Double covers of EPW-sextics

Kieran G. O'Grady*

"Sapienza" Università di Roma

June 8 2011

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0 Introduction

EPW-sextics are defined as follows. Let V be a 6-dimensional complex vector space. Choose a volume-form $\mathrm{vol}\colon \bigwedge^6 V \xrightarrow{\sim} \mathbb{C}$ and equip $\bigwedge^3 V$ with the symplectic form

$$(\alpha, \beta)_V := \mathrm{vol}(\alpha \wedge \beta). \tag{0.0.1}$$

Let $\mathbb{LG}(\bigwedge^3 V)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\bigwedge^3 V$ - of course $\mathbb{LG}(\bigwedge^3 V)$ does not depend on the choice of volume-form. Let $F \subset \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ be the sub vector-bundle with fiber

$$F_v := \{\alpha \in \bigwedge^3 V \mid v \wedge \alpha = 0\} \tag{0.0.2}$$

over $[v] \in \mathbb{P}(V)$. Notice that $(,)_V$ is zero on F_v and $2\dim(F_v) = 20 = \dim \bigwedge^3 V$; thus F is a Lagrangian sub vector-bundle of the trivial symplectic vector-bundle on $\mathbb{P}(V)$ with fiber $\bigwedge^3 V$. Next choose $A \in \mathbb{LG}(\bigwedge^3 V)$. Let

$$F \xrightarrow{\lambda_A} (\bigwedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{0.0.3}$$

*Supported by PRIN 2007

be the composition of the inclusion $F \subset \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ followed by the quotient map. Since $\operatorname{rk} F = \dim(V/A)$ the determinat of λ_A makes sense. Let

$$Y_A := V(\det \lambda_A).$$

A straightforward computation gives that $\det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6)$ and hence $\det \lambda_A \in H^0(\mathcal{O}_{\mathbb{P}(V)}(6))$. It follows that if $\det \lambda_A \neq 0$ then Y_A is a sextic hypersurface. As is easily checked $\det \lambda_A \neq 0$ for generic $A \in \mathbb{LG}(\bigwedge^3 V)$ (notice that there exist "pathological" A's such that $\lambda_A = 0$ e.g. $A = F_{v_0}$). An *EPW-sextic* (after Eisenbud, Popescu and Walter [5]) is a sextic hypersurface in $\mathbb{P}^5$ which is projectively equivalent to Y_A for some $A \in \mathbb{LG}(\bigwedge^3 V)$. Let Y_A be an EPW-sextic. One constructs a coherent sheaf ξ_A on Y_A and a multiplication map $\xi_A \times \xi_A \to \mathcal{O}_{Y_A}$ which gives $\mathcal{O}_{Y_A} \oplus \xi_A$ a structure of $\mathcal{O}_{Y_A}$-algebra - this is known to experts, see [4] - we will give the construction in **Subsection 1.2**. The *double EPW-sextic* associated to A is $X_A := \operatorname{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A)$; we let $f_A \colon X_A \to Y_A$ be the structure morphism. In [12] we considered X_A for generic A and we proved that it is a Hyperkähler deformation of $(K3)^{[2]}$ (the blow-up of the diagonal in the symmetric square of a $K3$ surface). In the present paper we will analyze X_A for A varying in a codimension-1 subset of $\mathbb{LG}(\bigwedge^3 V)$. In order to state our main results we will introduce some notation. Given $A \in \mathbb{LG}(\bigwedge^3 V)$ we let

$$Y_A(k) = \quad \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) = k\}, \tag{0.0.4}$$
$$Y_A[k] = \quad \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) \geq k\}. \tag{0.0.5}$$

Thus $Y_A(0) = (\mathbb{P}(V) \setminus Y_A)$ and $Y_A = Y_A[1]$. Double EPW-sextics come with a natural polarization; we let

$$\mathcal{O}_{X_A}(n) := f_A^* \mathcal{O}_{Y_A}(n), \quad H_A \in |\mathcal{O}_{X_A}(1)|. \tag{0.0.6}$$

The following closed subsets of $\mathbb{LG}(\bigwedge^3 V)$ play a key rôle in the present paper:

$$\Sigma := \quad \{A \in \mathbb{LG}(\textstyle\bigwedge^3 V) \mid \exists W \in \mathbb{G}r(3, V) \text{ s. t. } \bigwedge^3 W \subset A\}, \tag{0.0.7}$$
$$\Delta := \quad \{A \in \mathbb{LG}(\textstyle\bigwedge^3 V) \mid Y_A[3] \neq \emptyset\}. \tag{0.0.8}$$

A straightforward computation, see [15], gives that Σ is irreducible of codimension 1. A similar computation, see **Proposition 2.2**, gives that Δ is irreducible of codimension 1 and distinct from Σ. Let

$$\mathbb{LG}(\bigwedge^3 V)^0 := \mathbb{LG}(\bigwedge^3 V) \setminus \Sigma \setminus \Delta \,. \tag{0.0.9}$$

Thus $\mathbb{LG}(\bigwedge^3 V)^0$ is open dense in $\mathbb{LG}(\bigwedge^3 V)$. In [12] we proved that if $A \in \mathbb{LG}(\bigwedge^3 V)^0$ then X_A is a hyperkähler (HK) 4-fold which can be deformed to $(K3)^{[2]}$. Moreover we showed that the family of polarized HK 4-folds (X_A, H_A) for A varying in$\mathbb{LG}(\bigwedge^3 V)^0$ is locally complete. Three other explicit locally complete families of projective HK's of dimension greater than 2 are known - see [2, 3, 8, 9]. In all of the examples the HK manifolds are deformations of the Hilbert square of a $K3$: they are distinguished by the value of the Beauville-Bogomolov form on the polarization class (it equals 2 in the case of double EPW-sextics and 6, 22 and 38 in the other cases). In the present paper we will analyze X_A for $A \in \Delta$, mainly under the hypothesis that $A \notin \Sigma$. Let $A \in (\Delta \setminus \Sigma)$. We will prove the following results

(1) $Y_A[3]$ is a finite set and it equals $Y_A(3)$. If A is generic in $(\Delta \setminus \Sigma)$ then $Y_A(3)$ is a singleton.

(2) One may associate to $[v_0] \in Y_A(3)$ a $K3$ surface $S_A(v_0) \subset \mathbb{P}^6$ of genus 6, well-defined up to projectivities. Conversely the generic $K3$ of genus 6 is projectively equivalent to $S_A(v_0)$ for some $A \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_A(3)$.

(3) The singular set of X_A is equal to $f_A^{-1} Y_A(3)$. There is a single $p_i \in X_A$ mapping to $[v_i] \in Y_A(3)$ and the cone of X_A at p_i is isomorphic to the cone over the set of incident couples $(x, r) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ (i.e. $\mathbb{P}(\Omega_{\mathbb{P}^2})$). Thus we have two standard small resolutions of a neighborhood of p_i in X_A, one with fiber $\mathbb{P}^2$ over p_i, the other with fiber $(\mathbb{P}^2)^\vee$. Making a choice ϵ of local small resolution at each p_i we get a resolution $X_A^\epsilon \to X_A$ with the following properties: There is

a birational map $X_A^\epsilon \dashrightarrow S_A(v_i)^{[2]}$ such that the pull-back of a holomorphic symplectic form on $S_A(v_i)^{[2]}$ is a symplectic form on X_A^ϵ. If $S_A(v_i)$ contains no lines (true for generic A by Item (2)) then there exists a choice of ϵ such that X_A^ϵ is isomorphic to $S_A(v_i)^{[2]}$.

(4) Given a sufficiently small open (classical topology) $\mathcal{U} \subset (\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$ containing A the family of double EPW-sextics parametrized by $\mathcal{U}$ has a simultaneous resolution of singularities (no base change) with fiber X_A^ϵ over A (for an arbitrary choice of ϵ).

A remark: if $Y_A(3)$ has more than one point we do not expect all the small resolutions to be projective (i.e. Kähler). Items (1)-(4) should be compared with known results on cubic 4-folds - recall that if $Z \subset \mathbb{P}^5$ is a smooth cubic hypersurface the variety $F(Z)$ parametrizing lines in Z is a HK 4-fold which can be deformed to $(K3)^{[2]}$ and moreover the primitive weight-4 Hodge structure of Z is isomorphic (after a Tate twist) to the primitive weight-2 Hodge structure of $F(Z)$, see [2]. Let $D \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the prime divisor parametrizing singular cubics. Let $Z \in D$ be generic: the following results are well-known.

(1') $\operatorname{sing} Z$ is a finite set.

(2') Given $p \in \operatorname{sing} Z$ the set $S_Z(p) \subset F(Z)$ of lines containing p is a $K3$ surface of genus 4 and viceversa the generic such $K3$ is isomorphic to $S_Z(p)$ for some Z and $p \in \operatorname{sing} Z$.

(3') $F(Z)$ is birational to $S_Z(p)^{[2]}$.

(4') After a local base-change of order 2 ramified along D the period map extends across Z.

Thus Items (1')-(2')-(3') are analogous to Items (1), (2) and (3) above, Item (4') is analogous to (4) but there is an important difference namely the need for a base-change of order 2. (Actually the paper [13] contains results showing that there is a statement valid for cubic hypersurfaces which is even closer to our result for double EPW-sextics, the rôle of Σ being played by the divisor parametrizing cubics containing a plane.) We explain the relevance of Items (1)-(4). Items (3) and (4) prove the theorem of ours mentioned above i.e. that if $A \in \mathbb{LG}(\bigwedge^3 V)^0$ then X_A is a HK deformation of $(K3)^{[2]}$ (the family of polarized double EPW-sextics is locally complete by a straightforward parameter count). The proof in this paper is independent of the proof in [12]. Beyond giving a new proof of an "old"theorem the above results show that away from Σ the period map is regular, it lifts (locally) to the relevant classifying space and the value at $A \in (\Delta \setminus \Sigma)$ may be identified with the period point of the Hilbert square $S_A(v_0)^{[2]}$. We remark that in [14] we had proved that the period map is as well-behaved as possible at the generic $A \in (\Delta \setminus \Sigma)$, however we did not have the exact statement about X_A^ϵ and we had no statement about an arbitrary $A \in (\Delta \setminus \Sigma)$.

The paper is organized as follows. In **Section 1** we will give formulae that describe double EPW-sextics locally. The formulae are known to experts, see [4], we will go through the proofs because we could not find a suitable reference. We will also perform the local computations needed to prove Item (4) above. In **Section 2** we will go through some standard computations involving Δ. In **Section 3** we will prove Items (1), (4) and the statements of Item (3) which do not involve the $K3$ surface $S_A(v_0)$. In **Section 4** we will prove Item (2) and the remaining statement of Item (3). **Section 5** contains auxiliary results on 3-dimensional linear sections of $\mathrm{Gr}(3, \mathbb{C}^5)$.

Notation and conventions: Throughout the paper V is a 6-dimensional complex vector space.

Let W be a finite-dimensional complex vector-space. The span of a subset $S \subset W$ is denoted by $\langle S \rangle$. Let $S \subset \bigwedge^q W$. The *support of* S is the smallest subspace $U \subset W$ such that $S \subset \operatorname{im}(\bigwedge^q U \longrightarrow \bigwedge^q W)$: we denote it by $\operatorname{supp}(S)$, if $S = \{\alpha\}$ is a singleton we let $\operatorname{supp}(\alpha) = \operatorname{supp}(\{\alpha\})$ (thus if $q = 1$ we have $\operatorname{supp}(\alpha) = \langle \alpha \rangle$). We define the support of a set of symmetric tensors analogously. If $\alpha \in \bigwedge^q W$ or $\alpha \in \operatorname{Sym}^d W$ the *rank of* α is the dimension of $\operatorname{supp}(\alpha)$. An element of $\operatorname{Sym}^2 W^\vee$ may be viewed either as a symmetric map or as a quadratic form: we will denote the former by $\widetilde{q}, \widetilde{r}, \ldots$ and the latter by $q, r, \ldots$ respectively.

Let $M = (M_{ij})$ be a $d \times d$ matrix with entries in a commutative ring R. We let $M^c = (M^{ij})$ be the matrix of cofactors of M, i.e. $M^{i,j}$ is $(-1)^{i+j}$ times the determinant of the matrix obtained from M by deleting its j-th row and i-th column. We recall the following interpretation of M^c. Suppose that $f\colon A\to B$ is a linear map between free R-modules of rank d and that M is the matrix associated to f by the choice of bases $\{a_1,\ldots,a_d\}$ and $\{b_1,\ldots,b_d\}$ of A and B respectively. Then $\bigwedge^{d-1} f$ may be viewed as a map

$$\bigwedge^{d-1} f\colon A^\vee\otimes\bigwedge^{d} A\cong\bigwedge^{d-1} A\longrightarrow\bigwedge^{d-1} B\cong B^\vee\otimes\bigwedge^{d} B. \tag{0.0.10}$$

(Here $A^\vee:=\mathrm{Hom}(A,R)$ and similarly for $B^\vee$.) The matrix associated to $\bigwedge^{d-1} f$ by the choice of bases $\{a_1^\vee\otimes(a_1\wedge\ldots\wedge a_d),\ldots,a_d^\vee\otimes(a_1\wedge\ldots\wedge a_d)\}$ and $\{b_1^\vee\otimes(b_1\wedge\ldots\wedge b_d),\ldots,b_d^\vee\otimes(b_1\wedge\ldots\wedge b_d)\}$ is equal to M^c.

Let W be a finite-dimensional complex vector-space. We will adhere to pre-Grothendieck conventions: $\mathbb{P}(W)$ is the set of 1-dimensional vector subspaces of W. Given a non-zero $w\in W$ we will denote the span of w by $[w]$ rather than $\langle w\rangle$; this agrees with standard notation. Suppose that $T\subset\mathbb{P}(W)$. Then $\langle T\rangle\subset\mathbb{P}(W)$ is the *projective span of* T i.e. the intersection of all linear subspaces of $\mathbb{P}(W)$ containing T.

Schemes are defined over $\mathbb{C}$, the topology is the Zariski topology unless we state the contrary. Let W be finite-dimensional complex vector-space: $\mathcal{O}_{\mathbb{P}(W)}(1)$ is the line-bundle on $\mathbb{P}(W)$ with fiber $L^\vee$ on the point $L\in\mathbb{P}(W)$. Let $F\in\mathrm{Sym}^d W^\vee$: we let $V(F)\subset\mathbb{P}(W)$ be the subscheme defined by vanishing of F. If $E\to X$ is a vector-bundle we denote by $\mathbb{P}(E)$ the projective fiber-bundle with fiber $\mathbb{P}(E(x))$ over x and we define $\mathcal{O}_{\mathbb{P}(W)}(1)$ accordingly. If Y is a subscheme of X we let $Bl_Y X\longrightarrow X$ be the blow-up of Y.

1 Symmetric resolutions and double covers

In **Subsection 1.1** we will describe a method (well-known to experts) for constructing double covers. In **Subsection 1.2** we will show how to implement the construction in order to construct double EPW-sextics. **Subsection 1.3** contains the main ingredients needed to construct the simultaneous desingularization described in Item (3) of **Section 0**.

1.1 Product formula and double covers

Let R be an integral Noetherian ring. Let N be an R-module with a free resolution

$$0\longrightarrow U_1\overset{\lambda}{\longrightarrow}U_0\overset{\pi}{\longrightarrow}N\longrightarrow 0,\qquad \mathrm{rk}\,U_1=\mathrm{rk}\,U_0=d>0. \tag{1.1.1}$$

Let $\{a_1,\ldots,a_d\}$ and $\{b_1,\ldots,b_d\}$ be bases of U_0 and U_1 respectively. Let M_λ be the matrix associated to λ by our choice of bases - notice that $\det M_\lambda$ annihilates N. Given a homomorphism

$$\beta\colon N\to\mathrm{Ext}^1(N,R) \tag{1.1.2}$$

one defines a product $m_\beta\colon N\times N\to R/(\det M_\lambda)$ as follows. Applying the $\mathrm{Hom}(\cdot,R)$-functor to (1.1.1) we get the exact sequence

$$0\longrightarrow U_0^\vee\overset{\lambda^t}{\longrightarrow}U_1^\vee\overset{\rho}{\longrightarrow}\mathrm{Ext}^1(N,R)\longrightarrow 0. \tag{1.1.3}$$

In particular $\det M_\lambda$ kills $\mathrm{Ext}^1(N,R)$. Now apply the functor $\mathrm{Hom}(N,\cdot)$ to the exact sequence

$$0\longrightarrow R\overset{\det M_\lambda}{\longrightarrow}R\longrightarrow R/(\det M_\lambda)\longrightarrow 0. \tag{1.1.4}$$

Since $\mathrm{Ext}^1(N,R)\to\mathrm{Ext}^1(N,R)$ is multiplication by $\det M_\lambda$ we get a coboundary isomorphism

$$\partial\colon\ \mathrm{Hom}(N,R/(\det M_\lambda))\overset{\sim}{\longrightarrow}\mathrm{Ext}^1(N,R). \tag{1.1.5}$$

We let

$$\begin{array}{ccc} N\times N & \overset{m_\beta}{\longrightarrow} & R/(\det M_\lambda)\\ (n,n') & \mapsto & (\partial^{-1}\beta(n))(n'). \end{array} \tag{1.1.6}$$

We will give an explicit formula for m_β. Let $\pi\colon U_0 \to N$ be as in (1.1.1). Then $\beta\circ\pi$ lifts to a homomorphism $\mu^t\colon U_0 \to U_1^\vee$ (the map is written as a transpose in order to conform to the notation for double EPW-sextics - see **Subsection 1.2**). It follows that there exists $\alpha\colon U_1 \to U_0^\vee$ such that

$$\begin{array}{ccccccccc} 0 & \to & U_1 & \xrightarrow{\lambda} & U_0 & \xrightarrow{\pi} & N & \to & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mu^t} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \to & U_0^\vee & \xrightarrow{\lambda^t} & U_1^\vee & \xrightarrow{\rho} & \mathrm{Ext}^1(N,R) & \to & 0 \end{array} \tag{1.1.7}$$

is a commutative diagram. Let $\{a_1^\vee,\ldots,a_d^\vee\}$ and $\{b_1^\vee,\ldots,b_d^\vee\}$ be the bases of $U_0^\vee$ and $U_1^\vee$ which are dual to the chosen bases of U_0 and U_1. Let M_{μ^t} be the matrix associated to μ^t by our choice of bases.

Proposition 1.1. *Keeping notation as above we have*

$$m_\beta(\pi(a_i),\pi(a_j)) \equiv (M_\lambda^c\cdot M_{\mu^t})_{ji} \mod (\det M_\lambda) \tag{1.1.8}$$

where M_λ^c is the matrix of cofactors of M_λ.

