circ \pi$. Let $\mathscr{F} := \psi^* \mathscr{E}$, where $\mathscr{E}$ is a tautological family on $S \times X$. Then the restriction $\mathscr{F}|_{\pi^{-1}(b)}$ is non-stable for all $b \neq 0$.
Thus we can apply Lemma (2.19) to the bundle $\mathscr{F}$ on $\mathscr{C}$. We conclude that there exist $1 \leqq i \leqq k$ and $[F] \in X$ such that $F|_{R_i}$ is not stable. This is absurd because $[R_i] \in (U_1 - Z_1)$. QED

Assume that C,X are as in the statement of Proposition (2.20). Let $D_k \in |knH|$ be a smooth curve as in the proposition. Let

$$\rho_k : X \to \mathscr{M}(D_k; \xi)$$

be the morphism given by restriction. Let Θ_k be the theta-divisor on $\mathscr{M}(D_k; \xi)$.

(2.21) Lemma. *Let notation be as above. Then there exists a positive rational number λ_k such that*

$$c_1(\rho_k^* \Theta_k) = \lambda_k c_1(\rho_1^* \Theta_1) .$$

Proof. First let's assume that there exists a tautological sheaf $\mathscr{E}$ on $S \times X$. Let $\mathscr{E}_k$ be its restriction to $D_k \times X$; by our hypotheses it is a vector bundle. Let M_k be a vector bundle on D_k such that

$$\chi(\mathscr{E}|_{D_k \times \{x\}} \otimes M_k) = 0 \qquad (*)$$

for $x \in X$. (As is easily verified, such an M_k always exists.) Then Grothendieck–Riemann–Roch on $D_k \times X$ gives

$$c(\pi_{X,!}(\mathscr{E}_k \otimes \pi_{D_k}^* M_k)) = \pi_{X,*}[\mathrm{ch}(\mathscr{E}_k \otimes \pi_{D_k}^* M_k) TdC] .$$

Considering the degree-one components of both sides of the above equality, and using $(*)$, one gets

$$-c_1(\det \pi_{X,!}(\mathscr{E}_k \otimes \pi_{D_k}^* M_k)) = r_{M_k} \cdot \pi_{X,*} \left(c_2(\mathscr{E}_k) - \frac{r_{\mathscr{E}_k} - 1}{2 r_{\mathscr{E}_k}} c_1(\mathscr{E}_k)^2 \right) .$$

Now assume that the rank of M_k is minimal. Then the left-hand side of the above equality is identified with the first Chern class of $\rho_k^* \Theta_k$ (see [DN]), while the right-hand side equals the slant product

$$k \cdot r_{M_k} \cdot \left(c_2(\mathscr{E}) - \frac{r_{\mathscr{E}} - 1}{2 r_{\mathscr{E}}} c_1(\mathscr{E})^2 \right) \Big/ [nH] .$$

This proves the lemma under the assumption that there is a tautological sheaf on $S \times X$. In general, by Theorem (A.5) in [Mu] there exists a *quasi-tautological* sheaf $\mathscr{F}$ on $S \times X$, i.e. such that for $[F] \in X$, the restriction $\mathscr{F}|_{S \times \{[F]\}}$ is isomorphic to $F^{\oplus \sigma}$ for some positive integer σ. One can repeat the proof above with $\mathscr{E}$ replaced by $\mathscr{F}$. QED

Proof of Proposition (2.18). By Serre's vanishing Theorem, if $k \gg 0$ then for all $[F_1], [F_2] \in X$ we have

$$H^1(F_1^* \otimes F_2(-knH)) = 0 . \qquad (*)$$

By Proposition (2.20) there exists a smooth $D_k \in |knH|$ such that $F|_{D_k}$ is stable for all $[F] \in X$. By $(*)$ the restriction map ρ_k is an injection. Since Θ_k is ample [DN], we conclude that

$$(\rho_k^* \Theta_k)^{\dim X} > 0 .$$

Proposition (2.18) follows at once from the above inequality and Lemma (2.21).

Proof of Propositions (2.1)–(2.2). The proof is by contradiction, so we assume that F is locally-free for all $[F] \in X$. Let $C \in |nH|$ be a smooth curve. We start by showing that there exists $[E] \in X$ such that $E|_C$ is not stable. This again we prove by reductio ad absurdum. Clearly we can assume X is irreducible. If $E|_C$ is stable for all $[E] \in X$, then we have the restriction morphism

$$\rho \colon X \to \mathcal{M}(C; \xi) .$$

Let g be the genus of C. By adjunction

$$g - 1 = \tfrac{1}{2} H^2 n^2 + \tfrac{1}{2} K \cdot Hn , \qquad (*)$$

and hence

$$\dim \mathcal{M}(C; \xi) = (r_\xi^2 - 1)(g - 1) = \tfrac{1}{2}(r_\xi^2 - 1)(H^2 n^2 + K \cdot Hn) .$$

By Item (1) of Proposition (2.1) we see that $\dim X > \dim \mathcal{M}(C; \xi)$, and thus

$$(\rho^* \Theta)^{\dim X} = 0 .$$

This contradicts Proposition (2.18). Hence we conclude that there exists $[E] \in X$ such that $E|_C$ is not stable. Let $X_C \subset X$ be the (closed) subset parametrizing sheaves whose restriction to C is not-stable; we have just proved that $X_C \neq \emptyset$. Let $X_C^\mu \subset X_C$ be the subset parametrizing sheaves which are μ-stable on S. We claim that $X_C^\mu \neq \emptyset$. For this we need the following

(2.22) Claim. *Let $x \in \mathcal{M}_\xi$, and let E be the sheaf, unique up to isomorphism, such that $x = [E]$ and* $\mathrm{Gr}\, E = E$. *Let $\lambda \colon \mathrm{Def}^0(E) \to \mathcal{M}_\xi$ be the classifying map induced by a tautological sheaf on $S \times \mathrm{Def}^0(E)$. Then λ surjects onto a neighborhood of x.*

Proof. The moduli space $\mathcal{M}_\xi$ is the G.I.T. quotient of a certain Quot-scheme P. Let

$$\pi_S^* \mathcal{O}_S(-nH)^{\oplus d} \to \mathcal{E} \to 0$$

be the tautological quotient on $S \times P$, and let $P^0 \subset P$ be the subset parametrizing semistable sheaves. Let $y \in P^0$ be a point in the unique closed orbit of P^0 mapping to x under the quotient map. Then $\mathrm{Gr}\, \mathcal{E}_y = \mathcal{E}_y$ and $[\mathcal{E}_y] = x$, hence $\mathcal{E}_y \cong E$. By Luna's étale slice Theorem [Lu] there exists a locally closed G_y-invariant (where G_y is the stabilizer of y) subset $V \subset P^0$ containing y such that the natural map $V/G_y \to \mathcal{M}_\xi$ is étale. In particular the morphism $\rho \colon V \to \mathcal{M}_\xi$, obtained composing with the projection, maps a neighborhood of

y surjectively onto a neighborhood of x. Since ρ is the classifying morphism for the restriction of $\mathscr{E}$ to $S \times V$, it factors through the map λ: $\mathrm{Def}^0(E) \to \mathscr{M}_\xi$. Thus λ must surject onto a neighborhood of x. QED

Now suppose that the "original" E (with $E|_C$ not stable) is not μ-stable. By (2.22) the map $\lambda : \mathrm{Def}^0(E) \to \mathscr{M}_\xi$ surjects onto a neighborhood of $[E]$, and hence

$$\dim(\lambda^{-1}X) \geqq \dim X . \qquad (**)$$

Obviously

$$\lambda^{-1}X_C = \{x \in \lambda^{-1}X | \mathscr{E}_x|_C \text{ is not stable}\} .$$

By Proposition (1.13) and Equation $(*)$ we have

$$\dim(\lambda^{-1}X_C) \geqq \dim \lambda^{-1}X - \frac{r_\xi^2}{4} g .$$

By Inequality $(**)$ and by Item (2) of Proposition (2.1) we conclude that

$$\dim(\lambda^{-1}X_C) > (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H) .$$

Hence by Proposition (1.10) there exists $x \in \lambda^{-1}X_C$ such that $\mathscr{E}_x$ is μ-stable. Then $\lambda(x) \in X_C^\mu$, and hence $X_C^\mu \neq \emptyset$. Let $[E'] \in X_C^\mu$; since E' is stable there is a neighborhood (in the analytic topology) of $[E']$ in $\mathscr{M}_\xi$ parametrizing a tautological family $\mathscr{F}$. Applying again Proposition (1.13), this time to $\mathscr{F}$, we get

$$\dim X_C^\mu \geqq \dim X - \frac{r_\xi^2}{8}(H^2 n^2 + K \cdot Hn) - \frac{r_\xi^2}{4} . \qquad (\dagger)$$

First let's finish the proof of Proposition (2.1). Item (3) in the statement of (2.1) and the above inequality imply that there exists $[F] \in X_C^\mu$ such that $[F] \notin \mathscr{M}_\xi(\alpha_C)$, where α_C is as in Proposition (2.6). At this point we apply Proposition (2.6) to X, C and F. We have just proved that Items (1)–(2) of that proposition are satisfied. Inequality (2.7) is satisfied because of Item (4) in the hypotheses of Proposition (2.1). Hence by (2.6) we conclude that $\partial X \neq \emptyset$, which contradicts our assumption. This proves Proposition (2.1). Now let's finish the proof of (2.2). By $(\dagger)$ and by Inequalities (2.3)–(2.4) there exists $[F] \in X_C^\mu$ such that $[F] \notin \mathscr{M}_\xi^C(\alpha_C)$ and $[F] \notin \mathscr{M}_\xi(K \cdot H/2\sqrt{H^2})$. By (1.8) we know that

$$h^0(F, F \otimes K)^0 \leqq h^0(2K) .$$

Again we apply Proposition (2.6) to X, C, F. We have proved that Items (1)–(2) of that proposition are satisfied. To verify (2.7) use the inequality above and Inequality (2.5). By Proposition (2.6) we conclude that $\partial X \neq \emptyset$, a contradiction. This proves Proposition (2.2).

3. Forcing intersection with the boundary

In this section we will prove Theorem A by applying Propositions (2.1) and (2.2): as should be clear, this is purely a computational problem. We will also give explicit conditions ensuring that every irreducible component of $\mathcal{M}_\xi$ has non-empty boundary (Proposition (3.11)), when $r_\xi = 2$. This last result will be the key ingredient in the proof of Theorem C. To simplify notation we will set $\varepsilon = \varepsilon(r_\xi, S, H)$, $e_K = e_K(r_\xi, S, H)$, and $\chi = \chi(\mathcal{O}_S)$.

Proof of Theorem A for $r_\xi > 2$. Let

$$\psi_1(\xi,n) := \tfrac{1}{2}(r_\xi^2 - 1)H^2n^2 + \tfrac{1}{2}(r_\xi^2 - 1)K \cdot Hn\,,$$

$$\psi_2(\xi,n) := \tfrac{1}{8}r_\xi^2H^2n^2 + \tfrac{1}{8}r_\xi^2K \cdot Hn + \tfrac{1}{4}r_\xi^2 + (2r_\xi - 1)\Delta_\xi + \varepsilon\,,$$

$$\begin{aligned}\psi_3(\xi,n) := {} & (2r_\xi - 1)\Delta_\xi + (2r_\xi^3 - \tfrac{39}{8}r_\xi^2 + 4r_\xi - 1)H^2n^2 \\ & + \left[\frac{r_\xi^3}{2}|K \cdot H| + r_\xi^5H^2\right] n + \frac{r_\xi^4}{5}\frac{(K \cdot H)^2}{H^2} + \frac{r_\xi^7}{2}H^2 \\ & + \frac{r_\xi^2}{2}\frac{(K \cdot H + r_\xi^2H^2)^2}{H^2} + r_\xi^2|\chi| + \frac{r_\xi^3}{8}|K^2|\,,\end{aligned}$$

$$\psi_4(\xi,n) := 2r_\xi\Delta_\xi - \tfrac{1}{2}(r_\xi - 1)H^2n^2 + \tfrac{1}{2}(r_\xi - 1)K \cdot Hn - (r_\xi^2 - 1)\chi + e_K + \frac{r_\xi^2}{4}\,.$$

If $i = 1, 2, 4$ then $\psi_i(\xi,n)$ equals the right-hand side of the inequality in Item (i) of Proposition (2.1). Proposition (1.2) and easy estimates show that $\psi_3(\xi,n)$ is an upper bound for the right-hand side of Item (3) in the same proposition. Thus if

$$\dim X > \max\{\psi_1(\xi,n), \psi_2(\xi,n), \psi_3(\xi,n), \psi_4(\xi,n)\}\,,$$

for some integer $n \geqq 1$, then by Proposition (2.1) we conclude that $\partial X \neq \emptyset$. As is easily checked

$$\max\{\psi_1(\xi,n), \psi_2(\xi,n)\} \leqq \psi_3(\xi,n) \tag{3.1}$$

for $n \geqq 1$. Hence we have

(3.2) *Keep notation as above. If*

$$\dim X > \max\{\psi_3(\xi,n), \psi_4(\xi,n)\}$$

for some integer $n \geqq 1$, then $\partial X \neq \emptyset$.

If Δ_ξ is sufficiently large, then the minimum of $\max\{\psi_3(\xi,n), \psi_4(\xi,n)\}$ for positive n is achieved by the solution of

$$\psi_3(\xi,n) = \psi_4(\xi,n)\,. \tag{\dagger}$$

So let x_0 be the positive root of the equation in n

$$(2r_\xi - 1)\Delta_\xi + \left(2r_\xi^3 - \frac{39}{8}r_\xi^2 + 4r_\xi - 1\right)H^2n^2 = 2r_\xi\Delta_\xi - \frac{r_\xi - 1}{2}H^2n^2\,,$$

obtained replacing the two sides of (†) by their dominant terms (that is dominant for Δ_ξ and n large). Thus

$$x_0 = \sqrt{\frac{\rho(r_\xi)}{H^2}}\sqrt{\Delta_\xi}\,.$$

Set $n_0 := [x_0]$. We will prove Theorem A by applying (3.2) with $n = n_0$. By the discussion above this choice of n is almost optimal if Δ_ξ is large (and with this choice the computations are relatively simple).

Lemma. *Let $X \subset \mathcal{M}_\xi$ be a closed irreducible subset such that*

$$\dim X > \max\{\psi_3(\xi,x_0), \psi_4(\xi,x_0-1)\}\,. \tag{3.3}$$

Then $\partial X \neq \emptyset$.

Proof. First notice that, since $\Delta_\xi \geqq \Delta_0$, we have $x_0 \geqq 1$, and hence n_0 is a positive integer. If ξ is fixed, the function $\psi_3(\xi,n)$ is increasing for positive n. Thus by (3.3) we have

$$\dim X > \psi_3(\xi,n_0)\,. \tag{$*$}$$

Now let's show that

$$\dim X > \psi_4(\xi,n_0)\,. \tag{†}$$

First we will prove that $\dim X > \psi_4(\xi,x_0-1)$ implies that

$$x_0 - 1 \geqq \frac{K\cdot H}{2H^2}\,, \tag{$**$}$$

or, in other words, if the hypotheses of Theorem A are satisfied by some $X \subset \mathcal{M}_\xi$, then

$$\Delta_\xi \geqq \rho^{-1}H^2\left(1+\frac{K\cdot H}{2H^2}\right)^2\,.$$

For this observe that, if ξ is fixed, then the unique critical point of the concave-down quadratic polynomial $\psi_4(\xi,n)$ is given by $(K\cdot H)/2H^2$. Hence if ($**$) does not hold then, since $0 \leqq (x_0-1)$, we have

$$\psi_4(\xi,x_0-1) \geqq \psi_4\left(\xi,\frac{K\cdot H}{H^2}\right) > 2r_\xi\Delta_\xi - (r_\xi^2-1)\chi + e_K\,.$$

By Inequality (0.2) we conclude that all points of X parametrize non-stable sheaves. On the other hand, by ($*$) and (3.1) we have $\dim X > \psi_2(\xi,n_0)$. As is easily checked this implies that X satisfies the hypotheses of Corollary (1.11); thus by this same corollary we get a contradiction. We conclude that ($**$)

holds. Now (†) follows at once from (3.3), (∗∗) and the fact that $\psi_4(\xi, n)$ is decreasing for $n \geqq (K \cdot H)/2H^2$. QED

Now we can finish the proof of Theorem A for $r_\xi > 2$. A straightforward computation gives

$$\psi_3(\xi, x_0) = \lambda_2 \Delta_\xi + \sqrt{\rho}\left[\frac{r_\xi^3}{2}\frac{|K \cdot H|}{\sqrt{H^2}} + r_\xi^5\sqrt{H^2}\right]\sqrt{\Delta_\xi} + \frac{r_\xi^4}{5}\frac{(K \cdot H)^2}{H^2}$$

$$+ \frac{r_\xi^7}{2}H^2 + \frac{r_\xi^2}{2}\frac{(K \cdot H + r_\xi^2 H^2)^2}{H^2} + r_\xi^2|\chi| + \frac{r_\xi^3}{8}|K^2|\,,$$

$$\psi_4(\xi, x_0 - 1) = \lambda_2 \Delta_\xi + \sqrt{\rho}\left[\frac{r_\xi - 1}{2}\frac{K \cdot H}{\sqrt{H^2}} + (r_\xi - 1)\sqrt{H^2}\right]\sqrt{\Delta_\xi}$$

$$- \frac{r_\xi - 1}{2}H^2 - \frac{r_\xi - 1}{2}K \cdot H - (r_\xi^2 - 1)\chi + e_K + \frac{r_\xi^2}{4}\,.$$

As is easily checked, if $\dim X$ satisfies (0.5) then it is greater than both these quantities. By the previous lemma we conclude that $\partial X \neq \emptyset$.

Proof of Theorem A when $r_\xi = 2$. The proof will be similar to the one given above, with the difference that instead of Proposition (2.1) we will use Proposition (2.2). We set $P_2 := h^0(2K)$.

