Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics

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Abstract

Eisenbud Popescu and Walter have constructed certain special sextic hypersurfaces in $\mathbb{P}^5$ as Lagrangian degeneracy loci. We prove that the natural double cover of a generic EPW-sextic is a deformation of the Hilbert square of a $K3$ surface $(K3)^{[2]}$ and that the family of such varieties is locally complete for deformations that keep the hyperplane class of type $(1,1)$ - thus we get an example similar to that (discovered by Beauville and Donagi) of the Fano variety of lines on a cubic 4-fold. Conversely suppose that X is a numerical $(K3)^{[2]}$, that H is an ample divisor on X of square 2 for Beauville's quadratic form and that the map $X \dashrightarrow |H|^\vee$ is the composition of the quotient $X \to Y$ for an anti-symplectic involution on X followed by an immersion $Y \hookrightarrow |H|^\vee$; then Y is an EPW-sextic and $X \to Y$ is the natural double cover.

1 Introduction

A compact Kähler manifold is irreducible symplectic if it is simply connected and it carries a holomorphic symplectic form spanning $H^{2,0}$ (see [1, 6]). An irreducible symplectic surface is nothing else but a $K3$ surface. Higher-dimensional irreducible symplectic manifolds behave like $K3$ surfaces in many respects [6, 7] however their classification up to deformation of complex structure is out of reach at the moment. Let S be a $K3$; the Hilbert square $S^{[2]}$ i.e. the blow-up of the diagonal in the symmetric square $S^{(2)}$ is the symplest example of an irreducible symplectic 4-fold. An irreducible symplectic 4-fold M is a *numerical* $(K3)^{[2]}$ if there exists an isomorphism of abelian groups

$$\psi\colon H^2(M;\mathbb{Z}) \xrightarrow{\sim} H^2(S^{[2]};\mathbb{Z}) \tag{1.1}$$

such that

$$\int_M \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4 \tag{1.2}$$

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for all $\alpha \in H^2(M;\mathbb{Z})$[1]. In [15] we studied numerical $(K3)^{[2]}$'s with the goal of classifying them up to deformation of complex structure. We proved that any numerical $(K3)^{[2]}$ is deformation equivalent to an X carrying an ample divisor H such that

$$\int_X c_1(H)^4 = 12, \quad \dim |H| = 5 \tag{1.3}$$

(the first equation is equivalent to $c_1(H)$ being of square 2 for the Beauville form), and such that the rational map

$$X \dashrightarrow |H|^\vee \tag{1.4}$$

satisfies one of the following two conditions:

(a) There exist an anti-symplectic involution $\phi\colon X \to X$ with quotient Y and an embedding $Y \hookrightarrow |H|^\vee$ such that (1.4) is the composition of the quotient map $f\colon X \to Y$ and the embedding $Y \hookrightarrow |H|^\vee$.

(b) Map (1.4) is birational onto a hypersurface of degree between 6 and 12.

In this paper we describe explicitely all the X occuring in Item (a) above. Notice that Y is singular because smooth hypersurfaces in $\mathbb{P}^5$ are simply connected. Moreover the singular locus is a surface because ϕ is anti-symplectic. Thus Y is far from being a generic sextic hypersurface; we will show that it belongs to a family of sextics constructed by Eisenbud, Popescu and Walter, see Example (9.3) of [4]. We will prove that conversely a generic EPW-sextic has a natural double cover which is a deformation of $(K3)^{[2]}$. Since EPW-sextics form an irreducible family we get that the X's satisfying (a) above are deformation equivalent. Actually if $(X_i, f_i^*\mathcal{O}_{Y_i}(1))$ are two polarized couples where $f_i\colon X_i \to Y_i$ satisfy (a) above for $i = 1, 2$ then we may deform $(X_1, f_1^*\mathcal{O}_{Y_1}(1))$ to $(X_2, f_2^*\mathcal{O}_{Y_2}(1))$ through polarized deformations. In particular all the explicit examples of $f\colon X \to Y$ satisfying (a) above that were constructed in [14] are equivalent through polarized deformations - this answers positively a question raised in Section (6) of [14]. We recall that no examples are known of X satisfying Item (b) above; in [15] we conjectured that they do not exist - one result in favour of the conjecture is that if X satisfies (a) above then all small deformations of X keeping $c_1(f^*\mathcal{O}_Y(1))$ of type (1,1) also satisfy (a) (see the proposition at the end of Section (4) of [15]). If our conjecture is true then the results of this paper together with the quoted results of [15] give that numerical $(K3)^{[2]}$'s are deformation equivalent to the Hilbert square of a $K3$.

