EPW-sextics: taxonomy

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0 Introduction

EPW-sextics are special sextic hypersurfaces in $\mathbb{P}^5$ which come equipped with a double cover ramified over their singular locus (generically a smooth surface). They were introduced by Eisenbud, Popescu and Walter [4] in order to give examples of a "quadratic sheaf"(on a hypersurface) which does not admit a symmetric resolution. We proved [15] that if the EPW-sextic is generic then the double cover is a hyperkähler (HK) 4-fold deformation deformation equivalent to the Hilbert square of a $K3$, moreover the family of (smooth) double EPW-sextics is a locally complete family of projective HK's. We recall that three other locally complete families of projective HK's of dimension greater than 2 are known, those introduced by Beauville and Donagi [2], Debarre and Voisin [3], Iliev and Ranestad [9, 10]; in all of the above examples the HK manifolds are deformations of the Hilbert square of a $K3$ and they are distinguished by the value of the Beauville-Bogomolov form on the polarization class (it equals 2 in the case of double EPW-sextics and 6, 22 and 38 in the other

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cases). EPW-sextics are defined as follows. Let V be a 6-dimensional complex vector space - this notation will be in force throughout the paper. We choose a volume-form on V

$$\mathrm{vol}\colon \bigwedge^6 V \xrightarrow{\sim} \mathbb{C} \tag{0.0.1}$$

and we equip $\bigwedge^3 V$ with the symplectic form

$$(\alpha,\beta)_V := \mathrm{vol}(\alpha\wedge\beta). \tag{0.0.2}$$

Let $\mathbb{LG}(\bigwedge^3 V)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\bigwedge^3 V$ - of course $\mathbb{LG}(\bigwedge^3 V)$ does not depend on the choice of volume-form. Given a non-zero $v\in V$ we let

$$F_v := \{\alpha\in \bigwedge^3 V \mid v\wedge\alpha = 0\} \tag{0.0.3}$$

be the sub-space of $\bigwedge^3 V$ consisting of multiples of v. Notice that $(,)_V$ is zero on F_v and $\dim(F_v) = 10$; thus $F_v\in\mathbb{LG}(\bigwedge^3 V)$. Let

$$F\subset \bigwedge^3 V\otimes\mathcal{O}_{\mathbb{P}(V)} \tag{0.0.4}$$

be the sub-vector-bundle with fiber F_v over $[v]\in\mathbb{P}(V)$. A straightforward computation gives that

$$\det F\cong \mathcal{O}_{\mathbb{P}(V)}(-6). \tag{0.0.5}$$

Given $A\in\mathbb{LG}(\bigwedge^3 V)$ we let

$$Y_A = \{[v]\in\mathbb{P}(V)\mid F_v\cap A\neq\{0\}\}. \tag{0.0.6}$$

Thus Y_A is the degeneracy locus of the map

$$F\xrightarrow{\lambda_A} (\bigwedge^3 V/A)\otimes\mathcal{O}_{\mathbb{P}(V)} \tag{0.0.7}$$

where λ_A is given by Inclusion (0.0.4) followed by the quotient map

$$\bigwedge^3 V\otimes\mathcal{O}_{\mathbb{P}(V)}\to(\bigwedge^3 V/A)\otimes\mathcal{O}_{\mathbb{P}(V)}.$$

Since the vector-bundles appearing in (0.0.7) have equal rank the determinat of λ_A makes sense and of course $Y_A = V(\det\lambda_A)$; this formula shows that Y_A has a natural structure of closed subscheme of $\mathbb{P}(V)$. By (0.0.5) we have $\det\lambda_A\in H^0(\mathcal{O}_{\mathbb{P}(V)}(6))$ and hence Y_A is either a sextic hypersurface or $\mathbb{P}(V)$. An *EPW-sextic* is a sextic hypersurface in $\mathbb{P}^5$ which is projectively equivalent to Y_A for some $A\in\mathbb{LG}(\bigwedge^3 V)$. One verifies readily that EPW-sextics exist; in fact given $[v]\in\mathbb{P}(V)$ there exists $A\in\mathbb{LG}(\bigwedge^3 V)$ such that $A\cap F_v=\{0\}$ and hence $[v]\notin Y_A$. (On the other hand there do exist $A\in\mathbb{LG}(\bigwedge^3 V)$ such that $Y_A=\mathbb{P}(V)$ e.g. $A=F_w$ for $[w]\in\mathbb{P}(V)$.) Let

$$\Sigma := \{A\in\mathbb{LG}(\bigwedge^3 V)\mid \exists W\in\mathbb{G}r(3,V)\text{ s. t. }\bigwedge^3 W\subset A\}, \tag{0.0.8}$$
$$\Delta := \{A\in\mathbb{LG}(\bigwedge^3 V)\mid \exists [v]\in\mathbb{P}(V)\text{ s. t. }\dim(A\cap F_v)\geq 3\}. \tag{0.0.9}$$

(We will denote Σ by $\Sigma(V)$ whenever we will need to keep track of V, and similarly for Δ). Then Σ and Δ are closed subsets of $\mathbb{LG}(\bigwedge^3 V)$; a straightforward computation shows that Σ and Δ are irreducible of codimension 1 - see **Section 2** for the case of Σ. Let

$$\mathbb{LG}(\bigwedge^3 V)^0 := \mathbb{LG}(\bigwedge^3 V)\setminus\Sigma\setminus\Delta. \tag{0.0.10}$$

Thus $\mathbb{LG}(\bigwedge^3 V)^0$ is open dense in $\mathbb{LG}(\bigwedge^3 V)$. In [15] we proved the following results. If $A\in\mathbb{LG}(\bigwedge^3 V)^0$ then $Y_A\neq\mathbb{P}(V)$ and there exists a finite degree-2 map $f_A\colon X_A\to Y_A$ unramified over the smooth locus of Y_A with X_A a HK 4-fold deformation equivalent to $(K3)^{[2]}$. For $A \in \mathbb{LG}(\bigwedge^3 V)^0$ let $h_A := c_1(f_A^* \mathcal{O}_{Y_A}(1))$. We proved that the family of polarized 4-folds

$$\{(X_A, h_A)\}_{A \in \mathbb{LG}(\bigwedge^3 V)^0}$$

is locally complete. Let us compare the family of double EPW-sextics and the family of HK 4-folds introduced by Beauville and Donagi [2]. Donagi and Beauville consider a cubic 4-fold $Z \subset \mathbb{P}^5$ and the family $F(Z)$ parametrizing lines in Z; they proved that if Z is smooth then $F(Z)$ is a HK 4-fold deformation equivalent to the Hilbert square of a $K3$. Moreover they showed that the primitive weight-2 integral Hodge structure of $F(Z)$ is isomorphic to the integral primitive weight-4 Hodge structure of Z (after a Tate twist) and that the isomorphism takes the Beauville-Bogomolov quadratic form on $H^2(F(Z))_{pr}$ to the opposite of the intersection from on $H^4(Z)_{pr}$. Thus the period map for the family $\{F(Z)\}$ may be studied via the period map for cubic 4-folds. Periods of cubic 4-folds were first studied by Voisin [18] who proved the Global Torelli Theorem. More recently Laza [11, 12] and Looijenga [13] proved various results, in particular they gave a complete description of the periods of smooth cubics.

This is the first in a series of papers on moduli and periods of double EPW-sextics. In order to present the results of the present paper we introduce the following notation: given $A \in \mathbb{LG}(\bigwedge^3 V)$ we let

$$\Theta_A := \{W \in \mathbb{G}r(3, V) \mid \bigwedge^3 W \subset A\}. \tag{0.0.11}$$

Our main result is a classification of those A such that Θ_A has strictly positive dimension (in particular $A \in \Sigma$). Why are we concerned with such A ? The period map $\mathbb{LG}(\bigwedge^3 V)^0 \to \mathbb{D}$ extends to a rational map $\mathbb{LG}(\bigwedge^3 V) \dashrightarrow \mathbb{D}^{BB}$ where $\mathbb{D}^{BB}$ is the Baily-Borel [1] compactification of $\mathbb{D}$: if $\dim \Theta_A > 0$ then either the period map is not regular at A or it goes to the boundary of $\mathbb{D}^{BB}$. Moreover many of the non-stable (in the sense of GIT) $A \in \mathbb{LG}(\bigwedge^3 V)$ have positive-dimensional Θ_A - the results of the present work will shed light on the description of the GIT-stable points in $\mathbb{LG}(\bigwedge^3 V)$ that will be appear in a forthcoming paper. Let us look at the analogous case of cubic 4-folds. We claim that the prime divisor $D \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ parametrizing singular cubics is analogous to Σ. As is well-known $F(Z)$ is smooth if and only if $Z \in (|\mathcal{O}_{\mathbb{P}^5}(3)| \setminus D)$ and if Z is a singular cubic 4-fold then $\operatorname{sing} F(Z)$ has dimension at least 2 (generically it is a $K3$ of degree 6). Moreover the period map extends across the generic $Z \in D$ but it does not lift to the relevant classifying space: in order to lift it one needs first to take a (local) double cover ramified over D. In the case of interest to us similar results hold. Let $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$; then X_A is either smooth (if $A \in \mathbb{LG}(\bigwedge^3 V)^0$) or the contraction of a finite union of (disjoint) copies of $\mathbb{P}^2$ in a 4-fold X_A^ϵ with a holomorphic symplectic form[1]. On the other hand if $A \in \Sigma$ (and $Y_A \neq \mathbb{P}(V)$) then X_A has singular locus of dimension at least 2 (generically a $K3$ of degree 2). What about periods ? The period map extends regularly on $(\Delta \setminus \Sigma)$ and it lifts to the classifying space. On the other hand let $A \in \Sigma$ be generic: the period map extends across A but in order to lift it to the relevant classifying space one needs first to take a (local) double cover ramified over Σ. Thus one might view the A such that $\dim \Theta_A > 0$ as analogues of cubic 4-folds whose singular locus is of strictly positive dimension - we notice that such cubics play a prominent rôle in Laza's papers [11, 12]. The following simple remark is very useful when analyzing cubics with positive dimensional singular locus: if $Z \subset \mathbb{P}^5$ is a cubic 4-fold and $p, q \in Z$ are distinct points then the line joining p and q is contained in Z. The elementary remark below might be considered as an analogue in our context.

Remark 0.1. Let $\Theta \subset \mathbb{G}r(3, V)$. The following statements are equivalent:

(1) $\dim(W_1 \cap W_2) > 0$ for any $W_1, W_2 \in \Theta$.

(2) The symplectic form $(,)_V$ vanishes on the subspace $\langle\langle \Theta \rangle\rangle \subset \bigwedge^3 V$ spanned by $\bigwedge^3 W$ for $W \in \Theta$.

In particular if $A \in \mathbb{LG}(\bigwedge^3 V)$ then $\mathbb{P}(W_1) \cap \mathbb{P}(W_2) \neq \emptyset$ for any $W_1, W_2 \in \Theta_A$.

[1]If A is generic in $(\Delta \setminus \Sigma)$ then X_A^ϵ is projective but it might not be Kähler for a particular A.

Morin [14] classified maximal families parametrizing pairwise incident planes in $\mathbb{P}^5$. Modulo projectivities there are 6 such families: 3 elementary (or Schubert) families, namely planes containing a fixed point, planes contained in a hyperplane and planes whose intersection with a fixed plane has dimension at least 1, and 3 non-elementary families, namely planes contained in a smooth quadric hypersurface, planes tangent to a Veronese surface and planes intersecting a Verones surface in a conic, see **Theorem 1.12**. The non-elementary families give rise to EPW-sextics which are a triple quadric (the first case) and a double discriminant cubic (the second and third case); they are in the indeterminacy locus of the period map and they correspond to double EPW-sextics approaching HK 4-folds with a (pseudo)polarization defining a map which is no longer 2-to-1 onto its image - see [5] for a discussion of the first case. Building on Morin's theorem we will classify the possible positive-dimensional irreducible components of Θ_A.

The paper is organized as follows. In the first section we will prove some basic results on EPW-sextics. In particular we will show that Θ_A determines how pathological Y_A might be - for example $Y_A = \mathbb{P}(V)$ if and only if the planes in Θ_A sweep out all of $\mathbb{P}(V)$. We will also show how to produce a triple smooth quadric, a double discriminant cubic and the union of 6 independent hyperplanes as EPW-sectics. In the last subsection we will show that EPW-sextics have a "classical" description as discriminant loci of certain linear systems of quadrics in $\mathbb{P}^9$ (see [8] for related results). The second section begins with some dimension counts for natural subsets of Σ and standard infinitesimal computations. The main body of that section is devoted to a classification of the elements of

$$\Sigma_\infty := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid \dim \Theta_A > 0\}. \tag{0.0.12}$$

In particular we will describe the irreducible components of Σ_∞ and we will compute their dimension. Going back to the analogy with the family of cubic 4-folds: the family of double EPW-sextics has a more elaborate geometry, in fact there are 12 irreducible components of Σ_∞ while the set of cubic 4-folds with positive dimensional singular locus has 5 irreducible components - see Theorem 6.1 of [11].

Notation and conventions: Let W be a finite-dimensional complex vector-space. The span of a subset $S \subset W$ is denoted by $\langle S\rangle$. Let $S \subset \bigwedge^q W$. The *support of* S is the smallest subspace $U \subset W$ such that $S \subset \operatorname{im}(\bigwedge^q U \longrightarrow \bigwedge^q W)$: we denote it by $\operatorname{supp}(S)$, if $S = \{\alpha\}$ is a singleton we let $\operatorname{supp}(\alpha) = \operatorname{supp}(\{\alpha\})$ (thus if $q = 1$ we have $\operatorname{supp}(\alpha) = \langle\alpha\rangle$). We define the support of a set of symmetric tensors analogously. If $\alpha \in \bigwedge^q W$ or $\alpha \in \operatorname{Sym}^d W$ the *rank of* α is the dimension of $\operatorname{supp}(\alpha)$. An element of $\operatorname{Sym}^2 W^\vee$ may be viewed either as a symmetric map or as a quadratic form: we will denote the former by $\widetilde{q}, \widetilde{r}, \ldots$ and the latter by $q, r, \ldots$ respectively.

Let U be a vector space. The *wedge subspace* of $\bigwedge^d U$ associated to a collection of subspaces $U_1, \ldots, U_\ell \subset U$ and a partition $i_1 + \cdots + i_\ell = d$ is defined as the span

$$(\bigwedge^{i_1} U_1) \wedge \cdots \wedge (\bigwedge^{i_\ell} U_\ell) := \langle \alpha_1 \wedge \cdots \wedge \alpha_\ell \mid \alpha_s \in \bigwedge^{i_s} U_s\rangle \tag{0.0.13}$$

Let W be a finite-dimensional complex vector-space. We will adhere to pre-Grothendieck conventions: $\mathbb{P}(W)$ is the set of 1-dimensional vector subspaces of W. Given a non-zero $w \in W$ we will denote the span of w by $[w]$ rather than $\langle w\rangle$; this agrees with standard notation. Suppose that $T \subset \mathbb{P}(W)$. Then $\langle T\rangle \subset \mathbb{P}(W)$ is the *projective span of* T i.e. the intersection of all linear subspaces of $\mathbb{P}(W)$ containing T while $\langle\langle T\rangle\rangle \subset W$ is the *vector-space span of* T i.e. the span of all $w \in (W \setminus \{0\})$ such that $[w] \in T$.

Schemes are defined over $\mathbb{C}$, the topology is the Zariski topology unless we state the contrary. Let W be finite-dimensional complex vector-space: $\mathcal{O}_{\mathbb{P}(W)}(1)$ is the line-bundle on $\mathbb{P}(W)$ with fiber $L^\vee$ on the point $L \in \mathbb{P}(W)$. Let $F \in \operatorname{Sym}^d W^\vee$: we let $V(F) \subset \mathbb{P}(W)$ be the subscheme defined by vanishing of F. If $E \to X$ is a vector-bundle we denote by $\mathbb{P}(E)$ the projective fiber-bundle with fiber $\mathbb{P}(E(x))$ over x and we define $\mathcal{O}_{\mathbb{P}(W)}(1)$ accordingly. If Y is a subscheme of X we let $Bl_Y X \longrightarrow X$ be the blow-up of Y.

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1 EPW-sextics

1.1 Symplectic Grassmannians

Let $\mathbf{H}$ be a complex vector-space of dimension $2n$ equipped with a symplectic form $(,)_{\mathbf{H}}$. We let $\mathbb{LG}(\mathbf{H}) \subset \mathbb{G}r(n, \mathbf{H})$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\mathbf{H}$. We will go through some well-known results regarding $\mathbb{LG}(\mathbf{H})$. Let $A \in \mathbb{LG}(\mathbf{H})$: the symplectic form gives an isomorphism

$$\begin{array}{ccc} \mathbf{H}/A & \xrightarrow{\sim} & A^{\vee} \\ \overline{x} & \mapsto & (a \mapsto (a, x)_{\mathbf{H}}) \end{array} \tag{1.1.1}$$

and hence we have a canonical inclusion

$$T_A \mathbb{LG}(\mathbf{H}) \subset T_A \mathrm{Gr}(n, \mathbf{H}) = \mathrm{Hom}(A, \mathbf{H}/A) = A^{\vee} \otimes A^{\vee} \tag{1.1.2}$$

Let $\mathbb{LG}(\mathbf{H}) \hookrightarrow \mathbb{P}(\bigwedge^n \mathbf{H})$ be the Plücker embedding: the pull-back of the ample generator of $\mathrm{Pic}(\mathbb{P}(\bigwedge^n \mathbf{H}))$ is the *Plücker line-bundle* on $\mathbb{LG}(\mathbf{H})$. The following result is well-known; one reason for providing a proof is to introduce notation that will be used throughout the paper.

Proposition 1.1. *Keep notation and hypotheses as above.*

(1) $\mathbb{LG}(\mathbf{H})$ *is smooth, irreducible and Inclusion* (1.1.2) *identifies* $T_A \mathbb{LG}(\mathbf{H})$ *with* $\mathrm{Sym}^2 A^{\vee}$.

(2) The Picard group of $\mathbb{LG}(\mathbf{H})$ *is generated by the class of the Plücker line-bundle.*

Proof. The symplectic group $Sp(\mathbf{H})$ acts transitively on $\mathbb{LG}(\mathbf{H})$ and hence $\mathbb{LG}(\mathbf{H})$ is smooth. Given $B \in \mathbb{LG}(\mathbf{H})$ we let

$$U_B := \{C \in \mathbb{LG}(\mathbf{H}) \mid B \cap C = \{0\}\}. \tag{1.1.3}$$

Clearly U_B is open in $\mathbb{LG}(\mathbf{H})$. One defines a (non canonical) isomorphism of varieties

$$\mathrm{Sym}^2 B \longrightarrow U_B \tag{1.1.4}$$

as follows. Choose $C \in U_B$. The direct-sum decomposition $\mathbf{H} = C \oplus B$ defines an isomorphism $C \xrightarrow{\sim} \mathbf{H}/B$; composing with the isomorphism $\mathbf{H}/B \xrightarrow{\sim} B^{\vee}$ (see (1.1.1)) we get an isomorphism $\iota\colon C \xrightarrow{\sim} B^{\vee}$. Let $\widetilde{q} \in \mathrm{Sym}^2 B$ and view $\widetilde{q}$ as a symmetric map $B^{\vee} \to B$; the graph $\Gamma_{\widetilde{q}}$ of $\widetilde{q}$ lies in $B^{\vee} \oplus B$ and hence

$$(\iota, \mathrm{Id}_B)^{-1} \Gamma_{\widetilde{q}} \subset C \oplus B = \mathbf{H}. \tag{1.1.5}$$

Moreover $(\iota, \mathrm{Id}_B)^{-1} \Gamma_{\widetilde{q}}$ is Lagrangian because $\widetilde{q}$ is symmetric and it belongs to U_B because $\Gamma_{\widetilde{q}}$ is a graph. We define (1.1.4) by sending $\widetilde{q}$ to $(\iota, \mathrm{Id}_B)^{-1} \Gamma_{\widetilde{q}}$. Now choose B transversal to A. Then $A \in U_B$ and hence we may choose $C = A$. We have defined an isomorphism $\iota\colon A \xrightarrow{\sim} B^{\vee}$ and hence (1.1.4) gives an isomorphism $\mathrm{Sym}^2 A^{\vee} \longrightarrow U_B$: the differential at 0 equals (1.1.2) and this proves that (1.1.2) identifies $T_A \mathbb{LG}(\mathbf{H})$ with $\mathrm{Sym}^2 A^{\vee}$. Irreducibility of $\mathbb{LG}(\mathbf{H})$ follows from the following two facts: first the open sets U_B for B varying in $\mathbb{LG}(\mathbf{H})$ form a covering of $\mathbb{LG}(\mathbf{H})$ and secondly $U_B \cap U_{B'}$ is non-empty for arbitrary $B, B' \in \mathbb{LG}(\mathbf{H})$. Let's prove Item (2). Given $A \in \mathbb{LG}(\mathbf{H})$ we let

$$D_A := \{B \in \mathbb{LG}(\mathbf{H}) \mid A \cap B \neq \{0\}\} = (\mathbb{LG}(\mathbf{H}) \setminus U_A). \tag{1.1.6}$$

One checks easily that D_A is of pure codimension 1 in $\mathbb{LG}(\mathbf{H})$ and hence it may be viewed as an effective divisor: in fact it belongs to the Plücker linear system. We have an exact sequence of Chow groups (see Proposition (1.8) of [6])

$$CH^0(D_A) \longrightarrow CH^1(\mathbb{LG}(\mathbf{H})) \longrightarrow CH^1(U_A) \longrightarrow 0. \tag{1.1.7}$$

Isomorphism (1.1.4) gives that $CH^1(U_A)=0$ and hence $CH^1(\mathbb{LG}(\mathbf{H}))$ is generated by the classes of irreducible components of D_A. Since D_A is irreducible and it it belongs to the Plücker linear system we get Item (2). $\square$

Corollary 1.2. *Let V be a 6-dimensional complex vector-space. Then $\mathbb{LG}(\bigwedge^3 V)$ is irreducible, smooth of dimension 55 and its Picard group is generated by the class of the Plücker line-bundle.*

1.2 Degeneracy loci attached to $A \in \mathbb{LG}(\bigwedge^3 V)$

Let $A \in \mathbb{LG}(\bigwedge^3 V)$. We let

$$Y_A[k] = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) \geq k\}. \tag{1.2.1}$$

Thus $Y_A[0] = \mathbb{P}(V)$ and $Y_A[1] = Y_A$. We will show that $Y_A[k]$ has a natural structure of closed sub-scheme of $\mathbb{P}(V)$. First we associate to $B \in \mathbb{LG}(\bigwedge^3 V)$ the open subset $\mathcal{U}_B \subset \mathbb{P}(V)$ defined by

$$\mathcal{U}_B := \{[v] \in \mathbb{P}(V) \mid F_v \cap B = \{0\}\}. \tag{1.2.2}$$

(In other words $\mathcal{U}_B$ is the intersection of U_B and $\mathbb{P}(V)$ embedded in $\mathbb{LG}(\bigwedge^3 V)$ by the map $[v] \mapsto F_v$.) Choose B transversal to A: we will write $Y_A[k] \cap \mathcal{U}_B$ as the k-th degeneracy locus of a symmetric map of vector-bundles. We have a direct-sum decomposition $\bigwedge^3 V = A \oplus B$ and for $[v] \in \mathcal{U}_B$ the Lagrangian subspace F_v is transversal to B; thus F_v is the graph of a symmetric map

$$\tau^B_A([v])\colon A \to B \cong A^\vee. \tag{1.2.3}$$

(The symplectic form $(,)_V$ together with the decomposition $\bigwedge^3 V = A \oplus B$ provides us with an isomorphism $B \cong A^\vee$ - see the proof of **Proposition 1.1**.) Since $\tau^B_A\colon \mathcal{U}_B \to \mathrm{Sym}^2 A^\vee$ is a regular map we may define a closed subscheme $Y^B_A[k] \subset \mathcal{U}_B$ by setting

$$Y^B_A[k] := V(\bigwedge^{(11-k)} \tau^B_A). \tag{1.2.4}$$

The support of $Y^B_A[k]$ is equal to $Y_A[k] \cap \mathcal{U}_B$. If $B' \subset \bigwedge^3 V$ is another Lagrangian subspace transversal to A then the restrictions of $Y^B_A[k]$ and $Y^{B'}_A[k]$ to $\mathcal{U}_B \cap \mathcal{U}_{B'}$ are equal. The open sets $\mathcal{U}_B$ with B transversal to A form a covering of $\mathbb{P}(V)$. Thus the collection of $Y^B_A[k]$'s glue together to give a closed subscheme of $\mathbb{P}(V)$ whose support is equal to $Y_A[k]$. It follows immediately from the definitions that the scheme $Y_A[1]$ is equal to the scheme Y_A defined in **Section 0**. By **Proposition 3.1** we have

$$\mathrm{cod}(Y_A[k], \mathbb{P}(V)) \leq \frac{k(k+1)}{2} \quad \text{if} \quad Y_A[k] \neq \emptyset. \tag{1.2.5}$$

We set

$$Y_A(k) := Y_A[k] \setminus Y_A[k+1]. \tag{1.2.6}$$

1.3 Local equation of Y_A

Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and $[v_0] \in \mathbb{P}(V)$: we will analyze Y_A in a neighborhood of $[v_0]$. Let $V_0 \subset V$ be a subspace complementary to $[v_0]$. We identify V_0 with the open affine $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ via the isomorphism

$$\begin{array}{ccc} V_0 & \xrightarrow{\sim} & \mathbb{P}(V) \setminus \mathbb{P}(V_0) \\ v & \mapsto & [v_0 + v]. \end{array} \tag{1.3.1}$$

(Thus $0 \in V_0$ corresponds to $[v_0]$.) Since Y_A is a sextic hypersurface we have

$$Y_A \cap V_0 = V(f_0 + f_1 + \cdots + f_6), \qquad f_i \in \mathrm{Sym}^i V_0^\vee, \tag{1.3.2}$$

where the f_i's are determined up to a common multiplicative non-zero constant. We will describe explicitly the polynomials f_i of (1.3.2) for $i \leq \dim(A \cap F_{v_0})$. First some preliminaries. Given $v \in V$

we define a quadratic form $\phi^{v_0}_v$ on F_{v_0} as follows. Let $\alpha\in F_{v_0}$; then $\alpha=v_0\wedge\beta$ for some $\beta\in\bigwedge^2 V$. We set

$$\phi^{v_0}_v(\alpha):=\mathrm{vol}(v_0\wedge v\wedge\beta\wedge\beta). \tag{1.3.3}$$

The above equation gives a well-defined quadratic form on F_{v_0} because β is determined up to addition by an element of F_{v_0}.