(3.4) Lemma. *Assume that H is effective. Let ξ be a set of sheaf data, with $r_\xi = 2$. Let $C \in |nH|$ be a smooth curve. Then*

$$\dim \mathcal{M}_\xi^C(n\sqrt{H^2}) \leqq 3\Delta_\xi + 2H^2n^2 + (K \cdot H + 2H^2 + 2)n$$

$$+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi - q_S\,.$$

Proof. By Proposition (1.4) we have

$$\dim \mathcal{M}_\xi^C(n\sqrt{H^2})$$

$$\leqq \max\{3\alpha^2 + ((K \cdot H + 2H^2 + 2)(H^2)^{-1/2} - n\sqrt{H^2})\alpha\}_{0 \leqq \alpha \leqq n\sqrt{H^2}} + 3\Delta_\xi$$

$$+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(\mathcal{O}_S) - q_S\,.$$

The above expression is a concave-up function of α, and hence its maximum is achieved at one of its end-points: a computation shows that the value at $\alpha = n\sqrt{H^2}$ is the greatest of the two values and that it equals the right-hand side of the inequality in the lemma. QED

Set

$$\phi_1(\xi,n) := \tfrac{3}{2}H^2n^2 + \tfrac{3}{2}K \cdot Hn\,,$$

$$\phi_2(\xi,n) := \tfrac{1}{2}H^2n^2 + \tfrac{1}{2}K \cdot Hn + 3\Delta_\xi + \varepsilon + \tfrac{3}{2}\,,$$

$$\phi_3(\xi,n) := 3\Delta_\xi + \tfrac{5}{2}H^2n^2 + (\tfrac{3}{2}K \cdot H + 2H^2 + 2)n + \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 4 - 3\chi - q_S\,,$$

$$\phi_4(\xi,n) := 3\Delta_\xi + \tfrac{1}{2}H^2n^2 + \tfrac{1}{2}K \cdot Hn + 1 + \tau(S,H)\,,$$

$$\phi_5(\xi,n) := 4\Delta_\xi - \tfrac{1}{2}H^2n^2 + \tfrac{1}{2}K \cdot Hn + P_2 - 3\chi + 1\,,$$

where

$$\tau(S,H) := \frac{5(K \cdot H)^2}{4H^2} + K \cdot H + \frac{K \cdot H}{H^2} + \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi - q_S \quad \text{if } K \cdot H \geqq 0$$

$$\tau(S,H) := 0 \quad \text{if } K \cdot H < 0\,.$$

For $i = 1,2,5$ the value of $\phi_i(\xi,n)$ equals the right-hand side of the inequality in Items (1)–(2) of Proposition (2.1) (with $r_\xi = 2$), and of Inequality (2.5), respectively. By Lemma (3.4) the right-hand side of (2.3) is bounded above by $\phi_3(\xi,n)$, and by Proposition (1.3) $\phi_4(\xi,n)$ is an upper bound for the right-hand side of (2.4). Hence by Proposition (2.2) it suffices to show that, for some integer $n \geqq 1$, we have

$$\dim X > \max\{\phi_1(\xi,n), \phi_2(\xi,n), \phi_3(\xi,n), \phi_4(\xi,n), \phi_5(\xi,n)\}\,.$$

As is easily checked,

$$\max\{\phi_1(\xi,n), \phi_2(\xi,n)\} \leqq \phi_3(\xi,n) \quad \text{for } n \geqq 0\,,$$

and thus we have

(3.5) *Keep notation as above. If*

$$\dim X > \max\{\phi_3(\xi,n), \phi_4(\xi,n), \phi_5(\xi,n)\} \tag{3.6}$$

for some integer $n \geqq 1$, then $\partial X \neq \emptyset$.

Proceeding as in the previous case, we let w_0 be the positive root of the equation in n

$$\tfrac{5}{2}H^2n^2 + 3\Delta_\xi = -\tfrac{1}{2}H^2n^2 + 4\Delta_\xi\,,$$

obtained by equating the two dominant terms of $\phi_3(\xi,n)$ and $\phi_5(\xi,n)$. Explicitly

$$w_0 = \frac{\sqrt{\Delta_\xi}}{\sqrt{3H^2}}\,.$$

(3.7) Lemma. *Keep notation as above. If*

$$\dim X > \max\{\phi_3(\xi,w_0), \phi_4(\xi,w_0), \phi_5(\xi,w_0 - 1)\}\,, \tag{3.8}$$

then $\partial X \neq \emptyset$.

Proof. Let $n_0 := [w_0]$. Since $\Delta_\xi > \Delta_0(2,S,H)$, we have $n_0 \geqq 1$. We claim that (3.6) holds with $n = n_0$; by (3.5) this will imply the lemma. If ξ is fixed, then

1. $\phi_3(\xi,n)$ is increasing for $n \geqq 1$,
2. $\phi_4(\xi,n)$ is increasing for $n \geqq -(K \cdot H)/2H^2$, and
3. $\phi_5(\xi,n)$ is decreasing for $n \geqq (K \cdot H)/2H^2$.

Since $n_0 \geqq 1$, $\phi_3(\xi,n)$ is increasing for $n \geqq n_0$, and since $\Delta_\xi > \Delta_0$, $\phi_5(\xi,n)$ is decreasing for $n \geqq (w_0 - 1)$. Hence (3.8) implies that

$$\dim X > \max\{\phi_3(\xi,n_0), \phi_5(\xi,n_0)\}\ .$$

If $K \cdot H \geqq 0$ then, by Item (2) above, Inequality (3.8) also implies $\dim X > \phi_4(\xi,n_0)$, so we are done. If $K \cdot H < 0$, then as is easily checked $\phi_4(\xi,n) \leqq \phi_3(\xi,n)$ for all $n \geqq 1$. Hence also in this case $\dim X > \phi_4(\xi,n_0)$. QED

Now one finishes the proof of Theorem A in the rank-two case by checking that if (0.5) holds, then the hypotheses of the above lemma are satisfied.

Another application of Proposition (2.2). Our goal in this subsection is to determine an effective Δ_1 with the property that, if $\Delta_\xi > \Delta_1$, then every closed subset of $\mathscr{M}_\xi$ whose dimension is at least the expected dimension of $\mathscr{M}_\xi$ has non-empty boundary. One such lower bound can be obtained by applying Theorem A. However, while Theorem A provides the best "asymptotic" result of its kind obtainable from Proposition (2.1), it is not sharp for Δ_ξ small; hence we proceed differently. We will limit ourselves to rank-two sheaves. Let z_0 be the positive root of the equation in n

$$4\Delta_\xi - 3\chi = \phi_5(\xi,n)\,, \tag{3.9}$$

i.e.

$$z_0 = \frac{1}{2H^2}[K \cdot H + \sqrt{(K \cdot H)^2 + 8H^2(P_2+1)}]\,. \tag{3.10}$$

(3.11) Proposition. *Assume that H is effective. Keeping notation as above, let*

$$\Delta_2(S,H) := (4K \cdot H + 7H^2 + 2)z_0 + 6H^2 + \frac{9}{2}K \cdot H + 3\frac{K \cdot H}{H^2} + \frac{7}{4}\frac{(K \cdot H)^2}{H^2} + \frac{3}{2H^2} + 14 + 5P_2(S) - \frac{K^2}{4} - q_S\,, \tag{3.12}$$

$$n_0 := \text{the least positive integer such that } n_0 > z_0\,. \tag{3.13}$$

Let ξ be a set of sheaf data, with $r_\xi = 2$. Assume that $\Delta_\xi > \Delta_2(S,H)$ and that the linear system $|n_0H|$ contains a smooth curve. Then the following hold:

1. *If Y is an irreducible component of $\mathscr{M}_\xi$, then the generic point of Y parametrizes a μ-stable sheaf, and hence*

$$\dim Y \geqq 4\Delta_\xi - 3\chi(\mathscr{O}_S)\,.$$

2. *If $X \subset \mathscr{M}_\xi$ is a closed irreducible subset such that*

$$\dim X \geqq 4\Delta_\xi - 3\chi(\mathscr{O}_S)\,, \tag{3.14}$$

then $\partial X \neq \emptyset$.

Proof. We start by proving Item (2). We will show that

$$4\Delta_\xi - 3\chi > \max\{\phi_3(\xi,n_0),\phi_4(\xi,n_0),\phi_5(\xi,n_0)\}\,. \qquad (*)$$

Item (2) will then follow from (3.14) and (3.5). Since z_0 is the positive root of (3.9), and since $\phi_5(\xi,n)$ is a concave-down function of n, we have

$$4\Delta_\xi - 3\chi > \phi_5(\xi,n_0)\,,$$

for any Δ_ξ (i.e. even if $\Delta_\xi \leqq \Delta_2$). Now let's first examine the case $K\cdot H \geqq 0$. In this case both $\phi_3(\xi,n)$ and $\phi_4(\xi,n)$ are increasing functions for $n \geqq 1$ (see the proof of Lemma (3.7)). Thus it suffices to show that

$$4\Delta_\xi - 3\chi > \phi_3(\xi,z_0+1)\,,$$
$$4\Delta_\xi - 3\chi > \phi_4(\xi,z_0+1)\,. \qquad (\dagger)$$

Computing, one gets that the two above inequalities are equivalent to

$$\Delta_\xi > \Delta_2(S,H)$$
$$\Delta_\xi > (K\cdot H + H^2)z_0 + 2H^2 + \frac{9}{2}K\cdot H + 4\frac{K\cdot H}{H^2} + \frac{3(K\cdot H)^2}{H^2} + \frac{3}{2H^2} + 8 + P_2 - \frac{K^2}{4} - q_S\,, \qquad (**)$$

respectively. As is easily checked using (3.10), the first inequality implies the second. This proves $(*)$ if $K\cdot H \geqq 0$. Now let's assume $K\cdot H < 0$. In this case $\phi_3(\xi,n)$ is again increasing for $n \geqq 1$ and $\phi_4(\xi,n_0) \leqq \phi_3(\xi,n_0)$ (see the proof of Lemma (3.7)). Hence it suffices to show that $(\dagger)$ holds. Since this inequality is equivalent to $(**)$ we are done. Now let's prove Item (1). As is easily checked one has

$$\phi_3(\xi,n_0) > 3\Delta_\xi + \varepsilon\,,$$

and hence

$$4\Delta_\xi - 3\chi > 3\Delta_\xi + \varepsilon\,.$$

Item (1) follows from the above inequality and Corollary (1.12). QED

We wish to rewrite $\Delta_2(S,H)$ when

$$c_1(K) = kc_1(H)$$

for some rational positive k. Let

$$N_2(S,H) := \frac{17}{2} + 6\sqrt{1+\frac{8(\chi+1)}{9K^2}} + \left[8 + \frac{4}{H^2} + \left(\frac{21}{2}+\frac{3}{H^2}\right)\sqrt{1+\frac{8(\chi+1)}{9K^2}}\right]k^{-1} + \left(6+\frac{14}{H^2}+\frac{3}{2(H^2)^2}\right)k^{-2}\,. \qquad (3.15)$$

A straightforward computation gives

(3.5) *(3.16)Let (S,H) be a polarized surface. Assume that $c_1(K) = kc_1(H)$ for a rational positive k. (In particular K is ample.) Then*

$$\Delta_2(S,H) := N_2(S,H)K^2 + 5\chi - q_S .$$

Comments

Theorem A naturally raises the question: what is the maximum dimension of closed subsets of $\mathcal{M}_\xi$ which do not intersect $\partial\mathcal{M}_\xi$? The following proposition gives a lower bound for this maximum which can be contrasted with the upper bound provided by Theorem A.

Proposition. *Let S be a surface such that $H^{1,1}_{\mathbf{Z}}(S) = \mathbf{Z}c_1(C)$, where C is a curve of (arithmetic) genus g. Fix an integer $r \geqq 2$ and, for $d \in \mathbf{Z}$, let*

$$\xi(d) = (r, -[C], d) .$$

If d is sufficiently large there exists a closed subvariety $X_d \subset \mathcal{M}_{\xi(d)}$ such that

$$\begin{aligned} &\partial X_d = \emptyset ,\\ &\dim X_d \geqq (r-2)d - (r-2)(r+g-1) - \dim \mathrm{Aut}(C) . \end{aligned} \tag{3.17}$$

Proof. Let L be a non-special line bundle on C. Set $d = \deg(L)$, $n := h^0(L) - 1$. We assume that:

1. The complete linear system $|L|$ defines an embedding

$$C \hookrightarrow \mathbf{P}^n = \mathbf{P}(H^0(L)^*) .$$

2. $n+1 \geqq r$ and $d > C \cdot C$.

Given an r-dimensional subspace $V \subset H^0(L)$ with no base-points, let F_V be the sheaf on S fitting into the exact sequence

$$0 \to F_V \to V \otimes \mathcal{O}_S \xrightarrow{e_V} \iota_* \mathcal{O}_C(L) \to 0 , \tag{$*$}$$

where e_V is the evaluation map, and $\iota : C \hookrightarrow S$ is the inclusion. Clearly F_V is locally-free. The Chern classes of F_V are given by

$$\begin{aligned} c_1(F_V) &= -[C] ,\\ c_2(F) &= d . \end{aligned} \tag{$\dagger$}$$

As is easily checked, it follows from $H^{1,1}_{\mathbf{Z}}(S) = \mathbf{Z}c_1(C)$ that F_V is μ-stable. One can identify the set of base-point free r-dimensional subspaces of $H^0(L)$ with the set U_C of $(n-r)$-dimensional linear subspaces of $\mathbf{P}^n$ not intersecting C (embedded by $|L|$). Let $\mathbf{P}^{n-2} \subset \mathbf{P}^n$ be disjoint from C, and set

$$B_d := \mathbf{Gr}(n-r, \mathbf{P}^{n-2}) .$$

Then B_d is a projective subset of U_C, and

$$\dim B_d = (r-2)(d-g-r+1)\,. \qquad (**)$$

Clearly one can construct a family $\mathscr{F}$ of vector bundles on S parametrized by B_d, with the property that if $[V] \in B_d$, then

$$\mathscr{F}|_{S\times[V]} \cong F_V\,.$$

Since the Chern classes of F_V are given by (†), and since F_V is stable for all V, the family $\mathscr{F}$ defines a morphism

$$\varphi\colon\ B_d \to \mathscr{M}_{\xi(d)}\,.$$

Let $X_d := \varphi(B_d)$. Clearly X_d is closed and $\partial X_d = \emptyset$. Now let's check that (3.17) holds. Dualizing $(*)$ one gets

$$0 \to V^* \otimes \mathscr{O}_S \to F_V^* \to \mathscr{O}_C(C) \otimes L^* \to 0\,.$$

By Item (2) above we have $V^* \cong H^0(F_V^*)$, and hence the isomorphism class of F_V determines V up to isomorphism. Formula (3.17) follows at once from this and Equation $(**)$. QED

4. Moduli of bundles with twisted endomorphisms

In this section we will prove Theorems B, C, and their corollaries. First some notation. If $X \subset \mathscr{M}_\xi$, we let $\overline{X} \supset X$ be the closure of X in $\mathscr{M}_\xi$, and $X^\mu \subset X$ be the subset parametrizing μ-stable sheaves. To simplify notation we will set

$$\partial\overline{X} = \partial(\overline{X}) \qquad \partial X^\mu := (\partial X)^\mu \qquad \partial\overline{X}^\mu := (\partial\overline{X})^\mu\,.$$

The proof of Theorem B goes roughly as follows. Assume that the dimension of W_ξ^L is large and that Δ_ξ is also large. Then by Theorem A we know that $\partial W_\xi^L \neq \emptyset$. Let $[E_0] \in \partial W_\xi^L$. The double-dual $F_1 = E_0^{**}$ is a locally-free sheaf with $r_{F_1} = r_\xi$, $\det F_1 \cong \det_\xi$, and $\Delta_{F_1} < \Delta_\xi$. The discriminant Δ_{F_1} depends on E_0, but for $[E_0]$ varying in an open non-empty subset $U_0 \subset \partial W_\xi^L$ the discriminant is constant. If we further restrict U_0, we can assume that F_1 is always μ-stable. Thus $[E_0] \mapsto [F_1]$ defines a map $U_0 \to \mathscr{M}_{\xi_1}$, whose image we denote by X_1. As is easily proved

$$\operatorname{cod}(\partial W_\xi^L, W_\xi^L) \leqq r_\xi - 1\,,$$

i.e. $\dim \partial W_\xi^L$ is large, and a dimension count shows that this implies $\dim X_1$ is also large. This last fact allows us to apply again Theorem A, this time to $\overline{X}_1$. Iterating this process we get a set of locally-closed $X_i \subset \mathscr{M}_{\xi_i}$, for $i = 0, 1, \ldots, n$, such that if $[E_i] \in \partial\overline{X}_i$ is generic then, setting $F_{i+1} = E_i^{**}$, we have $[F_{i+1}] \in X_{i+1}$. Applying Theorem A one proves that $\Delta_{\xi_n} \leqq \Delta_0$; a dimension count shows that this implies n is of the order of Δ_ξ. The point of this construction is the following. First, for a torsion-free sheaf E on S, there is a natural inclusion

$$H^0(E, E\otimes L)^0 \stackrel{\iota}{\hookrightarrow} H^0(E^{**}, E^{**}\otimes L)^0 .$$

Secondly, consider the canonical exact sequence

$$0 \to E \to E^{**} \stackrel{\phi}{\to} Q_E \to 0 ,$$

where Q_E is an Artinian sheaf. Let $f \in H^0(E^{**}, E^{**}\otimes L)^0$ be non-zero. Since f is not a scalar endomorphism, if we choose the map ϕ generically then f is not in the image of ι: put differently, if $h^0(E^{**}, E^{**}\otimes L)^0 \neq 0$ then ι is an isomorphism only for special choices of ϕ. Hence, if the $[F_i] \in X_i$ are generic we have

$$0 < h^0(F_0, F_0\otimes L)^0 \leqq h^0(F_1, F_1\otimes L)^0 \leqq \cdots h^0(F_n, F_n\otimes L)^0 , \qquad (*)$$

and whenever there is an equality we know that the map ϕ_{i+1} appearing in the exact sequence

$$0 \to E_i \to F_{i+1} \stackrel{\phi_{i+1}}{\to} Q_{i+1} \to 0$$

is not generic. Third, by Simpson (1.6) there is a bound for $h^0(F_i, F_i\otimes L)^0$ independent of ξ and thus, since n is of the order of Δ_ξ, we see that almost all inequalities of $(*)$ are in fact equalities. Hence for almost all i the map ϕ_i is not generic. Since $\mathrm{cod}(\partial\overline{X}_i, X_i) \leqq (r_\xi - 1)$ one concludes that the restrictions on the moduli of the ϕ_i's must be compensated by the moduli of the F_i's being "large". Going all the way down to X_n we progressively "inflate" the dimension of X_i, until we get that

$$\dim X_n > 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S) + e_K(r_\xi, S, H) .$$

Since the sheaves parametrized by X_n are μ-stable, this contradicts deformation theory (0.3). The proof of Theorem C is similar.

The double-dual construction

Assume that $X \subset \mathcal{M}_\xi$ and that $\partial X^\mu \neq \emptyset$. Let $[F] \in \partial X^\mu$; we have the canonical exact sequence

$$0 \to F \to F^{**} \stackrel{\psi_F}{\to} Q_F \to 0 , \qquad (4.1)$$

where Q_F is an Artinian sheaf of length

$$\ell(Q_F) = h^0(Q_F) > 0 .$$

Since F is μ-stable, so is F^{**}, and hence it determines a point $[F^{**}] \in \mathcal{M}_{\xi'}$, where

$$\xi' = (r_\xi, \det_\xi, c_2(\xi) - \ell(Q_F)) . \qquad (4.2)$$

The sheaves F^{**}, for $[F]$ varying in ∂X^{μ}, do not fit together to give a family of sheaves (their second Chern class might not be constant). However there is a (maximal) stratification of ∂X^{μ} by locally closed subsets with the property that the double duals of sheaves parametrized by points of the same stratum fit together to give (locally) a family of vector bundles. We call this the *double-dual stratification* of ∂X^{μ}. We will be interested in the open strata: we start by giving a lower bound for their dimension.

