Double covers of EPW-sextics

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0 Introduction

EPW-sextics are defined as follows. Let $V$ be a 6-dimensional complex vector space. Choose a volume-form $\text{vol}: \bigwedge^6 V \sim \mathbb{C}$ and equip $\bigwedge^3 V$ with the symplectic form

$$(\alpha, \beta)_V := \text{vol}(\alpha \wedge \beta).$$

(0.0.1)

Let $L_G(\bigwedge^3 V)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\bigwedge^3 V$ - of course $L_G(\bigwedge^3 V)$ does not depend on the choice of volume-form. Let $F \subset \bigwedge^3 V \otimes \mathcal{O}_{P(V)}$ be the sub vector-bundle with fiber

$$F_v := \{ \alpha \in \bigwedge^3 V \mid v \wedge \alpha = 0 \}$$

(0.0.2)

over $[v] \in P(V)$. Notice that $(,)_V$ is zero on $F_v$ and $2 \dim(F_v) = 20 = \dim \bigwedge^3 V$; thus $F$ is a Lagrangian sub vector-bundle of the trivial symplectic vector-bundle on $P(V)$ with fiber $\bigwedge^3 V$. Next choose $A \in L_G(\bigwedge^3 V)$. Let

$$F \xrightarrow{\lambda_A} \bigwedge^3 V/A \otimes \mathcal{O}_{P(V)}$$

(0.0.3)

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be the composition of the inclusion $F \subset \bigwedge^3 V \otimes \mathcal{O}_\mathbb{P}(V)$ followed by the quotient map. Since $\text{rk } F = \dim(V/A)$ the determinant of $\lambda_A$ makes sense. Let

$$Y_A := V(\det \lambda_A).$$

A straightforward computation gives that $\det F \cong \mathcal{O}_\mathbb{P}(V)(-6)$ and hence $\det \lambda_A \in H^0(\mathcal{O}_\mathbb{P}(V)(6)).$ It follows that if $\det \lambda_A \neq 0$ then $Y_A$ is a sextic hypersurface. As is easily checked $\det \lambda_A \neq 0$ for generic $A \in LG(\bigwedge^3 V)$ (notice that there exist “pathological” $A$‘s such that $\lambda_A = 0$ e.g. $A = F_{v_0}$).

An EPW-sextic (after Eisenbud, Popescu and Walter [5]) is a sextic hypersurface in $\mathbb{P}^5$ which is projectively equivalent to $Y_A$ for some $A \in LG(\bigwedge^3 V)$. Let $Y_A$ be an EPW-sextic. One constructs a coherent sheaf $\lambda_A$ on $Y_A$ and a multiplication map $\xi_A \times \lambda_A \to \mathcal{O}_{Y_A}$ which gives $\mathcal{O}_{Y_A} \oplus \xi_A$ a structure of $\mathcal{O}_{Y_A}$-algebra - this is known to experts, see [4] - we will give the construction in Subsection 1.2.

The double EPW-sextic associated to $A$ is $X_A := \text{Spec} (\mathcal{O}_{Y_A} \oplus \xi_A)$; we let $f_A: X_A \to Y_A$ be the structure morphism. In [12] we considered $X_A$ for generic $A$ and we proved that it is a Hyperkähler deformation of $(K3)^{[2]}$ (the blow-up of the diagonal in the symmetric square of a $K3$ surface). In the present paper we will analyze $X_A$ for $A$ varying in a codimension-$1$ subset of $LG(\bigwedge^3 V)$. In order to state our main results we will introduce some notation. Given $A \in LG(\bigwedge^3 V)$ we let

$$Y_A(k) = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) = k\}, \quad (0.0.4)$$

$$Y_A[k] = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) \geq k\}. \quad (0.0.5)$$

Thus $Y_A(0) = (\mathbb{P}(V) \setminus Y_A)$ and $Y_A = Y_A[1]$. Double EPW-sextics come with a natural polarization; we let

$$\mathcal{O}_{X_A}(n) := f_A^* \mathcal{O}_{Y_A}(n), \quad H_A \in |\mathcal{O}_{X_A}(1)|. \quad (0.0.6)$$

The following closed subsets of $LG(\bigwedge^3 V)$ play a key rôle in the present paper:

$$\Sigma := \{A \in LG(\bigwedge^3 V) \mid \exists W \in \mathbb{G}r(3, V) \text{ s. t. } \bigwedge^3 W \subset A\}, \quad (0.0.7)$$

$$\Delta := \{A \in LG(\bigwedge^3 V) \mid Y_A[3] \neq \emptyset\}. \quad (0.0.8)$$

A straightforward computation, see [15], gives that $\Sigma$ is irreducible of codimension $1$. A similar computation, see Proposition 2.2, gives that $\Delta$ is irreducible of codimension $1$ and distinct from $\Sigma$. Let

$$LG(\bigwedge^3 V)^0 := LG(\bigwedge^3 V) \setminus \Sigma \setminus \Delta. \quad (0.0.9)$$

Thus $LG(\bigwedge^3 V)^0$ is open dense in $LG(\bigwedge^3 V)$. In [12] we proved that if $A \in LG(\bigwedge^3 V)^0$ then $X_A$ is a hyperkähler (HK) $4$-fold which can be deformed to $(K3)^{[2]}$. Moreover we showed that the family of polarized HK $4$-folds $(X_A, H_A)$ for $A$ varying in $LG(\bigwedge^3 V)^0$ is locally complete. Three other explicit locally complete families of projective HK’s of dimension greater than $2$ are known - see [2, 3, 8, 9]. In all of the examples the HK manifolds are deformations of the Hilbert square of a $K3$; they are distinguished by the value of the Beauville-Bogomolov form on the polarization class (it equals $2$ in the case of double EPW-sextics and $6$, $22$ and $38$ in the other cases). In the present paper we will analyze $X_A$ for $A \in \Delta$, mainly under the hypothesis that $A \not\in \Sigma$. Let $A \in (\Delta \setminus \Sigma)$. We will prove the following results

1. $Y_A[3]$ is a finite set and it equals $Y_A(3)$. If $A$ is generic in $(\Delta \setminus \Sigma)$ then $Y_A(3)$ is a singleton.

2. One may associate to $[v_0] \in Y_A(3)$ a $K3$ surface $S_A(v_0) \subset \mathbb{P}^6$ of genus $6$, well-defined up to projectivities. Conversely the generic $K3$ of genus $6$ is projectively equivalent to $S_A(v_0)$ for some $A \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_A(3)$.

3. The singular set of $X_A$ is equal to $f_A^{-1} Y_A(3)$. There is a single $p_i \in X_A$ mapping to $[v_i] \in Y_A(3)$ and the cone of $X_A$ at $p_i$ is isomorphic to the cone over the set of incident couples $(x, r) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ (i.e. $\mathbb{P}(\Omega_{2\mathbb{P}^2})$). Thus we have two standard small resolutions of a neighborhood of $p_i$ in $X_A$, one with fiber $\mathbb{P}^2$ over $p_i$, the other with fiber $(\mathbb{P}^2)^\vee$. Making a choice $\epsilon$ of local small resolution at each $p_i$ we get a resolution $X_A^\epsilon \to X_A$ with the following properties: There is
a birational map $X_A \dashrightarrow S_A(v_1)[2]$ such that the pull-back of a holomorphic symplectic form on $S_A(v_1)[2]$ is a symplectic form on $X_A$. If $S_A(v_1)$ contains no lines (true for generic $A$ by Item (2)) then there exists a choice of $\epsilon$ such that $X_A$ is isomorphic to $S_A(v_1)[2]$.

(4) Given a sufficiently small open (classical topology) $U \subset (\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$ containing $A$ the family of double EPW-sextics parametrized by $U$ has a simultaneous resolution of singularities (no base change) with fiber $X_A$ over $A$ (for an arbitrary choice of $\epsilon$).

A remark: if $Y_A(3)$ has more than one point we do not expect all the small resolutions to be projective (i.e. Kähler). Items (1)-(4) should be compared with known results on cubic 4-folds - recall that if $Z$ is a smooth cubic hypersurface the variety $F(Z)$ parametrizing lines in $Z$ is a HK 4-fold which can be deformed to $(K3)[2]$ and moreover the primitive weight-4 Hodge structure of $Z$ is isomorphic (after a Tate twist) to the primitive weight-2 Hodge structure of $F(Z)$, see [2]. Let $D \subset |O_{\mathbb{P}^2}(3)|$ be the prime divisor parametrizing singular cubics. Let $Z \in D$ be generic: the following results are well-known.

(1') Sing $Z$ is a finite set.

(2') Given $p \in \text{Sing} Z$ the set $S_Z(p) \subset F(Z)$ of lines containing $p$ is a $K3$ surface of genus 4 and vice versa the generic such $K3$ is isomorphic to $S_Z(p)$ for some $Z$ and $p \in \text{Sing} Z$.

(3') $F(Z)$ is birational to $S_Z(p)[2]$.

(4') After a local base-change of order 2 ramified along $D$ the period map extends across $Z$.

Thus Items (1')-(2')-(3') are analogous to Items (1), (2) and (3) above, Item (4') is analogous to (4) but there is an important difference namely the need for a base-change of order 2. (Actually the paper [13] contains results showing that there is a statement valid for cubic hypersurfaces which is even closer to our result for double EPW-sextics, the rôle of $\Sigma$ being played by the divisor parametrizing cubics containing a plane.) We explain the relevance of Items (1)-(4). Items (3) and (4) prove the theorem of ours mentioned above i.e. that if $A \in \mathbb{LG}(\bigwedge^3 V)^0$ then $X_A$ is a HK deformation of $(K3)[2]$ (the family of polarized double EPW-sextics is locally complete by a straightforward parameter count). The proof in this paper is independent of the proof in [12]. Beyond giving a new proof of an "old" theorem, the above results show that away from $\Sigma$ the period map is as well-behaved as possible at the generic $A \in (\Delta \setminus \Sigma)$, however we did not have the exact statement about $X_A$ and we had no statement about an arbitrary $A \in (\Delta \setminus \Sigma)$.

The paper is organized as follows. In Section 1 we will give formulæ that describe double EPW-sextics locally. The formulæ are known to experts, see [4], we will go through the proofs because we could not find a suitable reference. We will also perform the local computations needed to prove Item (4) above. In Section 2 we will go through some standard computations involving $\Delta$. In Section 3 we will prove Items (1), (4) and the statements of Item (3) which do not involve the $K3$ surface $S_A(v_0)$. In Section 4 we will prove Item (2) and the remaining statement of Item (3).

Section 5 contains auxiliary results on 3-dimensional linear sections of $\text{Gr}(3, \mathbb{C}^5)$.

Notation and conventions: Throughout the paper $V$ is a 6-dimensional complex vector space.

Let $W$ be a finite-dimensional complex vector-space. The span of a subset $S \subset W$ is denoted by $\langle S \rangle$. Let $S \subset \bigwedge^q W$. The support of $S$ is the smallest subspace $U \subset W$ such that $S \subset \text{im}(\bigwedge^q U \rightarrow \bigwedge^q W)$: we denote it by $\text{supp}(S)$, if $S = \{\alpha\}$ is a singleton we let $\text{supp}(\alpha) = \text{supp}(\{\alpha\})$ (thus if $q = 1$ we have $\text{supp}(\alpha) = \langle \alpha \rangle$). We define the support of a set of symmetric tensors analogously. If $\alpha \in \bigwedge^q W$ or $\alpha \in \text{Sym}^q W$ the rank of $\alpha$ is the dimension of $\text{supp}(\alpha)$. An element of $\text{Sym}^q W^\ast$ may be viewed either as a symmetric map or as a quadratic form: we will denote the former by $\tilde{q}, \tilde{r}, \ldots$ and the latter by $q, r, \ldots$ respectively.

Let $M = (M_{ij})$ be a $d \times d$ matrix with entries in a commutative ring $R$. We let $M^{\epsilon} = (M^{ij})$ be the matrix of cofactors of $M$, i.e. $M^{i+j} = (-1)^{i+j}$ times the determinant of the matrix obtained
from $M$ by deleting its $j$-th row and $i$-th column. We recall the following interpretation of $M^c$. Suppose that $f: A \to B$ is a linear map between free $R$-modules of rank $d$ and that $M$ is the matrix associated to $f$ by the choice of bases $\{a_1, \ldots, a_d\}$ and $\{b_1, \ldots, b_d\}$ of $A$ and $B$ respectively. Then $\bigwedge^{d-1} f$ may be viewed as a map

$$\bigwedge^{d-1} f: A^\vee \otimes \bigwedge^{d-1} A \cong \bigwedge^{d-1} B \cong B^\vee \otimes \bigwedge^d B.$$  \hfill (0.0.10)

(Here $A^\vee := \text{Hom}(A, R)$ and similarly for $B^\vee$.) The matrix associated to $\bigwedge^{d-1} f$ by the choice of bases $\{a_1^\vee \otimes (a_1 \wedge \ldots \wedge a_d), \ldots, a_d^\vee \otimes (a_1 \wedge \ldots \wedge a_d)\}$ and $\{b_1^\vee \otimes (b_1 \wedge \ldots \wedge b_d), \ldots, b_d^\vee \otimes (b_1 \wedge \ldots \wedge b_d)\}$ is equal to $M^c$.

Let $W$ be a finite-dimensional complex vector-space. We will adhere to pre-Grothendieck conventions: $\mathbb{P}(W)$ is the set of 1-dimensional vector subspaces of $W$. Given a non-zero $w \in W$ we will denote the span of $w$ by $[w]$ rather than $\langle w \rangle$; this agrees with standard notation. Suppose that $T \subset \mathbb{P}(W)$. Then $(T) \subset \mathbb{P}(W)$ is the projective span of $T$ i.e. the intersection of all linear subspaces of $\mathbb{P}(W)$ containing $T$.

Schemes are defined over $\mathbb{C}$, the topology is the Zariski topology unless we state the contrary. Let $W$ be finite-dimensional complex vector-space: $\mathcal{O}_{\mathbb{P}(W)}(1)$ is the line-bundle on $\mathbb{P}(W)$ with fiber $L^\vee$ on the point $L \in \mathbb{P}(W)$. Let $F \in \text{Sym}^d W^\vee$: we let $V(F) \subset \mathbb{P}(W)$ be the subscheme defined by vanishing of $F$. If $E \to X$ is a vector-bundle we denote by $\mathbb{P}(E)$ the projective-fiber-bundle with fiber $\mathbb{P}(E(x))$ over $x$ and we define $\mathcal{O}_{\mathbb{P}(W)}(1)$ accordingly. If $Y$ is a subscheme of $X$ we let $Bl_Y X \to X$ be the blow-up of $Y$.

## 1 Symmetric resolutions and double covers

In **Subsection 1.1** we will describe a method (well-known to experts) for constructing double covers. In **Subsection 1.2** we will show how to implement the construction in order to construct double EPW-sextics. **Subsection 1.3** contains the main ingredients needed to construct the simultaneous desingularization described in Item (3) of **Section 0**.

### 1.1 Product formula and double covers

Let $R$ be an integral Noetherian ring. Let $N$ be an $R$-module with a free resolution

$$0 \to U_1 \xrightarrow{\lambda} U_0 \xrightarrow{\pi} N \to 0, \quad \operatorname{rk} U_1 = \operatorname{rk} U_0 = d > 0. \hfill (1.1.1)$$

Let $\{a_1, \ldots, a_d\}$ and $\{b_1, \ldots, b_d\}$ be bases of $U_0$ and $U_1$ respectively. Let $M_\lambda$ be the matrix associated to $\lambda$ by our choice of bases - notice that $\det M_\lambda$ annihilates $N$. Given a homomorphism

$$\beta: N \to \text{Ext}^1(N, R) \hfill (1.1.2)$$

one defines a product $m_\beta: N \times N \to R/(\det M_\lambda)$ as follows. Applying the $\text{Hom}(\cdot, R)$-functor to (1.1.1) we get the exact sequence

$$0 \to U_0^\vee \xrightarrow{\lambda^!} U_1^\vee \xrightarrow{\rho} \text{Ext}^1(N, R) \to 0. \hfill (1.1.3)$$

In particular $\det M_\lambda$ kills $\text{Ext}^1(N, R)$. Now apply the functor $\text{Hom}(N, \cdot)$ to the exact sequence

$$0 \to R \xrightarrow{\det M_\lambda} R \to R/(\det M_\lambda) \to 0. \hfill (1.1.4)$$

Since $\text{Ext}^1(N, R) \to \text{Ext}^1(N, R)$ is multiplication by $\det M_\lambda$ we get a coboundary isomorphism

$$\partial: \text{Hom}(N, R/(\det M_\lambda)) \xrightarrow{\sim} \text{Ext}^1(N, R). \hfill (1.1.5)$$

We let

$$N \times N \xrightarrow{m_\beta} R/(\det M_\lambda) \quad (n, n') \mapsto (\partial^{-1}\beta(n))(n'). \hfill (1.1.6)$$
We will give an explicit formula for \( m_\beta \). Let \( \pi: U_0 \rightarrow N \) be as in (1.1.1). Then \( \beta \circ \pi \) lifts to a homomorphism \( \mu^t: U_0 \rightarrow U_1^\vee \) (the map is written as a transpose in order to conform to the notation for double EPW-sextics - see Subsection 1.2). It follows that there exists \( \alpha: U_1 \rightarrow U_0^\vee \) such that

\[
\begin{array}{cccc}
0 & \rightarrow & U_1 & \xrightarrow{\lambda} & U_0 & \xrightarrow{\pi} & N & \rightarrow & 0 \\
\downarrow{\alpha} & & \downarrow{\mu^t} & & \downarrow{\beta} & & \downarrow{\beta} & & \downarrow{0} \\
0 & \rightarrow & U_0^\vee & \xrightarrow{\theta} & U_1^\vee & \xrightarrow{\rho} & \text{Ext}^1(N,R) & \rightarrow & 0
\end{array}
\]  

(1.1.7)

is a commutative diagram. Let \( \{a_1^\vee, \ldots, a_d^\vee\} \) and \( \{b_1^\vee, \ldots, b_d^\vee\} \) be the bases of \( U_0^\vee \) and \( U_1^\vee \) which are dual to the chosen bases of \( U_0 \) and \( U_1 \). Let \( M_{\mu^t} \) be the matrix associated to \( \mu^t \) by our choice of bases.

**Proposition 1.1.** Keeping notation as above we have

\[
m_\beta(\pi(a_i), \pi(a_j)) \equiv (M^t_{\lambda} \cdot M_{\mu^t})_{ji} \mod (\det M_{\lambda})
\]  

(1.1.8)

where \( M^t_{\lambda} \) is the matrix of cofactors of \( M_{\lambda} \).