Proposition 1.3. *Let $A\in\mathbb{LG}(\bigwedge^3 V)$. Let $[v_0]\in\mathbb{P}(V)$ and $V_0\subset V$ be a subspace complementary to $[v_0]$. Let $f_i\in\mathrm{Sym}^i V_0^\vee$ for $i=0,\ldots,6$ be the polynomials appearing in* (1.3.2)*. Let $K:=A\cap F_{v_0}$ and $k:=\dim K$. Then*

(1) $f_i=0$ for $i<k$, and

(2) there exists $\mu\in\mathbb{C}^*$ such that

$$f_k(v)=\mu\det(\phi^{v_0}_v|_K),\quad v\in V_0, \tag{1.3.4}$$

where $\phi^{v_0}_v$ is the quadratic form defined by (1.3.3).

Proof. Let $B\in\mathbb{LG}(\bigwedge^3 V)$ be transversal both to A and F_{v_0}. Let $\mathcal{V}\subset V_0$ be the open subset of v such that $[v_0+v]\in\mathcal{U}_B$ where $\mathcal{U}_B$ is given by (1.2.2). Notice that $0\in\mathcal{V}$. For $v\in\mathcal{V}$ we let $\widetilde{q}(v):=\tau^B_A([v_0+v])$ where τ^B_A is given by (1.2.3). Let $q(v)\colon A\to\mathbb{C}$ be the quadratic form associated to $\widetilde{q}(v)$. By definition of Y_A we have

$$Y_A\cap\mathcal{V}=V(\det q)\,. \tag{1.3.5}$$

We have $\ker q(0)=A\cap F_{v_0}=K$; by **Proposition 3.2** it follows that $\det q\in\mathfrak{m}_0^k$ where $\mathfrak{m}_0\subset\mathcal{O}_{\mathcal{V},0}$ is the maximal ideal. This proves Item (1). Let's prove Item (2). Let $(\det q)_k\in\left(\mathfrak{m}_0^k/\mathfrak{m}_0^{k+1}\right)\cong\mathrm{Sym}^k V_0^\vee$ be the "initial" term of $\det q$; by (1.3.5) we have

$$f_k=c(\det q)_k,\qquad c\in\mathbb{C}^*\,. \tag{1.3.6}$$

By **Proposition 3.2** there exists $\theta\in\mathbb{C}^*$ such that

$$(\det q)_k(v)=\theta\det\left(\left.\frac{d\,(q(vt)|_K)}{dt}\right|_{t=0}\right). \tag{1.3.7}$$

A straighforward computation (see Equation (2.26) of [15]) gives that

$$\left.\frac{d\,(q(vt)|_K)}{dt}\right|_{t=0}=\phi^{v_0}_v|_K. \tag{1.3.8}$$

Item (2) follows from (1.3.6), (1.3.7) and 1.3.8. □

In order to apply the above proposition we will need a geometric description of the right-hand side of (1.3.4). Let $[v_0]\in\mathbb{P}(V)$ and $V_0\subset V$ be complementary to $[v_0]$; we let

$$\begin{array}{rccc}\lambda^{v_0}_{V_0}\colon & \bigwedge^2 V_0 & \xrightarrow{\sim} & F_{v_0}\\ & \beta & \mapsto & v_0\wedge\beta\end{array} \tag{1.3.9}$$

Without choossing a complementary subspace we get an isomorphism

$$\begin{array}{rccc}\lambda^{v_0}\colon & \bigwedge^2(V/[v_0]) & \xrightarrow{\sim} & F_{v_0}\\ & \overline{\beta} & \mapsto & v_0\wedge\beta\end{array} \tag{1.3.10}$$

(Here $\overline{\beta}$ is the class represented by β; the point being that $v_0\wedge\beta$ is indeopendent of the representative.) Taking inverses we get isomorphisms

$$F_{v_0}\xrightarrow{\rho^{v_0}_{V_0}}\bigwedge^2 V_0,\qquad F_{v_0}\xrightarrow{\rho^{v_0}}\bigwedge^2(V/[v_0])\,. \tag{1.3.11}$$

Via $\rho^{v_0}_{V_0}$ we may view $\phi^{v_0}_v$ as a Plücker quadratic form on $\bigwedge^2 V_0$. More precisely: given $v\in V_0$ let q_v be the quadratic form on $\bigwedge^2 V_0$ defined by

$$\begin{array}{ccc} \bigwedge^2 V_0 & \overset{q_v}{\longrightarrow} & \mathbb{C} \\ \alpha & \mapsto & \mathrm{vol}(v_0\wedge v\wedge\alpha\wedge\alpha) \end{array} \tag{1.3.12}$$

Then q_v is a Plücker quadratic form and we have an isomorphism

$$\begin{array}{ccc} V_0 & \overset{\sim}{\longrightarrow} & H^0(\mathcal{I}_{\mathrm{Gr}(2,V_0)}(2)) \\ v & \mapsto & q_v \end{array} \tag{1.3.13}$$

Remark 1.4. We may view q_v as a (Plücker) quadratic form on $V/[v_0]$ because given $\overline{\alpha}\in\bigwedge^2(V/[v_0])$ the value $\mathrm{vol}(v_0\wedge v\wedge\alpha\wedge\alpha)$ is independent of the representative $\alpha\in\bigwedge^2 V$ of $\overline{\alpha}$.

Clearly we have the following relation between $\phi^{v_0}_v$ and q_v:

$$\mathrm{Sym}^2(\rho^{v_0}_{V_0})(\phi^{v_0}_v)=q_v,\qquad v\in V_0\,. \tag{1.3.14}$$

Since $\mathrm{Gr}(2,V_0)$ is cut out by quadrics we get that

$$\mathbb{P}(\rho^{v_0}_{V_0}(\bigcap_{v\in V_0} V(\phi^{v_0}_v|_K)))=\mathbb{G}r(2,V_0)\cap\mathbb{P}(\rho^{v_0}_{V_0}(K)). \tag{1.3.15}$$

Corollary 1.5. *Keep hypotheses as in* **Proposition 1.3**. *Then the following hold:*

(1) Suppose that $A\cap F_{v_0}$ does not contain a non-zero decomposable element of $\bigwedge^3 V$. Then $Y_A\neq\mathbb{P}(V)$ and $\mathrm{mult}_{[v_0]}Y_A=\dim(A\cap F_{v_0})$. If moreover $A\cap F_{v_0}$ is one-dimensional, say $A\cap F_{v_0}=\langle v_0\wedge\beta\rangle$, then the projective tangent space of Y_A at $[v_0]$ is

$$T_{[v_0]}Y_A=\mathbb{P}(\mathrm{supp}(v_0\wedge\beta))\,. \tag{1.3.16}$$

(2) If $A\cap F_{v_0}$ contains a non-zero decomposable element of $\bigwedge^3 V$ then Y_A is singular at $[v_0]$ unless $Y_A=\mathbb{P}(V)$.

Proof. Let's prove Item (1). Let $K:=A\cap F_{v_0}$ and $k:=\dim K$; we let f_k be the degree-k polynomial appearing in (1.3.2). By hypothesis $\mathrm{Gr}(2,V_0)\cap\mathbb{P}(\rho^{v_0}_{V_0}(K))=\emptyset$. By (1.3.15), Bertini's Theorem and (1.3.13) we get that if $v\in V_0$ is generic the quadratic form $\phi^{v_0}_v|_K$ is non-degenerate. Thus $f_k\neq 0$ by (1.3.4). If $\dim(A\cap F_{v_0})=1$ Formula (1.3.4) gives (1.3.16). Let's prove Item (2). Suppose that $Y_A\neq\mathbb{P}(V)$. Then one at least of the polynomials f_i appearing in (1.3.2) is non-zero; thus it suffices to prove that $f_1=0$. If $k\geq 2$ then we are done by Item (1) of **Proposition 1.3**. Now assume that $k=1$. Then $\mathbb{P}(\rho^{v_0}_{V_0}(K))$ is a point contained in $\mathrm{Gr}(2,V_0)$ and hence $f_1=0$ by (1.3.15) and (1.3.13). $\square$

In order to prove sharper results we will analyze the rational map

$$\mathbb{P}(\bigwedge^2 V_0)\overset{\Phi}{\dashrightarrow}|\mathcal{I}_{\mathrm{Gr}(2,V_0)}(2)|^\vee\cong\mathbb{P}(V_0^\vee)\,. \tag{1.3.17}$$

Let $Z\subset\mathbb{P}(\bigwedge^2 V_0)\times\mathbb{P}(V_0^\vee)$ be the incidence subvariety defined by

$$Z:=\{([\alpha],[\phi])\mid\phi(\mathrm{supp}\,\alpha)=0\}\,. \tag{1.3.18}$$

We have a triangle

$$\begin{tikzcd} & Z \arrow[dl, "\Psi"'] \arrow[dr, "\widetilde{\Phi}"] & \\ \mathbb{P}(\bigwedge^2 V_0) \arrow[rr, dashed, "\Phi"] & & \mathbb{P}(V_0^\vee) \end{tikzcd} \tag{1.3.19}$$

where Ψ and $\widetilde{\Phi}$ are the restrictions to Z of the two projections of $\mathbb{P}(\bigwedge^2 V_0)\times\mathbb{P}(V_0^\vee)$. We will be using the following result; the easy proof is left to the reader.

Lemma 1.6. *Keep notation as above - in particular* $\dim V_0 = 5$. *Then the following hold:*

(1) The map Ψ *appearing in* (1.3.19) *is the blow-up of* $\mathrm{Gr}(2, V_0)$.

(2) (1.3.19) *is a commutative diagram; in fact* Z *is the graph of the rational map* Φ.

In particular the lemma above states the following: if $\alpha \in \bigwedge^2 V_0$ is not decomposable then $\Phi([\alpha]) = \operatorname{supp} \alpha$.

Lemma 1.7. *Keep notation as above - in particular* $\dim V_0 = 5$. *Let* $\mathbf{K} \subset \mathbb{P}(\bigwedge^2 V_0)$ *be a projective subspace which does not intersect* $\mathrm{Gr}(2, V_0)$. *The map*

$$\begin{array}{ccc} \mathbf{K} & \longrightarrow & \Phi(\mathbf{K}) \\ {[\alpha]} & \mapsto & \Phi([\alpha]) \end{array} \tag{1.3.20}$$

is an isomorphism.

Proof. Let $U \in \Phi(\mathbf{K})$ where $U \subset V_0$ is a subspace of codimension 1. Then $\mathbb{P}(\bigwedge^2 U) \subset \mathbb{P}(\bigwedge^2 V_0)$. We claim that

$$\mathbb{P}(\bigwedge^2 U) \cap \mathbf{K} \text{ is a point.} \tag{1.3.21}$$

In fact suppose the contrary. Then there exists a line $L \subset (\mathbb{P}(\bigwedge^2 U) \cap \mathbf{K})$. Since $\mathrm{Gr}(2, U) \subset \mathbb{P}(\bigwedge^2 U)$ is a (quadric) hypersurface we get that $L \cap \mathrm{Gr}(2, U) \neq \emptyset$ and this contradicts the hypothesis that $\mathbf{K} \cap \mathrm{Gr}(2, V_0) = \emptyset$. This proves (1.3.21)). By commutativity of (1.3.19) we get that Map (1.3.20) is bijective with injective differential; it follows that Map (1.3.20) is an isomorphism. $\square$

Lemma 1.8. *Keep notation as above - in particular* $\dim V_0 = 5$. *Let* $\mathbf{K} \subset \mathbb{P}(\bigwedge^2 V_0)$ *be a projective subspace intersecting* $\mathrm{Gr}(2, V_0)$ *in a unique point* p_0 *and such that*

$$\mathbf{K} \cap T_{p_0} \mathrm{Gr}(2, V_0) = \{p_0\}\,. \tag{1.3.22}$$

The restriction of Φ *to* $\mathbf{K}$ *is identified with the natural map*

$$\mathbf{K} \dashrightarrow |\mathcal{I}_{p_0}(2)|^\vee\,. \tag{1.3.23}$$

Proof. Suppose the contrary. Then there exists a proper projective subspace $\mathbf{P} \subset |\mathcal{I}_{p_0}(2)|$ such that the restriction of Φ to $\mathbf{K}$ is identified with the natural map $\mathbf{K} \dashrightarrow \mathbf{P}^\vee$. It follows that there exists a subscheme $\mathcal{Z} \subset (\mathbf{K} \setminus \{p_0\})$ of length 2 over which Φ is constant. Let $L \subset \mathbf{K}$ be the line containing $\mathcal{Z}$. Arguing as in the proof of **Lemma 1.7** we get that L intersects $\mathrm{Gr}(2, V_0)$ in two points or is tangent to $\mathrm{Gr}(2, V_0)$; that contradicts our hypothesis. $\square$

Proposition 1.9. *Let* $A \in \mathbb{LG}(\bigwedge^3 V)$. *Suppose that* $[v_0] \in Y_A(2)$ *and that* $A \cap F_{v_0}$ *does not contain a non-zero decomposable element of* $\bigwedge^3 V$. *Then* $Y_A \neq \mathbb{P}(V)$ *and the following hold:*

1 $\mathrm{mult}_{[v_0]} Y_A = 2$ *and the quadric cone* $C_{[v_0]} Y_A$ *has rank* 3.

2 $Y_A[2]$ *is smooth two-dimensional at* $[v_0]$.

Proof. By **Corollary 1.5** we know that $Y_A \neq \mathbb{P}(V)$. Let $K := A \cap F_{v_0}$. We claim that the map

$$\begin{array}{ccc} V_0 & \longrightarrow & \mathrm{Sym}^2 K^\vee \\ v & \mapsto & \phi_v^{v_0}|_K \end{array} \tag{1.3.24}$$

is surjective. In fact let $\mathbf{K} := \mathbb{P}(\rho_{V_0}^{v_0}(K))$. By hypothesis $\mathbf{K}$ does not intersect the indeterminacy locus of the map Φ given by (1.3.17). Thus pull-back by Φ defines a map

$$H^0(\mathcal{O}_{\mathbb{P}(V_0^\vee)}(1)) \xrightarrow{\Phi^*} H^0(\mathcal{O}_{\mathbf{K}}(2))\,. \tag{1.3.25}$$

By **Lemma 1.7** the restriction of Φ to $\mathbf{K}$ is injective; since $\mathbf{K} \cong \mathbb{P}^1$ it follows that (1.3.25) is surjective. By Isomorphism (1.3.13) we get that (1.3.24) is surjective. Items (1), (2) follow from surjectivity of (1.3.24) together with (1.3.4), and (1.3.8), 3.0.3 respectively. $\square$

The following result was proved in [15].

Corollary 1.10 (Proposition (2.8) of [15]). *If $A \in \mathbb{LG}(\bigwedge^3 V)^0$ then $Y_A \neq \mathbb{P}(V)$ and the following hold:*

(1) $\operatorname{sing} Y_A$ *is a smooth pure-dimensional surface of degree* 40.

(2) $\operatorname{mult}_{[v]} Y_A = 2$ *for every* $[v] \in \operatorname{sing} Y_A$ *and the quadric cone* $C_{[v]}Y_A$ *has rank* 3.

Proof. **Corollary 1.5** gives that $Y_A \neq \mathbb{P}(V)$ and that $\operatorname{sing} Y_A = Y_A[2]$. Since Y_A is a global Lagrangian degeneracy locus there is a Porteous formula that gives the expected cohomology class of $Y_A[k]$ for every integer k - see [7]. In fact Formula (6.7) of [7] and the equation

$$c(F) = 1 - 6c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) + 18c_1(\mathcal{O}_{\mathbb{P}(V)}(1))^2 - 34c_1(\mathcal{O}_{\mathbb{P}(V)}(1))^3 + \dots \tag{1.3.26}$$

give that the expected cohomology class of $Y_A[2]$ is

$$\text{exp.class of } Y_A[2] = 2c_3(F) - c_1(F)c_2(F) = 40c_1(\mathcal{O}_{\mathbb{P}(V)}(1))^3 \tag{1.3.27}$$

Since the above class is non-zero it follows that $Y_A[2] \neq \emptyset$. By **Proposition 1.9** we get that $Y_A[2]$ is a smooth pure-dimensional surface. Surjectivity of Map (1.3.24) gives that the expected class of $Y_A[2]$ is the cohomology class of the smooth surface $Y_A[2]$; thus $\deg Y_A[2] = 40$. This proves Item (1). Item (2) follows from **Proposition 1.9**. □

We notice that the converse of **Corollary 1.10** holds but we will not prove it here.

We close the present subsection by showing how to detect the most pathological $A \in \mathbb{LG}(\bigwedge^3 V)$. Let

$$\mathbb{N}(V) := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid Y_A = \mathbb{P}(V)\}\,. \tag{1.3.28}$$

We say that a closed $Z \subset \mathbb{G}r(3, V)$ is *invasive* if

$$\bigcup_{W \in Z} W = V. \tag{1.3.29}$$

Claim 1.11. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$. Then $A \in \mathbb{N}(V)$ if and only if Θ_A is invasive. In particular if $\dim \Theta_A \leq 2$ then $Y_A \notin \mathbb{P}(V)$.*

Proof. Suppose that $A \in \mathbb{N}(V)$; then Θ_A is invasive by Item (1) of **Corollary 1.5**. Now suppose that Θ_A is invasive. Let $[v] \in \mathbb{P}(V)$. Since Θ_A is invasive there exists $W \in \Theta_A$ containing v. Then $\bigwedge^3 W \subset (A \cap F_v)$ and hence $[v] \in Y_A$. Since $[v]$ was arbitrary we get that $Y_A = \mathbb{P}(V)$. This proves that $A \in \mathbb{N}(V)$ if and only if Θ_A is invasive. Now suppose that $\dim \Theta_A \leq 2$. Then $\dim(\cup_{W \in \Theta_A} W) \leq 5$ and hence Θ_A is not invasive; thus $Y_A \notin \mathbb{P}(V)$ by the first part of the claim. □

1.4 Morin's Theorem

Suppose that $\Theta \subset \Theta_A$ for some $A \in \mathbb{LG}(\bigwedge^3 V)$; by **Remark 0.1** any two $W_1, W_2 \in \Theta$ have non-trivial intersection or equivalently $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ are incident. Ugo Morin [14] classified maximal irreducible families of pairwise incident planes in a projective space. In order to recall Morin's result we introduce certain subsets of $\mathrm{Gr}(3, V)$. The first three are Schubert cycles, namely

$$J_{v_0} := \{W \in \mathrm{Gr}(3, V) \mid v_0 \in W\}, \quad [v_0] \in \mathbb{P}(V) \tag{1.4.1}$$
$$I_U := \{W \in \mathrm{Gr}(3, V) \mid \dim(W \cap U) \geq 2\}, \quad U \in \mathrm{Gr}(3, V), \tag{1.4.2}$$

and $\mathrm{Gr}(3, E)$ where $E \in \mathrm{Gr}(5, V)$. Next let $\mathcal{Q}, \mathcal{V} \subset \mathbb{P}(V)$ be a smooth quadric hypersurface and a Veronese surface respectively; set

$$F_+(\mathcal{Q}), F_-(\mathcal{Q}) := \text{irred. compt.'s of } \{W \in \mathrm{Gr}(3, V) \mid \mathbb{P}(W) \subset \mathcal{Q}\}, \tag{1.4.3}$$
$$C(\mathcal{V}) := \{W \in \mathrm{Gr}(3, V) \mid \mathbb{P}(W) \cap \mathcal{V} \text{ is a conic}\}, \tag{1.4.4}$$
$$T(\mathcal{V}) := \{W \in \mathrm{Gr}(3, V) \mid \mathbb{P}(W) = T_p\mathcal{V} \text{ for some } p \in \mathcal{V}\}. \tag{1.4.5}$$

(Here $T_p\mathcal{V}$ is the projective tangent plane to $\mathcal{V}$ at p.) As is easily checked Item (1) of **Remark 0.1** holds for Θ equal to one of the six subsets listed above (of course there is no intrinsic difference between $F_+(\mathcal{Q})$ and $F_-(\mathcal{Q})$) - the first three are the *elementary complete systems* in Morin's terminology.

Theorem 1.12. [U. Morin [14]] *Let $\Theta \subset \mathrm{Gr}(3,V)$ be a closed irreducible subset such that* $\dim(W_1 \cap W_2) > 0$ *for every $W_1, W_2 \in \Theta$. Then one of the following holds:*

(a) There exists a smooth quadric $\mathcal{Q} \subset \mathbb{P}(V)$ such that $\Theta \subset F_{\pm}(\mathcal{Q})$.

(b) There exists a Veronese surface $\mathcal{V} \subset \mathbb{P}(V)$ such that Θ is contained in $C(\mathcal{V})$ or in $T(\mathcal{V})$.

(c) $\Theta \subset J_{v_0}$ for a certain $[v_0] \in \mathbb{P}(V)$.

(d) $\Theta \subset \mathrm{Gr}(3,E)$ for a certain $E \in \mathrm{Gr}(5,V)$.

(e) $\Theta \subset I_U$ for a certain $U \in \mathrm{Gr}(3,V)$.

Let us examine J_{v_0} and I_U in greater detail. Let

$$\overline{\rho}^{v_0}_{V_0} \colon \mathbb{P}(F_{v_0}) \xrightarrow{\sim} \mathbb{P}(\bigwedge^2 V_0) \tag{1.4.6}$$

be the isomorphism induced by (1.3.9). Restricting $\overline{\rho}_{v_0}$ to $J_{v_0} \subset \mathbb{P}(F_{v_0})$ we get an isomorphism

$$\begin{array}{ccc} J_{v_0} & \xrightarrow{\sim} & \mathrm{Gr}(2,V_0) \\ W & \mapsto & W \cap V_0 \end{array} \tag{1.4.7}$$

Next we examine I_U. Given a subspace $U \subset V$ of arbitrary dimension we let

$$S_U := (\bigwedge^2 U) \wedge V. \tag{1.4.8}$$

Of course $S_U = 0$ if $\dim U \le 1$ and thus S_U is of interest only for $\dim U \ge 2$. Notice that $S_U \supset \bigwedge^3 U$. We let

$$T_U := S_U / \bigwedge^3 U \cong \bigwedge^2 U \otimes (V/U), \tag{1.4.9}$$

and

$$\rho_U \colon S_U \longrightarrow T_U \tag{1.4.10}$$

be the quotient map. Now assume that $\dim U = 3$ and hence I_U is defined and $I_U \subset \mathbb{P}(S_U)$. Let

$$\overline{\rho}_U \colon \mathbb{P}(S_U) \dashrightarrow P(T_U) \tag{1.4.11}$$

be the rational map corresponding to (1.4.10) i.e. projection from $\mathbb{P}(\bigwedge^3 U)$. The following claim gives an explicit description of I_U; the easy proof is left to the reader.