(4.3) Proposition. *Let $\mathscr{F}$ be a family of rank-r torsion-free sheaves on S parametrized by an equidimensional variety B. Let $\partial B \subset B$ be the subset parametrizing singular sheaves. Then ∂B is closed, and*

$$\operatorname{cod}(\partial B, B) \leqq r - 1 \, .$$

Proof. Since $\mathscr{F}$ is a family of rank-r torsion-free sheaves on a smooth surface it has a short locally-free resolution

$$0 \to \mathscr{E}_1 \xrightarrow{\phi} \mathscr{E}_0 \to \mathscr{F} \to 0 \, .$$

Let $\mathrm{D}(\phi) \subset S \times B$ be the degeneracy locus of ϕ (i.e. the locus where ϕ drops rank). Clearly $\mathrm{D}(\phi)$ is closed, and since $r_{\mathscr{E}_0} - r_{\mathscr{E}_1} = r$ we have

$$\operatorname{cod}(\mathrm{D}(\phi), S \times B) \leqq r + 1 \, .$$

The result follows because $\partial B = \pi_B(\mathrm{D}(\phi))$. QED

(4.4) Corollary. *Let $X \subset \mathscr{M}_{\xi}$ be a locally closed equidimensional subset. Assume that $\partial X^{\mu} \neq \emptyset$. Let $Y \subset \partial X^{\mu}$ be an irreducible component of an open stratum of the double-dual stratification. Then*

$$\dim Y \geqq \dim X - (r_{\xi} - 1) \, .$$

Let Y be as in the above corollary, and set

$$Y^{**} := \{[F^{**}] \mid [F] \in Y\} \, .$$

Thus $Y^{**} \subset \mathscr{M}_{\xi'}$, where ξ' is given by (4.2). We will relate the dimensions of X and Y^{**}. For this we need to consider certain Quot-schemes. If E is a vector bundle on S, and ℓ is a positive integer, let $\mathrm{Quot}(E;\ell)$ be the parameter space for quotients of E of finite length equal to ℓ. Let $\mathrm{Quot}_0(E;\ell)$ be the (open) subset parametrizing quotients

$$E \to \bigoplus_{i=1}^{\ell} \mathbf{C}_{P_i} \, ,$$

where $\mathbf{C}_{P_i}$ is the skyscraper sheaf at P_i. The following result is due to J. Li, see Proposition (6.4) of [Li] (the proof there is given for $r_E = 2$, but in fact it carries over to any rank, see Proof of Lemma (4.2) of [GL2]):

(4.5) Theorem (Jun Li). *Let notation be as above. Then* $\mathrm{Quot}_0(E;\ell)$ *is dense in* $\mathrm{Quot}(E;\ell)$. *In particular* $\dim \mathrm{Quot}(E;\ell) = (r_E + 1)\ell$.

(4.6) Corollary. *Let $X \subset \mathcal{M}_\xi$, and assume that $\partial X^\mu \neq \emptyset$. Let $Y \subset \partial X^\mu$ be an irreducible component of an open stratum of the double-dual stratification. Let $\ell := \ell(Q_F)$, where $[F] \in Y$. Then*

$$\dim Y^{**} \geqq \dim X - (r_\xi + 1)\ell - r_\xi + 1 \,. \tag{4.7}$$

If (4.7) *is an equality, the following holds: Let $[E] \in Y^{**}$ be* **generic**, *and let $\phi \in \mathrm{Quot}_0(E;\ell)$ be a* **generic** *quotient. Then the sheaf F fitting into the exact sequence*

$$0 \to F \to E \xrightarrow{\phi} \bigoplus_{i=1}^{\ell} \mathbf{C}_{P_i} \to 0$$

is parametrized by a point of Y.

Proof. Let $\mathrm{Quot}(Y^{**};\ell) \to Y^{**}$ be the relative Quot-scheme, with fiber $\mathrm{Quot}(E;\ell)$ over $[E] \in Y^{**}$, and let $\mathrm{Quot}_0(Y^{**};\ell)$ be the open subset with fiber $\mathrm{Quot}_0(E;\ell)$ over $[E]$. We have an injection

$$f : Y \hookrightarrow \mathrm{Quot}(Y^{**};\ell)\,,$$

mapping $[F]$ to the canonical quotient ψ_F (see (4.1)). By Theorem (4.5) we conclude that

$$\dim Y \leqq \dim Y^{**} + (r_\xi + 1)\ell \,. \tag{$*$}$$

Inequality (4.7) follows from the above inequality and from Corollary (4.4). Now suppose that (4.7) is an equality then we must have equality also in $(*)$. By Theorem (4.5) we conclude that $f(Y) \cap \mathrm{Quot}_0(Y^{**};\ell)$ is dense in $\mathrm{Quot}_0(Y^{**};\ell)$; this proves the second statement of the corollary. QED

Now let L be a line bundle on S. If $X \subset \mathcal{M}_\xi$ is an irreducible locally closed subset with $X^\mu \neq \emptyset$, we set

$$h_L(X) := \min\{h^0(F, F \otimes L)^0 \,|\, [F] \in X^\mu\}\,.$$

By semicontinuity of cohomology dimension, if $[F] \in X^\mu$ is a generic point then $h^0(F, F \otimes L)^0 = h_L(X)$. The following proposition contains the observation that will allow us to deduce Theorems B and C from Theorem A.

(4.8) Proposition. *Let notation be as above. Let $X \subset \mathcal{M}_\xi$ be a locally closed irreducible subset such that $\partial X^\mu \neq \emptyset$. Let $Y \subset \partial X^\mu$ be an irreducible component of an open stratum of the double-dual stratification, and let $\ell := \ell(Q_F)$ for $[F] \in Y$. Then*:

1. $h_L(Y^{**}) \geqq h_L(X)$.
2. $\dim Y^{**} \geqq \dim X - (2r_\xi - 1)\ell - 1$.
3. *If $h_L(X) > 0$ then one at least of the inequalities in Items* (1)–(2) *is strict.*

Proof. If F is a torsion-free sheaf on S there is a natural inclusion

$$\imath : H^0(F, F \otimes L)^0 \hookrightarrow H^0(F^{**}, F^{**} \otimes L)^0\,. \tag{$*$}$$

Let's consider the case when $[F^{**}] \in Y^{**}$ is a generic point. Since the dimension of the right-hand side of the above inequality is equal to $h_L(Y^{**})$, and since the dimension of the left-hand side is not smaller than $h_L(X)$, we conclude that Item (1) holds. A straightforward computation shows that Item (2) is a consequence of Corollary (4.6). To prove Item (3) let's assume that Item (2) is an equality. The same computation that proves Item (2) also shows that if Item (2) is an equality then either $r_\xi = 2$ or $\ell = 1$. In each of these cases the right-hand side of Item (2) equals the right-hand side of (4.7), and thus we conclude that (4.7) is an equality. Hence, by Corollary (4.6), if $[E] \in Y^{**}$ is generic and F fits into an exact sequence

$$0 \to F \to E \xrightarrow{\phi} \bigoplus_{i=1}^{\ell} \mathbf{C}_{P_i} \to 0 \qquad (**)$$

with ϕ generic, then $[F] \in Y$.

Claim. *Let E be a locally-free sheaf on S, and assume that* $h^0(E, E \otimes L)^0 > 0$. *Let F be a sheaf fitting into Exact sequence* $(**)$, *with* ϕ **generic** (*in particular the* P_i*'s are generic*). *Then*

$$h^0(F, F \otimes L)^0 < h^0(E, E \otimes L)^0 .$$

Proof of the claim. Clearly $E = F^{**}$. Let ι be the inclusion of $(*)$ with $E = F^{**}$. Then

$$f \in \iota(H^0(F, F \otimes L)^0) \quad \text{if and only if } f^*\phi_i = \lambda_i \phi_i ,$$

where ϕ_i is the restriction of ϕ to the fiber E_{P_i}, and $\lambda_i \in \mathbf{C}$. Now choose a non-zero $f \in H^0(E, E \otimes L)^0$. Since f is not a scalar morphism, for a generic choice of ϕ the morphism f does not satisfy the condition above. Hence the inclusion ι is not surjective. This proves the claim. QED

Going back to the proof of Item (3), since $[E] \in Y^{**}$ is generic

$$h^0(E, E \otimes L)^0 = h_L(Y^{**})$$

Notice that we are assuming $h_L(X) > 0$ and hence by Item (1) $h_L(Y^{**}) > 0$. Let F fit into exact sequence $(**)$ with ϕ generic. We know that $[F] \in Y$, and furthermore by the above claim $h^0(F, F \otimes L)^0 < h_L(Y^{**})$. Since $h_L(X)$ is not greater than $h^0(F, F \otimes L)^0$ we conclude that Item (3) holds. QED

Proof of Theorem B. The proof will be by contradiction. So we assume that $\Delta_\xi > \Delta_1$, and that Inequality (0.6) is violated, i.e. there exists an irreducible component of W^L_ξ, call it X_0, such that

$$\dim X_0 > \lambda_2 \Delta_\xi + \lambda_1 \sqrt{\Delta_\xi} + \lambda'_0 + e_L . \qquad (4.9)$$

By the above inequality and Theorem A we have $\partial \overline{X}_0 \neq \emptyset$. The following lemma will show that in fact $\partial \overline{X}^\mu_0 \neq \emptyset$.

(4.10) Lemma. *Let* $X \subset \mathcal{M}_\xi$ *be an equidimensional locally closed subset such that*:

1. $\partial X \neq \emptyset$, *and*
2. $\dim X > (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H) + r_\xi - 1$.

Then $\partial X^\mu \neq \emptyset$.

Proof. Let $[F] \in \partial X$. If F is μ-stable there is nothing to prove, so assume F is not μ-stable. Let $\mathscr{F}$ be the family of sheaves on S parametrized by $\mathrm{Def}^0(\mathrm{Gr}\,F)$. By Claim (2.22) the map

$$\lambda\colon \mathrm{Def}^0(\mathrm{Gr}\,F) \to \mathscr{M}_\xi$$

induced by $\mathscr{F}$ is surjective onto a neighborhood of $[F]$, and hence

$$\dim(\lambda^{-1}X) \geqq \dim X\ .$$

Let $\partial(\lambda^{-1}X) \subset \lambda^{-1}X$ be the subset parametrizing singular sheaves. By Proposition (4.3) and by Item (2) we conclude that

$$\dim \partial(\lambda^{-1}X) > (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H)\ .$$

By (1.10) the subset of $\partial(\lambda^{-1}X)$ parametrizing μ-stable sheaves is non-empty. Since its image is contained in ∂X^μ, we conclude that $\partial X^\mu \neq \emptyset$. QED

An easy computation shows that (4.9) together with $\Delta_\xi > \Delta_1$ gives

$$\dim X_0 > (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H) + r_\xi - 1\ .$$

Hence by Lemma (4.10) we have $\partial\overline{X}_0^\mu \neq \emptyset$. Let $Y_0 \subset \partial\overline{X}_0^\mu$ be an irreducible component of an open stratum of the double-dual stratification. Let

$$\xi_1 := (r_\xi, \det_\xi, c_2(\xi) - \ell_0)\ ,$$

where $\ell_0 := \ell(Q_{F_0})$ for $[F_0] \in Y_0$. Set $X_1 := Y_0^{**}$. Thus

$$X_1 \subset \mathscr{M}_{\xi_1}\ .$$

Now consider $\overline{X}_1$ and, if $\partial\overline{X}_1^\mu \neq \emptyset$ continue in the same fashion. We will get a sequence

$$X_i \subset \mathscr{M}_{\xi_i}, \qquad Y_i \subset \partial\overline{X}_i^\mu, \qquad X_{i+1} = Y_i^{**}\ , \tag{4.11}$$

for $i = 0, \ldots, n$ (with $\xi_0 := \xi$), until we reach a point when $\partial\overline{X}_n^\mu = \emptyset$. We will show that the dimension of X_n is "too big", and thus get a contradiction. Let $\ell_i := \ell(Q_{F_i})$, where $[F_i] \in Y_i$, and set

$$\ell := \ell_0 + \cdots + \ell_{n-1}\ .$$

Lemma. *Keeping notation as above, we have*

$$\dim X_n \geqq \dim X_0 - (2r_\xi - 1)\ell - e_L\ .$$

Proof. By Item (1) of (4.8) we have

$$h_L(X_0) \leqq h_L(X_1) \leqq \cdots \leqq h_L(X_n)\ . \tag{$*$}$$

In particular, since $h_L(X_0) > 0$, we have $h_L(X_i) > 0$ for all i. Since by (1.6) $h_L(X_n) \leqq e_L$ there can be at most $(e_L - 1)$ strict inequalities in $(*)$. By

Proposition (4.8) we conclude that

$$\dim X_{i+1} \geqq \dim X_i - (2r_\xi - 1)\ell_i - \delta_i ,$$

where $\delta_i = 0$ or $\delta_i = 1$, and δ_i is equal to 1 for at most $(e_L - 1)$ values of i. The result follows at once from this. QED

The above Lemma together with (4.9) gives

$$\begin{aligned} \dim X_n &> \lambda_2 \Delta_\xi + \lambda_1 \sqrt{\Delta_\xi} + \lambda_0' - (2r_\xi - 1)\ell \\ &= (2r_\xi - 1)\Delta_{\xi_n} + [\lambda_2 - (2r_\xi - 1)]\Delta_\xi + \lambda_1\sqrt{\Delta_\xi} + \lambda_0' \qquad (4.12) \\ &> \max\{\lambda_2\Delta_{\xi_n} + \lambda_1\sqrt{\Delta_{\xi_n}} + \lambda_0', (2r_\xi - 1)\Delta_{\xi_n} + \varepsilon + r_\xi - 1\} . \qquad (4.13) \end{aligned}$$

Lemma. *Keeping notation as above, we have*

$$\Delta_{\xi_n} \leqq \Delta_0(r_\xi, S, H) . \qquad (4.14)$$

Proof. The proof is by contradiction, so we assume that (4.14) is violated. Then Inequality (4.13) and Theorem A give that $\partial \overline{X}_n \neq \emptyset$. By (4.13) and Lemma (4.10) we conclude that $\partial \overline{X}_n^\mu \neq \emptyset$. This contradicts the definition of X_n. QED

Now we can finish the proof of Theorem B. We will show that

$$\dim X_n > 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi + e_K . \qquad (*)$$

From this one concludes as follows: Since $X_n \subset \mathcal{M}_{\xi_n}$ the above inequality together with (0.3) implies that all sheaves parametrized by X_n are non-stable. But this is absurd by Inequality (4.13) and by (1.11). Now let's prove $(*)$. By (4.12) it suffices to show that

$$[\lambda_2 - (2r_\xi - 1)]\Delta_\xi + \lambda_1\sqrt{\Delta_\xi} + \lambda_0' > \Delta_{\xi_n} - (r_\xi^2 - 1)\chi + e_K .$$

By Inequality (4.14) it suffices to check that

$$[\lambda_2 - (2r_\xi - 1)]\Delta_\xi + \lambda_1\sqrt{\Delta_\xi} + \lambda_0' > \Delta_0 - (r_\xi^2 - 1)\chi + e_K .$$

This follows at once from $\Delta_\xi > \Delta_1$. QED

Proof of Corollary B'. Let $\tilde{\Delta}_1(r_\xi, S, H)$ be the smallest number such that:
- $\tilde{\Delta}_1 \geqq \Delta_1$, and
- if $\Delta_\xi > \tilde{\Delta}_1$ then

$$2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S) > (2r_\xi - 1)\Delta_\xi + \varepsilon(r_\xi, S, H) , \qquad (4.15)$$

$$2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S) > \lambda_2\Delta_\xi + \lambda_1\Delta_\xi + \lambda_0' + e_K . \qquad (4.16)$$

Set

$$\Delta_1'(r_\xi, S, H) := \tilde{\Delta}_1(r_\xi, S, H) + (r_\xi - 1)^{-1} e_K(r_\xi, S, H) .$$

As is easily checked Δ_1' depends only on $r_\xi, K^2, K \cdot H, H^2$ and $\chi(\mathcal{O}_S)$.

(4.17) Claim. *Let* (S,H) *be a polarized surface, with* H *satisfying* (0.4). *Let* ξ *be a set of sheaf data such that* $\Delta_\xi > \tilde{\Delta}_1(r_\xi,S,H)$. *Then* $\mathscr{M}_\xi$ *is good, and the generic point of any of its irreducible components parametrizes a* μ*-stable sheaf.*

Proof. Let X be an irreducible component of $\mathscr{M}_\xi$. Inequality (4.15) and Corollary (1.12) give that the generic point of X parametrizes a μ-stable sheaf. Since $\Delta_\xi > \Delta_1$, we conclude by Theorem B and Inequality (4.16) that

$$\dim W_\xi^K < 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathscr{O}_S)\,.$$

Thus by Inequality (0.2) we have

$$\dim X > W_\xi^K\,,$$

i.e. $h^0(F, F\otimes K)^0 = 0$ for a (stable) sheaf F parametrized by the generic point of X. Thus $\mathscr{M}_\xi$ is good. QED

Now assume that $\Delta_\xi > \Delta_1'$. Let X be an irreducible component of $\mathscr{M}_\xi$. By (4.17) we have $X^\mu \neq \emptyset$. We will show that if $[F] \in X^\mu$ is generic then F is locally-free. Assume the contrary, i.e. $X^\mu \subset \partial\mathscr{M}_\xi$. Let $Y \subset X^\mu$ be an irreducible component of an open stratum of the double-dual stratification. Then

$$Y^{**} \subset \mathscr{M}_{\xi'} \quad c_2(\xi) - c_2(\xi') > 0\,.$$

Theorem (4.5) gives that

$$\dim X^\mu \leqq \dim Y^{**} + (r_\xi + 1)(c_2(\xi) - c_2(\xi'))\,. \tag{$*$}$$

We distinguish between two cases:

1. If $\Delta_{\xi'} > \tilde{\Delta}_1$, then $\dim Y^{**} \leqq 2r_\xi\Delta_{\xi'} - (r_\xi^2 - 1)\chi(\mathscr{O}_S)$ by Claim (4.17).
2. If $\Delta_{\xi'} \leqq \tilde{\Delta}_1$, then $\dim Y^{**} \leqq 2r_\xi\Delta_{\xi'} - (r_\xi^2 - 1)\chi(\mathscr{O}_S) + e_K$ by (0.3).

In both cases, Inequality $(*)$ gives

$$\dim X^\mu < 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathscr{O}_S)\,.$$

This is absurd because $\mathscr{M}_\xi$ is good. Hence the generic point of X^μ parametrizes a locally-free sheaf. This finishes the proof of Corollary B$'$.