Proof. Equation (1.1.3) gives an isomorphism

$$\nu\colon \mathrm{Ext}^1(N,R) \xrightarrow{\sim} U_1^\vee/\lambda^t(U_0^\vee). \tag{1.1.9}$$

Let $\det(U_\bullet) := \bigwedge^d U_1^\vee \otimes \bigwedge^d U_0$. We will define an isomorphism

$$\theta\colon U_1^\vee/\lambda^t(U_0^\vee) \xrightarrow{\sim} \mathrm{Hom}\,(N,\det(U_\bullet)/(\det\lambda))\,. \tag{1.1.10}$$

First let

$$\begin{array}{ccccccc} U_1^\vee = & \bigwedge^{d-1} U_1 \otimes \bigwedge^d U_1^\vee & \xrightarrow{\widehat{\theta}} & \bigwedge^{d-1} U_0 \otimes \bigwedge^d U_1^\vee & = & \mathrm{Hom}(U_0,\det(U_\bullet)) \\ & \zeta\otimes\xi & \mapsto & \bigwedge^{d-1}(\lambda)(\zeta)\otimes\xi\,. & & \end{array} \tag{1.1.11}$$

We claim that

$$\mathrm{im}(\widehat{\theta}) = \{\phi\in \mathrm{Hom}(U_0,\det(U_\bullet)) \mid \phi\circ\lambda(U_1)\subset (\det\lambda)\}. \tag{1.1.12}$$

In fact by Cramer's formula

$$M_\lambda^c\cdot M_\lambda^t = M_\lambda^t\cdot M_\lambda^c = \det M_\lambda\cdot 1 \tag{1.1.13}$$

and Equation (1.1.12) follows. Thus $\widehat{\theta}$ induces a surjective homomorphism

$$\widetilde{\theta}\colon U_1^\vee \longrightarrow \mathrm{Hom}\,(N,\det(U_\bullet)/(\det\lambda))\,. \tag{1.1.14}$$

One checks easily that $\lambda^t(U_0^\vee) = \ker\widetilde{\theta}$ - use Cramer again. We define θ to be the homomorphism induced by $\widetilde{\theta}$; we have proved that it is an isomorphism. We claim that

$$\theta\circ\nu = \partial^{-1}, \qquad \partial \text{ as in (1.1.5).} \tag{1.1.15}$$

In fact let K be the fraction field of R and $0\to R\xrightarrow{\iota} I^0\to I^1\to\ldots$ be an injective resolution of R with $I^0 = \det(U_\bullet)\otimes K$ and $\iota(1) = \det\lambda\otimes 1$. Then $\mathrm{Ext}^\bullet(N,R)$ is the cohomology of the double complex $\mathrm{Hom}(U_\bullet, I^\bullet)$ and of course also of the single complexes $\mathrm{Hom}(U_\bullet,R)$ and $\mathrm{Hom}(N,I^\bullet)$. One checks easily that the isomorphism ∂ of (1.1.5) is equal to the isomorphism $H^1(\mathrm{Hom}(N,I^\bullet))\xrightarrow{\sim} H^1(\mathrm{Hom}(U_\bullet,I^\bullet))$ i.e.

$$\partial\colon\ \mathrm{Hom}(N,\det(U_\bullet)/(\det\lambda)) = \mathrm{Hom}(N,I^0/\iota(R)) \xrightarrow{\sim} H^1(\mathrm{Hom}(U_\bullet,I^\bullet)). \tag{1.1.16}$$

Let $f\in\mathrm{Hom}(N,\det(U_\bullet)/(\det\lambda))$; a representative of $\partial(f)$ in the double complex $\mathrm{Hom}(U_\bullet,I^\bullet)$ is given by $g^{0,1} := f\circ\pi\in\mathrm{Hom}(U_0,I^1)$. Let $g^{0,0}\in\mathrm{Hom}(U_0,\det(U_\bullet))$ be a lift of $g^{0,1}$ and $g^{1,0}\in\mathrm{Hom}(U_1,\det(U_\bullet))$ be defined by $g^{1,0} := g^{0,0}\circ\lambda$. One checks that $\mathrm{im}(g^{1,0})\subset(\det\lambda)$ and hence there exists $g\in\mathrm{Hom}(U_1,R)$ such that $g^{1,0} = \iota\circ g$. By construction g represents a class $[g]\in$

$H^1(\mathrm{Hom}(U_\bullet, R)) = U_1^\vee/\lambda^t(U_0^\vee)$ and $[g] = \nu\circ\partial(f)$. An explicit computation shows that $[g] = \theta^{-1}(f)$. This proves (1.1.15). Now we prove Equation (1.1.8). By (1.1.15) we have

$$m_\beta(\pi(a_i), \pi(a_j)) = (\partial^{-1}\beta\pi(a_i))(\pi(a_j)) = (\theta\nu\beta\pi(a_i))(\pi(a_j)). \tag{1.1.17}$$

Unwinding the definition of θ one gets that the right-hand side of the above equation equals the right-hand side of (1.1.8). □

Let m_β be given by (1.1.6): we define a product on $R/(\det M_\lambda)\oplus N$ as follows. Let $(r,n), (r',n') \in R/(\det M_\lambda)\oplus N$: we set

$$(r,n)\cdot(r',n') := (rr' + m_\beta(n,n'), rn' + r'n). \tag{1.1.18}$$

In general the above product is neither associative nor commutative. We will give an example in which the product is both associative and commutative. Suppose that we have

$$0 \longrightarrow U^\vee \xrightarrow{\gamma} U \xrightarrow{\pi} N \longrightarrow 0, \qquad \gamma^t = \gamma \tag{1.1.19}$$

with U a free R-module of rank $d > 0$ and the sequence is supposed to be exact. We get a commutative diagram (1.1.7) by letting

$$U_0 := U, \quad U_1 := U^\vee, \quad \lambda = \gamma, \quad \alpha = \mathrm{Id}_{U^\vee}, \quad \mu^t = \mathrm{Id}_U,$$

and $\beta = \beta(\gamma)\colon N \to \mathrm{Ext}^1(N,R)$ the map induced by Id_U. Abusing notation we let $m_\gamma\colon N\times N \to R/(\det M_\gamma)$ be the map defined by $m_{\beta(\gamma)}$.

Proposition 1.2. *Suppose that we have Exact Sequence* (1.1.19). *The product on* $R/(\det M_\gamma)\oplus N$ *defined by* m_γ *is associative and commutative.*

Proof. Let $d := \mathrm{rk}\, U > 0$. Let $\{a_1,\ldots,a_d\}$ be a basis of U and $\{a_1^\vee,\ldots,a_d^\vee\}$ be the dual basis of $U^\vee$. Let $M = M_\gamma$ i.e. the matrix associated to γ by our choice of bases. By (1.1.8) we have

$$m_\gamma(\pi(a_i),\pi(a_j)) \equiv M^c_{ji} \mod (\det M). \tag{1.1.20}$$

Since γ is a symmetric map M is a symmetric matrix. Thus M^c is a symmetric matrix. By (1.1.20) we get that m_γ is symmetric. It remains to prove that m_γ is associative. For $1 \le i < k \le d$ and $1 \le h \ne j \le d$ let $M^{i,k}_{h,j}$ be the $(d-2)\times(d-2)$-matrix obtained by deleting from M rows i,k and columns h,j. Let $X_{ijk} = (X^h_{ijk}) \in R^d$ be defined by

$$X^h_{ijk} := \begin{cases} (-1)^{i+k+j+h}\det M^{i,k}_{j,h} & \text{if } h < j, \\ 0 & \text{if } h = j. \\ (-1)^{i+k+j+h-1}\det M^{i,k}_{j,h} & \text{if } j < h. \end{cases} \tag{1.1.21}$$

A tedious but straightforward computation gives that

$$M^c_{ij}a_k - M^c_{jk}a_i = \gamma(\sum_{h=1}^d X^h_{ijk}a_h^\vee). \tag{1.1.22}$$

The above equation proves associativity of m_γ. □

Keep hypotheses as in **Proposition 1.2**. We let

$$X_\gamma := \mathrm{Spec}(R/(\det M_\lambda)\oplus N), \qquad Y_\gamma := \mathrm{Spec}(R/(\det M_\lambda)). \tag{1.1.23}$$

Let $f_\gamma\colon X_\gamma \to Y_\gamma$ be the structure map. We realize X_γ as a subscheme of $\mathrm{Spec}(R[\xi_1,\ldots,\xi_d])$ as follows. Since the ring $R/(\det M_\gamma)\oplus N$ is associative and commutative there is a well-defined surjective morphism of R-algebras

$$R[\xi_1,\ldots,\xi_d] \longrightarrow R/(\det M_\gamma)\oplus N \tag{1.1.24}$$

mapping ξ_i to a_i. Thus we have an inclusion

$$X_\gamma \hookrightarrow \mathrm{Spec}(R[\xi_1,\ldots,\xi_d]). \tag{1.1.25}$$

Claim 1.3. *Referring to Inclusion (1.1.25) the ideal of X_γ is generated by the entries of the matrices*

$$M_\gamma \cdot \xi, \qquad \xi \cdot \xi^t - M_\gamma^c . \tag{1.1.26}$$

(We view ξ as a column matrix.)

Proof. By (1.1.20) the ideal of X_γ is generated by $\det M_\gamma$ and the entries of the matrices in (1.1.26). By Cramer's formula $\det M_\gamma$ belongs to the ideal generated by the entries of the two matrices. This proves that the ideal of X_γ is as claimed. □

Now we suppose in addition that R is a finitely generated $\mathbb{C}$-algebra. Let $p \in \operatorname{Spec} R$ be a closed point: we are interested in the localization of X_γ at points in $f_\gamma^{-1}(p)$. Let $J \subset U^\vee(p)$ be a subspace complementary to $\ker \gamma(p)$. Let $\mathbf{J} \subset U^\vee$ be a free submodule whose fiber over p is equal to J. Let $\mathbf{K} \subset U^\vee$ be the submodule orthogonal to $\mathbf{J}$ i.e.

$$\mathbf{K} := \{u \in U^\vee \mid \gamma(a)(u) = 0 \quad \forall a \in \mathbf{J}\} . \tag{1.1.27}$$

The localization of $\mathbf{K}$ at p is free. Let $K := \mathbf{K}(p)$ be the fiber of $\mathbf{K}$ at p; clearly $K = \ker \gamma(p)$. Localizing at p we have

$$U_p^\vee = \mathbf{K}_p \oplus \mathbf{J}_p . \tag{1.1.28}$$

Corresponding to (1.1.28) we may write $\gamma_p = \gamma_{\mathbf{K}} \oplus_\perp \gamma_{\mathbf{J}}$ where $\gamma_{\mathbf{K}} \colon \mathbf{K}_p \to \mathbf{K}_p^\vee$ and $\gamma_J \colon \mathbf{J}_p \to \mathbf{J}_p^\vee$ are symmetric maps. Notice that we have an equality of germs

$$(Y_\gamma, p) = (Y_{\gamma_{\mathbf{K}}}, p). \tag{1.1.29}$$

We claim that there is a compatible isomorphism of germs $(X_{\gamma_{\mathbf{K}}}, f_{\gamma_{\mathbf{K}}}^{-1}(p)) \cong (X_\gamma, f_\gamma^{-1}(p))$. In fact let $k := \dim K$ and $d := \operatorname{rk} U$. Choose bases of $\mathbf{K}_p$ and $\mathbf{J}_p$; by (1.1.28) we get a basis of $U_p^\vee$. The dual bases of $\mathbf{K}_p^\vee$, $\mathbf{J}_p^\vee$ and $U_p^\vee$ are compatible with respect to the decomposition dual to (1.1.28). Corresponding to the chosen bases we have embeddings $X_{\gamma_K} \hookrightarrow Y_{\gamma_K} \times \mathbb{C}^k$ and $X_\gamma \hookrightarrow Y_\gamma \times \mathbb{C}^d$. The decomposition dual to (1.1.28) gives an embedding $j \colon Y_{\gamma_K} \times \mathbb{C}^k \hookrightarrow Y_\gamma \times \mathbb{C}^d$.

Claim 1.4. *Keep notation as above. The composition*

$$X_{\gamma_K} \hookrightarrow (Y_{\gamma_K} \times \mathbb{C}^k) \xrightarrow{j} (Y_\gamma \times \mathbb{C}^d) \tag{1.1.30}$$

defines an isomorphism of germs in the analytic topology

$$(X_{\gamma_{\mathbf{K}}}, f_{\gamma_{\mathbf{K}}}^{-1}(p)) \xrightarrow{\sim} (X_\gamma, f_\gamma^{-1}(p)) \tag{1.1.31}$$

which commutes with the maps $f_{\gamma_{\mathbf{K}}}$ and f_γ.

Proof. This follows by writing $\gamma_p = \gamma_{\mathbf{K}} \oplus_\perp \gamma_{\mathbf{J}}$ and by recalling (1.1.20). We pass to the analytic topology in order to be able to extract the square root of a regular non-zero function. □

Proposition 1.5. *Assume that R is a finitely generated $\mathbb{C}$-algebra. Suppose that we have Exact Sequence* (1.1.19)*. Then the following hold:*

(1) $f_\gamma^{-1} Y_\gamma(1) \to Y_\gamma(1)$ is a topological covering of degree 2.

(2) Let $p \in (Y_\gamma \setminus Y_\gamma(1))$ be a closed point. The fiber $f_\gamma^{-1}(p)$ consists of a single point q. Let ξ_i be the coordinates on X_γ associated to Embedding (1.1.25)*; then $\xi_i(q) = 0$ for $i = 1, \ldots, d$.*

Proof. (1): Localizing at $p \in Y_\gamma(1)$ and applying **Claim 1.4** we get Item (1). (2): Since $\operatorname{cork} M_\gamma(p) \geq 2$ we have $M_\gamma^c(p) = 0$. Thus Item (2) follows from **Claim 1.3**. □

We may associate a double cover $f_\gamma \colon X_\gamma \to Y_\gamma$ to a map β which is symmetric in the derived category.

Hypothesis 1.6. *We have* (1.1.7) *with α an isomorphism and in addition $\alpha = \mu$.*

Proposition 1.7. *Assume that* **Hypothesis 1.6** *holds. Then* $R/(\det M_\lambda) \oplus N$ *equipped with the product given by* (1.1.18) *is a commutative (associative) ring.*

Proof. Let $\gamma := \lambda \circ \mu^{-1}$ and $U := U_0$. Then (1.1.19) holds and the product defined by m_β is equal to the product defined by m_γ. By **Proposition 1.2** we get that $R/(\det M_\lambda) \oplus N$ is a commutative associative ring. □

Definition 1.8. Suppose that **Hypothesis 1.6** holds: the *symmetrization of* (1.1.7) is Exact Sequence (1.1.19) with γ and U as in the proof of **Proposition 1.7**.

1.2 Structure sheaf of double EPW-sextics

Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that $Y_A \neq \mathbb{P}(V)$. We will define the associated double cover $X_A \to Y_A$ by applying the results of **Subsection 1.1**. Since A is Lagrangian the symplectic form defines a canonical isomorphism $\bigwedge^3 V/A \cong A^\vee$; thus (0.0.3) defines a map of vector-bundles $\lambda_A \colon F \to A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}$. Let $i \colon Y_A \hookrightarrow \mathbb{P}(V)$ be the inclusion map: since a local generator of $\det \lambda_A$ annihilates $\mathrm{coker}(\lambda_A)$ there is a unique sheaf ζ_A on Y_A such that we have an exact sequence

$$0 \longrightarrow F \xrightarrow{\lambda_A} A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow i_*\zeta_A \longrightarrow 0. \tag{1.2.1}$$

Choose $B \in \mathbb{LG}(\bigwedge^3 V)$ transversal to A. Thus we have a direct-sum decomposition $\bigwedge^3 V = A \oplus B$ and hence a projection map $\bigwedge^3 V \to A$ inducing a map $\mu_{A,B} \colon F \to A \otimes \mathcal{O}_{\mathbb{P}(V)}$. We claim that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & F & \xrightarrow{\lambda_A} & A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_A & \to & 0 \\ & & \big\downarrow{\scriptstyle \mu_{A,B}} & & \big\downarrow{\scriptstyle \mu^t_{A,B}} & & \big\downarrow{\scriptstyle \beta_A} & & \\ 0 & \to & A \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda^t_A} & F^\vee & \longrightarrow & Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) & \to & 0\,. \end{array} \tag{1.2.2}$$

In fact the second row is obtained by applying the $Hom(\,\cdot\,, \mathcal{O}_{\mathbb{P}(V)})$-functor to (1.2.1) and the equality $\mu^t_{A,B} \circ \lambda_A = \lambda^t_A \circ \mu_{A,B}$ holds because F is a Lagrangian sub-bundle of $\bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$. Lastly β_A is defined to be the unique map making the diagram commutative; it exists because the rows are exact. Notice that the map β_A is independent of the choice of B as suggested by the notation. Next by applying the $Hom(i_*\zeta_A, \,\cdot\,)$-functor to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(6) \longrightarrow \mathcal{O}_{Y_A}(6) \longrightarrow 0 \tag{1.2.3}$$

we get the exact sequence

$$0 \longrightarrow i_*Hom(\zeta_A, \mathcal{O}_{Y_A}(6)) \xrightarrow{\partial} Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{n} Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}(6)) \tag{1.2.4}$$

where n is locally equal to multiplication by $\det \lambda_A$. Since the second row of (1.2.2) is exact a local generator of $\det \lambda_A$ annihilates $Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)})$; thus $n = 0$ and hence we get a canonical isomorphism

$$\partial^{-1} \colon Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{\sim} i_*Hom(\zeta_A, \mathcal{O}_{Y_A}(6)). \tag{1.2.5}$$

We define $\widetilde{m}_A$ by setting

$$\begin{array}{ccc} \zeta_A \times \zeta_A & \xrightarrow{\widetilde{m}_A} & \mathcal{O}_{Y_A}(6) \\ (\sigma_1, \sigma_2) & \mapsto & (\partial^{-1} \circ \beta_A(\sigma_1))(\sigma_2). \end{array} \tag{1.2.6}$$

Let $\xi_A := \zeta_A(-3)$. Tensorizing both sides of (1.2.6) by $\mathcal{O}_{Y_A}(-6)$ we get a multiplication map

$$\xi_A \times \xi_A \xrightarrow{m_A} \mathcal{O}_{Y_A}. \tag{1.2.7}$$

Thus we have defined a multiplication map on $\mathcal{O}_{Y_A} \oplus \xi_A$. The following result is well-known to experts.

Proposition 1.9. *Let* $A \in \mathbb{LG}(\bigwedge^3 V)$ *and suppose that* $Y_A \neq \mathbb{P}(V)$. *Let notation be as above. Then:*

(1) β_A is an isomorphism.

(2) The multiplication map m_A is associative and commutative.

Proof. Let $[v_0] \in \mathbb{P}(V)$. Choose $B \in \mathbb{LG}(\bigwedge^3 V)$ transversal to F_{v_0} (and to A of course). Then $\mu_{A,B}$ is an isomorphism in an open neighborhood U of $[v_0]$. It follows that β_A is an isomorphism in a neighborhood of $[v_0]$. This proves Item (1). Let's prove Item (2). Let $B \in \mathbb{LG}(\bigwedge^3 V)$ and U be as above; we may assume that U is affine. Let $N := H^0(i_*\zeta_A|_U)$ and $\beta := H^0(\beta_A|_U)$. Thus $\beta\colon N \to \operatorname{Ext}^1(N, \mathbb{C}[U])$. By Commutativity of Diagram (1.2.2) and by **Proposition 1.7** we get that the multiplication map m_β is associative and commutative. On the other hand m_β is the multiplication induced by m_A on N; since $[v_0]$ is an arbitrary point of $\mathbb{P}(V)$ it follows that m_A is associative and commutative. □

We let $X_A := \operatorname{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A)$ and we let $f_A\colon X_A \to Y_A$ be the structure morphism. Then X_A is the *double EPW-sextic* associated to A and f_A is its structure map. The *covering involution* of X_A is the automorphism $\phi_A\colon X_A \to X_A$ corresponding to the involution of $\mathcal{O}_{Y_A} \oplus \xi_A$ with (-1)-eigensheaf equal to ξ_A.

1.3 Local models of double covers

In the present subsection we assume that R is a finitely generated $\mathbb{C}$-algebra. Let $\mathcal{W}$ be a finite-dimensional complex vector-space. We will suppose that we have an exact sequence

$$0 \longrightarrow R\otimes\mathcal{W}^\vee \xrightarrow{\gamma} R\otimes\mathcal{W} \longrightarrow N \longrightarrow 0, \qquad \gamma = \gamma^t. \tag{1.3.1}$$

Thus we have a double cover $f_\gamma\colon X_\gamma \to Y_\gamma$. Let $p \in Y_\gamma$ be a closed point. We will examine X_γ in a neighborhood of $f_\gamma^{-1}(p)$ when the corank of $\gamma(p)$ is small. We may view γ as a regular map $\operatorname{Spec} R \to \operatorname{Sym}^2\mathcal{W}$; thus it makes sense to consider the differential

$$d\gamma(p)\colon T_p \operatorname{Spec} R \to \operatorname{Sym}^2\mathcal{W}. \tag{1.3.2}$$

Let $K(p) := \ker\gamma(p) \subset \mathcal{W}^\vee$; we will consider the linear map

$$\begin{array}{ccc} T_p \operatorname{Spec} R & \overset{\delta_\gamma(p)}{\longrightarrow} & \operatorname{Sym}^2 K(p)^\vee \\ \tau & \mapsto & d\gamma(p)(\tau)|_{K(p)}\,. \end{array} \tag{1.3.3}$$

Let $d := \dim\mathcal{W}$; choosing a basis of $\mathcal{W}$ we realize X_γ as a subscheme of $\operatorname{Spec} R \times \mathbb{C}^d$ with ideal given by **Claim 1.3**. Since $\operatorname{cork}\gamma(p) \geq 2$ **Proposition 1.5** gives that $f_\gamma^{-1}(p)$ consists of a single point q - in fact the ξ_i-coordinates of q are all zero. Throughout this subsection we let

$$f_\gamma^{-1}(p) = \{q\}\,. \tag{1.3.4}$$

Claim 1.10. *Keep notation as above. Suppose that $d = \dim\mathcal{W} = 2$ and that $\gamma(p) = 0$. Then $I(X_\gamma)$ is generated by the entries of $\xi\cdot\xi^t - M_\gamma^c$.*

Proof. **Claim 1.3** together with a straightforward computation. □

Example 1.11. Let $R = \mathbb{C}[x, y, z]$, $\mathcal{W} = \mathbb{C}^2$. Suppose that the matrix associated to γ is

$$M_\gamma = \begin{pmatrix} x & y \\ y & z \end{pmatrix}. \tag{1.3.5}$$

Then $f_\gamma\colon X_\gamma \to Y_\gamma$ is identified with

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & V(xz - y^2) \\ (\xi_1, \xi_2) & \mapsto & (\xi_2^2,\, -\xi_1\xi_2,\, \xi_1^2) \end{array} \tag{1.3.6}$$

i.e. the quotient map for the action of $\langle -1\rangle$ on $\mathbb{C}^2$.