**Proof.** Equation (1.1.3) gives an isomorphism

\[
\nu: \text{Ext}^1(N,R) \xrightarrow{\sim} U_1^\vee/\lambda(U_0^\vee).
\]  

(1.1.9)

Let \( \det(U_\bullet) := \bigwedge^d U_1^\vee \otimes \bigwedge^d U_0 \). We will define an isomorphism

\[
\theta: U_1^\vee/\lambda(U_0^\vee) \xrightarrow{\sim} \text{Hom}(N, \det(U_\bullet)/(\det \lambda)).
\]  

(1.1.10)

First let

\[
U_1^\vee = \bigwedge^{d-1} U_1 \otimes \bigwedge^d U_1^\vee \xrightarrow{\zeta \otimes \xi} \bigwedge^{d-1} U_0 \otimes \bigwedge^d U_1^\vee = \text{Hom}(U_0, \det(U_\bullet))
\]  

(1.1.11)

We claim that

\[
\text{im}(\tilde{\theta}) = \{ \phi \in \text{Hom}(U_0, \det(U_\bullet)) \mid \phi \circ \lambda(U_1) \subset (\det \lambda) \}.
\]  

(1.1.12)

In fact by Cramer’s formula

\[
M^t_{\lambda} \cdot M^t_{\lambda} = M^t_{\lambda} \cdot M^t_{\lambda} = \det M_{\lambda} \cdot 1
\]  

(1.1.13)

and Equation (1.1.12) follows. Thus \( \tilde{\theta} \) induces a surjective homomorphism

\[
\tilde{\theta}: U_1^\vee \longrightarrow \text{Hom}(N, \det(U_\bullet)/(\det \lambda)).
\]  

(1.1.14)

One checks easily that \( \lambda(U_0^\vee) = \ker \tilde{\theta} \) - use Cramer again. We define \( \theta \) to be the homomorphism induced by \( \tilde{\theta} \); we have proved that it is an isomorphism. We claim that

\[
\theta \circ \nu = \partial^{-1}, \quad \partial \text{ as in (1.1.5)}.
\]  

(1.1.15)

In fact let \( K \) be the fraction field of \( R \) and \( 0 \rightarrow R \xrightarrow{\iota} I^0 \rightarrow I^1 \rightarrow \ldots \) be an injective resolution of \( R \) with \( I^0 = \det(U_\bullet) \otimes K \) and \( \iota(1) = \det \lambda \otimes 1 \). Then \( \text{Ext}^\bullet(N,R) \) is the cohomology of the double complex \( \text{Hom}(U_\bullet, I^\bullet) \) and of course also of the single complexes \( \text{Hom}(U_\bullet, R) \) and \( \text{Hom}(N,I^\bullet) \). One checks easily that the isomorphism \( \partial \) of (1.1.5) is equal to the isomorphism \( H^1(\text{Hom}(N,I^\bullet)) \xrightarrow{\sim} H^1(\text{Hom}(U_\bullet, I^\bullet)) \) i.e.

\[
\partial: \text{Hom}(N, \det(U_\bullet)/(\det \lambda)) = \text{Hom}(N, I^\bullet/\iota(R)) \xrightarrow{\sim} H^1(\text{Hom}(U_\bullet, I^\bullet)).
\]  

(1.1.16)

Let \( f \in \text{Hom}(N, \det(U_\bullet)/(\det \lambda)) \); a representative of \( \partial(f) \) in the double complex \( \text{Hom}(U_\bullet, I^\bullet) \) is given by \( g^{0,1} := f \circ \pi \in \text{Hom}(U_0, I^1) \). Let \( g^{0,0} \in \text{Hom}(U_0, \det(U_\bullet)) \) be a lift of \( g^{0,1} \) and \( g^{1,0} \in \text{Hom}(U_1, \det(U_\bullet)) \) be defined by \( g^{1,0} := g^{0,0} \circ \lambda \). One checks that \( \text{im}(g^{1,0}) \subset (\det \lambda) \) and hence there exists \( g \in \text{Hom}(U_1, R) \) such that \( g^{1,0} = \iota \circ g \). By construction \( g \) represents a class \( [g] \in \).
This proves (1.1.15). Now we prove Equation (1.1.8). By (1.1.15) we have
\[
m_{\beta}(\pi(a_i), \pi(a_j)) = (\theta^{-1}(\beta \pi(a_i)))\pi(a_j)) = (\theta^{-1}\beta \pi(a_i))\pi(a_j)).
\] (1.1.17)
Unwinding the definition of \(\theta\) one gets that the right-hand side of the above equation equals the right-hand side of (1.1.8).

Let \(m_{\beta}\) be given by (1.1.6): we define a product on \(R/(\text{det } M_{\gamma}) \oplus N\) as follows. Let \((r, n), (r', n') \in R/(\text{det } M_{\gamma}) \oplus N\): we set
\[
(r, n) \cdot (r', n') := (rr' + m_{\beta}(n, n'), rn' + r'n).
\] (1.1.18)
In general the above product is neither associative nor commutative. We will give an example in which the product is both associative and commutative. Suppose that we have
\[
0 \longrightarrow U^\vee \xrightarrow{\gamma} U \xrightarrow{\lambda} N \longrightarrow 0, \quad \gamma^t = \gamma
\] (1.1.19)
with \(U\) a free \(R\)-module of rank \(d > 0\) and the sequence is supposed to be exact. We get a commutative diagram (1.1.7) by letting
\[
U_0 := U, \quad U_1 := U^\vee, \quad \lambda = \gamma, \quad \alpha = \text{Id}_{U^\vee}, \quad \mu^t = \text{Id}_U,
\]
and \(\beta = \beta(\gamma): N \to \text{Ext}^1(N, R)\) the map induced by \(\text{Id}_U\). Abusing notation we let \(m_{\gamma}: N \times N \to R/(\text{det } M_{\gamma})\) be the map defined by \(m_{\beta(\gamma)}\).

**Proposition 1.2.** Suppose that we have Exact Sequence (1.1.19). The product on \(R/(\text{det } M_{\gamma}) \oplus N\) defined by \(m_{\gamma}\) is associative and commutative.

**Proof.** Let \(d := \text{rk } U > 0\). Let \(\{a_1, \ldots, a_d\}\) be a basis of \(U\) and \(\{a_1^\vee, \ldots, a_d^\vee\}\) be the dual basis of \(U^\vee\). Let \(M = M_{\gamma}\) i.e. the matrix associated to \(\gamma\) by our choice of bases. By (1.1.8) we have
\[
m_{\gamma}(\pi(a_i), \pi(a_j)) \equiv M_{j,i}^c \mod (\text{det } M).
\] (1.1.20)
Since \(\gamma\) is a symmetric map \(M\) is a symmetric matrix. Thus \(M^c\) is a symmetric matrix. By (1.1.20) we get that \(m_{\gamma}\) is associative. For \(1 \leq i < k \leq d\) and \(1 \leq h \neq j \leq d\) let \(M_{i,j,h}^k\) be the \((d - 2) \times (d - 2)\)-matrix obtained by deleting from \(M\) rows \(i, k\) and columns \(h, j\). Let \(X_{i,j,h}^k = (X_{i,j,h}^k) \in R^d\) be defined by
\[
X_{i,j,h}^k := \begin{cases} 
(-1)^{i+k+j+h} \det M_{j,h}^{i,k} & \text{if } h < j, \\
0 & \text{if } h = j, \\
(-1)^{i+k+j+h-1} \det M_{j,h}^{i,k} & \text{if } j < h.
\end{cases}
\] (1.1.21)
A tedious but straightforward computation gives that
\[
M_{i,j,h}^c a_k - M_{j,h}^c a_i = \gamma\left(\sum_{h=1}^{d} X_{i,j,h}^k a_h^\vee\right).
\] (1.1.22)
The above equation proves associativity of \(m_{\gamma}\).

Keep hypotheses as in **Proposition 1.2.** We let
\[
X_{\gamma} := \text{Spec}(R/(\text{det } M_{\gamma}) \oplus N), \quad Y_{\gamma} := \text{Spec}(R/(\text{det } M_{\gamma})).
\] (1.1.23)
Let \(f_{\gamma}: X_{\gamma} \to Y_{\gamma}\) be the structure map. We realize \(X_{\gamma}\) as a subscheme of \(\text{Spec}(R[\xi_1, \ldots, \xi_d])\) as follows. Since the ring \(R/(\text{det } M_{\gamma}) \oplus N\) is associative and commutative there is a well-defined surjective morphism of \(R\)-algebras
\[
R[\xi_1, \ldots, \xi_d] \longrightarrow R/(\text{det } M_{\gamma}) \oplus N
\] (1.1.24)
mapping \(\xi_i\) to \(a_i\). Thus we have an inclusion
\[
X_{\gamma} \hookrightarrow \text{Spec}(R[\xi_1, \ldots, \xi_d]).
\] (1.1.25)
Claim 1.3. Referring to Inclusion (1.1.25) the ideal of $X_{\gamma}$ is generated by the entries of the matrices

$$M_{\gamma} \cdot \xi, \quad \xi \cdot \xi^t - M_{\gamma}^t.$$  

(We view $\xi$ as a column matrix.)

Proof. By (1.1.20) the ideal of $X_{\gamma}$ is generated by $\det M_{\gamma}$ and the entries of the matrices in (1.1.26). By Cramer’s formula $\det M_{\gamma}$ belongs to the ideal generated by the entries of the two matrices. This proves that the ideal of $X_{\gamma}$ is as claimed. □

Now we suppose in addition that $R$ is a finitely generated $\mathbb{C}$-algebra. Let $p \in \text{Spec} R$ be a closed point: we are interested in the localization of $X_{\gamma}$ at points in $f_{\gamma}^{-1}(p)$. Let $J \subset U_{\gamma}^{-1}(p)$ be a subspace complementary to $\ker \gamma(p)$. Let $J \subset U_{\gamma}^{-1}$ be a free submodule whose fiber over $p$ is equal to $J$. Let $K \subset U_{\gamma}^{-1}$ be the submodule orthogonal to $J$ i.e.

$$K := \{ u \in U_{\gamma}^{-1} \mid \gamma(a)(u) = 0 \ \forall a \in J \}.$$  

(1.1.27)

The localization of $K$ at $p$ is free. Let $K := K(p)$ be the fiber of $K$ at $p$; clearly $K = \ker \gamma(p)$. Localizing at $p$ we have

$$U_{\gamma}^{-1} = K_p \oplus J_p.$$  

(1.1.28)

Corresponding to (1.1.28) we may write $\gamma_p = \gamma_K \oplus J_1 \gamma_{1,2}$ where $\gamma_K : K_p \rightarrow K_{1,2}^\gamma$ and $\gamma_{1,2} : J_p \rightarrow J_{1,2}^\gamma$ are symmetric maps. Notice that we have an equality of germs

$$(Y_{\gamma}, p) = (Y_{\gamma_K}, p).$$  

(1.1.29)

We claim that there is a compatible isomorphism of germs $(X_{\gamma_K}, f_{\gamma_K}^{-1}(p)) \cong (X_{\gamma}, f_{\gamma}^{-1}(p))$. In fact let $k := \dim K$ and $d := \rk U$. Choose bases of $K_p$ and $J_p$; by (1.1.28) we get a basis of $U_{\gamma}^{-1}$. The dual bases of $K_p^\gamma$, $J_p^\gamma$ and $U_{\gamma}^\gamma$ are compatible with respect to the decomposition dual to (1.1.28). Corresponding to the chosen bases we have embeddings $X_{\gamma_K} \hookrightarrow Y_{\gamma_K} \times \mathbb{C}^k$ and $X_{\gamma} \hookrightarrow Y_{\gamma} \times \mathbb{C}^d$. The decomposition dual to (1.1.28) gives an embedding $j : Y_{\gamma_K} \times \mathbb{C}^k \hookrightarrow Y_{\gamma} \times \mathbb{C}^d$. $

Claim 1.4. Keep notation as above. The composition

$$X_{\gamma_K} \hookrightarrow (Y_{\gamma_K} \times \mathbb{C}^k) \overset{j}{\longrightarrow} (Y_{\gamma} \times \mathbb{C}^d)$$  

(1.1.30)

defines an isomorphism of germs in the analytic topology

$$(X_{\gamma_K}, f_{\gamma_K}^{-1}(p)) \cong (X_{\gamma}, f_{\gamma}^{-1}(p))$$  

(1.1.31)

which commutes with the maps $f_{\gamma_K}$ and $f_{\gamma}$.

Proof. This follows by writing $\gamma_p = \gamma_K \oplus J_1 \gamma_{1,2}$ and by recalling (1.1.20). We pass to the analytic topology in order to be able to extract the square root of a regular non-zero function. □

Proposition 1.5. Assume that $R$ is a finitely generated $\mathbb{C}$-algebra. Suppose that we have Exact Sequence (1.1.19). Then the following hold:

1. $f_{\gamma}^{-1}Y_{\gamma}(1) \rightarrow Y_{\gamma}(1)$ is a topological covering of degree 2.

2. Let $p \in (Y_{\gamma} \setminus Y_{\gamma}(1))$ be a closed point. The fiber $f_{\gamma}^{-1}(p)$ consists of a single point $q$. Let $\xi_i$ be the coordinates on $X_{\gamma}$ associated to Embedding (1.1.25); then $\xi_i(q) = 0$ for $i = 1, \ldots, d$.

Proof. (1): Localizing at $p \in Y_{\gamma}(1)$ and applying Claim 1.4 we get Item (1). (2): Since $\rk M_{\gamma}(p) \geq 2$ we have $M_{\gamma}(p) = 0$. Thus Item (2) follows from Claim 1.3. □

We may associate a double cover $f_{\gamma} : X_{\gamma} \rightarrow Y_{\gamma}$ to a map $\beta$ which is symmetric in the derived category.

Hypothesis 1.6. We have (1.1.7) with $\alpha$ an isomorphism and in addition $\alpha = \mu$. 

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Proposition 1.7. Assume that Hypothesis 1.6 holds. Then \( R/(\det M_\lambda) \oplus N \) equipped with the product given by (1.1.18) is a commutative (associative) ring.

Proof. Let \( \gamma := \lambda \circ \mu^{-1} \) and \( U := U_0 \). Then (1.1.19) holds and the product defined by \( m_\beta \) is equal to the product defined by \( m_\gamma \). By Proposition 1.2 we get that \( R/(\det M_\lambda) \oplus N \) is a commutative associative ring. \( \square \)

Definition 1.8. Suppose that Hypothesis 1.6 holds: the symmetrization of (1.1.7) is Exact Sequence (1.1.19) with \( \gamma \) and \( U \) as in the proof of Proposition 1.7.

1.2 Structure sheaf of double EPW-sextics

Let \( A \in \mathbb{LG}(\Lambda^3 V) \) and suppose that \( Y_A \neq \mathbb{P}(V) \). We will define the associated double cover \( X_A \rightarrow Y_A \) by applying the results of Subsection 1.1. Since \( A \) is Lagrangian the symplectic form defines a canonical isomorphism \( \Lambda^3 V/A \cong A^3 \); thus (0.0.3) defines a map of vector-bundles \( \lambda_A : F \rightarrow A^* \otimes \mathcal{O}_{\mathbb{P}(V)} \). Let \( i : Y_A \hookrightarrow \mathbb{P}(V) \) be the inclusion map: since a local generator of \( \det \lambda_A \) annihilates \( \text{coker}(\lambda_A) \) there is a unique sheaf \( \zeta_A \) on \( Y_A \) such that we have an exact sequence

\[
0 \rightarrow F \xrightarrow{\lambda_A} A^* \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow i_* \zeta_A \rightarrow 0. \tag{1.2.1}
\]

Choose \( B \in \mathbb{LG}(\Lambda^3 V) \) transversal to \( A \). Thus we have a direct-sum decomposition \( \Lambda^3 V = A \oplus B \) and hence a projection map \( \Lambda^3 V \rightarrow A \) inducing a map \( \mu_{A,B} : F \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(V)} \). We claim that there is a commutative diagram with exact rows

\[
\begin{array}{c}
0 \rightarrow F \xrightarrow{\lambda_A} A^* \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow i_* \zeta_A \rightarrow 0 \\
\mu_{A,B} \downarrow \quad \quad \quad \quad \quad \quad \downarrow \mu_{A,B}^* \quad \quad \quad \quad \quad \quad \downarrow \beta_A \\
0 \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(V)} \xrightarrow{\lambda_B^*} F^* \rightarrow \text{Ext}^1(i_* \zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \rightarrow 0. \tag{1.2.2}
\end{array}
\]

In fact the second row is obtained by applying the \( \text{Hom}(\cdot, \mathcal{O}_{\mathbb{P}(V)}) \)-functor to (1.2.1) and the equality \( \mu_{A,B}^* \circ \lambda_A = \lambda_B^* \circ \mu_{A,B} \) holds because \( F \) is a Lagrangian sub-bundle of \( \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \). Lastly \( \beta_A \) is defined to be the unique map making the diagram commutative; it exists because the rows are exact. Notice that the map \( \beta_A \) is independent of the choice of \( B \) as suggested by the notation. Next by applying the \( \text{Hom}(i_* \zeta_A, \cdot) \)-functor to the exact sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{O}_{\mathbb{P}(V)}(6) \rightarrow \mathcal{O}_{Y_A}(6) \rightarrow 0 \tag{1.2.3}
\]

we get the exact sequence

\[
0 \rightarrow i_* \text{Hom}(\zeta_A, \mathcal{O}_{Y_A}(6)) \xrightarrow{\partial} \text{Ext}^1(i_* \zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{n} \text{Ext}^1(i_* \zeta_A, \mathcal{O}_{\mathbb{P}(V)}(6)) \tag{1.2.4}
\]

where \( n \) is locally equal to multiplication by \( \det \lambda_A \). Since the second row of (1.2.2) is exact a local generator of \( \det \lambda_A \) annihilates \( \text{Ext}^1(i_* \zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \); thus \( n = 0 \) and hence we get a canonical isomorphism

\[
\partial^{-1} : \text{Ext}^1(i_* \zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{\sim} i_* \text{Hom}(\zeta_A, \mathcal{O}_{Y_A}(6)). \tag{1.2.5}
\]

We define \( \tilde{m}_A \) by setting

\[
\zeta_A \times \zeta_A \xrightarrow{\tilde{m}_A} \mathcal{O}_{Y_A}(6) \xrightarrow{(\partial^{-1} \circ \beta_A)} (\sigma_1, \sigma_2) \rightarrow (\partial^{-1} \circ \beta_A(\sigma_1))(\sigma_2). \tag{1.2.6}
\]

Let \( \xi_A := \zeta_A(-3) \). Tensorizing both sides of (1.2.6) by \( \mathcal{O}_{Y_A}(-6) \) we get a multiplication map

\[
\xi_A \times \xi_A \xrightarrow{m_A} \mathcal{O}_{Y_A}. \tag{1.2.7}
\]

Thus we have defined a multiplication map \( \mathcal{O}_{Y_A} \oplus \xi_A \). The following result is well-known to experts.

Proposition 1.9. Let \( A \in \mathbb{LG}(\Lambda^3 V) \) and suppose that \( Y_A \neq \mathbb{P}(V) \). Let notation be as above. Then:
(1) $\beta_A$ is an isomorphism.

(2) The multiplication map $m_A$ is associative and commutative.

Proof. Let $[v_0] \in \mathbb{P}(V)$. Choose $B \in LG(\Lambda^3 V)$ transversal to $F_{v_0}$ (and to $A$ of course). Then $\mu_{A,B}$ is an isomorphism in an open neighborhood $U$ of $[v_0]$. It follows that $\beta_A$ is an isomorphism in a neighborhood of $[v_0]$. This proves Item (1). Let’s prove Item (2). Let $B \in LG(\Lambda^3 V)$ and $U$ be as above; we may assume that $U$ is affine. Let $N := H^0(\iota_* \Omega_A|_U)$ and $\beta := H^0(\beta_A|_U)$. Thus $\beta : N \to \text{Ext}^1(N, \mathbb{C}[U])$. By Commutativity of Diagram (1.2.2) and by Proposition 1.7 we get that the multiplication map $m_\beta$ is associative and commutative. On the other hand $m_A$ is associative and commutative.

We let $X_A := \text{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A)$ and we let $f_A : X_A \to Y_A$ be the structure morphism. Then $X_A$ is the double EPW-sextic associated to $A$ and $f_A$ is its structure map. The covering involution of $X_A$ is the automorphism $\phi_A : X_A \to X_A$ corresponding to the involution of $\mathcal{O}_{Y_A} \oplus \xi_A$ with $(-1)$-eigensheaf equal to $\xi_A$.