Proof of Theorem C. The essential step is provided by the following

(4.18) Proposition. *Let* (S,H) *be a polarized surface satisfying the hypotheses of Theorem C. Let* ξ *be a set of sheaf data with* $r_\xi = 2$. *Assume that*

$$\Delta_\xi > \Delta_2(S,H) + 2h^0(2K)\,.$$

If $X_0 \subset \mathscr{M}_\xi$ *is an irreducible component then there exists* $[F] \in \partial X_0^\mu$ *such that*

$$h^0(F^{**}, F^{**}\otimes K)^0 = 0\,. \tag{4.19}$$

Proof. By Proposition (3.11) we have $\partial X_0 \neq \emptyset$. As is easily checked we have

$$4\Delta_2 - 3\chi > 3\Delta_2 + \varepsilon + 1\,. \tag{4.20}$$

Thus by Lemma (4.10) we conclude that $\partial X_0^\mu \neq \emptyset$. Let $Y_0 \subset \partial X_0^\mu$ be an irreducible component of an open stratum of the double-dual stratification, and set $X_1 := Y_0^{**}$. Assume that

$$h_K(X_1) > 0\,. \tag{4.21}$$

We will arrive at a contradiction, and thus we will conclude that $h_K(X_1) = 0$; this will prove the proposition. Proceeding as in the proof of Theorem B, we construct a series of locally closed irreducible subsets (as in (4.11)); the only difference is that in the present case we define n by requiring that:

1. $\Delta_{\xi_{n-1}} > \Delta_2$, and
2. either $\partial \overline{X}_n^\mu = \emptyset$, or $\Delta_{\xi_n} \leqq \Delta_2$.

Claim. *Keeping notation as above, we have*

$$\Delta_{\xi_n} \leqq \Delta_2\,, \tag{4.22}$$

$$h_K(X_1) \leqq h^0(2K) \quad \text{for } 0 \leqq i \leqq n-1\,. \tag{4.23}$$

Proof of the claim. Since $\Delta_\xi > \Delta_2$ we have $\dim X_0 \geqq 4\Delta_\xi - 3\chi$ (Proposition (3.11)). Thus, by Item (2) of Proposition (4.8) we have

$$\dim X_i \geqq 4\Delta_{\xi_i} - 3\chi \quad \text{for all } 0 \leqq i \leqq n\,. \tag{4.24}$$

Now assume that $\Delta_{\xi_n} > \Delta_2$. Then by the above inequality and by Proposition (3.11) we have $\partial \overline{X}_n \neq \emptyset$. By Inequality (4.20) and Lemma (4.10) we conclude that $\partial \overline{X}_n^\mu \neq \emptyset$. This contradicts the definition of n, and thus we conclude that (4.22) holds. In order to prove (4.23) we need the following easily checked inequality:

$$\begin{aligned} 4\Delta_2 - 3\chi > {} & 3\Delta_2 + 3\frac{(K\cdot H)^2}{4H^2} + \frac{(K\cdot H)(K\cdot H + 2H^2 + 2)}{2H^2} \\ & + \frac{3(K\cdot H + H^2 + 1)^2}{2H^2} + \frac{(K\cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(\mathcal{O}_S) - q_S\,. \end{aligned} \tag{4.25}$$

The right-hand side of (4.25) equals the right-hand side of the inequality in Proposition (1.3) with $\alpha := (K\cdot H)/2\sqrt{H^2}$, and Δ_ξ replaced by Δ_2. The above inequality, together with (4.24) and Item (1) preceding the claim, gives

$$\dim X_i > \dim \mathcal{M}_{\xi_i}\left(\frac{K\cdot H}{2\sqrt{H^2}}\right) \quad \text{for } 0 \leqq i \leqq n-1\,.$$

By (1.8) we conclude that (4.23) holds. QED

Now we will use the full strength of Proposition (4.8) to give a lower bound for $\dim X_n$ which is stronger than (4.24). By Item (1) of (4.8) we have

$$h_K(X_0) \leqq h_K(X_1) \leqq \cdots \leqq h_K(X_n)\,.$$

By (4.23) there are at most $h^0(2K)$ strict inequalities, i.e. there are at least $(n - h^0(2K))$ equalities. Since we are assuming $h_K(X_1) > 0$, each time there is an equality we can apply Item (3) of Proposition (4.8), and thus

$$\dim X_n \geqq \dim X_0 - 3(c_2(\xi) - c_2(\xi_n)) - h^0(2K) \geqq 3\Delta_{\xi_n} + \Delta_\xi - 3\chi - h^0(2K)\,.$$

Since $\Delta_\xi > \Delta_2 + 2h^0(2K)$ we conclude that

$$\dim X_n > 3\Delta_{\xi_n} - 3\chi + \Delta_2 + h^0(2K)\,. \qquad (*)$$

By construction the generic point of X_n parametrizes a μ-stable vector bundle. Hence:

1. If $e_K(X_n) \leqq h^0(2K)$ then by (0.3) we have $\dim X_n \leqq 4\Delta_{\xi_n} - 3\chi + h^0(2K)$, and

2. if $e_K(X_n) > h^0(2K)$, then by (1.8) $\dim X_n \leqq \dim \mathscr{M}_{\xi_n}((K \cdot H)/2\sqrt{H^2})$.

By (4.22) and by $(*)$ we see that Item (1) is impossible. On the other hand also Item (2) is impossible, by (1.3), $(*)$, (4.22) and (4.25). Thus we have a contradiction. This proves that (4.21) can not hold, and hence proves the proposition. QED

Now we can prove Theorem C. Let $X \subset \mathscr{M}_\xi$ be an irreducible component. By Proposition (4.18) there exists $[F] \in \partial X^\mu$ such that

$$h^0(F^{**}, F^{**} \otimes K)^0 = 0\,. \qquad (\dagger)$$

By openness of μ-stability the generic point of X parametrizes a μ-stable sheaf. Furthermore, since we have an inclusion

$$H^0(F, F \otimes K)^0 \hookrightarrow H^0(F^{**}, F^{**} \otimes K)^0\,.$$

we conclude by $(\dagger)$ that $\mathscr{M}_\xi$ is good at $[F]$. This proves that $\mathscr{M}_\xi$ is good. The only thing left to prove is that the generic point of X parametrizes a locally-free sheaf. Since F is μ-stable, a neighborhood of $[F]$ in $\mathscr{M}_\xi$ is isomorphic to $\mathrm{Def}^0(F)$, which is smooth because $\mathscr{M}_\xi$ is good at $[F]$. Therefore it suffices to show that there exists $x \in \mathrm{Def}^0(F)$ parametrizing a locally-free sheaf. This is equivalent to the existence of $x \in \mathrm{Def}(F)$ parametrizing a locally-free sheaf. As is well-known this follows from $(\dagger)$; we will recall the proof. Let

$$\mathrm{Supp}(F^{**}/F) = \{P_1, \ldots, P_\ell\}\,.$$

Let F_{m_i} be the localization of F at P_i. The versal deformation space of F_{m_i}, is smooth, and F_{m_i}, deforms to a free $\mathscr{O}$-module [Fr]. Hence, since $\mathrm{Def}(F)$ is smooth, it suffices to show that the map of tangent spaces

$$T_0\,\mathrm{Def}(F) \xrightarrow{\rho} \bigoplus_{i=1}^{\ell} \mathrm{Def}(F_{m_i})$$

induced by a versal sheaf on $S \times \mathrm{Def}(F)$, is surjective. The map ρ is part of the local-to-global exact sequence coming from the spectral sequence abutting to $\mathrm{Ext}^1(F,F)$. The piece of interest to us is

$$\mathrm{Ext}^1(F,F) \xrightarrow{\rho} H^0\left(\bigoplus_{i=1}^{\ell} \mathcal{E}xt^1(F_{m_i},F_{m_i})\right) \to H^2(\mathcal{H}om(F,F)) \xrightarrow{\theta} \mathrm{Ext}^2(F,F) \to 0\,.$$

We have an exact sequence

$$0 \to \mathcal{H}om(F,F) \to \mathcal{H}om(F^{**},F^{**}) \to R \to 0\,,$$

where R is an Artinian sheaf (supported at the P_i's). Hence (†) implies that the map θ is an isomorphism, and therefore ρ is surjective. This completes the proof of Theorem C.

Surfaces with ample canonical bundle

Theorem C gives the following

(4.26) Proposition. *Let S be a surface with K ample. Assume that there exists an effective divisor H on S such that $c_1(K) = kc_1(H)$ for some rational positive k, and such that $|n_0H|$ contains a smooth curve, where n_0 is given by* (3.13)*. Let $N_2(S,H)$ be as in* (3.15)*. Let ξ be a set of sheaf data with $r_\xi = 2$, and let $\mathcal{M}_\xi$ be the corresponding moduli space of sheaves on S, polarized by K. If*

$$\Delta_\xi > N_2(S,H)K^2 + 2K^2 + 7\chi(\mathcal{O}_S)\,,$$

then $\mathcal{M}_\xi$ is good.

Proof of Corollaries C′ and C″. For Corollary C′ notice that the hypotheses of Proposition (3.11) are satisfied by $H = K$. In fact K is effective by hypothesis. Furthermore an easy computation gives $z_0 > 2$, hence $n_0 \geqq 3$. Since K^2 is large ($K^2 \geqq 6$ suffices), n_0K is very ample by [Bo], in particular there exists a smooth curve in the linear system $|n_0K|$. The result follows from Proposition (4.26) and the easy estimate

$$N_2(S,K) < 40 + \frac{22(\chi+1)}{3K^2}\,,$$

valid for $K^2 > 100$. Similarly Corollary C″ follows from Proposition (4.26) and the estimate

$$N_2(S,H) < 15 + \frac{8(\chi+1)}{3K^2}\,, \tag{4.27}$$

valid for $k > 100$.

Examples of non-good moduli spaces. We will show that the lower bound given in Corollaries C′–C″ is, if not sharp, at least of the right form. Assume H is effective. We will consider non-trivial extensions

$$0 \to \mathcal{O}_S \to F \to I_Z(H) \to 0\,, \tag{4.28}$$

where Z is a zero-dimensional subscheme of S of length ℓ. These extensions are parametrized by

$$\mathrm{Ext}^1(I_Z(H), \mathcal{O}_S) \cong H^1(I_Z(K+H))^* .$$

From this one easily gets

(4.29) *Keep notation as above. Assume that*

$$\ell > h^0(K+H) = \chi(\mathcal{O}_S) + \tfrac{1}{2}K \cdot H + \tfrac{1}{2}H^2 . \tag{4.30}$$

Then if Z is generic we have

$$\dim \mathrm{Ext}^1(I_Z(H), \mathcal{O}_S) = \ell - \chi(\mathcal{O}_S) - \tfrac{1}{2}K \cdot H - \tfrac{1}{2}H^2 ,$$

and the generic non-trivial extension (4.28) *is locally-free.*

The following is also an easy exercise.

(4.31) *Keeping notation as above, assume that* (4.30) *is satisfied and that*

$$\ell > \tfrac{9}{8}H^2 + 2 + q_S .$$

Then if Z is generic the generic non-trivial extension (4.28) *is μ-stable, and furthermore its space of global sections is one-dimensional.*

Let

$$\xi(\ell) := (2, \mathcal{O}_S(H), \ell) .$$

By (4.29)–(4.31) if

$$\ell > \max\{\chi(\mathcal{O}_S) + \tfrac{1}{2}K \cdot H + \tfrac{1}{2}H^2, \tfrac{9}{8}H^2 + 2 + q_S\} \tag{4.32}$$

then there is an irreducible subset $\Sigma_\ell \subset \mathcal{M}_{\xi(\ell)}$ parametrizing extensions (4.28) with Z generic, and we have

$$\dim \Sigma_\ell = 3\ell - \chi(\mathcal{O}_S) - \tfrac{1}{2}K \cdot H - \tfrac{1}{2}H^2 - 1 .$$

Since $\Delta_{\xi(\ell)} = \ell - H^2/4$ we conclude that

(4.33) *Let notation be as above. Assume that ℓ satisfies* (4.32) *and that*

$$\ell < 2\chi(\mathcal{O}_S) - \tfrac{1}{2}K \cdot H + \tfrac{1}{2}H^2 .$$

Then $\mathcal{M}_{\xi(\ell)}$ is not good. More precisely it contains a subset Σ_ℓ whose dimension is greater than the expected dimension of $\mathcal{M}_{\xi(\ell)}$. Furthermore the generic point of Σ_ℓ parametrizes a μ-stable vector bundle.

One can easily check that if $q_S = 0$ and if $c_1(K) = kc_1(H)$ for $k \gg 0$, then the hypotheses of (4.33) are satisfied. Hence letting

$$\ell_0 := 2\chi(\mathcal{O}_S) - \tfrac{1}{2}K \cdot H + \tfrac{1}{2}H^2 - 1 .$$

one gets

(4.34) *Let notation be as above. Assume that $c_1(K) = kc_1(H)$ for $k \gg 0$. Then*

1. *The moduli space $\mathcal{M}_{\xi(\ell_0)}$ is not good, and*
2. $\Delta_{\xi(\ell_0)} = 2\chi(\mathcal{O}_S) - \tfrac{1}{2}K \cdot H + \tfrac{1}{4}H^2 - 1.$

By considering surfaces with arbitrarily large k, we conclude that a bound of the form given in Corollaries C′ – C″ is the best we can hope for.

5. Irreducibility

We will first derive Theorem D from Theorem A and Corollary B′; the argument is due to Gieseker–Li [GL1]. Then we will obtain Theorem E by making explicit Gieseker–Li's proof in the case considered by the theorem.

Proof of Theorem D. We will prove the following

(5.1) Proposition. *Let (S,H) be a polarized surface. For any integer $r \geqq 2$ and any line bundle M on S, there exists a number $\Delta_3(r,M,S,H)$ such that the following holds. Let ξ be a set of sheaf data with* $\det_\xi \cong M$, *and such that*

$$\Delta_\xi > \Delta_3(r_\xi, M, S, H)\,. \tag{5.2}$$

Then $\mathcal{M}_\xi$ is irreducible (and the generic point parametrizes a μ-stable vector bundle).

This result implies Theorem D. In fact, since tensorization by a line bundle N identifies $\mathcal{M}_\xi$ with $M_{\xi\otimes N}$ (with the obvious notation), Theorem D will hold if we set

$$\Delta_3(r,S,H) := \max\{\Delta_3(r,M,S,H)\}_{M\in\mathcal{S}}\,,$$

where $\mathcal{S}$ is any set of line bundles on S whose first Chern classes form a complete set of representatives for the finite group $H^{1,1}_{\mathbf{Z}}(S)/r_\xi H^{1,1}_{\mathbf{Z}}(S)$. The following lemma is the key ingredient in the proof of Proposition (5.1).

(5.3) Lemma. *Let (S,H) be a polarized surface. For any integer $r \geqq 2$ there exists a number $\hat{\Delta}_1(r,S,H)$ (with $\hat{\Delta}_1 \geqq \Delta'_1$, where Δ'_1 is as in Corollary B') such that the following holds. Let ξ, ℓ be a set of sheaf data and a positive integer respectively, such that*

$$\Delta_\xi \geqq \hat{\Delta}_1(r_\xi,S,H) + \ell\,.$$

Set

$$\xi' := (r_\xi,\ \det_\xi, c_2(\xi) - \ell)\,,$$

Let X be an irreducible component of $\mathcal{M}_\xi$. Then there exist a locally closed **non-empty** *subset $Y \subset \partial X$ and an open subset V of an irreducible component of $\mathcal{M}_{\xi'}$, with the following properties. If $[E] \in V$ then E is locally-free, μ-stable, and*

$$h^0(E, E\otimes K)^0 = 0\,.$$

Furthermore $[F] \in Y$ if and only if F fits into an exact sequence

$$0 \to F \to E \xrightarrow{\phi} \bigoplus_{i=1}^{\ell} \mathbf{C}_{P_i} \to 0\,, \tag{5.4}$$

for some $[E] \in V$ (here $\{P_1,\dots,P_\ell\}$ is a set of distinct points of S).

Proof. First one proves the lemma in the case $\ell = 1$, then the general case follows easily from this. The case $\ell = 1$ follows from Theorem A, Corollary B′ and dimension counts (use Proposition (4.3) and Theorem (4.5)). The argument is similar to that used in the proof of Corollary B′; we leave the details to the reader. QED

Now we are ready to prove Proposition (5.1). Set $r = r_\xi$. Choose a set of sheaf data

$$\xi_0 = (r, M, c_2(\xi_0)) ,$$

such that

$$\Delta_{\xi_0} \geqq \hat{\Delta}_1(r_\xi, S, H) \text{ with } c_2(\xi_0) \text{ minimal} . \tag{5.5}$$

Notice that, since by [HL, LQ] the moduli space $\mathscr{M}_\xi$ is non-empty for $\Delta_\xi \gg 0$, Lemma (5.3) shows that also $\mathscr{M}_{\xi_0}$ is non-empty. Let $U_{\xi_0} \subset \mathscr{M}_{\xi_0}$ be the open subset parametrizing μ-stable locally-free sheaves E such that

$$h^0(E, E \otimes K)^0 = 0 .$$

Since $\hat{\Delta}_1 \geqq \Delta'_1$, the subset U_{ξ_0} is dense in $\mathscr{M}_{\xi_0}$, and hence non-empty. There exists an integer n such that for all $[E] \in U_{\xi_0}$ the bundle $E(nH)$ has $(r-1)$ independent sections, and such that the degeneracy locus of the corresponding map

$$\mathcal{O}_S^{(r-1)} \to E(nH)$$

is (at most) zero-dimensional. (For example if $E(nH)$ is generated by global sections.) Hence if $[E] \in U_{\xi_0}$ then E fits into an exact sequence

$$0 \to \mathcal{O}_S(-nH)^{(r-1)} \to E \to I_Z \otimes M \otimes [(r-1)nH] \to 0 , \tag{5.6}$$

where I_Z is the ideal sheaf of a zero-dimensional subscheme Z. We can, and will, assume that

$$h^1(M \otimes [rnH + K]) = 0 . \tag{5.7}$$

Set

$$\Delta_3(r, M, S, H) := \hat{\Delta}_1(r, S, H) + h^0(M \otimes [rnH + K]) + 1 . \tag{5.8}$$

We will prove that Proposition (5.1) holds with this value of Δ_3. Thus we assume that Δ_ξ satisfies Inequality (5.2). Set

$$\ell := [\Delta_\xi - \hat{\Delta}_1] . \tag{\#}$$

Let X be an irreducible component of $\mathscr{M}_\xi$. By Lemma (5.3) there exists $[E] \in U_{\xi_0}$ such that X contains all the isomorphism classes of sheaves F fitting into (5.4), with E the chosen vector-bundle, and ℓ given by (#). Since E fits into Exact Sequence (5.6), if we choose ϕ appropriately we can arrange that $F := \ker \phi$ fit into the exact sequence

$$0 \to \mathcal{O}_S(-nH)^{(r-1)} \to F \to I_W \otimes M \otimes [(r-1)nH] \to 0 , \tag{$*$}$$

with $W = Z \cup \{P_1, \ldots, P_s\}$, for $s = (\Delta_\xi - \Delta_{\xi_0})$. Furthermore, since by hypotheses

$$\Delta_\xi - \Delta_{\xi_0} > h^0(M(rnH + K)) ,$$

we can also assume (using (5.7)) that

$$h^1(I_W \otimes M \otimes [rnH + K]) = \ell(W) - h^0(M \otimes [rnH + K]),$$

and hence by Serre duality

$$\begin{aligned}&\dim \operatorname{Ext}^1(I_W \otimes M \otimes [(r-1)nH], \mathcal{O}_S(-nH)^{(r-1)})\\ &\quad = (r-1) \cdot (\ell(W) - h^0(M \otimes [rnH + K])). \qquad (\dagger)\end{aligned}$$

Now let Σ_ℓ be the space parametrizing non-trivial extensions $(*)$, such that $(\dagger)$ holds. Then Σ_ℓ fibres over a non-empty open subset of $\mathrm{Hilb}^\ell(S)$, with projective spaces as fibres. In particular Σ_ℓ is irreducible. Since E is μ-stable we conclude that the subset $\Sigma_\ell^\mu \subset \Sigma_\ell$ parametrizing μ-stable extensions is non-empty. Let $\Omega_\ell \subset \mathcal{M}_\xi$ be the image of Σ_ℓ^μ under the classifying map; since Σ_ℓ is irreducible, so is Ω_ℓ. We have proved that X contains a point $[F] \in \Omega_\ell$. Since $h^0(F^{**}, F^{**} \otimes K)^0$ vanishes, $\mathcal{M}_\xi$ is smooth at $[F]$. Hence we conclude that X contains all of Ω_ℓ. To sum up: every irreducible component of $\mathcal{M}_\xi$ contains the irreducible (non-empty) subset Ω_ℓ, and $\mathcal{M}_\xi$ is smooth at the generic point of Ω_ℓ. This implies that $\mathcal{M}_\xi$ is irreducible.