Proposition 1.12. *Keep notation as above. Suppose that the following hold:*

(a) cork $\gamma(p) = 2$,

(b) the localization R_p is regular.

Then X_γ is smooth at q if and only if $\delta_\gamma(p)$ is surjective.

Proof. Applying **Claim 1.4** we get that we may assume that $d = 2$. Let

$$M_\gamma = \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{1.3.7}$$

By **Claim 1.10** the ideal of X_γ in $\operatorname{Spec} R \times \mathbb{C}^2$ is generated by the entries of $\xi \cdot \xi^t - M_\gamma^c$ i.e.

$$I(X_\gamma) = (\xi_1^2 - c,\ \xi_1\xi_2 + b,\ \xi_2^2 - a). \tag{1.3.8}$$

Thus

$$\operatorname{cod}(T_q X_\gamma, T_q(\operatorname{Spec} R \times \mathbb{C}^2)) = \dim\langle da(p)\,, db(p)\,, dc(p)\rangle. \tag{1.3.9}$$

On the other hand $\operatorname{cod}_q(X_\gamma\,, \operatorname{Spec} R \times \mathbb{C}^2) = 3$ and hence we get that X_γ is smooth at q if and only if $\delta_\gamma(p)$ is surjective. $\square$

Claim 1.13. *Keep notation and hypotheses as above. Suppose that* cork $\gamma(p) \geq 3$. *Then X_γ is singular at q.*

Proof. Let I be the ideal of X_γ in $\operatorname{Spec} R[\xi_1, \ldots, \xi_d]$. By **Claim 1.3** we get that I is non-trivial but the differential at q of an arbitrary $g \in I$ is zero. $\square$

Next we will discuss in greater detail those X_γ whose corank at $f_\gamma^{-1}(p)$ is equal to 3. First we will identify the "universal"example (the universal example for corank 2 is **Example 1.11**). Let $\mathcal{V}$ be a 3-dimensional complex vector space. We view $\operatorname{Sym}^2 \mathcal{V}$ as an affine (6-dimensional) space and we let $R := \mathbb{C}[\operatorname{Sym}^2 \mathcal{V}]$ be its ring of regular functions. We identify $R \otimes_{\mathbb{C}} \mathcal{V}$ and $R \otimes_{\mathbb{C}} \mathcal{V}^\vee$ with the space of $\mathcal{V}$-valued, respectively $\mathcal{V}^\vee$-valued, regular maps on $\operatorname{Sym}^2 \mathcal{V}$. Let

$$R \otimes_{\mathbb{C}} \mathcal{V}^\vee \xrightarrow{\gamma} R \otimes_{\mathbb{C}} \mathcal{V} \tag{1.3.10}$$

be the map induced on the spaces of global sections by the tautological map of vector-bundles $\operatorname{Spec} R \times \mathcal{V}^\vee \longrightarrow \operatorname{Spec} R \times \mathcal{V}$. The map γ is symmetric. Let N be the cokernel of γ: thus

$$0 \longrightarrow R \otimes_{\mathbb{C}} \mathcal{V}^\vee \xrightarrow{\gamma} R \otimes_{\mathbb{C}} \mathcal{V} \longrightarrow N \longrightarrow 0 \tag{1.3.11}$$

is an exact sequence. Since γ is symmetric it defines a double cover $f\colon X(\mathcal{V}) \to Y(\mathcal{V})$ where

$$Y(\mathcal{V}) := \{\alpha \in \operatorname{Sym}^2 \mathcal{V} \mid \operatorname{rk}\, \alpha < 3\} \tag{1.3.12}$$

is the variety of degenerate quadratic forms. We let

$$\phi\colon X(\mathcal{V}) \to X(\mathcal{V}) \tag{1.3.13}$$

be the covering involution of f. One describes explicitly $X(\mathcal{V})$ as follows. Let

$$(\mathcal{V} \otimes \mathcal{V})_1 := \{\mu \in (\mathcal{V} \otimes \mathcal{V}) \mid \operatorname{rk}\, \mu \leq 1\}. \tag{1.3.14}$$

Thus $(\mathcal{V} \otimes \mathcal{V})_1$ is the cone over the Segre variety $\mathbb{P}(\mathcal{V}) \times \mathbb{P}(\mathcal{V})$. We have a finite degree-2 map

$$\begin{array}{ccc} (\mathcal{V} \otimes \mathcal{V})_1 & \xrightarrow{\sigma} & Y(\mathcal{V}) \\ \mu & \mapsto & \mu + \mu^t\,. \end{array} \tag{1.3.15}$$

Proposition 1.14. *Keep notation as above. There exists a commutative diagram*

$$
\begin{array}{ccc}
(\mathcal{V}\otimes\mathcal{V})_1 & \xrightarrow{\ \tau\ } & X(\mathcal{V}) \\
 & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle f} & \\
 & Y(\mathcal{V}) &
\end{array}
\tag{1.3.16}
$$

where τ is an isomorphism. Let ϕ be Involution (1.3.13)*: then*

$$\phi\circ\tau(\mu)=\tau(\mu^t),\qquad \forall\mu\in(\mathcal{V}\otimes\mathcal{V})_1. \tag{1.3.17}$$

Proof. In order to define τ we will give a coordinate-free version of Inclusion (1.1.25) in the case of $X(\mathcal{V})$. Let

$$
\begin{array}{ccc}
\operatorname{Sym}^2\mathcal{V}\times(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}) & \xrightarrow{\Psi} & (\mathcal{V}\otimes\bigwedge^3\mathcal{V})\times(\mathcal{V}^\vee\otimes\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}\otimes\bigwedge^3\mathcal{V}) \\
(\alpha,\xi) & \mapsto & (\alpha\circ\xi,\ \xi^t\circ\xi-\bigwedge^2\alpha)\,.
\end{array}
\tag{1.3.18}
$$

A few words of explanation. In the definition of the first component of $\Psi(\alpha,\xi)$ we view ξ as belonging to $\operatorname{Hom}(\bigwedge^3\mathcal{V}^\vee,\mathcal{V}^\vee)$, in the definition of the second component of $\Psi(\alpha,\xi)$ we view ξ as belonging to $\operatorname{Hom}(\mathcal{V}\otimes\bigwedge^3\mathcal{V}^\vee,\mathbb{C})$. Moreover we make the obvious choice of isomorhpism $\mathbb{C}\cong\mathbb{C}^\vee$. Secondly

$$\bigwedge^2\alpha\in\operatorname{Hom}(\bigwedge^2\mathcal{V}^\vee,\ \bigwedge^2\mathcal{V})=\operatorname{Hom}(\mathcal{V}\otimes\bigwedge^3\mathcal{V}^\vee,\ \mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V})=\mathcal{V}^\vee\otimes\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}\otimes\mathcal{V}\,. \tag{1.3.19}$$

Choosing a basis of $\mathcal{V}$ we get an embedding $X(\mathcal{V})\subset\operatorname{Sym}^2\mathcal{V}\times\mathbb{C}^3$, see (1.1.25). **Claim 1.3** gives equality of pairs

$$(\operatorname{Sym}^2\mathcal{V}\times(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}),\ \Psi^{-1}(0))=(\operatorname{Sym}^2\mathcal{V}\times\mathbb{C}^3,\ X(\mathcal{V}))\,, \tag{1.3.20}$$

where $\Psi^{-1}(0)$ is the scheme-theoretic fiber of Ψ. Now notice that we have an isomorphism

$$
\begin{array}{ccc}
\mathcal{V}\otimes\mathcal{V} & \overset{T}{\underset{\sim}{\longrightarrow}} & \operatorname{Sym}^2\mathcal{V}\times(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}) \\
\epsilon & \mapsto & (\epsilon+\epsilon^t,\epsilon-\epsilon^t)\,.
\end{array}
\tag{1.3.21}
$$

Let $\tau:=T|_{(\mathcal{V}\otimes\mathcal{V})_1}$: thus we have an embedding

$$\tau\colon(\mathcal{V}\otimes\mathcal{V})_1\hookrightarrow\operatorname{Sym}^2\mathcal{V}\times(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}). \tag{1.3.22}$$

We will show that we have equality of schemes

$$\operatorname{im}(\tau)=\Psi^{-1}(0)(=X(\mathcal{V})). \tag{1.3.23}$$

First let

$$
\begin{array}{ccc}
\mathcal{V}\oplus\mathcal{V} & \xrightarrow{\rho} & (\mathcal{V}\otimes\mathcal{V})_1 \\
(\eta,\beta) & \mapsto & \eta^t\circ\beta.
\end{array}
\tag{1.3.24}
$$

Notice that ρ is the quotient map for the $\mathbb{C}^\times$-action on $\mathcal{V}\oplus\mathcal{V}$ defined by $t(\eta,\beta):=(t\eta,t^{-1}\beta)$. We have

$$\tau\circ\pi=(\eta^t\circ\beta+\beta^t\circ\eta,\eta\wedge\beta). \tag{1.3.25}$$

Let's prove that

$$\Psi^{-1}(0)\supset\operatorname{im}(\tau)\,. \tag{1.3.26}$$

Notice that $\mathrm{Gl}(\mathcal{V})$ acts on $(\mathcal{V}\otimes\mathcal{V})_1$ with a unique dense orbit namely $\{\eta^t\circ\beta \mid \eta\wedge\beta\neq 0\}$. An easy computation shows that $\tau(\eta^t\circ\beta)\in\Psi^{-1}(0)$ for a conveniently chosen $\eta^t\circ\beta$ in the dense orbit of $(\mathcal{V}\otimes\mathcal{V})_1$; it follows that (1.3.26) holds. On the other hand T defines an isomorphism of pairs

$$(\mathcal{V}\otimes\mathcal{V},\,(\mathcal{V}\otimes\mathcal{V})_1)\cong(\mathrm{Sym}^2\mathcal{V}^\vee\times(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}),\,\mathrm{im}(\tau))\,. \tag{1.3.27}$$

Since the ideal of $(\mathcal{V}\otimes\mathcal{V})_1$ in $\mathcal{V}\otimes\mathcal{V}$ is generated by 9 linearly independent quadrics we get that the ideal of $\mathrm{im}(\tau)$ in $\mathrm{Sym}^2\mathcal{V}^\vee\times(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V})$ is generated by 9 linearly independent quadrics. The ideal of $\Psi^{-1}(0)$ in $\mathrm{Sym}^2\mathcal{V}\times(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V})$ is likewise generated by 9 linearly independent quadrics - see (1.3.18). Since $\Psi^{-1}(0)\supset\mathrm{im}(\tau)$ we get that the ideals of $\Psi^{-1}(0)$ and of $\mathrm{im}(\tau)$ are the same and hence (1.3.23) holds. This proves that τ is an isomorphism between $(\mathcal{V}\otimes\mathcal{V})_1$ and $X(\mathcal{V})$. Diagram (1.3.16) is commutative by construction. Equation (1.3.17) is equivalent to the equality

$$\phi(\tau\circ\rho(\beta,\eta))=\tau\circ\rho(\eta,\beta)). \tag{1.3.28}$$

The above equality holds because $\beta\wedge\eta=-\eta\wedge\beta$. □

The following result is an immediate consequence of **Proposition 1.14**.

Corollary 1.15. $\mathrm{sing}\,X(\mathcal{V})=\tau(0)=f^{-1}(0)$.

2 The divisor Δ

2.1 Parameter counts

Let $\Delta_+\subset\mathbb{LG}(\bigwedge^3 V)$ and $\widetilde{\Delta}_+,\widetilde{\Delta}_+(0)\subset\mathbb{LG}(\bigwedge^3 V)\times\mathbb{P}(V)^2$ be

$$\begin{aligned}\Delta_+ &:= \{A\in\mathbb{LG}(\textstyle\bigwedge^3 V)\mid |Y_A[3]|>1\}\,, &(2.1.1)\\ \widetilde{\Delta}_+ &:= \{(A,[v_1],[v_2])\mid [v_1]\neq[v_2],\quad \dim(A\cap F_{v_i})\geq 3\}\,, &(2.1.2)\\ \widetilde{\Delta}_+(0) &:= \{(A,[v_1],[v_2])\mid [v_1]\neq[v_2],\quad \dim(A\cap F_{v_i})=3\}\,. &(2.1.3)\end{aligned}$$

Notice that $\widetilde{\Delta}_+$ and $\widetilde{\Delta}_+(0)$ are locally closed.

Lemma 2.1. *Keep notation as above. The following hold:*

(1) $\widetilde{\Delta}_+$ *is irreducible of dimension* 53.

(2) Δ_+ *is constructible and* $\mathrm{cod}(\Delta_+,\mathbb{LG}(\bigwedge^3 V))\geq 2$.

Proof. (1): Let's prove that $\widetilde{\Delta}_+(0)$ is irreducible of dimension 53. Consider the map

$$\begin{array}{ccc}\widetilde{\Delta}_+(0) & \overset{\eta}{\longrightarrow} & \mathrm{Gr}(3,\bigwedge^3 V)^2\times\mathbb{P}(V)^2\\ (A,[v_1],[v_2]) & \mapsto & (A\cap F_{v_1},A\cap F_{v_2},[v_1],[v_2])\,.\end{array} \tag{2.1.4}$$

We have

$$\mathrm{im}\,\eta=\{(K_1,K_2,[v_1],[v_2])\mid K_i\in\mathrm{Gr}(3,F_{v_i}),\quad K_1\bot K_2,\quad [v_1]\neq[v_2]\}\,. \tag{2.1.5}$$

We stratify $\mathrm{im}\,\eta$ according to $i:=\dim(K_1\cap F_{v_2})$ and to $j:=\dim(K_1\cap K_2)$; of course $j\leq i$. Let $(\mathrm{im}\,\eta)_{i,j}\subset\mathrm{im}\,\eta$ be the stratum corresponding to i,j. A straightforward computation gives that

$$\begin{aligned}\dim\eta^{-1}(\mathrm{im}\,\eta)_{i,j}&=10+7(3-i)+j(i-j)+(3-j)(4+i)+\frac{1}{2}(j+5)(j+4)=\\ &=53-4i-\frac{1}{2}j(j-1)\,.\end{aligned} \tag{2.1.6}$$

Since $0\leq i,j$ one gets that the maximum is achieved for $i=j=0$ and that it equals 53. It follows that $\widetilde{\Delta}_+(0)$ is irreducible of dimension 53. On the other hand $\widetilde{\Delta}_+(0)$ is dense in $\widetilde{\Delta}_+$ (easy) and hence we get that Item (1) holds. (2): Let $\pi_+\colon\widetilde{\Delta}_+\to\mathbb{LG}(\bigwedge^3 V)$ be the forgetful map: $\pi_+([v_1],[v_2],A)=A$. Then $\pi_+(\widetilde{\Delta}_+)=\Delta_+$. By Item (1) we get that $\dim\Delta_+\leq 53$: since $\dim\mathbb{LG}(\bigwedge^3 V)=55$ we get that Item (2) holds. □

Proposition 2.2. *The following hold:*

(1) Δ *is closed irreducible of codimension* 1 *in* $\mathbb{LG}(\bigwedge^3 V)$ *and not equal to* Σ.

(2) If $A \in \Delta$ *is generic then* $Y_A[3] = Y_A(3)$ *and it consists of a single point.*

Proof. (1): Let

$$\widetilde{\Delta} := \{(A, [v]) \mid \dim(F_v \cap A) \geq 3\}, \qquad \widetilde{\Delta}(0) := \{(A, [v]) \mid \dim(F_v \cap A) = 3\}. \tag{2.1.7}$$

Then $\widetilde{\Delta}$ is a closed subset of $\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V)$ and $\widetilde{\Delta}(0)$ is an open subset of $\widetilde{\Delta}$. Let $\pi\colon \widetilde{\Delta} \to \mathbb{LG}(\bigwedge^3 V)$ be the forgetful map. Thus $\pi(\widetilde{\Delta}) = \Delta$: since π is projective it follows that Δ is closed. Projecting $\widetilde{\Delta}(0)$ to $\mathbb{P}(V)$ we get that $\widetilde{\Delta}(0)$ is smooth irreducible of dimension 54. A standard dimension count shows that $\widetilde{\Delta}(0)$ is open dense in $\widetilde{\Delta}$; thus $\widetilde{\Delta}$ is irreducible of dimension 54. It follows that Δ is irreducible. By **Lemma 2.1** we know that $\dim \widetilde{\Delta}_+ \leq 53$. It follows that the generic fiber of $\widetilde{\Delta} \to \Delta$ is a single point, in particular $\dim \Delta = 54$ and hence $\operatorname{cod}(\Delta, \mathbb{LG}(\bigwedge^3 V)) = 1$ because $\dim \mathbb{LG}(\bigwedge^3 V) = 55$. A dimension count shows that $\dim(\Delta \cap \Sigma) < 54$ and hence $\Delta \neq \Sigma$. This finishes the proof of Item (1). (2): Let $A \in \Delta$ be generic: we already noticed that there exists a unique $[v] \in \mathbb{P}(V)$ such that $([v], A) \in \widetilde{\Delta}$, i.e. $Y_A[3]$ consists of a single point. Since $\widetilde{\Delta}(0)$ is dense in $\widetilde{\Delta}$ and $\dim \widetilde{\Delta} = \dim \Delta$ we get that $([v], A) \in \widetilde{\Delta}(0)$, i.e. $Y_A[3] = Y_A(3)$. This finishes the proof of Item (2). □

2.2 First order computations

Let $(A, [v_0]) \in \widetilde{\Delta}(0)$. We will study the differential of $\pi\colon \widetilde{\Delta} \to \mathbb{LG}(\bigwedge^3 V)$ at $(A, [v_0])$. First we will give a local description of $\widetilde{\Delta}$ as degeneracy locus. Let

$$\mathbb{N}(V) := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid Y_A = \mathbb{P}(V)\}. \tag{2.2.1}$$

Notice that $\mathbb{N}(V)$ is closed. Let $\mathcal{Y}$ be the tautological family of EPW-sextics i.e.

$$\mathcal{Y} := \{(A, [v]) \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V)) \times \mathbb{P}(V) \mid \dim(A \cap F_v) > 0\}. \tag{2.2.2}$$

Of course $\mathcal{Y}$ has a description as a determinantal variety and hence it has a natural scheme structure. For $\mathcal{U} \subset (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ open we let $\mathcal{Y}_{\mathcal{U}} := \mathcal{Y} \cap (\mathcal{U} \times \mathbb{P}(V))$. Given $B \in \mathbb{LG}(\bigwedge^3 V)$ let

$$U_B := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid A \pitchfork B\} \setminus \mathbb{N}(V). \tag{2.2.3}$$

(Here $A \pitchfork B$ means that A intersects transversely B i.e. $A \cap B = \{0\}$.) Let $i_{U_B} : \mathcal{Y}_{U_B} \hookrightarrow U_B \times \mathbb{P}(V)$ be the inclusion and let $\mathcal{A}$ be the tautological rank-10 vector-bundle on $\mathbb{LG}(\bigwedge^3 V)$ (the fiber of $\mathcal{A}$ over A is A itself). Going through the argument that produced Commutative Diagram (1.2.2) we get that there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{U_B} \boxtimes F & \xrightarrow{\lambda_{U_B}} & (\mathcal{A}^\vee|_{U_B}) \boxtimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_{U_B,*}\zeta_{U_B} & \to & 0 \\
 & & \Big\downarrow{\scriptstyle \mu_{U_B}} & & \Big\downarrow{\scriptstyle \mu^t_{U_B}} & & \Big\downarrow{\scriptstyle \beta_{U_B}} & & \\
0 & \to & (\mathcal{A}|_{U_B}) \boxtimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda^t_{U_B}} & \mathcal{O}_{U_B} \boxtimes F^\vee & \longrightarrow & Ext^1(i_{U_B,*}\zeta_{U_B}, \mathcal{O}_{U_B \times \mathbb{P}(V)}) & \to & 0
\end{array} \tag{2.2.4}$$

Now let $(A, [v_0]) \in \mathcal{Y}$. Choose $B \in \mathbb{LG}(\bigwedge^3 V)$ such that $B \pitchfork A$ and $B \pitchfork F_{v_0}$. Let $\mathcal{N} \subset \mathbb{P}(V)$ be an open neighborhood of $[v_0]$ such that $B \pitchfork F_w$ for all $w \in \mathcal{N}$. The restriction to U_B of $\mathcal{A}$ is trivial and the restriction to $\mathcal{N}$ of F is likewise trivial. Moreover the restriction of μ_{U_B} to $U_B \times \mathcal{N}$ is an isomorphism. Let

$$\gamma := (\lambda_{U_B}|_{U_B \times \mathcal{N}}) \circ (\mu_{U_B}|_{U_B \times \mathcal{N}})^{-1}. \tag{2.2.5}$$

We have an exact sequence

$$0 \longrightarrow (\mathcal{A}|_{U_B}) \boxtimes \mathcal{O}_{\mathcal{N}} \xrightarrow{\gamma} (\mathcal{A}^\vee|_{U_B}) \boxtimes \mathcal{O}_{\mathcal{N}} \longrightarrow i_{U_B,*}\zeta_{U_B}|_{U_B \times \mathcal{N}} \longrightarrow 0 \tag{2.2.6}$$

The map γ is symmetric, in fact it is the symmetrization of the restriction of (2.2.4) to $U_B \times \mathcal{N}$ - see **Definition 1.8**. Then $\widetilde{\Delta} \cap (U_B \times \mathcal{N})$ is the symmetric degeneration locus

$$\widetilde{\Delta} \cap (U_B \times \mathcal{N}) = \{(A', [v]) \in (U_B \times \mathcal{N}) \mid \operatorname{cork} \gamma(A', [v]) \geq 3\} \tag{2.2.7}$$

and hence it inherits a natural structure of closed subscheme of $\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V)$. In order to study the differential of the forgetful map $\widetilde{\Delta} \to \mathbb{P}(V)$ we will introduce some notation. Given $v \in V$ we define a quadratic form $\phi_v^{v_0}$ on F_{v_0} as follows. Let $\alpha \in F_{v_0}$; then $\alpha = v_0 \wedge \beta$ for some $\beta \in \bigwedge^2 V$. We set

$$\phi_v^{v_0}(\alpha) := \operatorname{vol}(v_0 \wedge v \wedge \beta \wedge \beta). \tag{2.2.8}$$

The above equation gives a well-defined quadratic form on F_{v_0} because β is determined up to addition by an element of F_{v_0}. Of course $\phi_v^{v_0}$ depends only on the class of v in $V/[v_0]$. Choose a direct-sum decomposition

$$V = [v_0] \oplus V_0. \tag{2.2.9}$$

We have the isomorphism

$$\begin{array}{rccc} \lambda_{V_0}^{v_0}\colon & \bigwedge^2 V_0 & \xrightarrow{\sim} & F_{v_0} \\ & \beta & \mapsto & v_0 \wedge \beta\,. \end{array} \tag{2.2.10}$$

Under the above identification the Plücker quadratic forms on $\bigwedge^2 V_0$ correspond to the quadratic forms $\phi_v^{v_0}$ for v varying in V_0. Let $K := A \cap F_{v_0}$ and

$$\begin{array}{ccc} V_0 & \xrightarrow{\tau_K^{v_0}} & \operatorname{Sym}^2 K^\vee \\ v & \mapsto & \phi_v^{v_0}|_K \end{array} \qquad \begin{array}{ccc} \operatorname{Sym}^2 A^\vee & \xrightarrow{\theta_K^A} & \operatorname{Sym}^2 K^\vee \\ q & \mapsto & q|_K\,. \end{array} \tag{2.2.11}$$

The isomorphism

$$\begin{array}{ccc} V_0 & \xrightarrow{\sim} & \mathbb{P}(V) \setminus \mathbb{P}(V_0) \\ v & \mapsto & [v_0 + v] \end{array}$$

defines an isomorphism $V_0 \cong T_{[v_0]}\mathbb{P}(V)$. Recall that the tangent space to $\mathbb{LG}(\bigwedge^3 V)$ at A is canonically identified with $\operatorname{Sym}^2 A^\vee$.