1.3 Local models of double covers

In the present subsection we assume that $R$ is a finitely generated $\mathbb{C}$-algebra. Let $W$ be a finite-dimensional complex vector-space. We will suppose that we have an exact sequence

$$0 \to R \otimes W^\vee \xrightarrow{\gamma} R \otimes W \to N \to 0, \quad \gamma = \gamma^t. \quad (1.3.1)$$

Thus we have a double cover $f_\gamma : X_\gamma \to Y_\gamma$. Let $p \in Y_\gamma$ be a closed point. We will examine $X_\gamma$ in a neighborhood of $f_\gamma^{-1}(p)$ when the corank of $\gamma(p)$ is small. We may view $\gamma$ as a regular map $\text{Spec} R \to \text{Sym}^2 W$; thus it makes sense to consider the differential

$$d\gamma(p) : T_p \text{Spec} R \to \text{Sym}^2 W. \quad (1.3.2)$$

Let $K(p) := \ker \gamma(p) \subset W^\vee$; we will consider the linear map

$$T_p \text{Spec} R \xrightarrow{\delta_\gamma(p)} \text{Sym}^2 K(p)^\vee \xrightarrow{\tau} d\gamma(p)(\tau)|_{K(p)}. \quad (1.3.3)$$

Let $d := \dim W$; choosing a basis of $W$ we realize $X_\gamma$ as a subscheme of $\text{Spec} R \times \mathbb{C}^d$ with ideal given by Claim 1.3. Since cork $\gamma(p) \geq 2$ Proposition 1.5 gives that $f_\gamma^{-1}(p)$ consists of a single point $q$ - in fact the $\xi_i$-coordinates of $q$ are all zero. Throughout this subsection we let

$$f_\gamma^{-1}(p) = \{q\}. \quad (1.3.4)$$

Claim 1.10. Keep notation as above. Suppose that $d = \dim W = 2$ and that $\gamma(p) = 0$. Then $I(X_\gamma)$ is generated by the entries of $\xi : \xi^t - M_\gamma$.

Proof. Claim 1.3 together with a straightforward computation.

Example 1.11. Let $R = \mathbb{C}[x, y, z]$, $W = \mathbb{C}^2$. Suppose that the matrix associated to $\gamma$ is

$$M_\gamma = \begin{pmatrix} x & y \\ y & z \end{pmatrix}. \quad (1.3.5)$$

Then $f_\gamma : X_\gamma \to Y_\gamma$ is identified with

$$\mathbb{C}^2 \to V(xz - y^2) \quad (\xi_1, \xi_2) \mapsto (\xi_2 - \xi_1, -
\xi_1, \xi_2). \quad (1.3.6)$$

i.e. the quotient map for the action of $(-1)$ on $\mathbb{C}^2$. 

Proposition 1.12. Keep notation as above. Suppose that the following hold:

(a) \( \text{cork } \gamma(p) = 2 \),
(b) the localization \( R_p \) is regular.

Then \( X_\gamma \) is smooth at \( q \) if and only if \( \delta_\gamma(p) \) is surjective.

Proof. Applying Claim 1.4 we get that we may assume that \( d = 2 \). Let

\[
M_\gamma = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.
\]

By Claim 1.10 the ideal of \( X_\gamma \) in \( \text{Spec } R \times \mathbb{C}^2 \) is generated by the entries of \( \xi \cdot \xi^t - M_\gamma \) i.e.

\[
I(X_\gamma) = (\xi_1^2 - c, \xi_1 \xi_2 + b, \xi_2^2 - a).
\]

Thus

\[
\text{cod}(T_qX_\gamma, T_q(\text{Spec } R \times \mathbb{C}^2)) = \dim \langle da(p), db(p), dc(p) \rangle.
\]

On the other hand \( \text{cod}_q(X_\gamma, \text{Spec } R \times \mathbb{C}^2) = 3 \) and hence we get that \( X_\gamma \) is smooth at \( q \) if and only if \( \delta_\gamma(p) \) is surjective.

Claim 1.13. Keep notation and hypotheses as above. Suppose that \( \text{cork } \gamma(p) \geq 3 \). Then \( X_\gamma \) is singular at \( q \).

Proof. Let \( I \) be the ideal of \( X_\gamma \) in \( \text{Spec } R[\xi_1, \ldots, \xi_d] \). By Claim 1.3 we get that \( I \) is non-trivial but the differential at \( q \) of an arbitrary \( g \in I \) is zero.

Next we will discuss in greater detail those \( X_\gamma \) whose corank at \( f_\gamma^{-1}(p) \) is equal to 3. First we will identify the “universal” example (the universal example for corank 2 is Example 1.11). Let \( V \) be a 3-dimensional complex vector space. We view \( \text{Sym}^2 V \) as an affine (6-dimensional) space and we let \( R := \mathbb{C}[\text{Sym}^2 V] \) be its ring of regular functions. We identify \( R \otimes \mathbb{C} V \) and \( R \otimes \mathbb{C} V^\vee \) with the space of \( V \)-valued, respectively \( V^\vee \)-valued, regular maps on \( \text{Sym}^2 V \). Let

\[
R \otimes \mathbb{C} V^\vee \to \text{Spec } R \times \mathbb{C}^2 \to \text{Spec } R \otimes \mathbb{C} V.
\]

be the map induced on the spaces of global sections by the tautological map of vector-bundles \( \text{Spec } R \times V^\vee \to \text{Spec } R \times V \). The map \( \gamma \) is symmetric. Let \( N \) be the cokernel of \( \gamma \): thus

\[
\begin{array}{c}
0 \\ R \otimes \mathbb{C} V^\vee \\
\gamma \\ R \otimes \mathbb{C} V \\
N \\
0
\end{array}
\]

is an exact sequence. Since \( \gamma \) is symmetric it defines a double cover \( f : X(V) \to Y(V) \) where

\[
Y(V) := \{ \alpha \in \text{Sym}^2 V \mid \text{rk } \alpha < 3 \}
\]

is the variety of degenerate quadratic forms. We let

\[
\phi : X(V) \to X(V)
\]

be the covering involution of \( f \). One describes explicitly \( X(V) \) as follows. Let

\[
(V \otimes V)_1 := \{ \mu \in (V \otimes V) \mid \text{rk } \mu \leq 1 \}.
\]

Thus \( (V \otimes V)_1 \) is the cone over the Segre variety \( P(V) \times P(V) \). We have a finite degree-2 map

\[
(V \otimes V)_1 \xrightarrow{\mu} Y(V) \xrightarrow{\mu + \mu^t}.
\]
Proposition 1.14. Keep notation as above. There exists a commutative diagram

\[
\begin{array}{ccc}
(V \otimes V)_1 & \overset{\tau}{\longrightarrow} & X(V) \\
\sigma \downarrow & & \downarrow f \\
Y(V) & \overset{\phi}{\longrightarrow} & (V \otimes V)
\end{array}
\]

where \( \tau \) is an isomorphism. Let \( \phi \) be Involution (1.3.13): then

\[
\phi \circ \tau(\mu) = \tau(\mu^t), \quad \forall \mu \in (V \otimes V)_1.
\]

Proof. In order to define \( \tau \) we will give a coordinate-free version of Inclusion (1.1.25) in the case of \( X(V) \). Let

\[
\text{Sym}^2 V \times (V^\vee \otimes \Lambda^3 V) \xrightarrow{\Psi} (V \otimes \Lambda^3 V) \times (V^\vee \otimes V^\vee \otimes \Lambda^3 V) \\
(\alpha, \xi) \mapsto (\alpha \circ \xi, \xi^t \circ \xi - \Lambda^2 \alpha).
\]

A few words of explanation. In the definition of the first component of \( \Psi(\alpha, \xi) \) we view \( \alpha \) as belonging to \( \text{Hom}(\Lambda^3 V^\vee, V^\vee) \), in the definition of the second component of \( \Psi(\alpha, \xi) \) we view \( \xi \) as belonging to \( \text{Hom}(V \otimes \Lambda^3 V^\vee, C) \). Moreover we make the obvious choice of isomorphism \( C \cong C^\vee \).

Secondly

\[
\bigwedge^2 \alpha \in \text{Hom}(\bigwedge^2 V^\vee, \bigwedge^2 V^\vee) = \text{Hom}(V \otimes \bigwedge^3 V^\vee, V^\vee \otimes \bigwedge^3 V^\vee) = V^\vee \otimes V^\vee \otimes \bigwedge^3 V \otimes V.
\]

Choosing a basis of \( V \) we get an embedding \( X(V) \subset \text{Sym}^2 V \times C^3 \), see (1.1.25). Claim 1.3 gives equality of pairs

\[
(\text{Sym}^2 V \times (V^\vee \otimes \Lambda^3 V), \Psi^{-1}(0)) = (\text{Sym}^2 V \times C^3, X(V)),
\]

where \( \Psi^{-1}(0) \) is the scheme-theoretic fiber of \( \Psi \). Now notice that we have an isomorphism

\[
\begin{array}{ccc}
V \otimes V & \overset{\tau}{\longrightarrow} & \text{Sym}^2 V \times (V^\vee \otimes \Lambda^3 V) \\
\epsilon & \mapsto & (\epsilon + \epsilon^t, \epsilon - \epsilon^t).
\end{array}
\]

Let \( \tau := T|_{(V \otimes V)_1} \); thus we have an embedding

\[
\tau: (V \otimes V)_1 \hookrightarrow \text{Sym}^2 V \times (V^\vee \otimes \Lambda^3 V).
\]

We will show that we have equality of schemes

\[
\text{im}(\tau) = \Psi^{-1}(0)(= X(V)).
\]

First let

\[
\begin{array}{ccc}
V \oplus V & \overset{\rho}{\longrightarrow} & (V \otimes V)_1 \\
(\eta, \beta) & \mapsto & \eta^t \circ \beta.
\end{array}
\]

Notice that \( \rho \) is the quotient map for the \( C^* \)-action on \( V \oplus V \) defined by \( t(\eta, \beta) := (t\eta, t^{-1}\beta) \). We have

\[
\tau \circ \pi = (\eta^t \circ \beta + \beta^t \circ \eta, \eta \wedge \beta).
\]

Let’s prove that

\[
\Psi^{-1}(0) \supset \text{im}(\tau).
\]
Notice that $\text{Gl}(V)$ acts on $(V \otimes V)_1$ with a unique dense orbit namely $\{\eta^t \circ \beta \mid \eta \wedge \beta \neq 0\}$. An easy computation shows that $\tau(\eta^t \circ \beta) \in \Psi^{-1}(0)$ for a conveniently chosen $\eta^t \circ \beta$ in the dense orbit of $(V \otimes V)_1$; it follows that (1.3.26) holds. On the other hand $T$ defines an isomorphism of pairs
\[
(V \otimes V, (V \otimes V)_1) \cong (\text{Sym}^2 V^* \times (V^* \otimes \bigwedge^3 V), \text{im}(\tau)).
\] (1.3.27)
Since the ideal of $(V \otimes V)_1$ in $V \otimes V$ is generated by 9 linearly independent quadrics we get that the ideal of $\text{im}(\tau)$ in $\text{Sym}^2 V^* \times (V^* \otimes \bigwedge^3 V)$ is generated by 9 linearly independent quadrics. The ideal of $\Psi^{-1}(0)$ in $\text{Sym}^2 V \times (V^* \otimes \bigwedge^3 V)$ is likewise generated by 9 linearly independent quadrics - see (1.3.18). Since $\Psi^{-1}(0) \supset \text{im}(\tau)$ we get that the ideals of $\Psi^{-1}(0)$ and of $\text{im}(\tau)$ are the same and hence (1.3.23) holds. This proves that $\tau$ is an isomorphism between $(V \otimes V)_1$ and $X(V)$.
Diagram (1.3.16) is commutative by construction. Equation (1.3.17) is equivalent to the equality
\[
\phi(\tau \circ \rho(\beta, \eta)) = \tau \circ \rho(\eta, \beta).
\] (1.3.28)
The above equality holds because $\beta \wedge \eta = -\eta \wedge \beta$.

The following result is an immediate consequence of Proposition 1.14.

**Corollary 1.15.** $\text{sing } X(V) = \tau(0) = f^{-1}(0)$.

2 The divisor $\Delta$

2.1 Parameter counts

Let $\Delta_+ \subset LG(\bigwedge^3 V)$ and $\tilde{\Delta}_+, \tilde{\Delta}_+(0) \subset LG(\bigwedge^3 V) \times \mathbb{P}(V)^2$ be
\[
\begin{align*}
\Delta_+ & := \{ A \in LG(\bigwedge^3 V) \mid |A[3]| > 1 \}, \\
\tilde{\Delta}_+ & := \{(A, [v_1], [v_2]) \mid [v_1] \neq [v_2], \text{ dim}(A \cap F_{v_1}) \geq 3 \}, \\
\tilde{\Delta}_+(0) & := \{(A, [v_1], [v_2]) \mid [v_1] \neq [v_2], \text{ dim}(A \cap F_{v_1}) = 3 \}.
\end{align*}
\] (2.1.1) (2.1.2) (2.1.3)

Notice that $\tilde{\Delta}_+$ and $\tilde{\Delta}_+(0)$ are locally closed.

**Lemma 2.1.** Keep notation as above. The following hold:

1. $\tilde{\Delta}_+$ is irreducible of dimension 53.

2. $\Delta_+$ is constructible and $\text{cod}(\Delta_+, LG(\bigwedge^3 V)) \geq 2$.

**Proof.** (1): Let’s prove that $\tilde{\Delta}_+(0)$ is irreducible of dimension 53. Consider the map
\[
\begin{array}{c}
\tilde{\Delta}_+(0) \\
(A, [v_1], [v_2])
\end{array} \quad \mapsto \quad \begin{array}{c}
\text{Gr}(3, \bigwedge^3 V)^2 \times \mathbb{P}(V)^2 \\
(A \cap F_{v_1}, A \cap F_{v_2}, [v_1], [v_2])
\end{array}.
\] (2.1.4)

We have
\[
\text{im} \eta = \{(K_1, K_2, [v_1], [v_2]) \mid K_1 \in \text{Gr}(3, F_{v_1}), \ K_1 \perp K_2, \ [v_1] \neq [v_2]\}.
\] (2.1.5)

We stratify $\text{im} \eta$ according to $i := \text{dim}(K_1 \cap F_{v_2})$ and to $j := \text{dim}(K_1 \cap K_2)$; of course $j \leq i$. Let $(\text{im} \eta)_{i,j} \subset \text{im} \eta$ be the stratum corresponding to $i,j$. A straightforward computation gives that
\[
\text{dim} \eta^{-1}(\text{im} \eta)_{i,j} = 10 + 7(3 - i) + j(i - j) + (3 - j)(4 + i) + \frac{1}{2}(j + 5)(j + 4) = 53 - 4i - \frac{1}{2}j(j - 1).
\] (2.1.6)

Since $0 \leq i,j$ one gets that the maximum is achieved for $i = j = 0$ and that it equals 53. It follows that $\tilde{\Delta}_+(0)$ is irreducible of dimension 53. On the other hand $\tilde{\Delta}_+(0)$ is dense in $\tilde{\Delta}_+$ (easy) and hence we get that Item (1) holds. (2): Let $\pi_+ : \Delta_+ \rightarrow LG(\bigwedge^3 V)$ be the forgetful map: $\pi_+([v_1], [v_2], A) = A$. Then $\pi_+(\tilde{\Delta}_+) = \Delta_+$. By Item (1) we get that $\text{dim} \Delta_+ \leq 53$: since $\text{dim} \text{LG}(\bigwedge^3 V) = 55$ we get that Item (2) holds.

\[\square\]
Proposition 2.2. The following hold:

(1) $\Delta$ is closed irreducible of codimension 1 in $\cal LG(\Lambda^3 V)$ and not equal to $\Sigma$.

(2) If $A \in \Delta$ is generic then $Y_A[3] = Y_A(3)$ and it consists of a single point.

Proof. (1): Let

$$\tilde{\Delta} := \{ (A, [v]) \mid \dim(F_v \cap A) \geq 3 \}, \quad \tilde{\Delta}(0) := \{ (A, [v]) \mid \dim(F_v \cap A) = 3 \}. \tag{2.1.7}$$

Then $\tilde{\Delta}$ is a closed subset of $\cal LG(\Lambda^3 V) \times \cal P(V)$ and $\tilde{\Delta}(0)$ is an open subset of $\tilde{\Delta}$. Let $\pi: \tilde{\Delta} \to \cal LG(\Lambda^3 V)$ be the forgetful map. Thus $\pi(\tilde{\Delta}) = \Delta$: since $\pi$ is projective it follows that $\tilde{\Delta}$ is closed. Projecting $\tilde{\Delta}(0)$ to $\cal P(V)$ we get that $\tilde{\Delta}(0)$ is smooth irreducible of dimension 54. A standard dimension count shows that $\tilde{\Delta}(0)$ is open dense in $\tilde{\Delta}$; thus $\tilde{\Delta}$ is irreducible of dimension 54. It follows that $\Delta$ is irreducible. By Lemma 2.1 we know that $\dim \Delta \leq 53$. It follows that the generic fiber of $\tilde{\Delta} \to \Delta$ is a single point, in particular $\dim \Delta = 54$ and hence $\operatorname{cod}(\Delta, \cal LG(\Lambda^3 V)) = 1$ because $\dim \cal LG(\Lambda^3 V) = 55$. A dimension count shows that $\dim(\Delta \cap \Sigma) < 54$ and hence $\Delta \neq \Sigma$. This finishes the proof of Item (1). (2): Let $A \in \Delta$ be generic: we already noticed that there exists a unique $[v] \in \cal P(V)$ such that $([v], A) \in \tilde{\Delta}$, i.e. $Y_A[3]$ consists of a single point. Since $\tilde{\Delta}(0)$ is dense in $\tilde{\Delta}$ and $\dim \tilde{\Delta} = \dim \Delta$ we get that $([v], A) \in \tilde{\Delta}(0)$, i.e. $Y_A[3] = Y_A(3)$. This finishes the proof of Item (2). \hfill \Box

2.2 First order computations

Let $(A, [v_0]) \in \tilde{\Delta}(0)$. We will study the differential of $\pi: \tilde{\Delta} \to \cal LG(\Lambda^3 V)$ at $(A, [v_0])$. First we will give a local description of $\tilde{\Delta}$ as degeneracy locus. Let

$$N(V) := \{ A \in \cal LG(\Lambda^3 V) \mid Y_A = \cal P(V) \}. \tag{2.2.1}$$

Notice that $N(V)$ is closed. Let $\cal Y$ be the tautological family of EPW-sextics i.e.

$$\cal Y := \{ (A, [v]) \in (\cal LG(\Lambda^3 V) \setminus N(V)) \times \cal P(V) \mid \dim(A \cap F_v) > 0 \}. \tag{2.2.2}$$

Of course $\cal Y$ has a description as a determinantal variety and hence it has a natural scheme structure. For $U \subset (\cal LG(\Lambda^3 V) \setminus N(V))$ open we let $\cal Y_U := \cal Y \cap (U \times \cal P(V))$. Given $B \in \cal LG(\Lambda^3 V)$ let

$$U_B := \{ A \in \cal LG(\Lambda^3 V) \mid A \cap B \} \setminus N(V). \tag{2.2.3}$$

(Here $A \cap B$ means that $A$ intersects transversely $B$ i.e. $A \cap B = \{ 0 \}$.) Let $i_{U_B}: \cal Y_U \hra U_B \times \cal P(V)$ be the inclusion and let $\cal A$ be the tautological rank-10 vector-bundle on $\cal LG(\Lambda^3 V)$ (the fiber of $\cal A$ over $A$ is $A$ itself). Going through the argument that produced Commutative Diagram (1.2.2) we get that there exists a commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & \cal O_{U_B} \boxtimes F \\
\big| \mu_{U_B} & & \lambda_{U_B} \\
\downarrow \nu_{U_B} & \rightarrow & \cal O_{\cal P(V)}^\wedge \cal O_{\cal P(V)} \\
0 & \rightarrow & \cal A|_{U_B} \boxtimes \cal O_{\cal P(V)} \end{array} \quad \Longrightarrow \quad \begin{array}{c}
i_{U_B} \bullet \zeta_{U_B} \\
\gamma_{U_B} \end{array} \quad \rightarrow \quad 0 \\
\begin{array}{ccc}
0 & \rightarrow & \cal O_{U_B} \boxtimes F \\
\big| \lambda_{U_B} & & \mu_{U_B} \\
\downarrow \nu_{U_B} & \leftarrow & \cal O_{\cal P(V)}^\wedge \cal O_{\cal P(V)} \\
0 & \rightarrow & \cal A|_{U_B} \boxtimes \cal O_{\cal P(V)} \end{array} \quad \Longrightarrow \quad \begin{array}{c}
i_{U_B} \bullet \zeta_{U_B} \\
\gamma_{U_B} \end{array} \quad \rightarrow \quad 0 \tag{2.2.4}
\end{array}
$$