Claim 1.13. *Let $U \in \mathrm{Gr}(3,V)$. Then I_U is the cone with vertex $\bigwedge^3 U$ over the Segre variety $\mathbb{P}(\bigwedge^2 U) \times \mathbb{P}(W) \subset \mathbb{P}(T_U)$.*

1.5 Menagerie

We show that the following are EPW-sextics: $3\mathcal{Q}$ where $\mathcal{Q}$ is a smooth quadric, $2\,\mathrm{chord}(\mathcal{V})$ where $\mathrm{chord}(\mathcal{V})$ is the chordal variety of a Veronese surface (i.e. the discriminant cubic parametrizing degenerate plane conics) and the union of six independent hyperplanes. These special EPW-sextics are analogues of certain cubic 4-folds and plane sextic curves which have a special rôle in the works of Laza [11, 12] and Shah [17] - Table (1) gives the dictionary between the three cases.

$\boxed{\textit{Triple smooth quadric}}$ Write $V = \bigwedge^2 U$ where U is a complex vector-space of dimension 4. Thus $\mathcal{Q} := \mathrm{Gr}(2,U) \subset \mathbb{P}(\bigwedge^2 U)$ is a smooth quadric hypersurface. We have embeddings

$$\begin{array}{ccccccc} \mathbb{P}(U) & \overset{i_+}{\hookrightarrow} & \mathrm{Gr}(3,\bigwedge^2 U) & \qquad & \mathbb{P}(U^\vee) & \overset{i_-}{\hookrightarrow} & \mathrm{Gr}(3,\bigwedge^2 U) \\ [u_0] & \mapsto & \{u_0 \wedge u \mid u \in U\} & & [f_0] & \mapsto & \bigwedge^2 ker(f_0). \end{array} \tag{1.5.1}$$

Table 1: Analogous special EPW-sextics, cubic 4-folds and plane sextics

EPW-sextic	cubic 4-fold	plane sextic
triple quadric, double discriminant cubic	discriminant cubic	triple conic
union of 6 independent hyperplanes	$V(x_0x_1x_2 + x_3x_4x_5)$	$V(x_0^2x_1^2x_2^2)$

Thus referring to (1.4.3) we may set

$$F_\pm(\mathcal{Q}) = \mathrm{im}(i_\pm)\,. \tag{1.5.2}$$

Let $A_\pm(U) \subset \bigwedge^3(\bigwedge^2 U)$ be the subspaces defined by

$$A_\pm(U) = \langle\langle \mathrm{im}(i_\pm)\rangle\rangle \tag{1.5.3}$$

Claim 1.14. *Keep notation as above. Then* $A_\pm(U) \in \mathbb{LG}(\bigwedge^3(\bigwedge^2 U))$ *and furthermore*

$$Y_{A_\pm(U)} = 3\mathcal{Q}. \tag{1.5.4}$$

Proof. Since any two planes in $\mathrm{im}(i_\pm)$ are incident we get that $A_\pm(U)$ is isotropic for $(,)_V$ - see **Remark 0.1**. Let $\mathcal{L}$ be the Plücker(ample) line-bundle on $\mathrm{Gr}(3, \bigwedge^2 U)$; one checks easily that

$$i_+^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(U)}(2), \qquad i_-^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(U^\vee)}(2). \tag{1.5.5}$$

Since $i_\pm$ is $SL(U)$-equivariant and $\mathrm{Sym}^2 U, \mathrm{Sym}^2 U^\vee$ are irreducible $SL(U)$-modules we get surjections

$$\begin{array}{ccccc} H^0(\mathcal{O}_{\mathbb{P}(U)}(2)) & \xleftarrow{i_+^*} & H^0(\mathrm{Gr}(3,\bigwedge^2 U)) & \xrightarrow{i_-^*} & H^0(\mathcal{O}_{\mathbb{P}(U^\vee)}(2)) \\ \| & & \| & & \| \\ \mathrm{Sym}^2 U^\vee & \longleftarrow & \bigwedge^3(\bigwedge^2 U^\vee) & \longrightarrow & \mathrm{Sym}^2 U \end{array} \tag{1.5.6}$$

It follows that we have isomorphisms of $SL(U)$-modules

$$A_+(U) \cong \mathrm{Sym}^2 U, \qquad A_-(U) \cong \mathrm{Sym}^2 U^\vee. \tag{1.5.7}$$

In particular we get that $2\dim A_\pm(U) = \dim \bigwedge^3(\bigwedge^2 U)$ and hence $A_\pm(U)$ is Lagrangian. Lastly we will prove (1.5.4). First we claim that

$$\Theta_{A_\pm(U)} = \mathrm{im}(i_\pm). \tag{1.5.8}$$

Suppose that $W \in \Theta_{A_+(U)}$; let's prove that $\mathbb{P}(W) \subset \mathcal{Q}$. If $\mathbb{P}(W) \not\subset \mathcal{Q}$ then

$$\dim(\mathbb{P}(W) \cap \mathcal{Q}) = 1. \tag{1.5.9}$$

On the other hand $\mathbb{P}(W)$ is incident to every plane parametrized by a point of $\mathrm{im}(i_+)$ (see **Remark 0.1**); by (1.5.9) we get that if $p \in \mathbb{P}(W) \cap \mathcal{Q}$ then

$$\dim\{\Lambda \in \mathrm{im}(i_+) \mid p \in \Lambda\} \geq 2. \tag{1.5.10}$$

That is absurd: if $p \in \mathcal{Q}$ the set of planes $\Lambda \subset \mathcal{Q}$ containing p is a line. This proves that $\mathbb{P}(W) \subset \mathcal{Q}$. Thus $W \in (\mathrm{im}(i_+) \cup \mathrm{im}(i_-))$. Since $\mathbb{P}(W)$ is incident to every plane parametrized by $\mathrm{im}(i_+)$ we get that $W \in \mathrm{im}(i_+)$. This proves (1.5.8) for $A_+(U)$ - the proof for $A_-(U)$ is the same mutatis mutandis. By (1.5.8) we get that $\Theta_{A_\pm(U)}$ is not invasive and hence $Y_{A_\pm(U)} \neq \mathbb{P}(\bigwedge^2 U)$ by **Claim 1.11**. On the other hand $Y_{A_\pm(U)}$ is $SL(U)$-invariant i.e. invariant for the action of $SO(\bigwedge^2 U, q)$ where the zero-locus of the quadratic form q is $\mathcal{Q}$. It follows that $Y_{A_\pm(U)}$ is a multiple of $\mathcal{Q}$; since $\deg Y_{A_\pm(U)} = 6$ we get (1.5.4). □

We notice that $A_+(U) \neq A_-(U)$ because if $(\Lambda_+, \Lambda_-) \in \mathrm{im}(i_+) \times \mathrm{im}(i_-)$ is generic then $\Lambda_+ \cap \Lambda_- = \emptyset$; it follows that the irreducible decomposition of the $SL(U)$-module $\bigwedge^3(\bigwedge^2 U)$ is given by

$$\bigwedge^3(\bigwedge^2 U) = A_+(U) \oplus A_-(U) \cong \mathrm{Sym}^2 U \oplus \mathrm{Sym}^2 U^\vee. \tag{1.5.11}$$

Remark 1.15. Referring to (1.5.11): the $SL(U)$-action on $\mathrm{Sym}^2 U^\vee$ is the inverse of the contragradient action.

Double discriminant cubic Write $V = \mathrm{Sym}^2 L$ where L is a complex vector-space of dimension 3. Let $(\mathrm{Sym}^2 L)_i \subset \mathrm{Sym}^2 L$ be the subset of symmetric tensors of rank at most i. Then $\mathcal{V} := \mathbb{P}((\mathrm{Sym}^2 L)_1)$ is a Veronese surface. The chordal variety of $\mathcal{V}$ is the discriminant cubic

$$\mathbb{P}((\mathrm{Sym}^2 L)_2) = \mathrm{chord}(\mathcal{V}) = \bigcup_{\ell \text{ chord of } \mathcal{V}} \ell. \tag{1.5.12}$$

(Tangents to $\mathcal{V}$ are included among chords of $\mathcal{V}$.) We have embeddings

$$\begin{array}{ccccccc} \mathbb{P}(L) & \overset{k}{\hookrightarrow} & \mathrm{Gr}(3, \mathrm{Sym}^2 L) & \qquad & \mathbb{P}(L^\vee) & \overset{h}{\hookrightarrow} & \mathrm{Gr}(3, \mathrm{Sym}^2 L) \\ [l_0] & \mapsto & \{l_0 \cdot l \mid l \in L\} & & [f_0] & \mapsto & \{q \mid f_0 \in kerq\}. \end{array} \tag{1.5.13}$$

Let $\mathcal{L}$ be the Plücker(ample) line-bundle on $\mathrm{Gr}(3, \bigwedge^2 U)$; one checks easily that

$$k^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(L)}(3), \qquad h^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(L^\vee)}(3). \tag{1.5.14}$$

Let $A_k(L), A_h(L) \subset \bigwedge^3(\mathrm{Sym}^2 L)$ be the subspaces defined by

$$A_k(L) = \langle\langle \mathrm{im}(k) \rangle\rangle, \qquad A_h(L) = \langle\langle \mathrm{im}(h) \rangle\rangle. \tag{1.5.15}$$

Arguing as in the proof of **Claim 1.14** one proves the following result.

Claim 1.16. *Keep notation as above. Then $A_k(L), A_h(L) \in \mathbb{LG}(\bigwedge^3(\mathrm{Sym}^2 L))$ and*

$$Y_{A_k(L)} = Y_{A_h(L)} = 2\,\mathrm{chord}(\mathcal{V})\,. \tag{1.5.16}$$

Moreover

$$\Theta_{A_k(L)} = \mathrm{im}(k), \qquad \Theta_{A_h(L)} = \mathrm{im}(h). \tag{1.5.17}$$

A remark: Equation (1.5.14) gives the irreducible decomposition of the $SL(L)$-module $\bigwedge^3(\mathrm{Sym}^2 L)$

$$\bigwedge^3(\mathrm{Sym}^2 L) \cong \mathrm{Sym}^3 L \oplus \mathrm{Sym}^3 L^\vee. \tag{1.5.18}$$

Union of six independent hyperplanes The following example was worked out together with C. Procesi. Let $\{v_0, \ldots, v_5\}$ be a basis of V and $\{X_0, \ldots, X_5\}$ be the dual basis of $V^\vee$. Our special A_{III} (III refers to Type III degeneration) is spanned by decomposable vectors $v_{i_1} \wedge v_{i_2} \wedge v_{i_3}$ for a suitable collection of $\{i_1, i_2, i_3\}$'s. To simplify notation we denote $v_{i_1} \wedge v_{i_2} \wedge v_{i_3}$ by the characteristic function of $\{i_1, i_2, i_3\}$, i.e. the string composed of three 0's and three 1's which has 1 at place j (we start counting from 0) if and only if $j \in \{i_1, i_2, i_3\}$. With the above notation A_{III} is given by

$$\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{array} \tag{1.5.19}$$

Notice that A_{III} is fixed by the maximal torus T of $SL(V)$ diagonalized by $\{v_0,\ldots,v_5\}$ and that T acts trivially on $\bigwedge^{10} A$. In particular $Y_{A_{III}} = V(cX_0 \cdot X_1 \cdots X_5)$ for a constant c. An explicit computation shows that $[v_0+v_1+\cdots+v_5] \notin Y_A$ and hence $Y_{A_{III}} = V(X_0 \cdot X_1 \cdots X_5)$. The reader can check that the following holds.

Claim 1.17. *Let T be a maximal torus of $SL(V)$. Suppose that $A \in \mathbb{LG}(\bigwedge^3 V)$ is fixed by T and that T acts trivially on $\bigwedge^{10} A$. Then A is $SL(V)$-equivalent to A_{III}.*

1.6 Dual of an EPW-sextic

The volume-form (0.0.1) defines a volume-form $\mathrm{vol}^\vee\colon \bigwedge^6 V^\vee \xrightarrow{\sim} \mathbb{C}$. Let $(,)_{V^\vee}$ be the symplectic form on $\bigwedge^3 V^\vee$ defined by $(\alpha,\beta)_{V^\vee} := \mathrm{vol}^\vee(\alpha\wedge\beta)$. We let $\mathbb{LG}(\bigwedge^3 V^\vee)$ be the symplectic Grassmannian relative to $(,)_{V^\vee}$. Let

$$\begin{array}{ccc} \bigwedge^3 V & \overset{\delta_V}{\xrightarrow{\sim}} & \bigwedge^3 V^\vee \\ \alpha & \mapsto & (\beta \mapsto (\alpha,\beta)_V) \end{array} \tag{1.6.1}$$

be the isomorphism induced by $(,)_V$. As is easily checked

$$(\alpha,\beta)_V = (\delta_V(\alpha),\delta_V(\beta))_{V^\vee}, \quad \alpha,\beta \in \bigwedge^3 V. \tag{1.6.2}$$

Thus we have an isomorphism

$$\begin{array}{ccc} \mathbb{LG}(\bigwedge^3 V) & \overset{\delta_V}{\xrightarrow{\sim}} & \mathbb{LG}(\bigwedge^3 V^\vee) \\ A & \mapsto & \delta_V(A) = \mathrm{Ann}\, A. \end{array} \tag{1.6.3}$$

The geometric meaning of $Y_{\delta_V(A)}$ is the following [15, 16]: if A is generic in $\mathbb{LG}(\bigwedge^3 V)$ then

$$Y_{\delta_V(A)} = Y_A^\vee, \tag{1.6.4}$$

where $Y_A^\vee$ is the dual of Y_A. We list below the images by δ_V of certain subspaces of $\bigwedge^3 V$.

$$\begin{aligned} \delta_V(F_{v_0}) &= \bigwedge^3 \mathrm{Ann}(v_0), && [v_0] \in \mathbb{P}(V) && (1.6.5)\\ \delta_V(\bigwedge^3 W) &= \bigwedge^3 \mathrm{Ann}(W), && W \in \mathrm{Gr}(3,V), && (1.6.6)\\ \delta_V(A_+(U) &= A_-(U^\vee), && \dim U = 4 && (1.6.7)\\ \delta_V(A_k(U) &= A_h(U^\vee). && \dim U = 3 && (1.6.8) \end{aligned}$$

(See **Subsection 1.5** for the notation in the last two lines.) Let $A \in \mathbb{LG}(\bigwedge^3 V)$. We notice that (1.6.5) gives the following description of $Y_{\delta_V(A)}$: given $E \in \mathrm{Gr}(5,V)$ then

$$E \in Y_{\delta_V(A)} \text{ if and only if } (\bigwedge^3 E) \cap A \neq \{0\}. \tag{1.6.9}$$

Let us examine the action of δ_V on $\Sigma(V)$. We have a canonical identification $\mathrm{Gr}(3,V) = \mathrm{Gr}(3,V^\vee)$ and (1.6.6) gives that

$$\Theta_A = \Theta_{\delta_V(A)}. \tag{1.6.10}$$

In particular

$$\delta_V(\Sigma(V)) = \Sigma(V^\vee), \quad \delta_V(\Sigma_\infty(V)) = \Sigma_\infty(V^\vee). \tag{1.6.11}$$

1.7 EPW-sextics as discriminant loci

Let $A \in \mathbb{LG}(\bigwedge^3 V)$. In **Subsection 1.2** we described Y_A locally around $[v_0] \in \mathbb{P}(V)$ as the discriminant locus of a symmetric map of vector-bundles. Recall that in order to do so we need to choose $B \in \mathbb{LG}(\bigwedge^3 V)$ transversal to A and to F_{v_0}. In this subsection we will write out explicitly the equation of Y_A by choosing B as follows. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$ and $\mathcal{D}$ be the direct-sum decomposition

$$V = [v_0] \oplus V_0. \tag{1.7.1}$$

Assume moreover that

$$(\bigwedge^3 V_0) \cap A = \{0\} \tag{1.7.2}$$

(notice that the above condition is equivalent to $\delta_V(A) \notin \mathbb{N}(V^\vee)$); then $B = \bigwedge^3 V_0$ is indeed a Lagrangian subspace transversal to A and to F_{v_0}. The open subset $\mathcal{U}_{\bigwedge^3 V_0} \subset \mathbb{P}(V)$ is readily seen to be equal to $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$; we identify it with V_0 via Map (1.3.1). Given $v \in V_0$ we have the map

$$\tau_A^{\bigwedge^3 V_0}([v_0 + v]) \colon A \to \bigwedge^3 V_0 \tag{1.7.3}$$

given by (1.2.3). We have the isomorphism

$$\begin{array}{ccc} \bigwedge^2 V_0 & \xrightarrow{\sim} & \bigwedge^3 V_0^\vee \\ \alpha & \mapsto & (\xi \mapsto \mathrm{vol}(v_0 \wedge \alpha \wedge \xi)) \,. \end{array} \tag{1.7.4}$$

On the other hand (1.1.1) gives an isomorphism $\bigwedge^3 V_0^\vee \cong A$; composing with (1.7.4) we get an isomorphism

$$\nu_A^{\mathcal{D}} \colon \bigwedge^2 V_0 \xrightarrow{\sim} A \,. \tag{1.7.5}$$

(The superscript $\mathcal{D}$ refers to Decomposition (1.7.1).) Let

$$\widetilde{q}_A^{\mathcal{D}}(v) := \tau_A^{\bigwedge^3 V_0}([v_0 + v]) \circ \nu_A^{\mathcal{D}} \colon \bigwedge^2 V_0 \to \bigwedge^3 V_0. \tag{1.7.6}$$

and $q_A^{\mathcal{D}}(v)$ be the associated quadratic form; thus

$$q_A^{\mathcal{D}}(v) \in \mathrm{Sym}^2 \bigwedge^2 V_0^\vee \,. \tag{1.7.7}$$

Identify $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ with V_0 via Map (1.3.1); by definition we have

$$Y_A \cap V_0 = V(\det(q_A^{\mathcal{D}}(v))) \tag{1.7.8}$$

where $v \in V_0$. We will write write out explicitly the maps introduced above. Given $\alpha \in \bigwedge^2 V_0$ we have $v_0 \wedge \alpha \in F_{v_0}$ and there exists a unique decomposition

$$v_0 \wedge \alpha = \beta + \gamma, \qquad \beta \in A, \quad \gamma \in \bigwedge^3 V_0 \,. \tag{1.7.9}$$

Wedging both sides of the above equation with elements of $\bigwedge^3 V_0$ we get that

$$\nu_A^{\mathcal{D}}(\alpha) = -\beta \,. \tag{1.7.10}$$

Moreover we get that

$$\widetilde{q}_A^{\mathcal{D}}(v)(\alpha) = -\gamma - v \wedge \alpha = \widetilde{q}_A^{\mathcal{D}}(0)(\alpha) + \widetilde{q}_v(\alpha) \tag{1.7.11}$$

where $\widetilde{q}_v$ is the symmetric map associated to the Plücker quadratic form (1.3.12). For future reference we record the following description of $\widetilde{q}_A^{\mathcal{D}}(0)$:

$$\widetilde{q}_A^{\mathcal{D}}(0)(\alpha) = \gamma \iff (v_0 \wedge \alpha + \gamma) \in A. \tag{1.7.12}$$

By (1.7.8) we have the following local description of Y_A.

Proposition 1.18. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and $[v_0] \in \mathbb{P}(V)$. Suppose that there exists a codimension-*1 *subspace $V_0 \subset V$ such that* (1.7.1)-(1.7.2) *hold. Identify $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ with V_0 via Map* (1.3.1). *Then*

$$Y_A \cap V_0 = V(\det(q_A^{\mathcal{D}}(0) + q_v)) \,. \tag{1.7.13}$$

2 EPW-sextics in Σ

2.1 Dimension computations for Σ

Let $\widetilde{\Sigma} \subset \mathrm{Gr}(3,V) \times \mathbb{LG}(\bigwedge^3 V)$ be defined by

$$\widetilde{\Sigma} := \{(W,A) \mid \bigwedge^3 W \subset A\}. \tag{2.1.1}$$

Given $d \geq 0$ we let $\widetilde{\Sigma}[d] \subset \mathrm{Gr}(3,V) \times \mathbb{LG}(\bigwedge^3 V)$ be defined by

$$\widetilde{\Sigma}[d] := \{(W,A) \in \widetilde{\Sigma} \mid \dim(A \cap S_W) \geq (d+1)\}. \tag{2.1.2}$$

Thus $\widetilde{\Sigma} := \widetilde{\Sigma}[0]$. Let $\pi\colon \mathrm{Gr}(3,V) \times \mathbb{LG}(\bigwedge^3 V) \to \mathbb{LG}(\bigwedge^3 V)$ be projection; we let

$$\Sigma[d] := \pi(\widetilde{\Sigma}[d]). \tag{2.1.3}$$

Thus $\Sigma := \Sigma[0]$. For a geometric interpretation of $\Sigma[1]$ see **Proposition 2.2**. Let

$$\Sigma_+ := \{A \in \Sigma \mid |\Theta_A| > 1\}. \tag{2.1.4}$$

Proposition 2.1. *Keep notation as above.*

(1) Let $0 \leq d \leq 9$. Then $\Sigma[d]$ is closed irreducible and

$$\mathrm{cod}(\Sigma[d], \mathbb{LG}(\bigwedge^3 V)) = (d^2+d+2)/2. \tag{2.1.5}$$

In particular Σ is closed irreducible of codimension 1.

(2) Σ_+ is an irreducible constructible subset of $\mathbb{LG}(\bigwedge^3 V)$ of codimension 2, moreover if $A \in \Sigma_+$ is generic then $|\Theta_A| = 2$ and $A \notin \Sigma[1]$.