Proposition 2.3. *Keep notation as above - in particular choose* (2.2.9). *Then*

$$T_{(A,[v_0])}\widetilde{\Delta} \subset T_{(A,[v_0])}\left(\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V)\right) = \operatorname{Sym}^2 A^\vee \oplus V_0 \tag{2.2.12}$$

is given by

$$T_{([v_0],A)}\widetilde{\Delta} = \{(q, v) \mid \theta_K^A(q) - \tau_K^{v_0}(v) = 0\}. \tag{2.2.13}$$

Proof. By the (local) degeneracy description (2.2.7) we get that $(q, v) \in T_{([v_0],A)}\widetilde{\Delta}$ if and only if

$$0 = d\gamma(A, [v_0])(q, v)|_K = d\gamma(A, [v_0])(q, 0)|_K + d\gamma(A, [v_0])(0, v)|_K.$$

It is clear that $d\gamma(A, [v_0])(q, 0)|_K = \theta_K^A(q)$. On the other hand Equation (2.26) of [12] gives that

$$d\gamma(A, [v_0])(0, v)|_K = -\tau_K^{v_0}(v). \tag{2.2.14}$$

The proposition follows. □

Corollary 2.4. $\widetilde{\Delta}(0)$ *is smooth (of codimension* 6 *in* $\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V)$*). Let* $(A, [v_0]) \in \widetilde{\Delta}(0)$ *and* $K := A \cap F_{v_0}$. *The differential* $d\pi(A, [v_0])$ *is injective if and only if* $\tau_K^{v_0}$ *is injective.*

Proof. Let $(A, [v_0]) \in \widetilde{\Delta}(0)$ and $K := A \cap F_{v_0}$. The map θ_K^A is surjective: by **Proposition 2.3** we get that $T_{(A,[v_0])}\widetilde{\Delta}(0)$ has codimension 6 in $T_{(A,[v_0])}(\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V))$. On the other hand the description of $\widetilde{\Delta}(0)$ as a symmetric degeneration locus - see (2.2.7) - gives that $\widetilde{\Delta}(0)$ has codimension at most 6 in $\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V)$: it follows that $\widetilde{\Delta}(0)$ is smooth of codimension 6 in $\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V)$. The statement about injectivity of $d\pi(A, [v_0])$ follows at once form **Proposition 2.3**. □

A comment regarding **Corollary 2.4**. The statement about smoothness of $\widetilde{\Delta}(0)$ is *not* contained in the proof of **Proposition 2.2** because in that proof we consider $\widetilde{\Delta}(0)$ with its reduced structure. Before stating the next result we give the following definition: given $A \in \mathbb{LG}(\bigwedge^3 V)$ we let

$$\Theta_A := \{W \in \mathrm{Gr}(3, V) \mid \bigwedge^3 W \subset A\}. \tag{2.2.15}$$

Proposition 2.5. *Let $(A, [v_0]) \in \widetilde{\Delta}(0)$ and let $K := A \cap F_{v_0}$. Then $\tau_K^{v_0}$ is injective if and only if*

(1) no $W \in \Theta_A$ contains v_0, or

(2) there is exactly one $W \in \Theta_A$ containing v_0 and moreover

$$A \cap F_{v_0} \cap (\bigwedge^2 W \wedge V) = \bigwedge^3 W. \tag{2.2.16}$$

If Item (1) holds then $\mathrm{im}\,\tau_K^{v_0}$ belongs to the unique open $\mathrm{PGL}(K)$-orbit of $\mathrm{Gr}(5, \mathrm{Sym}^2 K^\vee)$, if Item (2) holds then $\mathrm{im}\,\tau_K^{v_0}$ belongs to the unique closed $\mathrm{PGL}(K)$-orbit of $\mathrm{Gr}(5, \mathrm{Sym}^2 K^\vee)$.

Proof. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$. Let

$$\rho_{V_0}^{v_0} \colon F_{v_0} \xrightarrow{\sim} \bigwedge^2 V_0 \tag{2.2.17}$$

be the inverse of Isomorphism (2.2.10). Let $\mathbf{K} := \mathbb{P}(\rho_{V_0}^{v_0}(K)) \subset \mathbb{P}(\bigwedge^2 V_0)$; then $\mathbf{K}$ is a projective plane. Isomorphism $\rho_{V_0}^{v_0}$ identifies the space of quadratic forms $\phi_v^{v_0}$, for $v \in V_0$, with the space of Plücker quadratic forms on $\bigwedge^2 V_0$. Since the ideal of $\mathrm{Gr}(2, V_0) \subset \mathbb{P}(\bigwedge^2 V_0)$ is generated by the Pl'ucker quadratic forms we get that $\tau_K^{v_0}$ is identified with the natural restriction map

$$V_0 = H^0(\mathcal{I}_{\mathrm{Gr}(2,V_0)}(2)) \xrightarrow{\tau_K^{v_0}} H^0(\mathcal{O}_{\mathbf{K}}(2)) = \mathrm{Sym}^2 K^\vee. \tag{2.2.18}$$

It follows that if the scheme-theoretic intersection $\mathbf{K} \cap \mathrm{Gr}(2, V_0)$ is not empty nor a single reduced point then $\tau_K^{v_0}$ is not injective. Now suppose that $\mathbf{K} \cap \mathrm{Gr}(2, V_0)$ is

(1′) empty i.e. Item (1) holds, or

(2′) a single reduced point, i.e. Item (2) holds.

Let

$$\mathbb{P}(\bigwedge^2 V_0) \overset{\Phi}{\dashrightarrow} |H^0(\mathcal{I}_{\mathrm{Gr}(2,V_0)}(2))|^\vee = \mathbb{P}(V_0^\vee) \tag{2.2.19}$$

be the natural map: it associates to $[\alpha] \notin \mathrm{Gr}(2, V_0)$ the projectivization of $\mathrm{supp}\,\alpha$. We have a tautological identification

$$\mathbf{K} \overset{\Phi|_{\mathbf{K}}}{\dashrightarrow} \mathbb{P}(\mathrm{im}\,\tau_K^{v_0})^\vee$$

and of course $\Phi|_{\mathbf{K}}$ is the Veronese embedding $\mathbf{K} \to |\mathcal{O}_{\mathbf{K}}(2)|^\vee$ followed by the projection with center $\mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0}))$. Notice that $\tau_K^{v_0}$ is not injective if and only if $\dim \mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0})) \geq 1$. Now suppose that (1′) holds. Then $\Phi|_{\mathbf{K}}$ is regular and in fact it is an isomorphism onto its image - see Lemma 2.7 of [15]. Since the chordal variety of the Veronese surface in $|\mathcal{O}_{\mathbf{K}}(2)|^\vee$ is a hypersurface it follows that $\dim \mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0})) < 1$ and hence $\tau_K^{v_0}$ is injective.We also get that $\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0})$ is a point in $|\mathcal{O}_{\mathbf{K}}(2)|^\vee$ which does not belong to the chordal variety of the Veronese surface and hence it belongs to unique open $\mathrm{PGL}(K)$-orbit. Now suppose that (2′) holds. Assume that $\tau_K^{v_0}$ is not injective. Then $\dim \mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0})) \geq 1$. It follows that there exist $[x] \neq [y] \in \mathbf{K}$ in the regular locus of $\Phi|_{\mathbf{K}}$ (i.e. neither x nor y is decomposable) such that $\Phi([x]) = \Phi([y])$. By the description of Φ given above in terms of supports we get that $\mathrm{supp}(x) = \mathrm{supp}(y) = U$ where $\dim U = 4$; since $\mathrm{Gr}(2, U)$ is a hypersurface in $\mathbb{P}(\bigwedge^2 U)$ we get that the line $\langle [x], [y] \rangle \subset \mathbb{P}(\bigwedge^2 V_0)$ intersects $\mathrm{Gr}(2, U)$ in a subscheme of length 2. Since $\langle [x], [y] \rangle \subset \mathbf{K}$ it follows that $\mathbf{K} \cap \mathrm{Gr}(2, V_0)$ contains a scheme of length 2, that contradicts Item (2′). This proves that if (2′) holds then $\tau_K^{v_0}$ is injective. It also follows that $\mathrm{Ann}(\tau_K^{v_0})$ belongs to the Veronese surface in $|\mathcal{O}_{\mathbf{K}}(2)|^\vee$ i.e. $\mathrm{im}(\tau_K^{v_0})$ belongs to the unique closed $\mathrm{PGL}(K)$-orbit. □

3 Simultaneous resolution

In the first subsection we will analyze families of double EPW-sextics and their singular locus. The second subsection shows how to construct the simultaneous desingularization described in Item (3) of **Section 0** (the relation with the Hilbert square of a $K3$ will be given in **Section 4**).

3.1 Families of double EPW-sextics

Let $\mathcal{U} \subset (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ (see (2.2.1)) be open. Suppose that there exist a scheme $\mathcal{X}_{\mathcal{U}}$ and a finite $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ such that for every $A \in \mathcal{U}$ the induced map $f^{-1}Y_A \to Y_A$ is identified with $f_A\colon X_A \to Y_A$: then we say that a *tautological family of double EPW-sextics parametrized by $\mathcal{U}$ exists* - often we simply state that $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. Composing $f_{\mathcal{U}}$ with the natural map $\mathcal{Y}_{\mathcal{U}} \to \mathcal{U}$ we get a map $\rho_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ such that $\rho_{\mathcal{U}}^{-1}(A) \cong X_A$.

Proposition 3.1. *Let $B \in \mathbb{LG}(\bigwedge^3 V)$. A tautological family of double EPW-sextics parametrized by U_B exists (U_B is given by (2.2.3)).*

Proof. Let $\nu\colon \mathcal{Y}_{U_B} \to \mathbb{P}(V)$ be projection. Let $\xi_{U_B} := \zeta_{U_B} \otimes \nu^*\mathcal{O}_{\mathbb{P}(V)}(-3)$ where ζ_{U_B} is the sheaf on $\mathcal{Y}_{U_B}$ fitting in (2.2.4). Look at Commutative Diagram (2.2.4): proceeding as in the definition of the multiplication on $\mathcal{O}_{Y_A} \oplus \xi_A$ we get that β_{U_B} defines a multiplication on $\mathcal{O}_{\mathcal{Y}_{U_B}} \oplus \xi_{U_B}$. By **Proposition 1.7** we get that $\mathcal{O}_{\mathcal{Y}_{U_B}} \oplus \xi_{U_B}$ is an associative commutative ring. Let $\mathcal{X}_{U_B} := \mathrm{Spec}(\mathcal{O}_{\mathcal{Y}_{U_B}} \oplus \xi_{U_B})$ and $f_{U_B}\colon \mathcal{X}_{U_B} \to \mathcal{Y}_{U_B}$ be the structure map. □

Let $\mathcal{U} \subset (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ be open and such that $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. We will determine the singular locus of $\mathcal{X}_{\mathcal{U}}$. Let

$$\mathcal{Y}[d] := \quad \{(A,[v]) \in (\mathbb{LG}(\textstyle\bigwedge^3 V) \setminus \mathbb{N}(V)) \times \mathbb{P}(V) \mid \dim(A \cap F_v) \ge d\}, \tag{3.1.1}$$

$$\mathcal{Y}(d) := \quad \{(A,[v]) \in (\mathbb{LG}(\textstyle\bigwedge^3 V) \setminus \mathbb{N}(V)) \times \mathbb{P}(V) \mid \dim(A \cap F_v) = d\}. \tag{3.1.2}$$

Then $\mathcal{Y}[d]$ has a natural structure of closed subscheme of $\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V)$ given by its local description as a symmetric determinantal variety - see **Subsection 2.2** of [15]. Let $\mathcal{U} \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ be open. We let $\mathcal{Y}_{\mathcal{U}}[d] := \mathcal{Y}[d] \cap \mathcal{Y}_{\mathcal{U}}$ and similarly for $\mathcal{Y}_{\mathcal{U}}(d)$. Suppose that $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ is defined; we let

$$\mathcal{W}_{\mathcal{U}} := f_{\mathcal{U}}^{-1}\mathcal{Y}[3]. \tag{3.1.3}$$

Notice that the restriction of $f_{\mathcal{U}}$ to $\mathcal{W}_{\mathcal{U}}$ defines an isomorphism $\mathcal{W}_{\mathcal{U}} \xrightarrow{\sim} \mathcal{Y}_{\mathcal{U}}[3]$. We will prove the following result.

Proposition 3.2. *Let $\mathcal{U} \subset (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ be open and suppose that $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. Then $\mathrm{sing}\,\mathcal{X}_{\mathcal{U}} = \mathcal{W}_{\mathcal{U}}$.*

Proof. We may assume that $\mathcal{U} = U_B \times \mathcal{N}$ where $B \in \mathbb{LG}(\bigwedge^3 V)$ and $\mathcal{N} \subset \mathbb{P}(V)$ is an open subset such that $B \pitchfork F_w$ for all $w \in \mathcal{N}$. Then (see the proof of **Proposition 3.1**)

$$f_{U_B}^{-1}(\mathcal{U}) = X_\gamma \text{ where } \gamma \text{ is given by (2.2.5).} \tag{3.1.4}$$

Thus it suffices to examine X_γ. Let $(A,[v]) \in \mathcal{U}$ and

$$\delta_\gamma(A,[v])\colon T_{(A,[v])}\mathbb{LG}(\bigwedge^3 V) \times \mathbb{P}(V) \longrightarrow \mathrm{Sym}^2(A \cap F_v)^\vee \tag{3.1.5}$$

be as in (1.3.3). The restriction of $\delta_\gamma(A,[v])$ to the tangent space to $\mathbb{LG}(\bigwedge^3 V)$ at A is surjective; thus

$$\delta_\gamma(A,[v]) \text{ is surjective.} \tag{3.1.6}$$

Let $q \in \mathcal{X}_\gamma$ and $f_{\mathcal{U}}(q) = (A,[v])$. Suppose that $q \notin \mathcal{W}_{\mathcal{U}}$ i.e. that $\mathrm{cork}\,\gamma(p) \le 2$. If $\mathrm{cork}\,\gamma(p) = 1$ then $Y_{\mathcal{U}} = Y_\gamma$ is smooth because the differential $\delta_\gamma(A,[v])$ is surjective: by **Proposition 1.5** we get that $\mathcal{X}_{\mathcal{U}}$ is smooth at q. If $\mathrm{cork}\,\gamma(p) = 2$ then $\mathcal{X}_{\mathcal{U}}$ is smooth at q by **Proposition 1.12** - recall that the differential $\delta_\gamma(A,[v])$ is surjective. This proves that $\mathrm{sing}\,\mathcal{X}_{\mathcal{U}} \subset \mathcal{W}_{\mathcal{U}}$. On the other hand $\mathcal{W}_{\mathcal{U}} \subset \mathrm{sing}\,\mathcal{X}_{\mathcal{U}}$ by **Claim 1.13**. □

We will close the present subsection by proving a few results about the individual X_A's.

Lemma 3.3. *Let $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$ and $[v] \in Y_A$. Suppose that $\dim(A \cap F_v) \leq 2$ and that there is no $W \in \Theta_A$ (see (2.2.15)) containing v. Then X_A is smooth at $f_A^{-1}([v])$.*

Proof. Let $q \in f_A^{-1}([v])$. Suppose that $\dim(A \cap F_v) = 1$. By Corollary 2.5 of [15] we get that Y_A is smooth at $[v]$, thus X_A is smooth at q by **Proposition 1.5**. Suppose that $\dim(A \cap F_v) = 2$. Locally around q the double cover $X_A \to Y_A$ is isomorphic to $X_{\overline{\gamma}} \to Y_{\overline{\gamma}}$ where $\overline{\gamma}$ is the symmetrization of the restriction of β_A to an affine neighoborhood $\operatorname{Spec} R$ of $[v]$. Thus we may consider the differential $\delta_{\overline{\gamma}}([v])$ - see (1.3.3). The differential is surjective by Proposition 2.9 of [15], thus X_A is smooth at q by **Proposition 1.12**. □

Proposition 3.4. *Let $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \mathbb{N}(V))$. Then X_A is smooth if and only if $A \in \mathbb{LG}(\bigwedge^3 V)^0$.*

Proof. If $A \in \mathbb{LG}(\bigwedge^3 V)^0$ then X_A is smooth by [12]. Suppose that X_A is smooth. Then $A \notin \Delta$ by **Claim 1.13**. Assume that $A \in \Sigma$; we will reach a contradiction. Let $W \in \Theta_A$ and $[v] \in \mathbb{P}(W)$ - notice that $\mathbb{P}(W) \subset Y_A$. Let $q \in f_A^{-1}([v])$. Since $A \notin \Delta$ we have $1 \leq \dim(A \cap F_v) \leq 2$. Suppose that $\dim(A \cap F_v) = 1$. Then Y_A is singular at $[v]$ by Corollary 2.5 of [15], thus X_A is singular at q by **Proposition 1.5**. Suppose that $\dim(A \cap F_v) = 2$. Let $\overline{\gamma}$ be as in the proof of **Lemma 3.3**. Then $\delta_{\overline{\gamma}}([v])$ is not surjective - see Proposition 2.3 of [15] - and hence X_A is singular at q by **Proposition 1.12**. □

3.2 The desingularization

Definition 3.5. Let $\mathbb{LG}(\bigwedge^3 V)^* \subset \mathbb{LG}(\bigwedge^3 V)$ be the set of A such that the following hold:

(1) $A \notin \mathbb{N}(V)$.

(2) $Y_A[3]$ is finite.

(3) $Y_A[3] = Y_A(3)$.

Remark 3.6. $\mathbb{LG}(\bigwedge^3 V)^*$ is an open subset of $\mathbb{LG}(\bigwedge^3 V)$.

Claim 3.7. $(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma) \subset \mathbb{LG}(\bigwedge^3 V)^*$.