(5.9) Remark. *The above proof works also if we only assume that for the* **generic** $[E]$ *in any irreducible component of* U_{ξ_0}, E *fits into* (5.6).

Complete intersections with Picard number one

The goal of this subsection is to prove the following

(5.10) Proposition. *Let* (S, H) *be as in the statement of Theorem E. Assume also that* $\mathrm{Pic}(S) = \mathbf{Z}[H]$. *Then the conclusion of Theorem E holds for* (S, H).

We begin by giving an explicit value for $\hat{\Delta}_1$ for rank-two sheaves.

(5.11) Proposition. *Let* (S, H) *be a polarized surface. Assume that* H *is effective and that the linear system* $|n_0 H|$ *contains a smooth curve, where* n_0 *is defined by* (3.13). *Then Lemma* (5.3) *holds* (*in rank two*) *with*

$$\hat{\Delta}_1(2, S, H) := \Delta_2(S, H) + 2h^0(2K),$$

where $\Delta_2(S, H)$ *is defined by* (3.12).

Proof. First we prove that (5.3) holds for $\ell = 1$. Thus we assume that $\Delta_\xi \geqq (\hat{\Delta}_1 + 1)$; in particular S, H and ξ satisfy the hypotheses of Proposition (4.18). Let X be an irreducible component of $\mathcal{M}_\xi$. Let $[F_0] \in \partial X^\mu$ be such that Proposition (4.18) holds with $F = F_0$. Let $\tilde{Y}_1 \subset \partial X^\mu$ be the irreducible component of the stratum of the double dual stratification which contains $[F_0]$. Let $Y_1 \subset \tilde{Y}_1$ be the open subset parametrizing sheaves F such that

$$h^0(F^{**}, F^{**} \otimes K)^0 = 0.$$

The subset Y_1 is non-empty because $[F_0] \in Y_1$. We have $Y_1^{**} \subset \mathcal{M}_{\xi'}$, where

$$\xi' = (2, \det_\xi, c_2(\xi')), \quad c_2(\xi') < c_2(\xi) . \tag{\sharp}$$

By Proposition (4.3) and Theorem (4.5)

$$\dim Y_1^{**} + 3(c_2(\xi) - c_2(\xi')) \geqq \dim Y_1 \geqq \dim X - 1 . \tag{*}$$

By (4.19) both $\mathcal{M}_\xi$ and $\mathcal{M}_{\xi'}$ are good at $[F_0]$ and $[F_0^{**}]$ respectively. Hence $(*)$ gives

$$\Delta_{\xi'} \geqq \Delta_\xi - 1 . \tag{\dagger}$$

By $(\sharp)$ we conclude that $c_2(\xi') = (c_2(\xi) - 1)$. Thus $(\dagger)$ is an equality, and hence also all the inequalities in $(*)$. The result is that $\mathcal{M}_{\xi'}$ is good (by Proposition (4.18)) and furthermore

$$\dim Y_1^{**} = \dim \mathcal{M}_{\xi'} . \tag{**}$$

Now set $V := Y_1^{**}$. By $(**)$ V is an open subset of $\mathcal{M}_{\xi'}$. By construction, if $[F] \in Y_1$ then F fits into an exact sequence (5.4) for some $[E] \in V$ (with $\ell = 1$). Conversely, since the inequalities of $(*)$ are equalities, if $[F]$ is the generic sheaf fitting into (5.4) for $[E] \in V$ (with $\ell = 1$) then $[F] \in Y_1$. We conclude that

$$Y := \{[F] \in \mathcal{M}_\xi |\ F \text{ fits into (5.4) for some } [E] \in V, \text{ with } \ell = 1\}$$

is contained in the closure of Y_1, and hence $Y \subset X$. This proves Lemma (5.3) if $\ell = 1$. When $\ell > 1$ one iterates this construction. Define $Y_1 \subset \partial X$ as above. Let $X_2 := Y_1^{**}$, and define $Y_2 \subset \partial \overline{X}_2^\mu$ in the same way as we defined $Y_1 \subset \partial X^\mu$. We continue this process up to Y_ℓ (that this is possible is guaranteed by Proposition (4.18)). Set $V := Y_\ell^{**}$, and let $Y \subset \mathcal{M}_\xi$ be the parameter space for all sheaves F fitting into (5.4) for $[E] \in V$. One checks easily that $Y \subset \partial X$ and hence that Lemma (5.3) holds. QED

(5.12) Corollary. *Let* (S, H) *be a polarized surface satisfying the hypotheses of Theorem E. Then Lemma* (5.3) *holds with*

$$\hat{\Delta}_1(2, S, H) = 17K^2 + 10\chi(\mathcal{O}_S) .$$

Proof. Immediate from Proposition (5.11) together with (3.16) and (4.27). QED

Set

$$\xi_0 := (2, \mathcal{O}_S, 17K^2 + 10\chi(\mathcal{O}_S)) .$$

Then Δ_{ξ_0} satisfies (5.5).

(5.13) Lemma. *Keep notation as above. Let* (S, H) *be a polarized surface satisfying the hypotheses of Proposition* (5.10). *Let* $X \subset \mathcal{M}_{\xi_0}$ *be an irreducible component. There is an open dense subset* $U \subset X$, *parametrizing locally-free sheaves, such that if* $[E] \in U$ *then* E *fits into an exact sequence*

$$0 \to \mathcal{O}_S(-6K) \to E \to I_Z \otimes [6K] \to 0 , \tag{5.14}$$

where Z is a zero-dimensional subscheme of S of length

$$\ell(Z) = 53K^2 + 10\chi(\mathcal{O}_S)\,.$$

Proof. We will prove that E fits into (5.14); once this is done the length of Z is obtained by computing $c_2(E)$. We begin by showing that if $[E] \in \mathcal{M}_{\xi_0}$ and E is locally-free, then $E \otimes [5K]$ fits into an exact sequence

$$0 \to \mathcal{O}_S(nH) \to E \otimes [5K] \to I_W \otimes [10K - nH] \to 0\,, \qquad (*)$$

where W is a zero-dimensional subscheme of S, and $n \geqq 0$. In fact by Serre duality and μ-semistability we have

$$h^2(E \otimes [5K]) = h^0(E \otimes [-4K]) = 0\,,$$

hence $h^0(E \otimes [5K]) \geqq \chi(E \otimes [5K])$. Applying the Hirzebruch–Riemann–Roch Theorem one gets that

$$\chi(E \otimes [5K]) = 3K^2 - 8\chi(\mathcal{O}_S)\,.$$

The right-hand side is positive for $k \gg 0$, e.g. if $k > 100$, and hence $h^0(E \otimes [5K]) > 0$. (Here k is as in the statement of Theorem E.) That E fits into $(*)$ follows from this and the hypothesis that $\mathrm{Pic}(S) = Z[H]$ (in fact this is the only place where we use this assumption). We need to bound n. Of course by semistability we have $n \leqq 5k$; we will show that if $[E]$ is generic there is a better bound.

Claim. *Assume that $[E] \in X$ is generic. Then $n < 4k$.*

Proof of the claim. Assume that for generic $[E] \in X$ the sheaf $E \otimes [5K]$ fits into $(*)$ with $n \geqq 4k$. Then

$$\dim \mathcal{M}_{\xi_0}\left(\frac{K \cdot H}{\sqrt{H^2}}\right) = 4\Delta_{\xi_0} - 3\chi(\mathcal{O}_S)\,.$$

In fact this follows by writing

$$\mu_{[nH]} = 5K \cdot H - \alpha\sqrt{H^2} \quad \text{for } \alpha \leqq \frac{K \cdot H}{\sqrt{H^2}}\,.$$

Applying Proposition (1.3) we get

$$3\frac{(K \cdot H)^2}{H^2} + \frac{(K \cdot H)^2 + 2(K \cdot H)H^2 + 2K \cdot H}{H^2} + \frac{3((K \cdot H) + H^2 + 1)^2}{2H^2} + 3 \geqq 17K^2 + 10\chi(\mathcal{O}_S)\,.$$

As is easily verified this is false as soon as $k > 0$, and hence we conclude that $n < 4k$ for generic $[E] \in X$ (with E locally-free). QED

Now we are ready to show that if $[E] \in X$ is generic and E is locally-free, then $E \otimes [6K]$ has a section with isolated zeroes. By the above claim the vector-bundle $E \otimes [6K]$ fits into an exact sequence

$$0 \to \mathcal{O}_S(K + nH) \to E \otimes [6K] \to I_W \otimes [11K - nH] \to 0\,,$$

with $n < 4k$. First we show that

$$H^0(I_W \otimes [11K - nH]) \neq 0\,.$$

For this it suffices to prove that

$$h^0(\mathcal{O}_S(11K - nH)) > \ell(W)\,. \qquad (\dagger)$$

The left-hand side equals $\chi(\mathcal{O}_S(11K - nH))$, hence is computed by Hirzebruch–Riemann–Roch. An easy computation then gives that for ($\dagger$) to hold we need that

$$(13k^2 - 2k - 2)H^2 > 9\chi(\mathcal{O}_S)\,.$$

This inequality is satisfied as soon as $k > 2$. Let

$$\sigma \in H^0(I_W \otimes [11K - nH])$$

be a non-zero section; it lifts to section $\tilde{\sigma}$ of $E \otimes [6K]$ because $h^1(K + nH) = 0$. If τ is any section of $\mathcal{O}_S(K + nH)$, then

$$(\tau + \tilde{\sigma}) \subset (\sigma)\,,$$

where $(\,\cdot\,)$ denotes "zero-locus". Since $\mathcal{O}_S(K + nH)$ is very ample one easily concludes that there exists τ such that $\theta := (\tau + \tilde{\sigma})$ is section with isolated zeroes. Thus E fits into

$$0 \to \mathcal{O}_S \xrightarrow{\theta} E \otimes [6K] \to I_Z \otimes [12K] \to 0\,,$$

where $Z = (\theta)$. Tensoring the above exact sequence with $\mathcal{O}_S(-6K)$ one obtains (5.14). QED

Proof of Proposition (5.10). By Remark (5.9), Formula (5.8), Lemma (5.13) and Corollary (5.12), we can set

$$\Delta_3(2, \mathcal{O}_S, S, H) = \hat{\Delta}_1 + h^0(13K) + 1\,,$$

where the value of $\hat{\Delta}_1$ is given by Corollary (5.12). Proposition (5.10) follows at once.

Proof of Theorem E. Let S be a surface satisfying the hypotheses of Theorem E. Then

$$S = V_1 \cap \cdots \cap V_n \subset \mathbf{P}^{n+2}\,,$$

where V_i is a degree-d_i hypersurface. Let

$$\rho : \mathcal{S} \to B$$

be the family of smooth complete intersections of n hypersurfaces in $\mathbf{P}^{n+2}$ of degrees of $d_1 \ldots, d_n$. Thus B is an open subset of a Grassmannian, and $S = \rho^{-1}(b_0)$ for a certain $b_0 \in B$. If $b \in B$ we set $S_b := \rho^{-1}(b)$. We let $B_1 \subset B$ be the subset parametrizing surfaces whose Picard group is generated the hyperplane class. We recall the following

(5.15) Noether–Lefschetz Theorem. *Keep notation as above. Assume that $p_g(S_b) > 0$ for $b \in B$. Then B_1 is dense in B.*

For $c \in \mathbf{Z}$ let $\mathscr{M}_c(S_b)$ be the moduli space of torsion-free sheaves F on S_b, semistable with respect to $\mathcal{O}_{S_b}(1)$, with

$$r_F = 2, \qquad \det F \cong \mathcal{O}_{S_b}, \qquad c_2(F) = c\,.$$

By Maruyama [Ma] there exists a relative moduli space

$$\pi\colon \mathscr{M}_c(\mathscr{S}) \to B$$

proper over B, such that $\pi^{-1}(b) \cong \mathscr{M}_c(S_b)$ for all $b \in B$.

(5.16) Proposition. *Keep notation as above. Assume that the integer k such that $K_{S_b} \sim kH$ is large (e.g. $k > 100$). If*

$$c > 95K_{S_b}^2 + 11\chi(\mathcal{O}_{S_b}) + 1$$

then $\mathscr{M}_c(\mathscr{S})$ is irreducible.

Proof. For $b \in B$ let $\mathscr{M}_c^0(S_b) \subset \mathscr{M}_c(S_b)$ be the (open) subset parametrizing sheaves F such that

1. F is locally-free and stable,
2. $h^0(F, F \otimes K_{S_b})^0 = 0$.

Let $\mathscr{M}_c^0(\mathscr{S}) := \bigcup_{b\in B} \mathscr{M}_c^0(S_b)$. By Corollary C″ $\mathscr{M}_c^0(S_b)$ is dense in $\mathscr{M}_c(S_b)$ for all b, and hence $\mathscr{M}_c^0(\mathscr{S})$ is dense in $\mathscr{M}_c(\mathscr{S})$. Thus it suffices to show that $\mathscr{M}_c^0(\mathscr{S})$ is irreducible. Let X be anyone of its irreducible components. Let $X^* \subset X$ be the complement of the intersection with all other irreducible components. We claim that $\pi(X^*)$ contains an open non-empty subset of B. Since $\pi(X^*)$ is constructible it suffices to prove that it is not contained in any proper subvariety of B. Let $b \in \pi(X^*)$, and let $v \in T_b(B)$. By deformation theory (see for example Proposition 2.1 in [G2]) there exist a curve $\imath : \Lambda \hookrightarrow X^*$ and a point $P \in \Lambda$ such that $\imath(P) = [F]$ and

$$v \in \operatorname{Im} D(\pi \circ \imath)(P)\,,$$

where D is the differential. This proves that $\pi(X^*)$ is not contained in any proper subvariety of B, and hence it contains an open non-empty subset. Now let Y be another irreducible component of $\mathscr{M}_c^0(\mathscr{S})$. Then

$$X^* \cap Y^* = \emptyset\,. \tag{$*$}$$

Since $\pi(X^*)$ and $\pi(Y^*)$ both contain a Zariski-open non-empty subset of the irreducible variety B, we conclude by (5.15) that there exists

$$b_0 \in \pi(X^*) \cap \pi(Y^*)$$

such that $\operatorname{Pic}(S_{b_0}) \cong \mathbf{Z}[\mathcal{O}_{S_{b_0}}]$. By Proposition (5.10) the moduli space $\mathscr{M}_c(S_{b_0})$ is irreducible. Since both X^* and Y^* must contain an open non-empty subset of $\mathscr{M}_c(S_{b_0})$, we conclude that

$$X^* \cap Y^* \neq \emptyset\,.$$

This contradicts ($*$), and hence $\mathscr{M}_c^0(\mathscr{S})$ is irreducible. QED

(5.17) Corollary. *Keep assumptions as in Proposition* (5.16). *Then* $\mathcal{M}_c(S_b)$ *is connected for all* $b \in B$.

Proof. Let $b \in B_1$. By Proposition (5.10), $\pi^{-1}(b) = \mathcal{M}_c(S_b)$ is irreducible, hence connected. Since B_1 is dense in B (by (5.15)), and since $\mathcal{M}_c(\mathcal{S})$ is irreducible (Proposition (5.16)) and proper over B, we conclude that $\mathcal{M}_c(S_b)$ is connected for all $b \in B$. QED

Now we are ready to prove Theorem E. Let S and H be as in the statement of the theorem, and let c be an integer such that

$$c > 95K^2 + \chi(\mathcal{O}_S) + 1 . \tag{5.18}$$

We must prove that $\mathcal{M}_c(S)$ is irreducible.