Now let $(A, [v_0]) \in \cal Y$. Choose $B \in \cal LG(\Lambda^3 V)$ such that $B \cap A$ and $B \cap F_{v_0}$. Let $N \subset \cal P(V)$ be an open neighborhood of $[v_0]$ such that $B \cap F_w$ for all $w \in N$. The restriction to $U_B \setminus A$ is trivial and the restriction to $N \cap F$ is likewise trivial. Moreover the restriction of $\mu_{U_B}$ to $U_B \times N$ is an isomorphism. Let

$$\gamma := (\lambda_{U_B}|_{U_B \times N}) \circ (\mu_{U_B}|_{U_B \times N})^{-1}. \tag{2.2.5}$$

We have an exact sequence

$$0 \longrightarrow \cal A|_{U_B} \boxtimes \cal O_N \quad \longrightarrow \quad \cal A|_{U_B} \boxtimes \cal O_N \quad \longrightarrow \quad \cal O_{U_B} \bullet \zeta_{U_B}|_{U_B \times N} \longrightarrow 0 \tag{2.2.6}$$

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The map $\gamma$ is symmetric, in fact it is the symmetrization of the restriction of (2.2.4) to $U_B \times \mathcal{N}'$ - see Definition 1.8. Then $\tilde{\Delta} \cap (U_B \times \mathcal{N})$ is the symmetric degeneration locus

$$\tilde{\Delta} \cap (U_B \times \mathcal{N}) = \{(A',[v]) \in (U_B \times \mathcal{N}) \mid \text{cork}(A',[v]) \geq 3\}$$

and hence it inherits a natural structure of closed subscheme of $\mathbb{P}G(\wedge^3 V) \times \mathbb{P}(V)$. In order to study the differential of the forgetful map $\tilde{\Delta} \to \mathbb{P}(V)$ we will introduce some notation. Given $v \in V$ we define a quadratic form $\phi_v$ on $F_v$ as follows. Let $\alpha \in F_v$; then $\alpha = v_0 \wedge \beta$ for some $\beta \in \wedge^2 V$. We set

$$\phi_v(\alpha) := \text{vol}(v_0 \wedge v \wedge \beta \wedge \beta).$$

(2.2.8)

The above equation gives a well-defined quadratic form on $F_v$ because $\beta$ is determined up to addition by an element of $F_v$. Of course $\phi_v$ depends only on the class of $v$ in $V/[v_0]$. Choose a direct-sum decomposition

$$V = [v_0] \oplus V_0.$$

(2.2.9)

We have the isomorphism

$$\lambda_v^0 : \wedge^2 V_0 \xrightarrow{\beta} F_v \xrightarrow{v_0 \wedge \beta}. (2.2.10)$$

Under the above identification the Plücker quadratic forms on $\wedge^2 V_0$ correspond to the quadratic forms $\phi_v$ for $v$ varying in $V_0$. Let $K := A \cap F_v$ and

$$V_0 \xrightarrow{V \to \wedge^2 K^\vee \xrightarrow{\phi_v^K}} \phi_v^K \quad \text{Sym}^2 K^\vee \xrightarrow{\theta^A_K} \phi_v^K \quad \text{Sym}^2 A^\vee \xrightarrow{q} [v_0 + v]$$

(2.2.11)

The isomorphism

$$V_0 \xrightarrow{v \mapsto \phi_v^K} \mathbb{P}(V) \setminus \mathbb{P}(V_0)$$

defines an isomorphism $V_0 \cong T_{[v_0]} \mathbb{P}(V)$. Recall that the tangent space to $\mathbb{P}G(\wedge^3 V)$ at $A$ is canonically identified with $\text{Sym}^2 A^\vee$.

**Proposition 2.3.** Keep notation as above - in particular choose (2.2.9). Then

$$T_{(A,[v_0])} \tilde{\Delta} \subset T_{(A,[v_0])} (\mathbb{P}G(\wedge^3 V) \times \mathbb{P}(V)) = \text{Sym}^2 A^\vee \oplus V_0$$

(2.2.12)

is given by

$$T_{([v_0],A)} \tilde{\Delta} = \{(q,v) \mid \theta_A^K(q) - \tau_v^K(v) = 0\}.$$  

(2.2.13)

**Proof.** By the (local) degeneracy description (2.2.7) we get that $(q,v) \in T_{([v_0],A)} \tilde{\Delta}$ if and only if

$$0 = d\gamma(A,[v_0])(q,v)|_K = d\gamma(A,[v_0])(q,0)|_K + d\gamma(A,[v_0])(0,v)|_K.$$ 

It is clear that $d\gamma(A,[v_0])(q,0)|_K = \theta_A^K(q)$. On the other hand Equation (2.26) of [12] gives that

$$d\gamma(A,[v_0])(0,v)|_K = -\tau_v^K(v).$$

(2.2.14)

The proposition follows. \hfill \Box

**Corollary 2.4.** $\tilde{\Delta}(0)$ is smooth (of codimension 6 in $\mathbb{P}G(\wedge^3 V) \times \mathbb{P}(V)$). Let $(A,[v_0]) \in \tilde{\Delta}(0)$ and $K := A \cap F_v$. The differential $d\pi(A,[v_0])$ is injective if and only if $\tau_v^K$ is injective.

**Proof.** Let $(A,[v_0]) \in \tilde{\Delta}(0)$ and $K := A \cap F_v$. The map $\theta_A^K$ is surjective: by Proposition 2.3 we get that $T_{(A,[v_0])} \tilde{\Delta}(0)$ has codimension 6 in $T_{(A,[v_0])} (\mathbb{P}G(\wedge^3 V) \times \mathbb{P}(V))$. On the other hand the description of $\tilde{\Delta}(0)$ as a symmetric degeneration locus - see (2.2.7) - gives that $\tilde{\Delta}(0)$ has codimension at most 6 in $\mathbb{P}G(\wedge^3 V) \times \mathbb{P}(V)$: it follows that $\tilde{\Delta}(0)$ is smooth of codimension 6 in $\mathbb{P}G(\wedge^3 V) \times \mathbb{P}(V)$. The statement about injectivity of $d\pi(A,[v_0])$ follows at once form Proposition 2.3. \hfill \Box
A comment regarding Corollary 2.4. The statement about smoothness of $\hat{\Delta}(0)$ is not contained in the proof of Proposition 2.2 because in that proof we consider $\hat{\Delta}(0)$ with its reduced structure. Before stating the next result we give the following definition: given $A \in LG(\Lambda^3 V)$ we let

$$\Theta_A := \{ W \in \text{Gr}(3, V) \mid \bigwedge^3 W \subset A \}. \quad (2.2.15)$$

**Proposition 2.5.** Let $\langle A, [v_0] \rangle \in \hat{\Delta}(0)$ and let $K := A \cap F_{v_0}$. Then $\tau_K^{v_0}$ is injective if and only if

1. no $W \in \Theta_A$ contains $v_0$, or
2. there is exactly one $W \in \Theta_A$ containing $v_0$ and moreover

$$A \cap F_{v_0} \cap \left( \bigwedge^2 W \wedge V \right) = \bigwedge^3 W. \quad (2.2.16)$$

If Item (1) holds then $\text{im}(\tau_K^{v_0})$ belongs to the unique open $\text{PGL}(K)$-orbit of $\text{Gr}(5, \text{Sym}^2 K^\vee)$, if Item (2) holds then $\text{im}(\tau_K^{v_0})$ belongs to the unique closed $\text{PGL}(K)$-orbit of $\text{Gr}(5, \text{Sym}^2 K^\vee)$.

**Proof.** Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$. Let

$$\rho_{v_0}^V : F_{v_0} \xrightarrow{\sim} \bigwedge^2 V_0 \quad (2.2.17)$$

be the inverse of Isomorphism (2.2.10). Let $K := \mathbb{P}(\rho_{v_0}^V(K)) \subset \mathbb{P}(\bigwedge^2 V_0)$; then $K$ is a projective plane. Isomorphism $\rho_{v_0}^V$ identifies the space of quadratic forms $\phi_{v_0}^V$, for $v \in V_0$, with the space of Pf\"ucker quadratic forms on $\bigwedge^2 V_0$. Since the ideal of $\text{Gr}(2, V_0) \subset \mathbb{P}(\bigwedge^2 V_0)$ is generated by the Pf\"ucker quadratic forms we get that $\tau_K^{v_0}$ is identified with the natural restriction map

$$V_0 = H^0(\mathcal{I}_{\text{Gr}(2, V_0)}(2)) \xrightarrow{\tau_K^{v_0}} H^0(\mathcal{O}_K(2)) = \text{Sym}^2 K^\vee. \quad (2.2.18)$$

It follows that if the scheme-theoretic intersection $K \cap \text{Gr}(2, V_0)$ is not empty nor a single reduced point then $\tau_K^{v_0}$ is not injective. Now suppose that $K \cap \text{Gr}(2, V_0)$ is

1. empty i.e. Item (1) holds, or
2. a single reduced point, i.e. Item (2) holds.

Let

$$\mathbb{P}(\bigwedge^2 V_0) \xrightarrow{\Phi} |H^0(\mathcal{I}_{\text{Gr}(2, V_0)}(2))|^\vee = \mathbb{P}(V_0') \quad (2.2.19)$$

be the natural map: it associates to $[\alpha] \notin \text{Gr}(2, V_0)$ the projectivization of $\text{supp}(\alpha)$. We have a tautological identification

$$K \xrightarrow{\Phi|_K} \mathbb{P}(\text{im}(\tau_K^{v_0}))^\vee$$

and of course $\Phi|_K$ is the Veronese embedding $K \to |\mathcal{O}_K(2)|^\vee$ followed by the projection with center $\mathbb{P}(\text{Ann}(\text{im}(\tau_K^{v_0})))$. Notice that $\tau_K^{v_0}$ is not injective if and only if $\dim \mathbb{P}(\text{Ann}(\text{im}(\tau_K^{v_0}))) \geq 1$. Now suppose that (1') holds. Then $\Phi|_K$ is regular and in fact it is an isomorphism onto its image - see Lemma 2.7 of [15]. Since the chordal variety of the Veronese surface in $|\mathcal{O}_K(2)|^\vee$ is a hypersurface it follows that $\dim \mathbb{P}(\text{Ann}(\text{im}(\tau_K^{v_0}))) < 1$ and hence $\tau_K^{v_0}$ is injective. We also get that $\text{Ann}(\text{im}(\tau_K^{v_0}))$ is a point in $|\mathcal{O}_K(2)|^\vee$ which does not belong to the chordal variety of the Veronese surface and hence it belongs to unique open $\text{PGL}(K)$-orbit. Now suppose that (2') holds. Assume that $\tau_K^{v_0}$ is not injective. Then $\dim \mathbb{P}(\text{Ann}(\text{im}(\tau_K^{v_0}))) \geq 1$. It follows that there exist $[x] \neq [y] \in K$ in the regular locus of $\Phi|_K$ (i.e. neither $x$ nor $y$ is decomposable) such that $\Phi([x]) = \Phi([y])$. By the description of $\Phi$ given above in terms of supports we get that $\text{supp}(x) = \text{supp}(y) = U$ where $\dim U = 4$; since $\text{Gr}(2, U)$ is a hypersurface in $\mathbb{P}(\bigwedge^2 V_0)$ we get that the line $\langle [x], [y] \rangle \subset \mathbb{P}(\bigwedge^2 V_0)$ intersects $\text{Gr}(2, U)$ in a subscheme of length 2. Since $\langle [x], [y] \rangle \subset K$ it follows that $K \cap \text{Gr}(2, V_0)$ contains a scheme of length 2, that contradicts Item (2'). This proves that if (2') holds then $\tau_K^{v_0}$ is injective. It also follows that $\text{Ann}(\text{im}(\tau_K^{v_0}))$ belongs to the Veronese surface in $|\mathcal{O}_K(2)|^\vee$ i.e. $\text{im}(\tau_K^{v_0})$ belongs to the unique closed $\text{PGL}(K)$-orbit.
3 Simultaneous resolution

In the first subsection we will analyze families of double EPW-sextics and their singular locus. The second subsection shows how to construct the simultaneous desingularization described in Item (3) of Section 0 (the relation with the Hilbert square of a $K3$ will be given in Section 4).

3.1 Families of double EPW-sextics

Let $U \subset (\mathbb{LG}(\wedge^3 V) \setminus \mathbb{N}(V))$ (see (2.2.1)) be open. Suppose that there exist a scheme $X_U$ and a finite $f_U: X_U \to Y_U$ such that for every $A \in U$ the induced map $f_A^{-1} Y_A \to Y_A$ is identified with $f_A: X_A \to Y_A$: then we say that a tautological family of double EPW-sextics parametrized by $U$ exists - often we simply state that $f_U: X_U \to Y_U$ exists. Composing $f_U$ with the natural map $\mathcal{Y}_U \to U$ we get a map $\rho_U: X_U \to U$ such that $\rho_U^{-1}(A) \cong X_A$.

Proposition 3.1. Let $B \in \mathbb{LG}(\wedge^3 V)$. A tautological family of double EPW-sextics parametrized by $U_B$ exists ($U_B$ is given by (2.2.3)).

Proof. Let $\nu: \mathcal{Y}_B \to \mathbb{P}(V)$ be projection. Let $\xi_B := \nu^* \mathcal{O}_{\mathbb{P}(V)}(-3)$ where $\xi_B$ is the sheaf on $\mathcal{Y}_B$ sitting in (2.2.4). Look at Commutative Diagram (2.2.4): proceeding as in the definition of the multiplication on $\mathcal{O}_{\mathcal{Y}_B} \oplus \xi_B$ we get that $\beta_B$ defines a multiplication on $\mathcal{O}_{\mathcal{Y}_B} \oplus \xi_B$. By Proposition 1.7 we get that $\mathcal{O}_{\mathcal{Y}_B} \oplus \xi_B$ is an associative commutative ring. Let $\mathcal{X}_B := \text{Spec}(\mathcal{O}_{\mathcal{Y}_B} \oplus \xi_B)$ and $f_B: \mathcal{X}_B \to \mathcal{Y}_B$ be the structure map.

Let $U \subset (\mathbb{LG}(\wedge^3 V) \setminus \mathbb{N}(V))$ be open and such that $f_U: X_U \to Y_U$ exists. We will determine the singular locus of $X_U$. Let

$\mathcal{Y}[d] := \{(A, [v]) \in (\mathbb{LG}(\wedge^3 V) \setminus \mathbb{N}(V)) \times \mathbb{P}(V) \mid \dim(A \cap F_v) \geq d\}$, \hspace{1cm} (3.1.1)

$\mathcal{Y}(d) := \{(A, [v]) \in (\mathbb{LG}(\wedge^3 V) \setminus \mathbb{N}(V)) \times \mathbb{P}(V) \mid \dim(A \cap F_v) = d\}$. \hspace{1cm} (3.1.2)

Then $\mathcal{Y}[d]$ has a natural structure of closed subscheme of $\mathbb{LG}(\wedge^3 V) \times \mathbb{P}(V)$ given by its local description as a symmetric determinantal variety - see Subsection 2.2 of [15]. Let $U \in (\mathbb{LG}(\wedge^3 V) \setminus \mathbb{N}(V))$ be open. We let $\mathcal{Y}_U[d] := \mathcal{Y}[d] \cap \mathcal{Y}_U$ and similarly for $\mathcal{Y}_U(d)$. Suppose that $f_U: X_U \to Y_U$ is defined; we let

$\mathcal{W}_U := f_U^{-1} \mathcal{Y}[3]$. \hspace{1cm} (3.1.3)

Notice that the restriction of $f_U$ to $\mathcal{W}_U$ defines an isomorphism $\mathcal{W}_U \cong \mathcal{Y}_U[3]$. We will prove the following result.

Proposition 3.2. Let $U \subset (\mathbb{LG}(\wedge^3 V) \setminus \mathbb{N}(V))$ be open and suppose that $f_U: X_U \to Y_U$ exists. Then $\text{sing}(X_U) = \mathcal{W}_U$.

Proof. We may assume that $U \cong U_B \times N$ where $B \in \mathbb{LG}(\wedge^3 V)$ and $N \subset \mathbb{P}(V)$ is an open subset such that $B \cap F_w$ for all $w \in N$. Then (see the proof of Proposition 3.1)

$f_U^{-1}(U) = X_\gamma$ where $\gamma$ is given by (2.2.5).

Thus it suffices to examine $X_\gamma$. Let $(A, [v]) \in U$ and

$\delta_\gamma(A, [v]): T_{(A, [v])} \mathbb{LG}(\wedge^3 V) \times \mathbb{P}(V) \to \text{Sym}^2(A \cap F_v)^\vee$ \hspace{1cm} (3.1.5)

be as in (1.3.3). The restriction of $\delta_\gamma(A, [v])$ to the tangent space to $\mathbb{LG}(\wedge^3 V)$ at $A$ is surjective; thus

$\delta_\gamma(A, [v])$ is surjective. \hspace{1cm} (3.1.6)

Let $q \in X_\gamma$ and $f_U(q) = (A, [v])$. Suppose that $q \notin \mathcal{W}_U$ i.e. that cork $\gamma(p) \leq 2$. If cork $\gamma(p) = 1$ then $Y_U = Y_\gamma$ is smooth because the differential $\delta_\gamma(A, [v])$ is surjective: by Proposition 1.5 we get that $X_U$ is smooth at $q$. If cork $\gamma(p) = 2$ then $X_U$ is smooth at $q$ by Proposition 1.12: recall that the differential $\delta_\gamma(A, [v])$ is surjective. This proves that $\text{sing}(X_U) \subset \mathcal{W}_U$. On the other hand $\mathcal{W}_U \subset \text{sing}(X_U)$ by Claim 1.13. \hspace{1cm} $\square$
We will close the present subsection by proving a few results about the individual $X_A$'s.

**Lemma 3.3.** Let $A \in (\mathbb{LG}(\Lambda^3 V) \setminus \mathbb{N}(V))$ and $[v] \in Y_A$. Suppose that $\dim(A \cap F_v) \leq 2$ and that there is no $W \in \Theta_A$ (see (2.2.15)) containing $v$. Then $X_A$ is smooth at $q$.

**Proof.** Let $q \in f_A^{-1}([v])$. Suppose that $\dim(A \cap F_v) = 1$. By Corollary 2.5 of [15] we get that $Y_A$ is smooth at $[v]$, thus $X_A$ is smooth at $q$ by Proposition 1.1.5. Suppose that $\dim(A \cap F_v) = 2$. Locally around $q$ the double cover $X_A \to Y_A$ is isomorphic to $X_{\bar{\tau}} \to Y_{\bar{\tau}}$ where $\bar{\tau}$ is the symmetrization of the restriction of $\tau$ to an affine neighborhood Spec $R$ of $[v]$. Thus we may consider the differential $d_{\bar{\tau}}([v])$ - see (1.3.3). The differential is surjective by Proposition 2.9 of [15], thus $X_A$ is smooth at $q$ by Proposition 1.1.12. \hfill \Box

**Proposition 3.4.** Let $A \in (\mathbb{LG}(\Lambda^3 V) \setminus \mathbb{N}(V))$. Then $X_A$ is smooth if and only if $A \in \mathbb{LG}(\Lambda^3 V)^0$.