Before stating precisely our main results we recall the construction of EPW-sextics. Let V be a 6-dimensional vector space and $\mathbb{P}(V)$ be the projective space of 1-dimensional sub vector spaces $\ell \subset V$. Choose an isomorphism $vol\colon \wedge^6 V \xrightarrow{\sim} \mathbb{C}$ and let σ be the symplectic form on $\wedge^3 V$ defined by wedge product, i.e. $\sigma(\alpha, \beta) := vol(\alpha \wedge \beta)$; thus $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ has the structure of a symplectic vector-bundle of rank 20. Let F be the sub-vector-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ with fiber over $\ell \in \mathbb{P}(V)$ equal to

$$F_\ell := Im\left(\ell \otimes \wedge^2(V/\ell) \hookrightarrow \wedge^3 V\right). \tag{1.5}$$

[1]For the experts: the Beauville quadratic form and the Fujiki constant of M are the same as those of $S^{[2]}$.

Thus we have an exact sequence

$$0 \to F \xrightarrow{\nu} \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{1.6}$$

We have $rk(F) = 10$ and $\sigma|_{F_\ell} = 0$; thus F is a Lagrangian sub-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$. Let $\mathbb{LG}(\wedge^3 V)$ be the symplectic Grassmannian parametrizing σ-Lagrangian subspaces of $\wedge^3 V$. For $A \in \mathbb{LG}(\wedge^3 V)$ we let

$$\lambda_A \colon F \longrightarrow (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{1.7}$$

be Inclusion (1.6) followed by the projection $(\wedge^3 V) \otimes \mathcal{O}_{\mathbb{P}(V)} \to (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)}$. Since the vector-bundles appearing in (1.7) have equal rank we have $\det(\lambda_A) \in H^0((\det F)^{-1})$. We let $Y_A \subset \mathbb{P}(V)$ be the zero-scheme of $\det(\lambda_A)$. Let $\omega := c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$; a straightforward computation gives that

$$c(F) = 1 - 6\omega + 18\omega^2 - 34\omega^3 + \dots \tag{1.8}$$

In particular $\det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6)$. Thus Y is always non-empty and if $Y \neq \mathbb{P}(V)$ then Y is a sextic hypersurface. An *EPW-sextic* is a hypersurface in $\mathbb{P}(V)$ which is equal to Y_A for some $A \in \mathbb{LG}(\wedge^3 V)$. In Section (2) we describe explicitely the non-empty Zariski-open $\mathbb{LG}(\wedge^3 V)^0 \subset \mathbb{LG}(\wedge^3 V)$ parametrizing A such that the following hold: Y_A is a sextic hypersurface smooth at all points where the map λ_A of (1.7) has corank 1, the analytic germ (Y_A, ℓ) at a point ℓ where λ_A has corank 2 is isomorphic to the product of a smooth 2-dimensional germ times an A_1-singularity and furthermore λ_A has corank at most 2 at all points of $\mathbb{P}(V)$. Let $A \in \mathbb{LG}(\wedge^3 V)^0$; then Y_A supports a quadratic sheaf as defined by Casnati and Catanese [3] and hence there is a natural double cover $X_A \to Y_A$ with X_A smooth - see Section (4). In Section (5) we will prove the following result.

Theorem 1.1. *Keep notation as above. Then the following hold.*

(1) Suppose that X, H are a numerical $(K3)^{[2]}$ and an ample divisor on X such that both (1.3) and Item (a) above hold. Then there exists $A \in \mathbb{LG}(\wedge^3 V)^0$ such that $f \colon X \to Y$ is identified with the natural double cover $X_A \to Y_A$.

(2) For $A \in \mathbb{LG}(\wedge^3 V)^0$ let $X_A \to Y_A$ be the natural double cover defined in Section (4) and let H_A be the pull-back to X_A of $\mathcal{O}_{Y_A}(1)$. Then X_A is an irreducible symplectic variety deformation equivalent to $(K3)^{[2]}$ and both (1.3) and Item (a) above hold with $X = X_A$ and $H = H_A$.