Proof. Since $\widetilde{\Sigma}[d]$ is closed and π is projective we get that $\Sigma[d]$ is closed by (2.1.3). Let $\rho_d\colon \widetilde{\Sigma}[d] \to \mathrm{Gr}(3,V)$ be (the restriction of) projection. Let $W_0 \in \mathrm{Gr}(3,V)$; one describes $\rho_d^{-1}(W_0)$ as follows. Given $W \in \mathrm{Gr}(3,V)$ let

$$\mathcal{E}_W := (\bigwedge^3 W)^{\perp} / \bigwedge^3 W \tag{2.1.6}$$

where orthogonality is with respect to $(,)_V$. The symplectic form $(,)_V$ induces a symplectic form on $\mathcal{E}_W$ and hence we have an associated symplectic grassmannain $\mathbb{LG}(\mathcal{E}_W)$; notice that $T_W \in \mathbb{LG}(\mathcal{E}_W)$ where T_W is defined by (1.4.9). Let $T_W[d] := \{B \in \mathbb{LG}(\mathcal{E}_W) \mid \dim(B \cap T_W) \geq d\}$. We have an isomorphism

$$\begin{array}{ccc} \rho_d^{-1}(W_0) & \overset{\sim}{\longrightarrow} & T_{W_0}[d] \\ (W_0, A) & \mapsto & A/\bigwedge^3 W_0 \end{array} \tag{2.1.7}$$

We claim that

$$\mathrm{cod}(T_W[d], \mathbb{LG}(\mathcal{E}_W)) = d(d+1)/2. \tag{2.1.8}$$

In fact let $B_0 \in T_W[d]$. Let $C \in \mathbb{LG}(\mathcal{E}_W)$ be transversal both to B_0 and T_W. Let $U_C \subset \mathbb{LG}(\mathcal{E}_W)$ be the open set given by (1.1.3) (beware: the rôles of B and C have been exchanged). Then $B_0 \in U_C$ and we have an isomorphism $U_C \cong \mathrm{Sym}^2 C$ given by (1.1.4). Via this isomorphism $T_W[d]$ is identified with the subset $(\mathrm{Sym}^2 C)_d \subset \mathrm{Sym}^2 C$ of symmetric tensors of corank at least d; by **Proposition 3.1** we have $\mathrm{cod}((\mathrm{Sym}^2 C)_d, \mathrm{Sym}^2 C) = d(d+1)/2$ and hence we get that (2.1.8) holds. By **Proposition 1.1** and (2.1.8) we get that

$$\begin{aligned} \dim \widetilde{\Sigma}[d] &= \dim T_W[d] + \dim \mathrm{Gr}(3,V) = \dim \mathbb{LG}(\mathcal{E}_W) - \frac{d^2+d}{2} + \dim \mathrm{Gr}(3,V) = \\ &= \dim \mathbb{LG}(\bigwedge^3 V) - 10 - \frac{d^2+d}{2} + 9 = \dim \mathbb{LG}(\bigwedge^3 V) - \frac{d^2+d+2}{2}. \end{aligned} \tag{2.1.9}$$

One checks also that $T_W[d]$ is irreducible (recall that $0 \le d \le 9$, in particular $T_W[d]$ is not empty); it follows that $\widetilde{\Sigma}[d]$ is irreducible and hence $\Sigma[d]$ is irreducible as well. Summing up: we have proved that $\Sigma[d]$ is irreducible and that its codimension is at least the right-hand side of (2.1.5). Moreover in order to finish the proof of Item (1) it suffices to show that the restriction of projection $\widetilde{\Sigma}[d] \to \Sigma[d]$ is birational. Let $U \subset \mathrm{Gr}(3,V) \times \mathrm{Gr}(3,V)$ be

$$U := \{(W_1, W_2) \mid 0 < \dim(W_1 \cap W_2) < 3\} \tag{2.1.10}$$

and $\widetilde{\Sigma}_+[d] \subset U \times \mathbb{LG}(\bigwedge^3 V)$ be

$$\widetilde{\Sigma}_+[d] := \{(W_1, W_2, A) \mid (W_i, A) \in \widetilde{\Sigma}[d] \quad i = 1, 2\}. \tag{2.1.11}$$

We laim that in order to prove that $\widetilde{\Sigma}[d] \to \Sigma[d]$ is birational it suffices to show that

$$\dim \widetilde{\Sigma}_+[d] < \dim \widetilde{\Sigma}[d]. \tag{2.1.12}$$

In fact let's grant (2.1.12) and let's suppose that $\widetilde{\Sigma}[d] \to \Sigma[d]$ is not birational. By **Remark 0.1** we get that there is an open dense subset of $\widetilde{\Sigma}[d]$ which is in the image of the forgetful map

$$\begin{array}{ccc} \widetilde{\Sigma}_+[d] & \xrightarrow{f_d} & \widetilde{\Sigma}[d] \\ (W_1, W_2, A) & \mapsto & (W_1, A) \end{array} \tag{2.1.13}$$

and that contradicts (2.1.12). Let's proceed to prove (2.1.12). Let $\eta_d \colon \widetilde{\Sigma}_+[d] \to U$ be the (restriction of) projection. Let $(W_1, W_2) \in U$; the fiber $\eta_d^{-1}(W_1, W_2)$ is described as follows. Let

$$\mathcal{E}_{W_1,W_2} := (\bigwedge^3 W_1 \oplus \bigwedge^3 W_2)^{\perp} / (\bigwedge^3 W_1 \oplus \bigwedge^3 W_2). \tag{2.1.14}$$

We have an inclusion

$$\begin{array}{ccc} \eta_d^{-1}(W_1, W_2) & \overset{\theta_d}{\hookrightarrow} & \mathbb{LG}(\mathcal{E}_{W_1,W_2}) \\ (W_1, W_2, A) & \mapsto & A/(\bigwedge^3 W_1 \oplus \bigwedge^3 W_2) \end{array} \tag{2.1.15}$$

The above map is bijective if and only if $d = 0$. In order to describe the image for $d > 0$ we let $\mathcal{T}^{W_i}_{W_1,W_2} \subset \mathcal{E}_{W_1,W_2}$ be defined as

$$\mathcal{T}^{W_i}_{W_1,W_2} := \mathrm{im}((S_{W_i} \cap (\bigwedge^3 W_1 \oplus \bigwedge^3 W_2)^{\perp}) \longrightarrow \mathcal{E}_{W_1,W_2}).$$

Let $\mathcal{T}_{W_1,W_2}[d] := \{B \in \mathbb{LG}(\mathcal{E}_{W_1,W_2}) \mid \dim(B \cap \mathcal{T}^{W_i}_{W_1,W_2}) \ge d \quad i = 1, 2\}$. Clearly $\mathrm{im}\,\theta_d = \mathcal{T}_{W_1,W_2}[d]$. As is easily checked $\mathcal{T}^{W_i}_{W_1,W_2} \in \mathbb{LG}(\mathcal{E}_{W_1,W_2})$; thus arguing as in the proof of (2.1.8) we get that

$$\mathrm{cod}(\mathrm{im}\,\theta_d, \mathbb{LG}(\mathcal{E}_{W_1,W_2})) = \mathrm{cod}(\mathcal{T}_{W_1,W_2}[d], \mathbb{LG}(\mathcal{E}_{W_1,W_2})) \ge d(d+1)/2.$$

We have $\dim U = 17$; by **Proposition 1.1** we get that

$$\begin{aligned} \dim \widetilde{\Sigma}_+[d] &\le \dim \mathbb{LG}(\mathcal{E}_{W_1,W_2})) - \frac{d(d+1)}{2} + \dim U = \\ &= \dim \mathbb{LG}(\bigwedge^3 V) - 19 - \frac{d(d+1)}{2} + 17 = \dim \mathbb{LG}(\bigwedge^3 V) - \frac{d^2+d+2}{2} - 1. \end{aligned} \tag{2.1.16}$$

Thus (2.1.12) follows from the above inequality and (2.1.9). This finishes the proof of Item (1). Let's prove Item (2). We have $\Sigma_+ = \pi \circ f_0(\widetilde{\Sigma}_+[0])$. Since $\widetilde{\Sigma}_+[0]$ is constructible we get that Σ_+ is constructible and by (2.1.16) we get that $\mathrm{cod}(\Sigma_+, \mathbb{LG}(\bigwedge^3 V)) \ge 2$. A dimension count similar to those performed above gives that $\mathrm{cod}(\Sigma_+, \mathbb{LG}(\bigwedge^3 V)) = 2$ and that the generic $A \in \Sigma_+$ has the properties stated in Item (2). $\square$

2.2 First order computations

In proving **Proposition 2.1** we have shown that $\widetilde{\Sigma}$ is a locally trivial fibration over $\mathrm{Gr}(3,V)$ with fiber $\mathbb{LG}(\mathcal{E}_W)$ over W; thus $\widetilde{\Sigma}$ is smooth. Let $\rho := \pi|_{\widetilde{\Sigma}}\colon \widetilde{\Sigma} \to \Sigma$. The differential of ρ at $(W,A) \in \widetilde{\Sigma}$ is expressed as follows. Of course $T_{(W,A)}\widetilde{\Sigma} \subset T_W \mathrm{Gr}(3,V) \oplus T_A \mathbb{LG}(\bigwedge^3 V)$. Choosing a volume form on W i.e. a generator α of $\bigwedge^3 W$ we have isomorphisms

$$T_W \mathrm{Gr}(3,V) = \mathrm{Hom}(W, V/W) = \bigwedge^2 W \otimes (V/W) = S_W / \bigwedge^3 W = T_W.$$

Let $\varphi\colon \bigwedge^3 V \to \bigwedge^3 V/A$ be the quotient map. Given $\tau \in S_W/\bigwedge^3 W$ we let $\widetilde{\varphi}(\tau) := \varphi(\widetilde{\tau})$ where $\widetilde{\tau} \in S_W$ is an element representing the equivalence class τ; this makes sense because $\bigwedge^3 W \subset A$. On the other hand the tangent space $T_A\mathbb{LG}$ is given by **Proposition 1.1**: we have a canonical identification

$$T_A \mathbb{LG}(\bigwedge^3 V) \cong \{\theta\colon A \to A^\vee \mid \theta^t = \theta\}.$$

Given the above identifications one has

$$T_{(W,A)}\widetilde{\Sigma} = \{(\tau,\theta) \in T_W \times \mathrm{Sym}^2 A^\vee \mid \theta(\alpha) = \widetilde{\varphi}(\tau)\}. \tag{2.2.1}$$

(The proof consists of a straightforward computation.) In particular we get that

$$\ker d\rho(W,A) = A \cap T_W. \tag{2.2.2}$$

Thus we have the following interpretation of $\widetilde{\Sigma}[1]$:

$$\widetilde{\Sigma}[1] = \{(W,A) \in \widetilde{\Sigma} \mid d\pi(W,A) \text{ is not injective}\} \tag{2.2.3}$$

Proposition 2.2. *The irreducible decomposition of* $\mathrm{sing}\,\Sigma$ *is equal to*

$$\mathrm{sing}\,\Sigma = \overline{\Sigma}_+ \cup \Sigma[1]\,. \tag{2.2.4}$$

Both irreducible components are of codimension 1 *in* Σ.

Proof. By **Proposition 2.1** we know that $\Sigma[1]$ is irreducible of dimension 53 and that $\overline{\Sigma}_+ \neq \Sigma[1]$. By Item (2) of **Proposition 2.1** the map $\rho\colon \widetilde{\Sigma} \longrightarrow \Sigma$ is birational. Thus 2.2.3 gives that $\Sigma[1] \subset \mathrm{sing}\,\Sigma$. Since $\Sigma_+ \not\subset \Sigma_\infty$ we also have $\overline{\Sigma}_+ \subset \mathrm{sing}\,\Sigma$. Lastly Σ is smooth away from $\overline{\Sigma}_+ \cup \Sigma[1]$ by (2.2.2). ☐

Proposition 2.3. *Let* $A \in \mathbb{LG}(\bigwedge^3 V)$ *and* $W_1, W_2 \in \mathrm{Gr}(3,V)$. *Suppose that for* $i = 1,2$

$$A \cap S_{W_i} = \bigwedge^3 W_i. \tag{2.2.5}$$

Then $\mathrm{im}(d\rho(W_1,A)$ *and* $\mathrm{im}(d\rho(W_2,A)$ *are codimension-*1 *transverse subspaces of* $T_A\mathbb{LG}(\bigwedge^3 V)$.

Proof. Both $\mathrm{im}(d\rho(W_1,A)$ and $\mathrm{im}(d\rho(W_2,A)$ are codimension-1 subspaces of $T_A\mathbb{LG}(\bigwedge^3 V)$ by (2.2.2); it remains to prove that they are distinct. Let α_i be a generator of $\bigwedge^3 W_i$; Formula (2.2.1) gives that

$$\mathrm{im}\, d\rho(W_i,A) = \{\theta \in \mathrm{Sym}^2 A^\vee \mid \theta(\alpha_i) \in \varphi(S_{W_i})\}.$$

It follows that $\mathrm{im}(d\rho(W_1,A) = \mathrm{im}(d\rho(W_2,A)$ if and only if $\varphi(S_{W_i}) = \ker(\alpha_{3-i})$. Since S_{W_i} is lagrangian that is possible only if $\alpha_{3-i} \in S_{W_i}$; that is absurd by (2.2.5). ☐

The following result is a straightforward consequence of **Proposition 2.1** and **Proposition 2.3**.

Corollary 2.4. *If* $A \in \Sigma_+$ *is generic then* Σ *has normal crossings at* A *with exactly two sheets.*

2.3 One-dimensional components of Θ_A

In this subsection we will classifiy couples (A,Θ) where $A\in\Sigma$ and Θ is a 1-dimensional irreducible component of Θ_A - of course our point of departure is Morin's **Theorem 1.12**.

Definition 2.5. Let $\Theta\subset \mathrm{Gr}(3,V)$ be closed: it is *isotropic* if $\langle\langle\Theta\rangle\rangle\subset\bigwedge^3 V$ is an isotropic subspace or equivalently $W_1\cap W_2\neq\{0\}$ for all $W_1,W_2\in\Theta$, it is *isolated isotropic* if in addition it is a union of irreducible components of $\langle\Theta\rangle\cap\mathrm{Gr}(3,V)$.

The following is an immediate consequence of **Remark 0.1**.

Remark 2.6. Suppose that $A\in\mathbb{LG}(\bigwedge^3 V)$ and that Θ is an irreducible component of Θ_A. Then Θ is isolated isotropic.

Before stating our main result on isolated isotropic curves in $\mathrm{Gr}(3,V)$ we go through some elementary remarks on projective families of planes in $\mathbb{P}(V)$. Let $\Theta\subset\mathrm{Gr}(3,V)$ be an irreducible closed subset. We let $\mathcal{E}_\Theta\to\Theta$ be the restriction of the tautological rank-3 vector-bundle on $\mathrm{Gr}(3,V)$ - thus the dual $\mathcal{E}_\Theta^\vee$ is globally generated. We let $R_\Theta\subset\mathbb{P}(V)$ be the variety swept out by the 2-dimensional projective spaces parametrized by Θ, i.e.

$$R_\Theta:=\bigcup_{W\in\Theta}\mathbb{P}(W). \tag{2.3.1}$$

Let

$$f_\Theta\colon\mathbb{P}(\mathcal{E}_\Theta)\to R_\Theta \tag{2.3.2}$$

be the tautological surjective map. We may factor f_Θ as follows. The surjective evaluation map $H^0(\mathcal{E}_\Theta^\vee)\otimes\mathcal{O}_\Theta\to\mathcal{E}_\Theta^\vee$ defines a map

$$h_\Theta\colon\mathbb{P}(\mathcal{E}_\Theta)\to\mathbb{P}(H^0(\mathcal{E}_\Theta^\vee)^\vee). \tag{2.3.3}$$

Let $T_\Theta:=\mathrm{im}(h_\Theta)$; there is a natural map

$$g_\Theta\colon T_\Theta\longrightarrow R_\Theta,\qquad f_\Theta=g_\Theta\circ h_\Theta. \tag{2.3.4}$$

The pull-back by g_Θ of the hyperplane line-bundle on $\mathbb{P}(V)$ is isomorphic to the hyperplane line-bundle on T_Θ; thus g_Θ is either an isomorphism or it may be identified with a projection of T_Θ. Now assume that $\dim\Theta=1$. Then $\dim R_\Theta=3$ and hence f_Θ is of finite degree; one has

$$\deg\Theta=c_1(\mathcal{E}_\Theta^\vee)=\deg f_\Theta\cdot\deg R_\Theta. \tag{2.3.5}$$

Proposition 2.7. *Suppose that $\Theta\subset\mathrm{Gr}(3,V)$ is an isolated isotropic irreducible curve. Then $\deg f_\Theta=\deg g_\Theta=1$ and $\deg\Theta=\deg R_\Theta=\deg T_\Theta$.*

Proof. Suppose that $\deg f_\Theta>1$; we will reach a contradiction. Since $\deg f_\Theta>1$ the generic point of R_Θ (i.e. the generic point on the generic plane parametrized by Θ) is contained in two distinct planes parametrized by Θ. Since $\dim\Theta=1$ it follows that two distinct planes parametrized by Θ meet in a line. Hence either all planes in Θ contain a fixed line or else they are all contained in a fixed 3-dimensional projective space $\mathbb{P}(U)$. If the former holds then $\deg f_\Theta=1$, that is a contradiction. Thus we may assume that $\Theta\subset\mathrm{Gr}(3,U)$ where $U\subset V$ is of dimension 3 and that Θ is not a line, in particular $2\le\dim\langle\Theta\rangle$. On the other hand

$$\langle\Theta\rangle\subset\mathbb{P}(\bigwedge^3 U)=\mathrm{Gr}(3,U)\subset\mathrm{Gr}(3,V). \tag{2.3.6}$$

Hence the linear space $\langle\Theta\rangle$ is contained in $\mathrm{Gr}(3,V)$; since it has dimension at least 2 we get that Θ is not an irreducible component of $\langle\Theta\rangle\cap\mathrm{Gr}(3,V)$, that contradicts **Definition 2.5**. This proves that $\deg f_\Theta=1$; since $f_\Theta=g_\Theta\circ h_\Theta$ it follows that $\deg g_\Theta=1$ as well. The equality $\deg\Theta=\deg R_\Theta=\deg T_\Theta$ follows from $\deg f_\Theta=\deg g_\Theta=1$ together with Equation (2.3.5). □

Table (2) lists families of curves in $\mathrm{Gr}(3,V)$ and assigns a Type to each family - notice that there are calligraphic and boldface Types, see **Remark 2.23** for an explanation of the difference. A few comments on Table (2). In the last four rows of Table (2) we refer to (1.5.1) and (1.5.13). We notice that Table (2) is preserved by duality. More precisely let

$$\begin{array}{ccc} \mathrm{Gr}(3,V) & \overset{\widetilde{\delta}_V}{\longrightarrow} & \mathrm{Gr}(3,V^\vee) \\ W & \mapsto & \mathrm{Ann}(W). \end{array} \tag{2.3.7}$$

If Θ belongs to one of the familes in Table (2) then $\widetilde{\delta}_V(\Theta)$ belongs to one of the familes as well - for this to make sense we choose an isomorphism $\mathbb{P}(V)\cong\mathbb{P}(V^\vee)$. Notation in Table (2) makes it clear what is the Type of $\widetilde{\delta}_V(\Theta)$ given the Type of Θ. All the asserted dualities are clear except possibly for the Types $\mathcal{E}_2,\mathcal{E}_2^\vee$. Let Θ_1 be of Type $\mathcal{E}_2$ and $\Theta_2:=\widetilde{\delta}_V(\Theta_1)$. We must check that $\mathcal{E}_{\Theta_2}\cong\mathcal{O}_{\mathbb{P}^1}(-1)^3$. We have a natural exact sequence

$$0\to\mathcal{E}_{\Theta_1}\overset{\alpha}{\longrightarrow}\mathcal{O}_{\mathbb{P}^1}^6\longrightarrow\mathcal{E}_{\Theta_2}^\vee\to 0.$$

(The map $\widetilde{\delta}_V$ identifies Θ_1 and Θ_2, moreover they are both isomorphic to $\mathbb{P}^1$.) On the other hand $\operatorname{coker}\alpha\cong\mathcal{O}_{\mathbb{P}^1}^3(1)$ and the result follows.

Claim 2.8. *Let X be one of the Types appearing in Table (2). Let Θ be of Type X: then Θ is isolated isotropic. Suppose in addition that Θ is generic of Type X (this makes sense because the relevant parameter spaces are irreducible): then the scheme-theoretic intersection $\langle\Theta\rangle\cap\mathrm{Gr}(3,V)$ is a smooth irreducible curve, set-theoretically equal to Θ.*

Proof. Let Θ be of Type X. Then Θ is contained in one of the maximal irreducible families of pairwise incident planes in $\mathbb{P}(V)$ listed in **Subsection 1.4** and hence it is isotropic. This is trivially verified except possibly for Θ of Type $\mathbf{Q}$: in that case notice that the projection from p of $\langle h_\Theta(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2))\rangle$ (a plane, call it $\mathbb{P}(U)$) intersects the projection of an arbitrary plane in T_Θ along a line and hence $\Theta\subset I_U$. If Θ is a line (Type $\mathcal{F}_1$) then the remaining statements of the claim are trivially true. From now on we may assume that X is one of the remaining Types. Let Θ be generic of Type X: we must prove that we have equality of sets

$$\langle\Theta\rangle\cap\mathrm{Gr}(3,V)=\Theta \tag{2.3.8}$$

and that the scheme-theoretic intersection on the left is reduced (it is clear that Θ is smooth). Suppose first that Θ is generic of Type $\mathcal{A}$, $\mathcal{A}^\vee$ or $\mathcal{C}_2$. Then (2.3.8) holds tautologically. Moreover let $[v_0]\in\mathbb{P}(V)$, $E\in\mathrm{Gr}(5,V)$ and $U\in\mathrm{Gr}(3,V)$; a straighforward computation with tangent spaces shows that the scheme-theoretic intersections $\mathbb{P}(F_{v_0})\cap\mathrm{Gr}(3,V)$, $\mathbb{P}(\bigwedge^3 E)\cap\mathrm{Gr}(3,V)$ and $\mathbb{P}(S_U)\cap\mathrm{Gr}(3,V)$ are smooth at every point except for the last case and the point U itself. From this we get that the scheme-theoretic intersection in (2.3.8) is reduced. Now suppose that Θ belongs to one of the remaining Types; then it belongs to one of $\mathrm{im}(i_+)$, $\mathrm{im}(k)$, $\mathrm{im}(h)$. More precisely one of the following holds:

(1) There exist an isomorphism $V\cong\bigwedge^2 U$ where $\dim U=4$ and a curve $C\subset\mathbb{P}(U)$ such that $\Theta=i_+(C)$. Moreover C is cut out scheme-theoretically by quadrics.

(2) There exist an isomorphism $V\cong\mathrm{Sym}^2 L$ where $\dim L=3$ and a curve $C\subset\mathbb{P}(L)$ such that $\Theta=k(C)$. Moreover C is cut out scheme-theoretically by cubics.

(3) There exist an isomorphism $V\cong\mathrm{Sym}^2 L^\vee$ where $\dim L=3$ and a curve $C\subset\mathbb{P}(L^\vee)$ such that $\Theta=h(C)$. Moreover C is cut out scheme-theoretically by cubics.

In fact one of the items above holds by definition if Θ is of Type $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$ or $\mathbf{T}^\vee$. If Θ is a conic then Item (1) holds with C a line. If Θ is of Type $\mathcal{E}_2$ then Item (2) holds with C a line, if Θ is of Type $\mathcal{E}_2^\vee$ then Item (3) holds with C a line. Lastly if Θ is of Type $\mathbf{Q}$ then Item (1) holds with C a conic. Now suppose that Item (1) holds: since i_+ is defined by the complete linear system of quadrics it follows that (2.3.8) holds. Similarly if Item (2) or (3) holds then we get (2.3.8) because k and h

Table 2: Types of one-dimensional components of Θ_A

Θ	$\deg\Theta$	$\mathcal{E}_\Theta$	R_Θ isomorphic to	$\dim\langle\Theta\rangle$	Type
line	1	$\mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$	T_Θ	1	$\mathcal{F}_1$
conic	2	$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^2$	T_Θ	2	$\mathcal{D}$
rat'l normal cubic	3	$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$	T_Θ	3	$\mathcal{E}_2$
rat'l normal cubic	3	$\mathcal{O}_{\mathbb{P}^1}(-1)^3$	proj. of T_Θ from a point	3	$\mathcal{E}_2^\vee$
rational normal quartic	4	$\mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$	proj. of T_Θ from $p \in \langle h_\Theta(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2))\rangle$	4	$\mathbf{Q}$
in J_{v_0}, $[v_0] \in \mathbb{P}(V)$	5			4	$\mathcal{A}$
in $\mathrm{Gr}(3, E)$, $E \in \mathrm{Gr}(5, V)$	5			4	$\mathcal{A}^\vee$
in $(I_U \setminus \{U\})$, $U \in \mathrm{Gr}(3, V)$	6			5	$\mathcal{C}_2$
$i_+(C)$, $C \subset \mathbb{P}(U)$ a rat'l normal cubic	6			6	$\mathbf{R}$
$i_+(C)$, $C \subset \mathbb{P}(U)$ a c.i. of 2 quadrics	8			7	$\mathbf{S}$
$k(C)$, $C \subset \mathbb{P}(L)$ a cubic	9			8	$\mathbf{T}$
$h(C)$, $C \subset \mathbb{P}(L^\vee)$ a cubic	9			8	$\mathbf{T}^\vee$

are defined by the complete linear system of cubics. It remains to show that the scheme-theoretic intersection (2.3.8) is reduced. Refering to Items (1), (2) and (3) above the reduced curve C is the scheme-theoretic intersection of quadrics if Item (1) holds and the scheme intersection of cubics if Item (2) or (3) holds: it follows that it suffices to show that the intersections $\mathbb{P}(A_+(U)) \cap \mathrm{Gr}(3, V)$, $\mathbb{P}(A_k(L)) \cap \mathrm{Gr}(3, V)$ and $\mathbb{P}(A_h(L^\vee)) \cap \mathrm{Gr}(3, V)$ are reduced. Consider the first intersection. Let $W = i_+([u_0]) \in \mathbb{P}(A_+(U)) \cap \mathrm{Gr}(3, V)$ and suppose that the intersection is not reduced at W. Acting with the stabilizer of $[u_0]$ in $PGL(U)$ we get that the tangent space at W of the scheme theoretic intersection $\mathbb{P}(A_+(U)) \cap \mathrm{Gr}(3, V)$ is all of the tangent space of $\mathbb{P}(A_+(U))$ at W. Since $\mathbb{P}(S_W)$ is the projective tangent space to $\mathrm{Gr}(3, V)$ (embedded in $\mathbb{P}(\bigwedge^3 V)$) we get that $\mathbb{P}(A_+(U)) \subset \mathbb{P}(S_W)$ and hence they are equal because they have the same dimension. This holds for each $W \in \mathrm{im}(i_+)$: that is absurd because if $W_1 \neq W_2$ then $S_{W_1} \neq S_{W_2}$. A similar argument shows that the scheme-theoretic intersections $\mathbb{P}(A_k(L)) \cap \mathrm{Gr}(3, V)$ and $\mathbb{P}(A_h(L^\vee)) \cap \mathrm{Gr}(3, V)$ are reduced. □

Below is one the main results of the present subsection.

Theorem 2.9. *An isolated isotropic irreducible curve $\Theta \subset \mathrm{Gr}(3, V)$ belongs to one of the Types of Table* (2).

Before proving **Theorem 2.9** we will give a series of preliminary results.

