DESINGULARIZED MODULI SPACES OF SHEAVES ON A $K3$.*

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Contents

0. Introduction.

Let X be a projective $K3$ surface and $\mathcal{M}(r,c_1,c_2)$ be the moduli space of semistable (with respect to the polarization $\mathcal{O}_X(1)$) torsion-free sheaves on X of rank r, with Chern classes c_1, c_2. If it so happens that every semistable sheaf is actually stable, e.g. when c_1 is non-divisible or when $r=2$, $c_1=0$ and c_2 is odd (the polarization must be "generic"), then $\mathcal{M}(r,c_1,c_2)$ is smooth and holomorphically symplectic [M1]. It has been proved [O1,O3] that in both of the above mentioned cases the moduli space is equivalent, up

* i.e. the union of "Desingularized ...", J. reine angew. Math. 512 (1999), and alg-geom/9708009

to deformation of complex structure and birational modifications, to a Hilbert scheme parametrizing zero-dimensional subschemes of X; one expects that the same statement is true whenever semistability implies stability (Yoshioka [Y] proved this in many cases). The present paper deals with the "opposite" case, that is when there do exist strictly semistable (i.e. non stable) sheaves. We will analyze the moduli space $\mathcal{M}_c$ of rank-two torsion-free sheaves with $c_1 = 0$ and $c_2 = c$ even. The sheaf $I_W \oplus I_Z$, where $2\ell(W) = 2\ell(Z) = c$, is strictly semistable for any choice of polarization; conversely if the polarization is generic (see (0.2) for the precise definition) these are the only strictly semistable sheaves. When $c = 0$ or $c = 2$ no stable sheaves exist, thus $\mathcal{M}_0$ is a point and $\mathcal{M}_2 \cong X^{(2)}$: in short not much is going on. Instead if $c \geq 4$ the moduli space $\mathcal{M}_c$ is interesting: it is singular exactly along the locus parametrizing strictly semistable sheaves, and the smooth locus is symplectic. A natural question to ask is the following:

(0.1) Does there exist a symplectic desingularization (or smooth model) of $\mathcal{M}_c$?

Our main result is the construction of a symplectic desingularization $\widetilde{\mathcal{M}}_4$ of $\mathcal{M}_4$, and the proof that $\widetilde{\mathcal{M}}_4$ is a new (ten-dimensional) irreducible symplectic variety. Explicitely, we show that $\widetilde{\mathcal{M}}_4$ is one-connected and that $h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$; this means $\widetilde{\mathcal{M}}_4$ is an irreducible symplectic variety. Furthermore we prove that $b_2(\widetilde{\mathcal{M}}_4) \geq 24$. Since all known ten-dimensional irreducible symplectic varieties have $b_2 = 7$ or $b_2 = 23$ [B,H], our results show that all deformations of $\widetilde{\mathcal{M}}_4$ are new irreducible symplectic varieties.

The paper is organized as follows. In §1 we explicitly describe Kirwan's desingularization $\widehat{\mathcal{M}}_c \to \mathcal{M}_c$, obtained by blowing up loci parametrizing strictly semistable sheaves. The desingularization $\widehat{\mathcal{M}}_c$ carries a regular two-form $\widehat{\omega}_c$ which degenerates on all three exceptional divisors if $c \geq 6$, while $\widehat{\omega}_4$ degenerates on a single divisor $\widehat{\Omega}_4$, namely the inverse image of the locus parametrizing sheaves equivalent to $I_Z \oplus I_Z$. In §2 we show that $\widehat{\Omega}_4$ is a $\mathbf{P}^2$-fibration, and the normal bundle has degree -1 on the $\mathbf{P}^2$'s. Hence we can contract $\widehat{\mathcal{M}}_4$ along this fibration, and we get a smooth complex manifold $\widetilde{\mathcal{M}}_4$ with a holomorphic symplectic form. We identify this contraction with the contraction of a certain $K_{\widehat{\mathcal{M}}_4}$-negative extremal ray, hence $\widetilde{\mathcal{M}}_4$ is projective by Mori theory. We also show that the a priori rational map $\widetilde{\mathcal{M}}_4 \cdots > \mathcal{M}_4$ is regular, thus $\widetilde{\mathcal{M}}_4$ is a symplectic desingularization of $\mathcal{M}_4$. (It is plausible that it equals the blow-up of $\mathcal{M}_4$ along the locus parametrizing strictly semistable sheaves.) In §3 we prove that $\widetilde{\mathcal{M}}_4$ is connected and $h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$. First, by a well-known deformation argument we can assume $\mathcal{O}_X(1)$ has degree 2. Let $C \in |\mathcal{O}_X(1)|$ be a smooth curve: we prove that

$$H^q(\widetilde{\mathcal{M}}_4; \mathbf{Z}) \cong H^q(\widetilde{\pi}^{-1}\{[F] \in \mathcal{M}_4 |\ F|_C \text{ is either singular or unstable}\}; \mathbf{Z})$$

for $q \leq 5$. (Here $\widetilde{\pi}: \widetilde{\mathcal{M}}_4 \to \mathcal{M}_4$ is the desingularization map.) The proof of this result is "copied" from [Li3]. Hence to get our result we must describe the above set. There are three components: the first is mapped by $\widetilde{\pi}$ to the locus parametrizing strictly semistable sheaves $I_Z \oplus I_W$ which are singular along C, the second is mapped by $\widetilde{\pi}$ to the closure of the locus parametrizing stable sheaves which are singular along C, the third is mapped to the closure of the locus parametrizing vector-bundles whose restriction to C is unstable. The first two sets are easily analyzed. The analysis of the third set requires more work: by considering elementary modifications (along C) we can describe this set in terms of the moduli space of rank-two semistable sheaves on X with $c_1 = c_1(\mathcal{O}_X(1))$, $c_2 = 3$. Once we have a birational description of these three sets, it is an easy matter to conclude that $h^0(\widetilde{\mathcal{M}}_4) = h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$. In §4 we prove that $\widetilde{\mathcal{M}}_4$ is simply-connected: first we show $\widetilde{\mathcal{M}}_4$ is birational to a certain Jacobian fibration over $|\mathcal{O}_X(2)|$, then we prove the Jacobian fibration is simply-connected via a monodromy argument. In §5 we show that the subspace of $H^2(\widetilde{\mathcal{M}}_4; \mathbf{Q})$ spanned by the (pulled-back) image of the μ-map together with two "geometric" divisors has dimension 24. In the final section we are concerned with Question (0.1) for $c \geq 6$. We do not have an answer, but we describe how to get a desingularization $\widetilde{\mathcal{M}}_c$ which is "closer" to being symplectic than $\widehat{\mathcal{M}}_c$ is by contracting extremal K-negative rays . (We suspect that there is no smooth symplectic model of $\mathcal{M}_c$ for $c \geq 6$.)

We close this discussion by mentioning one of our motivations for posing Question (0.1). Vafa and Witten [VW] have proposed formulae for the Euler characteristics (suitably interpreted?) of moduli spaces of semistable sheaves on surfaces. If the answer to (0.1) is affirmative, the Euler characteristic of any smooth symplectic model of $\mathcal{M}_c$ (which is independent of the model chosen) should be equal to Vafa-Witten's characteristic. If on the contrary the answer to (0.1) is negative it is not clear which mathematical Euler characteristic should equal the physicists' characteristic.

Notation used throughout the paper.

We let c be an even integer with $c \geq 4$, and we set $c = 2n$.

We let X be a projective $K3$ surface and $H = \mathcal{O}_X(1)$ be a *c-generic polarization*, i.e. an ample divisor class such that for $D \in \mathrm{Div}(S)$,

$$\text{if } D \cdot H = 0 \text{ and } -c \leq D \cdot D, \text{ then } D \sim 0. \tag{0.2}$$

There exist c-generic polarizations for any choice of c, because the collection of hyperplanes $D^{\perp}$, for $D \in \mathrm{Div}(X)$ with

$$-c \leq D \cdot D < 0,$$

defines a set of locally finite walls in the ample cone of X.

A torsion-free sheaf F on X is *Gieseker-Maruyama semistable* (with respect to the polarization $\mathcal{O}_X(1)$) if for every exact sequence

$$0 \to L \to F \to Q \to 0,$$

$$(\mathrm{rk}Q) \cdot \chi(L(n)) \leq (\mathrm{rk}L) \cdot \chi(Q(n)), \text{ for } n \gg 0.$$

If strict inequality holds (when $n \gg 0$) for all such sequences with $\mathrm{rk}L \neq 0 \neq \mathrm{rk}Q$ then F is *Gieseker-Maruyama stable.* A semistable non-stable sheaf is *strictly semistable.*

We let $\mathcal{M}_c$ be the moduli space of rank-two (Gieseker-Maruyama) semistable torsion-free sheaves F on X with $c_1(F) = 0$, $c_2(F) = c$; this is a projective scheme whose closed points are in one-to-one correspondence with S-equivalence classes of such sheaves [G,Ma].

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1. Kirwan's desingularization of $\mathcal{M}_c$.

F. Kirwan [K] defined a procedure for partially desingularizing G.I.T. quotients: in short one blows up loci parametrizing strictly semistable points, until all semistable points are actually stable, and then one takes the quotient of this blown up space. The aim of this section is to desingularize $\mathcal{M}_c$ by applying Kirwan's method. In (1.1) we recall how one can realize the moduli space $\mathcal{M}_c$ as the G.I.T. quotient of a Quot-scheme Q_c acted on by $\mathrm{PGL}(N)$, and we give a detailed description of Kirwan's procedure as applied to our case. In order to describe the resulting space (in particular to show it is smooth) we must analyze the local structure of Q_c at points corresponding to semistable sheaves $F \cong I_W \oplus I_Z$; in (1.2) we show that, by Luna'étale slice Theorem, this is equivalent to studying the versal deformation space of such sheaves. In (1.4)-(1.5) we give the required results about these deformation spaces; the main results are (1.4.1)-(1.5.1). In particular we will see that Q_c is singular at such points; thus the general results of [K] do not apply to our case, because Kirwan assumes that the semistable locus is smooth. However we are lucky: we show that Kirwan's blow-ups, dictated by the need to eliminate strictly semistable points, give a $\mathrm{PGL}(N)$-space S_c which is smooth along its semistable locus. This result is stated in (1.8.10), but the main ingredients of its proof are the results of (1.6)-(1.7). The quotient $S_c//\mathrm{PGL}(N)$ will not be smooth (except when $c = 4$), it has quotient singularities, but a last blow-up gives a space with smooth quotient $\widehat{\mathcal{M}}_c$; this is the main result of (1.8). In Subsection (1.9) we show that there is a regular two-form on $\widehat{\mathcal{M}}_c$ extending the symplectic form on the smooth locus of $\mathcal{M}_c$.

1.1. The Quot-scheme and Kirwan's desingularization.

We briefly recall the construction of $\mathcal{M}_c$ according to Simpson [S,Le]. By Serre's F.A.C. theorems, the following holds if $k \gg 0$. Let F be a sheaf parametrized by $\mathcal{M}_c$; then $H^p(F(k)) = 0$ for $p > 0$, and F can be realized as a quotient

$$\mathcal{O}_X(-k)^{(N)} \to F, \tag{1.1.1}$$

in such a way that the induced map $\mathbf{C}^N \to H^0(F(k))$ is an isomorphism. Let $Quot(k)$ be the Quot-scheme parametrizing quotients (1.1.1) whose Hilbert polynomial is that of rank-two sheaves with $c_1 = 0$, $c_2 = c$; if $x \in Quot(k)$ we let F_x be the quotient sheaf parametrized by x. Then $\mathrm{PGL}(N)$ acts on $Quot(k)$ and also on some positive multiple of the "Plücker" line-bundle over $Quot(k)$, i.e. the action is linearized. Hence it makes sense to speak of $\mathrm{PGL}(N)$-(semi)stable points: let $Q_c^{ss}, Q_c^s \subset Quot(k)$ be the open subsets consisting of $\mathrm{PGL}(N)$-semistable (respectively stable) points x such that F_x is torsion-free and $\mathrm{rk}(F_x) = 2$, $c_1(F_x) = 0$, $c_2(F_x) = c$. Let Q_c be the schematic closure of Q_c^{ss} in $Quot(k)$. Simpson proves that for k sufficiently large a point $x \in Q_c$ is $\mathrm{PGL}(N)$-semistable (stable) if and only if F_x is Gieseker-Maruyama semistable (respectively stable), and that

$$\mathcal{M}_c = Q_c//\mathrm{PGL}(N).$$

Kirwan's partial desingularization will be the $\mathrm{PGL}(N)$-quotient of a variety obtained by successively blowing up Q_c along (the closure of) loci parametrizing strictly semistable points: the idea is that strictly semistable points will gradually disappear and in the end all semistable points will be stable (in particular their stabilizers will be finite). A key ingredient is a theorem of Kirwan relating stability on a G-scheme to stability on the blow-up of a G-invariant subscheme. More precisely, let G be a reductive group acting linearly on a complex projective scheme Y (linearly means: the G-action has been lifted to an action on $\mathcal{O}_Y(1)$), let V be a G-invariant closed subscheme of Y, and $\pi\colon \widetilde{Y} \to Y$ be the blow-up of V. Then G acts on $\widetilde{Y}$, and also on

$$D_\ell := \pi^*\mathcal{O}_Y(\ell) \otimes \mathcal{O}_{\widetilde{Y}}(-E),$$

where E is the exceptional divisor of π. Thus the action on $\widetilde{Y}$ is linearized. Let $Y^{ss} \subset Y$, $Y^s \subset Y$ be the loci of semistable (stable) points with respect to $\mathcal{O}_Y(1)$, and let $\widetilde{Y}^{ss}(\ell) \subset \widetilde{Y}$ and $\widetilde{Y}^s(\ell) \subset \widetilde{Y}$ be the loci of semistable (stable) points with respect to D_ℓ.

(1.1.2) Theorem (Kirwan [K, 3.1-3.2-3.11]). *Keep notation as above. For $\ell \gg 0$ the loci $\widetilde{Y}^{ss}(\ell)$ and $\widetilde{Y}^s(\ell)$ are independent of ℓ: denote them by $\widetilde{Y}^{ss}$ and $\widetilde{Y}^s$ respectively. The following holds:*

$$\pi(\widetilde{Y}^{ss}) \subset Y^{ss} \tag{1.1.3}$$

$$\pi^{-1}(Y^s) \subset \widetilde{Y}^s. \tag{1.1.4}$$

In particular π induces a morphism

$$\overline{\pi}\colon \widetilde{Y}//G \to Y//G.$$

If ℓ is also sufficiently divisible, this morphism is identified with the blow-up of $V//G$.

We will blow up (the closure of) loci parametrizing strictly semistable sheaves.

(1.1.5) Lemma. *A point $x \in Q_c$ is strictly semistable (i.e. $x \in Q_c^{ss} \setminus Q_c^s$) if and only if F_x fits into an exact sequence*

$$0 \to I_Z \to F_x \to I_W \to 0, \tag{1.1.6}$$

where Z, W are zero-dimensional subschemes of X of length $\ell(Z) = \ell(W) = n$, and I_Z, I_W are their ideal sheaves. Furthermore the orbit $\mathrm{PGL}(N)x$ is closed in Q_c^{ss} if and only if the exact sequence above is split.

Proof. A straightforward computation shows that if F_x fits into Exact sequence (1.1.6) then it is strictly Gieseker-Maruyama semistable, hence x is strictly semistable. Now assume $x \in Q_c$ is strictly semistable. Then by Simpson F_x is strictly Gieseker-Maruyama semistable, i.e. it fits into an exact sequence

$$0 \to I_Z(D) \to F_x \to I_W(-D) \to 0, \tag{$*$}$$

where Z, W are zero-dimensional subschemes of X, and $D \in \mathrm{Div}(X)$, with

$$(\dagger) \qquad \chi(I_Z(D) \otimes \mathcal{O}_X(n)) = \chi(I_W(-D) \otimes \mathcal{O}_X(n)) \text{ for } n \gg 0.$$

Applying Whitney's formula to $(*)$ and writing out explicitely $(\dagger)$ we get

$$-c + \ell(Z) + \ell(W) = D \cdot D \text{ and } D \cdot H = 0,$$

respectively. Since H is c-generic (see (0.2)) we conclude that $D \sim 0$, hence the destabilizing subsheaf and quotient sheaf are I_Z and I_W respectively. Equality $(\dagger)$ then gives $\ell(Z) = \ell(W)$, hence F_x fits into Exact sequence (1.1.6). Let $e \in \mathrm{Ext}^1(I_W, I_Z)$ be the extension class of (1.1.6), and assume $e \neq 0$. One can construct a family of extensions $\{\mathcal{E}_t\}_{t \in \mathbf{A}^1}$ of I_W by I_Z with extension class te: for $t \neq 0$ the sheaves $\mathcal{E}_t$ are all isomorphic non-split extensions, while $\mathcal{E}_0 \cong I_Z \oplus I_W$. From this it follows that if (1.1.6) is non-split the orbit $\mathrm{PGL}(N)x$ is not closed. Since there must be a closed orbit in Q_c^{ss} which corresponds to the S-equivalence class of the semistable sheaf appearing in (1.1.6), this orbit must parametrize split extensions. **q.e.d.**

Let

$$\Omega_Q^0 := \{x \in Q |\ F_x \cong I_Z \oplus I_Z,\ [Z] \in X^{[n]}\},$$
$$\Gamma_Q^0 := \{x \in Q |\ F_x \text{ is a non-trivial extension of } I_Z \text{ by } I_Z,\ [Z] \in X^{[n]}\}$$
$$\Sigma_Q^0 := \{x \in Q |\ F_x \cong I_Z \oplus I_W,\ [Z],[W] \in X^{[n]},\ Z \neq W\},$$
$$\Lambda_Q^0 := \{x \in Q |\ F_x \text{ is a non-trivial extension of } I_Z \text{ by } I_W,\ [Z],[W] \in X^{[n]},\ Z \neq W\}$$

Here and in the rest of the paper we drop the subscript c from $\mathcal{M}_c$, Q_c, etc. whenever this causes no confusion. We let Ω_Q, Γ_Q, Σ_Q, Λ_Q be the closures in Q of Ω_Q^0, Γ_Q^0, Σ_Q^0 and Λ_Q^0 respectively. By Lemma (1.1.5),

$$(1.1.7) \qquad Q^{ss} \setminus Q^s = \Omega_Q^0 \amalg \Gamma_Q^0 \amalg \Sigma_Q^0 \amalg \Lambda_Q^0.$$

If G is a group acting on a set A, and $x \in A$, we let $St(x)$ be the stabilizer of x.

(1.1.8) Corollary. *Let $x \in Q^{ss}$. Then*

$$St(x) \cong \begin{cases} \mathrm{PGL}(2) & \text{if } x \in \Omega_Q^0, \\ (\mathbf{C}, +) & \text{if } x \in \Gamma_Q^0, \\ \mathbf{C}^* & \text{if } x \in \Sigma_Q^0, \\ \{1\} & \text{if } x \in \Lambda_Q^0 \amalg Q^s. \end{cases}$$

Proof. For $x \in Q$, one has $St(x) \cong Aut(F_x)/\text{scalars}$; the result follows easily. **q.e.d.**

By the above corollary the points of Q_c^{ss} with non-trivial reductive stabilizers are parametrized by Ω_Q^0 and Σ_Q^0. Thus Kirwan's method for (partially) desingularizing $\mathcal{M}_c$ is the following. Let

$$(1.1.9) \qquad \pi_R \colon R \to Q$$

be the blow-up of Ω_Q. We let $\Sigma_R \subset R$ be the strict transform of Σ_Q. (Notice that $\Sigma_Q \supset \Omega_Q$.) Let

$$(1.1.10) \qquad \pi_S \colon S \to R$$

be the blow-up of Σ_R. The linear action of $\mathrm{PGL}(N)$ on Q lifts to linear actions on R and S. Applying Theorem (1.1.2) to π_R and π_S we get a morphism

$$S_c//\mathrm{PGL}(N) \to Q_c//\mathrm{PGL}(N) = \mathcal{M}_c.$$

We will prove that $\widehat{\mathcal{M}}_4 := S_4//\mathrm{PGL}(N)$ is smooth. If $c \geq 6$ then $S_c//\mathrm{PGL}(N)$ has quotient singularities; a last blow up will produce a smooth desingularization of $\mathcal{M}_c$, which we denote $\widehat{\mathcal{M}}_c$. Both of these statements will be proved in Subsection (1.8).

1.2. Luna's étale slice.

If W is a subscheme of a scheme Z we let $C_W Z$ be the *normal cone to W in Z* [Fu]. Since the exceptional divisor of π_R is equal to $\mathbf{Proj}(C_{\Omega_Q} Q)$, we will need to determine the normal cone to Ω_Q in Q (at semistable points); similarly we will need to know $C_{\Sigma_R} R$. Luna's étale slice theorem reduces this to a problem about deformations of sheaves. We recall Luna's theorem. Let G be a reductive group acting linearly on a quasi-projective scheme Y. For $y \in Y$ we let $O(y)$ be its orbit. If $y \in Y^{ss}$ and $O(y)$ is closed (in Y^{ss}) then $St(y)$ is reductive.

(1.2.1) Luna's étale slice Theorem [Lu]. *Keeping notation as above, suppose $y_0 \in Y^{ss}$ and $O(y_0)$ is closed in Y^{ss}. Then there exists a slice normal to $O(y_0)$, i.e. an affine subscheme $\mathcal{V} \hookrightarrow Y^{ss}$, containing y_0 and invariant under the action of $St(y_0)$, such that the following holds. The (multiplication) morphism*

$$G \times_{St(y_0)} \mathcal{V} \xrightarrow{\phi} Y^{ss}$$

has open image, and is étale over its image. (Here $St(y_0)$ acts on $G \times \mathcal{V}$ by $h(g,y) := (gh^{-1}, hy)$.) Furthermore ϕ is G-equivariant (the G-action on $G \times_{St(y_0)} \mathcal{V}$ is induced by left multiplication on the first factor). The quotient map

$$\overline{\phi}: \mathcal{V}//St(y_0) \to Y^{ss}//G$$

has open image and is étale over its image. If Y^{ss} is smooth at y_0, then $\mathcal{V}$ is also smooth at y_0.

Now assume $W \subset Y^{ss}$ is a locally closed subset containing y_0, stable for the action of G; for example W could be $\Omega^0_Q \subset Q^{ss}$. Set

$$\mathcal{W} := W \cap \mathcal{V}.$$

(1.2.2) Corollary. *Keep notation and hypotheses as above. There is a $St(y_0)$-equivariant isomorphism*

$$(C_W Y^{ss})_{y_0} \cong (C_{\mathcal{W}} \mathcal{V})_{y_0}.$$

Proof. We work throughout in neighborhoods of $(1, y_0) \in G \times_{St(y_0)} \mathcal{V}$ and of $y_0 \in Y$. Set

$$\widetilde{W} := G \times_{St(y_0)} \mathcal{W}.$$

Since $\phi^{-1} W = \widetilde{W}$, and since ϕ is étale there is an isomorphism

$$C_W Y^{ss} \cong C_{\widetilde{W}} \left(G \times_{St(y_0)} \mathcal{V}\right).$$

The projections

$$\left(G \times_{St(y_0)} \mathcal{V}\right) \to G/St(y_0) \qquad \widetilde{W} = \left(G \times_{St(y_0)} \mathcal{W}\right) \to G/St(y_0)$$

are fibrations étale locally trivial, with fibers $\mathcal{V}$ and $\mathcal{W}$ respectively. Taking the fiber over the coset $[St(y_0)]$ we get the corollary. **q.e.d.**

Let's go back to our case: $\mathrm{PGL}(N)$ acting on Q^{ss}. The following result identifies the normal slice with a versal deformation space. (See also Wehler [W].)

(1.2.3) Proposition. *Let $x \in Q^{ss}$ be a point such that $O(x)$ is closed (in Q^{ss}). Let $\mathcal{V}$ be a normal slice (see (1.2.1)), and $(\mathcal{V}, x)$ be the germ of $\mathcal{V}$ at x. Let $\mathcal{F}$ be the restriction to $X \times (\mathcal{V}, x)$ of the tautological quotient sheaf on $X \times Q$. The couple $((\mathcal{V}, x), \mathcal{F})$ is a versal deformation space of F_x.*

Proof. We must prove two things: that the family $\mathcal{F}$ is complete and that the Kodaira-Spencer map

$$\overline{\kappa}: T_x \mathcal{V} \to \mathrm{Ext}^1(F_x, F_x)$$

is an isomorphism. Completeness follows easily from the universal property of the Quot-scheme. Let's prove that $\overline{\kappa}$ is an isomorphism. Since $\mathcal{F}$ is complete $\overline{\kappa}$ is surjective, thus it suffices to show that $\overline{\kappa}$ is injective. Letting

$$\kappa: T_x Q \to \mathrm{Ext}^1(F_x, F_x)$$

be the Kodaira-Spencer map at x of the tautological quotient on $X \times Q$, we must show that

$$\ker(\kappa) = T_x O(x).$$

Letting E_x be the kernel of the map (1.1.1) for $F = F_x$, we have

$$(*) \qquad 0 \to E_x \to \mathcal{O}_X(-k)^{(N)} \to F_x \to 0.$$

This gives the exact sequence

$$0 \to \mathrm{Hom}(F_x, F_x) \to \mathrm{Hom}(\mathcal{O}_X(-k)^{(N)}, F_x) \xrightarrow{\alpha} \mathrm{Hom}(E_x, F_x) \xrightarrow{\kappa} \mathrm{Ext}^1(F_x, F_x),$$

where $\mathrm{Hom}(E_x, F_x) = T_xQ$ and κ is Kodaira-Spencer. Thus we are reduced to proving that $\mathrm{Im}\alpha = T_xO(x)$. Let

$$\beta: Hom(\mathcal{O}_X(-k)^{(N)}, \mathcal{O}_X(-k)^{(N)}) \to \mathrm{Hom}(\mathcal{O}_X(-k)^{(N)}, F_x)$$

be the obvious map. Since $T_xO(x) = \mathrm{Im}(\alpha \circ \beta)$, it suffices to check that β is surjective. This follows at once from the isomorphism $H^0(\mathcal{O}_X^{(N)}) \xrightarrow{\sim} H^0(F_x(k))$. **q.e.d.**

1.3. Normal cones and deformations of sheaves.

We will need to describe $C_\Omega Q^{ss}$ and similar normal cones. By Corollary (1.2.2) and Proposition (1.2.3) this will be equivalent to describing the normal cone of certain loci in the versal deformation space of semistable sheaves; in this subsection we provide the necessary tools.

The Hessian cone. Let Y be a scheme, and $B \hookrightarrow Y$ a locally-closed subscheme such that

(1.3.1) $$B \text{ is smooth and } \dim T_bY \text{ is constant for every } b \in B.$$

By (1.3.1) we have a normal vector-bundle N_BY. Let I_B be the ideal sheaf of B in Y: the graded surjection

$$\bigoplus_{d=0}^{\infty} S^d \left(I_B/I_B^2\right) \to \bigoplus_{d=0}^{\infty} \left(I_B^d/I_B^{d+1}\right)$$

defines an embedding of cones $\iota: C_BY \hookrightarrow N_BY$. The (homogeneous) ideal $I\left(\iota(C_BY)\right)$ contains no linear terms. We let the *Hessian cone* of B in Y be the subscheme $H_BY \hookrightarrow N_BY$ whose homogeneous ideal is generated by the quadratic terms $I\left(\iota(C_BY)\right)_2$. Thus we have a chain of cones over B:

$$C_BY \subset H_BY \subset N_BY.$$

Notice that if $b \in B$, then

(1.3.2) $$\mathbf{P}(H_bY) \text{ is the cone over } \mathbf{P}\left(H_BY\right)_b \text{ with vertex } \mathbf{P}(T_bB).$$

Let $I_m := \mathrm{Spec}\left(\mathbf{C}[t]/(t^{m+1})\right)$; thus tangent vectors to Y at b are identified with pointed maps $I_1 \to (Y, b)$. Then

(1.3.3) $$\left(H_bY\right)_{red} = \{f_1: I_1 \to (Y, b) |\text{ there exists } f_2: I_2 \to (Y, b) \text{ extending } f_1\}.$$

Deformations of sheaves. Let $\mathcal{E}$ be a coherent sheaf on a projective scheme, and let $(\mathrm{Def}(\mathcal{E}), 0)$ be the parameter space of the versal deformation space of $\mathcal{E}$. Thus

(1.3.4) $$T_0\mathrm{Def}(\mathcal{E}) \cong \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}).$$

We will give explicit equations of the reduced Hessian cone of $\mathrm{Def}(\mathcal{E})$ at the origin. Let

$$\mathrm{Ext}^p(\mathcal{E}, \mathcal{E})^0 := \ker\left(\mathrm{Tr}: \mathrm{Ext}^p(\mathcal{E}, \mathcal{E}) \to H^p(\mathcal{O}_X)\right),$$

where the trace Tr is defined as in [DL]. The composition of Trace with Yoneda product

$$\mathrm{Ext}^p(\mathcal{E}, \mathcal{E}) \times \mathrm{Ext}^q(\mathcal{E}, \mathcal{E}) \xrightarrow{\mathrm{Yon}} \mathrm{Ext}^{p+q}(\mathcal{E}, \mathcal{E}) \xrightarrow{\mathrm{Tr}} H^{p+q}(\mathcal{O}_Y)$$

is a bilinear map, symmetric if $(-1)^{pq}=1$, anti-symmetric if $(-1)^{pq}=-1$. We are particularly interested in the *Yoneda square* map

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) & \xrightarrow{\Upsilon_{\mathcal{E}}} & \mathrm{Ext}^2(\mathcal{E},\mathcal{E})^0 \\ e & \longrightarrow & e\cup e \end{array}$$

Proposition. *Keep notation as above. Then*

$$(H_0\mathrm{Def}(\mathcal{E}))_{red} = \left(\Upsilon_{\mathcal{E}}^{-1}(0)\right)_{red}. \tag{1.3.5}$$

Proof. An *m-th order deformation* of $\mathcal{E}$ is a sheaf $\mathcal{E}_m$ on $Y\times I_m$, flat over I_m, such that $\mathcal{E}_m\otimes\mathbf{C}\cong\mathcal{E}$. By (1.3.3), the left-hand side of (1.3.5) consists of first order deformations of $\mathcal{E}$ which can be extended to second order deformations. Let $e\in\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$, and let

$$0\to t\mathcal{E}\xrightarrow{\alpha}\mathcal{E}_1\xrightarrow{\beta}\mathcal{E}\to 0 \tag{$*$}$$

be the first-order deformation of $\mathcal{E}$ corresponding to e. Here $t\mathcal{E}$ means that the $\mathbf{C}[t]/(t^2)$-module structure of $\mathcal{E}_1$ is described as follows: if σ is a local section of $\mathcal{E}_1$, then $t\sigma:=\alpha(t\beta(\sigma))$. From $(*)$ we get

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}_1)\longrightarrow\mathrm{Ext}^1(\mathcal{E},\mathcal{E})\xrightarrow{\partial}\mathrm{Ext}^2(\mathcal{E},\mathcal{E}).$$

Since ∂ is Yoneda product with e we must prove that $\mathcal{E}_1$ can be extended to a second order deformation if and only if $\partial(e)=0$. By the above exact sequence $\partial(e)=0$ if and only if Extension $(*)$ is the push-out (via β) of an extension

$$0\to t\mathcal{E}_1\xrightarrow{\gamma}\mathcal{F}\to\mathcal{E}\to 0. \tag{$\dagger$}$$

So let' assume that $\mathcal{E}_1$ can be extended to a second order deformation $\mathcal{E}_2$: thus $\mathcal{E}_2$ is an extension

$$0\to t\mathcal{E}_1\xrightarrow{\iota}\mathcal{E}_2\to\mathcal{E}\to 0.$$

Let's check that the push-out of $\mathcal{E}_2$ is isomorphic to $(*)$. Since β is surjective and $\ker(\beta)=\alpha(t\mathcal{E})$,

$$\text{push-out of }\mathcal{E}_2=\mathcal{E}_2/\iota\circ(t\alpha(t\mathcal{E}))=\mathcal{E}_2/t^2\mathcal{E}\cong\mathcal{E}_1.$$

Now let's prove the converse: assuming there is an extension $(\dagger)$ whose push-out is $(*)$ we will give $\mathcal{F}$ a structure of $\mathbf{C}[t]/(t^3)$-module making it a second order extension of $\mathcal{E}_1$. Since the push-out of $\mathcal{F}$ is equal to $\mathcal{F}/\gamma(t\alpha(t\mathcal{E}))$, we have

$$0\to t^2\mathcal{E}\xrightarrow{\gamma o t\alpha}\mathcal{F}\xrightarrow{\delta}\mathcal{E}_1\to 0.$$

If σ is a local section of $\mathcal{F}$ we set $t\sigma:=\gamma(t\delta(\sigma))$. **q.e.d.**

We will need the following result.

(1.3.6) Proposition. *Let $x\in Q^{ss}$ be a point with closed orbit, and $\mathcal{V}\ni x$ be a slice normal to the orbit. Then*

$$\dim_x\mathcal{V}\geq\dim\mathrm{Ext}^1(F_x,F_x)-\dim\mathrm{Ext}^2(F_x,F_x)^0.$$

More precisely, every irreducible component of the germ $(\mathcal{V},x)$ has dimension at least equal to the right-hand side of the above inequality.

Proof. By Proposition (1.2.3) there is an identification between $(\mathcal{V},x)$ and $\mathrm{Def}(F_x)$. It is known that $\mathrm{Def}(F_x)$ is the zero-locus of an obstruction map with domain a smooth germ of dimension $\dim\mathrm{Ext}^1(F_x,F_x)$, and codomain $\mathrm{Ext}^2(F_x,F_x)^0$: this implies the proposition. Alternatively, one can use the following result about the local structure of the Hilbert scheme Q at x. Let E_x be the sheaf on X defined by the exact sequence

$$0\to E_x\to\mathcal{O}_X(-k)^{(N)}\to F_x\to 0. \tag{1.3.7}$$

(See (1.1.1).) Then by Lemmas (1.4)-(1.8) of [Li2] we have

$$\dim_x Q \geq \dim T_x Q - \dim \mathrm{Ext}^2(F_x, F_x)^0 = \dim \mathrm{Hom}(E_x, F_x) - \dim \mathrm{Ext}^2(F_x, F_x)^0,$$

or more precisely every irreducible component of the germ (Q, x) has dimension at least equal to the right-hand side of the above inequality. Applying the functor $\mathrm{Hom}(\cdot, F_x)$ to (1.3.7) one gets

$$\dim \mathrm{Hom}(E_x, F_x) = \dim \mathrm{Ext}^1(F_x, F_x) + (N^2 - 1) - \dim St(x).$$

On the other hand, by Luna's étale slice Theorem (1.2.1) we have

$$\dim_x Q = \dim \mathrm{PGL}(N) + \dim_x \mathcal{V} - \dim St(x).$$

Putting together the above (in)equalities one gets the proposition. **q.e.d.**

1.4. The normal cone of Σ_Q^0.

Let $x \in \Sigma_Q^0$, and set

$$F_x = I_{Z_1} \oplus I_{Z_2}, \qquad \ell(Z_i) = n, \qquad Z_1 \neq Z_2.$$

Recall (1.1.8) that $St(x) \cong \mathbf{C}^*$. To simplify notation we often set $\Sigma = \Sigma_Q^0$.