**Proof.** If $A \in \mathbb{LG}(\Lambda^3 V)^0$ then $X_A$ is smooth by [12]. Suppose that $X_A$ is smooth. Then $A \notin \Delta$ by Claim 1.1.3. Assume that $A \in \Sigma$; we will reach a contradiction. Let $W \in \Theta_A$ and $[v] \in \mathbb{P}(W)$ - notice that $\mathbb{P}(W) \subseteq Y_A$. Let $q \in f_A^{-1}([v])$. Since $A \notin \Delta$ we have $1 \leq \dim(A \cap F_v) \leq 2$. Suppose that $\dim(A \cap F_v) = 1$. Then $Y_A$ is singular at $[v]$ by Corollary 2.5 of [15], thus $X_A$ is singular at $q$ by Proposition 1.1.5. Suppose that $\dim(A \cap F_v) = 2$. Let $\bar{\tau}$ be as in the proof of Lemma 3.3. Then $d_{\bar{\tau}}([v])$ is not surjective - see Proposition 2.3 of [15] - and hence $X_A$ is singular at $q$ by Proposition 1.1.12. \hfill \Box

### 3.2 The desingularization

**Definition 3.5.** Let $\mathbb{LG}(\Lambda^3 V)^* \subseteq \mathbb{LG}(\Lambda^3 V)$ be the set of $A$ such that the following hold:

1. $A \notin \mathbb{N}(V)$.

**Remark 3.6.** $\mathbb{LG}(\Lambda^3 V)^*$ is an open subset of $\mathbb{LG}(\Lambda^3 V)$.

**Claim 3.7.** $(\mathbb{LG}(\Lambda^3 V) \setminus \Sigma) \subseteq \mathbb{LG}(\Lambda^3 V)^*$.

**Proof.** Item (1) of Definition 3.5 holds by Claim 2.11 of [15]. Let’s prove that Item (2) of Definition 3.5 holds. Suppose that $Y_A[3] \neq Y_A(3)$ i.e. there exists $[v_0] \in \mathbb{P}(V)$ such that $\dim(A \cap F_{v_0}) \geq 4$. Let $V_0 \subseteq V$ be a codimension-1 subspace transversal to $[v_0]$ and let $\rho_{v_0}^{V_0}$ be as in (2.2.17). Let $K := \mathbb{P}(\rho_{v_0}^{V_0}(A \cap F_{v_0}))$. Then $\dim K \geq 3$; since $Gr(2, V_0)$ has codimension $3$ in $\mathbb{P}(\Lambda^2 V_0)$ it follows that there exists $[\alpha] \in K \cap Gr(2, V_0)$. Let $\tilde{\alpha} \in (A \cap F_{v_0})$ such that $\rho_{v_0}^{V_0}(\tilde{\alpha}) = \alpha$. Then $\tilde{\alpha}$ is non-zero and decomposable, that is a contradiction because $A \notin \Sigma$. Lastly let’s prove that Item (3) of Definition 3.5 holds. Let $[v_0] \in Y_A[3] = Y_A(3)$. Then $(A, [v_0]) \in \bar{\Delta}(0)$. Let $K := A \cap F_{v_0}$ and $\tau_K^{v_0}$ be as in (2.2.11). We have $T[v_0]Y_A[3] = T[v_0]Y_A(3) = \ker \tau_K^{v_0}$.

By Proposition 2.5 the map $\tau_K^{v_0}$ is injective. Thus $[v_0]$ is an isolated point of $Y_A[3]$. \hfill \Box

Let $A \in \mathbb{LG}(\Lambda^3 V)^*$. Let $U \subseteq \mathbb{LG}(\Lambda^3 V)^*$ be a small open (either in the Zariski or in the classical topology) subset containing $A$. In particular $\rho_U : X_U \to \mathcal{X}_U$ exists. Let $\pi_U : \tilde{X}_U \to X_U$ be the blow-up of $W_U$ and $E_U$ be the exceptional set of $\pi_U$.

**Claim 3.8.** Keep notation as above. Then $\tilde{X}_U$ is smooth. If $U$ is open and sufficiently small in the classical topology then we have a locally-trivial fibration

$$E_U \longrightarrow Y_U[3].$$

(3.2.1)

Let $(A, [v]) \in Y_U[3]$. The fiber of (3.2.1) over $(A, [v])$ is isomorphic to $\mathbb{P}(A \cap F_v)^V \times \mathbb{P}(A \cap F_v)^V$ and the restriction of $N_{E_U/X_U}$ to the fiber is isomorphic to $\mc{O}_{\mathbb{P}(A \cap F_v)^V}(-1) \boxtimes \mc{O}_{\mathbb{P}(A \cap F_v)^V}(-1)$. 

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Proof. By Proposition 3.2 we know that \( \widetilde{X}_U \) is smooth outside \( E_U \). It remains to examine \( \widetilde{X}_U \) over \( W_U \cong Y_U[3] \). We may assume that \( U = U_B \times N \) is as in the proof of Proposition 3.2. We will adopt the notation of that proof. Let \( q \in X \), and \( f_\nu(q) = (A, [v]) = p \). A neighborhood of \( q \) in \( X_B \) is isomorphic to \( X_1 \) where \( \gamma \) is given by (2.2.5) - see (3.1.4). We are assuming that \( q \in W_U \) and hence \( \text{cork} \, (p) = 3 \). Let \( f: X(V) \to Y(V) \) be as in Subsection 1.3 i.e. \( f \) is the universal double covering of corank 3 at the origin. We claim that there exists a map \( \nu: X_\gamma \to X(V) \) such that the following diagram commutes

\[
\begin{array}{c}
X_\gamma & \overset{\nu}{\longrightarrow} & X(V) \\
f_\gamma \downarrow & & \downarrow f \\
Y_\gamma & \overset{\mu}{\longrightarrow} & Y(V)
\end{array}
\]

and \( X_\gamma \) is identified with the fibered product \( Y_\gamma \times_{Y(V)} X(V) \). In fact it suffices to apply the reduction procedure of Subsection 1.1 that leads to Claim 1.4. Let \( K \) be as in Claim 1.4: by (1.1.29) we have \( (Y_\nu, p) = (Y_\gamma, p) \) and by Claim 1.4 we have a natural isomorphism \( (X_\gamma, f_\gamma^{-1}(p)) \to (X_\nu, f_\nu^{-1}(p)) \) commuting with \( f_\gamma \) and \( f_\nu \). Let \( U = \text{Spec} \, R \), we are free to replace \( U \) by any affine open subset containing \((A, [v])\). Thus we may assume that \( K \) is a trivial \( R \)-module i.e. \( K = V \otimes R \) where \( V \) is a complex 3-dimensional vector-space. Hence we may view \( \gamma_K \) as a map \( \gamma_K: \text{Spec} \, R \to \text{Sym}^3 V^\vee \). Notice that we have equality of schemes \( Y_\gamma = \gamma_K^{-1}(Y(V)) \); thus the restriction of \( \gamma_K \) to \( Y_\gamma \) defines a map \( \mu: Y_\gamma \to Y(V) \). The claim follows. By surjectivity of \( \delta_i(A, [v]) \) - see (3.1.6) - we get that the germ \((X_\gamma, f_\gamma^{-1}(p))\) is the product of a smooth germ (of dimension 54) and the germ \((X(V), f^{-1}(0))\). For each the explicit description of \( X(V) \) given by Proposition 1.14 we get right away that \( \widetilde{X}_U \) is smooth over \( q \) and the remaining statements as well. We need to assume that \( U \) is a small open subset in the classical topology in order to ensure that \( \text{Map} \, (3.2.1) \) is a locally-trivial fibration.

\[ \Box \]

Remark 3.9. Let \( A \in \text{Lg}(\wedge^3 V)^* \) and let \( Y_A[3] = \{ [v_1], \ldots, [v_s] \} \). Let \( U \subset \text{Lg}(\wedge^3 V)^* \) be a small open (in the classical topology) subset containing \( A \). For each \( 1 \leq i \leq s \) choose a projection

\[
E_\nu([v_i]) \longrightarrow \mathbb{P}(A \cap F_s)^V.
\]

There exists a unique \( \mathbb{P}^2 \)-fibration

\[
\epsilon: E_\nu \longrightarrow * \tag{3.2.3}
\]

where \( * \) is itself a fibration over \( Y_U[3] \) with fiber \( \mathbb{P}(A \cap F_s)^V \) over \( (A, [v]) \). We say that (3.2.3) is a choice of \( \mathbb{P}^2 \)-fibration \( \epsilon \) for \( X_A \).

Let \( A \in \text{Lg}(\wedge^3 V)^* \) and choose a \( \mathbb{P}^2 \)-fibration \( \epsilon \) for \( X_A \). Let \( U \subset \text{Lg}(\wedge^3 V)^* \) be a small open (in the classical topology) subset containing \( A \). By Claim 3.8 the normal bundle of \( E_U \) along the fibers of (3.2.4) is \( \mathcal{O}_{\mathbb{P}^2}(-1) \). Thus there exists a contraction \( \nu: \widetilde{X}_U \to X_U \) in the category of complex manifolds fitting into a commutative diagram

\[
\begin{array}{ccc}
\widetilde{X}_U & \overset{\nu}{\longrightarrow} & X_U \\
\varepsilon_U \downarrow & & \downarrow \varepsilon_U \\
X_U & \overset{\rho_U}{\longrightarrow} & \mathcal{O}_U
\end{array}
\]

Let \( f_U' = f_U \circ \rho_U: X_U \to Y_U \) and \( \rho_U: X_U \to U \) be the map \( f_U \) followed by \( Y_U \to U \). Let

\[
X_A := (\rho_U)^{-1}(A), \quad g_A := g_U|_{X_A}, \quad f_A := f_U'|_{X_A}, \quad \mathcal{O}_{X_A}(1) := (f_A)^* \mathcal{O}_{Y_A}(1), \quad H_A \in |\mathcal{O}_{X_A}(1)|.
\]

Our notation does not make any reference to \( U \) because the isomorphism class of the polarized couple \((X_A, \mathcal{O}_{X_A}(1))\) does not depend on the open set \( U \) containing \( A \). Notice that if \( A \in \Delta \) then \( \mathcal{O}_{X_A}(1) \) is not ample, in fact it is trivial on \( s \) copies of \( \mathbb{P}^2 \) where \( s = |Y_A[3]| \). Of course

\[
(X_A, \mathcal{O}_{X_A}(1)) \cong (X_A, \mathcal{O}_{X_A}(1)) \text{ if } A \in (\text{Lg}(\wedge V) \setminus \Delta). \tag{3.2.6}
\]
Proposition 3.10. Let $A \in \mathbb{L}G(\Lambda^3 V)^*$ and let $\epsilon$ be a choice of $\mathbb{P}^2$-fibration for $X_A$.

1) $X'_A$ is smooth away from $(f_A')^{-1}(\bigcup_{W \in \Theta_A} \mathbb{P}(W))$.

2) If $[v_i] \in Y_A[3]$ then $(f_A')^{-1}[v_i] \cong \mathbb{P}(A \cap F_{v_i})^\vee$.

3) If $\epsilon'$ is another choice of $\mathbb{P}^2$-fibration for $X_A$ there exists a commutative diagram

$$
\begin{array}{ccc}
X'_A & \longrightarrow & X''_A \\
\downarrow & & \downarrow \\
Y_A & \rightarrow & Y_A
\end{array}
$$

where the birational map is the flop of a collection of $(f_A')^{-1}[v_i]$'s. Conversely every flop of a collection of $(f_A')^{-1}[v_i]$'s is isomorphic to one $X_A'$.

Proof. Let’s prove Item (1). $X'_A$ is smooth away from $(f_A')^{-1}(Y_A[3] \cup \bigcup_{W \in \Theta_A} \mathbb{P}(W))$ by Lemma 3.3. It remains to prove that $X_A'$ is smooth at every point of $(f_A')^{-1}([v_1], \ldots, [v_s])$ where

$$
\{[v_1], \ldots, [v_s]\} = Y_A[3] \setminus \bigcup_{W \in \Theta_A} \mathbb{P}(W).
$$

Let $\mathcal{U} \subset \mathbb{L}G(\Lambda^3 V)^*$ be a small open (in the classical topology) subset containing $A$. Let $\tilde{\rho}_U := \rho_U \circ \tau_U$; thus $\tilde{\rho}_U : \tilde{X}_U \rightarrow \mathcal{U}$. For $1 \leq i \leq s$ the fiber over $(A, [v_i])$ of Fibration (3.2.1) is canonically isomorphic to $\mathbb{P}(A \cap F_{v_i})^\vee \times \mathbb{P}(A \cap F_{v_i})^\vee$. Let $\tilde{X}_A \subset \tilde{X}_U$ be the strict transform of $X_A$. Abusing notation we write

$$
\tilde{\rho}_U^{-1}(A) = \tilde{X}_A \cup \bigcup_{i=1}^s \mathbb{P}(A \cap F_{v_i})^\vee \times \mathbb{P}(A \cap F_{v_i})^\vee.
$$

(Of course $\mathbb{P}(A \cap F_{v_i})^\vee \times \mathbb{P}(A \cap F_{v_i})^\vee$ denotes the fiber over $(A, [v_i])$ of Fibration (3.2.1).) The components $\mathbb{P}(A \cap F_{v_i})^\vee \times \mathbb{P}(A \cap F_{v_i})^\vee$ are pairwise disjoint. We claim that for $i = 1, \ldots, s$ the intersection

$$
E_{A,i} := \tilde{X}_A \cap (\mathbb{P}(A \cap F_{v_i})^\vee \times \mathbb{P}(A \cap F_{v_i})^\vee)
$$

is a smooth symmetric divisor in the linear system $|\mathcal{O}_{\mathbb{P}(A \cap F_{v_i})^\vee}(1) \boxtimes \mathcal{O}_{\mathbb{P}(A \cap F_{v_i})^\vee}(1)|$. In order to prove this we go back to Map (1.3.15) - recall that $V$ is a 3-dimensional complex vector space. Pull-back by $\sigma$ defines an isomorphism

$$
\text{Sym}^2 V^\vee \xrightarrow{\sigma^*} (V^\vee \otimes V^\vee)^{\mathbb{Z}/2} := \text{Sym}_2 V^\vee
$$

which is $\text{Gl}(V)$-equivariant. Isomorphism $\sigma^*$ induces a PGL($V$)-equivariant isomorphism of projective spaces $\mathbb{P}(\text{Sym}^2 V^\vee) \xrightarrow{\tilde{\rho}^{-1}} \mathbb{P}(\text{Sym}_2 V^\vee)$. Of course $p$ maps a point in the unique open PGL($V$)-orbit of $\mathbb{P}(\text{Sym}^2 V^\vee)$ to a point in the unique open PGL($V$)-orbit of $\mathbb{P}(\text{Sym}_2 V^\vee)$. Now let $V = (A \cap F_{v_i})^\vee$. Let $K_i := (A \cap F_{v_i})$ and $\tau_{iK}^v$ be as in (2.2.11). By Proposition 2.5 we have that $\text{im}(\tau_{iK}^v)$ belongs to the unique open PGL($K_i$)-orbit of $\mathbb{P}(\text{Sym}^2 (A \cap F_{v_i}))$. Commutative Diagram (1.3.16) gives that $E_{A,i}$ is a symmetric smooth divisor in $|\mathcal{O}_{\mathbb{P}(A \cap F_{v_i})^\vee}(1) \boxtimes \mathcal{O}_{\mathbb{P}(A \cap F_{v_i})^\vee}(1)|$.

Thus we have described $\tilde{\rho}_U^{-1}(A)$. Since $X'_U$ is obtained from $\tilde{X}_U$ by contracting $E_U$ along the $\mathbb{P}^2$-fibration $\epsilon$ it follows that $X'_A$ is smooth at every point of $(f_A')^{-1}([v_1], \ldots, [v_s])$. This proves Item (1). Since $X'_A$ is obtained from $\tilde{X}_A$ by contracting each of the divisors $E_{A,i}$ along the fibration $\mathbb{P}^1 \rightarrow E_{A,i} \rightarrow (A \cap F_{v_i})^\vee$ determined by $\epsilon$ (and similarly for $\epsilon'$) we also get Items (2) and (3).

Corollary 3.11. Let $A \in (\mathbb{L}G(\Lambda^3 V) \setminus \Sigma)$. Then $g'_A: X'_A \rightarrow X_A$ is a desingularization for every choice of $\mathbb{P}^2$-fibration $\epsilon$ for $X_A$.

Proof. By Claim 3.7 we know that $A \in \mathbb{L}G(\Lambda^3 V)^*$: thus Proposition 3.10 applies to $X_A'$. Since $A \notin \Sigma$ we get that $X_A'$ is smooth by Item (1) of Proposition 3.10.

Corollary 3.12. Let $A, A' \in (\mathbb{L}G(\Lambda^3 V) \setminus \Sigma)$ and $\epsilon, \epsilon'$ be choices of $\mathbb{P}^2$-fibration for $X_A$. The quasi-polarized 4-folds $(X_A', H_A')$ and $(X_{A'}, H_{A'})$ are deformation equivalent.
4 Double EPW-sextics parametrized by $\Delta$

Let $A \in \Delta$ and $[v_0] \in Y_A(3)$. In the first subsection we will associate to $(A, [v_0])$ (under some hypotheses which are certainly satisfied if $A \notin \Sigma$) a K3 surface $S_A(v_0)$ of genus 6, meaning that it comes equipped with a big and nef divisor class $D_A(v_0)$ of square 10. We will also prove a converse: given a generic such pseudo-polarized K3 surface $S$ there exist $A \in \Delta$ and $[v_0] \in Y_A(3)$ such that the pseudo-polarized surfaces $S$ and $S_A(v_0)$ are isomorphic. In the second subsection we will assume that $A \in (\Delta \setminus \Sigma)$ - with this hypothesis $D_A(v_0)$ is very ample. We will prove that there exists a birational map $\psi: S_A(v_0) \dasharrow X_A^\epsilon$ where $\epsilon$ is an arbitrary choice of $\mathbb{P}^2$-fibration for $X_A$. That such a map exists for generic $A \in \Delta$ could be proved by invoking the results of [14]. Here we will present a direct proof (we will not appeal to [14] nor to [12]). Moreover we will prove that if $S_A(v_0)$ contains no lines (this will be the case for generic $A$) then there exists a choice of $\epsilon$ for which $\psi$ is regular - in particular $X_A^\epsilon$ is projective for such $\epsilon$. Lastly we will notice that the above results show that a smooth double cover of an EPW-sextic is a deformation of the Hilbert square of a K3 (and that the family of double EPW-sextics is a locally versal family of projective Hyperkähler manifolds): the proof is more direct than the proof of [12].

4.1 EPW-sextics and K3 surfaces

**Assumption 4.1.** $A \in \mathbb{L}\mathbb{G}(\Lambda^3 V)$, $[v_0] \in Y_A(3)$ and the following hold:

(a) There exists a codimension-1 subspace $V_0 \subset V$ such that $\Lambda^3 V_0 \cap A \neq \{0\}$.
(b) There exists at most one $W \in \Theta_A$ containing $v_0$.
(c) If $W \in \Theta_A$ contains $v_0$ then $A \cap (\Lambda^2 W \wedge V) = \Lambda^3 W$.

**Remark 4.2.** Let $A \in (\Delta \setminus \Sigma)$. Let $[v_0] \in Y_A(3) (= Y_A[3]$ by **Claim 3.7**). Then **Assumption 4.1** holds. In fact Items (b) and (c) hold trivially while Item (a) holds by Claim 2.11 and Equation (2.81) of [15].