Lemma 2.10. *Let $\Theta \subset \mathrm{Gr}(3, V)$ be isolated isotropic. If we have an inclusion of vector-bundles $\mathcal{O}_\Theta^2 \subset \mathcal{E}_\Theta$ then Θ is a linear space.*

Proof. By hypothesis there exists $U \in \mathrm{Gr}(2, V)$ such that

$$\Theta \subset \{W \in \mathrm{Gr}(3, V) \mid U \subset W\} \cong \mathbb{P}(V/U). \tag{2.3.9}$$

It follows that $\langle\Theta\rangle \subset \mathrm{Gr}(3,V)$. By **Definition 2.5** we get that $\Theta = \langle\Theta\rangle$. □

Proposition 2.11. *Let $\Theta \subset \mathrm{Gr}(3,V)$ be isolated isotropic and suppose that it is a conic; then it is of Type $\mathcal{D}$.*

Proof. By hypothesis $\Theta \cong \mathbb{P}^1$ and hence $\mathcal{E}_\Theta \cong \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_3)$ where $0 \le a_i$ and $\sum_i a_i = 2$. By **Lemma 2.10** we get that

$$\mathcal{E}_\Theta \cong \mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}. \tag{2.3.10}$$

It follows that R_Θ is isomorphic either to T_Θ (a 3-dimensional quadric of rank 4) or to a projection of such a quadric. If the latter holds then $\deg f_\Theta = 2$ contradicting **Proposition 2.7**. □

Proposition 2.12. *Let $\Theta \subset \mathrm{Gr}(3,V)$ be isolated isotropic and suppose that it is a cubic rational normal curve; then it is either of Type $\mathcal{E}_2$ or of Type $\mathcal{E}_2^\vee$.*

Proof. By hypothesis $\Theta \cong \mathbb{P}^1$. Arguing as in the proof of **Proposition 2.11** and invoking **Lemma 2.10** we get that

$$\mathcal{E}_\Theta \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2), \text{ or} \\ \mathcal{O}_{\mathbb{P}^1}(-1)^3. \end{cases} \tag{2.3.11}$$

Suppose that the first isomorphism holds. Then R_Θ is isomorphic either to T_Θ or to a projection of T_Θ. If the former holds then Θ is of Type $\mathcal{E}_2$. Suppose that the latter holds; we will reach a contradiction. In fact the trivial addend in (2.3.11) gives that $\Theta \subset J_{v_0}$ for some $[v_0] \in \mathbb{P}(V)$. Thus we have an embedding

$$\iota\colon \Theta \hookrightarrow \mathrm{Gr}(2, V/[v_0]). \tag{2.3.12}$$

We have $\dim\langle T_\Theta\rangle = 5$ and by assumption R_Θ is isomorphic to a projection of T_Θ; thus $\dim\langle R_\Theta\rangle = 4$ i.e. there exists $U \subset (V/[v_0])$ with $\dim U = 4$ such that

$$\iota(\Theta) \subset \mathrm{Gr}(2,U) \subset \mathbb{P}(\bigwedge^2 U). \tag{2.3.13}$$

By hypothesis $\Theta \cong \iota(\Theta)$ is a cubic rational normal curve and hence $\dim\langle\iota(\Theta)\rangle = 3$; since $\mathrm{Gr}(2,U)$ is a quadric hypersurface in $\mathbb{P}(\bigwedge^2 U)$ we get that $\langle\iota(\Theta)\rangle \cap \mathrm{Gr}(2,U)$ has pure dimension 2. It follows that $\langle\Theta\rangle \cap \mathrm{Gr}(3,V)$ has pure dimension 2 as well. Thus Θ is not a component of $\langle\Theta\rangle \cap \mathrm{Gr}(3,V)$ contradicting **Definition 2.5**. This proves that if the first isomorphism of (2.3.11) holds then Θ is of Type $\mathcal{E}_2$. Now suppose that the second isomorphism of (2.3.11) holds. Then R_Θ is not isomorphic to T_Θ (which is $\mathbb{P}^1 \times \mathbb{P}^2$ embedded by the Segre map) because any two distinct planes in T_Θ are disjoint. Hence R_Θ is isomorphic to a projection of T_Θ. Since $\dim\langle T_\Theta\rangle = 5$ the center of projection is either a point or a line. If the latter holds then $\deg g_\Theta = 3$, that contradicts **Proposition 2.7**. Thus R_Θ is isomorphic to a projection of T_Θ with center of projection a point i.e. Θ is of type $\mathcal{E}_2^\vee$. □

Proposition 2.13. *Let $\Theta \subset \mathrm{Gr}(3,V)$ be isolated isotropic and suppose that it is a quartic rational normal curve. Suppose in addition that R_Θ is not a cone and is non-degenerate (i.e. $\dim\langle R_\Theta\rangle = 5$). Then Θ is of Type* **Q**.

Proof. By hypothesis $\Theta \cong \mathbb{P}^1$ and hence $\mathcal{E}_\Theta \cong \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_3)$ where $0 \le a_i$ and $\sum_i a_i = 4$. Since R_Θ is not a cone $a_i > 0$ for $i = 1,2,3$. Thus

$$\mathcal{E}_\Theta \cong \mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2). \tag{2.3.14}$$

Since R_Θ is non-degenerate it is isomorphic to the projection of T_Θ from a point p. In order to prove the proposition it remains to show that

$$p \in \langle h_\Theta(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2))\rangle. \tag{2.3.15}$$

One verifies easily that if $p \notin \langle h_\Theta(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2))\rangle$ then the projections of two generic planes in T_Θ are disjoint - thus (2.3.15) holds. □

Remark 2.14. Suppose that Θ is of Type **Q**. The proof of **Proposition 2.13** provides the following description of R_Θ. There exists disjoint planes $P_1, P_2 \subset \mathbb{P}(V)$ and embeddings $\iota_1\colon \Theta \hookrightarrow P_1^\vee$, $\iota_2\colon \Theta \hookrightarrow P_2$ with $\mathrm{im}(\iota_1)$ and $\mathrm{im}(\iota_2)$ a conic such that

$$R_\Theta = \bigcup_{x\in\Theta} \langle \iota_1(x), \iota_2(x)\rangle. \tag{2.3.16}$$

(Of course $\langle \iota_1(x), \iota_2(x)\rangle$ is the plane corresponding to x.)

The following result shows that there are (at least) three interesting constructions of a Θ of Type **R**.

Claim 2.15. *Let U be a 4-dimensional vector space and $V := \bigwedge^2 U$. Let i_+ be as in (1.5.1). Let $\Theta \subset \mathrm{Gr}(3, V)$ be given by $\Theta = i_+(C)$ where $C \subset \mathbb{P}(U)$ is a rational normal cubic. There exist an isomorphism $V = \mathrm{Sym}^2 L$ where $\dim L = 3$ and conics $C \subset \mathbb{P}(L)$, $C' \subset \mathbb{P}(L^\vee)$ such that $\Theta = k(C) = h(C')$. (Here k, h are given by (1.5.13).)*

Proof. We have a map

$$\begin{array}{ccc} C^{(2)} & \xrightarrow{f} & \mathrm{Gr}(2, U) \subset \mathbb{P}(V) \\ P+Q & \mapsto & \langle P+Q\rangle \end{array} \tag{2.3.17}$$

Of course $C^{(2)} \cong \mathbb{P}^2$ and one may identify f (up to projectivities) with the natural map $\mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^2}(2)|^\vee$; thus $\mathrm{im}(f)$ is the Veronese surface $\mathcal{V}$. Let $P \in C$; the plane $i_+(p)$ intersects $\mathcal{V}$ in a conic. Thus $i_+(R) \subset C(\mathcal{V})$. It follows that there exist an isomorphism $V = \mathrm{Sym}^2 L$ where $\dim L = 3$ and a conic $C' \subset \mathbb{P}(L^\vee)$ such that $\Theta = h(C')$. The analogous result with h replaced by k follows by duality - see (1.6.7)-(1.6.8). $\square$

Lemma 2.16. *Let C be an irreducible projective curve with $\omega_C \cong \mathcal{O}_C$ (an elliptic curve, possibly singular). Let $\mathcal{F}$ be a rank-2 vector-bundle on C such that*

(a) $\deg \mathcal{F} = 4$,

(b) $\mathcal{F}$ is globally generated,

(c) there is no splitting $\mathcal{F} \cong \mathcal{O}_C \oplus \mathcal{L}$.

Then $h^0(\mathcal{F}) = 4$.

Proof. By Riemann-Roch we have $\chi(\mathcal{F}) = 4$ hence it suffices to prove that $h^1(\mathcal{F}) = 0$. Suppose that $h^1(\mathcal{F}) > 0$; we will reach a contradiction. By Serre duality we get that $h^0(\mathcal{F}^\vee) > 0$ and hence there exists a non-zero $\phi\colon \mathcal{F} \to \mathcal{O}_C$. Since $\mathcal{F}$ is globally generated $\mathrm{im}(\phi)$ is globally generated; it follows that $\mathrm{im}(\phi) = \mathcal{O}_C$. Let $\mathcal{K} := \ker(\phi)$; thus we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{2.3.18}$$

By Serre duality $h^1(\mathcal{K}) = h^0(\mathcal{K}^\vee)$ and furthermore $h^0(\mathcal{K}^\vee) = 0$ because $\mathcal{K}$ is an invertible sheaf of degree 4; thus $h^1(\mathcal{K}) = 0$ and hence (2.3.18) splits. That contradicts Item (c). $\square$

Proposition 2.17. *Let $\Theta \subset \mathrm{Gr}(3, V)$ be an isolated isotropic irreducible curve. Then Items (α),(β) below cannot both hold:*

(α) $\Theta \subset J_{v_0}$ for some $[v_0] \in \mathbb{P}(V)$,

(β) $\dim\langle\Theta\rangle = 3$ and Θ is the intersection of two quadric surfaces in $\langle\Theta\rangle$.

Proof. Let $V_0 \in \mathrm{Gr}(5, V)$ be transversal to $[v_0]$. Let $C := \overline{\rho}_{v_0}(\Theta)$; then $\overline{\rho}_{v_0}$ gives an isomorphism $g\colon \Theta \xrightarrow{\sim} C$. Let $\mathcal{F}^\vee$ be the restriction to C of the tautological rank-2 vector-bundle on $\mathrm{Gr}(2, V_0)$. We have an isomorphism $\mathcal{E}_\Theta \cong \mathcal{O}_\Theta \oplus g^*\mathcal{F}^\vee$. It follows from **Lemma 2.10** that there is no splitting $\mathcal{F} \cong \mathcal{O}_C \oplus \mathcal{L}$. Furthermore $\deg \mathcal{F} = 4$ by Item (β) and of course $\mathcal{F}$ is globally generated. By Item (β) we have $\omega_C \cong \mathcal{O}_C$. Thus **Lemma 2.16** gives that $h^0(\mathcal{F}) = 4$. It follows that there exists $U \in \mathrm{Gr}(4, V_0)$ such that $C \subset \mathrm{Gr}(2, U)$. Since $\mathrm{Gr}(2, U)$ is a a smooth quadric in $\mathbb{P}(\bigwedge^2 U)$ we get that $\langle C\rangle \cap \mathrm{Gr}(2, U)$ has pure dimension 2. It follows that $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$ has pure dimension 2 contradicting **Definition 2.5**. $\square$

Proof of **Theorem 2.9**. Suppose that $\dim\langle\Theta\rangle = 1$; then Θ is of Type $\mathcal{F}_1$. Thus we may assume that $2 \le \dim\langle\Theta\rangle$. By definition Θ is an irreducible component of $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$. Since $\mathrm{Gr}(3, V)$ is cut out by Plücker quadrics (in $\mathbb{P}(\bigwedge^3 V)$) it follows that:

(i) If $\dim\langle\Theta\rangle = 2$ then Θ is a smooth conic.

(ii) If $\dim\langle\Theta\rangle = 3$ then Θ is either a cubic rational normal curve or the complete intersection of 2 quadrics.

If (i) holds then Θ is of Type $\mathcal{D}$ by **Proposition 2.11**. Suppose that (ii) holds: if Θ is a cubic rational normal curve then it is of Type $\mathcal{E}_2$ or of Type $\mathcal{E}_2^\vee$ by **Proposition 2.12**. Thus from now on we may assume that

(I) $\dim\langle\Theta\rangle = 3$ and Θ is the complete intersection of two quadrics, or

(II) $4 \le \dim\langle\Theta\rangle$.

By Morin one of (a) - (e) of **Theorem 1.12** holds. We will perform a case-by-case analysis.

(a): $\Theta \subset F_\pm(\mathcal{Q})$. Let U be a 4-dimensional complex vector-space and identify V with $\bigwedge^2 U$ so that $\mathcal{Q}$ gets identified with $\mathrm{Gr}(2, U)$. We may assume that $\Theta \subset F_+(\mathcal{Q})$; thus $\Theta := i_+(C)$ for an irreducible curve $C \subset \mathbb{P}(U)$. By definition Θ is an irreducible component of $\langle\Theta\rangle \cap F_+(\mathcal{Q})$. Since i_+ is given by the complete linear system of quadrics in $\mathbb{P}(U)$ we get that C is a component of a complete intersection of quadrics. Thus C is a rational curve of degree at most 3 or the complete intersection of two quadrics. By (I)-(II) above we have $3 \le \dim\langle i_+(C)\rangle$ and hence C is not a line. Suppose that C is a conic; as is easily verified R_Θ is not a cone, and by duality we get that it is non-degenerate as well. Since Θ is a degree-4 rational normal curve we get that it is of Type **Q** by **Proposition 2.13**. If C is a cubic rational normal curve then Θ is of Type **R**. Lastly if C is the complete intersection of two quadrics then Θ is of Type **S**.

(b): $\Theta \subset C(\mathcal{V})$ or $\Theta \subset T(\mathcal{V})$. Let L be a 3-dimensional complex vector-space and identify V with $\mathrm{Sym}^2 L$ so that $\mathcal{V}$ gets identified with $\mathbb{P}((\mathrm{Sym}^2 L)_1)$. We discuss the case $\Theta \subset C(\mathcal{V})$, the other case will follow by duality. There exists a curve $C \subset \mathbb{P}(L^\vee)$ such that $\Theta = h(C)$. We recall that h is identified (up to projectivities) with $|\mathcal{O}_{\mathbb{P}(L^\vee)}(3)|$. Arguing as in Case (a) we get that $\deg C \le 3$. If C is a line then Θ is a cubic rational normal curve and hence it is of Type $\mathcal{E}_2$ or of Type $\mathcal{E}_2^\vee$ by **Proposition 2.12** - in fact of Type $\mathcal{E}_2$. If C is a smooth conic then Θ is of Type **R** by **Claim 2.15**. If C is a cubic then Θ if of Type $\mathbf{T}^\vee$.

(c): $\Theta \subset J_{v_0}$. By assumption one of Items (I), (II) above holds. In fact Item (I) cannot hold by **Proposition 2.17**. Hence Item (II) holds. We claim that

$$\dim\langle\Theta\rangle = 4. \tag{2.3.19}$$

In fact if $4 < \dim\langle\Theta\rangle$ then every irreducible component of $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$ has dimension at least 2, that contradicts **Definition 2.5**. By (2.3.19) we get that Θ is of Type $\mathcal{A}$.

(d): $\Theta \subset \mathrm{Gr}(3, E)$ where $E \in \mathrm{Gr}(5, V)$. Then $\widetilde{\delta}_V(\Theta) \subset J_\phi$ where $\langle\phi\rangle = \mathrm{Ann}(E)$; by the previous case we get that Θ is of Type $\mathcal{A}^\vee$.

(e): $\Theta \subset I_U$ where $U \in \mathrm{Gr}(3, V)$. Suppose first that $U \in \Theta$. Then Θ is an irreducible component of $\langle\Theta\rangle \cap I_U$, and since the latter is a cone with vertex U it follows that Θ is a cone with vertex U. Thus Θ is a line and hence it is of Type $\mathcal{F}_1$. From now on we may assume that $U \notin \Theta$; since Θ is an irreducible component of $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$ it follows that

$$U \notin \langle\Theta\rangle. \tag{2.3.20}$$

Let $\overline{\rho}_U$ be the (rational) map of (1.4.11); by (2.3.20) the restriction of $\overline{\rho}_U$ to Θ is a regular isomorphism onto

$$C := \overline{\rho}_U(\Theta) \subset \mathbb{P}(U) \times \mathbb{P}(V/U) \subset \mathbb{P}((U) \otimes \mathbb{P}(V/U)). \tag{2.3.21}$$

By assumption one of (I), (II) above holds. We claim that (I) cannot hold. In fact let $f\colon C \to \mathbb{P}(U)$ and $g\colon C \to \mathbb{P}(V/U)$ be the two projections. One easily checks that neither f nor g is constant. We have

$$\deg f^*\mathcal{O}_{\mathbb{P}(U)}(1) + \deg f^*\mathcal{O}_{\mathbb{P}(V/U)}(1) = 4. \tag{2.3.22}$$

Since C has arithmetic genus 1 we get that

$$2 = \deg f^*\mathcal{O}_{\mathbb{P}(U)}(1) = \deg f^*\mathcal{O}_{\mathbb{P}(V/U)}(1) \tag{2.3.23}$$

and moreover $\mathrm{im}(f), \mathrm{im}(g)$ are lines, say $\mathrm{im}(f) = \mathbb{P}(U_2)$ and $\mathrm{im}(g) = \mathbb{P}(W_2)$. It follows that

$$\langle C\rangle \supset (\mathbb{P}(U_2) \times \mathbb{P}(W_2)) \tag{2.3.24}$$

where $\langle C\rangle$ is the span of C in $\mathbb{P}((U) \otimes \mathbb{P}(V/U))$. Thus C is not an irreducible component of $\langle C\rangle \cap (\mathbb{P}(U) \times \mathbb{P}(V/U))$ and hence Θ is not an irreducible component of $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$; that contradicts **Definition 2.5**. Hence Item (II) above holds i.e.

$$4 \le \dim\langle\Theta\rangle. \tag{2.3.25}$$

On the other hand

$$\dim\langle\Theta\rangle \le 5. \tag{2.3.26}$$

In fact suppose that $5 < \dim\langle\Theta\rangle$. By (2.3.20) we get that $5 < \dim\langle C\rangle$ and hence every irreducible component of $\langle C\rangle \cap (\mathbb{P}(U) \times \mathbb{P}(V/U)$ has dimension at least 2. It follows that Θ is not an irreducible component of $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$, contradiction. This proves (2.3.26). Assume that $\dim\langle\Theta\rangle = 4$. Then

$$\dim\langle C\rangle = 4 \tag{2.3.27}$$

by (2.3.20). Since $\deg(\mathbb{P}(U) \times \mathbb{P}(V/U)) = 6$ it follows that $4 \le \deg C \le 5$. If $\deg C = 4$ then C is a quartic rational normal curve; as is easily verified R_Θ is not a cone and is non-degenerate thus Θ is of Type $\mathbf{Q}$ by **Proposition 2.13**. If $\deg C = 5$ we will reach a contradiction. First let's prove that C is of arithmetic genus 1. In fact the intersection of $\mathbb{P}(U) \times \mathbb{P}(V/U)$ with a generic 5-dimensional projective space containing C is a curve of degree 6 and arithmetic genus 1 and the component different from C is a line meeting C in a single point (and not tangent to C); it follows that $p_a(C) = 1$. Arguing as for C satisfying Item (I) we get that

$$C \subset (\mathbb{P}(U_2) \times \mathbb{P}(V/U)), \qquad U_2 \in \mathrm{Gr}(2, U) \tag{2.3.28}$$

or

$$C \subset (\mathbb{P}(U) \times \mathbb{P}(W_2)), \qquad W_2 \in \mathrm{Gr}(2, V/U). \tag{2.3.29}$$

Suppose that (2.3.28) holds; since $\dim\langle(\mathbb{P}(U_2) \times \mathbb{P}(V/U))\rangle = 5$ we get that

$$2 \le \dim\langle C\rangle \cap (\mathbb{P}(U_2) \times \mathbb{P}(V/U)) \tag{2.3.30}$$

by (2.3.27). It follows that Θ is not an irreducible component of $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$, contradiction. If (2.3.29) holds we argue similarly and again we get a contradiction. Thus we are left with the case $\dim\langle\Theta\rangle = 5$; then Θ is of Type $\mathcal{C}_2$. $\square$

Definition 2.18. Let X be one of the types listed in Table (2): we let $\mathbb{B}_X \subset \mathbb{LG}(\bigwedge^3 V)$ be the closure of the set of A such that Θ_A contains an irreducible component of Type X.

Proposition 2.19. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that there exists a 1-dimensional irreducible component of Θ_A. There exists a Type X in Table (2) such that $A \in \mathbb{B}_X$.*

Proof. Let $\Theta \subset \Theta_A$ be a 1-dimensional irreducible component. By **Remark 2.6** we know that $\Theta \subset \mathrm{Gr}(3, V)$ is an isolated isotropic irreducible curve and hence the proposition follows from **Theorem 2.9**. $\square$

Proposition 2.20. *Let X be one of the Types appearing in Table (2) and Θ be generic of Type X (this makes sense because the relevant parameter spaces are irreducible). There exists $A \in \mathbb{LG}(\bigwedge^3 V)$ such that $\Theta_A = \Theta$ and moreover Θ_A is generically reduced. In particular $\mathbb{B}_X \neq \emptyset$.*

Proof. By **Claim 2.8** we may assume that

$$\langle\Theta\rangle \cap \mathrm{Gr}(3, V) = \Theta \tag{2.3.31}$$

and the scheme-theoretic intersection is reduced. Let $L := \langle\langle\Theta\rangle\rangle$ and $\ell := \dim L$. We have a bijection

$$\begin{array}{ccc} \{A \in \mathbb{LG}(\bigwedge^3 V) \mid A \supset \langle\langle\Theta\rangle\rangle\} & \xrightarrow{\sim} & \mathbb{LG}(L^{\perp}/L) \\ A & \mapsto & A/L \end{array} \tag{2.3.32}$$

We will show that there exists $B \in \mathbb{LG}(L^{\perp}/L)$ such that the corresponding $A \in \mathbb{LG}(\bigwedge^3 V)$ has the stated properties. Let

$$Z := \{W_0 \in \mathrm{Gr}(3, V) \mid W_0 \cap W \neq \{0\}\}.$$

If $A \in \mathbb{LG}(\bigwedge^3 V)$ contains $\langle\langle\Theta\rangle\rangle$ and $W_0 \in \Theta_A$ then $W_0 \in Z$. On the other hand let $W_0 \in)Z \setminus \Theta)$. By (2.3.31) we have $\bigwedge^3 W_0 \notin L$ and hence its class is non-zero in $\mathbb{P}(L^{\perp}/L)$; it follows that

$$\mathrm{cod}(\{B \in \mathbb{LG}(L^{\perp}/L) \mid B \supset \bigwedge^3 W_0\}, \mathbb{LG}(L^{\perp}/L)) = 10 - \ell. \tag{2.3.33}$$

(We abuse notation: $\bigwedge^3 W_0$ is actually the image of $\bigwedge^3 W_0$ in $L^{\perp}/L$.) Let $\varphi\colon (Z \setminus \Theta) \to \mathbb{P}(L^{\perp}/L)$ which maps W_0 to the image of $\bigwedge^3 W_0$ in $L^{\perp}/L$. By (2.3.33) it suffices to prove that

$$\dim \varphi(Z \setminus \Theta) < (10 - \ell). \tag{2.3.34}$$

Let $Z = Z_1 \cup \ldots \cup Z_r$ be the decomposition into irreducible components. We must prove that

$$\dim \varphi(Z_i \setminus \Theta) < (10 - \ell) \tag{2.3.35}$$

for all i. For each Type X and for Θ generic of that Type we will describe the irreducible components Z_i and we will check that (2.3.35) holds.

$\boxed{\Theta \textit{ of Type } \mathcal{F}_1}$ Notice that R_Θ is a 3-dimensional linear space. Let $M \subset \mathbb{P}(V)$ be the intersection of all $\mathbb{P}(W)$ for $W \in \Theta$, thus M is a line. The decomposition into irreducibles of Z is the following:

$$Z = \{W_0 \mid \dim \mathbb{P}(W_0) \cap R_\Theta \geq 1\} \cup \{W_0 \mid \mathbb{P}(W_0) \cap M \neq \emptyset\}.$$

Let Z_i be an irreducible component; then $\dim Z_i = 7$ and hence $\dim \varphi(Z_i) \leq 7$. Since $\ell = 2$ we get that (2.3.35) holds.

$\boxed{\Theta \textit{ of Type } \mathcal{D}}$ In this case R_Θ is a 3-dimensional quadric with one singular point $[v_0]$. The variety $F_1(R_\Theta)$ parametrizing lines on R_Θ has two connected components (see **Claim 2.29** for a detailed description),call them $F_1(R_\Theta)^{\pm}$. The lines contained in the planes parametrized by Θ belong to one of the two components, say $F_1(R_\Theta)^{+}$. The decomposition into irreducibles of Z is the following:

$$Z = J_{v_0} \cup \mathrm{Gr}(2, R_\Theta) \cup \{W_0 \mid \mathbb{P}(W_0) \cap R_\Theta \text{ contains a line } J \in F_1(R_\Theta)^{-}\}.$$

Let Z_i be an irreducible component; then $\dim Z_i = 6$ and hence $\dim \varphi(Z_i) \leq 6$. Since $\ell = 3$ we get that (2.3.35) holds.