(1.4.1) Proposition. *Keep notation as above. Then Σ_Q^0 is smooth, and its normal cone in Q is a locally-trivial fibration over Σ_Q^0, with fiber the affine cone over a smooth quadric in $\mathbf{P}^{2c-5}$. More precisely, for $x \in \Sigma_Q^0$ there is a canonical isomorphism*

$$(C_\Sigma Q)_x \cong \{(e_{12}, e_{21}) \in \mathrm{Ext}^1(I_{Z_1}, I_{Z_2}) \oplus \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}) | \; e_{12} \cup e_{21} = 0\}. \tag{1.4.2}$$

Furthermore the action of $St(x)$ on $(C_\Sigma Q)_x$ is given by

$$\lambda(e_{12}, e_{21}) = (\lambda e_{12}, \lambda^{-1} e_{21}). \tag{1.4.3}$$

The proposition will be proved in several steps.

I. Yoneda square. Since

$$\mathrm{Ext}^p(F_x, F_x) = \bigoplus_{i,j} \mathrm{Ext}^p(I_{Z_i}, I_{Z_j})$$

we can write Yoneda square $\Upsilon := \Upsilon_{F_x}$ as

$$\Upsilon(\sum_{i,j} e_{ij}) = \sum_k e_{1k} \cup e_{k1} + \sum_k e_{1k} \cup e_{k2} + \sum_k e_{2k} \cup e_{k1} + \sum_k e_{2k} \cup e_{k2} \qquad e_{ij} \in \mathrm{Ext}^1(I_{Z_i}, I_{Z_j}).$$

By Serre duality

$$\mathrm{Ext}^2(I_{Z_i}, I_{Z_j}) \cong \mathrm{Hom}(I_{Z_j}, I_{Z_i})^\vee,$$

hence

$$\mathrm{Ext}^2(I_{Z_i}, I_{Z_j}) = 0 \text{ if } i \neq j \text{ and } \mathrm{Tr}\colon \mathrm{Ext}^2(I_{Z_i}, I_{Z_i}) \to H^2(\mathcal{O}_X) \text{ is an isomorphism.} \tag{1.4.4}$$

In particular

$$\mathrm{Ext}^2(F_x, F_x)^0 = \{(f_1, f_2) \in \mathrm{Ext}^2(I_{Z_1}, I_{Z_1}) \oplus \mathrm{Ext}^2(I_{Z_2}, I_{Z_2}) | \;\; f_1 + f_2 = 0\}. \tag{1.4.5}$$

Furthermore

$$\Upsilon(e) = (e_{11} \cup e_{11} + e_{12} \cup e_{21}, e_{21} \cup e_{12} + e_{22} \cup e_{22}).$$

By skew-commutativity (see (1.3)) $\mathrm{Tr}(e_{ii} \cup e_{ii}) = 0$, hence (1.4.4) implies that $e_{ii} \cup e_{ii} = 0$. This gives

$$\Upsilon(e) = (e_{12} \cup e_{21}, e_{21} \cup e_{12}). \tag{1.4.6}$$

Let

$$\begin{array}{rcl} \Psi\colon \mathrm{Ext}^1(F_x, F_x) & \to & \mathrm{Ext}^2(I_{Z_1}, I_{Z_1}) \\ e & \mapsto & e_{12} \cup e_{21}. \end{array}$$

By (1.4.6)-(1.4.5) we can identify Υ with Ψ, in particular $\Upsilon^{-1}(0) = \Psi^{-1}(0)$. Let

$$\begin{array}{rcl} \overline{\Psi}\colon \mathrm{Ext}^1(I_{Z_1}, I_{Z_2}) \oplus \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}) & \to & \mathrm{Ext}^1(I_{Z_1}, I_{Z_1}) \\ (e_{12}, e_{21}) & \mapsto & e_{12} \cup e_{21}. \end{array} \tag{1.4.7}$$

Thus $\overline{\Psi}$ is identified with the map induced by Ψ on $\mathrm{Ext}^1(F_x, F_x)/\ker\Psi$.

(1.4.8) Claim. $\mathbf{P}\overline{\Psi}^{-1}(0)$ *is a smooth quadric hypersurface in* $\mathbf{P}^{2c-5}$*. In particular, since* $c \geq 4$, $\mathbf{P}\Psi^{-1}(0)$ *is a reduced irreducible quadric.*

Proof. By Serre duality, Yoneda product

$$\mathrm{Ext}^1(I_{Z_1}, I_{Z_2}) \times \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}) \to \mathrm{Ext}^2(I_{Z_1}, I_{Z_1}) \tag{1.4.9}$$

is a perfect pairing. Hence

$$\mathbf{P}\overline{\Psi}^{-1}(0) \subset \mathbf{P}\left(\mathrm{Ext}^1(I_{Z_1}, I_{Z_2}) \oplus \mathrm{Ext}^1(I_{Z_2}, I_{Z_1})\right)$$

is a smooth quadric hypersurface. The claim follows from the equalities

$$-\dim \mathrm{Ext}^1(I_{Z_1}, I_{Z_2}) = \chi(I_{Z_1}, I_{Z_2}) = \chi(I_{Z_1}, I_{Z_1}) = 2 - c.$$

q.e.d.

II. The cone at the origin of the deformation space. Let $\mathcal{V}$ be a slice normal to the (closed) orbit $\mathrm{PGL}(N)x$: by Proposition (1.2.3) there is a natural isomorphism of germs $(\mathcal{V}, x) \cong \mathrm{Def}(F_x)$. In particular we have an embedding

$$C_x\mathcal{V} \subset \mathrm{Ext}^1(F_x, F_x).$$

(1.4.10) Proposition. *Keep notation as above. There are natural isomorphisms of schemes*

$$C_x\mathcal{V} = H_x\mathcal{V} \cong \Psi^{-1}(0).$$

Proof. By (1.3.5) and Claim (1.4.8), we have

$$(\mathbf{P}H_x\mathcal{V})_{red} = \mathbf{P}\Upsilon^{-1}(0) = \mathbf{P}\Psi^{-1}(0).$$

Since $\mathbf{P}\Psi^{-1}(0)$ is a reduced irreducible quadric hypersurface and $\mathbf{P}H_x\mathcal{V}$ is cut out by quadrics, it follows that

$$\mathbf{P}H_x\mathcal{V} = \mathbf{P}\Psi^{-1}(0).$$

Next, consider the inclusion

$$C_x\mathcal{V} \subset H_x\mathcal{V} = \Psi^{-1}(0).$$

By Proposition (1.3.6) and by (1.4.5)-(1.4.4) we have

$$\dim C_x\mathcal{V} = \dim \mathcal{V} \geq \dim \mathrm{Ext}^1(F_x, F_x) - 1 = \dim \Psi^{-1}(0).$$

Since $\Psi^{-1}(0)$ is reduced irreducible, we must have $C_x\mathcal{V} = \Psi^{-1}(0)$. **q.e.d.**

III. The normal cone. Let

$$\mathcal{W} := \mathcal{V} \cap \Sigma_Q^0.$$

By Corollary (1.2.2) there is a $St(x)$-equivariant isomorphism

$$(C_\Sigma Q)_x \cong (C_\mathcal{W}\mathcal{V})_x\,. \tag{1.4.11}$$

(1.4.12) Claim. *Keeping notation as above,* $\mathcal{W}$ *is smooth at* x *and*

$$T_x\mathcal{W} \cong \mathrm{Ext}^1(I_{Z_1}, I_{Z_1}) \oplus \mathrm{Ext}^1(I_{Z_2}, I_{Z_2}). \tag{1.4.13}$$

Furthermore, shrinking $\mathcal{V}$ *if necessary, we can assume that*

$$\dim T_{x'}\mathcal{V} = \dim T_x\mathcal{V} \text{ for all } x' \in \mathcal{W}. \tag{1.4.14}$$

Proof. We continue identifying $(\mathcal{V},x)$ with $\mathrm{Def}(F_x)$. First we prove (1.4.13). Let $\mathcal{F}$ be a first order deformation of F_x and let $e=\sum_{i,j}e_{ij}\in\mathrm{Ext}^1(F_x,F_x)$ be the corresponding extension class. Then e is tangent to $\mathcal{W}$ if and only if the (two) exact sequences

$$0\to I_{Z_i}\to F_x\to I_{Z_j}\to 0,\qquad i\neq j,$$

lift to $\mathcal{F}$. This condition is equivalent [O2, (1.17)] to

$$e_{12}=e_{21}=0.$$

This proves (1.4.13). To prove smoothness of $\mathcal{W}$, notice that $\mathcal{W}$ parametrizes all sheaves of the form $I_{Z'}\oplus I_{W'}$, for Z' near Z and W' near W; this implies that $\dim_x\mathcal{W}\geq 2c$. On the other hand the right-hand side of (1.4.13) has dimension $2c$, hence $\mathcal{W}$ is smooth at x. To prove the last statement, notice that the family $\mathcal{F}$ of sheaves parametrized by $\mathcal{V}$ is complete at all x' in a neighborhood of x, and that $\dim\mathrm{Ext}^1(F_{x'},F_{x'})$ is constant for $x'\in\mathcal{W}$. **q.e.d.**

Now let's prove Proposition (1.4.1), except for (1.4.3). First we show Σ^0_Q is smooth. Let $x\in\Sigma^0_Q$, let $\mathcal{V}$ be a slice normal to $O(x)$, and $\mathcal{W}:=\mathcal{V}\cap\Sigma^0_Q$. By the Étale slice Theorem (1.2.1), a neighborhood of x in Σ^0_Q is isomorphic to a neighborhood of $(1,x)$ in $\mathrm{PGL}(N)\times_{St(x)}\mathcal{W}$. This last space is smooth at $(1,x)$ because by (1.4.12) $\mathcal{W}$ is smooth at x. Thus x is a smooth point of Σ^0_Q. Now let's prove (1.4.2). By (1.4.11), in order to prove (1.4.2) we must give an isomorphism

$$(C_{\mathcal{W}}\mathcal{V})_x\cong\overline{\Psi}^{-1}(0), \tag{1.4.15}$$

where $\overline{\Psi}$ is as in (1.4.7). By Claim (1.4.12) the normal bundle $N_{\mathcal{W}}\mathcal{V}$ is defined, hence we have inclusions of cones

$$(C_{\mathcal{W}}\mathcal{V})_x\subset(H_{\mathcal{W}}\mathcal{V})_x\subset(N_{\mathcal{W}}\mathcal{V})_x\cong\mathrm{Ext}^1(I_{Z_1},I_{Z_2})\oplus\mathrm{Ext}^1(I_{Z_2},I_{Z_1}).$$

(The last isomorphism follows from (1.4.13).) Equality (1.3.2) together with Proposition (1.4.10) gives an isomorphism

$$(H_{\mathcal{W}}\mathcal{V})_x\cong\overline{\Psi}^{-1}(0).$$

Arguing as in the proof of (1.4.10), we conclude that $(C_{\mathcal{W}}\mathcal{V})_x=(H_{\mathcal{W}}\mathcal{V})_x$. In detail:

$$\dim(C_{\mathcal{W}}\mathcal{V})_x\geq\dim\mathcal{V}-\dim\mathcal{W}\geq\dim(H_{\mathcal{W}}\mathcal{V})_x=\dim\overline{\Psi}^{-1}(0),$$

where the second inequality follows from (1.3.6). Since $(C_{\mathcal{W}}\mathcal{V})_x\subset(H_{\mathcal{W}}\mathcal{V})_x=\overline{\Psi}^{-1}(0)$ and $\overline{\Psi}^{-1}(0)$ is reduced irreducible, we get (1.4.15). The rest of Proposition (1.4.1), except for (1.4.3), follows at once from (1.4.2). **q.e.d.**

IV. Action of $St(x)$. Let $x\in Q^{ss}$ be a point with closed orbit (x need not be in Σ^0_Q), and let $\mathcal{V}$ be a slice normal to $O(x)$. Since $St(x)=\mathrm{Aut}(F_x)/\{\text{scalars}\}$, the action of $St(x)$ on $\mathcal{V}$ defines also an action of $\mathrm{Aut}(F_x)$ on $\mathcal{V}$. For $g\in\mathrm{Aut}(F_x)$ we let

$$g_*\colon T_x\mathcal{V}\to T_x\mathcal{V}$$

be the differential at x of the map corresponding to g.

(1.4.16) Lemma. *Keeping notation as above, let*

$$e\in T_x\mathcal{V}\cong T_0\mathrm{Def}(F_x)\cong\mathrm{Ext}^1(F_x,F_x).$$

Then $g_*(e)=g\cup e\cup g^{-1}$.

Proof. Set $F_0=F_x$. Let F_1 and E_1 be the first order deformations of F_0 corresponding to e and g_*e respectively. The lemma follows from the existence of an isomorphism $\alpha_g\colon F_1\to E_1$ fitting into a commutative diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&tF_0&\longrightarrow&F_1&\longrightarrow&F_0&\longrightarrow&0\\ &&\uparrow{\scriptstyle g^{-1}}&&\downarrow{\scriptstyle\alpha_g}&&\downarrow{\scriptstyle g}&&\\ 0&\longrightarrow&tF_0&\longrightarrow&E_1&\longrightarrow&F_0&\longrightarrow&0.\end{array}$$

The isomorphism α_g exists because $\mathrm{Aut}(F_0)$ also acts on the restriction to $X \times \mathcal{V}$ of the tautological quotient on $X \times Q$, compatibly with the action on $\mathcal{V}$. **q.e.d.**

Now let's assume $x \in \Sigma_Q^0$, and let

$$g = (\alpha, \beta) \in \mathbf{C}^* \times \mathbf{C}^* = \mathrm{Aut}(F_x) \qquad e = \sum_{i,j} e_{ij} \in \mathrm{Ext}^1(F_x, F_x).$$

By Lemma (1.4.16) we have

$$g_*(e) = geg^{-1} = e_{11} + \alpha\beta^{-1} e_{12} + \alpha^{-1}\beta e_{12} + e_{22}.$$

Equation (1.4.3) follows at once from the above formula.

1.5. The normal cone of Ω_Q^0.

Let $x \in \Omega_Q^0$. If V is a two-dimensional complex vector space, then

$$F_x \cong I_Z \otimes_{\mathbf{C}} V, \qquad \ell(Z) = n \qquad St(x) = \mathrm{Aut}(F_x)/\{\text{scalars}\} \cong \mathrm{PGL}(V).$$

Let $W := sl(V)$ be the Lie algebra of $\mathrm{PGL}(V)$, and let

$$(m, n) = 4\mathrm{Tr}(mn)$$

be the Killing form. The adjoint representation gives an identification

$$\mathrm{PGL}(V) \cong \mathrm{SO}(W).$$

Let

$$E_Z = \mathrm{Ext}^1(I_Z, I_Z).$$

Choose a non-zero two-form $\omega \in H^0(K_X)$: the composition

$$E_Z \times E_Z \overset{\mathrm{Yon}}{\to} \mathrm{Ext}^2(I_Z, I_Z) \overset{\mathrm{Tr}}{\to} H^2(\mathcal{O}_X) \overset{\cup\omega}{\to} H^2(K_X) \overset{\int_X}{\to} \mathbf{C}$$

defines a skew-symmetric form $\langle\ ,\ \rangle$, non-degenerate by Serre duality. Set

$$\mathrm{Hom}^\omega(W, E_Z) := \{\varphi\colon W \to E_Z |\ \varphi^*\langle\ ,\ \rangle \equiv 0\}.$$

Then $St(x) = \mathrm{SO}(W)$ acts (by composition) on $\mathrm{Hom}^\omega(W, E_Z)$. In this subsection we will prove the following result.

(1.5.1) Proposition. *Keep notation as above. Then Ω_Q^0 is a smooth locally closed subset of Q, and its normal cone is a locally trivial bundle over Ω_Q^0. If $x \in \Omega_Q^0$ there is a natural $St(x)$-equivariant isomorphism*

(1.5.2) $$(C_\Omega Q)_x \cong \mathrm{Hom}^\omega(W, E_Z).$$

I. Yoneda square. There is a natural isomorphism

$$\mathrm{Ext}^p(F_x, F_x) \cong \mathrm{Ext}^p(I_Z, I_Z) \otimes gl(V),$$

and Yoneda product is the tensor product of Yoneda product on $\mathrm{Ext}^p(I_Z, I_Z)$ times the composition on $gl(V)$. Hence if $\Upsilon\colon E_Z \otimes gl(V) \to sl(V)$ is Yoneda square,

$$\Upsilon(\sum_i e_i \otimes m_i) = \sum_{i,j} (e_i \cup e_j) \otimes (m_i m_j).$$

Since the composition

$$\mathrm{Ext}^2(I_Z, I_Z) \xrightarrow{\mathrm{Tr}} H^2(\mathcal{O}_X) \xrightarrow{\cup\omega} H^2(K_X) = \mathbf{C}$$

is an isomorphism we can write, with a slight abuse of notation,

$$\Upsilon\left(\sum_i e_i \otimes m_i\right) = \frac{1}{2}\sum_{i,j}\langle e_i, e_j\rangle [m_i, m_j], \tag{1.5.3}$$

where $[\cdot,\cdot]$ is the commutator (bracket). Now consider the decomposition

$$E_Z \otimes gl(V) = E_Z \otimes sl(V) \oplus E_Z \otimes \mathbf{C}\mathrm{Id}_V = E_Z \otimes W \oplus E_Z \otimes \mathbf{C}\mathrm{Id}_V, \tag{1.5.4}$$

and let $\overline{\Upsilon} := \Upsilon|_{E_Z \otimes W}$. Since bracket in $sl(V)$ corresponds to wedge product in W, we have

$$\overline{\Upsilon}(\sum_i e_i \otimes v_i) = \frac{1}{2}\sum_{i,j}\langle e_i, e_j\rangle v_i \wedge v_j.$$

The map $\overline{\Upsilon}$ has the following geometric interpretation. The Killing form $(\ ,\ )$ on W gives isomorphisms

$$E_Z \otimes W \cong \mathrm{Hom}(W, E_Z) \qquad W \cong \bigwedge^2 W^*.$$

Let $\varphi \in \mathrm{Hom}(W, E_Z)$; a straightforward computation shows that, via the above identifications,

$$\begin{array}{rccc} \overline{\Upsilon}\colon & \mathrm{Hom}(W, E_Z) & \to & \bigwedge^2 W^* \\ & \varphi & \mapsto & 2\varphi^*\langle\ ,\ \rangle. \end{array}$$

In particular

$$\overline{\Upsilon}^{-1}(0) = \mathrm{Hom}^{\omega}(W, E_Z). \tag{1.5.5}$$

(1.5.6) Lemma. $\mathbf{P}\overline{\Upsilon}^{-1}(0)$ *is a reduced irreducible complete intersection of three quadrics in* $\mathbf{P}(E_Z \otimes W)$. *Since* $\mathbf{P}\Upsilon^{-1}(0)$ *is a cone over* $\mathbf{P}\overline{\Upsilon}^{-1}(0)$ *with vertex* $\mathbf{P}(E_Z \otimes \mathbf{C}\mathrm{Id}_V)$, *it follows that* $\mathbf{P}\Upsilon^{-1}(0)$ *is a reduced irreducible complete intersection of three quadrics in* $\mathbf{P}(E_Z \otimes gl(V))$.

Proof. The symmetric bilinear form

$$\widetilde{\Upsilon}(\sum_i e_i \otimes v_i, \sum_j f_j \otimes w_j) := \frac{1}{2}\sum_{i,j}\langle e_i, f_j\rangle v_i \wedge w_j$$

is the polarization of $\overline{\Upsilon}$, hence the differential of $\overline{\Upsilon}$ at $\varphi = \sum_i e_i \otimes v_i$ is given by

$$\begin{array}{rccc} d\overline{\Upsilon}(\varphi)\colon & \mathrm{Hom}(W, E_Z) & \to & \bigwedge^2 W^* \cong W \\ & \sum_j f_j \otimes w_j & \mapsto & \frac{1}{2}\sum_{i,j}\langle e_i, f_j\rangle v_i \wedge w_j. \end{array} \tag{1.5.7}$$

From the above formula one easily gets

$$\mathrm{rk}(d\overline{\Upsilon}(\varphi)) = \begin{cases} 3 & \text{if } \mathrm{rk}\varphi \geq 2, \\ 2 & \text{if } \mathrm{rk}\varphi = 1, \\ 0 & \text{if } \varphi = 0. \end{cases} \tag{1.5.8}$$

Let $\mathrm{cr}\overline{\Upsilon}$ be the set of critical points of $\overline{\Upsilon}$. Since $c \geq 4$, Equation (1.5.8) gives

$$\dim \mathbf{P}(\mathrm{cr}\overline{\Upsilon}) = c + 2 < 3c - 4 = \dim \mathbf{P}\,(E_Z \otimes W) - 3.$$

This proves $\mathbf{P}\overline{\Upsilon}^{-1}(0)$ is a reduced complete intersection of three quadrics. To prove irreducibility, notice that by the above formulae

$$\operatorname{cod}\left(\operatorname{sing}\mathbf{P}\overline{\Upsilon}^{-1}(0),\mathbf{P}\overline{\Upsilon}^{-1}(0)\right)\geq 2. \tag{1.5.9}$$

On the other hand, Equation (1.5.8) shows that

$$\dim T_p\mathbf{P}\overline{\Upsilon}^{-1}(0)=\dim\mathbf{P}\overline{\Upsilon}^{-1}(0)+1$$

at every $p\in\operatorname{sing}\mathbf{P}\overline{\Upsilon}^{-1}(0)$. Now assume $\mathbf{P}\overline{\Upsilon}^{-1}(0)$ is reducible. Since $\mathbf{P}\overline{\Upsilon}^{-1}(0)$ is connected, there are two irreducible components which meet. The above equality shows that the intersection of these components is locally the intersection of two divisors in a smooth ambient space, hence it has codimension one in $\mathbf{P}\overline{\Upsilon}^{-1}(0)$. This contradicts (1.5.9). **q.e.d.**

II. The cone at the origin of the deformation space. Let $\mathcal{V}$ be a slice normal to the (closed) orbit $\mathrm{PGL}(N)x$: by Proposition (1.2.3) there is a natural isomorphism of germs $(\mathcal{V},x)\cong\operatorname{Def}(F_x)$.

(1.5.10) Proposition. *Keeping notation as above, there are natural isomorphisms of schemes*

$$C_x\mathcal{V}=H_x\mathcal{V}\cong\Upsilon^{-1}(0).$$

Proof. The proof is similar to that of Proposition (1.4.10). By (1.3.5) and Lemma (1.5.6), we have

$$(\mathbf{P}H_x\mathcal{V})_{red}=\mathbf{P}\Upsilon^{-1}(0).$$

By (1.5.6), $\mathbf{P}\Upsilon^{-1}(0)$ is a reduced irreducible complete intersection of quadrics. Since $\mathbf{P}H_x\mathcal{V}$ is cut out by quadrics, it follows that

$$\mathbf{P}H_x\mathcal{V}=\mathbf{P}\Upsilon^{-1}(0).$$

Next, consider the inclusion

$$C_x\mathcal{V}\subset H_x\mathcal{V}=\Psi^{-1}(0).$$

By Proposition (1.3.6) and (1.5.6) we have

$$\dim C_x\mathcal{V}=\dim\mathcal{V}\geq\dim\operatorname{Ext}^1(F_x,F_x)-3=\dim\Upsilon^{-1}(0).$$

By (1.5.6) $\Upsilon^{-1}(0)$ is reduced irreducible, hence $C_x\mathcal{V}=\Upsilon^{-1}(0)$. **q.e.d.**

III. Proof of Proposition (1.5.1). Let $\mathcal{V}$ be a slice normal to the (closed) orbit $\mathrm{PGL}(N)x$, and let

$$\mathcal{W}:=\mathcal{V}\cap\Omega_Q^0.$$

By Corollary (1.2.2) there is a $St(x)$-equivariant isomorphism

$$\left(C_\Omega Q\right)_x\cong\left(C_{\mathcal{W}}\mathcal{V}\right)_x. \tag{1.5.11}$$

The following result is analogous to Claim (1.4.12).

(1.5.12) Claim. *Keeping notation as above, $\mathcal{W}$ is smooth at x and*

$$T_x\mathcal{W}=E_Z\otimes\mathbf{C}\mathrm{Id}_V. \tag{1.5.13}$$

(See Decomposition (1.5.4).) Furthermore, shrinking $\mathcal{V}$ if necessary, we can assume that

$$\dim T_{x'}\mathcal{V}=\dim T_x\mathcal{V}\ \textit{for all}\ x'\in\mathcal{W}.$$

Proof. Identifying the germ $(\mathcal{V}, x)$ with $(\mathrm{Def}(F_x), 0)$ (see Proposition (1.2.3)), we see that a neighborhood of $x \in \mathcal{W}$ parametrizes all sheaves of the form $I_{Z'} \oplus I_{Z'}$, for Z' near Z. In particular

$$\dim \mathcal{W} \geq c = \dim E_Z.$$

Hence Equation (1.5.13) implies the first statement. We prove (1.5.13). Let

$$\epsilon \in T_0 \mathrm{Def}(F_x) = \mathrm{Ext}^1(F_x, F_x) = E_Z \otimes gl(V),$$

and let $\mathcal{F}$ be the corresponding first order deformation of F_x. Then ϵ is tangent to $\mathcal{W}$ if and only if, for every exact sequence

$$(*) \qquad 0 \to \mathbf{C} \to V \to \mathbf{C} \to 0$$

the following holds: the exact sequence

$$(\dagger) \qquad 0 \to I_Z \to F_x \to I_Z \to 0$$

obtained tensoring $(*)$ by I_Z, lifts to $\mathcal{F}$. Choosing a basis of V adapted to $(*)$, write ϵ as a 2×2-matrix with entries in E_Z: then by [O2, (1.17)] Exact sequence $(\dagger)$ lifts to $\mathcal{F}$ if and only if ϵ is uppertriangular. Since this must hold for any choice of $(*)$, the matrix ϵ must be a scalar, i.e. $\epsilon \in E_Z \otimes \mathbf{C}\mathrm{Id}_V$. To prove the second statement, notice that the family $\mathcal{F}$ of sheaves parametrized by $\mathcal{V}$ is complete at all x' in a neighborhood of x, and that $\dim \mathrm{Ext}^1(F_{x'}, F_{x'})$ is constant for $x' \in \mathcal{W}$. **q.e.d.**

Using (1.5.12) and arguing as in the proof of Proposition (1.4.1) we see that Ω_Q^0 is smooth. Let's prove Isomorphism (1.5.2) (we leave it to the reader to verify that the normal cone of Ω_Q^0 is locally trivial). By (1.5.11)-(1.5.5) we must give an isomorphism

$$(1.5.14) \qquad (C_{\mathcal{W}}\mathcal{V})_x \cong \overline{\Upsilon}^{-1}(0).$$

We proceed as in the proof of (1.4.15). By (1.5.12) the normal bundle $N_{\mathcal{W}}\mathcal{V}$ is defined, hence we have inclusions of cones

$$(C_{\mathcal{W}}\mathcal{V})_x \subset (H_{\mathcal{W}}\mathcal{V})_x \subset (N_{\mathcal{W}}\mathcal{V})_x \cong E_Z \otimes W.$$

(The last isomorphism follows from (1.5.13).) By (1.3.2) and (1.5.10) we have

$$(H_{\mathcal{W}}\mathcal{V})_x \cong \overline{\Upsilon}^{-1}(0).$$

Furthermore

$$\dim (C_{\mathcal{W}}\mathcal{V})_x \geq \dim \mathcal{V} - \dim \mathcal{W} \geq \dim (H_{\mathcal{W}}\mathcal{V})_x = \dim \overline{\Upsilon}^{-1}(0),$$

where the second inequality follows from (1.3.6). Since $(C_{\mathcal{W}}\mathcal{V})_x \subset (H_{\mathcal{W}}\mathcal{V})_x = \overline{\Upsilon}^{-1}(0)$ and $\overline{\Upsilon}^{-1}(0)$ is reduced irreducible, we get (1.5.14). Finally, to prove that Isomorphism (1.5.2) is $St(x)$-equivariant, we apply Lemma (1.4.16) to $(\mathcal{V}, x) = \mathrm{Def}(F_x)$. If $g \in \mathrm{Aut}(F_x) = \mathrm{GL}(V)$, and $\epsilon \in \mathrm{Ext}^1(F_x, F_x) = E_Z \otimes gl(V)$,

$$g_*(\epsilon) = g \cup \epsilon \cup g^{-1}.$$

Since Yoneda products are given by composition in $gl(V)$, the automorphism group $\mathrm{GL}(V)$ acts by conjugation on $E_Z \otimes gl(V)$, i.e. we get the standard action of $\mathrm{SO}(W)$ on W.

1.6. Description of Ω_R^{ss}.

Recall that $\pi_R \colon R \to Q$ is the blow-up of Ω_Q. Let $\Omega_R \subset R$ be the exceptional divisor. By Theorem (1.1.2) we have

$$\pi_R(\Omega_R^{ss}) \subset \Omega_Q^{ss} = \Omega_Q^0.$$

In order to describe Ω_R^{ss} we will need a general result in the style of Theorem (1.1.2). Let G be a reductive group acting linearly on a projective scheme Y, let $W \hookrightarrow Y$ be a G-invariant closed subscheme, let $\pi \colon \widetilde{Y} \to Y$ be the blow-up of W. Then G acts also on $D_\ell := \pi^* \mathcal{O}_Y(\ell)(-E)$, where E is the exceptional divisor. The following lemma can be extracted from [K]; we sketch a proof for the reader's convenience.

(1.6.1) Lemma. *Keep notation as above. If $\ell \gg 0$ then the following holds. Let $\widetilde{y} \in \widetilde{Y}$ be such that $y = \pi(\widetilde{y})$ is semistable with orbit closed in Y^{ss}. Then $\widetilde{y}$ is G-(semi)stable if and only if it is (semi)stable for the action of $St(y)$ on $\pi^{-1}(y)$ (with the linearization obtained by restriction).*

Proof. First we prove the lemma when G is a torus T. Let ℓ_0 be such that $\pi^*\mathcal{O}_Y(\ell_0)(-E)$ is very ample. Let $\{\sigma_i\}$, $\{\tau_j\}$ be diagonal bases for the action of T on $H^0(D_{\ell_0})$ and $H^0(\mathcal{O}_Y(1))$ respectively. Let $\{p_i\}$, $\{q_j\}$ be the corresponding sets of characters of T. Letting Φ be the lattice of characters of T, and $\Phi_{\mathbf{R}} := \Phi \otimes \mathbf{R}$, we set

$$\begin{aligned} \Delta_\sigma &:= \text{convex hull in } \Phi_{\mathbf{R}} \text{ of the } p_i \text{ such that } \sigma_i(\widetilde{y}) \neq 0, \\ \Delta_\tau &:= \text{convex hull in } \Phi_{\mathbf{R}} \text{ of the } q_j \text{ such that } \tau_j(y) \neq 0, \\ \Lambda &:= \text{linear span of } \Delta_\tau. \end{aligned}$$

Since $O(y)$ is closed in Y^{ss}, there is an open $\mathcal{U} \subset \Lambda$ such that $0 \in \mathcal{U} \subset \Delta_\tau$. Thus there exists m_0 such that the following holds for all $m \geq m_0$. If $V \subset \Phi_{\mathbf{R}}$ is a codimension-one subspace not containing Λ, the image of $(\Delta_\sigma + m\Delta_\tau)$ in $\Phi_{\mathbf{R}}/V$ contains an open neighborhood of the origin. We claim that if $\ell = (\ell_0 + m)$, with $m \geq m_0$, the lemma holds for $\widetilde{y}$. By the numerical criterion for (semi)stability [Mm, (2.1)] it suffices to show that if λ is a one-parameter subgroup of G not contained in $St(y)$ then λ does not desemistabilize $\widetilde{y}$. Semistability with respect to the complete linear system $H^0(D_\ell)$ is the same as with respect to the sublinear system

$$H^0(D_{\ell_0}) \otimes H^0(\pi^*\mathcal{O}_Y(m)).$$

Identifying one-parameter subgroups of T with $\mathrm{Hom}(\Phi, \mathbf{Z})$, we get a one-to-one correspondence

$$\{\lambda\colon \mathbf{C}^* \to T \mid \lambda(\mathbf{C}^*)y = y\} \leftrightarrow \{f \in \mathrm{Hom}(\Phi, \mathbf{Z}) \mid \Delta_\tau \subset \ker(f \otimes \mathbf{R})\}.$$

Hence, with our choice of m, if λ does not fix y then λ does not desemistabilize $\widetilde{y}$ with respect to the above sublinear system. This proves the result for a single y, but in fact we can choose an m_0 which works for every y. Now let G be an arbitrary reductive group. Let $T < G$ be a maximal torus. Since the lemma holds for the action of T, and since every one-parameter subgroup of G is conjugate to a subgroup of T, the numerical criterion for (semi)stability shows that the lemma holds also for the action of G. **q.e.d.**

Remark. If y is stable, the proof above is exactly Kirwan's proof of (1.1.4).

By Proposition (1.5.1) there is a locally trivial fibration

$$\begin{array}{ccc} \mathbf{P}\mathrm{Hom}^\omega(W, E_Z) & \longrightarrow & \pi_R^{-1}(\Omega_Q^0) \\ \downarrow & & \downarrow \\ x & \in & \Omega_Q^0. \end{array}$$

(1.6.2) Proposition. *Keeping notation as above, a point $[\varphi] \in \mathbf{P}\mathrm{Hom}^\omega(W, E_Z)$ is $\mathrm{PGL}(N)$-semistable if and only if:*

$$\mathrm{rk}\varphi \begin{cases} \geq 2, & \textit{or} \\ = 1 & \textit{and } \ker\varphi^\perp \textit{ is non-isotropic.} \end{cases}$$

Proof. By Lemma (1.6.1) $[\varphi]$ is $\mathrm{PGL}(N)$-semistable if and only if it is semistable for the $\mathrm{SO}(W)$-action on $\mathbf{P}\mathrm{Hom}^\omega(W, E_Z)$. The proposition follows easily from the numerical criterion for semistability. **q.e.d.**

1.7. Description of Σ_R^{ss}.

Recall that $\Sigma_R \subset R$ is the strict transform of Σ_Q under the blow-up $\pi_R\colon R \to Q$. We are interested in Σ_R^{ss}, a locally closed subset of R. The main result of this subsection is the following.

(1.7.1) Proposition. *Keeping notation as above,*

(1) *Σ_R^{ss} is smooth,*
(2) *The scheme-theoretic intersection of Σ_R^{ss} and Ω_R is smooth and reduced.*

(3) *The normal cone of* Σ_R^{ss} *in* R *is a locally trivial bundle over* Σ_R^{ss}*, with fiber the cone over a smooth quadric in* $\mathbf{P}^{2c-5}$.

I. Description of $\Sigma_R^{ss} \setminus \Omega_R$. We will prove that

$$\Sigma_R^{ss} \setminus \Omega_R = \pi_R^{-1}(\Sigma_Q^0). \tag{1.7.2}$$

By Theorem (1.1.2) we know

$$\Sigma_R^{ss} \setminus \Omega_R \subset \pi_R^{-1}(\Sigma_Q^{ss} - \Omega_Q).$$

From (1.1.7) one gets

$$\Sigma_Q^0 \subset \left(\Sigma_Q^{ss} - \Omega_Q\right) \subset \Sigma_Q^0 \cup \Gamma_Q^0.$$

In fact the middle term is equal to the third term, but we will not need this because

$$\pi_R^{-1}(\Gamma_Q^0) \cap R^{ss} = \emptyset. \tag{1.7.3}$$

Before proving (1.7.3) we give a general lemma. We assume G is a reductive group acting linearly on a complex projective scheme Y, and $V \subset Y$ is a G-invariant closed subscheme. Let $\pi: \widetilde{Y} \to Y$ be the blow-up of V.