Proof. Item (1) of **Definition 3.5** holds by Claim 2.11 of [15]. Let's prove that Item (2) of **Definition 3.5** holds. Suppose that $Y_A[3] \neq Y_A(3)$ i.e. there exists $[v_0] \in \mathbb{P}(V)$ such that $\dim(A \cap F_{v_0}) \geq 4$. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$ and let $\rho^{v_0}_{V_0}$ be as in (2.2.17). Let $\mathbf{K} := \mathbb{P}(\rho^{v_0}_{V_0}(A \cap F_{v_0}))$. Then $\dim \mathbf{K} \geq 3$; since $\operatorname{Gr}(2, V_0)$ has codimension 3 in $\mathbb{P}(\bigwedge^2 V_0)$ it follows that there exists $[\alpha] \in \mathbf{K} \cap \operatorname{Gr}(2, V_0)$. Let $\widetilde{\alpha} \in (A \cap F_{v_0})$ such that $\rho^{v_0}_{V_0}(\widetilde{\alpha}) = \alpha$. Then $\widetilde{\alpha}$ is non-zero and decomposable, that is a contradiction because $A \notin \Sigma$. Lastly let's prove that Item (3) of **Definition 3.5** holds. Let $[v_0] \in Y_A[3] = Y_A(3)$. Then $(A, [v_0]) \in \widetilde{\Delta}(0)$. Let $K := A \cap F_{v_0}$ and $\tau^{v_0}_K$ be as in (2.2.11). We have

$$T_{[v_0]} Y_A[3] = T_{[v_0]} Y_A(3) = \ker \tau^{v_0}_K.$$

By **Proposition 2.5** the map $\tau^{v_0}_K$ is injective. Thus $[v_0]$ is an isolated point of $Y_A[3]$. □

Let $A \in \mathbb{LG}(\bigwedge^3 V)^*$. Let $\mathcal{U} \subset \mathbb{LG}(\bigwedge^3 V)^*$ be a small open (either in the Zariski or in the *classical* topology) subset containing A. In particular $\rho_{\mathcal{U}} \colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. Let $\pi_{\mathcal{U}} \colon \widetilde{\mathcal{X}}_{\mathcal{U}} \to \mathcal{X}_{\mathcal{U}}$ be the blow-up of $\mathcal{W}_{\mathcal{U}}$ and $E_{\mathcal{U}}$ be the exceptional set of $\pi_{\mathcal{U}}$.

Claim 3.8. *Keep notation as above. Then $\widetilde{\mathcal{X}}_{\mathcal{U}}$ is smooth. If $\mathcal{U}$ is open and sufficiently small in the* classical *topology then we have a locally-trivial fibration*

$$E_{\mathcal{U}} \longrightarrow Y_{\mathcal{U}}[3]. \tag{3.2.1}$$

Let $(A, [v]) \in Y_{\mathcal{U}}[3]$. The fiber of (3.2.1) over $(A, [v])$ is isomorphic to $\mathbb{P}(A \cap F_v)^\vee \times \mathbb{P}(A \cap F_v)^\vee$ and the restriction of $N_{E_{\mathcal{U}}/\widetilde{\mathcal{X}}_{\mathcal{U}}}$ to the fiber is isomorphic to $\mathcal{O}_{\mathbb{P}(A \cap F_v)^\vee}(-1) \boxtimes \mathcal{O}_{\mathbb{P}(A \cap F_v)^\vee}(-1)$.

Proof. By **Proposition 3.2** we know that $\widetilde{\mathcal{X}}_{\mathcal{U}}$ is smooth outside $E_{\mathcal{U}}$. It remains to examine $\widetilde{\mathcal{X}}_{\mathcal{U}}$ over $\mathcal{W}_{\mathcal{U}} \cong \mathcal{Y}_{\mathcal{U}}[3]$. We may assume that $\mathcal{U} = U_B \times \mathcal{N}$ is as in the proof of **Proposition 3.2**. We will adopt the notation of that proof. Let $q \in \mathcal{X}_\gamma$ and $f_{\mathcal{U}}(q) = (A, [v]) = p$. A neighborhood of q in $X_{\mathcal{U}}$ is isomorphic to X_γ where γ is given by (2.2.5) - see (3.1.4). We are assuming that $q \in \mathcal{W}_{\mathcal{U}}$ and hence $\operatorname{cork}\gamma(p) = 3$. Let $f\colon X(\mathcal{V}) \to Y(\mathcal{V})$ be as in **Subsection 1.3** i.e. f is the universal double covering of corank 3 at the origin. We claim that there exists a map $\nu\colon X_\gamma \to X(\mathcal{V})$ such that the following diagram commutes

$$\begin{array}{ccc} X_\gamma & \xrightarrow{\nu} & X(\mathcal{V}) \\ {\scriptstyle f_\gamma}\downarrow & & \downarrow{\scriptstyle f} \\ Y_\gamma & \xrightarrow{\mu} & Y(\mathcal{V}) \end{array} \tag{3.2.2}$$

and X_γ is identified with the fibered product $Y_\gamma \times_{Y(\mathcal{V})} X(\mathcal{V})$. In fact it suffices to apply the reduction procedure of **Subsection 1.1** that leads to **Claim 1.4**. Let $\mathbf{K}$ be as in **Claim 1.4**: by (1.1.29) we have $(Y_{\gamma_{\mathbf{K}}}, p) = (Y_\gamma, p)$ and by **Claim 1.4** we have a natural isomorphism $(X_{\gamma_{\mathbf{K}}}, f^{-1}_{\gamma_{\mathbf{K}}}(p)) \overset{\sim}{\to} (X_\gamma, f_\gamma^{-1}(p))$ commuting with $f_{\gamma_{\mathbf{K}}}$ and f_γ. Let $\mathcal{U} = \operatorname{Spec} R$: we are free to replace $\mathcal{U}$ by any affine open subset containing $(A, [v])$. Thus we may assume that $\mathbf{K}$ is a trivial R-module i.e. $\mathbf{K} = \mathcal{V} \otimes R$ where $\mathcal{V}$ is a complex 3-dimensional vector-space. Hence we may view $\gamma_{\mathbf{K}}$ as a map $\gamma_{\mathbf{K}}\colon \operatorname{Spec} R \to \operatorname{Sym}^2 \mathcal{V}^\vee$. Notice that we have equality of schemes $Y_\gamma = \gamma_{\mathbf{K}}^{-1} Y(\mathcal{V})$; thus the restriction of $\gamma_{\mathbf{K}}$ to Y_γ defines a map $\mu\colon Y_\gamma \to Y(\mathcal{V})$. The claim follows. By surjectivity of $\delta_\gamma(A, [v])$ - see (3.1.6) - we get that the germ $(X_\gamma, f_\gamma^{-1}(p))$ is the product of a smooth germ (of dimension 54) and the germ $(X(\mathcal{V}), f^{-1}(0))$. Looking at the explicit description of $X(\mathcal{V})$ given by **Proposition 1.14** we get right away that $\widetilde{\mathcal{X}}_{\mathcal{U}}$ is smooth over q and the remaining statements as well. We need to assume that $\mathcal{U}$ is a small open subset in the classical topology in order to ensure that Map (3.2.1) is a locally-trvial fibration. □

Remark 3.9. Let $A \in \mathbb{LG}(\bigwedge^3 V)^*$ and let $Y_A[3] = \{[v_1], \ldots, [v_s]\}$. Let $\mathcal{U} \subset \mathbb{LG}(\bigwedge^3 V)^*$ be a small open (in the *classical* topology) subset containing A. For each $1 \le i \le s$ choose a projection

$$E_{\mathcal{U}}([v_i]) \longrightarrow \mathbb{P}(A \cap F_v)^\vee. \tag{3.2.3}$$

There exists a unique $\mathbb{P}^2$-fibration

$$\epsilon\colon E_{\mathcal{U}} \longrightarrow \star \tag{3.2.4}$$

where $\star$ is itself a fibration over $Y_{\mathcal{U}}[3]$ with fiber $\mathbb{P}(A \cap F_v)^\vee$ over $(A, [v])$. We say that (3.2.3) is a *choice of* $\mathbb{P}^2$*-fibration* ϵ *for* X_A.

Let $A \in \mathbb{LG}(\bigwedge^3 V)^*$ and choose a $\mathbb{P}^2$-fibration ϵ for X_A. Let $\mathcal{U} \subset \mathbb{LG}(\bigwedge^3 V)^*$ be a small open (in the *classical* topology) subset containing A. By **Claim 3.8** the normal bundle of $E_{\mathcal{U}}$ along the fibers of (3.2.4) is $\mathcal{O}_{\mathbb{P}^2}(-1)$. Thus there exists a contraction $c_{\mathcal{U},\epsilon}\colon \widetilde{\mathcal{X}}_{\mathcal{U}} \to \mathcal{X}^\epsilon_{\mathcal{U}}$ in the category of complex manifolds fitting into a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{X}}_{\mathcal{U}} & \xrightarrow{c_{\mathcal{U},\epsilon}} & \mathcal{X}^\epsilon_{\mathcal{U}} \\ & {\scriptstyle \pi^\epsilon_{\mathcal{U}}}\searrow \quad \swarrow{\scriptstyle g^\epsilon_{\mathcal{U}}} & \\ & \mathcal{X}_{\mathcal{U}} & \end{array} \tag{3.2.5}$$

Let $f^\epsilon_{\mathcal{U}} = f_{\mathcal{U}} \circ g^\epsilon_{\mathcal{U}}\colon \mathcal{X}^\epsilon_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ and $\rho^\epsilon_{\mathcal{U}}\colon \mathcal{X}^\epsilon_{\mathcal{U}} \to \mathcal{U}$ be the map $f^\epsilon_{\mathcal{U}}$ followed by $\mathcal{Y}_{\mathcal{U}} \to \mathcal{U}$. Let

$$X^\epsilon_A := (\rho^\epsilon_{\mathcal{U}})^{-1}(A), \quad g^\epsilon_A := g^\epsilon_{\mathcal{U}}|_{X^\epsilon_A}, \quad f^\epsilon_A := f^\epsilon_{\mathcal{U}}|_{X^\epsilon_A}, \quad \mathcal{O}_{X^\epsilon_A}(1) := (f^\epsilon_A)^* \mathcal{O}_{Y_A}(1), \quad H^\epsilon_A \in |\mathcal{O}_{X^\epsilon_A}(1)|.$$

Our notation does not make any reference to $\mathcal{U}$ because the isomorphism class of the polarized couple $(X^\epsilon_A, \mathcal{O}_{X^\epsilon_A}(1))$ does not depend on the open set $\mathcal{U}$ containing A. Notice that if $A \in \Delta$ then $\mathcal{O}_{X^\epsilon_A}(1)$ is not ample, in fact it is trivial on s copies of $\mathbb{P}^2$ where $s = |Y_A[3]|$. Of course

$$(X^\epsilon_A, \mathcal{O}_{X^\epsilon_A}(1)) \cong (X_A, \mathcal{O}_{X_A}(1)) \text{ if } A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \Delta). \tag{3.2.6}$$

Proposition 3.10. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^*$ and let ϵ be a choice of $\mathbb{P}^2$-fibration for X_A.*

(1) X_A^ϵ is smooth away from $(f_A^\epsilon)^{-1}(\bigcup_{W\in\Theta_A}\mathbb{P}(W))$.

(2) If $[v_i] \in Y_A[3]$ then $(f_A^\epsilon)^{-1}[v_i] \cong \mathbb{P}(A\cap F_{v_i})^\vee$.

(3) If ϵ' is another choice of $\mathbb{P}^2$-fibration for X_A there exists a commutative diagram

$$\begin{array}{ccc} X_A^{\epsilon} & \dashrightarrow & X_A^{\epsilon'} \\ & {\scriptstyle f_A^{\epsilon}}\searrow \quad \swarrow{\scriptstyle f_A^{\epsilon'}} & \\ & Y_A & \end{array} \tag{3.2.7}$$

where the birational map is the flop of a collection of $(f_A^\epsilon)^{-1}[v_i]$'s. Conversely every flop of a collection of $(f_A^\epsilon)^{-1}[v_i]$'s is isomorphic to one $X_A^{\epsilon'}$.

Proof. Let's prove Item (1). X_A^ϵ is smooth away from $(f_A^\epsilon)^{-1}(Y_A[3]\cup\bigcup_{W\in\Theta_A}\mathbb{P}(W))$ by **Lemma 3.3**. It remains to prove that X_A^ϵ is smooth at every point of $(f_A^\epsilon)^{-1}\{[v_1],\ldots,[v_s]\}$ where

$$\{[v_1],\ldots,[v_s]\} = Y_A[3] \setminus \bigcup_{W\in\Theta_A}\mathbb{P}(W). \tag{3.2.8}$$

Let $\mathcal{U} \subset \mathbb{LG}(\bigwedge^3 V)^*$ be a small open (in the *classical* topology) subset containing A. Let $\widetilde{\rho}_{\mathcal{U}} := \rho_{\mathcal{U}}\circ\pi_{\mathcal{U}}$; thus $\widetilde{\rho}_{\mathcal{U}}\colon \widetilde{X}_{\mathcal{U}} \to \mathcal{U}$. For $1\le i\le s$ the fiber over $(A,[v_i])$ of Fibration (3.2.1) is canonically isomorphic to $\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee$. Let $\widehat{X}_A\subset\widetilde{X}_{\mathcal{U}}$ be the strict transform of X_A. Abusing notation we write

$$\widetilde{\rho}_{\mathcal{U}}^{-1}(A) = \widehat{X}_A \cup \bigcup_{i=1}^{s}\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee. \tag{3.2.9}$$

(Of course $\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee$ denotes the fiber over $(A,[v_i])$ of Fibration (3.2.1).) The components $\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee$ are pairwise disjoint. We claim that for $i=1,\ldots,s$ the intersection

$$E_{A,i} := \widehat{X}_A \cap (\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee) \tag{3.2.10}$$

is a smooth symmetric divisor in the linear system $|\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)\boxtimes\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)|$. In order to prove this we go back to Map (1.3.15) - recall that $\mathcal{V}$ is a 3-dimensional complex vector space. Pull-back by σ defines an isomorphism

$$\mathrm{Sym}^2\mathcal{V}^\vee \xrightarrow{\sigma^*} (\mathcal{V}^\vee\otimes\mathcal{V}^\vee)^{\mathbb{Z}/(2)} =: \mathrm{Sym}_2\mathcal{V}^\vee \tag{3.2.11}$$

which is $\mathrm{Gl}(\mathcal{V})$-equivariant. Isomorphism σ^* induces a $\mathrm{PGL}(\mathcal{V})$-equivariant isomorphism of projective spaces $\mathbf{p}\colon \mathbb{P}(\mathrm{Sym}^2\mathcal{V}^\vee)\xrightarrow{\sim}\mathbb{P}(\mathrm{Sym}_2\mathcal{V}^\vee)$. Of course $\mathbf{p}$ maps a point in the unique open $\mathrm{PGL}(\mathcal{V})$-orbit of $\mathbb{P}(\mathrm{Sym}^2\mathcal{V}^\vee)$ to a point in the unique open $\mathrm{PGL}(\mathcal{V})$-orbit of $\mathbb{P}(\mathrm{Sym}_2\mathcal{V}^\vee)$. Now let $\mathcal{V}=(A\cap F_{v_i})^\vee$. Let $K_i := (A\cap F_{v_i})$ and $\tau_{K_i}^{v_i}$ be as in (2.2.11). By **Proposition 2.5** we have that $\mathrm{im}(\tau_{K_i}^{v_i})$ belongs to the unique open $\mathrm{PGL}(K_i)$-orbit of $\mathbb{P}(\mathrm{Sym}^2(A\cap F_{v_i}))$. Commutative Diagram (1.3.16) gives that $E_{A,i}$ is a symmetric smooth divisor in $|\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)\boxtimes\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)|$. Thus we have described $\widetilde{\rho}_{\mathcal{U}}^{-1}(A)$. Since $X_{\mathcal{U}}^\epsilon$ is obtained from $\widetilde{X}_{\mathcal{U}}$ by contracting $E_{\mathcal{U}}$ along the $\mathbb{P}^2$-fibration ϵ it follows that X_A^ϵ is smooth at every point of $(f_A^\epsilon)^{-1}\{[v_1],\ldots,[v_s]\}$. This proves Item (1). Since X_A^ϵ is obtained from $\widehat{X}_A$ by contracting each of the divisors $E_{A,i}$ along the fibration $\mathbb{P}^1\to E_{A,i}\to\mathbb{P}(A\cap F_{v_i})^\vee$ determined by ϵ (and similarly for ϵ') we also get Items (2) and (3). □

Corollary 3.11. *Let $A\in(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)$. Then $g_A^\epsilon\colon X_A^\epsilon\to X_A$ is a desingularization for every choice of $\mathbb{P}^2$-fibration ϵ for X_A.*

Proof. By **Claim 3.7** we know that $A\in\mathbb{LG}(\bigwedge^3 V)^*$: thus **Proposition 3.10** applies to X_A^ϵ. Since $A\notin\Sigma$ we get that X_A^ϵ is smooth by Item (1) of **Proposition 3.10**. □

Corollary 3.12. *Let $A,A'\in(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)$ and ϵ,ϵ' be choices of $\mathbb{P}^2$-fibration for X_A. The quasi-polarized 4-folds $(X_A^\epsilon,H_A^\epsilon)$ and $(X_{A'}^\epsilon,H_{A'}^\epsilon)$ are deformation equivalent.*

4 Double EPW-sextics parametrized by Δ

Let $A \in \Delta$ and $[v_0] \in Y_A(3)$. In the first subsection we will associate to $(A, [v_0])$ (under some hypotheses which are certainly satisfied if $A \notin \Sigma$) a $K3$ surface $S_A(v_0)$ of genus 6, meaning that it comes equipped with a big and nef divisor class $D_A(v_0)$ of square 10. We will also prove a converse: given a generic such pseudo-polarized $K3$ surface S there exist $A \in \Delta$ and $[v_0] \in Y_A(3)$ such that the pseudo-polarized surfaces S and $S_A(v_0)$ are isomorphic. In the second subsection we will assume that $A \in (\Delta \setminus \Sigma)$ - with this hypothesis $D_A(v_0)$ is very ample. We will prove that there exists a bimeromorphic map $\psi\colon S_A^{[2]}(v_0) \dashrightarrow X_A^{\epsilon}$ where ϵ is an arbitrary choice of $\mathbb{P}^2$-fibration for X_A. That such a map exists for generic $A \in \Delta$ could be proved by invoking the results of [14]. Here we will present a direct proof (we will not appeal to [14] nor to [12]). Moreover we will prove that if $S_A(v_0)$ contains no lines (this will be the case for generic A) then there exists a choice of ϵ for which ψ is regular - in particular X_A^{ϵ} is projective for such ϵ. Lastly we will notice that the above results show that a smooth double cover of an EPW-sextic is a deformation of the Hilbert square of a $K3$ (and that the family of double EPW-sextics is a locally versal family of projective Hyperkähler manifolds): the proof is more direct than the proof of [12].

4.1 EPW-sextics and $K3$ surfaces

Assumption 4.1. $A \in \mathbb{LG}(\bigwedge^3 V)$, $[v_0] \in Y_A(3)$ and the following hold:

(a) There exists a codimension-1 subspace $V_0 \subset V$ such that $\bigwedge^3 V_0 \pitchfork A$ i.e. $\bigwedge^3 V_0 \cap A = \{0\}$.

(b) There exists at most one $W \in \Theta_A$ containing v_0.

(c) If $W \in \Theta_A$ contains v_0 then $A \cap (\bigwedge^2 W \wedge V) = \bigwedge^3 W$.

Remark 4.2. Let $A \in (\Delta \setminus \Sigma)$. Let $[v_0] \in Y_A(3)$ ($= Y_A[3]$ by **Claim 3.7**). Then **Assumption 4.1** holds. In fact Items (b) and (c) hold trivially while Item (a) holds by Claim 2.11 and Equation (2.81) of [15].