$\boxed{\Theta \textit{ of Type } \mathcal{E}_2 \textit{ or } \mathcal{E}_2^{\vee}}$ Suppose first that Θ is of Type $\mathcal{E}_2$. Notice that R_Θ is a cone over a smooth normal rational cubic scroll in a 4-dimensional linear space. Let $[v_0] \in R_\Theta$ be the vertex and $\mathbb{P}(U) \subset R_\Theta$ be the plane joining $[v_0]$ to the (-1)-line of the cubic scroll. The decomposition into irreducibles of Z is $Z = Z_1 \cup I_U \cup J_{v_0}$ where the generic plane in Z_1 intersects R_Θ in a smooth conic. The generic plane in Z_1 corresponds to an injection $\mathcal{O}_\Theta(-2) \hookrightarrow \mathcal{E}_\Theta$ and hence $\dim Z_1 = \dim \mathbb{P}(H^0(\mathcal{E}_\Theta(2))) = 5$. We also have $\dim J_{v_0} = 5$; since $\ell = 4$ we get that (2.3.35) holds for Z_1 and for I_U. Lastly consider J_{v_0} which has dimension 6. We notice that $F_{v_0} \supset L$ and that $\varphi(J_{v_0} \setminus \Theta) \subset \mathbb{P}(F_{v_0}/L)$; since $\dim \mathbb{P}(F_{v_0}/L) = 5$ we see that (2.3.35) holds in this case as well. If Θ is of Type $\mathcal{E}_2^\vee$ the result follows by duality from the case when Θ is of Type $\mathcal{E}_2$.

$\boxed{\Theta \textit{ of Type } \mathbf{Q}}$ We may choose an isomorphism $V \cong \bigwedge^2 U$ where $\dim U = 4$ and a conic $C \subset \mathbb{P}(U)$ such that $\Theta = i_+(C)$. Recall that we have an immersion $i_- \colon \mathbb{P}(U^\vee) \hookrightarrow \mathrm{Gr}(3, V)$. Every plane parametrized by Θ intersects the plane $i_-(\langle C \rangle)$. Let $i_-(\langle C \rangle) = \mathbb{P}(H)$. The decomposition into irreducibles of Z is $Z = Z_1 \cup I_H \cup \Theta_{A_+(U)}$ where the generic plane in Z_1 is spanned by the images via i_+ of the lines in one of the two rulings of a smooth quadric $Q \in |\mathcal{I}_C(2)|$. We have $\dim Z_1 = 4$; since $\ell = 5$ we get that (2.3.35) holds for Z_1. On the other hand $S_H \supset L$, $A_+(U) \supset L$ and we have that $\varphi(I_H \setminus \Theta) \subset \mathbb{P}(S_H/L)$, $\varphi(\Theta_{A_+(U)} \setminus \Theta) \subset \mathbb{P}(A_+(U)/L)$; since $4 = \dim \mathbb{P}(S_H/L) = \dim \mathbb{P}(A_+(U)/L)$ we see that (2.3.35) holds in this case as well.

$\boxed{\Theta \textit{ of Type } \mathcal{A} \textit{ or } \mathcal{A}^\vee}$ By duality it suffices to consider Θ of Type $\mathcal{A}$. Then R_Θ is a cone with vertex $[v_0]$ over a degree-5 surface $\overline{R}_\Theta$ ruled over an elliptic curve and spanning a 4-dimensional linear subspace. Clearly $J_{v_0} \subset Z$, we analyze $\Lambda \in (Z \setminus J_{v_0})$. Let $\mathcal{O}_\Theta \subset \mathcal{E}_\Theta$ be the sub line-bundle corresponding to the vertex $[v_0]$ and $\overline{\mathcal{E}}_\Theta := \mathcal{E}_\Theta/\mathcal{O}_\Theta$; by genericity of Θ we may assume that $\overline{\mathcal{E}}_\Theta$ is a stable (rank-2) vector-bundle. The projection of Λ from $[v_0]$ is a plane $\overline{\Lambda} \subset \langle \overline{R}_\Theta \rangle$ intersecting each line of the ruling of $\overline{R}_\Theta$. Thus $\overline{\Lambda}$ defines a section of $\mathbb{P}(\overline{\mathcal{E}}_\Theta)$ i.e. a sub line-bundle

$$\mathcal{L} \hookrightarrow \overline{\mathcal{E}}_\Theta. \tag{2.3.36}$$

By stability of $\overline{\mathcal{E}}_\Theta$ we have $\deg \mathcal{L} \le -3$. On the other hand $-4 \le \deg \mathcal{L}$ because $\deg \overline{R}_\Theta = 5$. We claim that we cannot have $\deg \mathcal{L} = -4$. In fact the transpose of (2.3.36) is a surjection $\overline{\mathcal{E}}_\Theta^\vee \to \deg \mathcal{L}^\vee$; since the map on global sections is surjective it follows that the section corresponding to (2.3.36) is a degree-4 elliptic curve spanning a 3-dimensional space, **not** a plane. Thus $\deg \mathcal{L} = -3$; conversely to each (2.3.36) with $\deg \mathcal{L} = -3$ there corresponds a plane $\overline{\Lambda} \subset \langle \overline{R}_\Theta \rangle$ intersecting each line of the ruling of $\overline{R}_\Theta$. Given such a plane all the planes in $\langle [v_0] \cup \overline{\Lambda} \rangle$ belong to Z. Let $Z_1 \subset \mathrm{Gr}(3, V)$ be the closure of the locus of such planes; we have proved that the irreducible decomposition of Z is $Z = Z_1 \cup J_{v_0}$. By stability of $\overline{\mathcal{E}}_\Theta$ we have $\dim \mathrm{Hom}(\mathcal{L}, \overline{\mathcal{E}}_\Theta) = 1$ for every line-bundle $\mathcal{L}$ of degree -3; it follows that $\dim Z_1 = 4$. Since $\ell = 5$ we get that (2.3.35) holds for $(Z_1 \setminus \Theta)$. The argument for the component J_{v_0} is the same as that given for Θ of Type $\mathcal{E}_2$.

$\boxed{\Theta \textit{ of Type } \mathcal{C}_2}$ Choose an isomorphism $V \cong \bigwedge^2 \mathbb{C}^4$. Let $H \subset \mathbb{C}^4$ be a subspace of codimension 1 and $C \subset \mathbb{P}(H)$ a cubic curve. Let $\Theta := i_+(C)$. Let $U \in \mathrm{Gr}(3, V)$ be the subspace such that $i_-(H) = \mathbb{P}(U)$. One checks easily that $\Theta \in (I_U \setminus \{U\})$ and that $\dim \langle \Theta \rangle = 5$. The proof of **Theorem 2.9** gives that Θ is of Type $\mathcal{C}_2$ - see Case (e). We will prove that there exists $A \in \mathbb{LG}(\bigwedge^3 V)$ such that $\Theta_A = \Theta$; it will follow that the same is true for a generic Θ of Type $\mathcal{C}_2$ (actually the generic Θ of Type $\mathcal{C}_2$ is equal to $i_+(C)$ as above). The decomposition into irreducibles of Z is $Z = I_U \cup A_+(\mathbb{C}^4)$; it follows that (2.3.34) holds.

$\boxed{\Theta \textit{ of Type } \mathbf{R} \textit{ or } \mathbf{S}}$ The decomposition into irreducibles of Z is $Z = Z_1 \cup \Theta_{A_+(U)}$ where the generic plane in Z_1 is spanned by the images via i_+ of the lines in one of the two rulings of a smooth quadric $Q \in |\mathcal{I}_C(2)|$. Suppose that Θ is of Type $\mathbf{R}$. Then $\dim Z_1 = 2$; since $\ell = 5$ we get that (2.3.35) holds for Z_1. One deals with the component $\Theta_{A_+(U)}$ as usual; that proves that (2.3.34) holds for Θ of Type $\mathbf{R}$. Suppose now that Θ is of Type $\mathbf{S}$. Then $\dim Z_1 = 1$; since $\ell = 8$ we get that (2.3.35) holds for Z_1 and (2.3.34) follows.

$\boxed{\Theta \textit{ of Type } \mathbf{T} \textit{ or } \mathbf{T}^\vee}$ By duality it suffices to consider Θ of Type $\mathbf{T}^\vee$. We have a rational map $\varphi \colon \mathrm{Gr}(3, V) \dashrightarrow \mathbb{P}(Sym^3 L^\vee)$ which assigns to a 3-dimensional subspace $\Lambda \subset \mathrm{Sym}^2 L$ the set of singular points of singular non-zero quadrics $q \in \Lambda$. Since $\dim \mathrm{Gr}(3, V) = \dim \mathbb{P}(Sym^3 L^\vee)$ either φ is not dominant or it is dominant with finite generic fiber (in fact it is dominant but we do not need this). Let $\Theta = h(C)$ where $C \subset \mathbb{P}(L^\vee)$ is a generic cubic. Then $Z = Z_1 \cup \Theta_{A_h(L)}$ where $Z_1 = \varphi^{-1}(C)$. Since Z_1 is finite and $\ell = 9$ we get that (2.3.35) holds for Z_1. One deals with the component $\Theta_{A_h(L)}$ as usual.

It remains to prove that $\Theta_A = \Theta$ is generically reduced. Let $L = \langle\langle \Theta \rangle\rangle$. Given $[W] \in \Theta$ we know that $\dim(L \cap S_W) = 2$ because by **Claim 2.8** the intersection $\mathbb{P}(L) \cap \mathrm{Gr}(3, V)$ is smooth 1-dimensional. We must prove that if $A \in \mathbb{LG}(\bigwedge^3 V)$ is generic in the left-hand side of (2.3.32) then

$$A \cap S_W = L \cap S_W. \tag{2.3.37}$$

Since S_W is lagrangian for $(,)_V$ the symplectic form defines an isomorphism $\bigwedge^3 V/S_W \xrightarrow{\sim} S_W^\vee$. Thus $(,)_V$ gives an injection $L/(L \cap S_W) \hookrightarrow S_W^\vee$. It follows that $L^\perp \cap S_W/(L \cap S_W)$ is a lagrangian subspace of $L^\perp/L$ and hence the generic $B \in \mathbb{LG}(L^\perp/L)$ intersects trivially $L^\perp \cap S_W/(L \cap S_W)$; the corresponding $A \in \mathbb{LG}(\bigwedge^3 V)$ in the left-hand side of (2.3.32) satisfies (2.3.37). $\square$

A straightforward parameter count gives the dimensions of the $\mathbb{B}_X$'s; we listed their codimensions in Table (4) (since δ_V preserves dimension we omitted writing out the codimension of $\mathbb{B}_{\mathcal{E}_2^\vee}, \mathbb{B}_{\mathcal{A}^\vee}, \mathbb{B}_{\mathbf{T}^\vee}$).

Corollary 2.21. *Let X be one of the types listed in Table (2). Then $\mathbb{B}_X$ is an irreducible component of Σ_∞. Moreover the $\mathbb{B}_X$'s are pairwise distinct.*

Proof. Irreducibility of $\mathbb{B}_X$ follows from irreducibility of the parameter space for curves of Type X. By **Proposition 2.20** we get that $\mathbb{B}_X$ is an irreducible component of Σ_∞. It remains to prove that if $X_1 \neq X_2$ then $\mathbb{B}_{X_1} \not\subset \mathbb{B}_{X_2}$. Suppose that $\mathbb{B}_{X_1} \subset \mathbb{B}_{X_2}$. Since for A generic in $\mathbb{B}_{X_i}$ the scheme Θ_A is a generically reduced curve we get that $\deg \Theta_1 = \Theta_2$ for Θ_i generic of Type X_i. Looking at the degrees of Θ appearing in Table (2) and the codimensions of $\mathbb{B}_X$'s in Table (4) we conclude that the inclusion in question is one of the followings: $\mathbb{B}_{\mathcal{E}_2} = \mathbb{B}_{\mathcal{E}_2^\vee}$, $\mathbb{B}_{\mathcal{A}} = \mathbb{B}_{\mathcal{A}^\vee}$, $\mathbb{B}_{\mathbf{R}} \subset \mathbb{B}_{\mathcal{C}_2}$ or $\mathbb{B}_{\mathbf{T}} = \mathbb{B}_{\mathbf{T}^\vee}$. One proves quickly that the listed inclusions do not hold except possibly the last one. Suppose that $\mathbb{B}_{\mathbf{T}} = \mathbb{B}_{\mathbf{T}^\vee}$. Then the following holds: for Θ generic of Type $\mathbf{T}$ i.e. such that $\Theta = k(C)$ for an isomorphism $V \cong \mathrm{Sym}^2 L$ and a cubic $C \subset \mathbb{P}(L)$ there exist an isomorphism $V \cong \mathrm{Sym}^2 L^\vee$ and a cubic $C' \subset \mathbb{P}(L^\vee)$ such that $\Theta = h(C')$. Thus $R_{k(C)} = R_{h(C')}$; this is absurd because the closure of the set of multibranch points of $R_{k(C)}$ is isomorphic to $C^{(2)}$ (points $l_0 \cdot l_1$ where $l_0, l_1 \in C$) while the closure of the set of multibranch points of $R_{h(C)}$ is isomorphic to $\mathbb{P}^2$ (points l^2). $\square$

Now assume that X is of calligraphic Type, denote it by $\mathcal{X}$. We will show that one may characterize the generic point of $\mathbb{B}_{\mathcal{X}}$ by a certain flag condition that one encounters when studying GIT-stability. Let

$$\mathsf{F} := \{v_0, \ldots, v_5\} \tag{2.3.38}$$

be a basis of V. For each calligraphic $\mathcal{X}$ appearing in Table (2) we define $\mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ to be the set of $A \in \mathbb{LG}(\bigwedge^3 V)$ satisfying the condition appearing on the second column of the corresponding row of Table (3); we adopt the notation

$$V_{ij} := \langle v_i, v_{i+1}, \ldots, v_j \rangle, \qquad 0 \le i < j \le 5. \tag{2.3.39}$$

Let

$$\mathbb{B}^*_{\mathcal{X}} := \bigcup_{\mathsf{F}} \mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$$

where F runs through the set of bases of V.

Claim 2.22. *Let $\mathcal{X}$ be one of the calligraphic Types in Table (2). Then $\mathbb{B}^*_{\mathcal{X}}$ is a constructible dense subset of $\mathbb{B}_{\mathcal{X}}$.*

Proof. It suffices to prove the following two results:

(I) Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that Θ_A contains an irreducible component Θ of Type $\mathcal{X}$ according to Table (2). There exists a basis F of V such that $A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$.

(II) Let F be a basis of V. If $A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{X}}$ is generic then Θ_A contains an irreducible component Θ of Type $\mathcal{X}$.

Table 3: Flag conditions, I

name	flag condition
$\mathbb{B}^{\mathsf{F}}_{\mathcal{A}}$	$\dim A \cap ([v_0] \wedge \bigwedge^2 V_{15}) \ge 5$
$\mathbb{B}^{\mathsf{F}}_{\mathcal{A}^\vee}$	$\dim A \cap (\bigwedge^3 V_{04}) \ge 5$
$\mathbb{B}^{\mathsf{F}}_{\mathcal{C}_2}$	$\dim A \cap (\bigwedge^3 V_{02} \oplus (\bigwedge^2 V_{02} \wedge V_{35})) \ge 6$
$\mathbb{B}^{\mathsf{F}}_{\mathcal{D}}$	$\dim A \cap ([v_0] \wedge \bigwedge^2 V_{14}) \ge 3$
$\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$	$\dim A \cap ([v_0] \wedge (\bigwedge^2 V_{12}) \oplus ([v_0] \wedge V_{12} \wedge V_{35})) \ge 4$
$\mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2^\vee}$	$\dim A \cap (\bigwedge^3 V_{02} \oplus (\bigwedge^2 V_{02} \wedge V_{34})) \ge 4$
$\mathbb{B}^{\mathsf{F}}_{\mathcal{F}_1}$	$A \supset (\bigwedge^2 V_{01} \wedge V_{23})$

Table 4: Codimension of the $\mathbb{B}_X$'s

$\mathbb{B}_\star$	$\mathbb{B}_{\mathcal{F}_1}$	$\mathbb{B}_{\mathcal{D}}$	$\mathbb{B}_{\mathcal{E}_2}$	$\mathbb{B}_{\mathbf{Q}}$	$\mathbb{B}_{\mathcal{A}}$	$\mathbb{B}_{\mathcal{C}_2}$	$\mathbb{B}_{\mathbf{R}}$	$\mathbb{B}_{\mathbf{S}}$	$\mathbb{B}_{\mathbf{T}}$
$\operatorname{cod}(\mathbb{B}_\star, \mathbb{LG}(\bigwedge^3 V))$	7	9	11	9	10	12	17	16	18

Items (I), (II) are obvious except possibly for $\mathcal{X} = \mathcal{E}_2$ or $\mathcal{X} = \mathcal{E}_2^\vee$. Suppose that $\mathcal{X} = \mathcal{E}_2$. Let's prove (I). We have $\mathcal{E}_\Theta \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and hence the Harder-Narasimhan filtration of $\mathcal{E}_\Theta$ gives rise to a filtration $U_1 \subset U_3 \subset U_6 = V$ where $\dim U_i = i$. Let F be a basis of V such that $[v_0] = U_1$. $[v_0] \oplus V_{12} = U_3$; then $A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$. Let's prove (II). Given $A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2}$ we let

$$F_A := A \cap ([v_0] \wedge (\bigwedge^2 V_{12}) \oplus ([v_0] \wedge V_{12} \wedge V_{35})).$$

Suppose that A is generic. Then $\dim F_A = 4$, moreover $\Theta_A = \mathbb{P}(F_A) \cap \mathrm{Gr}(3, V)$ and the latter is a curve of Type $\mathcal{E}_2$. Next suppose that $\mathcal{X} = \mathcal{E}_2^\vee$. Let's prove (I). By Table (2) we may describe R_Θ as follows. Let

$$\sigma\colon \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow |\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)|^\vee \cong \mathbb{P}(V)$$

be Segre's embedding followed by a suitable isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^3 \cong V$. Let $\mathcal{S} \subset \mathbb{P}(V)$ be the image of σ. Then R_Θ is the projection of $\mathcal{S}$ from a point $p \notin \mathcal{S}$; of course the planes that sweep out R_Θ are the projections of the planes $\sigma(\{x\} \times \mathbb{P}^2)$ for $x \in \mathbb{P}^1$. Let $H \subset \mathbb{P}(V)$ be the hyperplane containing R_Θ i.e. the hyperplane to which we project from p. One checks easily that there exists a (unique) line $L \subset \mathbb{P}^2$ such that the span $M := \langle \sigma(\mathbb{P}^1 \times L) \rangle$ contains p; notice that $\dim M = 3$ because $\sigma(\mathbb{P}^1 \times L)$ is a smooth quadric surface. Let $P \subset H$ be the projection of $(M \setminus \{p\})$ from p; thus P is a plane. It follows from the definitions that each plane in Θ intersects P in a line. Let F be a basis of V such that $\mathbb{P}(V_{02}) = P$ and $\mathbb{P}(V_{04}) = H$; then $A \in \mathbb{B}^{\mathsf{F}}_{\mathcal{E}_2^\vee}$. The proof of (II) is analogous to the proof of (II) for $\mathcal{X} = \mathcal{E}_2$, we omit details. □

Remark 2.23. Let X be one of the Types appearing in Table (2). In a forthcoming paper we will show that if X is calligraphic and $A \in \mathbb{B}_X$ then A is not GIT-stable (in general it is properly semistable) - calligraphic Types have been ordered according to the complexity of the destabilizing 1-PS for generic lagrangians of that Type. On the other hand we will show that if X is boldface and $A \in \mathbb{B}_X$ is generic then it is GIT-stable.

Table 5: Flag conditions, II

name	flag condition
$\mathbb{X}^{\mathsf{F}}_{\mathcal{A}_+}$	$\dim A\cap([v_0]\wedge\bigwedge^2 V_{15})\geq 6$
$\mathbb{X}^{\mathsf{F}}_{\mathcal{A}^{\vee}_+}$	$\dim A\cap(\bigwedge^3 V_{04})\geq 6$
$\mathbb{X}^{\mathsf{F}}_{\mathcal{C}_{1,+}}$	$A\supset\bigwedge^3 V_{02}$ and $\dim A\cap(\bigwedge^2 V_{02}\wedge V_{35})\geq 4$
$\mathbb{X}^{\mathsf{F}}_{\mathcal{C}_{2,+}}$	$\dim A\cap(\bigwedge^3 V_{02}\oplus\bigwedge^2 V_{02}\wedge V_{35})\geq 7$
$\mathbb{X}^{\mathsf{F}}_{\mathcal{D}_+}$	$\dim A\cap([v_0]\wedge\bigwedge^2 V_{14})\geq 4$
$\mathbb{X}^{\mathsf{F}}_{\mathcal{E}_{2,+}}$	$\dim A\cap([v_0]\wedge\bigwedge^2 V_{12}\oplus[v_0]\wedge V_{12}\wedge V_{35})\geq 5$
$\mathbb{X}^{\mathsf{F}}_{\mathcal{E}^{\vee}_{2,+}}$	$\dim A\cap(\bigwedge^3 V_{02}\oplus\bigwedge^2 V_{02}\wedge V_{34})\geq 5$
$\mathbb{X}^{\mathsf{F}}_{\mathcal{F}_{1,+}}$	$A\supset(\bigwedge^2 V_{01}\wedge V_{23})$ and $\dim A\cap(\bigwedge^2 V_{01}\wedge V_{45}\oplus V_{01}\wedge\bigwedge^2 V_{23})\geq 1$

2.4 Two-dimensional components of Θ_A

We will analyze those $A\in\mathbb{LG}(\bigwedge^3 V)$ such that Θ_A contains a 2-dimensional irreducible component. In order to state our result we will introduce certain closed subsets of $\mathbb{LG}(\bigwedge^3 V)$. First we associate to each of a collection of calligraphic Types $\mathcal{X}$ (containing all those appearing in Table (3)) a constructible subset $\mathbb{X}_{\mathcal{X},+}\subset\mathbb{LG}(\bigwedge^3 V)$. Let F be a basis of V as in (2.3.38). We let $\mathbb{X}^{\mathsf{F}}_{\mathcal{X},+}$ be the set of $A\in\mathbb{LG}(\bigwedge^3 V)$ satisfying the condition appearing on the second column of the corresponding row of Table (5) (Notation (2.3.39) is in force). Let

$$\mathbb{X}^{*}_{\mathcal{X},+}:=\bigcup_{\mathsf{F}}\mathbb{B}^{\mathsf{F}}_{\mathcal{X},+},\qquad \mathbb{X}_{\mathcal{X},+}:=\overline{\mathbb{X}}^{*}_{\mathcal{X},+}.$$

(F runs through the set of bases of V.)

Definition 2.24. Let U be a complex vector-space of dimension 4 and choose an isomorphism $V\cong\bigwedge^2 U$. Let $i_+\colon\mathbb{P}(U)\hookrightarrow\mathrm{Gr}(3,V)$ be the map given by (1.5.1).

(a) Let $\mathbb{X}_{\mathcal{W}}\subset\mathbb{LG}(\bigwedge^3 V)$ be the closure of the set of $PGL(V)$-translates of those A such that $\mathbb{P}(A)$ contains $i_+(Z)$ where Z is a smooth quadric.

(b) Let $\mathbb{X}_{\mathcal{Y}}\subset\mathbb{LG}(\bigwedge^3 V)$ be the set of $PGL(V)$-translates of those A such that $\mathbb{P}(A)$ contains $i_+(Z)$ where Z is either a quadric cone or a plane.

Lastly let L be a complex vector-space of dimension 3. Choose an isomorphism $V\cong\mathrm{Sym}^2 L$ and let $A_k(L),A_h(L)\in\mathbb{LG}(\bigwedge^3 V)$ be the lagrangians defined by (1.5.15). We let

$$\mathbb{X}_k:=\overline{PGL(V)A_k(L)},\qquad \mathbb{X}_h:=\overline{PGL(V)A_h(L)}.$$

Remark 2.25. If $\mathcal{X}$ appears in Table (3) then $\mathbb{X}^{*}_{\mathcal{X},+}\subset\mathbb{B}^{*}_{\mathcal{X}}$. Moreover $\mathbb{X}_{\mathcal{Y}},\mathbb{X}_{\mathcal{W}}\subset(\mathbb{B}_{\mathbf{R}}\cap\mathbb{B}_{\mathbf{S}})$, $\mathbb{X}_k\subset\mathbb{B}_{\mathbf{T}}$ and $\mathbb{X}_h\subset\mathbb{B}_{\mathbf{T}^{\vee}}$.