(1.7.4) Lemma. *Keep notation as above. Let* $\widetilde{x} \in \widetilde{Y}$ *be a point such that* $x := \pi(\widetilde{x})$ *satisfies:*

$$x \notin V, \qquad \overline{O(x)} \cap V^{ss} \neq \emptyset.$$

Then $\widetilde{x}$ *is not semistable.*

Proof. It follows from our hypothesis that there exists a one-parameter subgroup $\lambda: \mathbf{C}^* \to G$ such that

$$\lim_{t \to 0} \lambda(t)x = y \in V^{ss}.$$

Let E be the exceptional divisor of π, and

$$\sigma_0, \dots, \sigma_r \in H^0\left(\pi^* \mathcal{O}_Y(\ell)(-E)\right)$$

be a diagonal basis for the action of λ; set $\lambda(t)\sigma_i = t^{n_i}\sigma_i$. We make the identification

$$H^0\left(\pi^* \mathcal{O}_Y(\ell)(-E)\right) \cong H^0\left(I_V(\ell)\right).$$

Since y is semistable, $n_i \geq 0$ for all i such that $\sigma_i(\widetilde{x}) = \sigma_i(x) \neq 0$. We will show that in fact $n_i > 0$ for all such i; this will prove $\widetilde{x}$ is not semistable. Suppose $n_i = 0$; then σ_i is λ-invariant, hence $\sigma_i(x) \neq 0$ implies $\sigma_i(y) \neq 0$. This is absurd because σ_i vanishes on V. **q.e.d.**

Let's prove (1.7.3). Assume $\pi_R(y) = x \in \Gamma_Q^0$. Then $\overline{O(x)} \cap \Omega_Q^0 \neq \emptyset$, hence by Lemma (1.7.4) y is not semistable. This proves the left-hand side of (1.7.2) is contained in the right-hand side. Let's show that $\pi_R^{-1}(\Sigma_Q^0)$ is contained in R^{ss}. For $y \in \pi_R^{-1}(\Sigma_Q^0)$, let $x = \pi_R(y)$. Since $O(x)$ is closed in Q^{ss}, and disjoint from the closed $\mathrm{PGL}(N)$-invariant subset Ω_Q^0, there exists a $\mathrm{PGL}(N)$-invariant section $\sigma \in H^0(\mathcal{O}_Q(\ell))$ such that $\sigma(x) \neq 0$ and σ vanishes on Ω_Q. Viewing σ as a (invariant) section of $\pi_R^*\mathcal{O}_Q(\ell)(-E)$ we see that y is semistable.

II. Description of $\Sigma_R^{ss} \cap \Omega_R$. Of course $\Sigma_R^{ss} \cap \Omega_R$ is contained in $\Omega_R^{ss} \subset \pi_R^{-1}\Omega_Q^{ss} = \pi_R^{-1}\Omega_Q^0$. For $x \in \Omega_Q^0$, set $F_x = (I_Z \oplus I_Z)$, where $[Z] \in X^{[n]}$, and let

$$\mathrm{Hom}_k(W, E_Z) := \{\varphi \in \mathrm{Hom}(W, E_Z) |\ \mathrm{rk}\varphi \leq k\},$$
$$\mathrm{Hom}_k^{\omega}(W, E_Z) := \mathrm{Hom}_k(W, E_Z) \cap \mathrm{Hom}^{\omega}(W, E_Z).$$

(1.7.5) Lemma. *Let $x \in \Omega_Q^0$, and set $F_x = (I_Z \oplus I_Z)$. Then*

$$\pi_R^{-1}(x) \cap \Sigma_R^{ss} = \mathbf{P}\mathrm{Hom}_1^{ss}(W, E_Z). \tag{1.7.6}$$

Proof. If $y \in \Sigma_Q^0$ then $St(y) \cong \mathbf{C}^*$. Thus $\dim St(\widetilde{y}) \geq 1$ for all $\widetilde{y} \in \Sigma_R$. In particular, if

$$[\varphi] \in \pi_R^{-1}(x) \cap \Sigma_R^{ss},$$

the stabilizer $St([\varphi])$ ($< \mathrm{SO}(W)$) is positive dimensional. An easy analysis shows that $St([\varphi])$ is positive dimensional if and only if $\mathrm{rk}\varphi = 1$. By Proposition (1.6.2) we conclude that the left-hand side of (1.7.6) is contained in the right-hand side. Let's prove inclusion in the other direction. Assume $[\varphi] \in \mathbf{P}\mathrm{Hom}_1^{ss}(W, E_Z)$. The identifications

$$W^* \cong sl(V)^\vee \cong sl(V)$$

(the second isomorphism is given by the Killing form), allow us to write

$$\varphi = m \otimes \alpha, \qquad m \in sl(V),\ \alpha \in E_Z,\ \mathrm{Tr}(m^2) \neq 0.$$

Since $\mathrm{Tr}(m^2) \neq 0$ we can diagonalize m, hence using a basis of eigenvectors we can write

$$\varphi = \begin{bmatrix} e & 0 \\ 0 & -e \end{bmatrix} \qquad e \in E_Z. \tag{1.7.7}$$

Since $X^{[n]}$ is smooth there exist sheaves $\mathcal{L}$, $\mathcal{L}'$ on $X \times C$, flat over C, where C is a smooth curve, such that the following holds. For a certain point $0 \in C$

$$\mathcal{L}_0 \cong \mathcal{L}'_0 \cong I_Z,$$

and furthermore, if κ, κ' are the Kodaira-Spencer maps of $\mathcal{L}$, $\mathcal{L}'$ at 0, respectively, then

$$\kappa(\partial/\partial t) = e \qquad \kappa'(\partial/\partial t) = -e, \qquad \partial/\partial t \in T_0 C.$$

Lastly we can assume $\mathcal{L}_p \not\cong \mathcal{L}'_p$ for all $p \neq 0$. Set $\mathcal{G} := \mathcal{L} \oplus \mathcal{L}'$. If $\mathcal{V}$ is a slice normal to $O(x)$ at x then, by versality of the tautological quotient on $X \times \mathcal{V}$ (Proposition (1.2.3)), there exists a map

$$f\colon C \to \mathcal{V} \qquad f(0) = x$$

such that $\mathcal{G}$ is the pull-back of the tautological family of quotient sheaves on $X \times \mathcal{V}$. (We might have to shrink $C \ni 0$.) Hence the differential of f at 0 has image spanned by φ (see (1.7.7)). Since $f^{-1}\Omega_Q = \{0\}$, there is a well-defined lift $\widetilde{f}\colon C \to R$ of f, and clearly $\widetilde{f}(C) \subset \Sigma_R$. Thus

$$[\varphi] = \widetilde{f}(0) \in \Sigma_R^{ss} \cap \Omega_R.$$

q.e.d.

III. Explicit construction of Σ_R^{ss} and proof of Items (1)-(2) of (1.7.1). Let

$$\beta\colon \mathcal{X}^{\{n\}} \to X^{[n]} \times X^{[n]}$$

be the blow-up of the diagonal. Set $N := h^0(I_Z(k) \oplus I_W(k))$, where $[Z], [W] \in X^{[n]}$. Let

$$\alpha\colon \widetilde{\mathcal{U}} \to \mathcal{X}^{\{n\}}$$

be the principal $\mathrm{PGL}(N)$-fibration whose fiber over $y \in \mathcal{X}^{\{n\}}$ is

$$\mathbf{P}\mathrm{Isom}\left(\mathbf{C}^N, H^0\left(I_Z(k) \oplus I_W(k)\right)\right), \tag{1.7.8}$$

where $([Z],[W])=\beta(y)$. Over $\widetilde{\mathcal{U}}\times X$ there is a tautological family of quotients

$$\mathcal{O}^{(N)}_{\widetilde{\mathcal{U}}\times X}\to\mathcal{L}_1(k)\oplus\mathcal{L}_2(k)\to 0. \tag{1.7.9}$$

Here $\mathcal{L}_i$ is defined as follows. Let

$$p_i\colon X^{[n]}\times X^{[n]}\times X\to X^{[n]}\times X,\quad i=1,2,$$

be the projection which forgets the i-th factor, and let $\mathcal{Z}\subset X^{[n]}\times X$ be the tautological subscheme; then

$$\mathcal{L}_i(k):=((\beta\circ\alpha)\times\mathrm{id}_X)^*p_i^*\left(I_{\mathcal{Z}}\otimes\mathcal{O}_X(k)\right).$$

The family of quotients (1.7.9) defines a morphism

$$\widetilde{h}\colon\widetilde{\mathcal{U}}\to Q\qquad\widetilde{h}(\widetilde{\mathcal{U}})=\left(\Sigma^0_Q\cup\Omega^0_Q\right).$$

There is an action of O(2) on $\widetilde{\mathcal{U}}$. In fact realize O(2) as the subgroup of PGL(2) generated by

$$\mathrm{SO}(2)=\left\{\theta_\alpha:=\begin{bmatrix}\alpha&0\\0&\alpha^{-1}\end{bmatrix}\right\}/\{\pm\mathrm{Id}\},\qquad\tau:=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Then each θ_α can be viewed as an isomorphism of $\mathcal{L}_1(k)\oplus\mathcal{L}_2(k)$, hence it acts on $\widetilde{\mathcal{U}}$. Similarly, let $\iota\colon\mathcal{U}\to\mathcal{U}$ be the map induced by the involution interchanging the factors of $X^{[n]}\times X^{[n]}$: viewing τ as an isomorphism between $\mathcal{L}_1(k)\oplus\mathcal{L}_2(k)$ and $\iota^*\left(\mathcal{L}_1(k)\oplus\mathcal{L}_2(k)\right)$ we get an action of τ on $\widetilde{\mathcal{U}}$. Consider the G.I.T. quotient

$$\mathcal{U}:=\widetilde{\mathcal{U}}//\mathrm{O}(2).$$

A word about the linearization. Taking first the quotient by SO(2) and then by O(2)/SO(2) we see that a choice of linearization is relevant only for the quotient $\widetilde{\mathcal{U}}//\mathrm{SO}(2)$. Since SO(2) preserves the fibers of α we need only specify what is the linearization on each fiber (1.7.8); but the fiber is affine, thus a linearization is "not needed", or more precisely we choose the trivial linearization of the trivial line-bundle. The action of O(2) is free, hence $\mathcal{U}$ is an orbit space, and since $\widetilde{\mathcal{U}}$ is smooth, also $\mathcal{U}$ is smooth. The map $\widetilde{h}$ is constant on $O(2)$-orbits, hence it factors through a map

$$h_Q\colon\mathcal{U}\to\left(\Sigma^0_Q\cup\Omega^0_Q\right).$$

The following proposition proves Items (1)-(2) of (1.7.1).

(1.7.10) Proposition. *Keep notation as above. The map h_Q lifts to a map $h_R\colon\mathcal{U}\to R$, whose image is Σ^{ss}_R. Furthermore*

$$h_R\colon\mathcal{U}\to\Sigma^{ss}_R$$

is an isomorphism, in particular Σ^{ss}_R is smooth. Finally, the scheme-theoretic intersection of Σ^{ss}_R and Ω_R is smooth and reduced.

Proof. First let's prove that h_Q lifts. Let $D^{[n]}\subset X^{[n]}\times X^{[n]}$ be the diagonal, and set

$$D^{\{n\}}:=\text{exc. div. of }\beta=\beta^{-1}D^{[n]},\qquad\widetilde{D}:=\alpha^*D^{\{n\}},\qquad D:=\widetilde{D}//\mathrm{O}(2).$$

Notice that $\widetilde{D}$ is an O(2) invariant divisor, hence D is a (Cartier) divisor. We will show that

$$\widetilde{h}^*I_{\Omega_Q}=\mathcal{O}_{\widetilde{\mathcal{U}}}(-\widetilde{D}). \tag{†}$$

This implies that $h_Q^*I_{\Omega_Q}=\mathcal{O}_{\mathcal{U}}(-D)$, and hence, by the universal property of blow-up, h_Q lifts. As sets $\widetilde{h}^{-1}\Omega_Q=\widetilde{D}$, hence to prove (†) it will suffice to show the following: for any $u\in\widetilde{D}$, there exists a tangent vector $v\in T_u\widetilde{\mathcal{U}}$ such that

$$(\widetilde{h})_*(v)\notin T_{\widetilde{h}(u)}\Omega_Q. \tag{$*$}$$

To prove it, let $y = \alpha(u)$, $([Z],[Z]) = \beta(y)$, and $x = \widetilde{h}(y)$. The normal bundle of $D^{[n]}$ in $X^{[n]} \times X^{[n]}$ is canonically identified with the (-1)-eigenbundle for the action on

$$\left(T_{X^{[n]}\times X^{[n]}}\right)|_{D^{[n]}}$$

of the involution interchanging the factors of $X^{[n]} \times X^{[n]}$, hence

$$\beta^{-1}([Z],[Z]) = \mathbf{P}\left\{(e,e') \in \mathrm{Ext}^1(I_Z,I_Z) \oplus \mathrm{Ext}^1(I_Z,I_Z)|\ e = -e'\right\}.$$

Thus, if $w \in T_y\mathcal{X}^{\{n\}}$ is transversal to $T_yD^{\{n\}}$,

$$\beta_*(w) = (e,-e) \qquad 0 \neq e \in \mathrm{Ext}^1(I_Z,I_Z).$$

Since α is a smooth fibration there exists $v \in T_u\widetilde{\mathcal{U}}$ such that $\alpha_*(v) = w$. Identifying $(C_\Omega Q)_x$ with $\mathrm{Hom}^\omega(W,E_Z)$ as in (1.5.2) we see that

$$\widetilde{h}_*(v) = e \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.7.11}$$

In particular, since the right-hand side is non-zero, we have proved $(*)$. Thus the lift h_R exists. Now let's prove that h_R is an isomorphism between $\mathcal{U}$ and Σ_R^{ss}. Clearly $h_R(\mathcal{U} - D) = \pi_R^{-1}\Sigma_Q^0$, and hence by (1.7.2) we get a surjective map

$$h_R|_{(\mathcal{U}-D)} \colon (\mathcal{U} - D) \to (\Sigma_R^{ss} \setminus \Omega_R).$$

The above map is clearly one-to-one, and since $(\Sigma_R^{ss} \setminus \Omega_R)$ is smooth it must be an isomorphism. Next, let's examine $h_R|_D$. The restriction

$$\widetilde{h}|_{\widetilde{D}} \colon \widetilde{D} \to \Omega_Q^0$$

is surjective; if $x \in \Omega_Q^0$, and $F_x = V \otimes I_Z$, then

$$\widetilde{h}^{-1}(x) = \mathrm{PGL}(V) \times \mathbf{P}\left(\mathrm{Ext}^1(I_Z,I_Z)\right),$$

and thus

$$h_Q^{-1}(x) = (\mathrm{PGL}(V)/\mathrm{O}(2)) \times \mathbf{P}(E_Z) = \mathbf{P}\left(\{\varphi \in W|\ \varphi \text{ is non-isotropic}\}\right) \times \mathbf{P}(E_Z).$$

On the other hand, by (1.7.6) the term on the right is identified, via the Segre embedding, with $\pi_R^{-1}(x) \cap \Sigma_R^{ss}$. It follows from (1.7.11) that

$$h_R|_{h_Q^{-1}(x)} \colon h_Q^{-1}(x) \to \pi_R^{-1}(x) \cap \Sigma_R^{ss}$$

is the Segre isomorphism. Since $\Sigma_R^{ss} \cap \Omega_R$ is smooth, we get that

$$h_R|_D \colon D \to \Sigma_R^{ss} \cap \Omega_R \tag{$\bullet$}$$

is an isomorphism. Now we can finish proving that h_R is an isomorphism onto Σ_R^{ss}. First notice that h_R is a homeomorphism, hence Σ_R^{ss} is unibranch at every point of $\Sigma_R^{ss} \cap \Omega_R$. Thus it remains only to show that the differential $dh_R(u)$ is an isomorphism for all $u \in D$. Since $(\bullet)$ is an isomorphism it is sufficient to verify that $(h_R)_*(v) \notin T_{h_R(u)}\Omega_R$ for some $v \in T_u\mathcal{U}$: look at $(*)$. The proof just given also shows that the scheme-theoretic intersection of Σ_R^{ss} and Ω_R is smooth and reduced. **q.e.d.**

IV. Proof of Item (3) of Proposition (1.7.1). Since π_R is an isomorphism outside Ω_R, it follows from (1.7.2) that the normal cone of $(\Sigma_R^{ss} \setminus \Omega_R)$ in R is isomorphic to that of Σ_Q^0 in Q. Thus "outside Ω_R" Item (3) of (1.7.1) follows from Proposition (1.4.1). Now let $y \in \Sigma_R^{ss} \cap \Omega_R$, and set $x = \pi_R(y)$. Following the notation of Lemma (1.7.5) we let $y = [\varphi]$, where $\varphi \in \mathrm{Hom}_1(W,E_Z)$. Let ω_φ be the symplectic form induced by ω on $\left(\mathrm{Im}\varphi^\perp/\mathrm{Im}\varphi\right)$. We let $\mathrm{Hom}^{\omega_\varphi}(\ker\varphi, \mathrm{Im}\varphi^\perp/\mathrm{Im}\varphi)$ be the set of homomorphisms whose image is ω_φ-isotropic.

Proposition. *Keep notation as above. If* $[\varphi]\in\Sigma^{ss}_R\cap\Omega_R$ *there is a* $St([\varphi])$*-equivariant isomorphism*

$$(C_\Sigma R)_{[\varphi]}\cong \mathrm{Hom}^{\omega_\varphi}(\mathrm{Ker}\varphi,\mathrm{Im}\varphi^{\perp}/\mathrm{Im}\varphi). \tag{1.7.12}$$

The above proposition implies that Item (3) of (1.7.1) holds also "over $\Sigma^{ss}_R\cap\Omega_R$". In fact the right-hand side of (1.7.12) embeds into the $2(c-2)$-dimensional vector space $\mathrm{Hom}(\mathrm{Ker}\varphi,\mathrm{Im}\varphi^{\perp}/\mathrm{Im}\varphi)$, and since ω_φ is non-degenerate the image is the affine cone over a smooth projective quadric. Let's prove (1.7.12). By Item (2) of (1.7.1) the Cartier divisor Ω_R intersects transversely Σ^{ss}_R, hence

$$(C_\Sigma R)_{[\varphi]}\cong (C_{\Sigma\cap\Omega}\Omega_R)_{[\varphi]}\,.$$

Furthermore $\Omega^{ss}_R\to\Omega^0_Q$ is a locally-trivial fibration with smooth base, hence we can replace the right-hand side of the above equation by the normal in the fiber through $[\varphi]$. More precisely, if $x=\pi_R([\varphi])$,

$$(C_{\Sigma\cap\Omega}\Omega_R)_{[\varphi]}\cong \left(C_{\mathbf{P}\mathrm{Hom}_1(W,E_Z)}\mathbf{P}\mathrm{Hom}^{\omega}(W,E_Z)\right)_{[\varphi]}\,.$$

Thus (1.7.12) follows from the following result.

Lemma. *Let* $[\varphi]\in\mathbf{P}\mathrm{Hom}_1(W,E_Z)$ *(not necessarily semistable). There is a* $St([\varphi])$*-equivariant isomorphism*

$$\left(C_{\mathbf{P}\mathrm{Hom}_1(W,E_Z)}\mathbf{P}\mathrm{Hom}^{\omega}(W,E_Z)\right)_{[\varphi]}\cong \mathrm{Hom}^{\omega_\varphi}(\mathrm{Ker}\varphi,\mathrm{Im}\varphi^{\perp}/\mathrm{Im}\varphi). \tag{1.7.13}$$

Proof. Clearly

$$\left(C_{\mathbf{P}\mathrm{Hom}_1(W,E_Z)}\mathbf{P}\mathrm{Hom}^{\omega}(W,E_Z)\right)_{[\varphi]}\cong \left(C_{\mathrm{Hom}_1(W,E_Z)}\mathrm{Hom}^{\omega}(W,E_Z)\right)_{\varphi},$$

and we will work with the right-hand side. We will show that the Hessian cone of $\mathrm{Hom}_1(W,E_Z)$ in $\mathrm{Hom}^{\omega}(W,E_Z)$ satisfies the hypothesis of Lemma (1.3.6), hence the normal cone will be equal to the Hessian cone. First we check that the Hessian cone is defined, i.e. that (1.3.1) is satisfied. First $\mathrm{Hom}_1(W,E_Z)$ is smooth, and secondly, since $\mathrm{Hom}^{\omega}(W,E_Z)$ is the zero-scheme of Υ_0, and the differential $d\Upsilon_0$ has constant rank along $\mathrm{Hom}_1(W,E_Z)$ by (1.5.6), the tangent space to $\mathrm{Hom}^{\omega}(W,E_Z)$ has constant rank along $\mathrm{Hom}_1(W,E_Z)$. By (1.3.5) the Hessian cone is given by the zeroes of the Hessian map along $\mathrm{Hom}^{\omega}_1(W,E_Z)$. Let's compute the Hessian cone. For this we will choose bases $\{e_1,\ldots,e_{2n}\}$ of E_Z and $\{v_1,v_2,v_3\}$ of W such that $\varphi=e_1\otimes v_1$, and so that

$$\langle e_i,e_j\rangle=\begin{cases}1 & \text{if } i=2q-1,\ j=2q,\\ -1 & \text{if } i=2q,\ j=2q-1,\\ 0 & \text{otherwise.}\end{cases}$$

Using Formula (1.5.7) for the differential $d\Upsilon_0$ we get

$$d\Upsilon_0(\varphi)\left(\sum_{ij}Z_{ij}e_i\otimes v_j\right)=-\frac{1}{2}Z_{2,2}v_1\wedge v_2-\frac{1}{2}Z_{2,3}v_1\wedge v_3, \tag{1.7.14}$$

hence

$$(T\mathrm{Hom}^{\omega}(W,E_Z))_{\varphi}=\left\{\sum_{ij}Z_{ij}e_i\otimes v_j|\ Z_{2,2}=Z_{2,3}=0\right\}.$$

Furthermore

$$(T\mathrm{Hom}_1(W,E_Z))_{\varphi}=\left\{\sum_{ij}Z_{ij}e_i\otimes v_j|\ Z_{ij}=0 \text{ if } i\geq 2,\ j\geq 2.\right\}.$$

Thus we have an isomorphism

$$(N_{\mathrm{Hom}_1}\mathrm{Hom}^{\omega}(W,E_Z))_{\varphi}\cong\left\{\sum_{\substack{3\leq i\\ 2\leq j}}Z_{ij}e_i\otimes v_j\right\}. \tag{1.7.15}$$

To give an intrinsic formulation notice that there is a natural isomorphism

$$\left(N_{\mathrm{Hom}_1}\mathrm{Hom}(W,E_Z)\right)_\varphi \cong \mathrm{Hom}\left(\mathrm{Ker}\varphi, E_Z/\mathrm{Im}\varphi\right),$$

hence Isomorphism (1.7.15) can be read as

$$\left(N_{\mathrm{Hom}_1}\mathrm{Hom}^\omega(W,E_Z)\right)_\varphi \cong \left\{\alpha\colon \mathrm{Ker}\varphi \to E_Z/\mathrm{Im}\varphi \mid \mathrm{Im}\alpha \subset \left(\mathrm{Im}\varphi^\perp/\mathrm{Im}\varphi\right)\right\}.$$

Referring to (1.3.4), it follows from (1.7.14) that $p_{\mathcal{K}}(\bullet) = (\bullet, v_1)$: a computation gives the following equation (see (1.3.4)) for the Hessian cone of $\mathrm{Hom}_1(W,E_Z)$ in $\mathrm{Hom}^\omega(W,E_Z)$ at φ:

$$\sum_{2\le q\le n}\left(Z_{2q-1,2}Z_{2q,3} - Z_{2q,2}Z_{2q-1,3}\right) = 0.$$

In particular the hypothesis of Lemma (1.3.6) is satisfied, hence the normal cone is equal to the Hessian cone. Since the equation above defines the right-hand side of (1.7.13), we have proved that (1.7.13) holds. The isomorphism is clearly $St([\varphi])$-equivariant.

The action of stabilizers. We are interested in the action of $St(y)$ on $(C_\Sigma R)_y$ at a point $y \in \Sigma_R^{ss}$. If y is outside Ω_R, then by (1.7.2) the action is described by (1.4.3). Now let's assume $y \in \Sigma_R^{ss} \cap \Omega_R$, and set $x = \pi_R(y)$. Apply Lemma (1.7.5) and write $y = [\varphi]$, where $\varphi \in \mathrm{Hom}_1^{ss}(W,E_Z)$; thus $\ker\varphi^\perp$ is non-isotropic by (1.6.2). We choose bases of E_Z and W as in the previous subsubsection, and we add the condition that

$$\begin{aligned}
&(v_1,v_i) = -\delta_{1i},\\
&(v_j,v_j) = 0, \qquad j = 2,3,\\
&(v_2,v_3) = 1,
\end{aligned}$$

and $v_1 \wedge v_2 \wedge v_3$ is the volume form. Easy considerations show that $St([\varphi]) = O(\mathrm{Ker}\varphi)$, and more precisely $St([\varphi])$ is generated by

$$\theta_\alpha := \begin{bmatrix} 1 & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha^{-1}\end{bmatrix} \qquad \tau := \begin{bmatrix} -1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}.$$

The action of $St([\varphi])$ on $(C_\Sigma R)_{[\varphi]}$ is given by:

$$\begin{aligned}
&\theta_\alpha\left(\sum_{3\le i}(Z_{i,2}e_i\otimes v_2 + Z_{i,3}e_i\otimes v_3)\right) = \sum_{3\le i}(\alpha Z_{i,2}e_i\otimes v_2 + \alpha^{-1}Z_{i,3}e_i\otimes v_3),\\
&\tau\left(\sum_{3\le i}(Z_{i,2}e_i\otimes v_2 + Z_{i,3}e_i\otimes v_3)\right) = \sum_{3\le i}(-Z_{i,3}e_i\otimes v_2 - Z_{i,2}e_i\otimes v_3).
\end{aligned} \tag{1.7.16}$$

1.8. Analysis of Kirwan's desingularization.

Recall that $\pi_S\colon S \to R$ is the blow-up of R along Σ_R (see (1.1.10)). Let $\Omega_S \subset S$ be the strict transform of Ω_R, and $\Sigma_S \subset S$ be the exceptional divisor (i.e. the inverse image of Σ_R). Let $x \in \Omega_Q^0$, and set $F_x = I_Z \oplus I_Z$. By Item (2) of (1.7.1) and by (1.7.6) we have

$$(\pi_R\circ\pi_S)^{-1}(x) = Bl_{\mathbf{P}\mathrm{Hom}_1}\mathbf{P}\mathrm{Hom}^\omega(W,E_Z). \tag{1.8.1}$$

Thus

$$\bigcup_{x\in\Omega_Q^0} Bl_{\mathbf{P}\mathrm{Hom}_1}\mathbf{P}\mathrm{Hom}_2^\omega(W,E_Z) \subset \Omega_S.$$

Let $\Delta_S \subset \Omega_S$ be the closure of the left-hand side. Notice that

$$\operatorname{cod}(\Delta_{S_c}, S_c) = c - 3. \tag{1.8.2}$$

In particular $\Delta_{S_c} = \Omega_{S_c}$ if and only if $c = 4$. Let $\pi_{T_c}: T_c \to S_c$ be the blow-up of Δ_{S_c}; of course $T_c \neq S_c$ only if $c > 4$. Let $\Delta_{T_c} \subset T_c$ be the exceptional divisor, and $\Omega_{T_c}, \Sigma_{T_c} \subset T_c$ be the proper transforms of Ω_{S_c}, Σ_{S_c} respectively. Let

$$\widehat{\mathcal{M}}_c := T_c//\mathrm{PGL}(N), \qquad \widehat{\pi}: \widehat{\mathcal{M}}_c \to \mathcal{M}_c,$$

where $\widehat{\pi}$ is induced by the $\mathrm{PGL}(N)$-equivariant map $(\pi_R \pi_S \pi_T)$. Set

$$\widehat{\Omega}_c := \Omega_{T_c}//\mathrm{PGL}(N), \qquad \widehat{\Sigma}_c := \Sigma_{T_c}//\mathrm{PGL}(N), \qquad \widehat{\Delta}_c := \Delta_{T_c}//\mathrm{PGL}(N),$$

$$\Omega_c := \Omega_{Q_c}//\mathrm{PGL}(N) \cong X^{[n]}, \qquad \Sigma_c := \Sigma_{Q_c}//\mathrm{PGL}(N) \cong \left(X^{[n]} \times X^{[n]}\right)/\text{involution}.$$

The following is the main result of this section.

(1.8.3) Proposition. *Keep notation as above. Then $\widehat{\mathcal{M}}_c$ is a desingularization of $\mathcal{M}_c$.*

The proof of the proposition will be given after some preliminary results.

Analysis of Ω_S^{ss}. We will prove the following result.

(1.8.4) Proposition.

(1) *Ω_S^{ss} is smooth.*
(2) *$\Omega_S^{ss} = \Omega_S^s$.*

First some lemmas.

(1.8.5) Lemma. *Let $[Z] \in X^{[n]}$. Then the blow-up*

$$Bl_{\mathbf{P}\mathrm{Hom}_1} \mathbf{P}\mathrm{Hom}^{\omega}(W, E_Z)$$

is smooth.

Proof. By (1.7.13) the exceptional divisor is a (locally-trivial) fibration over $\mathbf{P}\mathrm{Hom}_1(W, E_Z)$, and the fiber over $[\varphi]$ is

$$\mathbf{P}\mathrm{Hom}^{\omega_\varphi}(\mathrm{Ker}\varphi, \mathrm{Im}\varphi^{\perp}/\mathrm{Im}\varphi),$$

i.e. a smooth quadric in $\mathbf{P}^{2c-5}$. Since $\mathbf{P}\mathrm{Hom}_1(W, E_Z)$ is smooth, it follows that the exceptional divisor is smooth. Thus the blow-up is smooth along the exceptional divisor. The complement of the exceptional divisor is smooth by (1.5.6). **q.e.d.**

(1.8.6) Lemma. *All $\mathrm{SO}(W)$-semistable points of $Bl_{\mathbf{P}\mathrm{Hom}_1} \mathbf{P}\mathrm{Hom}^{\omega}(W, E_Z)$ are $\mathrm{SO}(W)$-stable. Explicitely:*

(1) *Semistable points in the exceptional divisor are given by (referring to (1.7.12))*

$$\{([\varphi], [\alpha]) |\ [\varphi] \in \mathbf{P}\mathrm{Hom}_1^{ss}(W, E_Z),\ [\alpha] \in \mathbf{P}\mathrm{Hom}^{\omega_\varphi}(\mathrm{Ker}\varphi, \mathrm{Im}\varphi^{\perp}/\mathrm{Im}\varphi)\ \alpha(v_2) \neq 0 \neq \alpha(v_3)\}.$$

Furthermore, for $([\varphi], [\alpha])$ in the above set,

$$St([\varphi], [\alpha]) \cong \begin{cases} \mathbf{Z}/(2) & \text{if } \mathrm{rk}\alpha = 2, \\ \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) & \text{if } \mathrm{rk}\alpha = 1. \end{cases}$$

(2) *Semistable points not in the exceptional divisor are given by*

$$\{[\varphi] \in \mathbf{P}\mathrm{Hom}^{\omega}(W, E_Z) |\ \mathrm{rk}\varphi = 3 \text{ or } \mathrm{rk}\varphi = 2 \text{ and } \ker\varphi \text{ non-isotropic}\}.$$

For $[\varphi]$ belonging to the above set, $St([\varphi])$ is trivial if $\mathrm{rk}(\varphi)=3$, *and* $St([\varphi])\cong\mathbf{Z}/(2)$ *if* $\mathrm{rk}(\varphi)=2$.

Proof. Let's prove Item (1). By (1.1.2) semistable points of the exceptional divisor are contained in the inverse image of $\mathbf{P}\mathrm{Hom}_1^{ss}(W,E_Z)$. Applying Lemma (1.6.1) we get that a point in the exceptional divisor lying over $[\varphi]\in\mathbf{P}\mathrm{Hom}_1$ is $\mathrm{SO}(W)$-(semi)stable if and only if it is $St([\varphi])$-(semi)stable. One easily verifies that the points described in Item (1) are exactly the $St([\varphi])$-semistable points, and that in fact they are all stable. The computation of stabilizers is an easy exercise. Let's prove Item (2). Applying the numerical criterion for (semi)stability one checks that the points described are $\mathrm{SO}(W)$-stable. By Proposition (1.1.2) they remain stable in the blow-up. Now let's show that if $\mathrm{rk}\varphi=2$ and $\ker\varphi$ is isotropic, then $[\varphi]$ is not semistable in the blow-up. Choose $v\in W$ with $v\bot\ker\varphi$ and $v\notin\ker\varphi$. Then there exists a one-parameter subgroup $\lambda\colon\mathbf{C}^*\to\mathrm{SO}(W)$ such that

$$\lim_{t\to0}\lambda(t)\varphi=\psi,$$

where $\mathrm{rk}\psi=1$ and $\ker\psi^{\perp}=v$. Thus $[\psi]\in\mathbf{P}\mathrm{Hom}_1^{ss}(W,E_Z)$. Since $\mathbf{P}\mathrm{Hom}_1$ is the center of the blow-up, Lemma (1.7.4) tells us that $[\varphi]$ becomes non-semistable in the blow-up, as claimed. The stabilizers are easily computed. **q.e.d.**

Proof of (1.8.4). By (1.1.2) we know that $(\pi_R\circ\pi_S)(\Omega_S^{ss})\subset\Omega_Q^0$. Let $x\in\Omega_Q^0$, and set $F_x=I_Z\oplus I_Z$. By (1.8.1)-(1.8.5) the fiber $(\pi_R\circ\pi_S)^{-1}(x)$ is smooth. Since, by (1.5.1), Ω_Q^0 is smooth, and since semistability is an open condition, we get that Ω_S^{ss} is smooth. This proves Item (1). The second item follows at once from (1.6.1) and (1.8.6).

Analysis of Σ_S^{ss}. We will prove the following result.

(1.8.7) Proposition.

(1) Σ_S^{ss} *is smooth.*
(2) $\Sigma_S^{ss}=\Sigma_S^{s}$.

Proof. By (1.1.2) we know that

$$\Sigma_S^{ss}\subset\pi_S^{-1}(\Sigma_R^{ss})=\mathbf{P}(C_{\Sigma_R^{ss}}R).$$

Let $y\in\Sigma_R^{ss}$, and let $x=\pi_R(y)$. By (1.7.2) either $x\in\Sigma_Q^0$ or $x\in\Omega_Q^0$. In the latter case $\pi_S^{-1}(y)^{ss}$ is described in (1.8.6). In the former case, an easy computation together with (1.6.1) gives the following.

(1.8.8) Claim. *Keep notation as above. If $x\in\Sigma_Q^0$ then (referring to (1.4.2))*

$$\Sigma_S^{ss}\cap(\pi_R\circ\pi_S)^{-1}(x)=\mathbf{P}\left\{(e_{12},e_{21})|\ e_{12}\cup e_{21}=0\quad e_{12}\neq0\neq e_{21}\right\},$$

and all semistable points are stable. The stabilizer of any point in the above set is $\mathbf{Z}/(2)$.