Let $(A, [v_0])$ be as in **Assumption 4.1**: we will define a surface $S_A(v_0)$ of genus 6. The condition that $\bigwedge^3 V_0$ is transverse to A is open: thus we may assume that we have a direct-sum decomposition

$$V = [v_0] \oplus V_0. \tag{4.1.1}$$

We will denote by $\mathcal{D}$ be the direct-sum decomposition of V appearing in (4.1.1). Let

$$K_A^{\mathcal{D}} := \rho_{V_0}^{v_0}(A \cap F_{v_0}). \tag{4.1.2}$$

where $\rho_{V_0}^{v_0}$ is given by (2.2.17). Choose a volume-form on V_0. Wedge-product followed by the volume-form defines an isomorphism $\bigwedge^3 V_0 \cong \bigwedge^2 V_0^{\vee}$ and hence it makes sense to let

$$F_A^{\mathcal{D}} := \mathbb{P}(\operatorname{Ann} K_A^{\mathcal{D}}) \cap \operatorname{Gr}(3, V_0). \tag{4.1.3}$$

By **Proposition 5.2** and **Proposition 5.3** (see the Appendix) we know that $F_A^{\mathcal{D}}$ is a Fano 3-fold with at most one singular point. Next we will define a quadratic form on $\operatorname{Ann} K_A^{\mathcal{D}}$. By Item (a) of **Assumption 4.1** the subspace A is the graph of a map $\widetilde{q}_A^{\mathcal{D}}\colon \bigwedge^2 V_0 \to \bigwedge^3 V_0$: explicitly

$$\widetilde{q}_A^{\mathcal{D}}(\alpha) = \beta \iff (v_0 \wedge \alpha + \beta) \in A. \tag{4.1.4}$$

The map $\widetilde{q}_A^{\mathcal{D}}$ is symmetric because A, $\bigwedge^2 V_0$ and $\bigwedge^3 V_0$ are lagrangian subspaces of $\bigwedge^3 V$. Clearly $\ker \widetilde{q}_A^{\mathcal{D}} = K_A^{\mathcal{D}}$: it follows that $\widetilde{q}_A^{\mathcal{D}}$ induces an isomorphism

$$\widetilde{r}_A^{\mathcal{D}}\colon \bigwedge^2 V_0 / K_A^{\mathcal{D}} \xrightarrow{\sim} \operatorname{Ann} K_A^{\mathcal{D}} \subset \bigwedge^3 V_0. \tag{4.1.5}$$

The inverse $(\widetilde{r}_A^{\mathcal{D}})^{-1}$ defines a non-degenerate quadratic form $(r_A^{\mathcal{D}})^{\vee}$ on $\operatorname{Ann} K_A^{\mathcal{D}}$. For future reference we unwind the definition of $(\widetilde{r}_A^{\mathcal{D}})^{-1}$ and $(r_A^{\mathcal{D}})^{\vee}$. Let $\beta \in \operatorname{Ann} K_A^{\mathcal{D}}$ i.e.

$$v_0 \wedge \alpha + \beta \in A, \qquad \alpha \in \bigwedge^2 V_0. \tag{4.1.6}$$

Then

$$(\widetilde{r}_A^{\mathcal{D}})^{-1}(\beta) \equiv \alpha \pmod{K_A^{\mathcal{D}}}, \qquad (r_A^{\mathcal{D}})^\vee(\beta) = \mathrm{vol}(v_0 \wedge \alpha \wedge \beta). \tag{4.1.7}$$

Let $V((r_A^{\mathcal{D}})^\vee) \subset \mathbb{P}(\mathrm{Ann}\, K_A^{\mathcal{D}})$ be the zero-scheme of $(r_A^{\mathcal{D}})^\vee$: a smooth 5-dimensional quadric. Let

$$S_A^{\mathcal{D}} := V((r_A^{\mathcal{D}})^\vee) \cap F_A^{\mathcal{D}}. \tag{4.1.8}$$

Our first goal is to show that $S_A^{\mathcal{D}}$ does not depend on the choice of the subspace $V_0 \subset V$ complementary to $[v_0]$ i.e. it depends only on A and $[v_0]$. First we notice that $F_A^{\mathcal{D}}$ is independent of V_0. In fact $\bigwedge^3 V_0$ is transversal to F_{v_0}; since both $\bigwedge^3 V_0$ and F_{v_0} are Lagrangians the volume vol induces an isomorphism

$$g_{V_0} \colon \bigwedge^3 V_0 \overset{\sim}{\longrightarrow} F_{v_0}^\vee. \tag{4.1.9}$$

Thus g_{V_0} defines an inclusion

$$F_A^{\mathcal{D}} \hookrightarrow \mathbb{P}(\mathrm{Ann}\, K_A). \tag{4.1.10}$$

Remark 4.3. The image of Map (4.1.10) does not depend on V_0 i.e. it depends exclusively on A and $[v_0] \in Y_A(3)$; we will denote it by $Z_A(v_0)$.

Similarly g_{V_0} defines an inclusion

$$\mathbf{g}_{V_0} \colon S_A^{\mathcal{D}} \hookrightarrow \mathbb{P}(\mathrm{Ann}\, K_A). \tag{4.1.11}$$

Lemma 4.4. *Keep notation and assumptions as above. Then $\mathbf{g}_{V_0}(S_A^{\mathcal{D}})$ is independent of V_0, in other words it depends exclusively on A and $[v_0] \in Y_A(3)$.*

Proof. Let $V_0' \subset V$ be a codimension-1 subspace complementary to $[v_0]$ and transverse to A. Let $\mathcal{D}'$ denote the corresponding direct-sum decomposition of V; we must show that

$$\mathbf{g}_{V_0}(S_A^{\mathcal{D}}) = \mathbf{g}_{V_0'}(S_A^{\mathcal{D}'}). \tag{4.1.12}$$

The subspace V_0' is the graph of a linear function

$$\begin{array}{ccc} V_0 & \longrightarrow & [v_0] \\ v & \mapsto & f(v)v_0 \end{array} \tag{4.1.13}$$

and hence we have an isomorphism

$$\begin{array}{ccc} V_0 & \overset{\psi}{\longrightarrow} & V_0' \\ v & \mapsto & v + f(v)v_0 . \end{array} \tag{4.1.14}$$

We notice that

$$\bigwedge^3 \psi(\beta) = \beta + v_0 \wedge (f \lrcorner \beta) \tag{4.1.15}$$

where $\lrcorner$ denotes contraction. In particular $g_{V_0'} \circ \bigwedge^3 \psi = g_{V_0}$. Moreover $\phi := \bigwedge^3 \psi|_{\mathrm{Ann}\, K_A^{\mathcal{D}}}$ is an isomorphism between $\mathrm{Ann}\, K_A^{\mathcal{D}} \subset \bigwedge^3 V_0$ and $\mathrm{Ann}\, K_{A'}^{\mathcal{D}'} \subset \bigwedge^3 V_0'$. Thus it suffices to prove that

$$\phi(S_A^{\mathcal{D}}) = S_A^{\mathcal{D}'}. \tag{4.1.16}$$

We claim that

$$\phi^*(r_A^{\mathcal{D}'})^\vee - (r_A^{\mathcal{D}})^\vee \in H^0(\mathcal{I}_{F_A^{\mathcal{D}}}(2)). \tag{4.1.17}$$

In fact let $\beta \in \mathrm{Ann}\, K_A^{\mathcal{D}} \subset \bigwedge^3 V_0$; then (4.1.6) holds. By (4.1.15) we get that

$$v_0 \wedge (\alpha - (f \lrcorner \beta)) + \phi(\beta) = v_0 \wedge \alpha + \beta \in A. \tag{4.1.18}$$

By (4.1.15) we get that

$$\begin{aligned} \phi^*(r_A^{\mathcal{D}'})^\vee(\beta) &= \mathrm{vol}(v_0 \wedge (\alpha - (f \lrcorner \beta)) \wedge \phi(\beta)) = \\ &\mathrm{vol}(v_0 \wedge \alpha \wedge \phi(\beta)) - \mathrm{vol}(v_0 \wedge (f \lrcorner \beta) \wedge \phi(\beta)) = \\ &\mathrm{vol}(v_0 \wedge \alpha \wedge \beta) - \mathrm{vol}(v_0 \wedge (f \lrcorner \beta) \wedge \beta) = \\ &(r_A^{\mathcal{D}})^\vee(\beta) - \mathrm{vol}(v_0 \wedge (f \lrcorner \beta) \wedge \beta). \end{aligned} \tag{4.1.19}$$

The second term in the last expression is the restriction to $\mathbb{P}(\mathrm{Ann}\, K_A^{\mathcal{D}})$ of a Plücker quadratic form and hence it vanishes on $F_A^{\mathcal{D}}$. This proves (4.1.17) and hence (4.1.16) holds. ☐

By the above lemma we may give the following definition.

Definition 4.5. Let $A \in \mathbb{LG}(\bigwedge^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that **Assumption 4.1** holds. Let $\mathcal{D}$ be the direct-sum decomposition (4.1.1). We set

$$S_A(v_0) := \mathbf{g}_{V_0}(S_A^{\mathcal{D}}). \tag{4.1.20}$$

Keep assumptions and notation as above. We single out special points of $S_A(v_0)$ as follows. Suppose that $W \in \Theta_A$ (see (2.2.15) for the definition of Θ_A) and assume that $v_0 \notin W$. Let γ be a generator of $\bigwedge^3 W$ i.e. γ is decomposable with $\operatorname{supp}(\gamma) = W$. By hypothesis $\bigwedge^3 V_0 \cap A = \{0\}$ and hence $W \not\subset V_0$; thus

$$\gamma = (v_0 + u_1) \wedge u_2 \wedge u_3, \qquad u_i \in V_0\,. \tag{4.1.21}$$

Since $v_0 \notin W$ we have $u_1 \wedge u_2 \wedge u_3 \neq 0$; thus $[u_1 \wedge u_2 \wedge u_3] \in F_A^{\mathcal{D}}$. Moreover $[u_1 \wedge u_2 \wedge u_3] \in V((r_A^{\mathcal{D}})^\vee)$ by (4.1.7) and hence $[u_1 \wedge u_2 \wedge u_3] \in S_A^{\mathcal{D}}$. We let

$$\begin{array}{ccc} \Theta_A \setminus \{W \mid v_0 \in W\} & \xrightarrow{\theta_A^{\mathcal{D}}} & S_A^{\mathcal{D}} \\ W & \mapsto & [u_1 \wedge u_2 \wedge u_3]\,. \end{array} \tag{4.1.22}$$

The map

$$\theta_A(v_0) := \mathbf{g}_{V_0} \circ \theta_A^{\mathcal{D}} \colon (\Theta_A \setminus \{W \mid v_0 \in W\}) \to S_A(v_0) \tag{4.1.23}$$

is independent of $\mathcal{D}$, i.e. it depends exclusively on A and $[v_0]$. Notice that $\theta_A(v_0)$ is injective.

Proposition 4.6. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that* **Assumption 4.1** *holds. Let $\mathcal{D}$ be the direct-sum decomposition* (4.1.1)*. The set of points at which the intersection $V((r_A^{\mathcal{D}})^\vee) \cap F_A^{\mathcal{D}}$ is not transverse is equal to*

$$\operatorname{im} \theta_A^{\mathcal{D}} \coprod (S_A^{\mathcal{D}} \cap \operatorname{sing} F_A^{\mathcal{D}}). \tag{4.1.24}$$

Proof. Let $[\beta] \in S_A^{\mathcal{D}}$. In particular β is non-zero decomposable; let $U := \operatorname{supp} \beta$. Moreover since $[\beta] \in F_A^{\mathcal{D}}$ we have that (4.1.6) holds; let $\alpha \in \bigwedge^2 V_0$ be as in (4.1.6). We claim that

$$V((r_A^{\mathcal{D}})^\vee) \pitchfork F_A^{\mathcal{D}} \text{ at } [\beta] \text{ unless } \langle \alpha,\, K_A^{\mathcal{D}} \rangle \cap \bigwedge^2 U \neq \emptyset. \tag{4.1.25}$$

In fact the projective tangent space to $\operatorname{Gr}(3, V_0)$ at $[\beta]$ is given by

$$\mathbf{T}_{[\beta]} \operatorname{Gr}(3, V_0) = \mathbb{P}(\operatorname{Ann}(\bigwedge^2 U))\,. \tag{4.1.26}$$

On the other hand (4.1.7) gives that

$$\mathbf{T}_{[\beta]} V((r_A^{\mathcal{D}})^\vee) = \mathbb{P}(\operatorname{Ann} \alpha) \cap \mathbb{P}(\operatorname{Ann} K_A^{\mathcal{D}})\,. \tag{4.1.27}$$

Statement (4.1.25) follows at once from (4.1.26) and (4.1.27). Next we prove that

$$\langle \alpha,\, K_A^{\mathcal{D}} \rangle \cap \bigwedge^2 U \neq \emptyset \text{ if and only if } [\beta] \in \operatorname{sing} F_A^{\mathcal{D}} \text{ or } [\beta] \in \operatorname{im} \theta_A^{\mathcal{D}}. \tag{4.1.28}$$

Suppose that $[\beta] \in \operatorname{sing} F_A^{\mathcal{D}}$; then Item (1) of **Proposition 5.3** gives that $K_A^{\mathcal{D}} \cap \bigwedge^2 U \neq \emptyset$. Next suppose that $[\beta] \in \operatorname{im} \theta_A^{\mathcal{D}}$; then $\alpha \in \bigwedge^2 U$ by (4.1.21). This proves the "if" implication of (4.1.28). Let us prove the "only if" implication. First assume that $K_A^{\mathcal{D}} \cap \bigwedge^2 U \neq \{0\}$. Let $0 \neq \kappa_0 \in K_A^{\mathcal{D}} \cap \bigwedge^2 U$. Then κ_0 is decomposable because $\dim U = 3$ and hence $[\kappa_0]$ is the unique point belonging to $\mathbb{P}(K_A^{\mathcal{D}}) \cap \operatorname{Gr}(2, V_0)$. We get that $[\beta]$ is the unique singular point of $F_A^{\mathcal{D}}$ by (5.0.8). Lastly assume that $K_A^{\mathcal{D}} \cap \bigwedge^2 U = \{0\}$. Then there exists $\kappa \in K_A^{\mathcal{D}}$ such that $(\alpha + \kappa) \in \bigwedge^2 U$. Since $\kappa \in K_A^{\mathcal{D}}$ we have $(v_0 \wedge (\alpha + \kappa) + \beta) \in A$. The tensor $(v_0 \wedge (\alpha + \kappa) + \beta) \in A$ is decomposable, let W be its support. Then $v_0 \notin W$ because $\beta \neq 0$ and hence $[\beta] = \theta_A^{\mathcal{D}}(W)$. This finishes the proof of (4.1.28) and of the proposition. $\square$

Corollary 4.7. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that* **Assumption 4.1** *holds. Asssume in addition that Θ_A is finite. Then $S_A(v_0)$ is a reduced and irreducible surface with*

$$\operatorname{sing} S_A(v_0) = \operatorname{im}\theta_A(v_0) \coprod (S_A(v_0) \cap \operatorname{sing} Z_A(v_0)) . \tag{4.1.29}$$

(See **Remark 4.3** *for the definition of $Z_A(v_0)$.)*

Proof. By **Proposition 4.6** we know that $S_A^{\mathcal{D}}$ is a smooth surface outside the right-hand side of (4.1.29). By hypothesis Θ_A is finite and hence the right-hand side of (4.1.29) is finite. On the other hand by **Proposition 5.3** we know that $Z_A(v_0)$ is a 3-fold with at most one singular point, necessarily an ordinary quadratic singularity, and $S_A^{\mathcal{D}}$ is the complete intersection of $Z_A(v_0)$ and a quadric hypersurface. It follows that $S_A^{\mathcal{D}}$ is reduced and irreducible with singular set as claimed. □

Corollary 4.8. *Let hypotheses be as in* **Corollary 4.7***. Suppose in addition that $S_A(v_0)$ has Du Val singularities. Let $\widehat{S}_A(v_0) \to S_A(v_0)$ be the minimal desingularization. Then $\widehat{S}_A(v_0)$ is a $K3$ surface.*

Proof. Let $\mathcal{O}_{Z_A(v_0)}(1)$ be the pull-back by Map (4.1.10) of the hyperplane line-bundle on $\mathbb{P}(\operatorname{Ann}(F_{v_0} \cap A))$. Then $S_A(v_0) \in |\mathcal{O}_{Z_A(v_0)}(2)|$. By **Proposition 5.2** and **Proposition 5.3** there exist smooth divisors in $|\mathcal{O}_{Z_A(v_0)}(2)|$ and they are $K3$ surfaces; by simultaneous resolution of Du Val singularities we get that $\widehat{S}_A(v_0)$ is a $K3$ surface. □

Corollary 4.9. *Let $A \in (\Delta \setminus \Sigma)$. Let $[v_0] \in Y_A(3)$ (and hence* **Assumption 4.1** *holds by* **Remark 4.2***). Then $S_A(v_0)$ is a (smooth) $K3$.*

Proof. Immediate consequence of **Corollary 4.8**. □

Under the hypotheses of **Corollary 4.8** let $\mathcal{O}_{S_A(v_0)}(1)$ be the restriction to $S_A(v_0)$ of $\mathcal{O}_{Z_A(v_0)}(1)$. Let $\mathcal{O}_{\widehat{S}_A(v_0)}(1)$ be the pull-back of $\mathcal{O}_{S_A(v_0)}(1)$ to $\widehat{S}_A(v_0)$. We set

$$D_A(v_0) \in |\mathcal{O}_{S_A(v_0)}(1)| \qquad \widehat{D}_A(v_0) \in |\mathcal{O}_{\widehat{S}_A(v_0)}(1)|. \tag{4.1.30}$$

Remark 4.10. Let hypotheses be as in **Corollary 4.8**. Then $(\widehat{S}_A(v_0), \widehat{D}_A(v_0))$ is a quasi-polarized $K3$ surface of genus 6. Moreover the composition

$$\widehat{S}_A(v_0) \longrightarrow S_A(v_0) \longrightarrow \mathbb{P}(\operatorname{Ann}(F_{v_0} \cap A)) \tag{4.1.31}$$

is identified (up to projectivities) with the map associated to the complete linear system $|\widehat{D}_A(v_0)|$.

Remark 4.10 has a converse; in order to formulate it we identify $F_{v_0} \cong \bigwedge^2(V/[v_0])$ (the identification is well-defined up to homothety).

Assumption 4.11. $K \in \operatorname{Gr}(3, F_{v_0})$ and

(1) $\mathbb{P}(K) \cap \operatorname{Gr}(2, V/[v_0]) = \emptyset$, or

(2) the scheme-theoretic intersection $\mathbb{P}(K) \cap \operatorname{Gr}(2, V/[v_0])$ is a single reduced point.

Let

$$W_K := \mathbb{P}(\operatorname{Ann} K) \cap \operatorname{Gr}(3, V/[v_0]). \tag{4.1.32}$$

(This makes sense because we have an isomorphism $\bigwedge^2(V/[v_0]) \xrightarrow{\sim} \bigwedge^3(V/[v_0])^\vee$ well-defined up to homothety). Let

$$S := W_K \cap Q, \qquad Q \subset \mathbb{P}(\operatorname{Ann} K) \text{ a quadric.} \tag{4.1.33}$$

If Q is generic then S is a linearly normal $K3$ surface of genus 6, see **Corollary 4.8**. In fact the family of such $K3$ surfaces is locally versal. More generally suppose that **Assumption 4.11** holds, that S is given by (4.1.33) and that S has DuVal singularities. Let $\widehat{S} \to S$ be the minimal desingularization - thus $\widehat{S}$ is a $K3$ surface. Let $D \in |\mathcal{O}_S(1)|$ and $\widehat{D}$ be the pull-back of D to $\widehat{S}$. Consider the family $\mathcal{S} \to B$ of deformations of (S, D) obtained by deforming slightly K and Q; by

Brieskorn and Tjurina there is a suitable base change $\widehat{B} \to B$ such that the pull-back of $\mathcal{S}$ to $\widehat{B}$ admits a simultaneous resolution of singularities $\widehat{\mathcal{S}} \to \widehat{B}$ with fiber $\widehat{S}$ over the point corresponding to S. Of course there is a divisor class $\widehat{\mathcal{D}}$ on $\widehat{\mathcal{S}}$ whose restriction to $\widehat{S}$ is $\widehat{D}$ - thus $\widehat{\mathcal{S}} \to \widehat{B}$ is a family of quasi-polarized $K3$ surfaces. The following result is well-known - we omit the (standard) proof.

Proposition 4.12. *Keep notation and hypotheses as above. The family $\widehat{\mathcal{S}} \to \widehat{B}$ is a versal family of quasi-polarized $K3$ surfaces.*

Lemma 4.13. *Suppose that* **Assumption 4.11** *holds. Let S be as in* (4.1.33) *and assume that Q is transversal to W_K outside a finite set - thus S is a surface with finite singular set. There exists a smooth quadric $Q' \subset \mathbb{P}(\operatorname{Ann} K)$ such that $S = W_K \cap Q'$.*

Proof. Since W_K is cut out by quadrics Bertini's Theorem gives that the generic quadric in $\mathbb{P}(\operatorname{Ann} K)$ containing S is smooth outside $\operatorname{sing} S$; let $Q_0 = V(P_0)$ be such a quadric. Let $p \in \operatorname{sing} S$. The generic quadric $Q' = V(P') \in |\mathcal{I}_{W_K}(2)|$ is smooth at p and hence $V(P_0 + P')$ is smooth at p. Since $\operatorname{sing} S$ is finite we get that the generic quadric Q containing S is smooth at all points of $\operatorname{sing} S$. It follows that the generic quadric Q containing S is smooth. □

The following corollary provides an inverse of the process which produces $S_A(v_0)$ out of $(A, [v_0]) \in \widetilde{\Delta}(0)$ (with the extra hypotheses in **Assumption 4.1**).