Theorem 2.26. *Suppose that $A\in\mathbb{LG}(\bigwedge^3 V)$ and that Θ_A contains a 2-dimensional irreducible component. Then*

$$A\in(\mathbb{X}_{\mathcal{A}+}\cup\mathbb{X}_{\mathcal{A}^{\vee}_+}\cup\mathbb{X}_{\mathcal{C}_{1,+}}\cup\mathbb{X}_{\mathcal{C}_{2,+}}\cup\mathbb{X}_{\mathcal{D}_+}\cup\mathbb{X}_{\mathcal{E}_{2,+}}\cup\mathbb{X}_{\mathcal{E}^{\vee}_{2,+}}\cup\mathbb{X}_{\mathcal{F}_{1,+}}\cup\mathbb{X}_{\mathcal{Y}}\cup\mathbb{X}_{\mathcal{W}}\cup\mathbb{X}_h\cup\mathbb{X}_k).\quad(2.4.1)$$

In a forthcoming paper we will prove that each of the $\mathbb{X}_{\mathcal{X}_+}$ appearing in (2.4.1) is contained in the GIT-unstable locus of $\mathbb{LG}(\bigwedge^3 V)$ and hence is irrelevant when considering moduli of (double)EPW-sextics. That is the reason why the statement of **Theorem 2.26** is not as detailed as that of **Theorem 2.9**. In fact the set in (2.4.1) contains strictly the locus of $A \in \mathbb{LG}(\bigwedge^3 V)$ such that Θ_A contains an irreducible component of dimension 2: if A is generic in $\mathbb{X}_{\mathcal{C}_{1,+}}$ then $\dim \Theta_A = 0$. The proof of **Theorem 2.26** will be given at the end of the present subsection: we will first prove a series of auxiliary results. For the rest of this subsection $V_0 \subset V$ will be a 5-dimensional subspace.

Proposition 2.27. *Keeping notation as above let $L \subset \mathbb{P}(\bigwedge^2 V_0)$ be a linear subspace not contained in $\mathrm{Gr}(2, V_0)$. Suppose that $L \cap \mathrm{Gr}(2, V_0)$ contains a hypersurface which is not a hyperplane. Then there exists $U \in \mathrm{Gr}(4, V_0)$ such that $L \subset \mathbb{P}(\bigwedge^2 U)$.*

Proof. Let $Z \subset (L \cap \mathrm{Gr}(2, V_0))$ be the hypersurface which exists by hypothesis. Then $L \cap \mathrm{Gr}(2, V_0)$ is an intersection of quadrics (in L) because $\mathrm{Gr}(2, V_0)$ is the intersection of Plücker quadrics in $\mathbb{P}(\bigwedge^2 V_0)$. Since $L \cap \mathrm{Gr}(2, V_0)$ contains the non-degenerate hypersurface Z we get that Z is a quadric and it equals the scheme-theoretic intersection $L \cap \mathrm{Gr}(2, V_0)$. Let us consider the rational map

$$f \colon \mathbb{P}(\bigwedge^2 V_0) \dashrightarrow |I_{\mathrm{Gr}(2,V_0)}(2)|^\vee = \mathbb{P}(V_0^\vee). \tag{2.4.2}$$

The restriction of f to L is regular and constant because Z is a quadric hypersurface in L. On the other hand f has the following geometric interpretation by **Lemma 1.6**: if $[\alpha] \in (\mathbb{P}(\bigwedge^2 V_0) \setminus \mathrm{Gr}(2, V_0))$ then $f([\alpha])$ is canonically equal to the span of α. Since f is constant on L the proposition follows. □

Given a quadric $Q_0 \subset \mathbb{P}(V_0)$ let

$$F_1(Q_0) := \{\ell \subset Q_0 \mid \ell \text{ a line}\} \tag{2.4.3}$$

be the variety parametrizing lines contained in Q_0. We will need an explicit description of $F_1(Q_0)$ for Q_0 of corank at most 1. We start by recalling how one describes $F_1(Q)$ for $Q \subset \mathbb{P}(V)$ a smooth quadric. Let U be a 4-dimensional complex vector-space; choose an isomorphism $\mathbb{P}(V) \cong \mathbb{P}(\bigwedge^2 U)$ taking Q to $\mathrm{Gr}(2, U)$ (embedded in $\mathbb{P}(\bigwedge^2 U)$ by Plücker). Let $Z \subset \mathbb{P}(U) \times \mathbb{P}(U^\vee)$ be the incidence subvariety of couples $([u], [f])$ such that $f(u) = 0$. We have an isomorphism

$$\begin{array}{ccc} Z & \xrightarrow{\mu} & F_1(Q) \\ ([u],[f]) & \mapsto & \{K \in \mathrm{Gr}(2,U) \mid u \in K \subset \ker(f)\} \end{array} \tag{2.4.4}$$

Furthermore

$$\mu^* \mathcal{O}_{F_1(Q)}(1) \cong \mathcal{O}_{\mathbb{P}(U)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(U^\vee)}(1) \tag{2.4.5}$$

Let

$$\mathbb{P}(U) \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} \mathbb{P}(U^\vee) \tag{2.4.6}$$

be the projections - via Isomorphism (2.4.4) we will also view π_1, π_2 as maps with domain $F_1(Q)$. Now we are ready to describe $F_1(Q_0)$. We have

$$Q_0 = Q \cap V(\sigma), \qquad \sigma \in \bigwedge^2 V^\vee \tag{2.4.7}$$

where

(a) σ is non-degenerate if Q_0 is smooth,

(b) $\dim \ker(\sigma) = 2$ if $\operatorname{cork} Q_0 = 1$.

The proof of the following result is an easy exercise that we leave to the reader.

Claim 2.28. *Keep notation as above and suppose that Q_0 is smooth. The restrictions of π_1 and π_2 to $F_1(Q_0)$ are isomorphisms onto $\mathbb{P}(U)$ and $\mathbb{P}(U^\vee)$ respectively; let $\nu_1 : \mathbb{P}(U) \xrightarrow{\sim} F_1(Q_0)$ and $\nu_2 : \mathbb{P}(U^\vee) \xrightarrow{\sim} F_1(Q_0)$ be the inverses. Then*

$$\nu_1^* \mathcal{O}_{F_1(Q_0)}(1) \cong \mathcal{O}_{\mathbb{P}(U)}(2), \qquad \nu_2^* \mathcal{O}_{F_1(Q_0)}(1) \cong \mathcal{O}_{\mathbb{P}(U^\vee)}(2). \tag{2.4.8}$$

Now suppose that $\operatorname{cork} Q_0 = 1$ i.e. Item (b) holds. Let

$$L_1 := \mathbb{P}(\ker \sigma) \subset \mathbb{P}(U), \qquad L_2 := \mathbb{P}(\operatorname{Ann}(\ker \sigma)) \subset \mathbb{P}(U^\vee), \tag{2.4.9}$$

where σ is as in (2.4.7). Thus L_1, L_2 are lines. Let $T_1, T_2 \subset Z$ be the closed subsets defined by

$$T_i = \pi_i^{-1} L_i. \tag{2.4.10}$$

We leave the easy proof of the following result to the reader.

Claim 2.29. *Keep notation as above and suppose that* $\operatorname{cork} Q_0 = 1$*. Then:*

(1) The irreducible components of $F_1(Q_0)$ are $\mu(T_1)$ and $\mu(T_2)$.

(2) Let $\rho_i := \pi_{3-i}|_{T_i}$. The map ρ_i is the blow-up of L_{3-i}.

(3) Let $E_i \subset T_i$ be the exceptional divisor of ρ_i; then

$$\mathcal{O}_{T_1}(1) \cong \quad \rho_1^* \mathcal{O}_{\mathbb{P}(U^\vee)}(2)(-E_1), \tag{2.4.11}$$
$$\mathcal{O}_{T_2}(1) \cong \quad \rho_2^* \mathcal{O}_{\mathbb{P}(U)}(2)(-E_2) \tag{2.4.12}$$

(We view T_i as a subset of $F_1(Q_0)$ via Item (1) and we let $\mathcal{O}_{T_i}(1)$ be the restriction to T_i of the Plücker line-bundle on $F_1(Q_0)$.)

(4) The maps between spaces of global sections induced by (2.4.11)-(2.4.12) are surjective.

In the following lemma we think of $\operatorname{Gr}(2, V_0)$ as embedded in $\mathbb{P}(\bigwedge^2 V_0)$ by Plücker: given $W \subset \operatorname{Gr}(2, V_0)$ we denote by $\langle W \rangle$ the span of W in $\mathbb{P}(\bigwedge^2 V_0)$.

Lemma 2.30. *Let $Q_0 \subset \mathbb{P}(V_0)$ be an irreducible quadric. Suppose that $S \subset F_1(Q_0)$ is a projective surface and that*

(a) S is a two-dimensional cubic rational normal scroll (possibly singular), or

(b) $\dim \langle S \rangle = 4$ and S is the intersection of two quadric surfaces in $\langle S \rangle$.

Then there exists a line $\ell_0 \subset \mathbb{P}(V_0)$ which intersects all the lines parametrized by S.

Proof. Since Q_0 is irreducible the corank of Q_0 is 0, 1 or 2. We claim that Q_0 can not be smooth. In fact suppose that Q_0 is smooth. By **Claim 2.28** we get that $\mathcal{O}_S(1)$ is divisible by 2; this is absurd because a surface S satisfying Item (a) or Item (b) contains lines. Next let us suppose that $\operatorname{cork} Q_0 = 1$. We adopt the notation of **Claim 2.29**. Since S is irreducible we have $S \subset \mu(T_i)$ for $i = 1$ or $i = 2$. By simmetry we may suppose that $S \subset \mu(T_2)$. Since $\dim \langle S \rangle = 4$ it follows easily from Item (4) of **Claim 2.29** that $\rho_2(S)$ is a plane not containing L_1 - in particular S is a smooth normal cubic scroll. Let $q := L_1 \cap \rho_2(S)$; then $(q, \rho_2(S)) \in T_1$ and hence $\mu(q, \rho_2(S))$ is a line $\ell_0 \subset Q_0$. By construction every line in S intersects ℓ_0. Finally suppose that $\operatorname{cork} Q_0 = 2$. Then $\operatorname{sing} Q_0$ is a line and every line contained in Q_0 intersetcts $\operatorname{sing} Q_0$; thus the lemma holds in this case as well. □

Proposition 2.31. *Keep notation as above. Suppose that $\Lambda \subset \mathbb{P}(\bigwedge^2 V_0)$ is a linear subspace and that there exists a 2-dimensional irreducible component S of $\Lambda \cap \operatorname{Gr}(2, V_0)$. Then one of the following holds:*

(1) S is a plane.

(2) $\dim\langle S\rangle = 3$ *and there exists* $W \in \mathrm{Gr}(4, V_0)$ *such that* $S \subset \mathrm{Gr}(2, W)$.

(3) $\dim\langle S\rangle = 4$ *and there exists a line* $\ell_0 \subset \mathbb{P}(V_0)$ *which intersects all the lines parametrized by* S.

(4) $5 \le \dim\langle S\rangle$.

Proof. Suppose that $\dim\langle S\rangle = 2$. Then S is a plane and hence Item (1) holds. On the other hand if $5 \le \dim\langle S\rangle$ holds then Item (4) holds. Thus from now we may assume that

$$3 \le \dim\langle S\rangle \le 4. \tag{2.4.13}$$

Suppose that $\deg S = 2$; then Item (2) holds by **Proposition 2.27**. Thus we may suppose that $\deg S \ge 3$. The intersection $\Lambda \cap \mathrm{Gr}(2, V_0)$ is cut out by quadrics and S is one of its irreducible components: it follows that

$$\dim\langle S\rangle = 4, \qquad \deg S \le 4. \tag{2.4.14}$$

Notice also that if $\deg S = 4$ then necessarily S is an intersection of two quadrics. Thus S is one of the following:

(α) A normal cubic scroll (possibly singular).

(β) An intersection of two quadrics (in Λ) which is not a cone.

(γ) An intersection of two quadrics (in Λ) which is a cone over a degree-4 curve of arithmetic genus 1.

If (γ) holds then Item (3) holds with ℓ_0 the line corresponding to the vertex of the cone. Thus we may suppose that either (α) or (β) holds. Let $T \to S$ be a desingularization of S; since either (α) or (β) holds T is rational. In particular we have

$$\chi(\mathcal{O}_T) = 1. \tag{2.4.15}$$

Composing the desingularization map $T \to S$ with the inclusion $S \hookrightarrow \mathrm{Gr}(2, V_0)$ we get a map $g\colon T \to \mathrm{Gr}(2, V_0)$. Let $\mathcal{E}_T$ be the pull-back to T of the tautological rank-2 vector-bundle on $\mathrm{Gr}(2, V_0)$, $f_T\colon \mathbb{P}(\mathcal{E}_T) \to \mathbb{P}(V_0)$ be the tautological map and $R_T = \mathrm{im}(f_T)$. Let $\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E}_T)}(1)$. Let $\theta\colon \mathbb{P}(\mathcal{E}_T) \to T$ be the bundle map. We let $F := \mathcal{E}_T^\vee$; thus F is globally generated. The relation

$$c_1(\xi)^2 - (\theta^* c_1(F))c_1(\xi) + \theta^* c_2(F) = 0 \tag{2.4.16}$$

gives the equation

$$\int_T (c_1(F)^2 - c_2(F)) = \int_{\mathbb{P}(\mathcal{E}_T)} c_1(\xi)^3. \tag{2.4.17}$$

Thus we get

$$\int_T (c_1(F)^2 - c_2(F)) = \begin{cases} \deg f_T \cdot \deg R_T & \text{if } \dim R_T = 3, \\ 0 & \text{if } \dim R_T < 3, \end{cases} \tag{2.4.18}$$

and in particular

$$0 \le c_2(F) \le c_1(F)^2. \tag{2.4.19}$$

(The first inequality holds because F is globally generated.) We notice that $\dim R_T = 3$ unless R_T is a plane. Since (α) or (β) holds R_T is not a plane and hence $\dim R_T = 3$. By (2.4.18)-(2.4.19) we have $\deg R_T \le 4$. We claim that $2 \le \deg R_T$. In fact suppose that $\deg R_T = 1$. Then $R_T = \mathbb{P}(W)$ where $W \in \mathrm{Gr}(4, V)$ and hence $S \subset \mathrm{Gr}(2, W)$. Since $\mathrm{Gr}(2, W)$ is a smooth quadric hypersurface in $\mathbb{P}(\bigwedge^2 W)$ and $\dim\langle S\rangle = 4$ the intersection $\langle S\rangle \cap \mathrm{Gr}(2, W)$ is a 3-dimensional irreducible quadric containing S; this contradicts the hypothesis that S is an irreducible component of $\Lambda \cap \mathrm{Gr}(2, V_0)$. If $\deg R_T = 2$ then R_T is an irreducible 3-dimensional quadric and hence Item (3) holds by **Lemma 2.30**. It remains to prove the proposition under the assumption that $\dim R_T = 3$ and $3 \leq \deg R_T \leq 4$. Let's prove that

$$h^0(EndF) \geq 3. \tag{2.4.20}$$

By (2.4.18) it follows that if S is a cubic normal scroll then $c_1(F)^2 = 3$, $c_2(F) = 0$ and if S is the intersection of two quadrics then $c_1(F)^2 = 4$, $0 \leq c_2(F) \leq 1$. In both cases we have

$$0 \leq c_2(F) \leq 1. \tag{2.4.21}$$

Thus Hirzebruch-Riemann-Roch gives that

$$\chi(EndF) = 4\chi(\mathcal{O}_T) - (4c_2(F) - c_1(F)^2) \geq 4\chi(\mathcal{O}_T) = 4. \tag{2.4.22}$$

(The last equality holds by (2.4.15).) By Serre duality we get that

$$h^0(EndF) + h^0(EndF(K_T)) \geq 4. \tag{2.4.23}$$

One easily checks that there exists a non-zero $\sigma \in H^0(-K_T) > 0$. Thus σ defines a map $F(K_T) \to F$ which is an isomorphism away from the (non-empty) zero-set of σ; it follows that we have an inclusion $H^0(EndF(K_T)) \subset H^0(EndF)$ which does not contain the subsapce of homotheties. Hence $h^0(EndF) \geq 1 + h^0(EndF(K_T))$ and thus (2.4.20) follows from (2.4.23).

Claim 2.32. *There exist divisors C, D on T and a zero-dimensional subscheme $Z \subset T$ such that F fits into an exact sequence*

$$0 \longrightarrow \mathcal{O}_T(C) \longrightarrow F \longrightarrow I_Z(D) \longrightarrow 0. \tag{2.4.24}$$

and the following hold:

(I) $\mathcal{O}_T(C)$ and $I_Z(D)$ are globally generated, in particular

$$C \cdot C \geq 0, \qquad C \cdot D \geq 0, \qquad D \cdot D \geq 0. \tag{2.4.25}$$

(II) $h^0(\mathcal{O}_T(C - D)) > 0$ but $\mathcal{O}_T(C - D) \not\cong \mathcal{O}_T$.

Proof of the claim. By (2.4.20) there exists $\psi \in H^0(EndF)$ which is not a scalar. Let λ be an eigen-value of ψ (notice that the characteristic polynomial of ψ has constant coefficients) and $\phi := (\psi - \lambda \operatorname{Id}_F)$. Let $\mathcal{K} := \ker \phi$; then $\mathcal{K}$ is a rank-1 subsheaf of F and hence $\mathcal{K} \cong \mathcal{O}_T(C')$ for a certain divisor (class) C'. The quotient $F/\mathcal{K}$ is locally-free away from a finite set and hence it is isomorphic to $I_Z(D')$ for a certain divisor (class) D'. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_T(C') \longrightarrow F \xrightarrow{\pi} I_Z(D') \longrightarrow 0 \tag{2.4.26}$$

and an inclusion of vector-bundles $\iota\colon \mathcal{O}_T(D') \hookrightarrow F$ (injective on fibers) such that $\phi = \iota \circ \pi$. Suppose that $\pi \circ \iota \circ \pi \neq 0$; then (2.4.26) splits and hence $F \cong \mathcal{O}_T(C') \oplus \mathcal{O}_T(D')$. By (2.4.20) we get that $h^0(\mathcal{O}_T(C' - D') > 0$ or $h^0(\mathcal{O}_T(D' - C') > 0$; if the former holds we let $C := C'$ and $D := D'$, if the latter holds we let $C := D'$ and $D := C'$. With these choices the claim holds except possibly for the assertion that $\mathcal{O}_T(C - D) \not\cong \mathcal{O}_T$. We have

$$\mathcal{O}_T(C + D) \cong g^*\mathcal{O}_S(1). \tag{2.4.27}$$

Suppose that $\mathcal{O}_T(C - D) \cong \mathcal{O}_T$; then $g^*\mathcal{O}_S(1)$ is divisible by two, this is absurd because S contains lines which are not contained in $\operatorname{sing} S$. This proves the claim under the assumption that $\pi \circ \iota \circ \pi \neq 0$. Now suppose that $\pi \circ \iota \circ \pi = 0$. Then ϕ defines a non-zero map $I_Z(D') \to \mathcal{O}_T(C')$ and hence we get that $h^0(\mathcal{O}_T(C' - D') > 0$. Let $C := C'$ and $D := D'$. The claim holds with these choices. $\square$

We resume the proof of **Proposition 2.31**. Let C, D and Z be as in **Claim 2.32**; we have (2.4.27) and by Whitney's formula we get that

$$C \cdot D + \ell(Z) = c_2(F), \tag{2.4.28}$$

$$C \cdot C + 2C \cdot D + D \cdot D = \deg S. \tag{2.4.29}$$

By (2.4.21), (2.4.25) and (2.4.28) we have $0 \le C \cdot D \le 1$. Suppose that $C \cdot D = 0$. By **Claim 2.32** and Hodge Index we get that $D = 0$ and $F \cong \mathcal{O}_T(C) \oplus \mathcal{O}_T$. Thus T is a cone and hence Item (3) holds with ℓ_0 the line corresponding to the vertex of T. Now suppose that $C \cdot D = 1$. By (2.4.21) and (2.4.28) we have $Z = \emptyset$. By (2.4.27), Item (II) of **Claim 2.32** and Hodge Index we have

$$0 < (C - D) \cdot (C + D) = C \cdot C - D \cdot D. \tag{2.4.30}$$

By (2.4.25) and (2.4.29) we get that one of the following holds:

(i) $C \cdot C = 1$ and $D \cdot D = 0$ ($\deg S = 3$).

(ii) $C \cdot C = 2$ and $D \cdot D = 0$ ($\deg S = 4$).

Since $\mathcal{O}_T(D)$ is globally generated and $C \cdot D = 1$ it follows that $h^0(\mathcal{O}_T(D)) = 2$. Furthermore we get surjectivity of the map $\epsilon\colon V_0^\vee \to H^0(\mathcal{O}_T(D))$ given by the composition $V_0^\vee \to H^0(F) \to H^0(\mathcal{O}_T(D))$. Thus $\operatorname{cod}(\ker \epsilon, V_0^\vee) = 2$. Let $\ell_0 := \mathbb{P}(\operatorname{Ann}(\ker \epsilon))$; then Item (3) holds with this choice of ℓ_0. $\square$

Below is the analogue of **Proposition 2.31** obtained upon replacing $\operatorname{Gr}(2, V_0)$ by $\mathbb{P}^2 \times \mathbb{P}^2$. Let W_1, W_2 be 3-dimensional complex vector spaces; then $\mathbb{P}(W_1) \times \mathbb{P}(W_2) \subset \mathbb{P}(W_1 \otimes W_2)$ via Segre's embedding.

Proposition 2.33. *Keep notation as above. Suppose that $\Lambda \subset \mathbb{P}(W_1 \otimes W_2)$ is a linear subspace such that there exists a 2-dimensional irreducible component S of $\Lambda \cap (\mathbb{P}(W_1) \times \mathbb{P}(W_2))$. Then one of the following holds possibly after exchanging W_1 with W_2:*

(1) $S = \{[v_0]\} \times \mathbb{P}(W_2)$ for some $[v_0] \in \mathbb{P}(W_1)$.

(2) $S = \mathbb{P}(U_1) \times \mathbb{P}(U_2)$ where $U_i \in \operatorname{Gr}(2, W_i)$ i.e. S is a smooth quadric surface.

(3) S is a smooth hyperplane section of $\mathbb{P}(U_1) \times \mathbb{P}(W_2)$ where $U_1 \in \operatorname{Gr}(2, W_1)$, i.e. S is a smooth normal cubic scroll.

(4) S is the graph of an isomorphism $\mathbb{P}(W_1) \xrightarrow{\sim} \mathbb{P}(W_2)$ (and hence is a Veronese surface).

(5) $6 \le \dim\langle S\rangle$.

Proof. Let $f_i\colon S \to \mathbb{P}(W_i)$ be the restriction of projection for $i = 1, 2$. For $i = 1, 2$ let C_i be a divisor on S such that $\mathcal{O}_S(C_i) \cong f_i^* \mathcal{O}_{\mathbb{P}(W_i)}(1)$. Then $\mathcal{O}_S(C_1 + C_2) \cong \mathcal{O}_S(1)$ and hence we have

$$C_1 \cdot C_1 + 2C_1 \cdot C_2 + C_2 \cdot C_2 = \deg S. \tag{2.4.31}$$

Since $\mathcal{O}_S(C_i)$ is globally generated we have

$$C_i \cdot C_j \ge 0, \qquad 1 \le i, j \le 2. \tag{2.4.32}$$

Suppose that $C_1 \cdot C_2 = 0$; applying Hodge index to a desingularization of S we get that one of C_1, C_2 is linearly equivalent to 0 and hence Item (1) holds (possibly after exchanging W_1 with W_2). Thus we may assume that

$$C_1 \cdot C_2 > 0, \qquad \deg S \ge 2. \tag{2.4.33}$$

Suppose that $\deg S = 2$. Then $C_1 \cdot C_2 = 1$ and $C_i \cdot C_i = 0$ for $i = 1, 2$; it follows easily that Item (2) holds. Thus we are left with the case $\deg S \ge 3$. Since $\Lambda \cap (\mathbb{P}(W_1) \times \mathbb{P}(W_2))$ is cut out by quadrics it follows that $4 \le \dim\langle S\rangle$. On the other hand if $6 \le \dim\langle S\rangle$ then Item (5) holds and hence from now on we may assume that

$$4 \le \dim\langle S\rangle \le 5 \tag{2.4.34}$$

We claim that

$$\deg S \le 5. \tag{2.4.35}$$

In fact by (2.4.34) there exists a 6-dimensional linear space $\widetilde{\Lambda} \subset \mathbb{P}(W_1 \otimes W_2)$ containing Λ. If $\widetilde{\Lambda}$ is a generic such linear space then $\widetilde{\Lambda} \cap (\mathbb{P}(W_1) \times \mathbb{P}(W_2))$ is of pure dimension 2 and it contains an irreducible component other than S; thus

$$6 = \deg \mathbb{P}(W_1) \times \mathbb{P}(W_2) = \deg(\widetilde{\Lambda} \cap (\mathbb{P}(W_1) \times \mathbb{P}(W_2))) > \deg S. \tag{2.4.36}$$

This proves (2.4.35). By (2.4.31) and (2.4.32) we get that one of the following holds possibly after exchanging W_1 with W_2:

(α_k) $C_1 \cdot C_1 = 0$, $C_1 \cdot C_2 = 1$ and $C_2 \cdot C_2 = k$ where $1 \leq k \leq 3$.