Thus for every $y\in\Sigma_R^{ss}$, $\pi_S^{-1}(y)$ is a smooth quadric in $\mathbf{P}^{2c-5}$ (see Item (3) of (1.7.1)). By Item (1) of (1.7.1) Σ_R^{ss} is smooth. Since semistability is an open condition, we conclude that Σ_S^{ss} is smooth. This proves Item (1) of (1.8.7). The second Item follows from (1.6.1) and (1.8.8). **q.e.d.**

Analysis of S^{ss}. Let's show that

$$S^{ss}=\Sigma_S^{ss}\cup\Omega_S^{ss}\cup(\pi_S\circ\pi_R)^{-1}(Q^s).\tag{1.8.9}$$

By (1.1.4) the term on the right is contained in the term on the left. On the other hand, by (1.1.2)

$$S^{ss}\subset\Sigma_S^{ss}\cup\Omega_S^{ss}\cup(\pi_S\circ\pi_R)^{-1}(Q^s\cup\Gamma_Q^0\cup\Lambda_Q^0).$$

By (1.7.3) and (1.1.3) there are no semistable points in $(\pi_S\circ\pi_R)^{-1}(\Gamma_Q^0)$. Now apply (1.7.4) to $Y=R$, $\widetilde{Y}=S$, and $V=\Sigma_R$; since for $y\in\pi_R^{-1}(\Lambda_Q^0)$ we have $O(y)\cap\Sigma_R^{ss}\neq\emptyset$, there are no semistable points in $(\pi_S\circ\pi_R)^{-1}(\Lambda_Q^0)$. This finishes the proof of (1.8.9).

(1.8.10) Claim.

(1) $S^{ss} = S^s$.
(2) S^s *is smooth.*

Proof. Let's prove Item (1). By (1.1.4) $(\pi_S \circ \pi_R)^{-1}(Q^s)$ is in the stable locus. By (1.8.7)-(1.8.4) $\Omega_S^{ss} = \Omega_S^s$ and $\Sigma_S^{ss} = \Sigma_S^s$. Thus Item (1) follows from (1.8.9). Let's prove Item (2). First of all we show Q^s is smooth: by (1.2.1)-(1.2.3) it suffices to prove that for $x \in Q^s$ the deformation space $\mathrm{Def}(F_x)$ is smooth, and this follows (see (1.3.8)) from

$$\mathrm{Ext}^2(F_x, F_x)^0 \cong \left(\mathrm{Hom}(F_x, F_x)^0\right)^\vee = 0.$$

Since $(\pi_S \circ \pi_R)$ is an isomorphism outside $\Sigma_S \cup \Omega_S$, we get that $(\pi_S \circ \pi_R)^{-1}(Q^s)$ is smooth. Secondly Σ_S^s and Ω_S^s are smooth by (1.8.7)-(1.8.4); since they are Cartier divisors S^s is smooth along Σ_S^s and Ω_S^s. By (1.8.9) we conclude that S^s is smooth. **q.e.d.**

Analysis of Δ_S^s. We will prove the following.

(1.8.11) Proposition. *Keeping notation as above,* Δ_S^s *is smooth.*

First we need a preliminary result. For $[Z] \in X^{[n]}$ let

$$\begin{aligned}\mathbf{Gr}^\omega(k, E_Z) &:= \{[A] \in \mathbf{Gr}(k, E_Z) |\ A \text{ is } \omega\text{-isotropic.}\},\\ \widetilde{\mathbf{P}}\mathrm{Hom}_2^\omega(W, E_Z) &:= \{([K], [A], [\varphi]) \in \mathbf{P}(W) \times \mathbf{Gr}^\omega(2, E_Z) \times \mathbf{P}\mathrm{Hom}_2^\omega(W, E_Z) |\ K \subset \mathrm{Ker}\varphi,\ \mathrm{Im}\varphi \subset A\},\end{aligned}$$

and let $g: \widetilde{\mathbf{P}}\mathrm{Hom}_2^\omega(W, E_Z) \to \mathbf{P}\mathrm{Hom}_2^\omega(W, E_Z)$ be the map which forgets the first two "entries".

(1.8.12) Lemma. *Keeping notation as above, there exists an* $\mathrm{SO}(W)$*-equivariant isomorphism*

$$f: \widetilde{\mathbf{P}}\mathrm{Hom}_2^\omega(W, E_Z) \xrightarrow{\sim} Bl_{\mathbf{P}\mathrm{Hom}_1}\mathbf{P}\mathrm{Hom}_2^\omega(W, E_Z).$$

The map g corresponds to the blow-down map.

Proof. The ideal $I_{\mathbf{P}\mathrm{Hom}_1}$ of $\mathbf{P}\mathrm{Hom}_1$ is generated by 2×2 minors (Second Fundamental Theorem of Invariant Theory). Thus $g^* I_{\mathbf{P}\mathrm{Hom}_1}$ is locally generated by the "determinant" of $\overline{\varphi}: W/K \to A$, hence it is locally principal. By the universal property of the blow-up, there exists a map f as in the statement of the claim. Let's prove f is an isomorphism. Choose bases of W and E_Z, and realize the blow-up as the closure in $\mathbf{P}\mathrm{Hom}_2^\omega(W, E_Z) \times \mathbf{P}^{17}$ of

$$\{([\varphi], \ldots, [m_{IJ}(\varphi)], \ldots) |\ \varphi \in (\mathrm{Hom}_2(W, E_Z) \setminus \mathrm{Hom}_1(W, E_Z)),\ m_{IJ}(\varphi) = (I \times J)\text{-minor of } \varphi,\ |I| = |J| = 2\}.$$

A computation shows that f is given by

$$([K], [A], [\varphi]) \mapsto ([\varphi], \ldots, [p_I(K) q_J(A)], \ldots),$$

where $p_I(K)$ are Plücker coordinates of $[K^\perp] \in \mathbf{Gr}(2, W^*)$, and $q_J(A)$ are Plücker coordinates of $[A]$. This proves f is an isomorphism. Clearly f is $\mathrm{SO}(W)$-equivariant. **q.e.d.**

Proof of (1.8.11). By (1.1.2) we have $\pi_R \pi_S(\Delta_S^s) \subset \Omega_Q^0$. If $x \in \Omega_Q^0$ and $F_x = I_Z \oplus I_Z$, then

$$\Delta_S^s \cap (\pi_R \pi_S)^{-1}(x) \subset Bl_{\mathbf{P}\mathrm{Hom}_1}\mathbf{P}\mathrm{Hom}_2^\omega(W, E_Z).$$

The right-hand side is smooth by (1.8.12). Since Ω_Q^0 is smooth by (1.5.1), and since stability is an open condition, we conclude that Δ_S^s is smooth. **q.e.d.**

Proof of Proposition (1.8.3). By Item (1) of (1.8.10) we have $S^{ss} = S^s$, hence (1.1.2) implies that

(1.8.13) $$T^{ss} = T^s = \pi_T^{-1}(S^s) = Bl_{\Delta_S^s}(S^s).$$

By (1.8.11) Δ^s_S is smooth; since S^s is smooth (by (1.8.10)) we get that T^s is smooth. Let $x\in\Omega^0_Q$ and let $F_x=I_Z\oplus I_Z$; one easily proves that

$$\Delta_S\cap\Sigma_S\cap(\pi_R\pi_S)^{-1}(x)=\{([\varphi],[\alpha])|\ [\varphi]\in\mathbf{P}\mathrm{Hom}_1(W,E_Z),\ [\alpha]\in\mathbf{P}\mathrm{Hom}^{\omega_\varphi}(\mathrm{Ker}\varphi,\mathrm{Im}\varphi^\perp/\mathrm{Im}\varphi),\ \mathrm{rk}\alpha=1\},$$

where notation is as in (1.8.6). Hence if $z\in T^s$ we get by (1.8.6)-(1.8.8) that

$$St(z)\cong\begin{cases}\{1\} & \text{if } z\notin\Sigma^s_T\cup\Delta^s_T,\\ \mathbf{Z}/(2) & \text{if } z\in(\Sigma^s_T\cup\Delta^s_T)\setminus(\Sigma^s_T\cap\Delta^s_T),\\ \mathbf{Z}/(2)\oplus\mathbf{Z}/(2) & \text{if } z\in(\Sigma^s_T\cap\Delta^s_T).\end{cases}.$$

Since Σ^s_T and Δ^s_T are divisors, we conclude that $\widehat{\mathcal{M}}=T//\mathrm{PGL}(N)$ is smooth.

1.9. The two-form on the moduli space.

Let B be a smooth scheme, and $\mathcal{E}$ be a sheaf on $X\times B$, flat over B. The *Mukai-Tyurin form* $\omega_{\mathcal{E}}\in\Gamma(\Omega^2_B)$ is defined as follows [M1,T]: if $v,w\in T_bB$

$$\langle\omega_{\mathcal{E}}(b),v\wedge w\rangle:=\int_X\mathrm{Tr}\left(\kappa_{\mathcal{E}}(b)(v)\cup\kappa_{\mathcal{E}}(b)(w)\right)\wedge\omega,$$

where $\kappa_{\mathcal{E}}(b)\colon T_bB\to\mathrm{Ext}^1(E_b,E_b)$ is the Kodaira-Spencer map at b. We apply this construction to $B=T^s_c$, and $\mathcal{E}$ the pull-back via $X\times T^s_c\to X\times Q_c$ of the tautological quotient sheaf on $X\times Q_c$.

(1.9.1) Claim. *The two-form $\omega_{\mathcal{E}}$ on T^s_c is* PGL(N)*-invariant.*

Proof. The group GL(N) acts on T^s_c and on $\mathcal{E}$. For $g\in\mathrm{GL}(N)$ we have $g^*\omega_{\mathcal{E}}=\omega_{g^*\mathcal{E}}$. Thus

$$\begin{aligned}\langle g^*\omega_{\mathcal{E}}(b),v\wedge w\rangle&=\langle\omega_{g^*\mathcal{E}}(b),v\wedge w\rangle\\&=\int_X\mathrm{Tr}\left(\kappa_{g^*\mathcal{E}}(b)(v)\cup\kappa_{g^*\mathcal{E}}(b)(w)\right)\wedge\omega\\&=\int_X\mathrm{Tr}\left(g^{-1}\kappa_{\mathcal{E}}(b)(v)\cup\kappa_{\mathcal{E}}(b)(w)g\right)\wedge\omega\\&=\int_X\mathrm{Tr}\left(\kappa_{\mathcal{E}}(b)(v)\cup\kappa_{\mathcal{E}}(b)(w)\right)\wedge\omega\\&=\langle\omega_{\mathcal{E}}(b),v\wedge w\rangle,\end{aligned}$$

where the third equality is proved similarly to Lemma (1.4.16). **q.e.d.**

Applying Claim (1.9.1) and the étale slice Theorem (1.2.1) one sees that the two-form $\omega_{\mathcal{E}}$ on T^s_c descends to a holomorphic two-form $\widehat{\omega}_c$ on $\widehat{\mathcal{M}}_c$. By a theorem of Mukai [Muk] we get the following.

(1.9.2) Proposition. *The two-form $\widehat{\omega}_c$ on $\widehat{\mathcal{M}}_c$ is non-degenerate outside $\widehat{\Omega}_c\cup\widehat{\Sigma}_c\cup\widehat{\Delta}_c$.*

Proof. The map $\widehat{\pi}$ gives an isomorphism between the complement of $\widehat{\Omega}_c\cup\widehat{\Sigma}_c\cup\widehat{\Delta}_c$ and $(\mathcal{M}_c\setminus\Omega_c\setminus\Sigma_c)$ If $[F]\in(\mathcal{M}_c\setminus\Omega_c\setminus\Sigma_c)$, then by (1.2.3) $T_{[F]}\mathcal{M}_c\cong\mathrm{Ext}^1(F,F)$, thus $\widehat{\omega}_c$ is non-degenerate at $[F]$ by Serre duality. **q.e.d.**

2. A symplectic desingularization of $\mathcal{M}_4$.

Let $\mathbf{Gr}^\omega(2,T_{X^{[2]}})$ be the relative symplectic Grassmannian over $X^{[2]}$, with fiber $\mathbf{Gr}^\omega(2,E_Z)$ over $[Z]\in X^{[2]}$, and let $\mathcal{A}$ be the tautological $\mathbf{C}^2$-bundle over $\mathbf{Gr}^\omega(2,T_{X^{[2]}})$. In subsection (2.1) we will prove the following.

(2.0.1) Proposition. *Keep notation as above.*

1. *There is an isomorphism* $\widehat{\Omega}_4 \cong \mathbf{P}(S^2\mathcal{A})$ *such that the map*

$$\widehat{\pi}|_{\widehat{\Omega}_4}: \widehat{\Omega}_4 \to \Omega_4 \cong X^{[2]}$$

corresponds to the composition of the maps $\mathbf{P}(S^2\mathcal{A}) \to \mathbf{Gr}^{\omega}(2, T_{X^{[2]}}) \to X^{[2]}$.

2. *Let* A *be a fiber of the* $\mathbf{C}^2$*-bundle* $\mathcal{A}$*, so that* $\mathbf{P}(S^2A) \hookrightarrow \widehat{\Omega}_4$ *by Item (1). Then*

$$[\widehat{\Omega}_4]|_{\mathbf{P}(S^2A)} \cong \mathcal{O}_{\mathbf{P}(S^2A)}(-1).$$

Let $\widehat{\epsilon}_4 \in NE_1(\widehat{\mathcal{M}}_4)$ be the class of a line in a fiber of the $\mathbf{P}^2$-fibration

$$\widehat{\Omega}_4 \to \mathbf{Gr}^{\omega}(2, T_{X^{[2]}}). \tag{2.0.2}$$

The main result of this section is the following.

(2.0.3) Proposition. *Keep notation as above.*

1. $\mathbf{R}^+\widehat{\epsilon}_4$ *is a* $K_{\widehat{\mathcal{M}}_4}$*-negative extremal ray.*
2. *The contraction* $\theta: \widehat{\mathcal{M}}_4 \to \widetilde{\mathcal{M}}_4$ *of* $\mathbf{R}^+\widehat{\epsilon}_4$ *is identified with the contraction of* $\widehat{\Omega}_4$ *along the fibers of Fibration (2.0.2). Thus* $\widetilde{\mathcal{M}}_4$ *is smooth (and projective by Mori theory).*
3. *The two-form* $\widetilde{\omega}_4$ *on* $\widetilde{\mathcal{M}}_4$ *induced by* $\widehat{\omega}_4$ *is non-degenerate.*
4. *The rational map* $\widetilde{\pi}: \widetilde{\mathcal{M}}_4 \cdots > \mathcal{M}_4$ *induced by* $\widehat{\pi}$ *is a morphism.*

Summarizing: $\widetilde{\mathcal{M}}_4$ *is a smooth projective symplectic desingularization of* $\mathcal{M}_4$.

2.1. Proof of Proposition (2.0.1).

Proof of Item (1). For $[Z] \in X^{[2]}$, set

$$\widehat{\Omega}_Z := \widehat{\pi}^{-1}([I_Z \oplus I_Z]).$$

Since $\widehat{\Omega}_4$ and Ω_4 are orbit spaces, we have (see (1.8.1)-(1.8.12))

$$\widehat{\Omega}_Z = Bl_{\mathbf{P}\mathrm{Hom}_1}\mathbf{P}\mathrm{Hom}_2^{\omega}(W, E_Z)//\mathrm{SO}(W) = \widetilde{\mathbf{P}}\mathrm{Hom}_2^{\omega}(W, E_Z)//\mathrm{SO}(W).$$

Since $\mathrm{SO}(W)$ acts trivially on $\mathbf{Gr}^{\omega}(2, E_Z)$, we get a map

$$f: \widetilde{\mathbf{P}}\mathrm{Hom}_2^{\omega}(W, E_Z)//\mathrm{SO}(W) \to \mathbf{Gr}^{\omega}(2, E_Z).$$

It follows easily from (1.8.12), (1.8.6) and (1.6.1) that

$$\widetilde{\mathbf{P}}\mathrm{Hom}_2^{\omega}(W, E_Z)^{ss} = \widetilde{\mathbf{P}}\mathrm{Hom}_2^{\omega}(W, E_Z)^{s} = \{([K], [A], [\varphi]) |\ [K] \text{ is non-isotropic}\}, \tag{2.1.1}$$

hence the projection $\widetilde{\mathbf{P}}\mathrm{Hom}_2^{\omega}(W, E_Z) \to \mathbf{P}(W)$ maps the stable locus to the complement of the isotropic conic, i.e. $\mathbf{P}(W)^{ss}$. Since $\mathrm{SO}(W)$ acts transitively on $\mathbf{P}(W)^{ss}$, we get that

$$f^{-1}([A]) \cong \mathbf{P}\mathrm{Hom}(K^{\perp}, A)//\mathrm{O}(K^{\perp}),$$

where $[K] \in \mathbf{P}(W)^{ss}$ is any chosen point. As is easily verified, the map

$$\begin{array}{ccc} \mathbf{P}\mathrm{Hom}(K^{\perp}, A) & \to & \mathbf{P}(S^2A) \\ [\alpha] & \mapsto & [\alpha \circ {}^t\alpha] \end{array} \tag{2.1.2}$$

is the quotient map for the $\mathrm{O}(K^{\perp})$-action. This proves Item (1) of Proposition (2.0.1).

Proof of Item (2). Recall that $T_4 = S_4$, hence $\widehat{\mathcal{M}}_4 = S_4//\mathrm{PGL}(N)$. We let $q: S_4^s \to \widehat{\mathcal{M}}_4$ be the quotient map.

Claim. *Keeping notation as above,*

(2.1.3) $$q^*\widehat{\Omega}_4 \sim 2\Omega^s_{S_4}.$$

Proof. Since $q^{-1}\widehat{\Omega}_4 = \Omega^s_{S_4}$, all we have to do is determine the multiplicity of $q^*\widehat{\Omega}_4$ at a generic point of $\Omega^s_{S_4}$. Let $z \in (\Omega^s_{S_4} \setminus \Sigma_{S_4})$; by (1.8.8) $St(z) = \mathbf{Z}/(2)$. Let $\mathcal{V} \subset S^s_4$ be a slice normal to $O(z)$. By (1.2.1)

$$\mathcal{V}//\,(\mathbf{Z}/(2)) \cong \text{neighborhood of } q(z) \in \widehat{\mathcal{M}}_4.$$

Since the fixed locus for the action of $\mathbf{Z}/(2)$ is $\Omega^s_{S_4} \cap \mathcal{V}$, it follows that Equation (2.1.3) holds on $\mathcal{V}$. This proves (2.1.3). **q.e.d.**

Let $[K] \in \mathbf{P}(W)^{ss}$. It follows from (2.1.1) that there exists a straight line

$$\Lambda \subset \mathbf{P}\mathrm{Hom}(K^\perp, A)^s \subset \widetilde{\mathbf{P}}\mathrm{Hom}^\omega_2(W, E_Z)^s \subset S^s_4.$$

Claim. *Keeping notation as above,*

(2.1.4) $$\Omega_{S_4} \cdot \Lambda = -1.$$

Proof. We have $\Omega_{S_4} \sim \pi^*_{S_4}\Omega_{R_4}$, and

$$[\Omega_{R_4}]|_{\mathbf{P}\mathrm{Hom}^\omega(W,E_Z)} \cong \mathcal{O}_{\mathbf{P}\mathrm{Hom}^\omega(W,E_Z)}(-1)\,.$$

Since the restriction of π_S to Λ is an isomorphism to a straight line in $\mathbf{P}\mathrm{Hom}^\omega(W, E_Z)$, Equation (2.1.4) follows at once. **q.e.d.**

Let $[\widehat{\Omega}_4]|_{\mathbf{P}^2(S^2A)} \cong \mathcal{O}_{\mathbf{P}^2(S^2A)}(\ell)$. By (2.1.2) q maps Λ one-to-one onto a conic $\Gamma \subset \mathbf{P}^2(S^2A)$. Using (2.1.3)-(2.1.4) we get

$$2\ell = \widehat{\Omega}_4 \cdot \Gamma = q^*\widehat{\Omega}_4 \cdot \Lambda = 2\Omega_{S_4} \cdot \Lambda = -2.$$

Thus $\ell = -1$; this proves Item (2) of Proposition (2.0.1).

2.2. Proof of Proposition (2.0.3).

Proof of Item(1). We will prove the following formula:

(2.2.1) $$K_{\widehat{\mathcal{M}}_4} \sim 2\widehat{\Omega}_4.$$

That $\widehat{\epsilon}_4$ is K-negative follows immediately from this together with Item (2) of Proposition (2.0.1). To prove (2.2.1) notice that by (1.9.2) the two-form $\widehat{\omega}_4$ is non-degenerate outside $\widehat{\Omega}_4 \cup \widehat{\Sigma}_4$, hence

(2.2.2) $$K_{\widehat{\mathcal{M}}_4} \sim \left(\wedge^5\widehat{\omega}_4\right) = x\widehat{\Omega}_4 + y\widehat{\Sigma}_4$$

for non-negative integers x, y.

(2.2.3) Claim. *The coefficient of $\widehat{\Sigma}_4$ in Formula (2.2.2) is equal to zero.*

Proof. For $[Z], [W] \in X^{[2]}$ with $[Z] \neq [W]$ we set

$$\widehat{\Sigma}_{Z,W} := \widehat{\pi}^{-1}([I_Z \oplus I_W]) \cap \widetilde{\Sigma}_4.$$

By (1.4.2),

$$\widehat{\Sigma}_{Z,W} = \mathbf{P}\{(e_{12}, e_{21})|\ e_{12} \cup e_{21} = 0\}//\mathbf{C}^*, \qquad (*)$$

where $e_{12} \in \mathrm{Ext}^1(I_Z, I_W)$, $e_{21} \in \mathrm{Ext}^1(I_W, I_Z)$, and $\mathbf{C}^*$ acts as in (1.4.3). Taking into account that, by Serre duality, the cup-product appearing in the above formula gives a perfect pairing between $\mathrm{Ext}^1(I_Z, I_W)$ and $\mathrm{Ext}^1(I_W, I_Z)$, one verifies easily that the the quotient $(*)$ is isomorphic to $\mathbf{P}^1$. Thus

$$\widehat{\Sigma}_{Z,W} \cong \mathbf{P}^1. \tag{2.2.4}$$

Let's prove that

$$\widehat{\Sigma}_4 \cdot \widehat{\Sigma}_{Z,W} = -2. \tag{2.2.5}$$

Let $f\colon \mathrm{Ext}^1(I_Z, I_W) \to \mathrm{Ext}^1(I_W, I_Z)$ be a skew-symmetric isomorphism, and let

$$\Lambda := \{[e, f(e)]\} \subset \mathbf{P}\{(e_{12}, e_{21})|\ e_{12} \cup e_{21} = 0\}^s.$$

As is easily checked $q|_\Lambda\colon \Lambda \to \widehat{\Sigma}_{Z,W}$ is an isomorphism. Thus

$$\widehat{\Sigma}_4 \cdot \widehat{\Sigma}_{Z,W} = q^*\widehat{\Sigma}_4 \cdot \Lambda.$$

Arguing as in the proof of (2.1.3), one sees that $q^*\widehat{\Sigma}_4 \sim 2\Sigma^s_{S_4}$. Furthermore, since Λ is a line in the quadric

$$\mathbf{P}\{(e_{12}, e_{21})|\ e_{12} \cup e_{21} = 0\},$$

we have $\Sigma_{S_4} \cdot \Lambda = -1$. Thus

$$\widehat{\Sigma}_4 \cdot \widehat{\Sigma}_{Z,W} = q^*\widehat{\Sigma}_4 \cdot \Lambda = 2\Sigma_{S_4} \cdot \Lambda = -2,$$

proving (2.2.5). Now we can prove the claim. We notice that an open dense subset of $\widehat{\Sigma}_4$ is a $\mathbf{P}^1$-fibration with base the symmetric product of $X^{[2]}$ minus the diagonal, and fiber $\widehat{\Sigma}_{Z,W}$ over $([Z],[W])$. Applying adjunction to $\widehat{\Sigma}_4$, and noticing that $\widehat{\Sigma}_{Z,W} \cap \widehat{\Omega}_4 = \emptyset$, we get that the coefficient of $\widehat{\Sigma}_4$ in (2.2.indet) is zero, as claimed. **q.e.d.**

Let's finish the proof of (2.2.1). By adjunction and Claim (2.2.3),

$$K_{\widehat{\Omega}_4} \cong [(x+1)\widehat{\Omega}_4]|_{\widehat{\Omega}_4}.$$

Since $\widehat{\Omega}_4$ is a $\mathbf{P}^2$ fibration over $\mathbf{Gr}^\omega(2, T_{X^{[2]}})$, and since $[\widehat{\Omega}_4]$ has degree -1 on the $\mathbf{P}^2$-fibers, it follows that $x = 2$. We have proved (2.2.1). Thus we are left with the task of proving $\mathbf{R}^+\widehat{\epsilon}_4$ is extremal. In order to accomplish this we introduce another class in $NE_1(\widehat{\mathcal{M}}_4)$ as follows. Let $[Z] \in X^{[2]}$. Choose $[L] \in \mathbf{P}(E_Z)$, and let $\{[A_t] \in \mathbf{Gr}^\omega(2, E_Z)\}_{t \in \mathbf{P}^1}$ be the line through $[L]$, i.e. the family of dimension-two subspaces $A_t \subset E_Z$ such that $L \subset A_t \subset L^\perp$. Let $\iota^t\colon S^2L \hookrightarrow S^2A_t$ be inclusion, and set

$$\widehat{\gamma}_Z := \text{class in } NE_1(\widehat{\Omega}_Z) \text{ of } \{([A_t], \iota_t(S^2L))\}.$$

Let $\widehat{\gamma}_4 \in NE_1(\widehat{\mathcal{M}}_4)$ be the image of $\widehat{\gamma}_Z$ for the map $\iota^Z_*\colon \overline{NE}_1(\widehat{\Omega}_Z) \to \overline{NE}_1(\widehat{\mathcal{M}}_4)$ induced by inclusion $\iota^Z\colon \widehat{\Omega}_Z \hookrightarrow \widehat{\mathcal{M}}_4$.

(2.2.6) Claim. *Keep notation as above.*

a. *The classes $\widehat{\epsilon}_4, \widehat{\gamma}_4 \in N_1(\widehat{\mathcal{M}}_4)$ are linearly independent.*
b. *Let $[Z] \in X^{[2]}$. The map ι^Z_* is injective, with image $\mathbf{R}^+\widehat{\epsilon}_4 \oplus \mathbf{R}^+\widehat{\gamma}_4$.*

Proof. From Item (2) of (2.0.1) we get $\widehat{\Omega}_4 \cdot \widehat{\epsilon}_4 = -1$. We claim that

$$\widehat{\Omega}_4 \cdot \widehat{\gamma}_4 = 0. \tag{2.2.7}$$

Clearly Item (a) follows from these two formulae. To prove (2.2.7) it suffices to show that

$$\mathbf{R}^+[\widehat{\Sigma}_{Z,W}] = \mathbf{R}^+\widehat{\gamma}_4, \tag{2.2.8}$$

because $\widehat{\Sigma}_{Z,W} \cap \widehat{\Omega}_4 = \emptyset$. Let $[W]$ approach $[Z]$: we see that $[\widehat{\Sigma}_{Z,W}]$ can be represented by a one-cycle Γ on $\widehat{\Omega}_Z \cap \widehat{\Sigma}$. The cycle Γ must be mapped to a single point by the map induced from π_{S_4}

$$\widehat{\Omega}_Z = \widetilde{\mathbf{P}}\mathrm{Hom}^\omega_2(W, E_Z)//\mathrm{SO}(W) \to \mathbf{P}\mathrm{Hom}^\omega(W, E_Z)//\mathrm{SO}(W) \subset R_4//\mathrm{PGL}(N).$$

This implies $[\Gamma] \in \mathbf{R}^+\widehat{\gamma}_Z$, proving (2.2.8). Let's prove Item (b). Let $\widehat{\epsilon}_Z \in NE_1(\widehat{\Omega}_Z)$ be the class of a line in a fiber of the $\mathbf{P}^2$-fibration $\widehat{\Omega}_Z \to \mathbf{Gr}^\omega(2, E_Z)$; thus $\widehat{\epsilon}_4 = \iota^Z_* \widehat{\epsilon}_Z$. By Item (a) it suffices to show that

$$\overline{NE}_1(\widehat{\Omega}_Z) = \mathbf{R}^+\widehat{\epsilon}_Z \oplus \mathbf{R}^+\widehat{\gamma}_Z.$$

We have two projections

$$\mathbf{Gr}^\omega(2, E_Z) \stackrel{f}{\leftarrow} \widehat{\Omega}_Z \stackrel{g}{\to} \mathbf{P}\mathrm{Hom}^\omega(W, E_Z)//\mathrm{SO}(W). \tag{2.2.9}$$

As is easily verified, the maps f, g are the contractions of $\mathbf{R}^+\widehat{\epsilon}_Z$, $\mathbf{R}^+\widehat{\gamma}_Z$ respectively. Thus each of $\mathbf{R}^+\widehat{\epsilon}_Z$, $\mathbf{R}^+\widehat{\gamma}_Z$ is an extremal ray. On the other hand, since f is a $\mathbf{P}^2$-fibration over a smooth quadric threefold, $N_1(\widehat{\Omega}_Z)$ has rank two. **q.e.d.**

Now let's prove that $\mathbf{R}^+\widehat{\epsilon}_4 \oplus \mathbf{R}^+\widehat{\gamma}_4$ is an extremal face of $\overline{NE}_1(\widehat{\mathcal{M}}_4)$. Assume

$$\Gamma := \sum_{\alpha \in I} m_\alpha [\Gamma_\alpha] \in \mathbf{R}^+\widehat{\epsilon}_4 \oplus \mathbf{R}^+\widehat{\gamma}_4,$$

where, for each $\alpha \in I$, $m_\alpha > 0$ and Γ_α is an irreducible curve on $\widehat{\mathcal{M}}_4$. We must show that

$$[\Gamma_\alpha] \in \mathbf{R}^+\widehat{\epsilon}_4 \oplus \mathbf{R}^+\widehat{\gamma}_4 \text{ for all } \alpha \in I. \tag{2.2.10}$$

From $\widehat{\pi}_*\widehat{\epsilon}_4 = \widehat{\pi}_*\widehat{\gamma}_4 = 0$ we get $\widehat{\pi}_*\Gamma \equiv 0$, and since $\mathcal{M}_4$ is projective we conclude that $\widehat{\pi}(\Gamma_\alpha)$ is a point, for each $\alpha \in I$. Hence we can partition the indexing set as $I = I_\Omega \amalg I_\Sigma$, so that

$$\text{if} \begin{cases} \alpha \in I_\Omega, & \text{then } \Gamma_\alpha \subset \widehat{\Omega}_{Z_\alpha} \text{ for some } Z_\alpha \in X^{[2]}, \\ \alpha \in I_\Sigma, & \text{then } \Gamma_\alpha \subset \widehat{\Sigma}_{Z_\alpha, W_\alpha} \text{ for } Z_\alpha, W_\alpha \in X^{[2]} \text{ with } Z_\alpha \neq W_\alpha. \end{cases}$$

Statement (2.2.10) follows from Claim (2.2.6)-Item (b) if $\alpha \in I_\Omega$, and from (2.2.8) if $\alpha \in I_\Sigma$. This concludes the proof of Item (1) of Proposition (2.0.3). **q.e.d.**

Proof of Item(2). We must prove that if $C \subset \widetilde{\mathcal{M}}_4$ is an irreducible curve such that $[C] \in \mathbf{R}^+\widehat{\epsilon}_4$, then C belongs to a fiber of (2.0.2). By Item (2) of (2.0.1), $C \cdot \widehat{\Omega}_4 < 0$, hence $C \subset \widehat{\Omega}_4$. Since $\widehat{\pi}_* C \equiv 0$, there exists $[Z] \in X^{[2]}$ such that $C \subset \widehat{\Omega}_Z$. By (2.2.6)-Item (b) the relation $[C] \in \mathbf{R}^+\widehat{\epsilon}_Z$ must hold in $N_1(\widehat{\Omega}_Z)$. This implies that C belongs to a fiber of the map f of (2.2.9), i.e. to a fiber of (2.0.2).

Proof of Item(3). Let $\widetilde{\Omega}_4 := \theta(\widehat{\Omega}_4)$. By (2.2.2) and Claim (2.2.3) we see that $\widetilde{\omega}_4$ is non-degenerate outside $\widetilde{\Omega}_4$. By Item (2) of Proposition (2.0.3)

$$\mathrm{cod}(\widetilde{\Omega}_4, \widetilde{\mathcal{M}}_4) = 3. \tag{2.2.11}$$

Since the degeneracy locus of $\widetilde{\omega}_4$ is a divisor, we conclude that $\widetilde{\omega}_4$ is non-degenerate everywhere.

Proof of Item(4). Let $\Gamma \subset \widetilde{\mathcal{M}}_4 \times \mathcal{M}_4$ be the graph of $\widetilde{\pi}$, and let $\rho \colon \Gamma \to \widetilde{\mathcal{M}}_4$ be the projection. Since $\widetilde{\mathcal{M}}_4$ is smooth, it suffices, by Zariski's Main Theorem, to prove that there are no exceptional divisors of ρ. The graph Γ is the image of

$$(\theta \times \widehat{\pi}) \colon \widehat{\mathcal{M}}_4 \to \widetilde{\mathcal{M}}_4 \times \mathcal{M}_4.$$

Since outside $\widehat{\Omega}_4$ the contraction θ is an isomorphism onto its image, the map $\widetilde{\pi}$ is regular on $(\widetilde{\mathcal{M}}_4 \setminus \widetilde{\Omega}_4))$. Thus an exceptional divisor of ρ must be contained in $(\theta \times \widehat{\pi})(\widehat{\Omega}_4)$. Since the $\mathbf{P}^2$-fibers of $\widehat{\Omega}_4 \to \widetilde{\Omega}_4$ are contained in the fibers of $\widehat{\pi}$, we have an embedding

$$(\theta \times \widehat{\pi})(\widehat{\Omega}_4) \subset \widetilde{\Omega}_4 \times_{\Omega_4} \Omega_4 = \widetilde{\Omega}_4.$$

By (2.2.11) there can be no exceptional divisors contained in $(\theta \times \widehat{\pi})(\widehat{\Omega}_4)$, hence $\widetilde{\pi}$ is a regular map.

(2.2.12) Remark. By Proposition (2.0.3), $\widetilde{\mathcal{M}}_4$ is obtained from $\widehat{\mathcal{M}}_4$ by contracting the extremal ray $\mathbf{R}^+\widehat{\epsilon}_4$. On the other hand, by (2.2.8) the morphism

$$\widehat{\mathcal{M}}_4 = S_4//\mathrm{PGL}(N) \to R_4//\mathrm{PGL}(N)$$

is identified with the contraction of the extremal ray $\mathbf{R}^+\widehat{\gamma}_4$. Let $\epsilon_4 \in NE_1(R_4//\mathrm{PGL}(N))$ be the image of $\widehat{\epsilon}_4$ under the map induced by the above contraction. One easily verifies that the morphism

$$R_4//\mathrm{PGL}(N) \to Q_4//\mathrm{PGL}(N) = \mathcal{M}_4$$

is identified with the contraction of the extremal ray $\mathbf{R}^+\epsilon_4$.

3. $\widetilde{\mathcal{M}}_4$ is irreducible and $h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$.