Let $(A, [v_0])$ be as in **Assumption 4.1**: we will define a surface $S_A(v_0)$ of genus 6. The condition that $\Lambda^3 V_0$ is transverse to $A$ is open: thus we may assume that we have a direct-sum decomposition

$$V = [v_0] \oplus V_0. \quad (4.1.1)$$

We will denote by $D$ be the direct-sum decomposition of $V$ appearing in (4.1.1). Let

$$K^D_A := \rho_{v_0}^0(A \cap F_{v_0}). \quad (4.1.2)$$

where $\rho_{v_0}^0$ is given by (2.2.17). Choose a volume-form on $V_0$. Wedge-product followed by the volume-form defines an isomorphism $\Lambda^3 V_0 \cong \Lambda^2 V_0^\vee$ and hence it makes sense to let

$$F_A^D := \mathbb{P}(\text{Ann } K^D_A) \cap \text{Gr}(3, V_0). \quad (4.1.3)$$

By **Proposition 5.2** and **Proposition 5.3** (see the Appendix) we know that $F_A^D$ is a Fano 3-fold with at most one singular point. Next we will define a quadratic form on Ann $K^D_A$. By Item (a) of **Assumption 4.1** the subspace $A$ is the graph of a map $q^D_A: \Lambda^2 V_0 \to \Lambda^3 V_0$: explicitly

$$q^D_A(\alpha) = \beta \iff (v_0 \wedge \alpha + \beta) \in A. \quad (4.1.4)$$

The map $q^D_A$ is symmetric because $A, \Lambda^2 V_0$ and $\Lambda^3 V_0$ are lagrangian subspaces of $\Lambda^3 V$. Clearly $\ker q^D_A = K^D_A$: it follows that $q^D_A$ induces an isomorphism

$$\tilde{r}^D_A: \Lambda^2 V_0/K^D_A \sim \text{Ann } K^D_A \subset \Lambda^3 V_0. \quad (4.1.5)$$

The inverse $(\tilde{r}^D_A)^{-1}$ defines a non-degenerate quadratic form $(r^D_A)^\vee$ on Ann $K^D_A$. For future reference we unwind the definition of $(\tilde{r}^D_A)^{-1}$ and $(r^D_A)^\vee$. Let $\beta \in \text{Ann } K^D_A$ i.e.

$$v_0 \wedge \alpha + \beta \in A, \quad \alpha \in \Lambda^2 V_0. \quad (4.1.6)$$
Then
\[(r_A^P)^{-1}(\beta) \equiv \alpha \quad \text{(mod } K_A^P), \quad (r_A^P)^\vee(\beta) = \text{vol}(v_0 \land \alpha \land \beta).\]  
\hspace{1cm} (4.1.7)
Let \(V((r_A^P)^\vee) \subset \mathbb{P}(\text{Ann } K_A^P)\) be the zero-scheme of \((r_A^P)^\vee\): a smooth 5-dimensional quadric. Let
\[S_A^P := V((r_A^P)^\vee) \cap F_A^P.\]  
\hspace{1cm} (4.1.8)
Our first goal is to show that \(S_A^P\) does not depend on the choice of the subspace \(V_0 \subset V\) complementary to \([v_0] \in Y_A(3)\), i.e., it depends only on \(A\) and \([v_0] \in Y_A(3)\). First we notice that \(F_A^P\) is independent of \(V_0\). In fact \(\Lambda^3 V_0\) is transversal to \(F_{v_0}\); since both \(\Lambda^3 V_0\) and \(F_{v_0}\) are Lagrangians the volume \(\text{vol}\) induces an isomorphism
\[g_{v_0}: \bigwedge^3 V_0 \xrightarrow{\sim} F_{v_0}^V.\]  
\hspace{1cm} (4.1.9)
Thus \(g_{v_0}\) defines an inclusion
\[F_A^P \hookrightarrow \mathbb{P}(\text{Ann } K_A)\].  
\hspace{1cm} (4.1.10)
Remark 4.3. The image of Map \((4.1.10)\) does not depend on \(V_0\) i.e. it depends exclusively on \(A\) and \([v_0] \in Y_A(3)\); we will denote it by \(Z_A(v_0)\).

Similarly \(g_{v_0}\) defines an inclusion
\[g_{v_0}: S_A^P \hookrightarrow \mathbb{P}(\text{Ann } K_A).\]  
\hspace{1cm} (4.1.11)
Lemma 4.4. Keep notation and assumptions as above. Then \(g_{v_0}(S_A^P)\) is independent of \(V_0\), in other words it depends exclusively on \(A\) and \([v_0] \in Y_A(3)\).

Proof. Let \(V'_0 \subset V\) be a codimension-1 subspace complementary to \([v_0]\) and transverse to \(A\). Let \(D'\) denote the corresponding direct-sum decomposition of \(V\); we must show that
\[g_{v_0}(S_A^P) = g_{v'_0}(S_{A}^{D'}).\]  
\hspace{1cm} (4.1.12)
The subspace \(V'_0\) is the graph of a linear function
\[V_0 \twoheadrightarrow [v_0] \quad v \mapsto f(v)v_0\]  
\hspace{1cm} (4.1.13)
and hence we have an isomorphism
\[V_0 \xrightarrow{\psi} V'_0 \quad v \mapsto v + f(v)v_0.\]  
\hspace{1cm} (4.1.14)
We notice that
\[\bigwedge^3 \psi(\beta) = \beta + v_0 \land (f \cdot \beta)\]  
\hspace{1cm} (4.1.15)
where \(\cdot\) denotes contraction. In particular \(g_{v_0} \circ \Lambda^3 \psi = g_{v_0}\). Moreover \(\phi := \Lambda^3 \psi|_{\text{Ann } K_A^P}\) is an isomorphism between \(\text{Ann } K_A^P \subset \Lambda^3 V_0\) and \(\text{Ann } K_{A}^{D'} \subset \Lambda^3 V'_0\). Thus it suffices to prove that
\[\phi(S_A^P) = S_{A}^{D'}.\]  
\hspace{1cm} (4.1.16)
We claim that
\[\phi^*(r_{A}^{D'})^\vee - (r_{A}^P)^\vee \in H^0(I_{F_A}(2)).\]  
\hspace{1cm} (4.1.17)
In fact let \(\beta \in \text{Ann } K_A^P \subset \Lambda^3 V_0\); then \((4.1.6)\) holds. By \((4.1.15)\) we get that
\[v_0 \land (\alpha - (f \cdot \beta)) + \phi(\beta) = v_0 \land \alpha + \beta \in A.\]  
\hspace{1cm} (4.1.18)
By \((4.1.15)\) we get that
\[\phi^*(r_{A}^P)^\vee(\beta) = \text{vol}(v_0 \land (\alpha - (f \cdot \beta)) \land \phi(\beta)) = \text{vol}(v_0 \land \alpha \land \phi(\beta)) - \text{vol}(v_0 \land (f \cdot \beta) \land \phi(\beta)) = \text{vol}(v_0 \land \alpha \land \beta) - \text{vol}(v_0 \land (f \cdot \beta) \land \beta) = (r_{A}^P)^\vee(\beta) - \text{vol}(v_0 \land (f \cdot \beta) \land \beta).\]  
\hspace{1cm} (4.1.19)
The second term in the last expression is the restriction to \(\mathbb{P}(\text{Ann } K_A^P)\) of a Plücker quadratic form and hence it vanishes on \(F_A^P\). This proves \((4.1.17)\) and hence \((4.1.16)\) holds. \(\square\)
By the above lemma we may give the following definition.

**Definition 4.5.** Let $A \in LG(\Lambda^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that **Assumption 4.1** holds. Let $\mathcal{D}$ be the direct-sum decomposition (4.1.1). We set

$$S_A(v_0) := g_{v_0}(S_A^\mathcal{D}). \quad (4.1.20)$$

Keep assumptions and notation as above. We single out special points of $S_A(v_0)$ as follows. Suppose that $W \in \Theta_A$ (see (2.2.15) for the definition of $\Theta_A$) and assume that $v_0 \notin W$. Let $\gamma$ be a generator of $\Lambda^3 W$ i.e. $\gamma$ is decomposable with $\text{supp}(\gamma) = W$. By hypothesis $\Lambda^3 V_0 \cap A = \{0\}$ and hence $W \not\subset V_0$; thus

$$\gamma = (v_0 + u_1) \cup u_2 \cup u_3, \quad u_i \in V_0. \quad (4.1.21)$$

Since $v_0 \notin W$ we have $u_1 \cup u_2 \cup u_3 \neq 0$; thus $[u_1 \cup u_2 \cup u_3] \in F_A^\mathcal{D}$. Moreover $[u_1 \cup u_2 \cup u_3] \in V((r_A^\mathcal{D})^\vee)$ by (4.1.7) and hence $[u_1 \cup u_2 \cup u_3] \in S_A^\mathcal{D}$. We let

$$\Theta_A \setminus \{W \mid v_0 \in W\} \xrightarrow{\theta_A^\mathcal{D}} S_A^\mathcal{D} \quad (4.1.22)$$

The map

$$\theta_A(v_0) := g_{v_0} \circ \theta_A^\mathcal{D} : (\Theta_A \setminus \{W \mid v_0 \in W\}) \to S_A(v_0) \quad (4.1.23)$$

is independent of $\mathcal{D}$, i.e. it depends exclusively on $A$ and $[v_0]$. Notice that $\theta_A(v_0)$ is injective.

**Proposition 4.6.** Let $A \in LG(\Lambda^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that **Assumption 4.1** holds. Let $\mathcal{D}$ be the direct-sum decomposition (4.1.1). The set of points at which the intersection $V((r_A^\mathcal{D})^\vee) \cap F_A^\mathcal{D}$ is not transverse is equal to

$$\text{im} \theta_A^\mathcal{D} \prod (S_A^\mathcal{D} \cap \text{sing} F_A^\mathcal{D}). \quad (4.1.24)$$

**Proof.** Let $[\beta] \in S_A^\mathcal{D}$. In particular $\beta$ is non-zero decomposable; let $U := \text{supp} \beta$. Moreover since $[\beta] \in F_A^\mathcal{D}$ we have that (4.1.6) holds; let $\alpha \in \Lambda^2 V_0$ be as in (4.1.6). We claim that

$$V((r_A^\mathcal{D})^\vee) \cap F_A^\mathcal{D} \subset [\beta] \text{ unless } \langle \alpha, K_A^\mathcal{D} \rangle \cap \bigwedge^2 U \neq \emptyset. \quad (4.1.25)$$

In fact the projective tangent space to $\text{Gr}(3, V_0)$ at $[\beta]$ is given by

$$T_{[\beta]}\text{Gr}(3, V_0) = \mathbb{P}(\text{Ann}(\bigwedge^2 U)). \quad (4.1.26)$$

On the other hand (4.1.7) gives that

$$T_{[\beta]}V((r_A^\mathcal{D})^\vee) = \mathbb{P}(\text{Ann} \alpha) \cap \mathbb{P}(\text{Ann} K_A^\mathcal{D}). \quad (4.1.27)$$

Statement (4.1.25) follows at once from (4.1.26) and (4.1.27). Next we prove that

$$\langle \alpha, K_A^\mathcal{D} \rangle \cap \bigwedge^2 U \neq \emptyset \text{ if and only if } [\beta] \in \text{sing} F_A^\mathcal{D} \text{ or } [\beta] \in \text{im} \theta_A^\mathcal{D}. \quad (4.1.28)$$

Suppose that $[\beta] \in \text{sing} F_A^\mathcal{D}$; then Item (1) of **Proposition 5.3** gives that $K_A^\mathcal{D} \cap \bigwedge^2 U \neq \emptyset$. Next suppose that $[\beta] \in \text{im} \theta_A^\mathcal{D}$; then $\alpha \in \bigwedge^2 U$ by (4.1.21). This proves the “if” implication of (4.1.28). Let us prove the “only if” implication. First assume that $K_A^\mathcal{D} \cap \bigwedge^2 U \neq \{0\}$. Let $0 \neq \kappa_0 \in K_A^\mathcal{D} \cap \bigwedge^2 U$. Then $\kappa_0$ is decomposable because $\dim U = 3$ and hence $[\kappa_0]$ is the unique point belonging to $\mathbb{P}(K_A^\mathcal{D}) \cap \text{Gr}(2, V_0)$. We get that $[\beta]$ is the unique singular point of $F_A^\mathcal{D}$ by (5.0.8). Lastly assume that $K_A^\mathcal{D} \cap \bigwedge^2 U = \{0\}$. Then there exists $\kappa \in K_A^\mathcal{D}$ such that $(\alpha + \kappa) \in \bigwedge^2 U$. Since $\kappa \in K_A^\mathcal{D}$ we have $(v_0 \cup (\alpha + \kappa) + \beta) \in A$. The tensor $(v_0 \cup (\alpha + \kappa) + \beta) \in A$ is decomposable, let $W$ be its support. Then $v_0 \notin W$ because $\beta \neq 0$ and hence $[\beta] = \theta_A^\mathcal{D}(W)$. This finishes the proof of (4.1.28) and of the proposition. \[\square\]
Corollary 4.7. Let $A \in LG(\wedge^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that Assumption 4.1 holds. Assume in addition that $\Theta_A$ is finite. Then $S_A(v_0)$ is a reduced and irreducible surface with
\[\operatorname{sing} S_A(v_0) = \operatorname{im} \theta_A(v_0) \bigcap (S_A(v_0) \cap \operatorname{sing} Z_A(v_0)).\] (4.1.29)

(See Remark 4.3 for the definition of $Z_A(v_0)$.)

Proof. By Proposition 4.6 we know that $S_A^P$ is a smooth surface outside the right-hand side of (4.1.29). By hypothesis $\Theta_A$ is finite and hence the right-hand side of (4.1.29) is finite. On the other hand by Proposition 5.3 we know that $Z_A(v_0)$ is a 3-fold with at most one singular point, necessarily an ordinary quadratic singularity, and $S_A^P$ is the complete intersection of $Z_A(v_0)$ and a quadric hypersurface. It follows that $S_A^P$ is reduced and irreducible with singular set as claimed. □

Corollary 4.8. Let hypotheses be as in Corollary 4.7. Suppose in addition that $S_A(v_0)$ has Du Val singularities. Let $\hat{S}_A(v_0) \to S_A(v_0)$ be the minimal desingularization. Then $\hat{S}_A(v_0)$ is a $K3$ surface.

Proof. Let $\mathcal{O}_{Z_A(v_0)}(1)$ be the pull-back by Map (4.1.10) of the hyperplane line-bundle on $\mathbb{P}((\operatorname{Ann}(F_v \cap A))$. Then $S_A(v_0) \in |\mathcal{O}_{Z_A(v_0)}(2)|$. By Proposition 5.2 and Proposition 5.3 there exist smooth divisors in $|\mathcal{O}_{Z_A(v_0)}(2)|$ and they are $K3$ surfaces; by simultaneous resolution of Du Val singularities we get that $\hat{S}_A(v_0)$ is a $K3$ surface. □

Corollary 4.9. Let $A \in (\Delta \setminus \Sigma)$. Let $[v_0] \in Y_A(3)$ (and hence Assumption 4.1 holds by Remark 4.2). Then $S_A(v_0)$ is a (smooth) $K3$.

Proof. Immediate consequence of Corollary 4.8. □

Under the hypotheses of Corollary 4.8 let $\mathcal{O}_{S_A(v_0)}(1)$ be the restriction to $S_A(v_0)$ of $\mathcal{O}_{Z_A(v_0)}(1)$. Let $\mathcal{O}_{\hat{S}_A(v_0)}(1)$ be the pull-back of $\mathcal{O}_{S_A(v_0)}(1)$ to $\hat{S}_A(v_0)$. We set
\[D_A(v_0) \in |\mathcal{O}_{S_A(v_0)(1)}|, \quad \hat{D}_A(v_0) \in |\mathcal{O}_{\hat{S}_A(v_0)}(1)|.\] (4.1.30)

Remark 4.10. Let hypotheses be as in Corollary 4.8. Then $(\hat{S}_A(v_0), \hat{D}_A(v_0))$ is a quasi-polarized $K3$ surface of genus 6. Moreover the composition
\[\hat{S}_A(v_0) \to S_A(v_0) \to \mathbb{P}(\operatorname{Ann}(F_v \cap A))\] (4.1.31)

is identified (up to projectivities) with the map associated to the complete linear system $|\hat{D}_A(v_0)|$.

Remark 4.10 has a converse; in order to formulate it we identify $F_v \cong \wedge^2(V/[v_0])$ (the identification is well-defined up to homothety).

Assumption 4.11. $K \in \operatorname{Gr}(3, F_v)$ and

1. $\mathbb{P}(K) \cap \operatorname{Gr}(2, V/[v_0]) = \emptyset$, or
2. the scheme-theoretic intersection $\mathbb{P}(K) \cap \operatorname{Gr}(2, V/[v_0])$ is a single reduced point.

Let
\[W_K := \mathbb{P}(\operatorname{Ann}(K) \cap \operatorname{Gr}(3, V/[v_0])).\] (4.1.32)

(This makes sense because we have an isomorphism $\wedge^2(V/[v_0]) \to \wedge^3(V/[v_0])$ well-defined up to homothety.) Let
\[S := W_K \cap Q, \quad Q \subset \mathbb{P}(\operatorname{Ann}(K)) \text{ a quadric.}\] (4.1.33)

If $Q$ is generic then $S$ is a linearly normal $K3$ surface of genus 6, see Corollary 4.8. In fact the family of such $K3$ surfaces is locally versal. More generally suppose that Assumption 4.11 holds, that $S$ is given by (4.1.33) and that $S$ has DuVal singularities. Let $\hat{S} \to S$ be the minimal desingularization - thus $\hat{S}$ is a $K3$ surface. Let $D \in |\mathcal{O}_S(1)|$ and $\hat{D}$ be the pull-back of $D$ to $\hat{S}$.

Consider the family $S \to B$ of deformations of $(S, D)$ obtained by deforming slightly $K$ and $Q$; by
If we replace the quadric $Q\Delta(0)$ (with the extra hypotheses in Assumption 4.1 Proposition 4.14. codimension-$\hat{\mathcal{S}}$ admits a simultaneous resolution of singularities $\hat{\mathcal{S}} \to \hat{\mathcal{B}}$ with fiber $\mathcal{S}$ over the point corresponding to $\mathcal{S}$. Of course there is a divisor class $\hat{\mathcal{D}}$ on $\hat{\mathcal{S}}$ whose restriction to $\mathcal{S}$ is $\mathcal{D}$ - thus $\hat{\mathcal{S}} \to \hat{\mathcal{B}}$ is a family of quasi-polarized $K3$ surfaces. The following result is well-known - we omit the (standard) proof.

**Proposition 4.12.** Keep notation and hypotheses as above. The family $\mathcal{S}\to \mathcal{B}$ is a versal family of quasi-polarized $K3$ surfaces.

**Lemma 4.13.** Suppose that Assumption 4.11 holds. Let $S$ be as in (4.1.33) and assume that $Q$ is transversal to $W_K$ outside a finite set - thus $S$ is a surface with finite singular set. There exists a smooth quadric $Q'\subset \mathbb{P}(Ann\ K)$ such that $S=W_K\cap Q'$.

**Proof.** Since $W_K$ is cut out by quadrics Bertini’s Theorem gives that the generic quadric in $\mathbb{P}(Ann\ K)$ containing $S$ is smooth outside sing $S$; let $Q_0=V(P_0)$ be such a quadric. Let $p\in$ sing $S$. The generic quadric $Q'=V(P')\in |I_{W_K}(2)|$ is smooth at $p$ and hence $V(P_0+P')$ is smooth at $p$. Since sing $S$ is finite we get that the generic quadric $Q$ containing $S$ is smooth at all points of sing $S$. It follows that the generic quadric $Q$ containing $S$ is smooth. $\square$

The following corollary provides an inverse of the process which produces $S_A(v_0)$ out of $(A, [v_0]) \in \tilde{\Delta}(0)$ (with the extra hypotheses in Assumption 4.1).

**Proposition 4.14.** Suppose that Assumption 4.11 holds. Let $S$ be as in (4.1.33) and assume that $Q$ is smooth and transversal to $W_K$ outside a finite set. There exist $A\in \Delta, [v_0] \in \mathbb{P}(V)$ and a codimension-1 subspace $V_0 \subset V$ transversal to $[v_0]$ such that the following hold:

1. $\bigwedge^3 V_0 \cap A = \{0\}$,
2. Items (c) and (d) of Assumption 4.1 hold,
3. the natural isomorphism $\mathbb{P}(\bigwedge^3(V/[v_0])) \sim \mathbb{P}(\bigwedge^3 V_0)$ maps $S$ to $S_A^D$ where $D$ is the direct-sum decomposition of $V$ appearing in (4.1.1).

