from the De Rham cohomology onto the cohomology with values in the constant sheaf $\mathbb{R}$. Instead of using $\mathscr{C}^\infty$ differential forms, one can consider the resolution of $\mathbb{R}$ given by the exterior derivative d acting on currents:

$$0 \to \mathbb{R} \to \mathscr{D}'_n \xrightarrow{d} \mathscr{D}'_{n-1} \to \cdots \to \mathscr{D}'_{n-q} \xrightarrow{d} \mathscr{D}'_{n-q-1} \to \cdots \to \mathscr{D}'_0 \to 0.$$

The sheaves $\mathscr{D}'_q$ are also $\mathscr{E}_X$-modules, hence acyclic. Thanks to (6.3), the inclusion $\mathscr{E}^q \subset \mathscr{D}'_{n-q}$ induces an isomorphism

$$H^q\big(\mathscr{E}^\bullet(X)\big) \simeq H^q\big(\mathscr{D}'_{n-\bullet}(X)\big), \tag{6.8}$$

both groups being isomorphic to $H^q(X, \mathbb{R})$. The isomorphism between cohomology of differential forms and singular cohomology (another topological invariant) was first established by [De Rham 1931]. The above proof follows essentially the method given by [Weil 1952], in a more abstract setting. As we will see, the isomorphism (6.7) can be put under a very explicit form in terms of Čech cohomology. We need a simple lemma.

(6.9) Lemma. *Let X be a paracompact differentiable manifold. There are arbitrarily fine open coverings $\mathcal{U} = (U_\alpha)$ such that all intersections $U_{\alpha_0 \ldots \alpha_q}$ are diffeomorphic to convex sets.*

Proof. Select locally finite coverings $\Omega'_j \subset\subset \Omega_j$ of X by open sets diffeomorphic to concentric euclidean balls in $\mathbb{R}^n$. Let us denote by τ_{jk} the transition diffeomorphism from the coordinates in Ω_k to those in Ω_j. For any point $a \in \Omega'_j$, the function $x \mapsto |x - a|^2$ computed in terms of the coordinates of Ω_j becomes $|\tau_{jk}(x) - \tau_{jk}(a)|^2$ on any patch $\Omega_k \ni a$. It is clear that these functions are strictly convex at a, thus there is a euclidean ball $B(a, \varepsilon) \subset \Omega'_j$ such that all functions are strictly convex on $B(a, \varepsilon) \cap \Omega'_k \subset \Omega_k$ (only a finite number of indices k is involved). Now, choose $\mathcal{U}$ to be a (locally finite) covering of X by such balls $U_\alpha = B(a_\alpha, \varepsilon_\alpha)$ with $U_\alpha \subset \Omega'_{\rho(\alpha)}$. Then the intersection $U_{\alpha_0 \ldots \alpha_q}$ is defined in Ω_k, $k = \rho(\alpha_0)$, by the equations

$$|\tau_{jk}(x) - \tau_{jk}(a_{\alpha_m})|^2 < \varepsilon^2_{\alpha_m}$$

where $j = \rho(\alpha_m)$, $0 \leqslant m \leqslant q$. Hence the intersection is convex in the open coordinate chart $\Omega_{\rho(\alpha_0)}$. □

Let Ω be an open subset of $\mathbb{R}^n$ which is starshaped with respect to the origin. Then the De Rham complex $\mathbb{R} \longrightarrow \mathscr{E}^\bullet(\Omega)$ is acyclic: indeed, Poincaré's lemma yields a homotopy operator $k^q : \mathscr{E}^q(\Omega) \longrightarrow \mathscr{E}^{q-1}(\Omega)$ such that

$$\begin{aligned} &k^q f_x(\xi_1, \ldots, \xi_{q-1}) = \int_0^1 t^{q-1} f_{tx}(x, \xi_1, \ldots, \xi_{q-1})\, dt, \quad x \in \Omega, \quad \xi_j \in \mathbb{R}^n, \\ &k^0 f = f(0) \in \mathbb{R} \quad \text{for} \quad f \in \mathscr{E}^0(\Omega). \end{aligned}$$

Hence $H^q_{\mathrm{DR}}(\Omega, \mathbb{R}) = 0$ for $q \geqslant 1$. Now, consider the resolution $\mathscr{E}^\bullet$ of the constant sheaf $\mathbb{R}$ on X, and apply the proof of the De Rham-Weil isomorphism theorem to Čech cohomology groups over a covering $\mathcal{U}$ chosen as in Lemma 6.9. Since the intersections $U_{\alpha_0 \ldots \alpha_s}$ are convex, all Čech cochains in $C^s(\mathcal{U}, \mathscr{Z}^q)$ are liftable in $\mathscr{E}^{q-1}$ by means of k^q. Hence