The proof goes as follows. First, by well-known arguments involving deformations of polarized $K3$'s [GH,O6 §2] we can assume the polarization has degree two, i.e. $H^2 = 2$; this will be assumed for the rest of the paper. We fix a smooth $C \in |H|$; since $K_C \cong [H]|_C$, the curve C has genus two. Let

$$\begin{aligned} J_C :=& \{[F] \in \mathcal{M}_4 |\ F|_C \text{ is not locally-free semistable}\} \\ \widetilde{J}_C :=& \widetilde{\pi}^{-1} J_C. \end{aligned}$$

(3.0.1) Proposition. *Let $\widetilde{\pi}: \widetilde{\mathcal{M}}_4 \to \mathcal{M}_4$ be the desingularization map (see Proposition (2.0.3)). For all $q \leq 5$ the inclusion $\widetilde{J}_C \hookrightarrow \widetilde{\mathcal{M}}_4$ induces an isomorphism*

$$H^q(\widetilde{\mathcal{M}}_4; \mathbf{Z}) \cong H^q(\widetilde{J}_C; \mathbf{Z}).$$

The proof, "copied" from [Li3], is as follows. Consider the *determinant map* $\phi: \mathcal{M}_4 \to \mathbf{P}^N$ associated to a high power of the determinant line-bundle [LP1,Li1], and let $\widetilde{\phi} := \phi \circ \widetilde{\pi}$. In (3.1) we will prove the following two propositions.

(3.0.2) Proposition. *Keep notation as above. The map $\widetilde{\phi}$ is semi-small, i.e. setting*

$$\Gamma_d := \{x \in \widetilde{\mathcal{M}}_4 |\ \dim \widetilde{\phi}^{-1}(\widetilde{\phi}(x)) = d\},$$

we have

$$\mathrm{cod}(\Gamma_d, \widetilde{\mathcal{M}}_4) \geq d.$$

(3.0.3) Proposition. *Keep notation as above. There exists a linear subspace $\Lambda \subset \mathbf{P}^N$, of codimension at most 4, such that $\widetilde{\phi}^{-1}\Lambda = \widetilde{J}_C$.*

(The hypothesis that $H^2 = 2$ is needed to prove this last proposition.) Granting these two propositions, (3.0.1) follows immediately by applying the generalized LHS Theorem [GM, p.150] to the map $\widetilde{\phi}$. Thus to prove our results about the topology of $\widetilde{\mathcal{M}}_4$ we must study $\widetilde{J}_C$. For simplicity we assume that X is a very general degree-two $K3$, i.e. that $Pic(X) = \mathbf{Z}[H]$. We start by describing J_C. Let

$$V_C^0 := \{[F] \in \mathcal{M}_4 |\ F|_C \text{ is locally-free but not semistable}\},$$

and let V_C be its closure. Since local freeness is an open condition, V_C^0 is an open subset of J_C, hence V_C is a union of irreducible components of J_C. In Subsection (3.3) we will prove the following.

(3.0.4) Proposition. *Keep notation and assumptions as above. Let $\mathcal{M}(H,3)$ be the moduli space of rank-two torsion-free semistable sheaves on X with $c_1 = c_1(H)$ and $c_2 = 3$. There exist an open dense subset $\mathcal{U} \subset \mathrm{Pic}^1(C) \times \mathcal{M}(H,3)$ and a $\mathbf{P}^1$-fibration $\alpha\colon P_C \to \mathcal{U}$ with the following properties:*

(1) *There is a birational morphism $\psi\colon P_C \to V_C$.*
(2) *There is a bisection $N_C \subset P_C$ of the $\mathbf{P}^1$-fibration α such that $\psi(N_C) \subset (\Sigma \setminus \Omega)$.*

(Here $\Sigma := \{[I_Z \oplus I_W]\}$.) In particular, since $\mathcal{M}(H,3)$ is irreducible and four-dimensional [O3], V_C is irreducible of dimension 7.

We explain briefly the idea behind the proof of (3.0.4). Let $[F] \in V_C^0$ be generic: we will prove that in the desemistabilizing sequence

$$0 \to L \to F|_C \to L^{-1} \to 0$$

(with L^{-1} locally-free) $degL = 1$ (see Proposition (3.2.8)). Let $\iota\colon C \hookrightarrow X$ be inclusion, and let G be the sheaf fitting into the exact sequence

$$0 \to G \to F \to \iota_* L^{-1} \to 0.$$

(Thus G is an elementary modification of F.) The sheaf $E := G(1)$ is stable (see Lemma (3.2.3)), and $c_1(E) = c_1(H)$, $c_2(E) = 3$. This defines a map $V_C^0 \to \mathcal{M}(H,3)$. Associating to $[F]$ the isomorphism class of L we get a map $V_C^0 \to \mathrm{Pic}^1(C)$. We will show that the product map $V_C^0 \to \mathrm{Pic}^1 \times \mathcal{M}(H,3)$ is dominant, and the fiber over $([L],[E])$ is identified with the open subset of $\mathbf{P}H^0(L \otimes E|_C)$ corresponding to sections which have no zeroes. One also proves that $h^0(L \otimes E|_C) = 2$ generically. This gives $\mathcal{U}$, P_C as in the proposition, and a morphism $\psi\colon P_C \to V_C$, birational onto an irreducible component of V_C. (The bisection N_C corresponds to $[\sigma] \in \mathbf{P}H^0(L \otimes E|_C)$ such that σ has a zero.) Finally a dimension count shows that there are no other components of V_C. Now let's pass to sheaves whose restriction to C is not locally-free. Let $B \subset \mathcal{M}_4$ be the locus parametrizing singular (i.e. not locally-free) sheaves, and let

$$B_C := \{[F] \in B |\ F \text{ is singular at some point of } C\}.$$

We state propositions describing open dense subsets of B and B_C; the proofs are in Subsection (3.4).

(3.0.5) Proposition. *Keep notation and assumptions as above. There is an open dense subset $B^0 \subset B$ such that the determinant map ϕ on B^0 is naturally identified with a $\mathbf{P}^1$-fibration*

$$B^0 \xrightarrow{\phi} X^{(4)}_{(4)} := \{q_1 + q_2 + q_3 + q_4 \in X^{(4)} |\ \text{the } q_i\text{'s are pairwise distinct}\}.$$

If $[F] \in B^0$ and $\phi([F]) = q_1 + q_2 + q_3 + q_4$, then F fits into an exact sequence

$$0 \to F \to \mathcal{O}_X \otimes \mathbf{C}^2 \xrightarrow{\Psi} \bigoplus_{i=1}^4 \mathbf{C}_{q_i} \to 0. \tag{3.0.6}$$

The fiber of ϕ over $q_1 + q_2 + q_3 + q_4$ is identified with the quotient

$$\left(\prod_{i=1}^4 \mathbf{P}(\mathrm{Hom}(\mathbf{C}^2, \mathbf{C}_{q_i})) \right) //\mathrm{PGL}(2) \cong \mathbf{P}^1.$$

(The $\mathrm{PGL}(2)$-linearization on $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ is the symmetric one.)

Keeping notation as above, let $B_C^0 \subset B^0$ be given by

$$B_C^0 := \phi^{-1}\{q_1 + q_2 + q_3 + q_4 |\ q_1 \in C,\ q_i \notin C \text{ for } 2 \le i \le 4\}. \tag{3.0.7}$$

It follows from Proposition (3.0.5) that B_C^0 is open in B_C. We will prove in Subsection (3.4) the following result.

(3.0.8) Proposition. *Keep notation as above. B_C^0 is dense in B_C.*

In particular B_C is irreducible of dimension 8. Since by Proposition (3.0.4) $\dim V_C = 7$, it follows that B_C is not in the closure of V_C; thus we have the following result.

(3.0.9) Lemma. *The decomposition of J_C into irreducible components is given by*

$$J_C = V_C \cup B_C.$$

Now let's describe the irreducible components of $\widetilde{J}_C$. Let $\widetilde{V}_C, \widetilde{B}_C \subset \widetilde{\mathcal{M}}_4$ be the proper transforms of V_C, B_C respectively. From (3.0.9) we get

$$\widetilde{J}_C = \widetilde{V}_C \cup \widetilde{B}_C \cup \widetilde{\pi}^{-1}(J_C \cap \Sigma).$$

(Recall that $\Sigma \subset \mathcal{M}_4$ is the indeterminacy locus of the inverse of $\widetilde{\pi}$.) By Lemma (1.1.5)

$$J_C \cap \Sigma = \Sigma_C := \{[I_Z \oplus I_W] \in \mathcal{M}_4 | \; Z \cap C \neq \emptyset \text{ or } W \cap C \neq \emptyset\}.$$

Let $\widetilde{\Sigma}_C := \widetilde{\pi}^{-1}\Sigma_C$.

(3.0.10) Claim. *$\widetilde{\Sigma}_C$ is irreducible of dimension 8.*

Proof. Consider the surjection

$$\widetilde{\Sigma} \xrightarrow{\widetilde{\pi}} \Sigma \cong X^{[2]} \times X^{[2]}/\text{involution}.$$

Let $\Omega := \{[I_Z \oplus I_Z]\}$, and set $\widetilde{\Omega} := \widetilde{\pi}^{-1}\Omega$. Since $\widetilde{\Omega} \subset \widetilde{\Sigma}$ (because under the isomorphism in Item (1) of Proposition (2.0.1) the intersection $\widehat{\Sigma}_4 \cap \widehat{\Omega}_4$ corresponds to degenerate conics),

$$\widetilde{\Sigma}_C = \left(\widetilde{\pi}|_{\widetilde{\Sigma}}\right)^{-1} \Sigma_C.$$

Since Σ has quotient singularities, the divisor Σ_C is $\mathbf{Q}$-Cartier, hence the above equality shows that every irreducible component of $\widetilde{\Sigma}_C$ has dimension 8. The map

$$\widetilde{\Sigma}_C \setminus \widetilde{\Omega} \xrightarrow{\widetilde{\pi}} \Sigma_C \setminus \Omega$$

is a $\mathbf{P}^1$-fibration, by (2.2.4). Thus $(\widetilde{\Sigma}_C \setminus \widetilde{\Omega})$ is an open irreducible subset of $\widetilde{\Sigma}_C$, of dimension 8: we will show that the closure of $(\widetilde{\Sigma}_C \setminus \widetilde{\Omega})$ is the whole of $\widetilde{\Sigma}_C$. The map

$$\widetilde{\Sigma}_C \cap \widetilde{\Omega} \xrightarrow{\widetilde{\pi}} \Sigma_C \cap \Omega$$

is a fibration with fiber $\mathbf{Gr}^{\omega}(2, E_Z)$ over $[I_Z \oplus I_Z]$ (see Proposition (2.0.3)). Since $\Sigma_C \cap \Omega$ has dimension 3, we get that $\dim(\widetilde{\Sigma}_C \cap \widetilde{\Omega}) = 6$. Hence $\widetilde{\Sigma}_C \cap \widetilde{\Omega}$ is in the closure of $\widetilde{\Sigma}_C \setminus \widetilde{\Omega}$ because every irreducible component of $\widetilde{\Sigma}_C$ has dimension 8. **q.e.d.**

Since $\dim \widetilde{V}_C = 7$ and $\dim \widetilde{B}_C = 8$, the above claim shows that $\widetilde{\Sigma}_C$ is an irreducible component of $\widetilde{J}_C$. Recalling (3.0.9) we get the following.

(3.0.11) Lemma. *The decomposition of $\widetilde{J}_C$ into irreducible components is the following:*

$$\widetilde{J}_C = \widetilde{V}_C \cup \widetilde{B}_C \cup \widetilde{\Sigma}_C.$$

To determine the topology of $\widetilde{J}_C$ we need to examine the intersections of its irreducible components. Item (2) of Proposition (3.0.4) will give us enough information regarding $\widetilde{V}_C \cup \widetilde{\Sigma}_C$. Let's examine $\widetilde{B}_C \cap \widetilde{\Sigma}_C$. First we show that $\widetilde{B}_C$ contains a subset isomorphic to B_C^0. To this end notice that by Proposition (3.0.5)

$$B^0 \cap \Sigma = \{([\Psi_1], \ldots, [\Psi_4]) | \text{ two (but not three) of the } [\Psi_i] \text{ coincide}\}//\mathrm{PGL}(2). \tag{3.0.12}$$

Thus $B^0 \cap \Sigma$ is open in Σ. Since $\widetilde{\pi}$ outside Ω is the blow-up of Σ, and $B^0 \cap \Omega = \emptyset$, we get that

$$\widetilde{\pi}^{-1} B^0 = Bl_{B^0 \cap \Sigma}(B^0). \tag{3.0.13}$$

Furthermore $B^0 \cap \Sigma$ is a three-section of $B^0 \to X_{(4)}^{(4)}$ (by (3.0.12)), hence it is a Cartier divisor in B^0. By (3.0.13) we get that the inclusion $B^0 \hookrightarrow \mathcal{M}_4$ lifts to an inclusion $\iota \colon B^0 \hookrightarrow \widetilde{\mathcal{M}}_4$. Let $\widetilde{B}_C^0 := \iota(B_C^0)$: then $\widetilde{B}_C^0$ is an open subset of $\widetilde{B}_C$ isomorphic to B_C^0. Set

$$\widetilde{S}_C := \widetilde{B}_C^0 \cap \widetilde{\Sigma}_C.$$

The following result follows at once from (3.0.12) and Definition (3.0.7).

(3.0.14) Lemma. *Keeping notation as above,*

$$\widetilde{\pi}(\widetilde{S}_C) = \{[I_Z \oplus I_W] |\ Z, W \text{ are reduced disjoint, } Z \cap C \text{ is a point, } W \cap C = \emptyset\}.$$

The map $\widetilde{S}_C \xrightarrow{\widetilde{\pi}} \widetilde{\pi}(\widetilde{S}_C)$ *is one-to-one.*

Now we are ready to prove that $b_0(\widetilde{\mathcal{M}}_4) = h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$. Let's show that $b_0(\widetilde{\mathcal{M}}_4) = 1$. By Proposition (3.0.1) we know that $b_0(\widetilde{\mathcal{M}}_4) = b_0(\widetilde{J}_C)$. Since, by Lemma (3.0.11), $\widetilde{V}_C$, $\widetilde{B}_C$, $\widetilde{\Sigma}_C$ are the irreducible components of $\widetilde{J}_C$ it suffices to verify that

$$\widetilde{V}_C \cap \widetilde{\Sigma}_C \neq \emptyset \neq \widetilde{\Sigma}_C \cap \widetilde{B}_C.$$

The second intersection is non-empty by Lemma (3.0.14). To show that $\widetilde{V}_C \cap \widetilde{\Sigma}_C \neq \emptyset$ we use Proposition (3.0.4). The morphism $\psi: P_C \to V_C$ gives rise to a rational map $\widetilde{\psi}: P_C \cdots > \widetilde{V}_C$. Since P_C is smooth, the indeterminacy locus I of $\widetilde{\psi}$ has codimension at least two, hence $N_C \setminus I$ is non-empty. By Item (2) of (3.0.4), $\psi(N_C) \subset \Sigma \cap V_C$, and Lemma (1.1.5) gives $\Sigma \cap V_C \subset \Sigma_C$. Thus $\psi(N_C) \subset \Sigma_C$, and hence $\widetilde{\psi}(N_C \setminus I) \subset \widetilde{\Sigma}_C$. Since $\mathrm{Im}\widetilde{\psi} \subset \widetilde{V}_C$ we get that $\widetilde{V}_C \cap \widetilde{\Sigma}_C \neq \emptyset$. Now let's show that $h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$. For a projective variety Y we define $H^{2,0}(Y)$ to be $H^{2,0}(Y')$, where Y' is any desingularization of Y. It will suffice to prove that

(3.0.15) $$\text{the restriction map } H^{2,0}(\widetilde{\mathcal{M}}_4) \to H^{2,0}(\widetilde{B}_C) \text{ is injective.}$$

(Of course restriction really means "pull-back to a resolution".) In fact by (3.0.5)-(3.0.7) any desingularization Y of $\widetilde{B}_C$ has a dominant rational map to $C \times X^{[3]}$, with generic fiber isomorphic to $\mathbf{P}^1$, hence $h^{2,0}(Y) = h^{2,0}(X^{[3]}) = 1$. Thus (3.0.15) implies $h^{2,0}(\widetilde{\mathcal{M}}_4) \leq 1$; since $h^{2,0}(\widetilde{\mathcal{M}}_4) \geq 1$ it follows that $h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$. So let τ be a two-form on $\widetilde{\mathcal{M}}_4$, and assume that

(3.0.16) $$[\tau]|_{\widetilde{B}_C} = 0,$$

i.e. the two-form $\tau_{\widetilde{B}_C}$ is zero (holomorphic forms on projective manifolds inject into cohomology !). Let's prove

(3.0.17) $$\tau|_{\widetilde{\Sigma}_C} = 0.$$

It follows from Lemma (3.0.14) that $\widetilde{\pi}^{-1}(\widetilde{\pi}(\widetilde{S}_C))$ is an open dense subset of $\widetilde{\Sigma}_C$, hence it suffices to show that the restriction of τ to $\widetilde{\pi}^{-1}(\widetilde{\pi}(\widetilde{S}_C))$ is zero. Now $\widetilde{\pi}^{-1}(\widetilde{\pi}(\widetilde{S}_C))$ is a $\mathbf{P}^1$-fibration over $\widetilde{\pi}(\widetilde{S}_C)$, hence every two-form is the pull-back of a form on $\widetilde{\pi}(\widetilde{S}_C)$. Since $\widetilde{S}_C \subset \widetilde{B}_C$, we know that the restriction of τ to $\widetilde{S}_C$ is zero, hence τ is zero on all of $\widetilde{\pi}^{-1}(\widetilde{\pi}(\widetilde{S}_C))$. Finally let's show that

(3.0.18) $$\tau|_{\widetilde{V}_C} = 0.$$

Let $\alpha: P_C \to \mathcal{U}$ be the $\mathbf{P}^1$-fibration of Proposition (3.0.4), and let

$$\widetilde{P}_C := \alpha^{-1}(\mathcal{U} \setminus \alpha(I)) \qquad \widetilde{N}_C := N_C \cap \widetilde{P}_C,$$

where I is the indeterminacy locus of $\widetilde{\psi}$. Since I is closed of codimension at least two, $\widetilde{P}_C$ is an open dense subset of P_C, and $\widetilde{N}_C$ is a non empty divisor on $\widetilde{P}_C$. Since $\widetilde{\psi}(\widetilde{N}_C) \subset \widetilde{\Sigma}_C$, we get from (3.0.17) that $\widetilde{\psi}^*\tau|_{\widetilde{N}_C} = 0$. Since $\widetilde{P}_C$ is a $\mathbf{P}^1$-fibration over $(\mathcal{U} \setminus \alpha(I))$, every two-form on $\widetilde{P}_C$ is the pull-back of a form on $(\mathcal{U} \setminus \alpha(I))$. Since the restriction of $\widetilde{\psi}^*\tau$ to the bisection $\widetilde{N}_C$ is zero we get that $\widetilde{\psi}^*\tau = 0$; this proves (3.0.18) because $\widetilde{\psi}(\widetilde{P}_C)$ is dense in $\widetilde{V}_C$. Now we can prove (3.0.15). If (3.0.16) holds, then by (3.0.17)-(3.0.18) $[\tau]$ is in the kernel of the map

$$H^{2,0}(\widetilde{\mathcal{M}}_4) \to H^{2,0}(\widetilde{V}_C) \oplus H^{2,0}(\widetilde{B}_C) \oplus H^{2,0}(\widetilde{\Sigma}_C)$$

defined by restrictions. By Proposition (3.0.1) the above map is injective, hence $[\tau] = 0$.

Summarizing: we have shown that to prove $b_0(\widetilde{\mathcal{M}}_4) = h^{2,0}(\widetilde{\mathcal{M}}_4) = 1$ it suffices to prove Propositions (3.0.2)-(3.0.5) and (3.0.8).

Remark. The procedure outlined above should allow us to determine the Betti numbers of $\widetilde{\mathcal{M}}_4$ up to b_5 included. However, even the determination of b_2 (which should be equal to 24) requires a much more detailed analysis of $\widetilde{V}_C \cup \widetilde{\Sigma}_C \cup \widetilde{B}_C$; the calculation of $h^{2,0}$ is simpler because $h^{2,0}$ is a birational invariant.

3.1. Proof of Propositions (3.0.2)-(3.0.3).

Let $\mathcal{M}$ be a moduli space of rank-two semistable torsion-free sheaves on a surface, with fixed Chern classes. Le Potier and Jun Li [LP1,Li1] have constructed a determinant line bundle $\mathcal{L}$ on $\mathcal{M}$. They proved that if $m \gg 0$ then $\mathcal{L}^{\otimes m}$ is base-point free [Li1, Thm.3]; furthermore Jun Li proved that the image of the *determinant map*

$$\phi_m\colon \mathcal{M} \to \mathbf{P}\left(H^0(\mathcal{M}, \mathcal{L}^{\otimes m})^*\right)$$

associated to the complete linear system $|\mathcal{L}^{\otimes m}|$ can be identified with the Uhlenbeck compactification in such a way that $\phi_m\colon \mathcal{M} \to \mathrm{Im}\phi_m$ is the standard map [Li1 Thm.4, FM]. We will study ϕ_m for our moduli space $\mathcal{M}_4$. We assume m is very large, so that Jun Li's results apply, and we set $\phi = \phi_m$.

Explicit description of ϕ. Let $\mathcal{M}_4^{lf}, \mathcal{M}_4^{st} \subset \mathcal{M}_4$ be the (open) subsets parametrizing stable and locally-free sheaves, respectively. By Lemma (1.1.5) we have $\mathcal{M}_4^{lf} \subset \mathcal{M}_4^{st}$. Since the restriction of the determinant map to the moduli space of stable vector-bundles is an isomorphism [Li1, Thm.4], the restriction of ϕ to $\mathcal{M}^{lf}$ is an isomorphism onto its image. Now let's examine the restriction of ϕ to $B = \mathcal{M}_4 \setminus \mathcal{M}_4^{lf}$, i.e. of the *boundary* of $\mathcal{M}_4$. We introduce the *singularity cycle* of a point $[F] \in \mathcal{M}_4$. Assuming that the representative F has been chosen so that it is isomorphic to the direct sum of the successive quotients of its Harder-Narasimhan filtration (hence if F is strictly semistable, $F \cong I_Z \oplus I_W$), we set

$$sing[F] := \sum_{p \in X} \ell_F(p),$$

where $\ell_F(p)$ is the length at p of the Artinian sheaf F^{**}/F. The following proposition will be used to describe the singularity cycle of points $[F] \in B$.

(3.1.1) Proposition. *Keeping notation as above, let* $[F] \in B \cap \mathcal{M}_4^{st}$. *Then* $F^{**} \cong \mathcal{O}_X^{(2)}$.

Proof. Since F is not locally-free, $c_2(F^{**}) < c_2(F) = 4$. Thus Hirzebruch-Riemann-Roch gives $\chi(F^{**}) > 0$. By Serre duality $h^2(F^{**}) = h^0(F^*) = h^0(F^{**})$, hence $h^0(F^{**}) > 0$. A non-zero section of F^{**} must have isolated zeroes by slope-semistability, hence we have an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\alpha} F^{**} \to I_Z \to 0,$$

where $\ell(Z) = c_2(F^{**})$. If $\ell(Z) = 0$ then $F^{**} \cong \mathcal{O}_X^{(2)}$, and we are done. By Cayley-Bacharach $\ell(Z) \neq 1$. Now assume $\ell(Z) \geq 2$: then we can find a subsheaf $I_W \subset \mathcal{O}_X$, with $\ell(W) = (4 - \ell(Z))$, such that $\alpha(I_W) \subset F$, and this contradicts stability of F. **q.e.d.**

Proposition (3.1.1) together with Lemma (1.1.5) implies that if $[F] \in B$, then $sing[F] \in X^{(4)}$. By [Li1, Thm.4] the restriction of ϕ to B is simply the singularity map

(3.1.2).
$$\begin{array}{ccc} B & \xrightarrow{\phi} & X^{(4)} \\ [F] & \mapsto & sing[F] \end{array}$$

The image $\phi(B)$ contains the dense subset of $X^{(4)}$ parametrizing cycles $q_1 + \cdots + q_4$ with the q_i pairwise distinct; in fact one easily verifies that a sheaf F fitting into an exact sequence

$$0 \to F \to \mathcal{O}_X \otimes \mathbf{C}^2 \xrightarrow{\Psi} \bigoplus_{i=1}^{4} \mathbf{C}_{q_i} \to 0,$$

where the kernels of Ψ at the points q_i are pairwise distinct elements of $\mathbf{P}(\mathbf{C}^2)$ is Gieseker-Maruyama stable. Since B is projective $\phi(B)$ is closed, hence $\phi(B) = X^{(4)}$. Thus the Uhlenbeck stratification [FM] is as follows:

$$\mathrm{Im}\phi = \phi(\mathcal{M}_4^{lf}) \coprod X^{(4)}.$$

Proof of (3.0.2). The reason why (3.0.2) holds is that the non-degenerate two-form $\widetilde{\omega}_4$ is in a "stratified" sense the pull-back of a two-form on the Uhlenbeck compactification (strictly speaking this is nonsense because the Uhlenbeck compactification is singular). The precise statement is as follows. There is a stratification

$$X^{(4)} = \coprod_p X_p^{(4)} \tag{3.1.3}$$

indexed by partitions of 4. If $p = (p_1, p_2, p_3, p_4)$ is a partition, i.e. $\sum_{i=1}^{4} ip_i = 4$, the stratum $X_p^{(4)}$ parametrizes cycles

$$\gamma = \sum_{j=1}^{p_1} q_j^1 + 2\sum_{j=1}^{p_2} q_j^2 + \cdots,$$

where the q_j^i are pairwise distinct. Letting $X_\delta^{(s)} := \big(X^{(s)} \setminus \mathit{diagonals}\big)$, we have an open inclusion

$$X_p^{(4)} \subset X_\delta^{(p_1)} \times X_\delta^{(p_2)} \times \cdots,$$

in particular $X_p^{(4)}$ is smooth. Let $\omega^{(s)} \in \Gamma(\Omega^2_{X_\delta^{(s)}})$ be the holomorphic two-form obtained from ω by symmetrization, and for a partition p, let $\omega_p^{(4)}$ be the two-form on $X_p^{(4)}$ given by

$$\omega_p^{(4)} := \sum_{i=1}^{4} i\pi_i^* \omega^{(p_i)},$$

where $\pi_i \colon X_p^{(4)} \to X_\delta^{(p_i)}$ is the i-th projection. Stratification (3.1.3) gives a stratification of $\mathrm{Im}\widetilde{\phi} = \mathrm{Im}\phi$, in which $\phi(\mathcal{M}_4^{lf})$ is the open stratum, and the $X_p^{(4)}$ are the remaining strata. The stratified pull-back formula is the following:

$$\widetilde{\omega}_4|_{\widetilde{\phi}^{-1}X_p^{(4)}} = -\frac{1}{4\pi^2}\widetilde{\phi}^*\omega_p^{(4)}. \tag{3.1.4}$$

The equality is an immediate consequence of [O5,(2-9)]. Now let's prove (3.0.2). Let $V \subset \Gamma_d$ be an irreducible component of maximum dimension. If $V \cap \mathcal{M}_4^{lf} \neq \emptyset$, then $d = 0$ and there is nothing to prove. So assume $V \subset \widetilde{\phi}^{-1}X^{(4)}$. Then $V \cap \widetilde{\phi}^{-1}X_p^{(4)}$ is dense in V for some partition p. Let $y \in V$ be such that V is smooth at y and the restriction $\widetilde{\phi}\colon V \to \widetilde{\phi}(V)$ is submersive at y; it follows from (3.1.4) that

$$\widetilde{\omega}_4\left(T_yV, T_y\widetilde{\phi}^{-1}(\widetilde{\phi}(y))\right) \equiv 0.$$

Thus $\widetilde{\omega}_4$ induces a map

$$T_y\widetilde{\mathcal{M}}_4/T_yV \to T_y\widetilde{\phi}^{-1}(\widetilde{\phi}(y))^*,$$

which is surjective because $\widetilde{\omega}_4$ is non-degenerate. Hence

$$\mathrm{cod}(\Gamma_d, \widetilde{\mathcal{M}}_4) = \mathrm{cod}(V, \widetilde{\mathcal{M}}_4) = \dim\left(T_y\widetilde{\mathcal{M}}_4/T_yV\right) \geq \dim T_y\widetilde{\phi}^{-1}(\widetilde{\phi}(y)) = d.$$

Proof of (3.0.3). Let θ be line bundle on C of degree one (i.e. half the canonical degree, since C has genus two); if $[F] \in \mathcal{M}_4$ then by Riemann-Roch

$$\chi(F|_C \otimes \theta) = 0. \tag{3.1.5}$$

There is a canonical section [FM,Li1] σ_θ of the determinant line-bundle $\mathcal{L}$ such that

$$supp(\sigma_\theta) = \{[F] \in \mathcal{M}_4 |\ h^0(F|_C \otimes \theta) > 0\}.$$

By (3.1.5) the right-hand side has codimension at most one in every component of $\mathcal{M}_4$, but a priori it might contain a whole component. In any case there exists a linear subspace $\Lambda_\theta \subset \mathbf{P}^N$ of codimension at most one such that $\phi^{-1}\Lambda_\theta = supp(\sigma_\theta)$. The moduli space of rank-two semistable vector-bundles on C with trivial determinant has dimension three, hence a theorem of Raynaud [R] shows that if $\theta_1, \ldots, \theta_4$ are generic line-bundles of degree one, there is no semistable vector-bundle V with trivial determinant such that

$$h^0(V \otimes \theta_i) > 0, \qquad i = 1, \ldots, 4.$$

On the other hand, if V is a degree zero rank-two sheaf on C which is either singular or locally-free non-semistable, then $h^0(V \otimes \theta) > 0$ for any choice of a degree-one line-bundle θ on C. Hence Proposition (3.0.3) holds with

$$\Lambda := \Lambda_{\theta_1} \cap \cdots \cap \Lambda_{\theta_4},$$

where $\theta_1, \ldots, \theta_4$ are chosen generically.

3.2. V_C and elementary modifications.

From now on we will assume that X is a very general genus two $K3$, i.e. that $Pic(X) = \mathbf{Z}[H]$. Let $[F] \in V_C^0$, and let

$$0 \to L \to F|_C \to L^{-1} \to 0 \tag{3.2.1}$$

be the desemistabilizing sequence, i.e. L is a line bundle of degree $d > 0$. Let $\iota: C \hookrightarrow X$ be inclusion, and let G be the elementary modification of F determined by (3.2.1), i.e. the sheaf fitting into the exact sequence

$$0 \to G \xrightarrow{\alpha} F \to \iota_* L^{-1} \to 0. \tag{3.2.2}$$

By (1.1.5) the sheaf F is stable, hence by Proposition (3.1.1) F is locally-free. Thus G is a rank-two locally-free sheaf. Furthermore $c_1(G) = -H$, $c_2(G) = (4-d)$. (First compute the Chern classes of $\iota_* L^{-1}$ by applying Grothendieck-Riemann-Roch to ι.)

(3.2.3) Lemma. *Keep notation as above. The vector-bundle G is slope-stable.*

Proof. Assume G is not slope-stable. Since $\mathrm{Pic}(X) = \mathbf{Z}[H]$, there exists an injection $\mathcal{O}_X(k) \hookrightarrow G$ with $k \geq 0$. Composing with α we get an injection $\mathcal{O}_X(k) \hookrightarrow F$, contradicting (Gieseker-Maruyama) semistability of F. **q.e.d.**

Corollary. *Keeping notation as above, either $d = 1$ or $d = 2$.*

Proof. By Serre duality $h^2(G^* \otimes G) = h^0(G^* \otimes G)$. Since G is slope-stable $h^0(G^* \otimes G) = 1$, hence $\chi(G^* \otimes G) \leq 2$. Hirzebruch-Riemann-Roch gives $\chi(G^* \otimes G) = (4d - 6)$, thus $d \leq 2$. **q.e.d.**

By the above corollary we have a decomposition into locally closed subsets

$$V_C^0 = V_C^0(1) \coprod V_C^0(2), \tag{3.2.4}$$

where $V_C^0(d)$ is the set of $[F] \in V_C^0$ for which the line-bundle L of (3.2.1) has degree d. We will describe $V_C^0(d)$ in terms of $\mathcal{M}(H, 4-d)$, where $\mathcal{M}(H, 4-d)$ is the moduli space of semistable rank-two torsion-free sheaves on X with $c_1 = H$, $c_2 = (4-d)$. Let $\mathcal{M}(H, 4-d)^{lf} \subset \mathcal{M}(H, 4-d)$ be the subset parametrizing locally-free sheaves. Let $\mathcal{E}_d$ be a tautological vector-bundle on $X \times \mathcal{M}^{lf}(H, 4-d)$ (it exists by [M2,(A.7)]). Let $\mathcal{Q}_C(d)$ be the relative quot-scheme of $\mathcal{E}_d|_{C\times\mathcal{M}^{lf}(H,4-d)}$ over $\mathcal{M}^{lf}(H, 4-d)$ parametrizing quotients $E|_C \to \xi$, where ξ is a rank-one sheaf of degree $(2+d)$, and let $\mathcal{Q}^0_C(d) \subset \mathcal{Q}_C(d)$ be the open subset parametrizing locally-free quotients. We will define an isomorphism

$$V^0_C(d) \xrightarrow{\sim} \mathcal{Q}^0_C(d). \tag{3.2.5}$$

Keeping notation as above, let $E := G(1)$; then $c_1(E) = H$, $c_2(E) = (4-d)$, so that by (3.2.3) $[E] \in \mathcal{M}(H, 4-d)^{lf}$. Notice also that the bundle $E|_C$ comes with a canonical rank-one subsheaf: in fact the long exact sequence of $Tor(\cdot, \mathcal{O}_C)$ associated to (3.2.2) gives

$$0 \to L^{-1} \otimes \mathcal{O}_C(-C) \xrightarrow{\beta} G|_C \to L \to 0,$$

and if we tensor the above sequence with $\mathcal{O}_C(1)$ we obtain

$$0 \to L^{-1} \to E|_C \xrightarrow{\gamma} L \otimes K_C \to 0. \tag{3.2.6}$$

Since $\deg(L \otimes K_C) = (2+d)$, the couple $([E], \gamma)$ is a point of $\mathcal{Q}^0_C(d)$. This defines the map (3.2.5). This map is an isomorphism onto its image because one recovers F from E and (3.2.6) as follows. First notice that $F(-C) \hookrightarrow G$, secondly that the restriction to C of this inclusion has image equal to $\mathrm{Im}\beta$, thus we have an exact sequence

$$0 \to F(-1) \to G \to \iota_* L \to 0.$$

Tensoring with $\mathcal{O}_X(1)$ we see that F is the elementary modification of E associated to (3.2.6); thus (3.2.5) is an isomorphism of $V^0_C(d)$ onto its image. The following result says that (3.2.5) is onto.

Lemma. *Let $d = 1, 2$, and let $[E] \in \mathcal{M}^{lf}(H, 4-d)$. Assume $E|_C$ fits into Exact Sequence (3.2.6), where L is a line-bundle with $\deg L = d$. Let F be the elementary modification of E associated to (3.2.6), i.e. we have*

$$0 \to F \xrightarrow{\delta} E \to \iota_*(L \otimes K_C) \to 0. \tag{3.2.7}$$

Then F is a rank-two slope-stable vector-bundle with $c_1(F) = 0$, $c_2(F) = 4$, and $[F] \in V^0_C(d)$. Thus (3.2.5) maps $[F]$ to the couple $([E], \beta)$.