If we replace the quadric $Q$ by a smooth quadric $Q'\subset \mathbb{P}(Ann\ K)$ such that $S=W_K\cap Q'$ and let $A' \in \Delta$ be the corresponding point, there exists a projectivity of $\mathbb{P}(V)$ fixing $[v_0]$ which takes $A$ to $A'$.

**Proof.** Let $Q=V(P)$. The dual of $Ann\ K$ is $\bigwedge^2(V/[v_0])/K$; thus the polarization of $P$ defines a non-degenerate symmetric map

$$Ann\ K \sim \bigwedge^2(V/[v_0])/K.$$  \hfill (4.1.34)

The inverse of the above map is non-degenerate symmetric map

$$\bigwedge^2(V/[v_0])/K \sim Ann\ K.$$ \hfill (4.1.35)

Composing on the right with $\bigwedge^2(V/[v_0]) \sim \bigwedge^2(V/[v_0])$ and the quotient map $\bigwedge^2(V/[v_0]) \to \bigwedge^2(V/[v_0])/K$ and on the left with $Ann\ K \leftarrow \bigwedge^3(V/[v_0])$ and $\bigwedge^3(V/[v_0]) \sim \bigwedge^3(V/[v_0])$ we get a symmetric map

$$\bigwedge^2 V_0 \to \bigwedge^3 V_0$$ \hfill (4.1.36)

with 3-dimensional kernel corresponding to $K$. The graph of the above map is a Lagrangian $A \in LG(\bigwedge^3 V)$. One checks easily that (1), (2) and (3) hold. One gets that the projective equivalence of $A$ does not depend on $Q$ by going through the proof of Lemma 4.4. $\square$
4.2 $X_A$ for $A \in (\Delta \setminus \Sigma)$

Let $S$ be a K3. Let $\Delta_{S}^{[2]} \subset S^{[2]}$ be the irreducible codimension 1 subset parametrizing non-reduced subschemes. There exists a square root of the line bundle $\mathcal{O}_{S^{[2]}}(\Delta_{S}^{[2]})$: we denote by $\xi$ its first Chern class. There is a natural morphism of integral Hodge structures $\mu: H^2(S) \to H^2(S^{[2]})$ such that $H^2(S^{[2]}, \mathbb{Z}) = \mu(H^2(S, \mathbb{Z})) \oplus \mathbb{Z} \xi$, see [1]. Let $(\cdot, \cdot)$ be the Beauville-Bogomolov bilinear symmetric form on $H^2(S^{[2]})$. It is known [1] that

$$\mu(\eta), \mu(\eta) = \int_{S} c_1(\eta)^2, \mu(H^2(S, \mathbb{Z})) \xi, \quad (\xi, \xi) = -2. \tag{4.2.1}$$

Since $S$ and $S^{[2]}$ are regular varieties we may identify their Picard groups with $H^{1,1}(S)$ and $H^{1,1}(S^{[2]})$ respectively. Let $C \in \text{Pic}(S)$; abusing notation we will denote by $\mu(C)$ the class in $\text{Pic}(S^{[2]})$ corresponding to $\mu(\mathcal{O}_{S}(C)) \in H^{1,1}(S)$: if $C$ is an integral curve it is represented by subschemes whose support intersects $C$. The following is the main result of the present subsection.

**Theorem 4.15.** Let $A \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_A[3] (= Y_A(3)$ by Claim 3.11 of [15]) - thus $S_A(v_0)$ is a K3 surface by Corollary 4.9. Then the following hold:

1. If $S_A(v_0)$ does not contain lines (true for generic $A$ by Proposition 4.12) then there exist a choice $\epsilon$ of $\mathbb{P}^2$-fibration for $X_A$ and an isomorphism

$$\psi: S_A(v_0)^{[2]} \dashrightarrow X_A^\epsilon \tag{4.2.2}$$

such that

$$\psi^*H_A^\epsilon \sim \mu(D_A(v_0)) - \Delta_{S_A(v_0)}^{[2]}. \tag{4.2.3}$$

2. Let $A$ and $\epsilon$ be arbitrary. There exists a bimeromorphic map

$$\psi: S_A(v_0)^{[2]} \dashrightarrow X_A^\epsilon \tag{4.2.4}$$

such that (4.2.3) holds.

**Remark 4.16.** Suppose that $S_A(v_0)$ contains a line $L$. The restriction of the right-hand side of (4.2.3) to $L^{[2]}$ (embedded in $S_A(v_0)^{[2]}$) is $\mathcal{O}_{L^{[2]}}(-1)$. Since $H_A^\epsilon$ is nef we get that in this case $\text{Map}(4.2.4)$ cannot be regular.

The proof of **Theorem 4.15** will be given after a series of auxiliary results. Let $S \subset \mathbb{P}^6$ be a linearly normal K3 surface of genus 6 such that $\mathcal{I}_S(2)$ is globally generated; then $S$ is projectively normal and hence Riemann-Roch gives that $\dim |\mathcal{I}_S(2)| = 5$. One defines a rational map $S^{[2]} \dashrightarrow |\mathcal{I}_S(2)|^\vee$ as follows. Given $[Z] \in S^{[2]}$ we let $\langle Z \rangle \subset \mathbb{P}^5$ be the line spanned by $Z$. We let

$$\left( S^{[2]} \setminus \bigcup_{L \subset S \text{ line}} L^{(2)} \right) \twoheadrightarrow |\mathcal{I}_S(2)|^\vee \cong \mathbb{P}^5 \tag{4.2.5}$$

$$\left. \begin{array}{c} \{ Z \} \\ \mapsto \end{array} \right\} \{ Q \in |\mathcal{I}_S(2)| \mid \text{s.t. } Q \supset \langle Z \rangle \}.$$

Let $D$ be a hyperplane divisor on $S$; one shows (see Claim (5.16) of [11]) that

$$g^*\mathcal{O}_{\mathbb{P}^5}(1) \cong \mu(D) - \Delta_{S}^{[2]}. \tag{4.2.6}$$

(Notice that the set of lines on $S$ is finite and hence $\bigcup_{L \subset S \text{ line}} L^{(2)}$ has codimension 2 in $S^{[2]}$.) In fact $g$ can be identified with the map associated to the complete linear system $|\mu(D) - \Delta_{S}^{[2]}|$. We will analyze $g$ under the assumption that $S$ is generic (in a precise sense).

**Assumption 4.17.** Item (1) of **Assumption 4.11** holds.

$$S := W_K \cap Q \tag{4.2.7}$$

where $Q \subset \mathbb{P} (\text{Ann } K)$ is a quadric intersecting transversely $W_K$. 

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Let \( S \subset \mathbb{P}(\text{Ann} \, K) \) be as in **Assumption 4.17**. Then \( S \) is a linearly normal K3 surface of genus 6 and \( \mathcal{I}_S(2) \) is globally generated. Thus the map \( g \) of (4.2.5) is defined. Let \( F(W_K) \) be the variety parametrizing lines in \( W_K \). Since the set of lines in \( S \) is finite (empty for generic \( S \) by **Proposition 4.12**) we have a map

\[
(F(W_K) \setminus \{L \mid L \subset S\}) \longrightarrow S^{[2]}_0
\]

\[
L \mapsto L \cap Q.
\]  

(4.2.8)

**Definition 4.18.** Let \( P^0_S \subset S^{[2]}_0 \) be the image of Map (4.2.8) and \( P_S \) be its closure in \( S^{[2]} \).

We recall that \( F(W_K) \cong \mathbb{P}^2 \) by Iskovskih’s **Proposition 5.2**.

**Claim 4.19.** Let \( S \subset \mathbb{P}(\text{Ann} \, K) \) be as in **Assumption 4.17**. Suppose moreover that \( S \) contains no lines. Let \( C_1, C_2, \ldots, C_s \) be the (smooth) conics contained in \( S \) (of course the generic \( S \) contains no conics). Then \( P_S, C^{(2)}_1, \ldots, C^{(2)}_s \) are pairwise disjoint subset of \( S^{[2]} \). Moreover there exists a biregular morphism

\[
c: S^{[2]}_0 \longrightarrow N(S),
\]

(4.2.9)

contracting each of \( P_S, C^{(2)}_1, \ldots, C^{(2)}_s \). Thus \( N(S) \) is a compact complex normal space with

\[
sing \, N(S) = \{c(P_S), \ldots, c(C^{(2)}_s), \ldots \mid C \subset S \text{ a conic}\}
\]

(4.2.10)

and \( c \) is an isomorphism of the complement of \( P_S \cup C^{(2)}_1 \cup \ldots \cup C^{(2)}_s \) onto the smooth locus of \( N(S) \). The map \( g \) (regular on all of \( S^{[2]} \) because \( S \) contains no lines) descends to a regular map

\[
\varphi: N(S) \rightarrow |\mathcal{I}_S(2)|^\vee, \quad \varphi \circ c = g.
\]  

(4.2.11)

**Proof.** \( P_S \) is isomorphic to \( \mathbb{P}^2 \) by Iskovskih’s **Proposition 5.2** and each \( C^{(2)}_i \) is isomorphic to \( \mathbb{P}^2 \) because \( C_i \) is a conic. Thus each of \( P_S, C_i \) can be contracted individually. Let’s show that \( P_{S}, C^{(2)}_1, \ldots, C^{(2)}_s \) are pairwise disjoint. Suppose that \( [Z] \in P_S \cap C^{(2)}_1 \). Let \( \Lambda \) be the plane containing \( C_i \). Then \( \Lambda \cap W_K \) contains the line \( (Z) \) and the smooth conic \( C_i \). Since \( W_K \) is cut out by quadrics it follows that \( \Lambda \subset W_K \), that is absurd because \( W_K \) contains no planes. This proves that \( P_S \cap C^{(2)}_1 = \emptyset \). On the other hand there does not exist \( [Z] \in C^{(2)}_1 \cap C^{(2)}_j \) by **Corollary 5.5**. that \( P_S, C^{(2)}_1, \ldots, C^{(2)}_s \) are pairwise disjoint. Thus the contraction (4.2.9) exists. It remains to prove that \( g \) is constant on each of \( P_S, C^{(2)}_1, \ldots, C^{(2)}_s \). In fact if \( [Z] \in P_S \) then \( g([Z]) = |\mathcal{I}_{W_K}(2)| \), if \( [Z] \in C^{(2)}_i \) then

\[
g([Z]) = \{Q \in |\mathcal{I}_S(2)| \mid Q \supset (C_i)\}.
\]

\[\square\]

Now we go back to the “general” case: we suppose that **Assumption 4.17** holds however \( S \) may very well contain lines. Let

\[
S^{[2]}_s := S^{[2]}_0 \setminus P_S \cup \bigcup_{R \subset S \text{ line or conic}} \text{Hilb}^2 R.
\]

(4.2.12)

(Notice that if \( R \subset S \) is a conic which is not smooth then we delete all \( [Z] \in S^{[2]}_s \) such that \( Z \) is contained in the scheme \( R \).) The following result is essentially **Lemma 3.7** of [14].

**Proposition 4.20.** Suppose that **Assumption 4.17** holds.

1. The fibers of \( g|_{S^{[2]}_s} \) are finite of cardinality at most 2 and the generic fiber has cardinality 2.

2. There exist an open dense subset \( A \subset S^{[2]}_s \) and an anti-symplectic (and hence non-trivial) involution \( \phi: A \rightarrow A \) such that

\[
(g|_A) \circ \phi = g|_A.
\]

The induced map

\[
A/\langle \phi \rangle \rightarrow g(A)
\]

is a bijection.
(3) If in addition \( S \) does not contain lines \( \phi \) descends to a regular involution \( \overline{\phi} : N(S) \to N(S) \) such that \( \overline{\phi} \circ \overline{\phi} = \overline{\phi} \) and the induced map

\[
j : N(S)/(\overline{\phi}) \longrightarrow g(S[2])
\]

is a bijection. Moreover

\[
\text{cod}(\text{Fix}(\overline{\phi}), N(S)) \geq 2
\]

where \( \text{Fix}(\overline{\phi}) \) is the fixed-locus of \( \overline{\phi} \).

Let \( A \) and \( [v_0] \) be as in the statement of \textbf{Theorem 4.15}: we will perform the key computation one needs to prove that theorem. Let \( V_0 \subset V \) be a codimension-1 subspace transversal to \( [v_0] \) and such that \( \Lambda^3 V_0 \cap A = \{0\} \). Let \( D \) be Decomposition \( V = [v_0] \oplus V_0 \) and \( S_0^A \) be given by (4.1.8) - thus \( S_0^A \) sits in \( \mathbb{P}(\text{Ann} K^P_A) \cap \text{Gr}(3, V_0) \) and is isomorphic to \( S_A(v_0) \). Let \( f \in V_0^\vee \); we let \( q_f \) be the quadratic form on \( \Lambda^3 V_0 \) defined by setting

\[
q_f(\omega) := \text{vol}_0((f \omega) \wedge \omega)
\]

where \( \text{vol}_0 \) is a volume-form on \( V_0 \). Then \( q_f \) is a Plücker quadric, in fact we have an isomorphism

\[
V_0^\vee \xrightarrow{\sim} H^0(\mathcal{I}_{\text{Gr}(3, V_0)}(2))
\]

\[
f \mapsto q_f.
\]

Let \( V^\vee = [v_0^\vee] \oplus V_0^\vee \) be the dual decomposition of \( D \); thus \( v_0^\vee \in \text{Ann} V_0 \) and \( v_0^\vee(v_0) = 1 \). We have an isomorphism

\[
[v_0^\vee] \oplus V_0^\vee \xrightarrow{\sim} H^0(\mathcal{I}_{S_A^1}(2))
\]

\[
x v_0^\vee + f \mapsto x(r_D^A)^{\vee} + q_f.
\]

We let

\[
i : [\mathcal{I}_{S_A^1}(2)]^{\vee} \xrightarrow{\sim} \mathbb{P}(V)
\]

be the projectivization of the transpose of (4.1.9).

\textbf{Proposition 4.21.} Let \( A \) and \( [v_0] \) be as in the statement of \textbf{Theorem 4.15} and keep notation as above. Let \( g \) be Map (4.2.5) for \( S_A^1 \) - this makes sense by \textbf{Corollary 4.9}. Then \( i(\text{im} \ g) \subset Y_A \).

\textbf{Proof.} Let \( [Z] \in ([S_A^1])^2 \setminus \Delta_{S_A^1} \setminus P_{S_A^1} \).

We will prove that

\[
i(g([Z])) \in Y_A.
\]

This will suffice to prove the lemma because the right-hand side of (4.2.21) is dense in \( (S_A^1)^2 \) and \( Y_A \) is closed. By hypothesis \( Z \) is reduced; thus \( Z = \{[\beta], [\beta']\} \) where \( \beta, \beta' \in \Lambda^3 V_0 \) are decomposable. The line \( \langle [\beta], [\beta'] \rangle \) spanned by \( [\beta] \) and \( [\beta'] \) is not contained in \( F^P \) because \( [Z] \not\in P_{S_A^1} \).

Thus \( \langle [\beta], [\beta'] \rangle \) is not contained in \( \text{Gr}(3, V_0) \) and it follows that the vector sub-spaces of \( V_0 \) supporting the decomposable vectors \( \beta \) and \( \beta' \) intersect in a 1-dimensional subspace. Thus there exists a basis \( \{v_1, \ldots, v_5\} \) of \( V_0 \) such that

\[
\beta = v_1 \wedge v_2 \wedge v_3, \quad \beta' = v_1 \wedge v_4 \wedge v_5.
\]

We may assume moreover that \( \text{vol}_0(v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge v_5) = 1 \). By (4.1.6) and (4.1.7) there exist \( \alpha, \alpha' \in \Lambda^2 V_0 \) such that

\[
v_0 \wedge \alpha + \beta, \quad v_0 \wedge \alpha' + \beta' \in A, \quad \alpha \wedge \beta = \alpha' \wedge \beta' = 0.
\]

Since \( A \) is Lagrangian we get that

\[
\text{vol}_0(\alpha \wedge \beta') = \text{vol}_0(\alpha' \wedge \beta) =: c.
\]
Let \( t_0, \ldots, t_5 \in \mathbb{C} \); a straightforward computation gives that
\[
(t_0(r_A^D)^\vee + \sum_{i=1}^5 t_i q_i)(\beta + \beta') = 2ct_0 + 2t_1. \tag{4.2.26}
\]
Thus
\[
\iota([Z]) = [cv_0 + v_1]. \tag{4.2.27}
\]
It remains to prove that
\[
(cv_0 + v_1) \in Y_A. \tag{4.2.28}
\]
Let \( K_A^v \) be as in (4.1.2); we claim that it suffices to prove that there exist \((x, x') \in (\mathbb{C}^2 \setminus \{(0, 0)\})\) and \( \kappa \in K_A^v \) such that
\[
(cv_0 + v_1) \cap (x(v_0 \wedge \alpha + \beta) + x'(v_0 \wedge \alpha' + \beta') + v_0 \wedge \kappa) = 0. \tag{4.2.29}
\]
In fact assume that (4.2.29) holds. Then
\[
0 \neq (x(v_0 + \alpha + \beta) + x'(v_0 \wedge \alpha' + \beta') + v_0 \wedge \kappa) \in A \cap F_{cv_0+v_1}. \tag{4.2.30}
\]
(The inequality holds because \( \beta, \beta' \) are linearly independent.) A straightforward computation gives that (4.2.29) is equivalent to
\[
x(c\beta - v_1 \wedge \alpha) + x'(c\beta' - v_1 \wedge \alpha') = v_1 \wedge \kappa. \tag{4.2.31}
\]
As is easily checked we have
\[
(c\beta - v_1 \wedge \alpha), (c\beta' - v_1 \wedge \alpha') \in ([v_1] \wedge (\bigwedge (v_2, v_3, v_4, v_5))) \cap \{v_2 \wedge v_3, v_4 \wedge v_5\}^\perp \tag{4.2.32}
\]
where perpendicularity is with respect to wedge-product followed by vol. Multiplication by \( v_1 \) gives an injection \( K_A^v \hookrightarrow ([v_1] \wedge (\bigwedge^2 (v_2, v_3, v_4, v_5))) \); in fact no non-zero element of \( K_A^v \) is decomposable because \( A \notin \Sigma \). Since the right-hand side of (4.2.32) has dimension 4 and \( \dim K_A^v = 3 \) we get that there exists \((x, x') \in (\mathbb{C}^2 \setminus \{(0, 0)\})\) such that (4.2.31) holds.

**Lemma 4.22.** Let \( A \in (\mathbb{L}G(A^3V) \setminus \Sigma) \). Then \( Y_A(1) \) is not empty, the topological double cover \( f_A^{-1}Y_A(1) \to Y_A(1) \) is not trivial and \( Y_A \) is integral.

**Proof.** By Claim 3.7 we know that \( Y_A[3] \) is finite. On the other hand \( (Y_A[2] \setminus Y_A[3]) \) is a smooth surface - see Proposition 2.8 of [12]. Since \( \text{sing} Y_A \subset Y_A[2] \) it follows that \( Y_A \) is integral and \( Y_A(1) \) is connected. Let \( [v_0] \in (Y_A[2] \setminus Y_A[3]) \). By Propostion 1.5 we know that \( f_A^{-1}([v_0]) \) is a singleton \( \{q\} \). Moreover \( X_A \) is smooth at \( q \) by Lemma 3.3. Thus there exists an open neighborhood \( U \) of \([v_0]\) in \( Y_A \) such that \( f_A^{-1}U \) is smooth. Moreover \( (f_A^{-1}Y_A[2]) \cap f_A^{-1}U \) is nowhere dense in \( f_A^{-1}U \). Since \( f_A^{-1}U \) is smooth the complement \( f_A^{-1}(Y_A(1) \cap U) \) is connected. Since \( Y_A(1) \) is connected it follows that \( f_A^{-1}Y_A(1) \) is connected.