Claim. *Keep notation and assumptions as above. Suppose also that* $\mathcal{M}_c(S)$ *is reducible. Then there exist two irreducible components* X_1, X_2 *such that* $X_1 \cap X_2$ *contains a point parametrizing a stable sheaf.*

Proof. By Corollary (5.17) there exist two irreducible components X_1, X_2 such that their intersection is non-empty. Let $[F] \in X_1 \cap X_2$. If F is stable there is nothing to prove, so assume F is non-stable. Let $E := \mathrm{Gr}(F)$. By Claim (2.22) the natural map

$$\lambda : \mathrm{Def}(E) \to \mathcal{M}_c(S)$$

surjects onto a neighborhood of $[F]$. (Since S is regular, $\mathrm{Def}^0(E) = \mathrm{Def}(E)$.) Hence $\lambda^{-1}X_i$ is a closed non-empty subset of $\mathrm{Def}(E)$, and

$$\dim \lambda^{-1}X_i \geqq \dim X_i = 4c - 3\chi(\mathcal{O}_S) . \tag{$*$}$$

On the other hand the dimension of the tangent space to Def(E) at the origin is given by

$$\dim T_0(\mathrm{Def}^0(E)) = h^1(E,E) = -\chi(E,E) + h^0(E,E) + h^0(E,E\otimes K) .$$

By Lemma (1.7) and by (1.6) we conclude that

$$\dim T_0 \,\mathrm{Def}(E) \leqq 4c - 3\chi(\mathcal{O}_S) + 3 + \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2 .$$

Thus, by Inequality ($*$), we have

$$\dim(\lambda^{-1}X_1 \cap \lambda^{-1}X_2) \geqq 4c - 3\chi(\mathcal{O}_S) - 3 - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2 .$$

As is easily checked

$$4c - 3\chi(\mathcal{O}_S) - 3 - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2 > 3c + \varepsilon(2, S, H) ,$$

if $k > 0$. Hence by Proposition (1.10) there exists a point in $x \in \lambda^{-1}X_1 \cap \lambda^{-1}X_2$ parametrizing a μ-stable sheaf. Then $\lambda(x) \in X_1 \cap X_2$ parametrizes a stable sheaf. QED

So let's assume that $\mathcal{M}_c(S)$ is reducible. By the above claim there exist two irreducible components X, Y, of $\mathcal{M}_c(S)$ and a point $[F]$ in their intersection such that F is stable. By (0.3) we conclude that

$$\dim_{[F]} X \cap Y \geqq 4c - 3\chi(\mathcal{O}_S) - h^0(F, F \otimes K)^0 .$$

Applying (1.6) we get

$$\dim_{[F]} X \cap Y \geqq 4c - 3\chi(\mathcal{O}_S) - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2 . \qquad (\dagger)$$

Since $X \cap Y$ is in the singular locus of $\mathcal{M}_c(S)$ and since F is stable, we have

$$\dim_{[F]} X \cap Y \leqq W_\xi^K ,$$

where $\xi = (2, \mathcal{O}_S, c)$. By Theorem B and by ($\dagger$) we get

$$4c - 3\chi(\mathcal{O}_S) - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2 \leqq \lambda_2 c + \lambda_1 \sqrt{c} + \lambda_0' + e_K .$$

A straightforward computation shows that this is impossible if c satisfies (5.18) and k is large, for example $k > 100$. This proves that $\mathcal{M}_c(S)$ is irreducible.

6. Proof of the technical results

We will prove the technical results stated in the first section. Let H be an ample divisor on the surface S. Unless otherwise stated, stability and μ-stability of sheaves on S is with respect to H.

Proof of (1.6), (1.7), (1.8). The key ingredient is provided by a certain bound on the number of sections of a semistable sheaf. This bound, in a more general context, is due to Simpson [S, Corollary (1.7)]. In the case of surfaces there is an effective version given by Le Potier [LP, Théorème (4.5)], and this is what we will use. For the reader's convenience we reproduce Le Potier's proof. First we need the following

Lemma. *Let C be a smooth connected projective curve, and let F be a semi-stable vector-bundle on C. Then*

$$h^0(F) \leqq r_F[\mu_F + 1]_+ ,$$

where $[x]_+ := \max\{x, 0\}$.

Proof. It suffices to show that

$$h^0(F) \leqq r_F(\mu_F + 1) = c_1(F) + r_F \qquad (*)$$

for $c_1(F) \geqq -r_F$. The proof is by induction on $c_1(F)$. If $c_1(F) < 0$ then $(*)$ holds by semistability. Now assume $c_1(F) \geqq 0$, and let $P \in C$. Considering the exact sequence

$$0 \to F(-P) \to F \to F|_P \to 0$$

we get $h^0(F) \leqq h^0(F(-P)) + r_F$. Applying the inductive hypothesis to $F(-P)$ we get that $(*)$ holds. QED

(6.1) Proposition (Simpson–Le Potier). *Keeping notation as above, assume that H satisfies* (0.4)*. Let G be a μ-semistable torsion-free sheaf on S. Then*

$$h^0(G) \leqq \frac{r_G}{2H^2}(\mu_G + (r_G+1)H^2 + 1)^2 .$$

Proof. Let $C \in |H|$ be a generic curve and let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = G|_C$$

be the Harder–Narasimhan filtration of $G|_C$. Let $\mathrm{Gr}_i := F_i/F_{i-1}$ and set

$$\mu_i := \mu(\mathrm{Gr}_i) \qquad r_i := \text{ rank of } \mathrm{Gr}_i .$$

By the previous lemma we have

$$h^0(G|_C) \leqq \sum_{i=1}^{\ell} h^0(\mathrm{Gr}_i) \leqq \sum_{i=1}^{\ell} r_i[\mu_i + 1]_+ . \qquad (\dagger)$$

Since H satisfies (0.4) one can apply a theorem of Flenner [Fl, Theorem (1.4)] stating, in this case, that

$$\mu_i - \mu_{i+1} \leqq H^2 \quad \text{for } i = 1, \ldots, \ell .$$

Thus $\mu_i \leqq \mu(G|_C) + r_G H^2$ for all i. By $(\dagger)$ we conclude that

$$h^0(G|_C) \leqq r_G[\mu(G|_C) + r_G H^2 + 1]_+ = r_G[\mu_G + r_G H^2 + 1]_+ .$$

Replacing G by $G(-j) := G \otimes [-jH]$ we get

$$h^0(G(-j)|_C) \leqq r_G[\mu_G + (r_G - j)H^2 + 1]_+ .$$

Since G is torsion-free $h^0(G(-j)) = 0$ for $j \gg 0$, and thus we have

$$h^0(G) \leqq \sum_{j=0}^{\infty} h^0(G(-j)|_C) \leqq r_G \sum_{j=0}^{\infty} [\mu_G + (r_G - j)H^2 + 1]_+ .$$

The last term on the right is bounded above by

$$r_G \int_{-1}^{x_1} (\mu_G + (r_G - x)H^2 + 1)dx, \quad \text{where } x_1 := r_G - 1 + (\mu_G + 1)/H^2 .$$

Computing the integral one gets the inequality of the proposition. QED

(6.2) Corollary. *Assume that H satisfies* (0.4)*. Let F, G be μ-semistable torsion-free sheaves on S. Then*

$$h^0(F, G) \leqq \frac{r_F \cdot r_G}{2H^2}(\mu_G - \mu_F + (r_F \cdot r_G + 1)H^2 + 1)^2 .$$

Proof. Since F and G are both torsion-free μ-semistable sheaves on S, so is $Hom(F,G)$. The corollary follows by applying Proposition (6.1) to $\mathrm{Hom}(F,G)$. QED

Proof of (1.6). The result follows from Corollary (6.2) with $G = F \otimes L$.

Proof of (1.7). The proof is by double induction on the lengths of μ-Jordan–Hölder filtrations for A and B. If the lengths are both one, then A and B are stable, hence $h^0(A,B) \leqq 1$, and the result is true. To prove the inductive step, let $A_1 \subset A$ and $B_1 \subset B$ be the first terms of Jordan–Hölder filtrations. Then

$$h^0(A,B) \leqq 1 + r_{A_1}(r_B - r_{B_1}) + r_{B_1}(r_A - r_{A_1}) + (r_A - r_{A_1})(r_B - r_{B_1}) .$$

The lemma follows by simplifying the right-hand side.

Proof of (1.8). The result follows immediately from the following

Proposition. *Let L be a line bundle on S. Let F be a rank-two torsion-free sheaf on S such that*

$$h^0(F, F \otimes L)^0 > h^0(L^{\otimes 2}) .$$

Then F is not *$(L \cdot H/2\sqrt{H^2})$-stable, i.e. there exists a rank-one subsheaf $A \subset F$ such that*

$$\mu_A \geqq \mu_F - \frac{L \cdot H}{2} .$$

Proof. Replacing F by F^{**} we can assume that F is locally-free. Let

$$\phi\colon\ H^0(F, F \otimes L)^0 \to H^0(L^{\otimes 2})$$

be the map defined by $\phi(\sigma) := \det \sigma$. Since $0 \in \phi^{-1}(0), \phi^{-1}(0)$ is non-empty, and hence by our hypothesis $\dim \phi^{-1}(0) > 0$. Thus there exists a non-zero map

$$f\colon\ F \to F \otimes L$$

with $\det f = \operatorname{tr} f = 0$, i.e. $f \circ f = 0$. The kernel of f is a rank-one torsion-free subsheaf of F, which we denote by A. We have

$$0 \to A \to F \xrightarrow{q} B \to 0 .$$

The sheaf B is also torsion-free, because it is isomorphic to $\operatorname{Im} f \subset F \otimes L$. Since $f \circ f = 0$, the map f is obtained as the composition

$$F \xrightarrow{q} B \xrightarrow{g} A \otimes L ,$$

for some non-zero map g. Thus

$$\mu_B - \mu_A \leqq \mu_L .$$

This implies the proposition. QED

Proof of (1.2), (1.3), (1.4). Let $[F] \in \mathcal{M}_\xi(\alpha)$, and let

$$0 \to A \to F \xrightarrow{f} B \to 0 . \qquad (6.3)$$

be an α-destabilizing sequence. Replacing A by $f^{-1}(\mathrm{Tor}(B))$, if necessary, we can assume that A and B are torsion-free. The inequalities

$$\mu_F - \frac{\alpha}{r_A}\sqrt{H^2} \leqq \mu_A \leqq \mu_F$$

are equivalent to the existence of $0 \leqq \alpha_0 \leqq \alpha$ such that

$$\mu_A = \mu_F - \frac{\alpha_0}{r_A}\sqrt{H^2} \, .$$

Hence we must bound the dimension of the open subset of the Quot-scheme parametrizing Exact sequences (6.3), where μ_A, α_0 are as above. A bound for this dimension is given by

$$h^1(A,A) + h^1(B,A) + h^1(B,B) \, .$$

If A, B are μ-semistable one bounds each h^1 above by expressing it in terms of the corresponding Euler characteristic and two h^0's (Serre duality), and applying Simpson's bound (6.2) to the h^0's. However we do not know that for the generic extension (6.3) the sheaves A, B are μ-semistable. All we can assume is that either A or B is μ-semistable: simply replace A (or B) by the first subsheaf (respectively the last quotient) of its H-N filtration. So let's assume A is μ-semistable: the H-N filtration of B induces a filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n+1} = F \qquad (6.4)$$

with $A = F_1, \ B = F/A$. Let

$$Q_i := F_i/F_{i-1}, \qquad r_i := r_{Q_i}, \qquad \mu_i := \mu_{Q_i}, \qquad \Delta_i := \Delta(Q_i) \, . \qquad (6.5)$$

By definition the above filtration satisfies the following:

(6.6)
1. *Q_i is torsion-free and μ-semistable, and*
2. $\mu_2 > \mu_3 > \cdots > \mu_{n+1}$.

As discussed above, in order to bound the dimension of $\mathcal{M}_\xi(\alpha)$ we need to estimate a certain sum of h^1's. This is done in the two propositions below: they are proved by writing down Euler characteristics, applying Simpson's bound (6.2) to take care of the h^0, h^2, and using Bogomolov's inequality.

(6.7) Proposition. *Keep notation as above. Assume that H satisfies* (0.4). *Suppose that F is μ-semistable and that* (6.6) *holds. Define α by setting*

$$\mu_A = \mu_F - \frac{\alpha}{r_A}\sqrt{H^2} \, ,$$

and let $T(r_F, S, H)$ *and* $s(r_F)$ *be as in* (1.1). *If* $K \cdot H \geqq 0$, *then*

$$\sum_{i \leqq j} h^1(Q_j, Q_i) \leqq (2r_F - 1)\Delta_F + \frac{1}{2}\left(1 + \frac{1}{r_A}\right)(2r_F - r_A)\alpha^2$$
$$+ [r_B(r_B - 1) + r_F]\frac{K \cdot H + 2s(r_F)}{2\sqrt{H^2}}\alpha$$
$$+ r_B(r_B^3 - r_B^2 - r_F r_B + r_F^2)\frac{(K \cdot H)^2}{8H^2} + (r_B^3 + r_A r_F)\frac{s(r_F)^2}{2H^2}$$
$$+ (r_F^2 - r_A r_B)\frac{(K \cdot H + s(r_F))^2}{2H^2} + T(r_F, S, H).$$

If $K \cdot H < 0$, *then*

$$\sum_{i \leqq j} h^1(Q_j, Q_i) \leqq (2r_F - 1)\Delta_F + \frac{1}{2}\left(1 + \frac{1}{r_A}\right)(2r_F - r_A)\alpha^2$$
$$+ \left\{[r_F + r_B(r_B - 1)]\frac{K \cdot H + 2s(r_F)}{2\sqrt{H^2}} - r_F(r_B - 1)\frac{K \cdot H}{\sqrt{H^2}}\right\}\alpha$$
$$+ [r_F r_B(r_B - 1)(r_F + 2r_A) + r_A r_B^2(r_A + 1)]\frac{(K \cdot H)^2}{8H^2}$$
$$+ (r_B^3 + r_A r_F)\frac{s(r_F)^2}{2H^2} + (r_F^2 - r_A r_B)\frac{(K \cdot H + s(r_F))^2}{2H^2}$$
$$+ \frac{r_F^2}{2}(1 - 2\chi(\mathcal{O}_S)) + \frac{r_F^3}{8}|K^2|.$$

(6.8) Proposition. *Keeping notation as above, assume that* H *is effective. Suppose that* F *is a rank-two* μ*-semistable sheaf with a filtration* (6.4), *and that* (6.6) *is satisfied. Define* α *by setting*

$$\mu_A = \mu_F - \frac{\alpha}{r_A}\sqrt{H^2}.$$

Then

$$\sum_{i \leqq j} h^1(Q_j, Q_i) \leqq 3\Delta_F + 3\alpha^2 + \frac{K \cdot H + 2H^2 + 2}{\sqrt{H^2}}\alpha$$
$$+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(\mathcal{O}_S).$$

Before proving the two propositions above we show why they imply (1.2), (1.3) and (1.4).

Proof of (1.2)–(1.3) assuming (6.7)–(6.8). Let $\mathcal{F}$ be a family of sheaves on S parametrized by a scheme B. Let $\overline{\chi} := (\chi_1, \ldots, \chi_{n+1})$ be a sequence of polynomials in the variable t. Drezet–Le Potier ([DL], Subsection (1.6)) have constructed a scheme

$$\mathrm{Drap}(\mathcal{F}; \overline{\chi}) \to B,$$

proper over B, whose fiber over $x \in B$ parametrizes all filtrations (6.4) with $F = \mathscr{F}_x$, and where

$$\chi(Q_i(tH)) = \chi_i \quad i = 1, \dots, (n+1) . \tag{$\sharp$}$$

This scheme represents an obvious functor; if $n = 1$ it reduces to Grothendieck's Quot-scheme. Now let P_ξ be a parameter space for semistable sheaves of which $\mathscr{M}_\xi$ is the geometric invariant theory quotient. Thus there is a family of semistable sheaves $\mathscr{F}_\xi$ on S parametrized by P_ξ inducing a surjection $P_\xi \to \mathscr{M}_\xi$. For $\alpha \in \mathbf{R}$ let $I(\alpha)$ be the set of sequences $\overline{\chi}$ such that if F fits into (6.4) and ($\sharp$) holds, then

$$\mu_A = \mu_F - \frac{\alpha}{r_A}\sqrt{H^2} . \tag{6.9}$$

(The definition of $I(\alpha)$ makes sense because the slope of a sheaf is determined by its Hilbert polynomial.) If $\overline{\chi}$ is as above, let $\mathrm{Drap}_0(\mathscr{F}_\xi; \overline{\chi})$ be the subset of $\mathrm{Drap}(\mathscr{F}_\xi; \overline{\chi})$ parametrizing filtrations such that (6.6) is satisfied. Since each of the properties of (6.6) is open, $\mathrm{Drap}_0(\mathscr{F}_\xi; \overline{\chi})$ is an open subset of $\mathrm{Drap}(\mathscr{F}_\xi; \overline{\chi})$. Set

$$\mathrm{Drap}_0(\mathscr{F}_\xi, \alpha) := \bigcup_{\overline{\chi} \in I(\alpha)} \mathrm{Drap}_0(\mathscr{F}_\xi; \overline{\chi}) .$$

Claim. *Let notation be as above. Then* $\mathrm{Drap}_0(\mathscr{F}_\xi, \alpha)$ *is a scheme of finite type.*

Proof. Since $\mathrm{Drap}_0(\mathscr{F}_\xi, \overline{\chi})$ is of finite type for every $\overline{\chi}$, all we have to show is that

$$I^*(\alpha) := \{\overline{\chi} \in I(\alpha) | \ \mathrm{Drap}_0(\mathscr{F}_\xi; \overline{\chi}) \neq \emptyset\}$$

is a finite set. So let F be a sheaf satisfying the hypotheses of Proposition (6.7), and let Q_i be as in (6.5). It suffices to check that there is a universal bound for the size of the coefficients of the Hilbert polynomials of the Q_i, if (S, H), α, r_F, $c_1(F)$ and $c_2(F)$ are fixed. This in turn amounts to bounding the size of

$$c_1(Q_i) \cdot H , \qquad c_1(Q_i)^2 , \qquad c_1(Q_i) \cdot K , \qquad c_2(Q_i) .$$

That $c_1(Q_i) \cdot H$ is bounded follows from (6.9) and from Item (2) of (6.6). Thus, by Hodge index, we also get that $c_1(Q_i)^2$ is bounded above. Let's show that it is also bounded below. By Item (1) of (6.6) and Bogomolov's Inequality we have

$$\frac{1}{2r_i} c_1(Q_i)^2 \geqq \frac{1}{2} c_1(Q_i)^2 - c_2(Q_i) .$$

Thus

$$\sum_{i=1}^{n+1} \frac{1}{2r_i} c_1(Q_i)^2 \geqq \sum_{i=1}^{n+1} \frac{1}{2} c_1(Q_i)^2 - c_2(Q_i) = \frac{1}{2} c_1(F)^2 - c_2(F) . \tag{$*$}$$

Since the values of $c_1(Q_i)^2$ are bounded above one concludes that they are also bounded below. Since $c_1(Q_i) \cdot H$ is bounded we conclude by the Hodge index

theorem that $c_1(Q_i) \cdot K$ is bounded. Finally boundedness of $c_2(Q_i)$ follows from boundedness of $c_1(Q_i)^2$, Bogomolov's Inequality and the equality in $(*)$.
QED

Let π be the composition of the projection $\mathrm{Drap}_0(\mathscr{F}_\xi, \alpha) \to P_\xi$ and the quotient map $P_\xi \to \mathscr{M}_\xi$. By construction we have

$$\mathscr{M}_\xi(\alpha_0) = \bigcup_{0 \leqq \alpha \leqq \alpha_0} \pi(\mathrm{Drap}_0(\mathscr{F}_\xi, \alpha)) . \qquad (\flat)$$

Hence, by the claim above, $\mathscr{M}_\xi(\alpha_0)$ is a constructible set. Now let's consider $\dim \mathscr{M}_\xi(\alpha_0)$. For convenience of exposition we will assume that there is a tautological family of sheaves $\mathscr{G}_\xi$ parametrized by $\mathscr{M}_\xi$; it will be clear how to modify the argument if this is not the case. Let $x \in \mathrm{Drap}(\mathscr{G}_\xi; \overline{\chi})$ correspond to (6.4). An easy inductive argument with extensions shows that

$$\dim \mathrm{Drap}(\mathscr{G}_\xi; \overline{\chi})_x \leqq \sum_{i \leqq j} h^1(Q_j, Q_i) .$$

By $(\flat)$ we conclude that $\dim \mathscr{M}_\xi(\alpha_0)$ is bounded above by the maximum of the values of the right-hand side of the inequality in (6.7) (or in (6.8)) for $r_F = r_\xi$, $\Delta_F = \Delta_\xi$, $0 \leqq \alpha \leqq \alpha_0$, and $1 \leqq r_A \leqq r_\xi$. This immediately gives (1.3). It also gives a slightly weaker version of (1.2). In order to get (1.2) one argues that if $[F] \in \mathscr{M}_\xi(\alpha_0)$ then at least one of F, F^{**} fits into a filtration (6.4) such that (6.6) is satisfied and furthermore $r_A \leqq r_\xi/2$. The arguments above together with Theorem (4.5) will show then that $\dim \mathscr{M}_\xi(\alpha_0)$ is bounded above by the maximum of the values of the right-hand side of the inequality in (6.7) for $r_F = r_\xi$, $\Delta_F = \Delta_\xi$, $0 \leqq \alpha \leqq \alpha_0$, and $1 \leqq r_A \leqq r_\xi/2$. Proposition (1.2) follows from this together with easy estimates.