Proof. The Chern classes of F are easily computed from the exact sequence defining F. Furthermore, applying the functor $Tor(\cdot, \mathcal{O}_C)$ to (3.2.7) we get Exact sequence (3.2.1), so all we have to prove is that F is slope-stable. Suppose F is not slope-stable. Since $\mathrm{Pic}(X) = \mathbf{Z}[H]$, there is an injection $\mathcal{O}_X(k) \hookrightarrow F$, with $k \geq 0$. Composing with δ we get $\mathcal{O}_X(k) \hookrightarrow E$; since E is semistable we must have $k = 0$, i.e. we have a non-zero section $\sigma \in H^0(E)$. The restriction of σ to C is a section of L^{-1}, hence zero because $\deg(L^{-1}) < 0$. Since $C \in |\mathcal{O}_X(1)|$ the section σ gives rise to an injection $\mathcal{O}_X(1) \hookrightarrow E$, contradicting semistability of E. **q.e.d.**

(3.2.8) Proposition. *$V^0_C(1)$ is (open) dense in V^0_C.*

Proof. By an argument similar to that proving Proposition (1.13) of [O6], one shows that every irreducible component of V^0_C has codimension at most $g(C)+1 = 3$. Since $\mathcal{M}_4$ is of pure dimension 10, every component of V^0_C has dimension at least 7. Hence by (3.2.4) and (3.2.5) it suffices to show that

$$\dim \mathcal{Q}^0_C(2) = 5. \tag{3.2.9}$$

The moduli space $\mathcal{M}(H, 2)$ has expected dimension zero: by a result of Mukai [M2,(3.6)] it consists of a single point $[W]$, parametrizing a slope-stable vector-bundle.

(3.2.10) Claim. *Keep notation as above. Then W is isomorphic to the pull-back by the double cover map*

$$X \to |\mathcal{O}_X(1)|^* \cong \mathbf{P}^2$$

of the bundle $T_{\mathbf{P}^2}(-1)$.

Proof of the claim. Computing Chern classes we get $c_1(W) = H$, $c_2(W) = 2$. Hence by Mukai's result all we have to do is show that W is slope-stable. Since $T_{\mathbf{P}^2}(-1)$ has sections with isolated zeroes, so does W, hence we have an exact sequence

$$0 \to \mathcal{O}_X \to W \to I_Z(1) \to 0,$$

where $Z \subset X$ is a zero-dimensional subscheme of length 2. Since $\mathrm{Pic}(X) = \mathbf{Z}[H]$, it follows immediately that W is slope-stable. **q.e.d.**

Now we notice that

$$W|_C \cong \mathcal{O}_C \oplus K_C. \tag{3.2.11}$$

In fact if R is a line in $\mathbf{P}^2$ the exact sequence

$$0 \to T_R \to T_{\mathbf{P}^2}|_R \to N_{R/\mathbf{P}^2} \to 0.$$

gives

$$T_{\mathbf{P}^2}(-1)|_R \cong \mathcal{O}_R \oplus \mathcal{O}_R(1),$$

which implies (3.2.11). Now let's prove (3.2.9). Claim (3.2.10) gives a morphism

$$\begin{array}{ccc} \mathcal{Q}^0_C(2) & \to & \mathrm{Pic}^2(C) \\ (W|_C \to L \otimes K_C \to 0) & \mapsto & [L] \end{array}.$$

The fiber over $[L]$ is the open subset of $\mathbf{P}H^0(L \otimes W|_C)$ corresponding to sections with no zeroes. By (3.2.11) we have

$$\dim \mathbf{P}H^0(L \otimes W|_C) = \begin{cases} 3 & \text{if } L \not\cong K_C, \\ 4 & \text{if } L \cong K_C. \end{cases}$$

Equality (3.2.9) follows immediately from this. This proves Proposition (3.2.8). **q.e.d.**

3.3. Proof of Proposition (3.0.4).

Given Isomorphism (3.2.5) and Proposition (3.2.8), it is clear that we must examine the restriction to C of vector-bundles parametrized by $\mathcal{M}(H,3)^{lf}$. By [M1] the moduli space $\mathcal{M}(H,3)$ is smooth symplectic of dimension 4, and by [O6] it is irreducible.

(3.3.1) Lemma. *If $[E] \in \mathcal{M}(H,3)$ then $h^0(E) = 2$.*

Proof. By Riemann-Roch $\chi(E) = 2$. By Serre duality $h^2(E) = \dim \mathrm{Hom}(E, \mathcal{O}_X)$, hence stability gives $h^2(E) = 0$. Thus $h^0(E) \geq 2$. Let $\sigma \in H^0(E)$ be non-zero. We claim the quotient $Q := E/\mathcal{O}_X\sigma$ is torsion-free. Suppose Q has torsion: then σ must vanish on a divisor, and since $\mathrm{Pic}(X) = \mathbf{Z}[H]$ this implies we have an injection $I_W(k) \hookrightarrow E$, where $k \geq 1$, and $W \subset X$ is a zero-dimensional subscheme. This contradicts semistability of E. Thus we have an exact sequence

$$0 \to \mathcal{O}_X \to E \to I_Z(1) \to 0,$$

where $Z \subset X$ is a zero-dimensional subscheme of length $c_2(E) = 3$. Since $H \cdot H = 2$, we have $h^0(I_Z(1)) \leq 1$, hence $h^0(E) \leq 2$. **q.e.d.**

By the lemma above we can define a morphism

$$\rho\colon \mathcal{M}(H,3) \to |\mathcal{O}_X(1)| \tag{3.3.2}$$

as follows. Choose a basis $\{\sigma,\tau\}$ of $h^0(E)$. We claim $\sigma\wedge\tau \not\equiv 0$. Assume the contrary: then σ, τ generate a rank-one subsheaf of E with two linearly independent global sections, i.e. a sheaf isomorphic to $I_W(k)$, where $k\geq 1$ and $W\subset X$ is a zero-dimensional subscheme. This contradicts semistability of E. Thus we can set

$$\rho([E]) := (\sigma\wedge\tau) \in |\mathcal{O}_X(1)|.$$

(3.3.3) Lemma. *For all $D\in|\mathcal{O}_X(1)|$, the fiber $\rho^{-1}D$ has pure dimension 2. In particular the map ρ is surjective.*

Proof. Since $\dim\mathcal{M}(H,3)=4$, every component of $\rho^{-1}D$ has dimension at least 2. It follows from [O5,(2-9)] that the symplectic form on $\mathcal{M}(H,3)$ is identically zero on $\rho^{-1}D$, hence every component of $\rho^{-1}D$ has dimension at most 2. **q.e.d.**

Now we are ready to examine the restriction to C of vector-bundles parametrized by $\mathcal{M}^{lf}(H,3)$.

(3.3.4) Proposition. *Let $[E]\in\mathcal{M}(H,3)^{lf}$. Then $E|_C$ is not semistable if and only if $\rho([E])=C$. In this case*

$$E|_C \cong \xi\oplus(K_C\otimes\xi^{-1}),$$

where ξ is degree 3 line-bundle.

Proof. Assume $\rho([E])=C$. Choose a basis $\{\sigma,\tau\}$ of $H^0(E)$, and consider the exact sequence

$$0\to\mathcal{O}_X^{(2)}\xrightarrow{(\sigma,\tau)}E\to\iota_*\eta\to 0.$$

Restricting to C we get

$$0\to\xi\to E|_C\to\eta\to 0, \tag{3.3.5}$$

where ξ, η are rank-one sheaves. Thus ξ is locally-free of rank-one, and a Chern class computation gives $\deg\xi=3$. Since $C=(\sigma\wedge\tau)$ is smooth, σ and τ have no zeroes in common, and this implies η is also locally-free. Since $\det(E|_C)\cong K_C$, we have $\eta\cong\xi^{-1}\otimes K_C$. Exact Sequence (3.3.5) splits because $H^1(\xi^{\otimes 2}\otimes K_C^{-1})=0$. Now suppose $E|_C$ is not semistable, and let (3.3.5) be the desemistabilizing sequence, i.e. η is a line-bundle and $d=\deg\xi\geq 2$. Let F be the elementary modification of E defined by (3.3.5), i.e. we have

$$0\to F\to E\to\iota_*\eta\to 0.$$

Then $\mathrm{rk}F=2$, $c_1(F)=0$ and $c_2(F)=(3-d)\leq 1$. Arguing as in the proof of Lemma (3.2.3) we see that F is slope-semistable. Since there are no slope-semistable rank-two vector-bundles on X with $c_1=0$, $c_2=1$ (see the proof of (1.1.1)), we conclude that $c_2(F)=0$, hence $F\cong\mathcal{O}_X^{(2)}$. This means that $\rho([E])=C$. **q.e.d.**

(3.3.6) Proposition. *Let $[E]\in\mathcal{M}(H,3)^{lf}$. Assume $D=\rho([E])$ is not equal to C, and set $D\cdot C=p+p'$. There is an exact sequence*

$$0\to\mathcal{O}_C(p)\to E|_C\to\mathcal{O}_C(p')\to 0, \tag{3.3.7}$$

which is split if $p\neq p'$.

Proof. Choose a basis $\{\sigma,\tau\}$ of $H^0(E)$. Since $p\in D$, the vectors $\sigma(p)$, $\tau(p)$ are linearly dependent. Hence there exists a non-trivial linear combination $\lambda\sigma+\mu\tau$ which is zero at p. Let

$$\epsilon := (\lambda\sigma+\mu\tau)|_C \in H^0(E|_C).$$

Since ϵ vanishes at p, it defines a non-zero map $\mathcal{O}_C(p) \to E|_C$. By Proposition (3.3.4) the bundle $E|_C$ is semistable, hence ϵ vanishes only at p and with multiplicity one. This gives Exact Sequence (3.3.7). Reversing the roles of p and p', we see that if $p \neq p'$ the sequence is split. **q.e.d.**

Now we can define the varieties $\mathcal{U}$, P_C, N_C appearing in the statement of Proposition (3.0.4). Let $([E], f) \in \mathcal{Q}_C(1)$; thus $[E] \in \mathcal{M}^{lf}(H, 3)$ and

$$E|_C \xrightarrow{f} \xi \to 0 \tag{3.3.8}$$

is a quotient with ξ a degree-3 rank-one sheaf on C. Since $E|_C$ is locally-free of degree 2, the kernel of f is a rank-one locally-free sheaf of degree (-1), say L^{-1}. Thus we can define a morphism

$$\begin{array}{ccc} \mathcal{Q}_C(1) & \xrightarrow{\alpha} & \mathrm{Pic}^1(C) \times \mathcal{M}(H,3)^{lf} \\ ([E], f) & \mapsto & ([L], [E]) \end{array}. \tag{3.3.9}$$

Let $\mathcal{U} \subset \mathrm{Pic}^1(C) \times \mathcal{M}(H,3)^{lf}$ be the open subset consisting of couples $([L], [E])$ such that $h^0(L) = 0$ and $\rho([E])$ intersects C in exactly two points. By Lemma (3.3.3) $\mathcal{U}$ is non-empty. Let

$$P_C := \alpha^{-1}(\mathcal{U}), \qquad P_C^0 := P_C \cap \mathcal{Q}_C^0(1), \qquad N_C := P_C \setminus \mathcal{Q}_C^0(1).$$

(3.3.10) Claim. *Keep notation as above. Then $\alpha: P_C \to \mathcal{U}$ is a $\mathbf{P}^1$-fibration, and N_C is a bisection of this fibration.*

Proof. Let $([L], [E]) \in \mathcal{U}$. Then

$$\begin{aligned} \alpha^{-1}([L], [E]) &\cong \mathbf{P}H^0(L \otimes E|_C), \\ N_C \cap \alpha^{-1}([L], [E]) &\cong \mathbf{P}\{\sigma \in H^0(L \otimes E|_C) |\ \ \sigma \text{ has a zero}\}. \end{aligned} \tag{3.3.11}$$

Now let $\rho([E]) \cap C = \{p, p'\}$. By (3.3.6)

$$H^0(L \otimes E|_C) = H^0(L(p)) \oplus H^0(L(p')). \tag{3.3.12}$$

Since $K_C \sim p + p'$, and since $L \ncong [p']$, we see that $h^0(L(p)) = 1$. Similarly $h^0(L(p')) = 1$, hence the space of sections in (3.3.12) is two-dimensional. By (3.3.11) we conclude that the fibers of α are isomorphic to $\mathbf{P}^1$. An easy argument identifies P_C with the projectivization of a direct image sheaf over $\mathcal{U}$ with fiber $H^0(L \otimes E|_C)$ over the point $([L], [E])$. This proves that $\alpha: P_C \to \mathcal{U}$ is a $\mathbf{P}^1$-fibration. To finish the proof we remark that, since $h^0(L) = 0$, a section

$$\sigma = (\sigma_1, \sigma_2) \in H^0(L(p)) \oplus H^0(L(p'))$$

has a zero if and only if $\sigma_1 = 0$ or $\sigma_2 = 0$. **q.e.d.**

(3.3.13) Lemma. *Keeping notation as above, P_C^0 is dense in $\mathcal{Q}_C^0(1)$.*

Proof. This is a dimension count. Stratify the complement of $\mathcal{U}$ in $\mathrm{Pic}^1(C) \times \mathcal{M}(H,3)^{lf}$ according to the dimension of the fibers of α (see (3.3.9)). Using (3.3.3), (3.3.4) and (3.3.6), one easily verifies that for each stratum $\mathcal{S}$,

$$\dim \alpha^{-1}(\mathcal{S}) < 7.$$

On the other hand (see the proof of (3.2.8)) every irreducible component of $V_C^0(1)$ has dimension at least 7. By (3.2.5) $\mathcal{Q}_C^0(1) \cong V_C^0(1)$, hence the above inequality shows that $\alpha^{-1}(\mathcal{S}) \cap \mathcal{Q}_C^0(1)$ is not dense in any component of $\mathcal{Q}_C^0(1)$. **q.e.d.**

Restricting the inverse of Isomorphism (3.2.5) to P_C^0, we get a morphism

$$\psi_0: P_C^0 \to V_C.$$

By Proposition (3.2.8) and Lemma (3.3.13) ψ_0 is birational. In order to prove Proposition (3.0.4) we must extend ψ_0 to all of P_C. Let $([E], f) \in P_C$, where f is as in (3.3.8), and let F be the associated elementary modification, i.e. we have

$$0 \to F \to E \to \iota_* \xi \to 0. \tag{3.3.14}$$

The sheaf F is torsion-free of rank two, with $c_1(F) = 0$, $c_2(F) = 4$. We know that if $([E], f) \in P_C^0$ then F is stable, and $\psi_0([E], f) = [F]$. To extend ψ_0 over N_C we must show that F is semistable also for $([E], f) \in N_C$.

(3.3.15) Claim. *Keep notation as above. Let* $([E], f) \in N_C$, *and let* F *be the sheaf fitting into (3.3.14). Then* F *is a strictly (Gieseker-Maruyama) semistable sheaf. More precisely there is an exact sequence*

$$0 \to I_Z \to F \to I_W \to 0, \tag{3.3.16}$$

where $Z, W \subset X$ *are zero-dimensional subschemes of length 2, with* $Z \neq W$.

Proof. Let $\alpha([E], f) = ([L], [E])$. Since $\rho([E]) \cap C$ consists of two distinct points, Proposition (3.3.6) gives

$$E|_C \cong \mathcal{O}_C(p) \oplus \mathcal{O}_C(p'),$$

where $p \neq p'$. Since $([E], f) \in N_C$ we can assume (see the proof of (3.3.10)) that f corresponds to the quotient of the injection given by the composition

$$L^{-1} \overset{\sigma}{\hookrightarrow} \mathcal{O}_C(p) \hookrightarrow E|_C.$$

The map σ vanishes on a divisor $Z \subset C$ of degree 2. There is a global section $\tau\colon \mathcal{O}_X \to E$ which restricted to C gives the non-zero section of $\mathcal{O}_C(p)$ (see the proof of (3.3.6)), hence

$$\tau(I_Z)|_C = \mathrm{Im}\sigma.$$

Thus $\tau(I_Z)$ is a subsheaf of the elementary modification F, i.e. we have an exact sequence

$$0 \to I_Z \to F \to \eta \to 0,$$

for some rank-one sheaf η. We claim η is torsion-free. First, a local computation shows that η is locally-free at points of C. Secondly, since outside C the sheaves F and E are isomorphic, also η and $E/\mathrm{Im}\tau$ are isomorphic outside C. By slope-stability, the section τ has isolated zeroes, hence $E/\mathrm{Im}\tau$ is torsion-free, and we conclude that η is torsion-free outside C. Thus $\eta \cong I_W(k)$ for some integer k. Since $c_1(F) = 0$, we have $k = 0$, and since $c_2(F) = 4$ the length of W is 2. This proves F fits into Exact Sequence (3.3.16). Finally, since η is locally-free on C, W does not intersect C. Since $Z \subset C$, we conclude $Z \neq W$. **q.e.d.**

By Claim (3.3.15) we get the desired extension of ψ_0 by setting

$$\begin{array}{ccc} P_C & \xrightarrow{\psi} & V_C \\ ([E], f) & \mapsto & [F] \end{array},$$

where notation is as above. Let's prove Proposition (3.0.4). Item (1) holds by Isomorphism (3.2.5) and Proposition (3.2.8). Item (2) holds by Claim (3.3.15).

3.4. Proof of Propositions (3.0.5) and (3.0.8).

Proof of Proposition (3.0.5). Consider the restriction of the determinant map ϕ to B, given by (3.1.2). Let $X^{(4)}_{(4)} \subset X^{(4)}$ be the open stratum of Stratification (3.1.3), i.e. the partition is $(4) := (4, 0, 0, 0)$, and set

$$B^0 := \phi^{-1} X^{(4)}_{(4)}.$$

Let $\gamma \in X^{(4)}_{(4)}$, so $\gamma = q_1 + \cdots + q_4$, where the q_j are pairwise distinct. Let $Quot(\mathcal{O}_X^{(2)}, \gamma)$ be the quot-scheme parametrizing quotients $\Psi\colon \mathcal{O}_X^{(2)} \to \bigoplus_{j=1}^{4} \mathbf{C}_{q_j}$. Thus

$$Quot(\mathcal{O}_X^{(2)}, \gamma) \cong \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1.$$

The group $\mathbf{P}\mathrm{Aut}(\mathcal{O}_X^{(2)}) \cong \mathrm{PGL}(2)$ acts on this quot-scheme (the diagonal action on $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$). Choose the symmetric linearization of this action: as is easily verified (semi)stability of $([\Psi_1], [\Psi_2], [\Psi_3], [\Psi_4]) \in (\mathbf{P}^1)^4$ is equivalent to (semi)stability of the sheaf F fitting into the exact sequence

$$0 \to F \to \mathcal{O}_X^{(2)} \xrightarrow{\Psi} \bigoplus_{j=1}^{4} \mathbf{C}_{q_j} \to 0.$$

Hence we get a morphism

$$\begin{array}{ccc} \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 // \mathrm{PGL}(2) & \xrightarrow{\iota} & B \\ \text{point representing } ([\Psi_1], [\Psi_2], [\Psi_3], [\Psi_4]) & \mapsto & [F]. \end{array}$$

As is easily checked this map is injective. Since every sheaf parametrized by $\phi^{-1}\gamma$ is equivalent to the kernel F of some quotient in $Quot(\mathcal{O}_X^{(2)}, \gamma)^{ss}$, the map is one-to-one onto $\phi^{-1}(\gamma)$. A local computation shows that the differential if ι is injective everywhere. This shows that B^0 is a $\mathbf{P}^1$-fibration as claimed. It remains to prove that B^0 is dense in B. As is well-known the locus $B_Q \subset Q_4$ in the Quot-scheme (see (1.1)) parametrizing singular sheaves has codimension one. It follows immediately that B^{st} is of pure codimension one in $\mathcal{M}_4$. Since Σ is in the closure of B^{st} (easy), we get that B has pure codimension one. Since $\dim \mathcal{M}_4 = 10$, we get $\dim B = 9$. Hence to prove that B^0 is dense in B it suffices to show that for all partitions p of 4 different from (4), we have

(3.4.1) $$\dim \phi^{-1} X_p^{(4)} < 9.$$

Since $\dim \phi^{-1} X_p^{(4)} \leq \dim \widetilde{\phi}^{-1} X_p^{(4)}$ ($\widetilde{\pi}$ is surjective !) it suffices to prove Inequality (3.4.1) with ϕ replaced by $\widetilde{\phi}$. This follows at once from the lemma below.

(3.4.2) Lemma. *Let $p = (p_1, \ldots, p_4)$ be a partition of 4. If $\gamma \in X_p^{(4)}$, then*

$$\dim \widetilde{\phi}^{-1}(\gamma) \leq 5 - \sum_{i=1}^{4} p_i.$$

Proof. The map

$$\widetilde{\phi}^{-1} X_p^{(4)} \xrightarrow{\widetilde{\phi}} X_p^{(4)}$$

is a locally trivial fibration in the analytic topology, hence the fibers have a constant dimension, say d. Thus $\widetilde{\phi}^{-1} X_p^{(4)} \subset \Gamma_d$, so that by Proposition (3.0.2)

$$d \leq \mathrm{cod}(\widetilde{\phi}^{-1} X_p^{(4)}, \widetilde{\mathcal{M}}_4) = 10 - (\dim X_p^{(4)} + d).$$

Since $\dim X_p^{(4)} = 2 \sum_{i=1}^{4} p_i$, the lemma follows . **q.e.d.**

Proof of Proposition (3.0.8). We claim that every irreducible component of B_C has dimension at least 8. Let

$$X_C^{(4)} := \{\gamma \in X^{(4)} | \ \mathrm{supp}\gamma \cap C \neq \emptyset\},$$

so that

(3.4.3). $$B_C = (\phi|_B)^{-1} X_C^{(4)}$$

Since $X^{(4)}$ is the Quotient of a smooth variety by a finite group, $X_C^{(4)}$ is a $\mathbf{Q}$-Cartier divisor in $X^{(4)}$; by (3.4.3) every component of B_C has codimension at most one in B, and this proves our claim. To prove that B_C^0 is dense in B_C one proceeds as in the proof that B^0 is dense in B. We consider the stratification of $\phi(B_C)$

obtained by intersecting with the strata $X_p^{(4)}$, and applying Lemma (3.4.2) we get that $\phi^{-1}(X^{(4)}_{(4)} \cap X^{(4)}_C)$ is dense in B_C. Proposition (3.0.8) follows immediately from this.

4. $\widetilde{\mathcal{M}}_4$ is simply-connected.

We assume that $H^2 = 2$ and that $\mathrm{Pic}(X) = \mathbf{Z}[H]$. We will show that $\widetilde{\mathcal{M}}_4$ is birational to (a component of) the Jacobian fibration parametrizing (stable) degree-six line-bundles on curves in $|\mathcal{O}_X(2)|$. This implies that $\pi_1(\mathcal{J})$ surjects onto $\pi_1(\widetilde{\mathcal{M}}_4)$, hence it will suffice to show that $\mathcal{J}$ is simply connected. The latter result is proved by a monodromy argument.

4.1. $\widetilde{\mathcal{M}}_4$ is birational to a Jacobian fibration.

Let $C \subset |\mathcal{O}_X(2)|$ be a reduced irreducible curve, $\iota\colon C \hookrightarrow X$ be inclusion, and L a degree-six line-bundle on C. We let $\mathcal{N}$ be Simpson's moduli space [LP2,S] of pure one-dimensional sheaves ξ on X, stable with respect to $\mathcal{O}_X(1)$, with

$$\chi(\xi(n)) = \chi(\iota_* L(n)).$$

Thus $[\iota_* L]$ is a typical point of $\mathcal{N}$. We will be interested in a certain open subset of $\mathcal{N}$ defined as follows. First, let $U \subset |\mathcal{O}_X(2)|$ be the open subset parametrizing reduced curves. The following result is a straightforward application of the definition of stability according to Simpson [LP2,S].

(4.1.1) Lemma. *Let $[C] \in U$, and let L be a line-bundle on C such that:*

1. *if C is irreducible, the degree of L is 6,*
2. *if $C = C_1 \cup C_2$ and $L_i := L|_{C_i}$, the degree of L_i is 3.*

Then $\iota_* L$ is a stable pure one-dimensional sheaf on X.

Definition. Let $\mathcal{J} \subset \mathcal{N}$ be the open set parametrizing sheaves $\iota_* L$, where $[C] \in U$, $\iota\colon C \hookrightarrow X$ is inclusion, and L is a line-bundle on C satisfying the hypotheses of Lemma (4.1.1). We will often denote by (C, L) the point $[\iota_* L] \in \mathcal{J}$.

(4.1.2) Proposition. *$\mathcal{J}$ is smooth of dimension* 10.

Proof. By results of Mukai [M1] we know that $\mathcal{N}$ is smooth of dimension 10. Since $\mathcal{J}$ is open in $\mathcal{N}$, the proposition follows. **q.e.d.**

Let

$$\mathcal{J}^0 := \{(C, L) \in \mathcal{J} |\ \ C \text{ is smooth, } h^0(L) = 2 \text{ and } L \text{ is globally generated}\}.$$

If $(C, L) \in \mathcal{J}$, then $\chi(L) = 2$, hence $h^0(L) \geq 2$; thus $\mathcal{J}^0$ is open in $\mathcal{J}$. Furthermore, as is easily verified, $\mathcal{J}^0$ is non-empty. Let $(C, L) \in \mathcal{J}^0$. Following Lazarsfeld [La] we will associate to (C, L) a stable rank-two vector-bundle on X with $c_1 = 0$, $c_2 = 4$; this construction will define an isomorphism between $\mathcal{J}^0$ and an open subset of $\widetilde{\mathcal{M}}_4$. Since L is globally generated, the evaluation map $H^0(L) \otimes \mathcal{O}_X \to \iota_* L$ is surjective: let E be the sheaf on X fitting into the exact sequence

$$0 \to E \xrightarrow{\epsilon} H^0(L) \otimes \mathcal{O}_X \to \iota_* L \to 0. \tag{4.1.3}$$

Lemma. *Keeping notation as above, E is a slope-stable rank-two vector bundle on X with Chern classes $c_1(E) = -2H$, $c_2(E) = 6$.*

Proof. The Chern classes are easily computed from (4.1.3). To show stability, consider the exact sequence

$$0 \to L^{-1} \to H^0(L) \otimes \mathcal{O}_C \to L \to 0.$$

By (4.1.3) we have

$$\mathrm{Im}(\epsilon|_C) = L^{-1}. \tag{4.1.4}$$

Now suppose E is not stable: since $\mathrm{Pic}(X) = \mathbf{Z}[H]$ there is an injection of sheaves $\mathcal{O}_X(k) \overset{\alpha}{\hookrightarrow} E$, where $k \geq -1$. By (4.1.4)

$$(\epsilon \circ \alpha)|_C \in H^0(L^{-1}(-k)).$$

Since $\deg L = 6$, we have $\deg L^{-1}(-k) \leq -2$. Thus $\epsilon \circ \alpha$ vanishes on C, i.e.

$$\epsilon \circ \alpha \in H^0(\mathcal{O}_X^{(2)}(-k)(-C)) = H^0(\mathcal{O}_X^{(2)}(-k-2)).$$

Since $k \geq -1$, this last group is zero, contradiction. **q.e.d.**

Set $F := E(1)$; by the above lemma F is a slope-stable rank-two vector-bundle on X with $c_1(F) = 0$, $c_2(F) = 4$. Thus we can define $\Phi^0\colon \mathcal{J}^0 \to \mathcal{M}_4$ by setting

$$\begin{array}{ccc} \mathcal{J}^0 & \xrightarrow{\Phi^0} & \mathcal{M}_4 \\ (C, L) & \mapsto & [F]. \end{array}$$

Let $\mathcal{M}_4^0 \subset \mathcal{M}_4^{lf}$ be the subset parametrizing sheaves F such that $h^0(F(1)) = 2$, and such that the locus where the evaluation map

$$H^0(F(1)) \otimes \mathcal{O}_X \xrightarrow{f} F(1),$$

drops rank is a smooth curve.

(4.1.5) Proposition. *Keeping notation as above, $\mathcal{M}_4^0$ is an open subset of $\mathcal{M}_4$. The map Φ^0 is an isomorphism of $\mathcal{J}^0$ onto $\mathcal{M}_4^0$.*

Proof. To simplify notation we set $G := F(1)$, where $[F] \in \mathcal{M}_4^{lf}$. By Riemann-Roch we get $\chi(G) = 2$. By Serre duality $h^2(G) = h^0(G^*)$, hence stability gives $h^2(G) = 0$. Thus $h^0(G) \geq 2$, so that the locus where $h^0(G) = 2$ is open in $\mathcal{M}_4^{lf}$ (which is open in $\mathcal{M}_4$). The other condition defining $\mathcal{M}_4^0$ is clearly open. Now let's prove $\Phi^0(\mathcal{J}^0) \subset \mathcal{M}_4^0$. Tensoring (4.1.3) by $\mathcal{O}_X(2)$ and observing that $G = E(2)$, we get

$$0 \to G \to \mathcal{O}_X^{(2)}(2) \to \iota_* L(2) \to 0. \tag{4.1.6}$$

The long exact sequence of $Tor(\cdot, \mathcal{O}_C)$ associated to (4.1.6) gives

$$0 \to L \to G|_C \to L^{-1}(2) \to 0.$$

From (4.1.6) we get a natural injection $\beta\colon \mathcal{O}_X^{(2)}(2)(-C) \hookrightarrow G$, and $\mathrm{Im}(\beta|_C) = L$. Since $C \in |\mathcal{O}_X(2)|$ we get an exact sequence

$$0 \to \mathcal{O}_X^{(2)} \xrightarrow{\gamma} G \to \iota_* L^{-1}(2) \to 0. \tag{4.1.7}$$

By adjunction $\mathcal{O}_C(2) \cong K_C$, hence Serre duality gives $h^0(L^{-1}(2)) = h^1(L)$. By hypothesis $h^0(L) = \chi(L)$, hence $h^1(L) = 0$. Thus the cohomology long exact sequence of (4.1.7) gives $h^0(G) = 2$. Furthermore the map γ drops rank along C, which is smooth, hence $[F] \in \mathcal{M}_4^0$. Now we define an inverse

$$(\Phi^0)^{-1}\colon \mathcal{M}_4^0 \to \mathcal{J}^0.$$

Let $[F] \in \mathcal{M}_4^0$. Since $[F] \in \mathcal{M}_4^0$, we have an exact sequence

$$0 \to H^0(G) \otimes \mathcal{O}_X \xrightarrow{f} G \to \iota_* \xi \to 0, \tag{4.1.8}$$

where $(\det f) \in |\mathcal{O}_X(2)|$ is smooth: set $C := (\det f)$. The map f is nowhere zero because C is smooth, hence ξ is a line-bundle on C: we can write $\xi = L^{-1} \otimes K_C$, where L is a line-bundle. By a Chern class computation one gets $\deg L = 6$. Since $h^0(G) = 2$, the cohomology long exact sequence of (4.1.8) gives $h^0(L^{-1} \otimes K_C) = 0$.

Thus $h^1(L) = 0$, so that $h^0(L) = 2$. Furthermore, the long exact sequence of $Tor(\cdot, \mathcal{O}_C)$ associated to (4.1.8) gives

$$0 \to L \to G|_C \to L^{-1} \otimes K_C \to 0.$$

Since $(\mathrm{Im} f)|_C = L$, the line-bundle L is generated by global sections. We have proved $(C, L) \in \mathcal{J}^0$. As is easily verified the map

$$\begin{array}{ccc} \mathcal{M}_4^0 & \to & \mathcal{J}^0 \\ [F] & \to & (C, L) \end{array}$$

is the inverse of Φ^0. **q.e.d.**

We can view Φ^0 as a map to $\widetilde{\mathcal{M}}_4$, because by Proposition (1.1.5) $\mathcal{M}_4^0$ is in the stable locus of $\mathcal{M}_4$. Since $\widetilde{\mathcal{M}}_4$ is irreducible, Proposition (4.1.5) implies that Φ^0 extends to a birational map

$$\Phi : \mathcal{J} \cdots > \widetilde{\mathcal{M}}_4.$$

Let $I(\Phi) \subset \mathcal{J}$ be the indeterminacy locus of Φ. By Proposition (4.1.2) $\mathcal{J}$ is smooth, hence $I(\Phi)$ has codimension at least two and the map induced by inclusion

$$\pi_1(\mathcal{J} \setminus I(\Phi)) \to \pi_1(\mathcal{J})$$

is an isomorphism. On the other hand, since Φ is a birational map and $\widetilde{\mathcal{M}}_4$ is smooth, the map

$$\pi_1(\mathcal{J} \setminus I(\Phi)) \to \pi_1(\widetilde{\mathcal{M}}_4)$$

is surjective. Thus we have proved the following result.

(4.1.9) Proposition. *The map Φ induces a surjection $\pi_1(\mathcal{J}) \to \pi_1(\widetilde{\mathcal{M}}_4)$.*

4.2. $\mathcal{J}$ is simply-connected.

Let $f : X \to |\mathcal{O}_X(1)|^* \cong \mathbf{P}^2$ be the two-to-one cover, and let $B \subset \mathbf{P}^2$ be the (sextic) branch curve. Let $V \subset U$ be the open subset parametrizing smooth curves $C \in |\mathcal{O}_X(2)|$. Then

$$V = U \setminus (\Delta \cup \Lambda), \qquad (4.2.1)$$

where

$$\begin{aligned} \Delta :=& \{C \in |\mathcal{O}_X(2)| \mid f(C) \text{ is a smooth conic tangent to } B\}, \\ \Lambda :=& \{C \in |\mathcal{O}_X(2)| \mid f(C) \text{ is a singular reduced conic}\}. \end{aligned}$$

Let

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{\rho} & |\mathcal{O}_X(2)| \cong \mathbf{P}^5 \\ \xi & \mapsto & \mathrm{supp}\xi, \end{array}$$

and set $\mathcal{J}_V := \rho^{-1}V$. By Proposition (4.1.2) $\mathcal{J}$ is smooth, hence the map

$$j_\# : \pi_1(\mathcal{J}_V) \to \pi_1(\mathcal{J})$$

induced by inclusion $j : \mathcal{J}_V \hookrightarrow \mathcal{J}$ is surjective. We will show $j_\#$ is trivial: this will prove $\mathcal{J}$ is simply-connected. The map $\mathcal{J}_V \to V$ is a fibration with fibers 5-dimensional Jacobians. The homotopy exact sequence of a fibration gives an exact sequence

$$\pi_1(Jac(C)) \to \pi_1(\mathcal{J}_V) \to \pi_1(V) \to \{1\},$$

where $C \in |\mathcal{O}_X(2)|$ is a fixed smooth curve.

(4.2.2) Lemma. *Keeping notation as above, the restriction of $j_\#$ to $\pi_1(Jac(C))$ is trivial.*

Proof. We have an isomorphism $\pi_1(Jac(C)) \cong H_1(C; \mathbf{Z})$. As is easily seen the vanishing cycles on C for the family of curves parametrized by V generate all of $H_1(C; \mathbf{Z})$. Since $j_\#$ is trivial on vanishing cycles, the lemma follows. **q.e.d.**

By the above lemma, $j_\#$ induces a surjective homomorphism $\bar{j}_\#: \pi_1(V) \to \pi_1(\mathcal{J})$. We will finish the proof by showing that $\bar{j}_\#$ is trivial. This is a consequence of the following easy result.