**Proposition 4.23.** Keep hypotheses and notation as in Proposition 4.21. Then \( \iota(\text{Im} g) = Y_A \).

**Proof.** By Item (1) of Proposition 4.20 the map \( g \) has finite generic fiber and hence \( \dim \text{Im} g = 4 \). By Proposition 4.21 we get that \( \iota(\text{Im} g) \) is an irreducible component of \( Y_A \). On the other hand \( Y_A \) is irreducible by Lemma 4.22; it follows that \( \iota(\text{Im} g) = Y_A \).

**Remark 4.24.** Keep notation as in Proposition 4.21; then
\[
\iota \circ g(P_{S_A}^D) = \iota(H^p_0(I_{d_A^D}(2))) = [v_0]. \tag{4.2.33}
\]
Proof of Theorem 4.15. Let’s prove that Item (1) holds. Let $A$ and $[v_0]$ be as in the statement of Theorem 4.15. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$ and such that $\Lambda^1 V_0 \cap A = \{0\}$. Let $D$ be Decomposition $V = [v_0] \oplus V_0$. In order to simplify notation we set $S = S^\Lambda_A$; thus $S \cong S_A(v_0)$ and by hypothesis $S$ does not contain lines. Let $j$ be the map of (4.2.15); by Proposition 4.21 the composition $\iota \circ j$ is a map

$$\iota \circ j : N(S)/\langle \phi \rangle \longrightarrow Y_A. \quad (4.2.34)$$

We claim that $\iota \circ j$ is an isomorphism: in fact it has finite fibers and is birational by Proposition 4.20, since dim$i(X_A) = 2$ (because $A \notin \Sigma$) the hypersurface $X_A$ is normal and hence $\iota \circ j$ is an isomorphism. Let $\pi : N(S) \rightarrow N(S)/\langle \phi \rangle$ be the quotient map. By (4.2.16) the singular locus of $N(S)/\langle \phi \rangle$ is the image of Fix($\phi$) (and thus isomorphic to Fix($\phi$)); since (4.2.34) is an isomorphism we get that

$$N(S) \setminus \text{Fix}(\phi) \quad \longrightarrow \quad Y_A^m \quad \xi \quad \iota \circ j \circ \pi(x) \quad (4.2.35)$$

is a topological covering of degree 2. We claim that

$$\pi_1(Y_A^m) \cong \mathbb{Z}/(2). \quad (4.2.36)$$

In fact $(N(S) \setminus \text{Fix}(\phi)) \cong (S^{[2]} \setminus (P_3 \cup \text{Fix}(\phi|_{S^{[2]} \setminus P_3})))$. Since $(P_3 \cup \text{Fix}(\phi|_{S^{[2]} \setminus P_3}))$ is of codimension 2 in the simply connected manifold $S^{[2]}$ we get that $(N(S) \setminus \text{Fix}(\phi))$ is simply connected. Thus (4.2.35) is the universal covering of $Y_A^m$ and we get (4.2.36). On the other hand $Y_A^m \subset Y_A(1)$ by Corollary 1.5 of [15] and thus by Lemma 4.22 we get that $f_A^{-1} Y_A^m \rightarrow Y_A^m$ is the universal covering of $Y_A^m$ as well. Hence both $X_A$ and $N(S)$ are normal completions of the universal cover of $Y_A^m$ such that the extended maps to $Y_A$ are finite; it follows that they are isomorphic (over $Y_A$). The singular locus of $N(S)$ is given by (4.2.10). On the other hand sing $X_A = Y_A[3]$. By Remark 4.24 we can order the set of (smooth) conics on $S$, say $C_1, \ldots, C_s$ and the set of points in $Y_A[3]$ different from $[v_0]$, say $[v_1], \ldots, [v_s]$ so that

$$\overline{\nu}(c(P_S)) = [v_0], \quad \overline{\nu}(c(C_i^{[2]})) = [v_i], \quad 1 \leq i \leq s. \quad (4.2.37)$$

(Recall Remark 4.24.) Let $\epsilon_0$ be a choice of $\mathbb{P}^2$-fibration for $X_A$; then $\overline{\psi}$ defines a birational map $\psi_0 : S^{[2]} \dasharrow X_A^{[2]}$ such that

$$\psi_0^* H_A^{[2]} \cong H_S - D^{[2]} \quad (4.2.38)$$

where $D$ is the hyperplane class of $S$ (thus $(S, D)$ is isomorphic to $(S_A(v_0), D_A(v_0))$). The birational map $\psi_0$ is an isomorphism away from

$$P_S \cup C_1^{[2]} \cup \ldots \cup C_s^{[2]}. \quad (4.2.39)$$

It follows that $\psi_0$ is the flop of a collection of irreducible components of (4.2.39). By Proposition 3.10 we get that there exists a choice $\epsilon$ of $\mathbb{P}^2$-fibration for $X_A$, call it $\psi$, such that the corresponding birational map $\psi : S^{[2]} \dasharrow X_A^{[2]}$ is birational. Equation (4.2.3) follows from (4.2.38). This finishes the proof that Item (1) holds. Item (2) follows from Item (1) and a specialization argument - we leave the details to the reader.

We close the present subsection by reproving a result of ours. Let $h_A := c_1(\mathcal{O}_{X_A}(H_A))$.

Theorem 4.25 (O’Grady [12]). Let $A \in \mathbb{L}G(\Lambda^3 V)^0$. Then $X_A$ is a deformation of $(K3)^{[2]}$ and $(h_A, h_A)_{X_A} = 2$. Any small deformation of $(X_A, H_A)$ (i.e. a small deformation of $X_A$ keeping $H_A$ of type $(1, 1)$) is isomorphic to $(X_B, H_B)$ for some $B \in \mathbb{L}G(\Lambda^3 V)^0$.

Proof. Let $A_0 \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_{A_0}[3]$. Suppose moreover that $S_{A_0}(v_0)$ does not contain lines. By Theorem 4.15 there exists a choice $\epsilon$ of $\mathbb{P}^2$-fibration for $X_{A_0}$ such that we have an isomorphism

$$\psi : S^{[2]} \dasharrow X_{A_0}^{[2]}, \quad \psi^* H_{A_0}^{[2]} \sim H_A(v_0) - \Delta_{S_{A_0}(v_0)}^{[2]}, \quad (4.2.40)$$
On the other hand \((X_A, H_A)\) is a deformation of \((X_{A_0}, H_{A_0})\) by Corollary 3.12. This proves that \((X_A, H_A)\) is a deformation of \((S^2, (\mu(D_A(v_0))) - \Delta^2_{\text{Sing}(v_0)})\). By (4.2.1) we get that \(h_A, h_A'\) are equal. Lastly we prove that an arbitrary small deformation of \((X_A, H_A)\) is isomorphic to \((X_A', H_A')\) for some \(A' \in \mathbb{L}G(\wedge^3 V)^0\). The deformation space of \((X_A, H_A)\) has dimension given by

\[
\dim \text{Def}(X_A, H_A) = h^{1,1}(X_A) - 1 = 20. \tag{4.2.41}
\]

On the other hand \(\mathbb{L}G(\wedge^3 V)^0\) is contained in the locus of points in \(\mathbb{L}G\) which are stable for the natural (linearized) \(PL(V)\)-action - this is proved in [12]. Thus by varying \(A \in \mathbb{L}G(\wedge^3 V)\) we get

\[
\dim \mathbb{L}G(\wedge^3 V) - \dim \mathbb{L}G(\wedge^3 V)^0 = 55 - 35 = 20 \tag{4.2.42}
\]

moduli of double EPW-sextics. Since (4.2.41) and (4.2.42) are equal we conclude that an arbitrary small deformation of \((X_A, H_A)\) is isomorphic to \((X_B, H_B)\) for some \(B \in \mathbb{L}G(\wedge^3 V)^0\). \hfill \Box

5 Appendix: Three-dimensional sections of \(\text{Gr}(3, \mathbb{C}^5)\)

In the present section \(V_0\) is a complex vector-space of dimension 5. Choose a volume form \(\text{vol}_0\) on \(V_0\): it defines an isomorphism

\[
\wedge^2 V_0 \overset{\alpha}{\rightarrow} \wedge^3 V_0 \quad \omega \mapsto \text{vol}_0(\alpha \wedge \omega) \tag{5.0.1}
\]

Let \(K \subset \wedge^2 V_0\) be a 3-dimensional subspace such that either

\[
\mathbb{P}(K) \cap \text{Gr}(2, V_0) = \emptyset \tag{5.0.2}
\]

or else

\[
\mathbb{P}(K) \cap \text{Gr}(2, V_0) = \{[\nu_0]\} = \mathbb{P}(K) \cap T_{\nu_0}\text{Gr}(2, V_0). \tag{5.0.3}
\]

In other words either \(\mathbb{P}(K)\) does not intersects \(\text{Gr}(2, V_0)\) or else the scheme-theoretic intersection is a single reduced point. We will describe

\[
W_K := \mathbb{P}(\text{Ann} K) \cap \text{Gr}(3, V_0) \tag{5.0.4}
\]

First we recall that the dual of \(\text{Gr}(3, V_0)\) is \(\text{Gr}(2, V_0)\). More precisely let \([\alpha] \in \mathbb{P}(\wedge^2 V_0)\): then

\[
\text{sing}(\mathbb{P}(\text{Ann} \alpha) \cap \text{Gr}(3, V_0)) = \{U \in \text{Gr}(3, V_0) \mid U \supset \text{supp} \alpha\}. \tag{5.0.5}
\]

In particular \(\mathbb{P}(\text{Ann} \alpha)\) is tangent to \(\text{Gr}(3, V_0)\) if and only if \([\alpha] \in \text{Gr}(2, V_0)\) (and if that is the case it is tangent along a \(\mathbb{P}^2\)). Secondly we record the following observation (the proof is an easy exercise).

**Lemma 5.1.** Let \(U \subset V_0\) be a codimension-1 subspace. Let \(\alpha \in \wedge^2 V_0\). Then

\[
\alpha \wedge (\wedge^3 U) = 0 \tag{5.0.6}
\]

if and only if \(\text{supp} \alpha \subset U\).

We recall the following result of Iskovskih.

**Proposition 5.2** (Iskovskih [10]). Keep notation as above. Let \(K \subset \wedge^2 V_0\) be a 3-dimensional subspace such that (5.0.2) holds. Then

1. \(W_K\) is a smooth Fano 3-fold of degree 5 with \(\omega_{W_K} \cong \mathcal{O}_{W_K}(-2)\).
2. the Fano variety \(\text{F}(W_K)\) parametrizing lines on \(W_K\) (reduced structure) is isomorphic to \(\mathbb{P}^2\),
3. the projective equivalence class of \(W_K\) does not depend on \(K\).
Proposition 5.3. Keep notation as above. Let $K \subset \bigwedge^2 V_0$ be a sub vector-space of dimension 3 such that (5.0.3) holds. Then $W_K$ is a singular Fano 3-fold of degree 5 with $\omega_{W_K} \cong O_{W_K}(-2)$ and one singular point which is ordinary quadratic and belongs to

$$\{ U \in \text{Gr}(3, V_0) \mid U \supset \text{supp} \kappa_0 \}. \quad (5.0.7)$$

Proof. If $\kappa \in (K \setminus [\kappa_0])$ then $\kappa$ is not decomposable and hence $\mathbb{P}(\text{Ann} \kappa)$ is transverse to $\text{Gr}(3, V_0)$; by (5.0.5) we get that

$$\text{sing} W_K = \{ U \in \text{Gr}(3, V_0) \mid U \supset \text{supp} \kappa_0 \} \cap \mathbb{P}(\text{Ann} K). \quad (5.0.8)$$

We claim that the above intersection consists of one point. First notice that we have a natural identification

$$\{ U \in \text{Gr}(3, V_0) \mid U \supset \text{supp} \kappa_0 \} \cong \mathbb{P}(V_0 / \text{supp} \kappa_0) \quad (5.0.9)$$

and a linear map

$$K \xrightarrow{\nu} (V_0 / \text{supp} \kappa_0)^{\vee} \quad (5.0.10)$$

where $v \in V_0$ and $\nu$ is its class in $V_0 / \text{supp} \kappa_0$. Given (5.0.8) and Identification (5.0.9) we get that

$$\text{sing} W_K = \mathbb{P}(\text{Ann} \nu). \quad (5.0.11)$$

Of course $\kappa_0 \in \ker \nu$ and hence in order to prove that $\text{sing} W_K$ is a singleton it suffices to prove that $\ker \nu = [\kappa_0]$. If $\kappa \in (K \setminus [\kappa_0])$ then $\kappa_0 \wedge \kappa \neq 0$; in fact this follows from (5.0.3) together with the equality

$$\mathbb{P}\{ \kappa \in \bigwedge^2 V_0 \mid \kappa_0 \wedge \kappa = 0 \} = T_{[\kappa_0]} \text{Gr}(2, V_0). \quad (5.0.12)$$

Since $\kappa_0 \wedge \kappa \neq 0$ we have $\nu(\kappa) \neq 0$. This proves that $\text{sing} W_K$ consists of a single point. The formula for the dualizing sheaf of $W_K$ follows at once from adjunction. It remains to prove that $W_K$ has a single singular point and that it is an ordinary quadratic point. Let $\widetilde{W}_K \subset \mathbb{P}(\text{supp} \kappa_0) \times \mathbb{P}(V_0 / \text{supp} \kappa_0) \times W_K$ be the closed subset defined by

$$\widetilde{W}_K := \text{Sing}(v, U, W) \mid v \in W \subset U. \quad (5.0.13)$$

The projection $\widetilde{W}_K \to \mathbb{P}(V_0 / \text{supp} \kappa_0)$ is a $\mathbb{P}^1$-fibration and hence $\widetilde{W}_K$ is smooth. One shows that the projection $\pi : \widetilde{W}_K \to W_K$ is the blow-up of $\text{sing} W_K$. Moreover $\pi^{-1}(\text{sing} W_K) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and one gets that the singularity of $W_K$ is ordinary quadratic. \hfill \square

Our last result is about the base-locus of 3-dimensional linear systems of quadrics containing $W_K$ for $K \subset \bigwedge^2 V_0$ a 3-dimensional subspace such that (5.0.2) holds. First we consider the analogous question for the Grassmannian $\text{Gr}(3, \bigwedge^3 V_0)$. Let’s consider the rational map

$$\mathbb{P}(\bigwedge^3 V_0) \xrightarrow{\Phi} \mathbb{P}(\text{Gr}(3, V_0)(2)^{\vee}) \cong \mathbb{P}(V_0) \quad (5.0.14)$$

where the last isomorphism is given by (4.2.18). Let $Z \subset \mathbb{P}(\bigwedge^3 V_0) \times \mathbb{P}(V_0)$ be the incidence subvariety defined by

$$Z := \{ ([\omega], [v]) \mid v \wedge \omega = 0 \}. \quad (5.0.15)$$

Then we have a commutative triangle

$$\begin{array}{ccc}
Z & \xrightarrow{\Phi} & \mathbb{P}(V_0) \\
\Psi \downarrow & & \downarrow \Phi \\
\mathbb{P}(\bigwedge^3 V_0) & \xrightarrow{\Phi} & \mathbb{P}(V_0)
\end{array} \quad (5.0.16)$$

where $\Psi$ and $\Phi$ are the restrictions to $Z$ of the two projections of $\mathbb{P}(\bigwedge^3 V_0) \times \mathbb{P}(V_0)$. Moreover $\Psi$ is the blow-up of $\text{Gr}(3, V_0)$. In particular the following holds: if $\omega \in \bigwedge^3 V_0$ is not decomposable then
there exists a unique \([v] \in \mathbb{P}(V_0)\) such that \(v \wedge \omega = 0\) and moreover \(\Phi([\omega]) = [v]\). Let \([v] \in \mathbb{P}(V_0)\); by \((4.2.18)\) we may view \(\text{Ann}(v) \subset V_0^\vee\) as a hyperplane in \([\mathcal{I}_{Gr(3,V_0)}(2)]\); by commutativity of \((5.0.16)\) we have

\[
\bigcap_{f \in \text{Ann}(v)} V(q_f) = \text{Gr}(3,V_0) \cup \{[\omega] \in \mathbb{P}(\bigwedge^3 V_0) \mid v \wedge \omega = 0\}. \tag{5.0.17}
\]

**Proposition 5.4.** Let \(K \subset \bigwedge^2 V_0\) be a 3-dimensional subspace such that \((5.0.2)\) holds. Let \(L \subset [\mathcal{I}_{W_K}(2)]\) be a hyperplane (here \(\mathcal{I}_{W_K}\) is the ideal sheaf of \(W_K\) in \(\mathbb{P}(\text{Ann} K)\)). Then

\[
\bigcap_{t \in L} Q_t = W_K \cup R_L \tag{5.0.18}
\]

where \(R_L\) is a plane. Moreover \(W_K \cap R_L\) is a conic.

**Proof.** Restriction to \(\mathbb{P}(\text{Ann} K)\) defines an isomorphism

\[
|\mathcal{I}_{Gr(3,V_0)}(2)| \sim |\mathcal{I}_{W_K}(2)|. \tag{5.0.19}
\]

By \((4.2.18)\) we get that we may identify \(L\) with \(\mathbb{P}(\text{Ann}(v))\) for a well-defined \([v] \in \mathbb{P}(V_0)\) and each quadric \(Q_t\) for \(t \in L\) with \(\mathbb{P}(\text{Ann} K) \cap V(q_f)\) for a suitable \([f] \in \mathbb{P}(\text{Ann}(v))\). By \((5.0.17)\) we have

\[
\bigcap_{f \in \text{Ann}(v)} (\mathbb{P}(\text{Ann} K) \cap V(q_f)) = W_K \cup R_L \tag{5.0.20}
\]

where

\[
R_L := \mathbb{P}(\text{Ann} K) \cap \{[\omega] \in \mathbb{P}(\bigwedge^3 V_0) \mid v \wedge \omega = 0\}. \tag{5.0.21}
\]

Thus \(R_L\) is a linear space of dimension at least 2. Now notice that we have an isomorphism

\[
\bigwedge^2 V_0/[v] \xrightarrow{\pi} \bigwedge^2 V_0 = \{[\omega] \in \mathbb{P}(\bigwedge^3 V_0) \mid v \wedge \omega = 0\} \tag{5.0.22}
\]

where \(\alpha \in \bigwedge^2 V_0\) is an element mapped to \(\pi\) by the quotient map \(\bigwedge^2 V_0 \to \bigwedge^2 V_0/[v]\). Since \(\text{dim}(V_0/[v]) = 4\) the Grassmannian \(\text{Gr}(2,V_0/[v])\) is a quadric hypersurface in \(\mathbb{P}(\bigwedge^3 V_0/[v])\); it follows that either \(R_L \subset W_K\) or \(R_L \cap W_K\) is a quadric hypersurface in \(R_L\). By Lefschetz Pic(\(W_K\)) is generated by the hyperplane class; it follows that \(W_K\) contains no planes and no quadric surfaces. Thus necessarily \(\text{dim} R_L = 2\), moreover \(R_L \not\subset W_K\) and the intersection \(R_L \cap W_K\) is a conic. \(\square\)

**Corollary 5.5.** Let \(K \subset \bigwedge^2 V_0\) be a 3-dimensional subspace such that \((5.0.2)\) holds and \(C(W_K)\) be the variety parametrizing conics on \(W_K\) (reduced structure). Then we have an isomorphism

\[
|\mathcal{I}_{W_K}(2)|^\vee \xrightarrow{\sim} C(W_K) \tag{5.0.23}
\]

where \(R_L\) as in **Proposition 5.4.** Moreover given \(Z \in W_K^{[2]}\) there exists a unique conic containing \(Z\) namely \(R_L \cap W_K\) where \(L \subset [\mathcal{I}_{W_K}(2)]^\vee\) is the hyperplane of quadrics containing \([Z]\).

**References**