Proposition 4.14. *Suppose that* **Assumption 4.11** *holds. Let S be as in* (4.1.33) *and assume that Q is smooth and transversal to W_K outside a finite set. There exist $A \in \Delta$, $[v_0] \in \mathbb{P}(V)$ and a codimension-*1 *subspace $V_0 \subset V$ transversal to $[v_0]$ such that the following hold:*

(1) $\bigwedge^3 V_0 \cap A = \{0\}$,

(2) Items (c) and (d) of **Assumption 4.1** *hold,*

(3) the natural isomorphism $\mathbb{P}(\bigwedge^3(V/[v_0])) \xrightarrow{\sim} \mathbb{P}(\bigwedge^3 V_0)$ maps S to $S_A^{\mathcal{D}}$ where $\mathcal{D}$ is the direct-sum decomposition of V appearing in (4.1.1).

If we replace the quadric Q by a smooth quadric $Q' \subset \mathbb{P}(\operatorname{Ann} K)$ such that $S = W_K \cap Q'$ and let $A' \in \Delta$ be the corresponding point, there exists a projectivity of $\mathbb{P}(V)$ fixing $[v_0]$ which takes A to A'.

Proof. Let $Q = V(P)$. The dual of $\operatorname{Ann} K$ is $\bigwedge^2(V/[v_0])/K$; thus the polarization of P defines a non-degenerate symmetric map

$$\operatorname{Ann} K \xrightarrow{\sim} \bigwedge^2(V/[v_0])/K. \tag{4.1.34}$$

The inverse of the above map is non-degenerate symmetric map

$$\bigwedge^2(V/[v_0])/K \xrightarrow{\sim} \operatorname{Ann} K. \tag{4.1.35}$$

Composing on the right with $\bigwedge^2(V/[v_0]) \xrightarrow{\sim} \bigwedge^2(V/[v_0])$ and the quotient map $\bigwedge^2(V/[v_0]) \to \bigwedge^2(V/[v_0])/K$ and on the left with $\operatorname{Ann} K \hookrightarrow \bigwedge^3(V/[v_0])$ and $\bigwedge^3(V/[v_0]) \xrightarrow{\sim} \bigwedge^3(V/[v_0])$ we get a symmetric map

$$\bigwedge^2 V_0 \longrightarrow \bigwedge^3 V_0 \tag{4.1.36}$$

with 3-dimensional kernel corresponding to K. The graph of the above map is a Lagrangian $A \in \mathbb{LG}(\bigwedge^3 V)$. One checks easily that (1), (2) and (3) hold. One gets that the projective equivalence of A does not depend on Q by going through the proof of **Lemma 4.4**. □

4.2 X_A^ϵ for $A \in (\Delta \setminus \Sigma)$

Let S be a $K3$. Let $\Delta_S^{[2]} \subset S^{[2]}$ be the irreducible codimension 1 subset parametrizing non-reduced subschemes. There exists a square root of the line bundle $\mathcal{O}_{S^{[2]}}(\Delta_S^{[2]})$: we denote by ξ its first Chern class. There is a natural morphism of integral Hodge structures $\mu\colon H^2(S) \to H^2(S^{[2]})$ such that $H^2(S^{[2]};\mathbb{Z}) = \mu(H^2(S;\mathbb{Z})) \oplus \mathbb{Z}\xi$, see [1]. Let $(\cdot,\cdot)$ be the Beauville-Bogomolov bilinear symmetric form on $H^2(S^{[2]})$. It is known [1] that

$$(\mu(\eta),\mu(\eta)) = \int_S c_1(\eta)^2, \quad \mu(H^2(S;\mathbb{Z}))\bot\mathbb{Z}\xi, \quad (\xi,\xi) = -2. \tag{4.2.1}$$

Since S and $S^{[2]}$ are regular varieties we may identify their Picard groups with $H^{1,1}_{\mathbb{Z}}(S)$ and $H^{1,1}_{\mathbb{Z}}(S^{[2]})$ respectively. Let $C \in \mathrm{Pic}(S)$; abusing notation we will denote by $\mu(C)$ the class in $\mathrm{Pic}(S^{[2]})$ corresponding to $\mu(\mathcal{O}_S(C)) \in H^{1,1}_{\mathbb{Z}}(S)$: if C is an integral curve it is represented by subschemes whose support intersects C. The following is the main result of the present subsection.

Theorem 4.15. *Let $A \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_A[3]$ ($= Y_A(3)$ by Claim 3.11 of [15]) - thus $S_A(v_0)$ is a $K3$ surface by* **Corollary 4.9**. *Then the following hold:*

(1) If $S_A(v_0)$ does not contain lines (true for generic A by **Proposition 4.12***) then there exist a choice ϵ of $\mathbb{P}^2$-fibration for X_A and an isomorphism.*

$$\psi\colon S_A(v_0)^{[2]} \dashrightarrow X_A^\epsilon \tag{4.2.2}$$

such that

$$\psi^* H_A^\epsilon \sim \mu(D_A(v_0)) - \Delta^{[2]}_{S_A(v_0)}. \tag{4.2.3}$$

(2) Let A and ϵ be arbitrary. There exists a bimeromorphic map

$$\psi\colon S_A(v_0)^{[2]} \dashrightarrow X_A^\epsilon \tag{4.2.4}$$

such that (4.2.3) *holds.*

Remark 4.16. Suppose that $S_A(v_0)$ contains a line L. The restriction of the right-hand side of (4.2.3) to $L^{(2)}$ (embedded in $S_A(v_0)^{[2]}$) is $\mathcal{O}_{L^{(2)}}(-1)$. Since H_A^ϵ is nef we get that in this case Map (4.2.4) cannot be regular.

The proof of **Theorem 4.15** will be given after a series of auxiliary results. Let $S \subset \mathbb{P}^6$ be a linearly normal $K3$ surface of genus 6 such that $\mathcal{I}_{S/\mathbb{P}^6}(2)$ is globally generated; then S is projectively normal and hence Riemann-Roch gives that $\dim |\mathcal{I}_S(2)| = 5$. One defines a rational map $S^{[2]} \dashrightarrow |\mathcal{I}_S(2)|^\vee$ as follows. Given $[Z] \in S^{[2]}$ we let $\langle Z \rangle \subset \mathbb{P}^5$ be the line spanned by Z. We let

$$\begin{array}{ccc} (S^{[2]} \setminus \bigcup\limits_{L\,\subset\, S \text{ line}} L^{(2)}) & \xrightarrow{g} & |\mathcal{I}_S(2)|^\vee \cong \mathbb{P}^5 \\ [Z] & \mapsto & \{Q \in |\mathcal{I}_S(2)| \mid \text{s.t. } Q \supset \langle Z \rangle\}\,. \end{array} \tag{4.2.5}$$

Let D be a hyperplane divisor on S; one shows (see Claim (5.16) of [11]) that

$$g^*\mathcal{O}_{\mathbb{P}^5}(1) \cong \mu(D) - \Delta_S^{[2]}. \tag{4.2.6}$$

(Notice that the set of lines on S is finite and hence $\bigcup_{L\subset S\text{line}} L^{(2)}$ has codimension 2 in $S^{[2]}$.) In fact g can be identified with the map associated to the complete linear system $|(\mu(D) - \Delta_S^{[2]})|$. We will analyze g under the assumption that S is generic (in a precise sense).

Assumption 4.17. Item (1) of **Assumption 4.11** holds.

$$S := W_K \cap Q \tag{4.2.7}$$

where $Q \subset \mathbb{P}(\mathrm{Ann}\, K)$ is a quadric intersecting transversely W_K.

Let $S \subset \mathbb{P}(\operatorname{Ann} K)$ be as in **Assumption 4.17**. Then S is a linearly normal $K3$ surface of genus 6 and $\mathcal{I}_S(2)$ is globally generated. Thus the map g of (4.2.5) is defined. Let $F(W_K)$ be the variety parametrizing lines in W_K. Since the set of lines in S is finite (empty for generic S by **Proposition 4.12**) we have a map

$$\begin{array}{ccc} (F(W_K) \setminus \{L \mid L \subset S\}) & \longrightarrow & S^{[2]} \\ L & \mapsto & L \cap Q\,. \end{array} \tag{4.2.8}$$

Definition 4.18. Let $P_S^0 \subset S^{[2]}$ be the image of Map (4.2.8) and P_S be its closure in $S^{[2]}$.

We recall that $F(W_K) \cong \mathbb{P}^2$ by Iskovskih's **Proposition 5.2**.

Claim 4.19. *Let $S \subset \mathbb{P}(\operatorname{Ann} K)$ be as in* **Assumption 4.17**. *Suppose moreover that S contains no lines. Let $C_1, C_2, \ldots, C_s$ be the (smooth) conics contained in S (of course the generic S contains no conics). Then $P_S, C_1^{(2)}, \ldots, C_s^{[2]}$ are pairwise disjoint subset of $S^{[2]}$. Moreover there exists a biregular morphism*

$$c\colon S^{[2]} \longrightarrow N(S). \tag{4.2.9}$$

contracting each of $P_S, C_1^{(2)}, \ldots, C_s^{[2]}$. Thus $N(S)$ is a compact complex normal space with

$$\operatorname{sing} N(S) = \{c(P_S), \ldots, c(C^{(2)}), \ldots \mid C \subset S \text{ a conic}\} \tag{4.2.10}$$

and c is an isomorphism of the complement of $P_S \cup C_1^{(2)} \cup \ldots \cup C_s^{[2]}$ onto the smooth locus of $N(S)$. The map g (regular on all of $S^{[2]}$ because S contains no lines) descends to a regular map

$$\overline{g}\colon N(S) \to |\mathcal{I}_S(2)|^\vee, \qquad \overline{g} \circ c = g\,. \tag{4.2.11}$$

Proof. P_S is isomorphic to $\mathbb{P}^2$ by Iskovskih's **Proposition 5.2** and each $C_i^{(2)}$ is isomorphic to $\mathbb{P}^2$ because C_i is a conic. Thus each of P_S, C_i can be contracted individually. Let's show that $P_S, C_1^{(2)}, \ldots, C_s^{[2]}$ are pairwise disjoint. Suppose that $[Z] \in P_S \cap C_i^{(2)}$. Let Λ be the plane containing C_i. Then $\Lambda \cap W_K$ contains the line $\langle Z \rangle$ and the smooth conic C_i. Since W_K is cut out by quadrics it follows that $\Lambda \subset W_K$, that is absurd because W_K contains no planes. This proves that $P_S \cap C_i^{(2)} = \emptyset$. On the other hand there does not exist $[Z] \in C_i^{(2)} \cap C_j^{(2)}$ by **Corollary 5.5**. that $P_S, C_1^{(2)}, \ldots, C_s^{[2]}$ are pairwise disjoint. Thus the contraction (4.2.9) exists. It remains to prove that g is constant on each of $P_S, C_1^{(2)}, \ldots, C_s^{[2]}$. In fact if $[Z] \in P_S$ then $g([Z]) = |\mathcal{I}_{W_K}(2)|$, if $[Z] \in C_i^{(2)}$ then

$$g([Z]) = \{Q \in |\mathcal{I}_S(2)| \mid Q \supset \langle C_i \rangle\}.$$

□

Now we go back to the "general" case: we suppose that **Assumption 4.17** holds however S may very well contain lines. Let

$$S_\star^{[2]} := S^{[2]} \setminus P_S \setminus \bigcup_{R \subset S \text{ line or conic}} \operatorname{Hilb}^2 R\,. \tag{4.2.12}$$

(Notice that if $R \subset S$ is a conic which is not smooth then we delete all $[Z] \in S^{[2]}$ such that Z is contained in the scheme R.) The following result is essentially **Lemma 3.7** of [14].

Proposition 4.20. *Suppose that* **Assumption 4.17** *holds.*

(1) The fibers of $g|_{S_\star^{[2]}}$ are finite of cardinality at most 2 and the generic fiber has cardinality 2.

(2) There exist an open dense subset $\mathcal{A} \subset S_\star^{[2]}$ and an anti-symplectic (and hence non-trivial) involution $\phi\colon \mathcal{A} \to \mathcal{A}$ such that

$$(g|_{\mathcal{A}}) \circ \phi = g|_{\mathcal{A}}\,. \tag{4.2.13}$$

The induced map

$$\mathcal{A}/\langle \phi \rangle \longrightarrow g(\mathcal{A}) \tag{4.2.14}$$

is a bijection.

(3) *If in addition S does not contain lines ϕ descends to a regular involution $\overline{\phi}\colon N(S)\to N(S)$ such that $\overline{g}\circ\overline{\phi}=\overline{g}$ and the induced map*

$$j\colon N(S)/\langle\overline{\phi}\rangle \longrightarrow g(S^{[2]}) \tag{4.2.15}$$

is a bijection. Moreover

$$\operatorname{cod}(\operatorname{Fix}(\overline{\phi}),N(S))\geq 2 \tag{4.2.16}$$

where $\operatorname{Fix}(\overline{\phi})$ *is the fixed-locus of* $\overline{\phi}$.

Let A and $[v_0]$ be as in the statement of **Theorem 4.15**: we will perform the key computation one needs to prove that theorem. Let $V_0\subset V$ be a codimension-1 subspace transversal to $[v_0]$ and such that $\bigwedge^3 V_0\cap A=\{0\}$. Let $\mathcal{D}$ be Decomposition $V=[v_0]\oplus V_0$ and $S_A^{\mathcal{D}}$ be given by (4.1.8) - thus $S_A^{\mathcal{D}}$ sits in $\mathbb{P}(\operatorname{Ann}K_A^{\mathcal{D}})\cap\operatorname{Gr}(3,V_0)$ and is isomorphic to $S_A(v_0)$. Let $f\in V_0^{\vee}$; we let q_f be the quadratic form on $\bigwedge^3 V_0$ defined by setting

$$q_f(\omega):=\operatorname{vol}_0((f\lrcorner\omega)\wedge\omega) \tag{4.2.17}$$

where vol_0 is a volume-form on V_0. Then q_f is a Plücker quadric, in fact we have an isomorphism

$$\begin{array}{ccc} V_0^{\vee} & \overset{\sim}{\longrightarrow} & H^0(\mathcal{I}_{\operatorname{Gr}(3,V_0)}(2)) \\ f & \mapsto & q_f\,. \end{array} \tag{4.2.18}$$

Let $V^{\vee}=[v_0^{\vee}]\oplus V_0^{\vee}$ be the dual decomposition of $\mathcal{D}$; thus $v_0^{\vee}\in\operatorname{Ann}V_0$ and $v_0^{\vee}(v_0)=1$. We have an isomorphism

$$\begin{array}{ccc} [v_0^{\vee}]\oplus V_0^{\vee} & \overset{\sim}{\longrightarrow} & H^0(\mathcal{I}_{S_A^{\vee}}(2)) \\ xv_0^{\vee}+f & \mapsto & x(r_A^{\mathcal{D}})^{\vee}+q_f\,. \end{array} \tag{4.2.19}$$

We let

$$\iota\colon |\mathcal{I}_{S_A^{\mathcal{D}}}(2)|^{\vee}\overset{\sim}{\longrightarrow}\mathbb{P}(V) \tag{4.2.20}$$

be the projectivization of the transpose of (4.2.19).

Proposition 4.21. *Let A and $[v_0]$ be as in the statement of* **Theorem 4.15** *and keep notation as above. Let g be Map* (4.2.5) *for $S_A^{\mathcal{D}}$ - this makes sense by* **Corollary 4.9**. *Then $\iota(\operatorname{im}g)\subset Y_A$.*

Proof. Let

$$[Z]\in((S_A^{\mathcal{D}})^{[2]}_{\star}\setminus\Delta^{[2]}_{S_A^{\mathcal{D}}}\setminus P_{S_A^{\mathcal{D}}})\,. \tag{4.2.21}$$

We will prove that

$$\iota(g([Z])\in Y_A\,. \tag{4.2.22}$$

This will suffice to prove the lemma because the right-hand side of (4.2.21) is dense in $(S_A^{\mathcal{D}})^{[2]}_{\star}$ and Y_A is closed. By hypothesis Z is reduced; thus $Z=\{[\beta],[\beta']\}$ where $\beta,\beta'\in\bigwedge^3 V_0$ are decomposable. The line $\langle[\beta],[\beta']\rangle$ spanned by $[\beta]$ and $[\beta']$ is not contained in $F_A^{\mathcal{D}}$ because $[Z]\notin P_{S_A^{\mathcal{D}}}$. Thus $\langle[\beta],[\beta']\rangle$ is not contained in $\operatorname{Gr}(3,V_0)$ and it follows that the vector sub-spaces of V_0 supporting the decomposable vectors β and β' intersect in a 1-dimensional subspace. Thus there exists a basis $\{v_1,\ldots,v_5\}$ of V_0 such that

$$\beta=v_1\wedge v_2\wedge v_3,\quad \beta'=v_1\wedge v_4\wedge v_5\,. \tag{4.2.23}$$

We may assume moreover that $\operatorname{vol}_0(v_1\wedge v_2\wedge v_3\wedge v_4\wedge v_5)=1$. By (4.1.6) and (4.1.7) there exist $\alpha,\alpha'\in\bigwedge^2 V_0$ such that

$$v_0\wedge\alpha+\beta,\ v_0\wedge\alpha'+\beta'\in A,\qquad \alpha\wedge\beta=\alpha'\wedge\beta'=0\,. \tag{4.2.24}$$

Since A is Lagrangian we get that

$$\operatorname{vol}_0(\alpha\wedge\beta')=\operatorname{vol}_0(\alpha'\wedge\beta)=:c\,. \tag{4.2.25}$$

Let $t_0,\ldots,t_5\in\mathbb{C}$; a straightforward computation gives that

$$(t_0(r_A^{\mathcal{D}})^\vee+\sum_{i=1}^{5}t_iq_{v_i^\vee})(\beta+\beta')=2ct_0+2t_1\,. \tag{4.2.26}$$

Thus

$$\iota(g([Z]))=[cv_0+v_1]\,. \tag{4.2.27}$$

It remains to prove that

$$[cv_0+v_1]\in Y_A\,. \tag{4.2.28}$$

Let $K_A^{\mathcal{D}}$ be as in (4.1.2); we claim that it suffices to prove that there exist $(x,x')\in(\mathbb{C}^2\setminus\{(0,0)\})$ and $\kappa\in K_A^{\mathcal{D}}$ such that

$$(cv_0+v_1)\wedge(x(v_0\wedge\alpha+\beta)+x'(v_0\wedge\alpha'+\beta')+v_0\wedge\kappa)=0\,. \tag{4.2.29}$$

In fact assume that (4.2.29) holds. Then

$$0\neq(x(v_0\wedge\alpha+\beta)+x'(v_0\wedge\alpha'+\beta')+v_0\wedge\kappa)\in A\cap F_{cv_0+v_1}\,. \tag{4.2.30}$$

(The inequality holds because β,β' are linearly independent.) A straightforward computation gives that (4.2.29) is equivalent to

$$x(c\beta-v_1\wedge\alpha)+x'(c\beta'-v_1\wedge\alpha')=v_1\wedge\kappa\,. \tag{4.2.31}$$

As is easily checked we have

$$(c\beta-v_1\wedge\alpha),\ (c\beta'-v_1\wedge\alpha')\in([v_1]\wedge(\bigwedge^2\langle v_2,v_3,v_4,v_5\rangle))\cap\{v_2\wedge v_3,\ v_4\wedge v_5\}^\perp \tag{4.2.32}$$

where perpendicularity is with respect to wedge-product followed by vol_0. Multiplication by v_1 gives an injection $K_A^{\mathcal{D}}\hookrightarrow([v_1]\wedge(\bigwedge^2\langle v_2,v_3,v_4,v_5\rangle))$; in fact no non-zero element of $K_A^{\mathcal{D}}$ is decomposable because $A\notin\Sigma$. Since the right-hand side of (4.2.32) has dimension 4 and $\dim K_A^{\mathcal{D}}=3$ we get that there exists $(x,x')\in(\mathbb{C}^2\setminus\{(0,0)\})$ such that (4.2.31) holds. □

Lemma 4.22. *Let $A\in(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)$. Then $Y_A(1)$ is not empty, the topological double cover $f_A^{-1}Y_A(1)\to Y_A(1)$ is not trivial and Y_A is integral.*

Proof. By **Claim 3.7** we know that $Y_A[3]$ is finite. On the other hand $(Y_A[2]\setminus Y_A[3])$ is a smooth surface - see Proposition 2.8 of [12]. Since $\mathrm{sing}\,Y_A\subset Y_A[2]$ it follows that Y_A is integral and $Y_A(1)$ is connected. Let $[v_0]\in(Y_A[2]\setminus Y_A[3])$. By **Proposition 1.5** we know that $f_A^{-1}([v_0])$ is a singleton $\{q\}$. Moreover X_A is smooth at q by **Lemma 3.3**. Thus there exists an open neighborhood U of $[v_0]$ in Y_A such that $f_A^{-1}U$ is smooth. Moreover $(f_A^{-1}Y_A[2])\cap f_A^{-1}U$ is nowhere dense in $f_A^{-1}U$. Since $f_A^{-1}U$ is smooth the complement $f_A^{-1}(Y_A(1)\cap U)$ is connected. Since $Y_A(1)$ is connected it follows that $f_A^{-1}Y_A(1)$ is connected. □