(4.2.3) Lemma. *Let $D \subset \mathbf{C}$ be a disc centered at 0, and let $\psi: D \hookrightarrow U$ be a (holomorphic) embedding such that:*

1. $\psi(D^0) \subset V$, *where* $D^0 := D \setminus \{0\}$,
2. $\psi(0)$ *is a smooth point of* Δ *(or* Λ*), and* D *intersects* Δ *(respectively* Λ*) transversely.*

Then, after shrinking D, we can assume there exists a lift $\widetilde{\psi}: D \to \mathcal{J}$ of ψ.

Proof. Let $q: \mathcal{C} \to D$ be the family of curves in $|\mathcal{O}_X(2)|$ parametrized by D. By Item (2) the analytic surface $\mathcal{C}$ is smooth. Shrinking D we can assume there exists a line-bundle $\mathcal{L}$ on $\mathcal{C}$ such that for each $t \in D$, the couple $(q^{-1}t, \mathcal{L}|_{q^{-1}t})$ satisfies the hypotheses of Lemma (4.1.1). By the modular property of $\mathcal{J}$ the couple $(\mathcal{C}, \mathcal{L})$ induces a morphism $\widetilde{\psi}: D \to \mathcal{J}$ lifting ψ. **q.e.d.**

To finish the proof that $\bar{j}_\#$ is trivial, let $R \subset U$ be a straight line (notice that $|\mathcal{O}_X(2)| \setminus U$ has codimension 3) transverse to Δ and Λ. The map induced by inclusion

$$\pi_1(R \setminus (\Delta \cup \Lambda)) \to \pi_1(V)$$

is a surjection by Bertini's Theorem [GM, p.151]. The fundamental group on the left is generated by $\pi_1(D^0_{x_i})$, where $R \cap (\Delta \cup \Lambda) = \{x_1, \dots, x_k\}$, and $D^0_{x_i} \subset (R \setminus (\Delta \cup \Lambda))$ is a small punctured disc centered at x_i. By Lemma (4.2.3) the map $\bar{j}_\#$ is trivial on each $\pi_1(D^0_{x_i})$. This shows $\bar{j}_\#$ is trivial, and concludes the proof that $\mathcal{J}$ is simply-connecetd.

5. Proof that $b_2(\widetilde{\mathcal{M}}_4) \geq 24$.

Let $\phi: \mathcal{M}_4 \to \mathbf{P}^N$ be the determinant map of Subsection (3.1). Composing Donaldson's map $\mu: H_2(X; \mathbf{Q}) \to H^2(\phi(\mathcal{M}_4); \mathbf{Q})$ (see [FM,Mo]) with pull-back by ϕ, we get

$$\phi^*\mu: H_2(X; \mathbf{Q}) \to H^2(\mathcal{M}_4; \mathbf{Q}).$$

This map is an injection, because by [FM VII.2.17,Mo]

$$\int_{\mathcal{M}_4} \wedge^{10}(\phi^*\mu(\alpha)) = \frac{10!}{5!2^5}\left(\int_X \wedge^2\alpha\right)^5. \tag{5.1}$$

(5.2) Claim. *Keep notation as above, and let $h \in H^2(\mathcal{M}_4)$ be the class of an ample divisor. Then*

$$\dim\left(\mathbf{Q}\widetilde{\pi}^*h + \widetilde{\pi}^*\phi^*\mu(H_2(X; \mathbf{Q}))\right) = 23.$$

Proof. Consider the boundary divisor $B \subset \mathcal{M}_4$ (see (3.1)). By Proposition (3.0.5) B contains $\mathbf{P}^1$'s contracted by ϕ. If $\widetilde{\mathbf{P}}^1 \subset \widetilde{\mathcal{M}}_4$ is the proper transform of a $\mathbf{P}^1 \subset B$ contracted by ϕ, then

$$\langle \widetilde{\pi}^*h, \widetilde{\mathbf{P}}^1 \rangle \neq 0, \qquad \langle \widetilde{\pi}^*\phi^*\mu(H_2(X; \mathbf{Q})), \widetilde{\mathbf{P}}^1 \rangle = 0.$$

Thus $\mathbf{Q}\widetilde{\pi}^*h$ is not contained in $\widetilde{\pi}^*\phi^*\mu(H_2(X; \mathbf{Q}))$. Furthermore the subspace $\widetilde{\pi}^*\phi^*\mu(H_2(X; \mathbf{Q}))$ has dimension 22 by (5.1). This proves the claim. **q.e.d.**

Finally let's show that $\mathbf{Q}c_1(\widetilde{\Sigma})$ and $\mathbf{Q}\widetilde{\pi}^*h + \widetilde{\pi}^*\phi^*\mu(H_2(X;\mathbf{Q}))$ span a 24-dimensional subspace of $H^2(\widetilde{\mathcal{M}}_4;\mathbf{Q})$. Let $\widetilde{\Sigma}_{Z,W} \subset \widetilde{\mathcal{M}}_4$ be the fiber of $\widetilde{\pi}$ over $[I_Z \oplus I_W]$, where $Z \neq W$. By (2.2.4) $\widetilde{\Sigma}_{Z,W} \cong \mathbf{P}^1$, and by (2.2.5)

$$\langle c_1(\widetilde{\Sigma}), \widetilde{\Sigma}_{Z,W}\rangle = -2, \qquad \langle \mathbf{Q}\widetilde{\pi}^*h \oplus \widetilde{\pi}^*\phi^*\mu(H_2(X;\mathbf{Q})), \widetilde{\Sigma}_{Z,W}\rangle = 0.$$

Thus $\mathbf{Q}c_1(\widetilde{\Sigma})$ is not contained in $\mathbf{Q}\widetilde{\pi}^*h + \widetilde{\pi}^*\phi^*\mu(H_2(X;\mathbf{Q}))$. By Claim (5.2) we conclude that

$$\dim\Big(\mathbf{Q}c_1(\widetilde{\Sigma}) + \mathbf{Q}\widetilde{\pi}^*h + \widetilde{\pi}^*\phi^*\mu(H_2(X;\mathbf{Q}))\Big) = 24.$$

6. Towards a smooth minimal model of $\mathcal{M}_c$, for $c \geq 6$.

We begin by stating some auxiliary results. If V is a three-dimensional vector space, let

$$\mathbf{CC}(V) := \text{closure of } \{(C,D) \in \mathbf{P}(S^2V) \times \mathbf{P}(S^2\check{V}) |\ C,\ D \text{ are smooth conics dual to each other}\},$$

i.e. the space of complete conics in $\mathbf{P}(\check{V})$. Let $\mathbf{Gr}^\omega(3, T_{X^{[n]}})$ be the relative symplectic Grassmannian over $X^{[n]}$, with fiber $\mathbf{Gr}^\omega(3, E_Z)$ over $[Z] \in X^{[n]}$. Let $\mathcal{B}$ be the tautological rank-three bundle over $\mathbf{Gr}^\omega(3, T_{X^{[n]}})$, and $\mathbf{CC}(\mathcal{B})$ be the tautological family of complete conics over $\mathbf{Gr}^\omega(3, T_{X^{[n]}})$. Thus we have a locally-trivial fibration

$$\begin{array}{ccc} \mathbf{CC}(B) & \hookrightarrow & \mathbf{CC}(\mathcal{B}) \\ \downarrow & & \downarrow \\ ([Z],[B]) & \in & \mathbf{Gr}^\omega(3, T_{X^{[n]}}). \end{array}$$

We will prove (later) the following.

(6.0.1) Proposition. *Keep notation as above, and assume $c \geq 6$. Then $\widehat{\Omega}_c$ is isomorphic to $\mathbf{CC}(\mathcal{B})$; under this isomorphism the map*

$$\widehat{\pi}|_{\widehat{\Omega}_c} : \widehat{\Omega}_c \to \Omega_c \cong X^{[n]}$$

corresponds to the natural projection $\mathbf{CC}(\mathcal{B}) \to X^{[n]}$.

For $[Z] \in X^{[n]}$, let

$$\widehat{\Omega}_Z := \mathbf{CC}(\mathcal{B}_Z) \cong \widehat{\pi}^{-1}([I_Z \oplus I_Z]) \cap \widehat{\Omega}_c,$$

where $\mathcal{B}_Z$ is the restriction of $\mathcal{B}$ to $\mathbf{Gr}^\omega(3, E_Z)$. Let's define classes $\widehat{\sigma}_Z, \widehat{\epsilon}_Z, \widehat{\gamma}_Z \in NE_1(\widehat{\Omega}_Z)$ as follows. If $[B] \in \mathbf{Gr}^\omega(3, E_Z)$ let $\widehat{\Omega}_B$ be the fiber over $[B]$ of the natural fibration $\widehat{\Omega}_Z \to \mathbf{Gr}^\omega(3, E_Z)$; thus $\widehat{\Omega}_B \cong \mathbf{CC}(B)$. We have two projections

$$\mathbf{P}(S^2B) \overset{\Phi_B}{\leftarrow} \mathbf{CC}(B) \overset{\check{\Phi}_B}{\to} \mathbf{P}(S^2\check{B}).$$

Each of Φ_B, $\check{\Phi}_B$ is the blow-up of the locus parametrizing conics of rank one. Set

$$\widehat{\sigma}_Z := \text{class in } N_1(\widehat{\Omega}_Z) \text{ of a line in a } \mathbf{P}^2\text{-fiber of } \check{\Phi}_B,$$

$$\widehat{\epsilon}_Z := \text{class in } N_1(\widehat{\Omega}_Z) \text{ of a line in a } \mathbf{P}^2\text{-fiber of } \Phi_B.$$

To define $\widehat{\gamma}_Z$, choose $[A] \in \mathbf{Gr}^\omega(2, E_Z)$, $q \in S^2A$ of rank two, and a line $\Lambda \subset \mathbf{P}(A^\perp/A)$. For $t \in \Lambda$, let $B_t \subset E_Z$ be the three-dimensional subspace containing A and projecting to the line corresponding to t; clearly $[B_t] \in \mathbf{Gr}^\omega(3, E_Z)$. Furthermore, let $q_t \in S^2B_t$ be the image of q under the inclusion $A \hookrightarrow B_t$. We set

$$\widehat{\gamma}_Z := \text{class in } N_1(\widehat{\Omega}_Z) \text{ of } \{\Phi_{B_t}^{-1}([q_t])\}_{t\in\Lambda}.$$

Notice that since q_t has rank two for all t, $[q_t]$ is never in the exceptional locus of Φ_{B_t}, and thus the right-hand side of the above equality is indeed a curve. Now let $i^Z : \widehat{\Omega}_Z \hookrightarrow \widehat{\mathcal{M}}_c$ be inclusion, and set

$$\widehat{\sigma}_c := i^Z_*\widehat{\sigma}_Z,$$

$$\widehat{\epsilon}_c := i^Z_*\widehat{\epsilon}_Z,$$

$$\widehat{\gamma}_c := i^Z_*\widehat{\gamma}_Z.$$

Notice that the right-hand sides of the above equalities are independent of Z, thus $\widehat{\sigma}_c$, $\widehat{\epsilon}_c$, $\widehat{\gamma}_c$ are well-defined elements of $NE_1(\widehat{\mathcal{M}}_c)$. Later we will prove the following.

(6.0.2) Proposition. *Keep notation as above, and assume $c \geq 6$. Then:*

1. *$\widehat{\sigma}_c$, $\widehat{\epsilon}_c$, $\widehat{\gamma}_c$ are linearly independent.*
2. *$\mathbf{R}^+\widehat{\sigma}_c \oplus \mathbf{R}^+\widehat{\epsilon}_c \oplus \mathbf{R}^+\widehat{\gamma}_c$ is a $K_{\widehat{\mathcal{M}}_c}$-negative extremal face of $\overline{NE}_1(\widehat{\mathcal{M}}_c)$.*

Let $\overline{\mathcal{M}}_c$ be the scheme obtained contracting the $K_{\widehat{\mathcal{M}}_c}$-negative extremal ray $\mathbf{R}^+\widehat{\sigma}_c$.

(6.0.3) Proposition. *Keep notation as above. Then $\overline{\mathcal{M}}_c$ is a smooth projective desingularization of $\mathcal{M}_c$.*

Let $\overline{\epsilon}_c \in N_1(\overline{\mathcal{M}}_c)$ be the image of $\widehat{\epsilon}_c$. We will prove the following.

(6.0.4) Proposition. *Keep notation as above. Then $\mathbf{R}^+\overline{\epsilon}_c$ is a $K_{\overline{\mathcal{M}}_c}$-negative extremal ray of $\overline{NE}_1(\overline{\mathcal{M}}_c)$. The scheme $\widetilde{\mathcal{M}}_c$ obtained contracting $\mathbf{R}^+\overline{\epsilon}_c$ is a smooth projective desingularization of $\mathcal{M}_c$. It carries a holomorphic two-form, degenerate on a single irreducible divisor (the image of $\widehat{\Sigma}_c$ under the map $\widehat{\mathcal{M}}_c \to \widetilde{\mathcal{M}}_c$).*

We stop at $\widetilde{\mathcal{M}}_c$: it is the best we can do in trying to find a smooth symplectic model of $\mathcal{M}_c$. If $\widetilde{\gamma}_c \in NE_1(\widetilde{\mathcal{M}}_c)$ denotes the image of $\widehat{\gamma}_c$, then $\mathbf{R}^+\widetilde{\gamma}_c$ is a $K_{\widetilde{\mathcal{M}}_c}$-negative extremal ray; let $\mathcal{M}_c^\flat$ be the scheme obtained contracting $\mathbf{R}^+\widetilde{\gamma}_c$. Then $\mathcal{M}_c^\flat$ is a minimal model (the canonical bundle is trivial) of $\mathcal{M}_c$, but it is not smooth. In fact

$$(\mathcal{M}_c \setminus \Omega_c) \cong \left(\mathcal{M}_c^\flat \setminus \Omega_c^\flat\right),$$

where $\Omega_c^\flat$ is the image of $\widehat{\Omega}_c$. As a last observation, we remark that $\mathcal{M}_c$ is obtained from $\widehat{\mathcal{M}}_c$ as follows: first we contract $\mathbf{R}^+\widehat{\gamma}_c$ to get $S_c//\mathrm{PGL}(N)$, then we contract the image of $\mathbf{R}^+\widehat{\epsilon}_c$ in $NE_1(S_c//\mathrm{PGL}(N))$ to get $R_c//\mathrm{PGL}(N)$, finally the contraction of the image of $\mathbf{R}^+\widehat{\sigma}_c$ in $NE_1(R_c//\mathrm{PGL}(N))$ gives $\mathcal{M}_c$. In other words, $\mathcal{M}_c^\flat$ is obtained from $\widehat{\mathcal{M}}_c$ by reversing the order of the contractions.

6.1. Proof of Proposition (6.0.1).

Let $x \in \Omega_Q^0$: set $F_x = I_Z \otimes V$ and $W = sl(V)$. (Here $V \cong \mathbf{C}^2$.) Consider the following fiber bundles over $\mathbf{Gr}^\omega(3, E_Z)$:

$$\begin{aligned}
\mathbf{P}\mathrm{Hom}(W, \mathcal{B}_Z) :=& \mathbf{P}\left(\check{W} \otimes \mathcal{B}_Z\right), \\
\mathbf{P}\mathrm{Hom}_k(W, \mathcal{B}_Z) :=& \{\varphi \in \mathbf{P}\mathrm{Hom}(W, \mathcal{B}_Z) |\ \mathrm{rk}\varphi \leq k\}, \\
\widehat{\mathbf{Hom}}(W, \mathcal{B}_Z) :=& \text{blow-up of } \mathbf{P}\mathrm{Hom}(W, \mathcal{B}_Z) \text{ along } \mathbf{P}\mathrm{Hom}_1(W, \mathcal{B}_Z).
\end{aligned}$$

Let $h \colon \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z) \to \mathbf{P}\mathrm{Hom}(W, \mathcal{B}_Z)$ be the blow-down map, and

$$f \colon \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z) \to \mathbf{P}\mathrm{Hom}^\omega(W, E_Z)$$

be the composition of h with the obvious map $\mathbf{P}\mathrm{Hom}(W, \mathcal{B}_Z) \to \mathbf{P}\mathrm{Hom}^\omega(W, E_Z)$. Proposition (6.0.1) will be a straightforward consequence of the following result.

(6.1.1) Proposition. *Let $x \in \Omega_Q^0$, and keep notation as above. There is an isomorphism*

$$\widehat{f} \colon \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z) \xrightarrow{\sim} (\pi_R \pi_S \pi_T)^{-1}(x)$$

such that $\pi_S \pi_T \widehat{f} = f$. (Recall (1.5.2) that $\pi_R^{-1}(x) \cong \mathbf{P}\mathrm{Hom}^\omega(W, E_Z)$.)

Proof. We brake up the proof into various steps.

I. The map $\overline{f}$. We will prove there exists a map

$$\overline{f} \colon \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z) \to (\pi_R \pi_S)^{-1}(x)$$

lifting f. Let $\mathbf{D}_1 \subset \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z)$ be the exceptional divisor of h. Equality (1.8.1) gives that $(\pi_R \pi_S)^{-1}(x)$ is the blow-up of $\pi_R^{-1}(x) \cong \mathbf{P}\mathrm{Hom}^\omega(W, E_Z)$ along $\mathbf{P}\mathrm{Hom}_1(W, E_Z)$; hence to prove $\overline{f}$ exists it is sufficient to verify that

$$f^* I_{\mathbf{P}\mathrm{Hom}_1(W, E_Z)} = \mathcal{O}_{\widehat{\mathbf{Hom}}(W, \mathcal{B}_Z)}(-\mathbf{D}_1). \tag{6.1.2}$$

Since we have an equality of sets $f^{-1}(\mathbf{P}\mathrm{Hom}_1(W,E_Z)) = \mathbf{D}_1$, we must show that given any $p \in \mathbf{D}_1$, there exists $w \in T_p\widehat{\mathbf{Hom}}(W,\mathcal{B}_Z)$ such that

$$f_*(w) \notin T_{f(p)}\mathbf{P}\mathrm{Hom}_1(W,E_Z). \tag{6.1.3}$$

Let $[B] \in \mathbf{Gr}^{\omega}(3,E_Z)$ be the image of p under the bundle projection. Thus $h(p) \in \mathbf{P}\mathrm{Hom}(W,B)$, and p is in the image of the inclusion

$$\iota\colon Bl_{\mathbf{P}\mathrm{Hom}_1(W,B)}\mathbf{P}\mathrm{Hom}(W,B) \hookrightarrow \widehat{\mathbf{Hom}}(W,\mathcal{B}_Z). \tag{6.1.4}$$

The intersection of $\mathbf{D}_1$ with the left-hand side is smooth, thus there exists

$$v \in T_p\left(Bl_{\mathbf{P}\mathrm{Hom}_1(W,B)}\mathbf{P}\mathrm{Hom}(W,B)\right)$$

transverse to $\mathbf{D}_1$. The vector $w := \iota_*(v)$ satisfies (6.1.3).

II. The restriction of $\overline{f}$ to $\mathbf{D}_1$. For vector spaces A, B, let $\mathrm{Hom}_k(A,B)$ be the determinantal variety of linear maps $A \to B$ of rank at most k, and let $\mathbf{P}\mathrm{Hom}_k(A,B)$ be its projectivization. Let $\varphi \in \mathrm{Hom}_h(A,B)$ be of rank exactly h; recall that the natural map

$$T_\varphi\mathrm{Hom}(A,B) = \mathrm{Hom}(A,B) \to \mathrm{Hom}(\mathrm{Ker}\varphi, B/\mathrm{Im}\varphi)$$

induces an isomorphism

$$\left(C_{\mathrm{Hom}_h}\mathrm{Hom}_k(A,B)\right)_\varphi \cong \mathrm{Hom}_{k-h}\left(\mathrm{Ker}\varphi, B/\mathrm{Im}\varphi\right),$$

Projectivizing we get an isomorphism

$$\left(C_{\mathbf{P}\mathrm{Hom}_h}\mathbf{P}\mathrm{Hom}_k(A,B)\right)_{[\varphi]} \cong \mathrm{Hom}_{k-h}(\mathrm{Ker}\varphi, B/\mathrm{Im}\varphi), \tag{6.1.5}$$

canonical up to scalars. Applying this isomorphism to $\mathbf{P}\mathrm{Hom}_1(W,B) \subset \mathbf{P}\mathrm{Hom}(W,B)$, for $[B] \in \mathbf{Gr}^{\omega}(3,E_Z)$, we get

$$\mathbf{D}_1 \cong \{([B],[\varphi],[\alpha]) \mid [B] \in \mathbf{Gr}^{\omega}(3,E_Z),\ [\varphi] \in \mathbf{P}\mathrm{Hom}_1(W,B),\ [\alpha] \in \mathbf{P}\mathrm{Hom}(\mathrm{Ker}\varphi, B/\mathrm{Im}\varphi)\}.$$

By (6.1.2) we know $\overline{f}$ maps $\mathbf{D}_1$ to Σ_S. By (1.7.12) there is a canonical isomorphism

$$\Sigma_S \cap (\pi_R\pi_S)^{-1}(x) \cong \{([\varphi],[\alpha]) \mid [\varphi] \in \mathbf{P}\mathrm{Hom}_1(W,E_Z),\ [\alpha] \in \mathbf{P}\mathrm{Hom}^{\omega_\varphi}(\mathrm{Ker}\varphi, \mathrm{Im}\varphi^\perp/\mathrm{Im}\varphi)\}.$$

For $([B],[\varphi],[\alpha]) \in \mathbf{D}_1$, let $j\colon B \hookrightarrow E_Z$, $\overline{j}\colon B/\mathrm{Im}\varphi \hookrightarrow \mathrm{Im}\varphi^\perp/\mathrm{Im}\varphi$ be the inclusion maps; one verifies easily that

$$\overline{f}([B],[\varphi],[\alpha]) = ([j\circ\varphi],[\overline{j}\circ\circ\alpha]). \tag{6.1.6}$$

This describes the restriction of $\overline{f}$ to $\mathbf{D}_1$. Let $\mathbf{D}_2 \subset \widehat{\mathbf{Hom}}^{\omega}(W,\mathcal{B}_Z)$ be the strict transform of $\mathbf{P}\mathrm{Hom}_2(W,\mathcal{B}_Z)$; applying (6.1.5) one gets

$$\begin{aligned}\mathbf{D}_1 \cap \mathbf{D}_2 &= \{([B],[\varphi],[\alpha]) \mid \mathrm{rk}\alpha = 1\},\\ (\Sigma_S \cap \Delta_S) \cap (\pi_R\pi_D)^{-1}(x) &= \{([\varphi],[\alpha]) \mid \mathrm{rk}\alpha = 1\}.\end{aligned}$$

In particular we have an isomorphism

$$\overline{f}|_{(\mathbf{D}_1\setminus\mathbf{D}_2)}\colon (\mathbf{D}_1 \setminus \mathbf{D}_2) \xrightarrow{\sim} (\Sigma_S \setminus \Delta_S) \cap (\pi_R\pi_S)^{-1}(x). \tag{6.1.7}$$

III. The map $\widehat{f}$. We will lift $\overline{f}$ to a map

$$\widehat{f}\colon \widehat{\mathbf{Hom}}(W,\mathcal{B}_Z) \to (\pi_R\pi_S\pi_T)^{-1}(x).$$

Let

$$\Delta := \Delta_S \cap (\pi_R \pi_S)^{-1}(x) \cong \widetilde{\mathbf{P}}\mathrm{Hom}_2^{\omega}(W, E_Z).$$

(See (1.8.12).) Since $(\pi_R \pi_S \pi_T)^{-1}(x)$ is the blow-up of $(\pi_R \pi_S)^{-1}(x)$ along Δ, the existence of a lift $\widehat{f}$ will follow from

$$\overline{f}^* I_\Delta = \mathcal{O}_{\widehat{\mathbf{Hom}}(W,\mathcal{B}_Z)}(-\mathbf{D}_2). \tag{6.1.8}$$

To prove this equality, first notice that set-theoretically $\overline{f}^{-1}(\Delta) = \mathbf{D}_2$. Thus it suffices to show that for any $p \in \mathbf{D}_2$, there exists $w \in T_p \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z)$ such that $\overline{f}_* w \notin T_\Delta$. Let $[B] \in \mathbf{Gr}^{\omega}(3, E_Z)$ be the image of p under the bundle projection; thus p is in the image of Inclusion (6.1.4). Since

$$\mathbf{D}_2 \cap Bl_{\mathbf{P}\mathrm{Hom}_1(W,B)} \mathbf{P}\mathrm{Hom}(W, B) = Bl_{\mathbf{P}\mathrm{Hom}_1(W,B)} \mathbf{P}\mathrm{Hom}_2(W, B),$$

and the right-hand side is smooth (see (1.8.12)), there exists $v \in T_p \left(Bl_{\mathbf{P}\mathrm{Hom}_1(W,B)} \mathbf{P}\mathrm{Hom}(W, B)\right)$ transverse to $\mathbf{D}_2$. As is easily checked $w := \iota_* v$ has the stated property.

IV. Proof that $\widehat{f}$ is an isomorphism. Clearly $\widehat{f}$ is birational. Since $(\pi_R \pi_S \pi_T)^{-1}(x)$ is smooth, it suffices by Zariski's Main Theorem to show that $\widehat{f}$ is an isomorphism in codimension one. One checks easily that the restriction of $\widehat{f}$ to the complement of $(\mathbf{D}_1 \cap \mathbf{D}_2)$ is an isomorphism onto its image. **q.e.d.**

Now we prove Proposition (6.0.1). Let $[Z] \in X^{[n]}$, and let $x \in \Omega^0_Q$ be such that $F_x \cong I_Z \otimes V$. By Proposition (6.1.1)

$$\widehat{\pi}^{-1}([I_Z \otimes V]) = \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z)//\mathrm{SO}(W).$$

The projection $\widehat{\mathbf{Hom}}(W, \mathcal{B}_Z) \to \mathbf{Gr}^{\omega}(3, E_Z)$ is $\mathrm{SO}(W)$-invariant, hence it descends to

$$\lambda \colon \widehat{\mathbf{Hom}}(W, \mathcal{B}_Z)//\mathrm{SO}(W) \to \mathbf{Gr}^{\omega}(3, E_Z).$$

For $[B] \in \mathbf{Gr}^{\omega}(3, E_Z)$ we have

$$\lambda^{-1}([B]) = Bl_{\mathbf{P}\mathrm{Hom}_1} \mathbf{P}\mathrm{Hom}(W, B)//\mathrm{SO}(W).$$

The map

$$\begin{array}{ccc} \mathbf{P}\mathrm{Hom}(W, B)^{ss} & \to & \mathbf{P}(S^2 B) \\ \alpha & \mapsto & \alpha \circ \alpha^t, \end{array}$$

identifies $\mathbf{P}\mathrm{Hom}(W, B)//\mathrm{SO}(W)$ with $\mathbf{P}(S^2 B)$. Since $\mathbf{P}\mathrm{Hom}_1(W, B)//\mathrm{SO}(W)$ is the locus of conics of rank one, and since taking the quotient commutes with blowing up (1.1.2), we conclude that $\lambda^{-1}([B]) = \mathbf{CC}(S^2 B)$. Proposition (6.0.1) follows at once.

6.2. Proof of Proposition (6.0.2)-Item (1).

Let $[Z] \in X^{[n]}$; we will introduce a basis of $N^1(\widehat{\Omega}_Z)$. Letting $\rho \colon \mathbf{P}(S^2 \mathcal{B}_Z) \to \mathbf{Gr}^{\omega}(3, E_Z)$ be bundle projection, and $\theta \colon \widehat{\Omega}_Z \to \mathbf{P}(S^2 \mathcal{B}_Z)$ be the blow-down map (see (6.0.1)), set

$$\begin{aligned} h &:= c_1(\check{\mathcal{B}}_Z), \\ x &:= c_1\left(\mathcal{O}_{\mathbf{P}(S^2 \mathcal{B}_Z)}(1)\right), \\ e &:= \text{class of the exceptional divisor of } \theta. \end{aligned}$$

Abusing notation we will denote with the same symbols the classes obtained pulling back h and x to $\widehat{\Omega}_Z$.

(6.2.1) Claim. *The classes h, x, e form a basis of $N^1(\widehat{\Omega}_Z)$.*

Proof. For $[B] \in \mathbf{Gr}^{\omega}(3, E_Z)$ the restrictions of x, e to $\widehat{\Omega}_B$ give a basis of $H^2(\widehat{\Omega}_B)$, hence it suffices to prove that h generates $H^2(\mathbf{Gr}^{\omega}(3, E_Z)$. This follows by applying Sommese's generalization of Lefschetz' hyperplane section theorem [La,(1.8)] to the embedding $\mathbf{Gr}^{\omega}(3, E_Z) \hookrightarrow \mathbf{Gr}(3, E_Z)$. **q.e.d.**

(6.2.2) Corollary. *Keeping notation as above,*

$$N_1(\widehat{\Omega}_Z) = \mathbf{R}^+\widehat{\sigma}_Z \oplus \mathbf{R}^+\widehat{\epsilon}_Z \oplus \mathbf{R}^+\widehat{\gamma}_Z.$$

Proof. As is easily checked the intersection matrix of $\{h, x, e\}$ with $\{\widehat{\sigma}_Z, \widehat{\epsilon}_Z, \widehat{\gamma}_Z\}$ is non-singular (see (6.4.8)). Hence the result follows from duality together with (6.2.1). **q.e.d.**

We will prove the following formulae:

$$c_1(\widehat{\Sigma}_c \cap \widehat{\Omega}_Z) = e, \tag{6.2.3}$$

$$c_1(\widehat{\Delta}_c \cap \widehat{\Omega}_Z) = -2e - 2h + 3x, \tag{6.2.4}$$

$$c_1(K_{\widehat{\Omega}_Z}) = 2e - (c-6)h - 6x. \tag{6.2.5}$$

Before proving the formulae we draw some consequences.

(6.2.6) Lemma. *The map $i^Z_*: N_1(\widehat{\Omega}_Z) \to N_1(\widehat{\mathcal{M}}_c)$ induced by inclusion is injective.*

Proof. By the adjunction formula $K_{\widehat{\Omega}_Z}$ is in the image of the restriction map $\mathrm{Pic}(\widehat{\mathcal{M}}_c) \to \mathrm{Pic}(\widehat{\Omega}_Z)$. By (6.2.3)-(6.2.5) we get that $N^1(\widehat{\mathcal{M}}_c) \to N^1(\widehat{\Omega}_Z)$ is surjective. Dualizing we conclude i^Z_* is injective. **q.e.d.**

In particular, since by (6.2.2) $\widehat{\sigma}_Z$, $\widehat{\epsilon}_Z$, $\widehat{\gamma}_Z$ are linearly independent, we see that $\widehat{\sigma}_c$, $\widehat{\epsilon}_c$, $\widehat{\gamma}_c$ are linearly independent; this proves Item (1) of Proposition (6.0.2). Now let's prove Formulae (6.2.3)-(6.2.5). The first one is the easiest: it follows immediately from (6.1.2).

Proof of (6.2.4). Let $D_Z \subset \mathbf{P}(S^2\mathcal{B}_Z)$ be the locus parametrizing singular conics. Then $\widehat{\Delta}_c \cap \widehat{\Omega}_Z$ is the strict transfrom of D_Z under θ. Since, for $[B] \in \mathbf{Gr}^\omega(3, E_Z)$, the locus of singular conics in $\mathbf{P}(S^2B)$ has multiplicity two along the locus of rank-one conics, we get that

$$c_1(\widehat{\Delta}_c \cap \widehat{\Omega}_Z) = \theta^* c_1(D_Z) - 2e.$$

hence Formula (6.2.4) will follow from the following equation:

$$c_1(D_Z) = -2h + 3x. \tag{6.2.7}$$

To prove it, we observe that D_Z is the degeneracy locus of the tautological map

$$\rho^*\check{\mathcal{B}}_Z \otimes \mathcal{O}_{\mathbf{P}(S^2\mathcal{B}_Z)}(-1) \xrightarrow{\Phi} \rho^*\mathcal{B}_Z.$$

Since $\mathrm{Det}\Phi \in \Gamma\left(\wedge^3\rho^*\mathcal{B}_Z \otimes \wedge^3\rho^*\mathcal{B}_Z \otimes \mathcal{O}_{\mathbf{P}(S^2\mathcal{B}_Z)}(3)\right)$, Equation (6.2.7) follows at once.

Proof of (6.2.5). First we prove that

$$c_1(K_{\mathbf{Gr}^\omega(3,E_Z)}) = -(c-2)h. \tag{6.2.8}$$

Consider the exact sequence

$$0 \to T_{\mathbf{Gr}^\omega} \to T_{\mathbf{Gr}}|_{\mathbf{Gr}^\omega} \xrightarrow{\rho_*} N_{\mathbf{Gr}^\omega/\mathbf{Gr}} \to 0.$$

Since $\mathbf{Gr}^\omega$ is the zero-locus of a section of $\wedge^2\mathcal{B}_Z^\vee$,

$$c_1(N_{\mathbf{Gr}^\omega/\mathbf{Gr}}) = c_1(\wedge^2\mathcal{B}_Z^\vee) = 2h.$$

Equation (6.2.8) follows from this together with the formula for the canonical bundle of a Grassmannian. Next, we get

$$c_1\left(K_{\mathbf{P}(S^2\mathcal{B}_Z)}\right) = -(c-6) - 6x \tag{6.2.9}$$

by considering the exact sequence

$$0 \to \mathrm{Ker}\rho_* \to T_{\mathbf{P}(S^2\mathcal{B}_Z)} \xrightarrow{\rho_*} \rho^* T_{\mathbf{Gr}^\omega} \to 0.$$

Finally, Formula (6.2.5) follows from (6.2.9) because $\widehat{\Omega}_Z$ is obtained by blowing up a codimension-two subset of $\mathbf{P}(S^2\mathcal{B}_Z)$, and e is the class of the exceptional divisor.

We close this subsection with a description of $\overline{NE}_1(\widehat{\Omega}_Z)$.

Claim. *Keeping notation as above, we have*

$$\overline{NE}_1(\widehat{\Omega}_Z) = \mathbf{R}^+\widehat{\sigma}_Z \oplus \mathbf{R}^+\widehat{\epsilon}_Z \oplus \mathbf{R}^+\widehat{\gamma}_Z. \tag{6.2.10}$$

Proof. By Corollary (6.2.2) it suffices to show that each of $\mathbf{R}^+\widehat{\sigma}_Z$, $\mathbf{R}^+\widehat{\epsilon}_Z$, $\mathbf{R}^+\widehat{\gamma}_Z$ is extremal. The maps

$$\mathbf{P}^2(S^2\mathcal{B}_Z) \leftarrow \mathbf{CC}(\mathcal{B}_Z) \rightarrow \mathbf{P}^2(S^2\check{\mathcal{B}}_Z)$$

can be identified with the contraction of $\mathbf{R}^+\widehat{\epsilon}_Z$ and $\mathbf{R}^+\widehat{\sigma}_Z$ respectively. Thus $\mathbf{R}^+\widehat{\epsilon}_Z$ and $\mathbf{R}^+\widehat{\sigma}_Z$ are extremal rays of $\overline{NE}_1(\widehat{\Omega}_Z)$. Next notice that the contraction of $\mathbf{R}^+\widehat{\gamma}_c$ can be identified with the map $\widehat{\mathcal{M}}_c \rightarrow S_c//\mathrm{PGL}(N)$, hence $\mathbf{R}^+\widehat{\gamma}_c$ is an extremal ray; by Lemma (6.2.6) we conclude that $\mathbf{R}^+\widehat{\gamma}_Z$ is an extremal ray. **q.e.d.**

6.3. Digression on $\widehat{\Sigma}_c$.

For $[Z],[W] \in X^{[n]}$, with $Z \neq W$, set

$$\widehat{\Sigma}_{Z,W} := \widehat{\pi}^{-1}([I_Z \oplus I_W]).$$

Thus

$$\widehat{\Sigma}_{Z,W} \in \widehat{\Sigma}_c \setminus (\widehat{\Omega}_c \cup \widehat{\Delta}_c). \tag{6.3.1}$$

For k a positive integer, let $I_k := \{(p,H) \in \mathbf{P}^k \times \check{\mathbf{P}}^k |\ p \in H\}$.