Proof of (1.4) assuming (6.8). We keep the notation introduced in the previous proof. For simplicity of exposition we assume that there exists a tautological family $\mathscr{G}_\xi$ of sheaves parametrized by $\mathscr{M}_\xi$. Let $\mathrm{Drap}_0^C(\mathscr{G}_\xi; \alpha)$ be the subset of $\mathrm{Drap}_0(\mathscr{G}_\xi; \alpha)$ parametrizing filtrations

$$0 \to A \xrightarrow{f} F \to B \to 0 , \qquad (6.10)$$

such that the restrictions $A|_C$ and $F|_C$ are locally-free and $A|_C$ spans a destabilizing subline bundle of $F|_C$. As is easily checked $\mathrm{Drap}_0^C(\mathscr{G}_\xi; \alpha)$ is locally-closed. Since

$$\bigcup_{0 \leqq \alpha \leqq \alpha_0} \pi(\mathrm{Drap}_0^C(\mathscr{G}_\xi; \alpha)) = \mathscr{M}_\xi^C(\alpha_0) ,$$

we conclude that $\mathscr{M}_\xi^C(\alpha_0)$ is constructible. Now let's prove the upper bound for its dimension. We start by examining the map of vector bundles $f|_C$. Let Ω be its zero-locus. Since $A|_C$ spans a destabilizing subline bundle of $F|_C$, we have

$$\mu(A|_C) + \deg \Omega \geqq \mu(F|_C) . \qquad (*)$$

Define α by setting

$$\mu_A = \mu_F - \alpha\sqrt{H^2} .$$

(Thus $0 \leqq \alpha \leqq \alpha_0$.) Since $C \in |nH|$ we conclude by $(*)$ that

$$\deg \Omega \geqq n\alpha\sqrt{H^2}\,. \tag{6.11}$$

Since B is a rank-one torsion-free sheaf we have $B = I_Z \otimes B^{**}$, for a zero-dimensional subscheme Z of S. We claim that for each point $P \in \Omega$ we have

$$\operatorname{mult}_P(Z) \geqq \operatorname{mult}_P(\Omega)\,. \tag{6.12}$$

In fact, let $f = (f_1, f_2)$ be an expression for f in a neighborhood of P, so that I_Z is locally generated by f_1, f_2. Let y be a local equation for C. Then

$$\operatorname{mult}_P(Z) = \dim_{\mathbf{C}} \mathcal{O}/(f_1, f_2), \qquad \operatorname{mult}_P(\Omega) = \dim_{\mathbf{C}} \mathcal{O}/(f_1, f_2, y)\,,$$

where $\mathcal{O}$ is the local ring at P; Inequality (6.12) follows at once from this. Putting together (6.12) and (6.11) we get

$$\sum_{P \in C} \operatorname{mult}_P(Z) \geqq n\alpha\sqrt{H^2}\,. \tag{6.13}$$

To conclude the proof of Proposition (1.4) we go back to the proof that (6.8) implies (1.3). Let $x \in \operatorname{Drap}(\mathcal{G}_\xi; \alpha)$ correspond to the filtration (6.10). We argued that

$$\dim \operatorname{Drap}(\mathcal{G}_\xi; \alpha)_x \leqq h^1(A,A) + h^1(B,A) + h^1(B,B)\,. \tag{$**$}$$

Here

$$h^1(B,B) = \dim \operatorname{Pic}(S) + \dim \operatorname{Hilb}^{\ell}(S)\,,$$

is the number of moduli of rank-one torsion-free sheaves with the same Chern classes as $B(\ell := \ell(Z) = c_2(B))$. For $m \in \mathbf{R}$ let $I_m(C) \subset \operatorname{Hilb}^{\ell}(S)$ be the locus parametrizing subschemes Z such that $\sum_{P \in C} \operatorname{mult}_P(Z) \geqq m$. By (6.13), Inequality $(**)$ is replaced in the present case by

$$\dim \operatorname{Drap}^C(\mathcal{G}_\xi; \alpha)_x \leqq h^1(A,A) + h^1(B,A) + \dim \operatorname{Pic}(S) + \dim I_{n\alpha\sqrt{H^2}}\,.$$

An easy application of a result of Iarrobino ([I], Corollary 2) shows that

$$\operatorname{cod}(I_m(C), \operatorname{Hilb}^{\ell}(S)) \geqq m\,,$$

and hence

$$\dim \operatorname{Drap}^C(\mathcal{G}_\xi; \alpha)_x \leqq h^1(A,A) + h^1(B,A) + h^1(B,B) - n\alpha\sqrt{H^2}\,.$$

Applying Proposition (6.8) one gets that the right-hand side of the inequality in (1.4) is an upper bound for the above sum of h^1's, if $0 \leqq \alpha \leqq \alpha_0$. This proves (1.4).

Proof of Proposition (6.7). By additivity of the Euler characteristic, and by Serre duality, we have

$$\sum_{i \leqq j} h^1(Q_j, Q_i) = -\chi(F,F) + \sum_{i<j} \chi(Q_i, Q_j) + \sum_{i \leqq j} h^0(Q_j, Q_i)$$
$$+ \sum_{i \leqq j} h^0(Q_i, Q_j \otimes K_S)\,.$$

For $x \in \mathrm{Pic}(S) \otimes \mathbf{Q}$, set $\chi(x) := \chi(\mathcal{O}_S) + (x^2 - x \cdot K)/2$. Applying the formula

$$\chi(F,G) = r_F \cdot r_G \left[\chi\left(\frac{c_1(G)}{r_G} - \frac{c_1(F)}{r_F}\right) - \frac{\Delta_F}{r_F} - \frac{\Delta_G}{r_G}\right],$$

valid for any couple of sheaves F, G (of positive rank) on S, we get:

$$\begin{aligned}\sum_{i \leqq j} h^1(Q_j, Q_i) = 2r_F\Delta_F - r_F^2\chi(\mathcal{O}_S) + \sum_{i<j} r_i r_j \left[\chi\left(\frac{c_1(Q_j)}{r_j} - \frac{c_1(Q_i)}{r_i}\right) - \frac{\Delta_i}{r_i} - \frac{\Delta_j}{r_j}\right] \\ + \sum_{1<j} h^0(A, Q_j \otimes K_S) + \sum_{2 \leqq i \leqq j} h^0(Q_j, Q_i) \\ + h^0(A,A) + h^0(A, A \otimes K_S) + \sum_{1<j} h^0(Q_j, A) \\ + \sum_{2 \leqq i \leqq j} h^0(Q_i, Q_j \otimes K_S)\,.\end{aligned}$$

Set

$$\begin{aligned}\Theta &:= \sum_{i<j} r_i r_j \left[\chi\left(\frac{c_1(Q_j)}{r_j} - \frac{c_1(Q_i)}{r_i}\right) - \frac{\Delta_i}{r_i} - \frac{\Delta_j}{r_j}\right], \\ \Lambda &:= \sum_{1<j} h^0(A, Q_j \otimes K_S)\,, \\ \Gamma &:= \sum_{2 \leqq i \leqq j} h^0(Q_j, Q_i)\,, \\ \Omega &:= h^0(A,A) + h^0(A, A \otimes K_S) + \sum_{1<j} h^0(Q_j, A) + \sum_{2 \leqq i \leqq j} h^0(Q_i, Q_j \otimes K_S)\,.\end{aligned} \tag{6.14}$$

Then the previous equation becomes

$$\sum_{i \leqq j} h^1(Q_j, Q_i) = 2r_F\Delta_F - r_F^2\chi(\mathcal{O}_S) + \Theta + \Lambda + \Gamma + \Omega\,.$$

Proposition (6.7) follows from the bounds for Θ, Λ, Γ and Ω given below, together with a straightforward computation.

(6.15) *Let notation be as above. If $K \cdot H \geqq 0$, then*

$$\begin{aligned}\Theta \leqq -\Delta_F + \frac{(r_A+1)r_B}{2r_A}\alpha^2 + [r_B(r_B-1) - r_F]\frac{K \cdot H}{2\sqrt{H^2}}\alpha \\ + [r_F r_B(r_B-1)(r_F - 2r_A) + r_A r_B^2(r_A+1)]\frac{(K \cdot H)^2}{8H^2} \\ + T(r_F, S, H) + r_F^2\chi(\mathcal{O}_S)\,,\end{aligned}$$

(6.16) *With notation as above, assume $K \cdot H < 0$. Then*

$$\Theta \leqq -\Delta_F + \frac{(r_A+1)r_B}{2r_A}\alpha^2 - [(r_B-1)(2r_F-r_B)+r_F]\frac{K\cdot H}{2\sqrt{H^2}}\alpha$$
$$+ [r_F r_B(r_B-1)(r_F+2r_A) + r_A r_B^2(r_A+1)]\frac{(K\cdot H)^2}{8H^2}$$
$$+\frac{r_F^2}{2}+\frac{r_F^3}{8}|K^2|\,.$$

In both cases (i.e. regardless of the sign of $K \cdot H$), we have

$$\Lambda \leqq \frac{1}{2r_A}(r_A^2+r_A+r_F)\alpha^2 + \frac{r_F}{\sqrt{H^2}}(K\cdot H+s(r_F))\alpha + \frac{r_A r_B}{2H^2}(K\cdot H+s(r_F))^2\,, \tag{6.17}$$

$$\Gamma \leqq \frac{r_B-1}{2}\alpha^2 + \frac{r_B(r_B-1)s(r_F)}{\sqrt{H^2}}\alpha + \frac{r_B^3}{2H^2}s(r_F)^2\,, \tag{6.18}$$

$$\Omega \leqq \frac{r_A r_F}{2H^2}s(r_F)^2 + \frac{r_A^2+r_B^2}{2H^2}(K\cdot H+s(r_F))^2\,. \tag{6.19}$$

Proof of (6.15)–(6.16). Let

$$r_\ell := \max\{r_i \mid 1 \leqq i \leqq n+1\}\,,$$
$$\sigma_i := -r_1 - \cdots - r_{i-1} + r_{i+1} + \cdots r_{n+1}\,. \tag{6.20}$$

For future reference we notice that

$$\sum_{i=1}^{n+1} r_i\sigma_i = 0\,. \tag{6.21}$$

(6.22) Lemma. *Keeping notation as above,*

$$\Theta \leqq -(r_F - r_\ell)\Delta_F + \frac{1}{2}\left(r_F^2 - \sum_{i=1}^{n+1} r_i^2\right)\chi(\mathcal{O}_S)$$
$$+ r_\ell \sum_{i=1}^{n+1}\frac{c_1(Q_i)^2}{2r_i} - r_\ell\frac{c_1(F)^2}{2r_F} + \frac{1}{2}\sum_{i=1}^{n+1}\sigma_i c_1(Q_i)\cdot K_S\,.$$

Proof. We brake the sum defining Θ (Equation (6.14)) into two pieces. First consider

$$\Theta' := \sum_{i<j} r_i r_j \chi\left(\frac{c_1(Q_j)}{r_j} - \frac{c_1(Q_i)}{r_i}\right).$$

A straightforward computation gives

$$\Theta' = r_F\sum_{i=1}^{n+1}\frac{c_1(Q_i)^2}{2r_i} - \frac{1}{2}c_1(F)^2 + \frac{1}{2}\sum_{i=1}^{n+1}\sigma_i c_1(Q_i)\cdot K_S + \frac{1}{2}\left(r_F^2 - \sum_{i=1}^{n+1} r_i^2\right)\chi(\mathcal{O}_S)\,.$$

Now consider

$$\Theta'' := \sum_{i<j}(r_j\Delta_i + r_i\Delta_j)\,.$$

Then

$$\Theta'' := \sum_{i=1}^{n+1} (r_F - r_i)\Delta_i .$$

Since $(r_F - r_i) \geqq (r_F - r_\ell)$, and since $\Delta_i \geqq 0$ (Bogomolov's theorem), we conclude that

$$\Theta'' \geqq (r_F - r_\ell) \sum_{i=1}^{n+1} \Delta_i = (r_F - r_\ell) \sum_{i=1}^{n+1} [c_2(Q_i) - \tfrac{1}{2}c_1(Q_i)^2]$$

$$+ (r_F - r_\ell) \sum_{i=1}^{n+1} \frac{1}{2r_i} c_1(Q_i)^2 .$$

Additivity of the Chern character gives

$$\sum_{i=1}^{n+1} \left[c_2(Q_i) - \tfrac{1}{2}c_1(Q_i)^2\right] = c_2(F) - \tfrac{1}{2}c_1(F)^2 = \Delta_F - \frac{1}{2r_F} c_1(F)^2 ,$$

and thus we have

$$-\Theta'' \leqq -(r_F - r_\ell)\Delta_F + (r_F - r_\ell)\frac{1}{2r_F} c_1(F)^2 - (r_F - r_\ell) \sum_{i=1}^{n+1} \frac{1}{2r_i} c_1(Q_i)^2 .$$

Since $\Theta = (\Theta' - \Theta'')$, the lemma follows. QED

Set

$$\Xi := r_\ell \sum_{i=1}^{n+1} \frac{c_1(Q_i)^2}{2r_i} - r_\ell \frac{c_1(F)^2}{2r_F} + \frac{1}{2} \sum_{i=1}^{n+1} \sigma_i c_1(Q_i) \cdot K_S . \tag{6.23}$$

(6.24) Lemma. *Let notation be as above. Then*

$$\Xi \leqq \frac{r_\ell + r_A r_\ell}{2r_A}\alpha^2 + \frac{1}{2\sqrt{H^2}} \{[r_B(r_B - 1) - r_F](K_S \cdot H)$$

$$+ r_F(r_B - 1)[|K_S \cdot H| - (K_S \cdot H)]\}\alpha$$

$$+ \frac{r_B}{8r_\ell H^2} \{[r_F^2(r_B - 1) + r_A r_B(r_A + 1)](K_S \cdot H)^2$$

$$- 2r_F r_A(r_B - 1)|K_S \cdot H|(K_S \cdot H)\} - \frac{K^2}{8r_\ell} \sum_{i=1}^{n+1} r_i \sigma_i^2 .$$

Proof. First of all notice that Ξ is left invariant if F and the Q_i are tensored by a line bundle ζ, or even if they are formally tensored by $\zeta \in \mathrm{Pic}(S) \otimes \mathbf{Q}$. (Use Equation (6.21).) Choosing ζ such that $c_1(F \otimes \zeta) = 0$, we can assume that $c_1(F) = 0$. Of course by doing this the classes $c_1(F)$ and $c_1(Q_i)$ become elements of $NS(S) \otimes \mathbf{Q}$. Now rewrite the right-hand side of (6.23) to get

$$\Xi = \frac{r_A}{2r_\ell} \left[\frac{r_\ell}{r_A} c_1(A) + \frac{r_B}{2} K\right]^2 + \frac{1}{2r_\ell} \sum_{i=2}^{n+1} r_i \left[\frac{r_\ell}{r_i} c_1(Q_i) + \frac{\sigma_i}{2} K\right]^2 - \frac{K^2}{8r_\ell} \sum_{i=1}^{n+1} r_i \sigma_i^2 . \tag{6.25}$$

In what follows we will use the inequality

$$L^2 \leqq \frac{(L \cdot H)^2}{H^2} ,$$

which, by the Hodge index theorem, holds for all $L \in NS(S) \otimes \mathbf{Q}$. It gives

$$\frac{r_A}{2r_\ell}\left[\frac{r_\ell}{r_A}c_1(A)+\frac{r_B}{2}K\right]^2 \leqq \frac{r_\ell}{2r_A}\alpha^2 - \frac{r_B(K\cdot H)}{2\sqrt{H^2}}\alpha + \frac{r_A r_B^2(K\cdot H)^2}{8r_\ell H^2}, \quad (6.26)$$

and

$$\frac{1}{2r_\ell}\sum_{i=2}^{n+1} r_i\left[\frac{r_\ell}{r_i}c_1(Q_i)+\frac{\sigma_i}{2}K\right]^2 \leqq \frac{1}{2r_\ell H^2}\sum_{i=2}^{n+1} r_i\left[r_\ell\mu_i+\frac{\sigma_i}{2}\mu_K\right]^2. \quad (6.27)$$

We will bound the right-hand side of the above inequality by applying the following lemma. Its (easy) proof is left to the reader.