Proposition 4.23. *Keep hypotheses and notation as in* **Proposition 4.21**. *Then $\iota(\overline{\mathrm{im}\,g})=Y_A$.*

Proof. By Item (1) of **Proposition 4.20** the map g has finite generic fiber and hence $\dim\overline{\mathrm{im}\,g}=4$. By **Proposition 4.21** we get that $\iota(\overline{\mathrm{im}\,g})$ is an irreducible component of Y_A. On the other hand Y_A is irreducible by **Lemma 4.22**; it follows that $\iota(\overline{\mathrm{im}\,g})=Y_A$. □

Remark 4.24. Keep notation as in **Proposition 4.21**; then

$$\iota\circ g(P^0_{S_A^{\mathcal{D}}})=\iota(H^0(\mathcal{I}_{F_A^{\mathcal{D}}}(2)))=[v_0]. \tag{4.2.33}$$

Proof of **Theorem 4.15**. Let's prove that Item (1) holds. Let A and $[v_0]$ be as in the statement of **Theorem 4.15**. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$ and such that $\bigwedge^3 V_0 \cap A = \{0\}$. Let $\mathcal{D}$ be Decomposition $V = [v_0] \oplus V_0$. In order to simplify notation we set $S = S_A^{\mathcal{D}}$; thus $S \cong S_A(v_0)$ and by hypothesis S does not contain lines. Let j be the map of (4.2.15); by **Proposition 4.21** the composition $\iota \circ j$ is a map

$$\iota \circ j \colon N(S)/\langle\overline{\phi}\rangle \longrightarrow Y_A\,. \tag{4.2.34}$$

We claim that $\iota \circ j$ is an isomorphism: in fact it has finite fibers and is birational by **Proposition 4.20**, since $\dim \operatorname{sing} Y_A = 2$ (because $A \notin \Sigma$) the hypersurface Y_A is normal and hence $\iota \circ j$ is an isomorphism. Let $\pi\colon N(S) \to N(S)/\langle\overline{\phi}\rangle$ be the quotient map. By (4.2.16) the singular locus of $N(S)/\langle\overline{\phi}\rangle$ is the image of $\operatorname{Fix}(\overline{\phi})$ (and thus isomorphic to $\operatorname{Fix}(\overline{\phi})$); since (4.2.34) is an isomorphism we get that

$$\begin{array}{ccc} N(S)\setminus \operatorname{Fix}(\overline{\phi}) & \longrightarrow & Y_A^{sm} \\ x & \mapsto & \iota\circ j\circ \pi(x) \end{array} \tag{4.2.35}$$

is a topological covering of degree 2. We claim that

$$\pi_1(Y_A^{sm}) \cong \mathbb{Z}/(2)\,. \tag{4.2.36}$$

In fact $(N(S)\setminus\operatorname{Fix}(\overline{\phi})) \cong (S^{[2]}\setminus(P_S \cup \operatorname{Fix}(\phi|_{S^{[2]}\setminus P_S}))$. Since $(P_S \cup \operatorname{Fix}(\phi|_{S^{[2]}\setminus P_S}))$ is of codimension 2 in the simply connected manifold $S^{[2]}$ we get that $(N(S)\setminus\operatorname{Fix}(\overline{\phi}))$ is simply connected. Thus (4.2.35) is the universal covering of Y_A^{sm} and we get (4.2.36). On the other hand $Y_A^{sm} \subset Y_A(1)$ by Corollary 1.5 of [15] and thus by **Lemma 4.22** we get that $f_A^{-1}Y_A^{sm} \to Y_A^{sm}$ is the universal covering of Y_A^{sm} as well. Hence both X_A and $N(S)$ are normal completions of the universal cover of Y_A^{sm} such that the extended maps to Y_A are finite; it follows that they are isomorphic (over Y_A). The singular locus of $N(S)$ is given by (4.2.10). On the other hand $\operatorname{sing} X_A = Y_A[3]$. By **Remark 4.24** we can order the set of (smooth) conics on S, say $C_1, \ldots, C_s$ and the set of points in $Y_A[3]$ different from $[v_0]$, say $[v_1], \ldots, [v_s]$ so that

$$\overline{\psi}(c(P_S)) = [v_0], \qquad \overline{\psi}(c(C_i^{(2)})) = [v_i], \quad 1 \le i \le s. \tag{4.2.37}$$

(Recall **Remark 4.24**.) Let ϵ_0 be a choice of $\mathbb{P}^2$-fibration for X_A; then $\overline{\psi}$ defines a birational map $\psi_0\colon S^{[2]} \dashrightarrow X_A^{\epsilon_0}$ such that

$$\psi_0^* H_A^{\epsilon_0} \cong \mu(D) - \Delta_S^{[2]} \tag{4.2.38}$$

where D is the hyperplane class of S (thus (S, D) is isomorphic to $(S_A(v_0), D_A(v_0))$). The birational map ψ_0 is an isomorphism away from

$$P_S \cup C_1^{(2)} \cup \ldots \cup C_s^{(2)}. \tag{4.2.39}$$

It follows that ψ_0 is the flop of a collection of irreducible components of (4.2.39). By **Proposition 3.10** we get that there exists a choice of $\mathbb{P}^2$-fibration for X_A, call it ϵ, such that the corresponding birational map $\psi\colon S^{[2]} \dashrightarrow X_A^{\epsilon}$ is biregular. Equation (4.2.3) follows from (4.2.38). This finishes the proof that Item (1) holds. Item (2) follows from Item (1) and a specialization argument - we leave the details to the reader. □

We close the present subsection by reproving a result of ours. Let $h_A := c_1(\mathcal{O}_{X_A}(H_A))$.

Theorem 4.25 (O'Grady [12])**.** *Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$. Then X_A is a deformation of $(K3)^{[2]}$ and $(h_A, h_A)_{X_A} = 2$. Any small deformation of (X_A, H_A) (i.e. a small deformation of X_A keeping h_A of type $(1,1)$) is isomorphic to (X_B, H_B) for some $B \in \mathbb{LG}(\bigwedge^3 V)^0$.*

Proof. Let $A_0 \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_{A_0}[3]$. Suppose moreover that $S_{A_0}(v_0)$ does not contain lines. By **Theorem 4.15** there exists a choice ϵ of $\mathbb{P}^2$-fibration for X_{A_0} such that we have an isomorphism

$$\psi\colon S^{[2]} \xrightarrow{\sim} X_{A_0}^{\epsilon}, \qquad \psi^* H_{A_0}^{\epsilon} \sim \mu(D_A(v_0)) - \Delta_{S_{A_0}(v_0)}^{[2]}. \tag{4.2.40}$$

On the other hand (X_A, H_A) is a deformation of $(X^{\epsilon}_{A_0}, H^{\epsilon}_{A_0})$ by **Corollary 3.12**. This proves that (X_A, H_A) is a deformation of $(S^{[2]}, (\mu(D_A(v_0)) - \Delta^{[2]}_{S_{A_0}(v_0)}))$. By (4.2.1) we get that $(h_A, h_A)_{X_A} = 2$. Lastly we prove that an arbitrary small deformation of (X_A, H_A) is isomorphic to $(X_{A'}, H_{A'})$ for some $A' \in \mathbb{LG}(\bigwedge^3 V)^0$. The deformation space of (X_A, H_A) has dimension given by

$$\dim \mathrm{Def}(X_A, H_A) = h^{1,1}(X_A) - 1 = 20\,. \tag{4.2.41}$$

On the other hand $\mathbb{LG}(\bigwedge^3 V)^0$ is contained in the locus of points in $\mathbb{LG}$ which are stable for the natural (linearized) $PL(V)$-action - this is proved in [12]. Thus by varying $A \in \mathbb{LG}(\bigwedge^3 V)$ we get

$$\dim \mathbb{LG}(\bigwedge^3 V) - \dim SL(V) = 55 - 35 = 20 \tag{4.2.42}$$

moduli of double EPW-sextics. Since (4.2.41) and (4.2.42) are equal we conclude that an arbitrary small deformation of (X_A, H_A) is isomorphic to (X_B, H_B) for some $B \in \mathbb{LG}(\bigwedge^3 V)^0$. □

5 Appendix: Three-dimensional sections of $\mathrm{Gr}(3, \mathbb{C}^5)$

In the present section V_0 is a complex vector-space of dimension 5. Choose a volume form vol_0 on V_0; it defines an isomorphism

$$\begin{array}{ccc} \bigwedge^2 V_0 & \xrightarrow{\sim} & \bigwedge^3 V_0^{\vee} \\ \alpha & \mapsto & \omega \mapsto \mathrm{vol}_0(\alpha \wedge \omega) \end{array} \tag{5.0.1}$$

Let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that either

$$\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0) = \emptyset \tag{5.0.2}$$

or else

$$\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0) = \{[\kappa_0]\} = \mathbb{P}(K) \cap T_{[\kappa_0]}\mathrm{Gr}(2, V_0)\,. \tag{5.0.3}$$

In other words either $\mathbb{P}(K)$ does not intersects $\mathrm{Gr}(2, V_0)$ or else the scheme-theoretic intersection is a single reduced point. We will describe

$$W_K := \mathbb{P}(\mathrm{Ann}\, K) \cap \mathrm{Gr}(3, V_0) \tag{5.0.4}$$

First we recall that the dual of $\mathrm{Gr}(3, V_0)$ is $\mathrm{Gr}(2, V_0)$. More precisely let $[\alpha] \in \mathbb{P}(\bigwedge^2 V_0)$: then

$$\mathrm{sing}(\mathbb{P}(\mathrm{Ann}\, \alpha) \cap \mathrm{Gr}(3, V_0)) = \{U \in \mathrm{Gr}(3, V_0) \mid U \supset \mathrm{supp}\, \alpha\}. \tag{5.0.5}$$

In particular $\mathbb{P}(\mathrm{Ann}\, \alpha)$ is tangent to $\mathrm{Gr}(3, V_0)$ if and only if $[\alpha] \in \mathrm{Gr}(2, V_0)$ (and if that is the case it is tangent along a $\mathbb{P}^2$). Secondly we record the following observation (the proof is an easy exercise).

Lemma 5.1. *Let $U \subset V_0$ be a codimension-1 subspace. Let $\alpha \in \bigwedge^2 V_0$. Then*

$$\alpha \wedge (\bigwedge^3 U) = 0 \tag{5.0.6}$$

if and only if $\mathrm{supp}\, \alpha \subset U$.

We recall the following result of Iskovskih.

Proposition 5.2 (Iskovskih [10]). *Keep notation as above. Let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that (5.0.2) holds. Then*

(1) W_K is a smooth Fano 3-fold of degree 5 with $\omega_{W_K} \cong \mathcal{O}_{W_K}(-2)$,

(2) the Fano variety $F(W_K)$ parametrizing lines on W_K (reduced structure) is isomorphic to $\mathbb{P}^2$,

(3) the projective equivalence class of W_K does not depend on K.

Proposition 5.3. *Keep notation as above. Let $K \subset \bigwedge^2 V_0$ be a sub vector-space of dimension 3 such that (5.0.3) holds. Then W_K is a singular Fano 3-fold of degree 5 with $\omega_{W_K} \cong \mathcal{O}_{W_K}(-2)$ and one singular point which is ordinary quadratic and belongs to*

$$\{U \in \mathrm{Gr}(3, V_0) \mid U \supset \mathrm{supp}\, \kappa_0\}. \tag{5.0.7}$$

Proof. If $\kappa \in (K \setminus [\kappa_0])$ then κ is not decomposable and hence $\mathbb{P}(\mathrm{Ann}\, \kappa)$ is transverse to $\mathrm{Gr}(3, V_0)$; by (5.0.5) we get that

$$\mathrm{sing}\, W_K = \{U \in \mathrm{Gr}(3, V_0) \mid U \supset \mathrm{supp}\, \kappa_0\} \cap \mathbb{P}(\mathrm{Ann}\, K)\,. \tag{5.0.8}$$

We claim that the above intersection consists of one point. First notice that we have a natural identification

$$\{U \in \mathrm{Gr}(3, V_0) \mid U \supset \mathrm{supp}\, \kappa_0\} \cong \mathbb{P}(V_0/\,\mathrm{supp}\, \kappa_0) \tag{5.0.9}$$

and a linear map

$$\begin{array}{ccc} K & \xrightarrow{\nu} & (V_0/\,\mathrm{supp}\, \kappa_0)^\vee \\ \kappa & \mapsto & (\overline{v} \mapsto \mathrm{vol}_0(v \wedge \kappa_0 \wedge \kappa)) \end{array} \tag{5.0.10}$$

where $v \in V_0$ and $\overline{v}$ is its class in $V_0/\,\mathrm{supp}\, \kappa_0$. Given (5.0.8) and Identification (5.0.9) we get that

$$\mathrm{sing}\, W_K = \mathbb{P}(\mathrm{Ann}\, \mathrm{im}\, \nu)\,. \tag{5.0.11}$$

Of course $\kappa_0 \in \ker \nu$ and hence in order to prove that $\mathrm{sing}\, W_K$ is a singleton it suffices to prove that $\ker \nu = [\kappa_0]$. If $\kappa \in (K \setminus [\kappa_0])$ then $\kappa_0 \wedge \kappa \neq 0$; in fact this follows from (5.0.3) together with the equality

$$\mathbb{P}\{\kappa \in \bigwedge^2 V_0 \mid \kappa_0 \wedge \kappa = 0\} = T_{[\kappa_0]}\mathrm{Gr}(2, V_0)\,. \tag{5.0.12}$$

Since $\kappa_0 \wedge \kappa \neq 0$ we have $\nu(\kappa) \neq 0$. This proves that $\mathrm{sing}\, W_K$ consists of a single point. The formula for the dualizing sheaf of W_K follows at once from adjunction. It remains to prove that W_K has a single singular point and that it is an ordinary quadratic point. Let $\widetilde{W}_K \subset \mathbb{P}(\mathrm{supp}\, \kappa_0) \times \mathbb{P}(V_0/\,\mathrm{supp}\, \kappa_0) \times W_K$ be the closed subset defined by

$$\widetilde{W}_K := \{([v], U, W) \mid v \in W \subset U\}\,. \tag{5.0.13}$$

The projection $\widetilde{W}_K \to \mathbb{P}(V_0/\,\mathrm{supp}\, \kappa_0)$ is a $\mathbb{P}^1$-fibration and hence $\widetilde{W}_K$ is smooth. One shows that the projection $\pi\colon \widetilde{W}_K \to W_K$ is the blow-up of $\mathrm{sing}\, W_K$. Moreover $\pi^{-1}(\mathrm{sing}\, W_K) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and one gets that the singularity of W_K is ordinary quadratic. □

Our last result is about the base-locus of 3-dimensional linear systems of quadrics containing W_K for $K \subset \bigwedge^2 V_0$ a 3-dimensional subspace such that (5.0.2) holds. First we consider the analogous question for the Grassmannian $\mathrm{Gr}(3, \bigwedge^3 V_0)$. Let's consider the rational map

$$\mathbb{P}(\bigwedge^3 V_0) \overset{\Phi}{\dashrightarrow} |\mathcal{I}_{\mathrm{Gr}(3,V_0)}(2)|^\vee \cong \mathbb{P}(V_0) \tag{5.0.14}$$

where the last isomorphism is given by (4.2.18). Let $Z \subset \mathbb{P}(\bigwedge^3 V_0) \times \mathbb{P}(V_0)$ be the incidence subvariety defined by

$$Z := \{([\omega], [v]) \mid v \wedge \omega = 0\}\,. \tag{5.0.15}$$

Then we have a commutative triangle

$$\begin{array}{ccccc} & & Z & & \\ & \overset{\Psi}{\swarrow} & & \overset{\widetilde{\Phi}}{\searrow} & \\ \mathbb{P}(\bigwedge^3 V_0) & & \overset{\Phi}{\dashrightarrow} & & \mathbb{P}(V_0) \end{array} \tag{5.0.16}$$

where Ψ and $\widetilde{\Phi}$ are the restrictions to Z of the two projections of $\mathbb{P}(\bigwedge^3 V_0) \times \mathbb{P}(V_0)$. Moreover Ψ is the blow-up of $\mathrm{Gr}(3, V_0)$. In particular the following holds: if $\omega \in \bigwedge^3 V_0$ is not decomposable then there exists a unique $[v] \in \mathbb{P}(V_0)$ such that $v \wedge \omega = 0$ and moreover $\Phi([\omega]) = [v]$. Let $[v] \in \mathbb{P}(V_0)$; by (4.2.18) we may view $\mathrm{Ann}(v) \subset V_0^\vee$ as a hyperplane in $|\mathcal{I}_{\mathrm{Gr}(3,V_0)}(2)|$; by commutativity of (5.0.16) we have

$$\bigcap_{f \in \mathrm{Ann}(v)} V(q_f) = \mathrm{Gr}(3, V_0) \cup \{[\omega] \in \mathbb{P}(\bigwedge^3 V_0) \mid v \wedge \omega = 0\}. \tag{5.0.17}$$

Proposition 5.4. *Let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that (5.0.2) holds. Let $L \subset |\mathcal{I}_{W_K}(2)|$ be a hyperplane (here $\mathcal{I}_{W_K}$ is the ideal sheaf of W_K in $\mathbb{P}(\mathrm{Ann}\, K)$). Then*

$$\bigcap_{t \in L} Q_t = W_K \cup R_L \tag{5.0.18}$$

where R_L is a plane. Moreover $W_K \cap R_L$ is a conic.

Proof. Restriction to $\mathbb{P}(\mathrm{Ann}\, K)$ defines an isomorphism

$$|\mathcal{I}_{\mathrm{Gr}(3,V_0)}(2)| \xrightarrow{\sim} |\mathcal{I}_{W_K}(2)|\,. \tag{5.0.19}$$

By (4.2.18) we get that we may identify L with $\mathbb{P}(\mathrm{Ann}(v))$ for a well-defined $[v] \in \mathbb{P}(V_0)$ and each quadric Q_t for $t \in L$ with $\mathbb{P}(\mathrm{Ann}\, K) \cap V(q_f)$ for a suitable $[f] \in \mathbb{P}(\mathrm{Ann}(v))$. By (5.0.17) we have

$$\bigcap_{f \in \mathrm{Ann}(v)} (\mathbb{P}(\mathrm{Ann}\, K) \cap V(q_f)) = W_K \cup R_L \tag{5.0.20}$$

where

$$R_L := \mathbb{P}(\mathrm{Ann}\, K) \cap \{[\omega] \in \mathbb{P}(\bigwedge^3 V_0) \mid v \wedge \omega = 0\}. \tag{5.0.21}$$

Thus R_L is a linear space of dimension at least 2. Now notice that we have an isomorphism

$$\begin{array}{ccc} \bigwedge^2 (V_0/[v]) & \xrightarrow{\sim} & \{[\omega] \in \mathbb{P}(\bigwedge^3 V_0) \mid v \wedge \omega = 0\} \\ \overline{\alpha} & \mapsto & v \wedge \alpha \end{array} \tag{5.0.22}$$

where $\alpha \in \bigwedge^2 V_0$ is an element mapped to $\overline{\alpha}$ by the quotient map $\bigwedge^2 V_0 \to \bigwedge^2 (V_0/[v])$. Since $\dim(V_0/[v]) = 4$ the Grassmannian $\mathrm{Gr}(2, V_0/[v])$ is a quadric hypersurface in $\mathbb{P}(\bigwedge^2 (V_0/[v]))$; it follows that either $R_L \subset W_K$ or $R_L \cap W_K$ is a quadric hypersurface in R_L. By Lefschetz $\mathrm{Pic}(W_K)$ is generated by the hyperplane class; it follows that W_K contains no planes and no quadric surfaces. Thus necessarily $\dim R_L = 2$, moreover $R_L \not\subset W_K$ and the intersection $R_L \cap W_K$ is a conic. □

Corollary 5.5. *Let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that (5.0.2) holds and $\mathcal{C}(W_K)$ be the variety parametrizing conics on W_K (reduced structure). Then we have an isomorphism*

$$\begin{array}{ccc} |\mathcal{I}_{W_K}(2)|^\vee & \xrightarrow{\sim} & \mathcal{C}(W_K) \\ L & \mapsto & R_L \cap W_K \end{array} \tag{5.0.23}$$

where R_L is as in **Proposition 5.4**. *Moreover given $Z \in W_K^{[2]}$ there exists a unique conic containing Z namely $R_L \cap W_K$ where $L \in |\mathcal{I}_{W_K}(2)|^\vee$ is the hypeprlane of quadrics containing $\langle Z \rangle$.*

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