(6.3.2) Proposition. *Keep notation as above. Let $[Z],[W] \in X^{[n]}$, with $Z \neq W$. There is an isomorphism*

$$\widehat{\Sigma}_{Z,W} \cong I_{c-3}. \tag{6.3.3}$$

Letting $r\colon \widehat{\Sigma}_{Z,W} \rightarrow \mathbf{P}^{c-3}$ and $\check{r}\colon \widehat{\Sigma}_{Z,W} \rightarrow \check{\mathbf{P}}^{c-3}$ be the maps determined by the above isomorphism,

$$[\widehat{\Sigma}_c]|_{\widehat{\Sigma}_{Z,W}} \cong r^*\mathcal{O}_{\mathbf{P}^{c-3}}(-1) \otimes \check{r}^*\mathcal{O}_{\check{\mathbf{P}}^{c-3}}(-1). \tag{6.3.4}$$

Proof. Isomorphism (6.3.3) is an easy consequence of Proposition (1.4.1). Let's prove (6.3.4). By a monodromy argument,

$$[\widehat{\Sigma}_c]|_{\widehat{\Sigma}_{Z,W}} \cong r^*\mathcal{O}_{\mathbf{P}^{c-3}}(a) \otimes \check{r}^*\mathcal{O}_{\check{\mathbf{P}}^{c-3}}(a)$$

for some integer a. Copying the proof of (2.2.5) one gets $a = -1$. **q.e.d.**

6.4. The canonical class, and intersection numbers.

We will prove the following formula:

$$K_{\widehat{\mathcal{M}}_c} \sim (3c-7)\widehat{\Omega}_c + (c-4)\widehat{\Sigma}_c + (2c-6)\widehat{\Delta}_c. \tag{6.4.1}$$

First notice that there exist non-negative integers α_c, β_c, γ_c such that

$$K_{\widehat{\mathcal{M}}_c} \sim \alpha_c\widehat{\Omega}_c + \beta_c\widehat{\Sigma}_c + \gamma_c\widehat{\Delta}_c.$$

In fact by (1.9.2) the canonical form $\wedge^{2c-3}\widehat{\omega}_c$ is non-zero on the complement of $(\widehat{\Omega}_c \cup \widehat{\Sigma}_c \cup \widehat{\Delta}_c)$.

(6.4.2) Lemma. *Keeping notation as above, we have* $\beta_c = (c-4)$.

Proof. Let $[Z], [W] \in X^{[n]}$, with $Z \neq W$. Applying adjunction to $\widehat{\Sigma}_c$ we get that

$$K_{\widehat{\Sigma}_{Z,W}} \cong [K_{\widehat{\mathcal{M}}_c} + \widehat{\Sigma}_c]|_{\widehat{\Sigma}_{Z,W}} = [(\beta_c+1)\widehat{\Sigma}_c]|_{\widehat{\Sigma}_{Z,W}}.$$

By (6.3.3) we know $\widehat{\Sigma}_{Z,W} \cong I_{c-3}$, hence

$$K_{\widehat{\Sigma}_{Z,W}} \cong r^*\mathcal{O}_{\mathbf{P}^{c-3}}(-(c-3)) \otimes \check{r}^*\mathcal{O}_{\check{\mathbf{P}}^{c-3}}(-(c-3)).$$

By (6.3.4) we conclude that $\beta_c = (c-4)$. **q.e.d.**

(6.4.3) Lemma. *Let* $[Z] \in X^{[n]}$ *and* $[B] \in \mathbf{Gr}^{\omega}(3, E_Z)$*. Then*

$$c_1(\widehat{\Omega}_c)|_{\widehat{\Omega}_B} = (-2x+e)|_{\widehat{\Omega}_B}.$$

(see (6.2) for the definition of x, e*.)*

Proof. Let $\widehat{\mathbf{Hom}}(W,B)$ be the blow up of $\mathbf{P}\mathrm{Hom}(W,B)$ along the locus of rank-one homomorphisms. Then

$$\widehat{\Omega}_B = \widehat{\mathbf{Hom}}(W,B)//\mathrm{SO}(W) = \widehat{\mathbf{Hom}}(W,B)^s/\mathrm{SO}(W).$$

Let $\widehat{f}\colon \widehat{\mathbf{Hom}}(W,B)^s \to \widehat{\Omega}_B$ be the quotient map; we have

$$\widehat{f}^*[\widehat{\Omega}]|_{\widehat{\Omega}_B} \cong [\Omega_T]|_{\widehat{\mathbf{Hom}}(W,B)^s}.$$

By (1.8.4)-(1.8.10)-(1.8.11), Ω_S^s, S^s, and Δ_S^s are all smooth, hence

$$\Omega_T^s \sim (\pi_S\pi_T)^*\Omega_R^s - \Delta_T^s.$$

Now let $\lambda\colon \widehat{\mathbf{Hom}}(W,B)^s \to \mathbf{P}\mathrm{Hom}(W,B)$ be the blow-down map, i.e. the restriction of $(\pi_S\pi_T)$ (see (6.1.1)). By the previous linear equivalence we have

$$[\Omega_T] \cong \lambda^*[\Omega_R] \otimes [-\Delta_T] \text{ in } \mathrm{Pic}\left(\widehat{\mathbf{Hom}}(W,B)^s\right). \tag{6.4.4}$$

Clearly $\lambda^*[\Omega_R] \cong \lambda^*\mathcal{O}_{\mathbf{P}\mathrm{Hom}(W,B)}(-1)$. On the other hand, $\Delta_T|_{\widehat{\mathbf{Hom}}(W,B)}$ is the strict transfrom under λ of the locus parametrizing morphisms of rank at most two; an easy computation gives

$$[\Delta_T]|_{\widehat{\mathbf{Hom}}(W,B)} \cong \lambda^*\mathcal{O}_{\mathbf{P}\mathrm{Hom}}(3) \otimes [-2E],$$

where E is the exceptional divisor of λ. Thus (6.4.4) becomes

$$\widehat{f}^*[\widehat{\Omega}] \cong \lambda^*\mathcal{O}_{\mathbf{P}\mathrm{Hom}}(-4) \otimes [2E]. \tag{6.4.5}$$

Now consider the commutative diagram

$$\begin{array}{ccc} \widehat{\mathbf{Hom}}(W,B)^s & \xrightarrow{\widehat{f}} & \widehat{\Omega}_B \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\theta_B} \\ \mathbf{P}\mathrm{Hom}(W,B)^{ss} & \xrightarrow{f} & \mathbf{P}(S^2B), \end{array}$$

where f is the quotient map, and θ_B is the blow-up of conics of rank one. Since $f([\alpha]) = [\alpha\alpha^t]$, we have

$$\widehat{f}^*\theta_B^* x = \lambda^* f^* x = \lambda^* c_1(\mathcal{O}_{\mathbf{P}\mathrm{Hom}}(2)).$$

Furthermore, since the generic point of E has stabilizer of order two, $\widehat{f}^*e = 2c_1(E)$. Feeding these equalities into (6.4.5) we get

$$\widehat{f}^*c_1(\widehat{\Omega}_c) \cong \widehat{f}^*(-2\theta_B^*x + e).$$

Since the pull-back map $\widehat{f}^*: \mathrm{Pic}(\widehat{\Omega}_B) \to \mathrm{Pic}(\widehat{\mathbf{Hom}}(W,B)^s)$ is injective [DN, Lemme (3.2)], this proves the lemma. **q.e.d.**

Now we prove Formula (6.4.1). Writing out adjunction for $\widehat{\Omega}_Z$ and applying Lemma (6.4.2), we get

(6.4.6) $$c_1(K_{\widehat{\Omega}_Z}) \cong \Big(c_1(K_{\widehat{\mathcal{M}}_c}) + c_1(\widehat{\Omega}_c)\Big)|_{\widehat{\Omega}_Z} = \Big((\alpha_c+1)c_1(\widehat{\Omega}_c) + (c-4)c_1(\widehat{\Sigma}_c) + \gamma_c c_1(\widehat{\Delta}_c)\Big)|_{\widehat{\Omega}_Z}.$$

By (6.2.1) together with Lemma (6.4.3) we can write

$$c_1(\widehat{\Omega}_c)|_{\widehat{\Omega}_Z} = k_c h - 2x + e,$$

for some integer k_c. Feeding this equality, together with (6.2.3)-(6.2.4)-(6.2.5), into Formula (6.4.6), we get three equations in the unknowns k_c, α_c, γ_c. The equations uniquely determine the unknowns, and we get (6.4.1). We also get the formula

(6.4.7) $$c_1(\widehat{\Omega}_c)|_{\widehat{\Omega}_Z} = h - 2x + e.$$

We close this subsection with tables of intersection numbers to be used later on. A straightforward computation gives the first table:

(6.4.8)

	h	x	e
$\widehat{\epsilon}_Z$	0	0	−1
$\widehat{\sigma}_Z$	0	1	2
$\widehat{\gamma}_Z$	1	0	0

Formulae (6.2.3)-(6.2.4)-(6.4.7), together with the table above, give the following intersection matrix:

(6.4.9)

	$\widehat{\Omega}_c$	$\widehat{\Sigma}_c$	$\widehat{\Delta}_c$
$\widehat{\epsilon}_c$	−1	−1	2
$\widehat{\sigma}_c$	0	2	−1
$\widehat{\gamma}_c$	1	0	−2

6.5. Digression on $\widehat{\Delta}_c$.

For $[Z] \in X^{[n]}$ set

$$\widehat{\Delta}_Z := \widehat{\pi}^{-1}([I_Z \oplus I_Z]) \cap \widehat{\Delta}_c.$$

We will describe $\widehat{\Delta}_Z$ quite explicitely. Let $\mathcal{A}_Z$ be the tautological rank-two vector bundle on $\mathbf{Gr}^\omega(2, E_Z)$.

(6.5.1) Proposition. *The image of the map $f: \widehat{\Delta}_Z \to S_c//\mathrm{PGL}(N)$ is naturally identified with $\mathbf{P}(S^2\mathcal{A}_Z)$. The map f is a $\mathbf{P}^{c-4}$-fibration. There is an identification*

(6.5.2) $$\widehat{\Delta}_Z \cap \widehat{\Omega}_c \cong \{([A],[B],[q]) |\ [A] \in \mathbf{Gr}^\omega(2,E_Z),\ [B] \in \mathbf{Gr}^\omega(3,E_Z),\ [q] \in \mathbf{P}(S^2A),\ A \subset B\},$$

such that the restriction of f is identified with the forgetful map $([A],[B],[q]) \mapsto ([A],[q])$.

Proof. By definition, $\widehat{\Delta}_c = \Delta_{T_c}//\mathrm{PGL}(N)$, where Δ_{T_c} is the exceptional divisor of $\pi_T: T_c \to S_c$, the blow-up of Δ_{S_c}. By (1.8.10)-(1.8.13) there are no strictly semistable points to consider, hence we get a map

$$\varphi: \widehat{\Delta}_c = \Delta^s_{T_c}/\mathrm{PGL}(N) \to \Delta^s_{S_c}/\mathrm{PGL}(N),$$

where the single slash is a reminder that the quotients are orbit spaces. Since S_c^s and $\Delta^s_{S_c}$ are both smooth (1.8.10)-(1.8.11), and since by (1.8.2) we have $\mathrm{cod}(\Delta^s_{S_c}, S_c^s) = (c-3)$, $\Delta^s_{T_c}$ is a $\mathbf{P}^{c-4}$ bundle over $\Delta^s_{S_c}$. If $x \in \Delta^s_{S_c}$, the stabilizer of x acts trivially on $\pi_T^{-1}(x)$, hence φ is also a $\mathbf{P}^{c-4}$-fibration. Now let's show that the fiber of

$$\psi\colon \Delta^s_{S_c}/\mathrm{PGL}(N) \to \Omega^{ss}_{Q_c}//\mathrm{PGL}(N) = \Omega_c \cong X^{[n]}$$

over $[I_Z \oplus I_Z]$ is isomorphic to $\mathbf{P}(S^2\mathcal{A}_Z)$. In fact, by (1.8.12)

$$\psi^{-1}([I_Z \oplus I_Z]) \cong \widetilde{\mathbf{P}}\mathrm{Hom}_2^\omega(W, E_Z)//\mathrm{SO}(W).$$

The projection $\widetilde{\mathbf{P}}\mathrm{Hom}_2^\omega(W, E_Z) \to \mathbf{Gr}^\omega(2, E_Z)$ is $\mathrm{SO}(W)$-invariant, hence it descends to a map

$$\psi^{-1}([I_Z \oplus I_Z]) \to \mathbf{Gr}^\omega(2, E_Z).$$

One checks easily that the fiber over $[A]$ is naturally isomorphic to $\mathbf{P}(S^2A)$; this gives the isomorphism

$$f(\widehat{\Delta}_Z) = \psi^{-1}([I_Z \oplus I_Z]) \cong \mathbf{P}(S^2\mathcal{A}_Z).$$

Hence $\widehat{\Delta}_Z$ is indeed a $\mathbf{P}^{c-4}$-fibration over $\mathbf{P}(S^2\mathcal{A}_Z)$. To finish the proof of the proposition we define a map from $\widehat{\Delta}_Z \cap \widehat{\Omega}_c$ to the right-hand side of (6.5.2). If $[B] \in \mathbf{Gr}^\omega(3, E_Z)$, then

$$\widehat{\Delta}_Z \cap \mathbf{CC}(B) = \text{closure of } \{(C, D) \in \mathbf{CC}(B) |\ C \text{ has rank two}\}.$$

Thus for every $(C, D) \in \widehat{\Delta}_Z \cap \mathbf{CC}(B)$ the conic $D \subset \mathbf{P}(B)$ has rank one, i.e. it is the projectivization of a codimension one linear subspace $A_D \subset B$. Thus we get a map

$$\begin{array}{ccc} \widehat{\Delta}_Z \cap \mathbf{CC}(B) & \to & \mathbf{Gr}(2, B) \\ (C, D) & \mapsto & [A_D]. \end{array}$$

The fiber over $[A_D]$ is naturally identified with $\mathbf{P}(S^2A_D)$; let $[q_{C,D}] \in \mathbf{P}(S^2A_D)$ be the point corresponding to (C, D). We set

$$\begin{array}{ccc} \widehat{\Delta}_Z \cap \widehat{\Omega}_c & \to & \text{right-hand side of (6.5.2)} \\ ([B], C, D) & \mapsto & ([A_D], [B], [q_{C,D}]). \end{array}$$

This gives Isomorphism (6.5.2). **q.e.d.**

We continue examining $\widehat{\Delta}_Z$. Let $[A] \in \mathbf{Gr}^\omega(2, E_Z)$ and consider $\mathbf{P}(S^2A) \hookrightarrow \mathbf{P}(S^2\mathcal{A}_Z)$; restricting the $\mathbf{P}^{c-4}$-fibration to $\mathbf{P}(S^2A)$ we get a fibration

$$\begin{array}{ccc} \mathbf{P}^{c-4} & \to & f^{-1}\mathbf{P}(S^2A) \\ & & \downarrow \\ & & \mathbf{P}(S^2A). \end{array} \quad (6.5.3)$$

(6.5.4) Lemma. *Fibration (6.5.3) is trivial.*

Proof. The intersection $f^{-1}\mathbf{P}(S^2A) \cap \widehat{\Omega}_c$ is a divisor restricting to a hyperplane section (embedded linearly) on each $\mathbf{P}^{c-4}$ fiber, and furthermore by (6.5.2) it is isomorphic to $\mathbf{P}(S^2A) \times \mathbf{P}^{c-5}$. This implies that

$$f^{-1}\mathbf{P}(S^2A) \cong \mathbf{P}(V),$$

where V is a vector-bundle fitting into an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}(S^2A)}(a)^{(c-4)} \to V \to \mathcal{O}_{\mathbf{P}(S^2A)}(b) \to 0, \quad (6.5.5)$$

with $f^{-1}\mathbf{P}(S^2A) \cap \widehat{\Omega}_c = \mathbf{P}(\mathcal{O}_{\mathbf{P}(S^2A)}(a)^{(c-4)})$. Now let $\mathbf{P}(S^2A) \times [B] \in \mathbf{P}(\mathcal{O}_{\mathbf{P}(S^2A)}(a)^{(c-4)})$. By (6.5.5) we have

$$[\widehat{\Omega}_c]|_{\mathbf{P}(S^2A)\times[B]} \cong \mathcal{O}_{\mathbf{P}(S^2A)}(b-a).$$

Hence to prove the lemma it suffices to check that the left-hand side of the above equality is trivial. Let $L \subset \mathbf{P}(S^2A) \times [B]$ be a line. In $N_1(\widehat{\mathcal{M}}_c)$ we have $[L] = \widehat{\sigma}_c$, so that by the entry on the first column and second row of (6.4.9) we get $[\widehat{\Omega}_c]|_L = \mathcal{O}_L$. This proves $a = b$. **q.e.d.**

We will need to know $N_1(\widehat{\Delta}_Z)$ and $\overline{NE}_1(\widehat{\Delta}_Z)$. Let $j^Z\colon \widehat{\Delta}_Z \hookrightarrow \widehat{\mathcal{M}}_c$ be Inclusion.

(6.5.6) Lemma. *The map* $j^Z_*: N_1(\widehat{\Delta}_Z) \to N_1(\widehat{\mathcal{M}}_c)$ *is injective, and its image equals* $\mathbf{R}\widehat{\sigma}_c \oplus \mathbf{R}\widehat{\epsilon}_c \oplus \mathbf{R}\widehat{\gamma}_c$.

Proof. The map $N_1(\widehat{\Delta}_Z \cap \widehat{\Omega}_c) \to N_1(\widehat{\Delta}_Z)$ is an isomorphism. Since also $N_1(\widehat{\Delta}_Z \cap \widehat{\Omega}_c) \to N_1(\widehat{\Omega}_Z)$ is an isomorphism, the result follows from (6.2.6). **q.e.d.**

By the above lemma we can define $\widehat{\sigma}'_Z, \widehat{\epsilon}'_Z, \widehat{\gamma}'_Z \in N_1(\widehat{\Delta}_Z)$ as the classes such that

$$j^Z_* \widehat{\sigma}'_Z = \widehat{\sigma}_c, \qquad j^Z_* \widehat{\epsilon}'_Z = \widehat{\epsilon}_c, \qquad j^Z_* \widehat{\gamma}'_Z = \widehat{\gamma}_c. \tag{6.5.7}$$

Let's give explicit representatives of the above classes. All representatives will be contained in $\widehat{\Delta}_Z \cap \widehat{\Omega}_c$, so we refer to (6.5.2) for the description of the latter. Choose $[L] \in \mathbf{P}(E_Z)$, $[A] \in \mathbf{Gr}^\omega(2, E_Z)$, $[B] \in \mathbf{Gr}^\omega(3, E_Z)$, $[q^L] \in \mathbf{P}(S^2 L)$, $[q^A] \in \mathbf{P}(S^2 A)$, with $A \subset B$. Let $\Lambda_1, \Lambda_2, \Lambda_3 \subset \widehat{\Delta}_Z \cap \widehat{\Omega}_c$ be the curves defined by

$$\begin{aligned}
\Lambda_1 :=& \{([A],[B],[q_t]) \mid [q_t] \in \mathbf{P}^2(S^2 A) \text{ varies in a line}\},\\
\Lambda_2 :=& \{([A],[B_t],[q^A]) \mid [B_t/A] \text{ varies in a line}\},\\
\Lambda_3 :=& \{([A_t],[B],[i^t_* q^L]) \mid i^t: L \hookrightarrow A_t,\ A_t/L \text{ varies in a line}\}.
\end{aligned}$$

It follows easily from (6.5.2) that

$$\begin{aligned}
\widehat{\sigma}'_Z =& [\Lambda_1],\\
\widehat{\gamma}'_Z =& [\Lambda_2],\\
2\widehat{\epsilon}'_Z =& [\Lambda_3].
\end{aligned}$$

(6.5.8) Lemma. *Keeping notation as above, we have*

$$\overline{NE}_1(\widehat{\Delta}_Z) = \mathbf{R}^+\widehat{\sigma}'_Z \oplus \mathbf{R}^+\widehat{\epsilon}'_Z \mathbf{R}^+\widehat{\gamma}'_Z.$$

Proof. By (6.5.6) it suffices to prove that each of $\mathbf{R}^+\widehat{\sigma}'_Z$, $\mathbf{R}^+\widehat{\epsilon}'_Z$, $\mathbf{R}^+\widehat{\gamma}'_Z$ is extremal. By Lemma (6.5.4) there is a fibration

$$\begin{array}{ccc}
\mathbf{P}^2 \times \mathbf{P}^{c-4} & \to & \widehat{\Delta}_Z \\
 & & \downarrow \\
 & & \mathbf{Gr}^\omega(2, E_Z).
\end{array} \tag{6.5.9}$$

Correspondingly we have two maps of $\widehat{\Delta}_Z$, the first contracting the $\mathbf{P}^2$'s, the second contracting the $\mathbf{P}^{c-4}$'s. As is easily checked the first map can be identified with the contraction of $\mathbf{R}^+\widehat{\sigma}'_Z$, and the second map can be identified with the contraction of $\mathbf{R}^+\widehat{\gamma}'_Z$. Thus $\mathbf{R}^+\widehat{\sigma}'_Z$ and $\mathbf{R}^+\widehat{\gamma}'_Z$ are both extremal rays. To prove $\mathbf{R}^+\widehat{\epsilon}'_Z$ is extremal, consider the natural map

$$\begin{array}{ccc}
\mathbf{P}(S^2\mathcal{A}_Z) & \xrightarrow{\phi} & \mathbf{P}(S^2 E_Z) \\
([A],[q]) & \mapsto & [i^A_* q],
\end{array}$$

where $[A] \in \mathbf{Gr}^\omega(2, E_Z)$, $[q] \in \mathbf{P}(S^2 A)$, and $i^A_*: S^2 A \to S^2 E_Z$ is the map induced by inclusion. As is easily checked, ϕ is the contraction of $\mathbf{R}^+[\Gamma]$, where $\Gamma \subset \mathbf{P}(S^2\mathcal{A}_Z)$ is defined as follows: fix $[L] \in \mathbf{P}(E_Z)$ $[q^L] \in \mathbf{P}(S^2 L)$, and set

$$\Gamma := \{[A_t], [i^t_* q_L] \mid i^t: L \hookrightarrow A_t,\ A_t/L \text{ varies linearly in } \mathbf{P}(L^\perp/L).\}$$

Thus $\mathbf{R}^+[\Gamma]$ is an extremal ray of $\overline{NE}_1(\mathbf{P}(S^2\mathcal{A}_Z))$. Now consider the map $f: \widehat{\Delta}_Z \to \mathbf{P}(S^2\mathcal{A}_Z)$; then

$$f_*\widehat{\epsilon}'_Z = f_*[\Lambda_3] = [\Gamma].$$

Since $[\Gamma]$ generates an extremal ray, and since f is the contraction of $\mathbf{R}^+\widehat{\gamma}'_Z$ we see that if $\mathbf{R}^+\widehat{\epsilon}'_Z$ is not extremal, there exists an irreducible curve $C \subset \widehat{\Delta}_Z$ such that in $N_1(\widehat{\Delta}_Z)$

$$[C] \equiv s\widehat{\epsilon}'_Z - t\widehat{\gamma}'_Z, \qquad s > 0, t > 0.$$

Intersecting with $\widehat{\Omega}_c$ and applying (6.4.9) we get that $C \cdot \widehat{\Omega}_c < 0$, hence $C \subset \widehat{\Omega}_Z$. By (6.5.7) and by (6.2.6) we conclude that for some $s > 0$, $t > 0$, $[C] = (s\widehat{\epsilon}_Z - t\widehat{\gamma}_Z)$ in $N_1(\widehat{\Omega}_Z)$, contradicting (6.2.10). **q.e.d.**

6.6. Proof of Proposition (6.0.2)-Item (2).

By Formula (6.4.1) and Table (6.4.9) we get

$$\begin{aligned} K_{\widehat{\mathcal{M}}_c} \cdot \widehat{\sigma}_c &= -2, \\ K_{\widehat{\mathcal{M}}_c} \cdot \widehat{\epsilon}_c &= -1, \\ K_{\widehat{\mathcal{M}}_c} \cdot \widehat{\gamma}_c &= -(c-5). \end{aligned}$$

Thus we are left with the task of proving $\mathbf{R}^+\widehat{\sigma}_c \oplus \mathbf{R}^+\widehat{\epsilon}_c \oplus \mathbf{R}^+\widehat{\gamma}_c$ is an extremal face. First we give a preliminary result. Let $[Z], [W] \in X^{[n]}$ with $Z \neq W$, and let $k^{Z,W} \colon \widehat{\Sigma}_{Z,W} \hookrightarrow \widehat{\mathcal{M}}_c$ be inclusion.

(6.6.1) Lemma. *Keeping notation as above,*

$$k_*^{Z,W}\left(\overline{NE}_1(\widehat{\Sigma}_{Z,W})\right) = \mathbf{R}^+(\widehat{\epsilon}_c + \widehat{\gamma}_c).$$

Proof. Letting $[W]$ approach $[Z]$ we see that

$$k_*^{Z,W}\left(\overline{NE}_1(\widehat{\Sigma}_{Z,W})\right) \subset i_*^Z \overline{NE}_1(\widehat{\Omega}_Z).$$

Furthermore by (6.3.1)

$$k_*^{Z,W}\left(\overline{NE}_1(\widehat{\Sigma}_{Z,W})\right) \perp \left(\mathbf{R}c_1(\widehat{\Omega}_c) \oplus \mathbf{R}c_1(\widehat{\Delta}_c)\right).$$

Applying Table (6.4.9) we get the lemma. **q.e.d.**

Now assume that

$$\sum_{\alpha \in I} m_\alpha [\Gamma_\alpha] \in \mathbf{R}^+\widehat{\sigma}_c \oplus \mathbf{R}^+\widehat{\epsilon}_c \oplus \mathbf{R}^+\widehat{\gamma}_c, \tag{6.6.2}$$

where, for each $\alpha \in I$, $m_\alpha > 0$ and Γ_α is an irreducible curve on $\widehat{\mathcal{M}}_c$; we must show that

$$[\Gamma_\alpha] \in \mathbf{R}^+\widehat{\sigma}_c \oplus \mathbf{R}^+\widehat{\epsilon}_c \oplus \mathbf{R}^+\widehat{\gamma}_c \text{ for each } \alpha \in I. \tag{6.6.3}$$

From $\widehat{\pi}_*\widehat{\sigma}_c = \widehat{\pi}_*\widehat{\epsilon}_c = \widehat{\pi}_*\widehat{\gamma}_c = 0$, we get $\widehat{\pi}_*\Gamma_\alpha \equiv 0$ for all α, and since $\mathcal{M}_c$ is projective this implies $\widehat{\pi}(\Gamma_\alpha)$ is a point. Thus we can partition the indexing set as $I = I_\Omega \amalg I_\Sigma \amalg I_\Delta$ so that

$$\text{if} \begin{cases} \alpha \in I_\Omega, & \text{then } \Gamma_\alpha \subset \widehat{\Omega}_{Z_\alpha} \text{ for some } Z_\alpha \in X^{[n]}, \\ \alpha \in I_\Sigma, & \text{then } \Gamma_\alpha \subset \widehat{\Sigma}_{Z_\alpha, W_\alpha} \text{ for } Z_\alpha, W_\alpha \in X^{[n]} \text{ with } Z_\alpha \neq W_\alpha, \\ \alpha \in I_\Delta, & \text{then } \Gamma_\alpha \subset \widehat{\Delta}_{Z_\alpha} \text{ for some } Z_\alpha \in X^{[n]}. \end{cases}$$

Statement (6.6.3) follows from (6.2.10) if $\alpha \in I_\Omega$, from (6.6.1) if $\alpha \in I_\Sigma$, and from (6.5.8) if $\alpha \in I_\Delta$.

6.7. Proof of Proposition (3.0.3).

By Proposition (6.0.2)-Item(2) and Mori theory, $\overline{\mathcal{M}}_c$ is projective. Let's prove $\overline{\mathcal{M}}_c$ is smooth. Fibration (6.5.9) shows that $\widehat{\Delta}_Z$ is a $\mathbf{P}^2$-fibration (with base a $\mathbf{P}^{c-4}$-bundle over $\mathbf{Gr}^\omega(2, E_Z)$), hence $\widehat{\Delta}_c$ is a $\mathbf{P}^2$-fibration

$$\begin{array}{ccc} \mathbf{P}^2 & \to & \widehat{\Delta}_c \\ & & \downarrow \\ & & \Lambda_c, \end{array} \tag{6.7.1}$$

where Λ_c fibers over $X^{[n]}$, the fiber over $[Z]$ being a $\mathbf{P}^{c-4}$-bundle over $\mathbf{Gr}^{\omega}(2,E_Z)$. Let $\mathbf{P}^2$ be a fiber of (6.7.1) and $L\subset\mathbf{P}^2$ be a line. By (6.5) we have $[L]=\widehat{\sigma}_c$ in $N_1(\widehat{\mathcal{M}}_c)$, hence (6.4.9) gives

$$[\widehat{\Delta}_c]|_{\mathbf{P}^2}\cong\mathcal{O}_{\mathbf{P}^2}(-1). \tag{6.7.2}$$

Claim. *Keep notation as above. The contraction of $\mathbf{R}^+\widehat{\sigma}_c$ is identified with the contraction of $\widehat{\mathcal{M}}_c$ along Fibration (6.7.1).*

Proof. If L is line in a fiber of (6.7.1), then $[L]=\widehat{\sigma}_c$. Hence we must prove that if $\Gamma\subset\widehat{\mathcal{M}}_c$ is an irreducible curve such that $[\Gamma]\in\mathbf{R}^+\widehat{\sigma}_c$, then Γ lies in a fiber of (6.7.1). Since $\Gamma\cdot\widehat{\Delta}_c<0$, Γ is contained in $\widehat{\Delta}_c$. Furthermore $\widehat{\pi}_*\Gamma\equiv 0$, hence there exists $[Z]\in X^{[n]}$ such that $\Gamma\subset\widehat{\Delta}_Z$. Applying Lemma (6.5.6), we get the following relation in $N_1(\widehat{\Delta}_Z)$:

$$[\Gamma]\in\mathbf{R}^+\widehat{\sigma}'_Z.$$

This implies Γ is contained in a fiber of (6.7.1). **q.e.d.**

The above claim together with (6.7.2) proves that $\overline{\mathcal{M}}_c$ is smooth. Finally we must show that the rational map $\overline{\mathcal{M}}_c\cdots>\mathcal{M}_c$ induced by $\widehat{\pi}$ is regular. One proceeds as in the proof that the analogous map $\widetilde{\mathcal{M}}_4\cdots>\mathcal{M}_4$ is regular (see the proof of (2.0.3)): the point is that $\widehat{\pi}$ is constant on the $\mathbf{P}^2$'s we have contracted.

6.8. Proof of Proposition (6.0.4).

Let $\widehat{\theta}\colon\widehat{\mathcal{M}}_c\to\overline{\mathcal{M}}_c$ be the contraction map, and $\overline{\pi}\colon\overline{\mathcal{M}}_c\to\mathcal{M}_c$ be the map induced by $\widehat{\pi}$; thus $\widehat{\pi}=\overline{\pi}\circ\widehat{\theta}$. Since, by Proposition (6.0.2), $\mathbf{R}^+\widehat{\epsilon}_c$ is extremal, so is $\mathbf{R}^+\overline{\epsilon}_c$. Applying (6.4.9) we get

$$K_{\overline{\mathcal{M}}_c}\cdot\overline{\epsilon}_c=\widehat{\theta}^*K_{\overline{\mathcal{M}}_c}\cdot\widehat{\epsilon}_c=\left(K_{\widehat{\mathcal{M}}_c}-2\widehat{\Delta}_c\right)\cdot\widehat{\epsilon}_c=-5. \tag{6.8.1}$$

Now let $\overline{\theta}\colon\overline{\mathcal{M}}_c\to\widetilde{\mathcal{M}}_c$ be the contraction of $\mathbf{R}^+\overline{\epsilon}_c$; by Mori theory $\widetilde{\mathcal{M}}_c$ is projective. To prove $\widetilde{\mathcal{M}}_c$ is smooth, consider $\overline{\Omega}_c:=\widehat{\theta}(\widehat{\Omega}_c)$. Clearly we have a fibration

$$\begin{array}{ccc}\mathbf{P}^5 & \to & \overline{\Omega}_c\\ & & \downarrow\\ & & \mathbf{Gr}^{\omega}(3,T_{X^{[n]}}),\end{array} \tag{6.8.2}$$

where the fiber over $([Z],B)$ is canonically identified with $\mathbf{P}(S^2\check{B})$. If L is a line in a fiber of the above fibration, then $[L]=\overline{\epsilon}_c$, hence by (6.8.1) together with adjunction we get

$$[\overline{\Omega}_c]|_{\mathbf{P}^5}\cong\mathcal{O}_{\mathbf{P}^5}(-1).$$

Claim. *Keeping notation as above, $\widetilde{\mathcal{M}}_c$ is obtained contracting $\overline{\mathcal{M}}_c$ along Fibration (6.8.2). In particular $\widetilde{\mathcal{M}}_c$ is smooth.*

Proof. For $[Z]\in X^{[n]}$, let $\overline{\Omega}_Z:=\overline{\pi}^{-1}([I_Z\oplus I_Z])$; we have a fibration

$$\begin{array}{ccc}\mathbf{P}(S^2\check{B}) & \to & \overline{\Omega}_Z\\ \downarrow & & \downarrow\\ [B] & \in & \mathbf{Gr}^{\omega}(3,E_Z).\end{array}$$

It follows from (6.2.6) that the map $N_1(\overline{\Omega}_Z)\to N_1(\overline{\mathcal{M}}_c)$ induced by inclusion is injective. Arguing as in (6.7) we get the claim. **q.e.d.**

The a priori rational map $\widetilde{\mathcal{M}}_c\cdots>\mathcal{M}_c$ is seen to be regular by an argument similar to that given in the proof of (2.0.3); the point is that $\overline{\pi}$ is constant on the $\mathbf{P}^5$'s which have been contracted. Finally, let $\widetilde{\omega}_c$ be the two-form on $\widetilde{\mathcal{M}}_c$ induced by $\widehat{\omega}_c$; clearly $\widetilde{\omega}_c$ is non-degenerate outside $\widetilde{\Sigma}_c:=(\overline{\theta}\circ\widehat{\theta})(\widehat{\Sigma}_c)$, in fact by (6.4.1)

$$\left(\wedge^{2c-3}\widetilde{\omega}_c\right)=(c-4)\widetilde{\Sigma}_c.$$

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