(6.28) Lemma. *Let $x_1,\dots,x_n$ be real numbers and $r_1,\dots,r_n$ be positive integers. Let $N := \sum_{i=1}^n r_i$. If $x_i \geqq a$ for $i = 1,\dots,n$, then*

$$\sum_{i=1}^n r_i x_i^2 \leqq \left[\sum_{i=1}^n r_i x_i - (N-1)a\right]^2 + (N-1)a^2.$$

We apply the lemma to the sum on the right-hand side of (6.27). Set $x_i := r_\ell\mu_{i+1} + \sigma_{i+1}\mu_K/2$. Then $N = r_B$, and

$$\sum_{i=2}^{n+1} r_i x_i = r_\ell\sqrt{H^2}\alpha - \tfrac{1}{2}r_A r_B \mu_K.$$

Since $\mu_2 \geqq \cdots \geqq \mu_{n+1} \geqq 0$, we can set $a := -\frac{1}{2}r_F|\mu_K|$. Lemma (6.28) gives

$$\begin{aligned}\frac{1}{2r_\ell H^2}\sum_{i=2}^{n+1} r_i\left[r_\ell\mu_i+\frac{\sigma_i}{2}\mu_K\right]^2 \leqq{}& \tfrac{1}{2}r_\ell\alpha^2 + \frac{1}{2\sqrt{H^2}}\left[r_F(r_B-1)|\mu_K| - r_A r_B\mu_K\right]\alpha \\ &+ \frac{1}{8r_\ell H^2}\{[r_F(r_B-1)|\mu_K| - r_A r_B\mu_K]^2 \\ &+ r_F^2(r_B-1)\mu_K^2\}.\end{aligned}$$

Lemma (6.24) now follows from (6.25), (6.26), (6.27), and the above inequality, together with a straightforward computation. QED

Inequalities (6.15)–(6.16) follow from Lemmas (6.22)–(6.24) and some easy estimates. The only estimate which is not completely trivial is provided by the following

Lemma. *Keeping notation as above, we have*

$$r_F \leqq \frac{1}{r_\ell}\sum_{i=1}^{n+1} r_i\sigma_i^2.$$

Proof. One checks easily that, if n is replaced by $(n+1)$, and r_{n+1} by t_{n+1}, t_{n+2} (with $t_{n+1}+t_{n+2} = r_{n+1}$), then the right-hand side of the above inequality increases. Thus the minimum, with a fixed $r_1 = r_A$, is given by $n = 1$. The lemma then follows by direct computation. QED

Proof of Inequality (6.17). By Corollary (6.2), we have

$$\Lambda \leqq \frac{r_A}{2H^2} \sum_{1<j} r_j(\mu_j + \mu_K - \mu_A + s(r_F))^2 .$$

Lemma (6.28) together with an easy computation gives (6.17).

Proof of Inequality (6.18). To simplify notation let $s := s(r_F)$. By Corollary (6.2) we have

$$\Gamma \leqq \frac{1}{2H^2} \sum_{1<i<j} r_i r_j(\mu_i - \mu_j + s)^2 + \sum_{1<i} r_i^2 s^2 .$$

Expanding the squares in the first sum on the right-hand side, we write

$$\Gamma = \frac{1}{2H^2}(\Gamma_1 + \Gamma_2 + \Gamma_3)\,, \tag{6.29}$$

where

$$\Gamma_1 := s^2 \sum_{1<i} r_i^2 + s^2 \sum_{1<i<j} r_i r_j\,,$$

$$\Gamma_2 := \sum_{1<i<j} (r_i r_j \mu_i^2 + r_i r_j \mu_j^2) - 2 \sum_{1<i<j} r_i r_j \mu_i \mu_j\,,$$

$$\Gamma_3 := 2s \sum_{1<i<j} (r_i r_j \mu_i - r_i r_j \mu_j)\,.$$

We rewrite Γ_2 as

$$\Gamma_2 = \sum_{1<i} r_i(r_B - r_i)\mu_i^2 - 2 \sum_{1<i<j} r_i r_j \mu_i \mu_j = r_B \sum_{1<i} r_i \mu_i^2 - r_B^2 \mu_B^2 .$$

For the second equality we have used the relation $r_B \mu_B = \sum_{1<i} r_i \mu_i$. We also write

$$\Gamma_3 = 2s \sum_{1<i} r_i \tau_i \mu_i\,,$$

where $\tau_i := (-r_2 - \cdots - r_{i-1} + r_{i+1} + \cdots + r_{n+1})$. (See (6.20).) Now write

$$\Gamma_2 + \Gamma_3 = r_B \sum_{1<i} r_i \left[\left(\mu_i + \frac{\tau_i s}{r_B}\right)^2 - \frac{\tau_i^2 s^2}{r_B^2}\right] - r_B^2 \mu_B^2$$

$$\leqq r_B \sum_{1<i} r_i \left(\mu_i + \frac{\tau_i s}{r_B}\right)^2 - r_B^2 \mu_B^2 . \tag{6.30}$$

We bound the sum of squares on the last line by applying Lemma (6.28). We let $a := (\mu_F - s)$. Clearly $N = r_B$. Furthermore

$$\sum_{1<i} r_i \left(\mu_i + \frac{\tau_i s}{r_B}\right) = r_B \mu_B + \frac{s}{r_B} \sum_{1<i} r_i \tau_i = r_B \mu_B .$$

Lemma (6.28) gives

$$\sum_{1<i} r_i \left(\mu_i + \frac{\tau_i s}{r_B}\right)^2 \leqq r_B^2 \mu_B^2 - 2r_B(r_B - 1)\mu_B(\mu_F - s) + r_B(r_B - 1)(\mu_F - s)^2 .$$

Putting this together with Inequality (6.30) we conclude that

$$\Gamma_2 + \Gamma_3 \leqq r_B^2(r_B - 1)(\mu_B - \mu_F + s)^2 = (r_B - 1)H^2\alpha^2 + 2r_B(r_B - 1)s\sqrt{H^2}\alpha + r_B^2(r_B - 1)s^2 .$$

Adding $\Gamma_1 \leqq r_B^2 s^2$, and using Equation (6.29), one gets (6.18).

Proof of Inequality (6.19). The inequality follows from Corollary (6.2) and some simple estimates.

Proof of Proposition (6.8). The proof follows that of Proposition (6.7), with the difference that we replace Corollary (6.2) by the following

(6.31) Proposition. *Assume that H is effective. Let A be a rank-one torsion-free sheaf on S. Then*

$$h^0(A) \leqq \frac{1}{2H^2}(\mu_A + H^2 + 1)^2 .$$

Before proving the above proposition we need a few preliminaries. If A is an effective divisor on S, let

$$A_f := \text{fixed part of } |A|, \qquad A_m := A - A_f = \text{moving part of } A .$$

(6.32) Lemma. *Let notation be as above. Assume that H is effective. Let* $C \in |H|$. *If A is a divisor on S, then*

$$h^0(\mathcal{O}_C(A_m)) \leqq [C \cdot A + 1]_+ ,$$

where $[x]_+ := \max\{x, 0\}$.

Proof. We might as well assume that A_m is effective non-zero. Since $|A_m|$ has no fixed part, there exists $D \in |A_m|$ such that the scheme-theoretic intersection of C and D, call it Z, is zero-dimensional. Then the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_Z \to 0$$

gives

$$h^0(\mathcal{O}_C(A_m)) \leqq h^0(\mathcal{O}_C) + C \cdot D \leqq h^0(\mathcal{O}_C) + C \cdot A .$$

As is easily checked C is 1-connected, hence $h^0(\mathcal{O}_C) = 1$ (see [BPV]). Hence the result follows from the above inequality. QED

Proof of Proposition (6.31). Let $C \in |H|$. By considering the exact sequence

$$0 \to \mathcal{O}_S(A - (i+1)H) \to \mathcal{O}_S(A - iH) \to \mathcal{O}_C(A - iH) \to 0$$

for all non-negative integers i, one easily concludes that

$$h^0(\mathcal{O}_S(A)) \leqq \sum_{i=0}^{N} h^0(\mathcal{O}_C((A - iH)_m)) ,$$

where N is the maximum value of i such that

$$h^0(\mathcal{O}_S(A - iH)) > 0 .$$

Thus by Lemma (6.32)

$$h^0(\mathcal{O}_S(A)) \leqq \sum_{i=0}^{N} [\mu_A - iH^2 + 1]_+ .$$

An easy estimate of the right-hand side gives the proposition. QED

Proof of (1.10), (1.11), (1.12). Proof of Proposition (1.10). Let $V(F) \subset \mathrm{Def}(F)$ be the locus parametrizing strictly μ-semistable sheaves. We will prove that

$$\dim V(F) \leqq (2r_F - 1) + \varepsilon(r_F, S, H) + q_S . \tag{6.33}$$

Clearly this is equivalent to Proposition (1.10). Let $\mathscr{F}$ be the family of sheaves parametrized by $\mathrm{Def}(F)$. Let $\mathrm{Quot}_0(\mathscr{F})$ be the quot-scheme parametrizing torsion-free quotients of $\mathscr{F}_x$, for x varying in $\mathrm{Def}(F)$, with slope equal to that of F (i.e. destabilizing quotients). By a theorem of Grothendieck [G], $\mathrm{Quot}_0(\mathscr{F})$ is of finite type. Let

$$\pi : \mathrm{Quot}_0(\mathscr{F}) \to \mathrm{Def}(F) ,$$

be the projection. Clearly

$$V(F) = \pi(\mathrm{Quot}_0(\mathscr{F})) , \tag{6.34}$$

and hence $\dim V(F) \leqq \dim \mathrm{Quot}_0(\mathscr{F})$. Now let $y \in \mathrm{Quot}_0(\mathscr{F}; P)$ correspond to the μ-destabilizing sequence

$$0 \to A \to \mathscr{F}_x \to B \to 0 . \tag{$*$}$$

Then there is an exact sequence ([DL], (5.44))

$$0 \to \mathrm{Hom}(A,B) \to T_y \mathrm{Quot}_0(\mathscr{F}_x; P) \xrightarrow{\pi_*} \mathrm{Ext}^1(\mathscr{F}_x, \mathscr{F}_x) \xrightarrow{\omega_+} \mathrm{Ext}^1(A,B) , \tag{6.35}$$

where ω_+ is induced by the inclusion $A \subset \mathscr{F}_x$ and the quotient $\mathscr{F}_x \to B$. By applying the functors $\mathrm{Hom}(\,\cdot\,, \mathscr{F}_x)$ and $\mathrm{Hom}(A, \,\cdot\,)$ to $(*)$ one concludes that

$$\dim \mathrm{Quot}_0(\mathscr{F}_x; P)_y \leqq h^0(A,B) + h^1(A,A) + h^1(B,B) + h^1(B,A) .$$

Now we estimate the sum of the h^1's by repeating the proof of Proposition (6.7). (We will get sharper results because the filtration consists of only two terms.) Adopting the notation used in that proof, we have

$$\Theta \leqq -r_A \Delta_F + \frac{r_A r_F}{8} \left[\frac{(K \cdot H)^2}{H^2} - K^2 \right] .$$

Then, by applying (6.2), (6.31), or (1.7) to bound Λ, Γ and Ω, we get

$$\dim \mathrm{Quot}_0(\mathscr{F}; P)_y \leqq (2r_F - 1) + \varepsilon(r_F, S, H) + q_S .$$

By (6.34) this proves Inequality (6.33), and hence the proposition.

Proof of Corollary (1.11). Let $[F] \in X$; we can assume F is strictly μ-semistable. Let $\mathscr{F}$ be the family of sheaves on S parametrized by $\mathrm{Def}^0(\mathrm{Gr}\, F)$. By Claim (2.22) the map

$$\lambda : \mathrm{Def}^0(\mathrm{Gr}\, F) \to \mathscr{M}_\xi\,,$$

induced by $\mathscr{F}$ is surjective onto a neighborhood of $[F]$. Thus $\dim \lambda^{-1}X$ satisfies the same inequality as $\dim X$. By (1.10) there exists $x \in \lambda^{-1}X$ parametrizing a μ-stable sheaf. Then $\lambda(x) \in X$ is a point parametrizing a μ-stable sheaf.

QED

Proof of Corollary (1.12). Let $X \subset \mathscr{M}_\xi$ be an irreducible component. Let $[F] \in X$ be a point not belonging to any other component of $\mathscr{M}_\xi$. If F is μ-stable there is nothing to prove, so assume F is not μ-stable. By deformation theory [Fr] we have

$$\dim \mathrm{Def}^0(F) \geqq 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi(\mathscr{O}_S)\,.$$

Hence by Proposition (1.10) we conclude that the generic point $x \in \mathrm{Def}^0(F)$ parametrizes a μ-stable sheaf. This implies that the generic point of X parametrizes a μ-stable sheaf. QED

Proof of Proposition (1.13). If $g = 0$, there are no stable vector bundles, and hence the proposition is trivially verified. Thus we can assume that $g > 0$. Let F be a non-stable vector bundle on C. Let $V(F) \subset \mathrm{Def}(F)$ be the subset parametrizing non-stable bundles. Since $\mathrm{Def}(F)$ is smooth, it suffices to show that, if V_i is an irreducible component of $V(F)$, then

$$\mathrm{cod}(V_i, \mathrm{Def}(F)) \leqq \frac{r^2}{4} g\,. \tag{6.36}$$

This is what we will prove. One can stratify $V(F)$ according to the ranks and slopes of the successive quotients of the Harder–Narasimhan filtration. Thus the stratum corresponding to the *type*

$$\mathbf{t} := ((r_1, \mu_1), \ldots, (r_n, \mu_n))\,,$$

where $\mu_1 > \cdots > \mu_n$, consists of the points $x \in \mathrm{Def}(F)$ such that there is a filtration

$$F_1 \subset F_2 \subset \cdots \subset F_n = \mathscr{F}_x\,,$$

with F_i/F_{i-1} a rank-r_i semistable bundle with slope μ_i. As is well-known [AB, LP], each $V_\mathbf{t}$ is smooth, and

$$\mathrm{cod}(V_\mathbf{t}, \mathrm{Def}(F)) = \sum_{i<j} r_i r_j (\mu_i - \mu_j + g - 1)\,. \tag{6.37}$$

In order to prove Inequality (6.36) we need the following

Lemma. *Keep notation as above. Assume that the genus g of C is positive. Let F be a non-stable vector bundle on C. Then every irreducible component* *of $V(F)$ contains an open dense subset parametrizing minimally non-stable bundles, i.e. bundles F fitting into an exact sequence*

$$0 \to A \to F \to B \to 0\,, \tag{6.38}$$

where A, B are semistable vector bundles such that

$$\mu_A \geqq \mu_B\,, \tag{6.39}$$

$$\mu_A - \frac{1}{r_A} < \mu_B + \frac{1}{r_B}\,. \tag{6.40}$$

Proof. Fix a component V_i of $V(F)$. If the generic point of V_i parametrizes a semistable bundle, then there is nothing to prove. Thus we can assume that the bundles parametrized by V_i are unstable. Let $\mathbf{t}$ be the type of the Harder–Narasimhan filtration for the generic bundle parametrized by V_i. Since versality is an open condition, we can replace F by $\mathscr{F}_x$ for a generic point $x \in V_i$, and thus we can assume that $V(F)$ is irreducible and that $V(F) = V_{\mathbf{t}}$. First let's show that the Harder–Narasimhan filtration corresponding to $\mathbf{t}$ consists of only two terms. Let A be the first term of the H.-N. filtration for F, and let

$$0 \to A \to F \to B \to 0\,, \tag{$*$}$$

be the corresponding exact sequence. Let P be the Hilbert polynomial of B, and let $\mathrm{Quot}(\mathscr{F}; P)$ be the Quot-scheme parametrizing quotients of $\mathscr{F}_x$, for $x \in \mathrm{Def}(F)$, with Hilbert polynomial equal to P. If $y \in \mathrm{Quot}(\mathscr{F}; P)$, then there is an exact sequence (6.35). The last map is surjective, because C is smooth of dimension one. By a criterion of Drezet–Le Potier [DL], $\mathrm{Quot}(\mathscr{F}; P)$ is smooth. Hence its dimension can be computed from (6.35). Furthermore, since $h^0(A, B) = 0$, the projection $\pi\colon \mathrm{Quot}(\mathscr{F}; P) \to \mathrm{Def}(F)$ is an embedding; let W be its image. An easy computation gives

$$\mathrm{cod}(W, \mathrm{Def}(F)) = r_A r_B(\mu_A - \mu_B + g - 1)\,.$$

Clearly $W \subset V(F)$, and hence $\mathrm{cod}(W, \mathrm{Def}(F)) \geqq (V(F), \mathrm{Def}(F))$. Since $V(F) = V_{\mathbf{t}}$, one easily concludes from Equation (6.37) that $\mathbf{t} = ((r_A, \mu_A), (r_B, \mu_B))$. Now let's show that $(*)$ is minimally destabilizing, i.e. that (6.40) is satisfied. We argue by contradiction. Assume that (6.40) is violated. Let A' be a sheaf fitting into an exact sequence

$$0 \to A' \to A \xrightarrow{\phi} k_P \to 0\,, \tag{6.41}$$

where k_P is the skyscraper sheaf at some point $P \in C$. Let $B' := B \oplus k_P$. Then there is an exact sequence

$$0 \to A' \to F \to B' \to 0\,. \tag{$\dagger$}$$

Let P' be the Hilbert polynomial of B', and let $\mathrm{Quot}(\mathscr{F}; P')$ be the Quot-scheme of quotients of $\mathscr{F}_x$ with Hilbert polynomial equal to P', for $x \in \mathrm{Def}(F)$.

As in the previous case, $\mathrm{Quot}(\mathscr{F};P')$ is smooth, and (6.35) gives

$$\dim \mathrm{Quot}(\mathscr{F};P') = h^1(F,F) - r_A r_B(\mu_A - \mu_B + g - 1) + r_F\,. \qquad (\sharp)$$

Let $\pi\colon \mathrm{Quot}(\mathscr{F};P') \to \mathrm{Def}(F)$ be the projection. Since (6.40) is violated,

$$\pi(\mathrm{Quot}(\mathscr{F};P')) \subset V(F)\,. \qquad (**)$$

As is easily checked, $\pi^{-1}(\mathbf{o})$ consists of all sequences (†), where A' fits into an exact sequence (6.41). (Here $\mathbf{o}$ is the point parametrizing F.) Letting P and ϕ vary, we get

$$\dim \pi^{-1}(\mathbf{o}) = r_A\,.$$

Since $r_A < r_F$, we conclude by ($\sharp$) that

$$\dim \pi(\mathrm{Quot}(\mathscr{F};P')) > h^1(F,F) - r_A r_B(\mu_A - \mu_B + g - 1)\,.$$

This inequality, together with ($**$), contradicts Formula (6.37). This proves that (6.40) is satisfied, and hence F is minimally non-stable. QED

Proof of (6.36). By the previous lemma we can assume that every sheaf parametrized by V_i fits into Exact sequence (6.38), with (6.39)–(6.40) satisfied. We distinguish two cases, according to whether V_i parametrizes unstable or semistable sheaves. In the first case Formula (6.37) gives

$$\mathrm{cod}(V_i, \mathrm{Def}(F)) = r_A r_B(\mu_A - \mu_B + g - 1)\,.$$

By (6.39)–(6.40) we conclude that (6.36) is satisfied. Now assume that F is semistable. Let $\mathrm{Quot}_0(\mathscr{F})$ be the Quot-scheme parametrizing quotients of $\mathscr{F}_x$ whose slope is equal to that of $\mathscr{F}_x$, for $x \in \mathrm{Def}(F)$. Let $\pi : \mathrm{Quot}_0(\mathscr{F}) \to \mathrm{Def}(F)$ be the projection. Exact sequence (6.35) and the smoothness of $\mathrm{Quot}_0(\mathscr{F})$ give

$$\dim \pi(\mathrm{Quot}_0(\mathscr{F})) \geqq h^1(F,F) + \chi(A,B) - h^0(A,B)\,.$$

Since $\pi(\mathrm{Quot}_0(\mathscr{F})) = V_i$ one concludes, by Lemma (1.7), that (6.36) is satisfied.

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