(β_m) $C_1 \cdot C_1 = 1$, $C_1 \cdot C_2 = 1$ and $C_2 \cdot C_2 = m$ where $1 \leq m \leq 2$.

(γ_n) $C_1 \cdot C_1 = 0$, $C_1 \cdot C_2 = 2$ and $C_2 \cdot C_2 = n$ where $0 \leq n \leq 1$.

Suppose that (α_k) holds. Then $f_1(S)$ is a curve in $\mathbb{P}(W_1)$, in fact it is a line L because $C_1 \cdot C_2 = 1$. Thus $S \subset L \times \mathbb{P}(W_2)$; since $C_1 \cdot C_2 = 1$ and $C_2 \cdot C_2 = k$ we have $S \in |\mathcal{O}_L(k) \boxtimes \mathcal{O}_{\mathbb{P}(W_2)}(1)|$. We claim that $\dim\langle S\rangle = 4$. In fact suppose the contrary; then $\dim\langle S\rangle = 5$ by (2.4.34) and hence $\langle S\rangle = L \times \mathbb{P}(W_2)$, contradicting the hypothesis that S is an irreducible component of $\Lambda \cap \mathbb{P}(W_1) \times \mathbb{P}(W_2)$. Since $\dim\langle S\rangle = 4$ we must have $k = 1$ and hence Item (2) holds. Next suppose that (β_m) holds: it follows that $m = 1$ and that Item (4) holds. Lastly suppose that (γ_n) holds; we will reach a contradiction. If $n = 0$ then $f_i(S)$ is a curve for $i = 1, 2$; since $C_1 \cdot C_2 = 2$ it follows that $S = L \times Z$ where $L \subset \mathbb{P}(W_1)$ is a line and $Z \subset \mathbb{P}(W_2)$ is a smooth conic, possibly after exchanging W_1 with W_2. Then $\langle S\rangle = L \times \mathbb{P}(W_2)$, this contradicts the hypothesis that S is an irreducible component of $\Lambda \cap \mathbb{P}(W_1) \times \mathbb{P}(W_2)$. If $n = 1$ then $f_1(S)$ is a curve and since $C_1 \cdot C_2 = 2$ we get that $f_1(S)$ is either a line or a smooth conic. Suppose that $f_1(S)$ is a line L: arguing as in Case (α_k) for $k = 2, 3$ one gets that $\langle S\rangle \supset L \times \mathbb{P}(W_2)$, this contradicts the hypothesis that S is an irreducible component of $\Lambda \cap \mathbb{P}(W_1) \times \mathbb{P}(W_2)$. Lastly suppose that $f_1(S)$ is a (smooth) conic. Then one gets that $f_2 \colon S \to \mathbb{P}(W_2)$ is the blow-up of a point and that the linear system cut out on S by $|\mathcal{O}_{\mathbb{P}(W_1 \otimes W_2)}(1)|$ is equal to $|f_2^* \mathcal{O}_{\mathbb{P}(W_2)}(3)(-2E)|$ where $E \subset S$ is the exceptional divisor of f_2. It follows that $\dim\langle S\rangle = 6$, that contradicts (2.4.34). $\square$

The following is our last preliminary result.

Claim 2.34. *Let $U \in \mathrm{Gr}(3, V)$. Suppose that $\Theta \subset I_U$ is a projective surface such that*

(a) $U \notin \Theta$,

(b) $\overline{\rho}_U(\Theta)$ is the graph of an isomorphism $\mathbb{P}(\bigwedge^2 U) \xrightarrow{\sim} \mathbb{P}(V/U)$. (See (1.4.11) *for the definition of $\overline{\rho}_U$.)*

Then there exist an identification $V = \bigwedge^2 \mathbb{C}^4$ and a plane $Z \subset \mathbb{P}(\mathbb{C}^4)$ such that $\Theta = i_+(Z)$ where i_+ is given by (1.5.1).

Proof. Let $Z \subset \mathbb{P}^3$ be a plane and $\Theta = i_+(Z)$. Let $U \subset V$ be such that $\mathbb{P}(U) = i_-(Z)$ (we recall that $i_- \colon (\mathbb{P}^3)^\vee \hookrightarrow \mathrm{Gr}(3, V)$); then Items (a) and (b) hold. The result follows because $SL(V)$ acts transitively on the family of couples (U, Θ) such that Items (a) and (b) hold. $\square$

Proof of **Theorem 2.26**. By Morin one of (a) - (e) of **Theorem 1.12** holds. We will perform a case-by-case analysis.

(a): $\Theta \subset F_\pm(\mathcal{Q})$. Let U be a 4-dimensional complex vector-space and identify V with $\bigwedge^2 U$ so that $\mathcal{Q}$ gets identified with $\mathrm{Gr}(2, U)$. We may assume that $\Theta \subset F_+(\mathcal{Q})$; thus $\Theta := i_+(Z)$ for an irreducible surface $Z \subset \mathbb{P}(U)$. By (1.5.5) the map i_+ is defined by the linear system of quadrics in $\mathbb{P}(U)$. It follows that Z is contained in a quadric of $\mathbb{P}(U)$ and hence is a plane or a quadric, thus $A \in (\mathbb{X}_\mathcal{Y} \cup \mathbb{X}_\mathcal{W})$.

(b): $\Theta \subset C(\mathcal{V})$ or $\Theta \subset T(\mathcal{V})$. Then $A \in (\mathbb{X}_k \cup \mathbb{X}_h)$.

(c): $\Theta\subset J_{v_0}$. Let $V_0\in\mathrm{Gr}(5,V)$ be transversal to $[v_0]$ and $\overline{\rho}_{v_0}$ be given by (1.4.6). Let $S:=\overline{\rho}_{v_0}(\Theta)\subset\mathrm{Gr}(2,V_0)$ and $\Lambda:=\langle S\rangle$. Then the hypotheses of **Proposition 2.31** are satisfied and hence one of Items (1)-(4) of that proposition holds. If Item (1) holds then Θ is plane: as is easily checked

$$\text{if } \Theta_A \text{ contains a plane then } A\in(\mathbb{X}_{\mathcal{F}_{1,+}}\cup\mathbb{X}_{\mathcal{F}_{2,+}}). \tag{2.4.37}$$

If Item (2) of **Proposition 2.31** holds then $A\in\mathbb{X}_{\mathcal{D}_+}$. Suppose that Item (3) of **Proposition 2.31** holds. Let $V_{12}\subset V_0$ be the subspace of dimension 2 such that $\ell_0=\mathbb{P}(V_{12})$ and $V_{35}\subset V_0$ be a subspace complementary to V_{12}. Then $S\subset\mathbb{P}(\bigwedge^2 V_{12}\oplus V_{12}\wedge V_{35})$ and hence

$$\Theta\subset\mathbb{P}([v_0]\wedge\bigwedge^2 V_{12}\oplus[v_0]\wedge V_{12}\wedge V_{35}).$$

Since $\dim\langle\Theta\rangle=\dim\langle S\rangle=4$ we get that $A\in\mathbb{X}_{\mathcal{E}_{2,+}}$. If Item (4) of **Proposition 2.31** holds then $A\in\mathbb{X}_{\mathcal{A}_+}$.

(d): $\Theta\subset\mathrm{Gr}(3,E)$ where $E\in\mathrm{Gr}(5,V)$. By Case (c) and duality we get that

$$A\in\mathbb{X}_{\mathcal{F}_{1,+}}\cup\mathbb{X}_{\mathcal{F}_{2,+}}\cup\mathbb{X}_{\mathcal{D}_+}\cup\mathbb{X}_{\mathcal{E}^\vee_{2,+}}\cup\mathbb{X}_{\mathcal{A}^\vee_+}.$$

(e): $\Theta\subset I_U$ where $U\in\mathrm{Gr}(3,V)$. Since $\dim\Theta=2$ we have $2\le\dim\langle\Theta\rangle$. If $\dim\langle\Theta\rangle=2$ then Θ is plane: by (2.4.37) we have $A\in(\mathbb{X}_{\mathcal{F}_{1,+}}\cup\mathbb{X}_{\mathcal{F}_{2,+}})$. Thus from now on we may suppose that

$$3\le\dim\langle\Theta\rangle. \tag{2.4.38}$$

We distinguish between the two cases:

(e1) $U\in\langle\Theta\rangle$,

(e2) $U\notin\langle\Theta\rangle$.

Suppose that (e1) holds; we will prove that

$$A\in(\mathbb{X}_{\mathcal{C}_{1,+}}\cup\mathbb{X}_{\mathcal{D}_+}). \tag{2.4.39}$$

Since $U\in\langle\Theta\rangle$ and Θ is an irreducible component of $\langle\Theta\rangle\cap I_U$ we get that Θ is a cone with vertex U. If $4\le\dim\langle\Theta\rangle$ then $A\in\mathbb{X}_{\mathcal{C}_{1,+}}$. Thus we may assume that

$$\dim\langle\Theta\rangle=3. \tag{2.4.40}$$

Let $\overline{\rho}_U$ be the map of (1.4.11). Then $C:=\overline{\rho}_U(\Theta)$ is a 1-dimensional irreducible component of $\langle C\rangle\cap(\mathbb{P}(\bigwedge^2 U)\times\mathbb{P}(V/U))$. By (2.4.40) we have $\dim\langle C\rangle=2$; thus C is a smooth conic because $\mathbb{P}(\bigwedge^2 U)\times\mathbb{P}(V/U)$ is cut out by quadrics (in $\mathbb{P}(\bigwedge^2 U\otimes(V/U))$). Let $f\colon C\to\mathbb{P}(\bigwedge^2 U)$ and $g\colon C\to\mathbb{P}(V/U)$ be the projection maps. Neither f nor g is constant because C is a 1-dimensional irreducible component of $\langle C\rangle\cap(\mathbb{P}(\bigwedge^2 U)\times\mathbb{P}(V/U))$; thus

$$f^*\mathcal{O}_{\mathbb{P}(\bigwedge^2 U)}(1)\cong g^*\mathcal{O}_{\mathbb{P}(V/U)}(1)\cong\mathcal{O}_C(1). \tag{2.4.41}$$

It follows that $\mathrm{im}(f)$ and $\mathrm{im}(g)$ are lines and hence there exist $[v_0]\in\mathbb{P}(U)$ and $W_2\in\mathrm{Gr}(2,V/U)$ such that

$$\mathrm{im}(f)=\mathbb{P}(\{v_0\wedge u\mid u\in U\}),\qquad \mathrm{im}(g)=\mathbb{P}(W_2). \tag{2.4.42}$$

Let $U_0\subset U$ be complementary to $[v_0]$. Let $V_{14}\subset V$ be the 4-dimensional subspace containing U_0 and projecting to W_2 under the quotient map $V\to V/U$. Let $\mathsf{F}:=\{v_0,v_1,\ldots,v_5\}$ be a basis of V adapted to V_{14} i.e. $V_{14}=\langle v_1,\ldots,v_4\rangle$: then $A\in\mathbb{X}^{\mathsf{F}}_{\mathcal{D}_+}$. This finishes the proof that if Item (e1) above holds then (2.4.39) holds. Next suppose that Item (e2) holds. Let $W\subset V$ be a subspace complementary to U; thus

$$\langle\Theta\rangle\subset\mathbb{P}(\bigwedge^3 U\oplus\bigwedge^2 U\wedge W). \tag{2.4.43}$$

Let $W_1 := \bigwedge^2 U$, $W_2 := W$, $S := \overline{\rho}_U(\Theta) \subset \mathbb{P}(W_1) \times \mathbb{P}(W_2)$ and $\Lambda := \langle S \rangle$; then the hypotheses of **Proposition 2.33** are satisfied. Moreover since Item (e2) holds we have $\dim \Lambda = \dim\langle\Theta\rangle$. By (2.4.38) one of Items (2)-(5) of **Proposition 2.33** holds. If Item (2) holds then $A \in \mathbb{X}_{\mathcal{D}_+}$. If Item (3) of **Proposition 2.33** holds then $A \in \mathbb{X}_{\mathcal{E}_{2,+}}$. If Item (4) of **Proposition 2.33** holds then $A \in \mathbb{X}_{\mathcal{Y}}$ by **Claim 2.34**. If Item (5) of **Proposition 2.33** holds then $A \in_{\mathcal{C}_{2,+}}$. $\square$

Proposition 2.35. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that Θ_A contains an irreducible component of dimension 2. Then there exists a Type X appearing in Table (2) such that $A \in \mathbb{B}_X$.*

Proof. By **Theorem 2.26** we know that A belongs to the right-hand side of (2.4.1). By **Remark 2.25** there exists a Type X in Table (2) such that $A \in \mathbb{B}_X$ except possibly if $A \in \mathbb{B}_{\mathcal{C}_{1,+}}$. It is not true that $\mathbb{B}_{\mathcal{C}_{1,+}}$ is contained in on of the $\mathbb{B}_X$ - in fact Θ_A is a singleton for generic $A \in \mathbb{B}_{\mathcal{C}_{1,+}}$. However going through the proof of **Theorem 2.26** we see that the only instance in which $A \in \mathbb{B}_{\mathcal{C}_{1,+}}$ corresponds to Item (e1), see (2.4.39). Let A_0 be as in Item (e1) with $A_0 \in \mathbb{B}_{\mathcal{C}_{1,+}}$; thus Θ_A contains a 2-dimensional irreducible component $\Theta \subset I_U$, where $U \in \mathrm{Gr}(3, V)$, Θ is a cone with vertex U and $4 \le \dim\langle\Theta\rangle$. Let $H \subset \langle\Theta\rangle$ be a generic codimension-1 linear subspace, in particular it does not contain U and $\dim(H \cap \Theta) = 1$. Then A_0 is in the family $\mathcal{F}$ of $A \in \mathbb{LG}(\bigwedge^3 V)$ containing $\langle\langle H \rangle\rangle$; if $A \in \mathcal{F}$ is generic then $H \cap \Theta$ is a 1-dimensional irreducible component of Θ_A and hence A belongs to right-hand side of (2.4.1). Since the $\mathbb{B}_X$ are closed we get that A_0 is contained in one of the $\mathbb{B}_X$. Actually going through the proof of**Theorem 2.9** we get that $A_0 \in \mathbb{B}_{\mathcal{E}_2} \cup \mathbb{B}_{\mathcal{E}_2^\vee} \cup \mathbb{B}_{\mathbf{Q}} \cup \mathbb{B}_{\mathcal{C}_2}$. $\square$

2.5 Higher-dimensional components of Θ_A

We let

$$\mathbb{X}_+ := \overline{PGL(V)A_+(U)}$$

where $A_+(U)$ is given by (1.5.3).

Theorem 2.36. *Suppose that $A \in \mathbb{LG}(\bigwedge^3 V)$ and that $\dim \Theta_A \ge 3$. then*

$$A \in (\mathbb{X}_{\mathcal{A}_+} \cup \mathbb{X}_{\mathcal{A}_+^\vee} \cup \mathbb{X}_{\mathcal{C}_{1,+}} \cup \mathbb{X}_{\mathcal{C}_{2,+}} \cup \mathbb{X}_{\mathcal{D}} \cup \mathbb{X}_{\mathcal{F}_{1,+}} \cup \mathbb{X}_+). \tag{2.5.1}$$

Proof. Since $3 \le \dim\Theta$ we have $3 \le \dim\langle\Theta\rangle$. If the latter is an equality then Θ is a 3-dimensional projective space and thus $A \in \mathbb{X}_{\mathcal{F}_{1,+}}$. Thus from now on we may suppose that

$$4 \le \dim\langle\Theta\rangle. \tag{2.5.2}$$

By Morin one of (a), (c), (d), (e) of **Theorem 1.12** holds (notice that $\dim C(\mathcal{V}) = \dim T(\mathcal{V}) = 2$). We will perform a case-by-case analysis.

(a): $\Theta \subset F_\pm(\mathcal{Q})$. In this case $A \in \mathbb{X}_+$.

(c): $\Theta \subset J_{v_0}$. If $5 \le \dim\langle\Theta\rangle$ then $A \in \mathbb{X}_{\mathcal{A}_+}$. Thus we may assume that $\dim\langle\Theta\rangle = 4$. Let $V_0 \in \mathrm{Gr}(5, V)$ be transversal to $[v_0]$ and $\overline{\rho}_{v_0}$ be as in (1.4.6). By (2.5.2) we have $\dim\langle\overline{\rho}_{v_0}(\Theta)\rangle = 4$. Since $\mathrm{Gr}(2, V_0)$ contains no projective space of dimension 4 we get that $\overline{\rho}_{v_0}(\Theta)$ is 3-dimensional. By **Proposition 2.27** there exists $U \in \mathrm{Gr}(4, V_0)$ such that $\overline{\rho}_{v_0}(\Theta) \subset \mathrm{Gr}(2, U)$; it follows that $A \in \mathbb{X}_{\mathcal{D}_+}$.

(d): $\Theta \subset \mathrm{Gr}(3, E)$ where $E \in \mathrm{Gr}(5, V)$. By duality and Case (d) we get that $\in (\mathbb{X}_{\mathcal{A}_+}^\vee \cup \mathbb{X}_{\mathcal{D}_+})$.

(e): $\Theta \subset I_U$ where $U \in \mathrm{Gr}(3, V)$. We distinguish the two cases:

(i) $U \in \langle\Theta\rangle$,

(ii) $U \notin \langle\Theta\rangle$.

Suppose that (i) holds. Since Θ is an irreducible component of $\langle\Theta\rangle \cap \mathrm{Gr}(3, V)$ we get that $U \in \Theta$ and moreover Θ is a cone with vertex U. Since we are under the assumption that (2.5.2) holds it follows that $A \in \mathbb{X}_{\mathcal{C}_{1,+}}$. Next suppose that (ii) holds. There exists a subspace $W \subset V$ complementary to U and such that

$$\langle\Theta\rangle \subset \mathbb{P}(\bigwedge^3 U \oplus \bigwedge^2 U \wedge W). \tag{2.5.3}$$

Moreover since $U \notin \langle\Theta\rangle$ we have $\dim\langle\overline{\rho}_U(\Theta)\rangle = \dim\langle\Theta\rangle$ where $\overline{\rho}_U$ be as in (1.4.11). If $6 \le \dim\langle\Theta\rangle$ we get that $A \in \mathbb{X}_{\mathcal{C}_{2,+}}$ by (2.5.3). Suppose that $\dim\langle\Theta\rangle \le 5$. Then $\overline{\rho}_U(\Theta)$ is an effective Cartier divisor on $PP(\bigwedge^2 U) \times \mathbb{P}(W)$ contained in 3 linearly independent divisors of the linear system $|\mathcal{O}_{\mathbb{P}(\bigwedge^2 U)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(W)}(1)|$. It follows that

(α) $\overline{\rho}_U(\Theta) \in |\mathcal{O}_{\mathbb{P}(\bigwedge^2 U)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(W)}|$, or

(β) $\overline{\rho}_U(\Theta) \in |\mathcal{O}_{\mathbb{P}(\bigwedge^2 U)} \boxtimes \mathcal{O}_{\mathbb{P}(W)}(1)|$.

Notice that in both cases Θ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ embedded by Segre and hence

$$5 = \dim\langle\overline{\rho}_U(\Theta)\rangle = \dim\langle\Theta\rangle. \tag{2.5.4}$$

Suppose that (α) holds. We have a canonical identification $\mathbb{P}(U) = |\mathcal{O}_{\mathbb{P}(\bigwedge^2 U)}(1)|$; let $[v_0] \in \mathbb{P}(U)$ be the point giving the divisor in $|\mathcal{O}_{\mathbb{P}(\bigwedge^2 U)}(1)|$ corresponding to Θ. Then $\Theta \subset J_{v_0}$; since (2.5.4) holds we get that $A \in \mathbb{X}_{\mathcal{A}_+}$. Similarly one shows that if (β) holds then $A \in \mathbb{X}_{\mathcal{A}_+^\vee}$. $\square$

Theorem 2.37. *The irredundant irreducible decomposition of Σ_∞ is given by*

$$\Sigma_\infty = \mathbb{B}_{\mathcal{A}} \cup \mathbb{B}_{\mathcal{A}^\vee} \cup \mathbb{B}_{\mathcal{C}_2} \cup \mathbb{B}_{\mathcal{D}} \cup \mathbb{B}_{\mathcal{E}_2} \cup \mathbb{B}_{\mathcal{E}_2^\vee} \cup \mathbb{B}_{\mathcal{F}_1} \cup \mathbb{B}_{\mathbf{Q}} \cup \mathbb{B}_{\mathbf{R}} \cup \mathbb{B}_{\mathbf{S}} \cup \mathbb{B}_{\mathbf{T}} \cup \mathbb{B}_{\mathbf{T}^\vee} \tag{2.5.5}$$

Proof. By **Corollary 2.21** all we have to prove is that Σ_∞ is contained in the right-hand side of (2.5.5). Let $A \in \Sigma_\infty$; thus $\dim\Theta_A \ge 1$. If $\dim\Theta_A = 1$ then A belongs to the right-hand side of (2.5.5) by **Theorem 2.9**. If $\dim\Theta_A = 2$ then A belongs to the right-hand side of (2.5.5) by **Theorem 2.26** and **Proposition 2.35**. Lastly suppose that $\dim\Theta_A \ge 3$; then A belongs to the right-hand side of (2.5.1) by **Theorem 2.36**. By **Remark 2.25** we get that A belongs to the right-hand side of (2.5.5) except possibly if $A \in \mathbb{B}_{\mathcal{C}_{1,+}}$; arguing as in the proof of **Proposition 2.35** we get that A belongs to the right-hand side of (2.5.5) in that case as well. $\square$

3 Appendix: Quadratic forms

Let U be a complex vector-space of finite dimension d. We view $\operatorname{Sym}^2 U^\vee$ as the vector-space of quadratic forms on U. We will recall a few standard results regarding the loci

$$(\operatorname{Sym}^2 U^\vee)_r := \{q \in \operatorname{Sym}^2 U^\vee \mid \operatorname{rk} q \le r\}, \qquad 0 \le r \le d. \tag{3.0.1}$$

The following is well-known.

Proposition 3.1. *Keep notation as above. Then $(\operatorname{Sym}^2 U^\vee)_r$ is a closed irreducible set of codimension $\binom{d-r+1}{2}$ in $\operatorname{Sym}^2 U^\vee$, smooth away from $(\operatorname{Sym}^2 U^\vee)_{r-1}$. Let $q_* \in ((\operatorname{Sym}^2 U^\vee)_r \setminus (\operatorname{Sym}^2 U^\vee)_{r-1})$ and $K := \ker q_*$; the (embedded) tangent space to $(\operatorname{Sym}^2 U^\vee)_r$ at q_* is*

$$T_{q_*}(\operatorname{Sym}^2 U^\vee)_r = \{q \in \operatorname{Sym}^2 U^\vee \mid q|_K = 0\}. \tag{3.0.2}$$

By **Proposition 3.1** the normal cone of $(\operatorname{Sym}^2 U^\vee)_r$ in $\operatorname{Sym}^2 U^\vee$ is a vector-bundle away from $(\operatorname{Sym}^2 U^\vee)_{r-1}$ and if $q_* \in ((\operatorname{Sym}^2 U^\vee)_r \setminus (\operatorname{Sym}^2 U^\vee)_{r-1})$ then we have a canonical isomorphism

$$\begin{array}{ccc} (C_{(\operatorname{Sym}^2 U^\vee)_r} \operatorname{Sym}^2 U^\vee)_{q_*} & \xrightarrow{\sim} & \operatorname{Sym}^2 K^\vee \\ {[q]} & \mapsto & q|_K. \end{array} \tag{3.0.3}$$

Next we consider $(\operatorname{Sym}^2 U^\vee)_{d-1}$ i.e. the locus defined by vanishing of the determinant of q. Given $q_* \in \operatorname{Sym}^2 U^\vee$ we let Φ be the polynomial on the vector-space $\operatorname{Sym}^2 U^\vee$ defined by $\Phi(q) := \det(q_* +$

q). Of course Φ is defined up to multiplication by a non-zero scalar, moreover it depends on q_* although that does not show up in the notation. Let

$$\Phi = \Phi_0 + \Phi_1 + \ldots + \Phi_d, \qquad \Phi_i \in \mathrm{Sym}^i(\mathrm{Sym}^2 U) \tag{3.0.4}$$

be the decomposition into homogeneous components. The result below follows from a straightforward computation.

Proposition 3.2. *Let $q_* \in \mathrm{Sym}^2 U^\vee$ and*

$$K := \ker(q_*), \qquad k := \dim K. \tag{3.0.5}$$

Let Φ_i be the polynomials appearing in (3.0.4)*. Then*

(1) $\Phi_i = 0$ for $i < k$, and

(2) there exists $c \neq 0$ such that $\Phi_k(q) = c \det(q|_K)$.

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