In particular by stability of (X, H) where X is a numerical $(K3)^{[2]}$ and H is an ample divisor satisfying (1.3) and (a) above (see the proposition at the end of Section (4) of [15]) the family of X_A's that one gets by letting A vary in $\mathbb{LG}(\wedge^3 V)^0$ is a locally complete family of polarized deformations of $(K3)^{[2]}$. This may be confirmed by the following computation. The tangent space to $\mathbb{LG}(\wedge^3 V)$ at a point A is isomorphic to $Sym_2 A^\vee$ and hence $\dim \mathbb{LG}(\wedge^3 V) = 55$. Since $\dim \mathbb{PGL}(V) = 35$ we get that $\dim(\mathbb{LG}(\wedge^3 V)^0 // \mathbb{PGL}(V)) = 20$ which is the number of moduli of a polarized deformation of $(K3)^{[2]}$. (We will show that EPW-sextics parametrized by $\mathbb{LG}(\wedge^3 V)^0$ are $\mathbb{PGL}(V)$-stable, see Proposition (6.1), hence the GIT-notation is appropriate.) Beauville and Donagi [2] described another locally complete family of polarized $(K3)^{[2]}$'s: if $Z \subset \mathbb{P}^5$ is a smooth cubic hypersurface the Fano variety $F(Z)$ parametrizing lines on

Z is a deformation of $(K3)^{[2]}$. A difference between our varieties and those of Beauville-Donagi is the following: the first Chern class of the Plücker line-bundle on $F(Z)$ has square 6 for Beauville's quadratic form while if X_A is as above and $f\colon X_A \to \mathbb{P}^5$ is the natural map then $c_1(f^*\mathcal{O}_{\mathbb{P}^5(1)}(1))$ has square 2. I know of no other explicit examples of a locally complete family of higher dimensional polarized irreducible symplectic varieties except possibly for the family described by A. Iliev - K. Ranestad [8]. We notice that the conjecture stated in [15] and mentioned above amounts to the statement that the familiy of X_A's is globally complete once we take into account the limiting X_A's one gets for $A \in \left(\mathbb{LG}(\wedge^3 V) \setminus \mathbb{LG}(\wedge^3 V)^0\right)$.

An interesting feature of EPW-sextics is that they are preserved by the duality map i.e. the dual of an EPW-sextic Y_A is an EPW-sextic, see Section (3). Thus duality defines a regular involution on an open dense subset of the moduli space of numerical $(K3)^{[2]}$'s X polarized by a divisor H satisfying (1.3)[2] and Item (a) above; this is discussed in Section (6).

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2 EPW-sextics

We will explicitely describe those EPW-sextics whose only singularities are the expected ones - the main result is Proposition (2.8). We start by recalling (see [4, 5]) how one defines natural subschemes $D_i(A,F) \subset \mathbb{P}(V)$ such that

$$suppD_i(A,F) = \{\ell \in \mathbb{P}(V) | \quad \dim(F_\ell \cap A) \geq i\}. \tag{2.1}$$

Definition 2.1. *Let $U \subset \mathbb{P}(V)$ be an open subset. A symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$ consists of a couple $(\mathcal{L},\mathcal{H})$ of trivial transversal Lagrangian sub-vector-bundles $\mathcal{L},\mathcal{H} \subset \wedge^3 V \otimes \mathcal{O}_U$.*

Let $(\mathcal{L},\mathcal{H})$ be a symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$. The symplectic form σ defines an isomorphism

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\sim} & \mathcal{L}^\vee \\ \alpha & \mapsto & \sigma(\alpha,\cdot) \end{array} \tag{2.2}$$

and hence we get a direct sum decomposition

$$\mathcal{L} \oplus \mathcal{L}^\vee = \wedge^3 V \otimes \mathcal{O}_U. \tag{2.3}$$

Conversely a direct sum decomposition (2.3) with $\mathcal{L},\mathcal{L}^\vee$ trivial Lagrangian sub-vector-bundles such that σ induces the tautological isomorphism $\mathcal{L}^\vee \xrightarrow{\sim} \mathcal{L}^\vee$ gives a symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$; this is how we usually present a symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$.

Claim 2.2. *Let $A \in \mathbb{LG}(\wedge^3 V)$ and $\ell_0 \in \mathbb{P}(V)$. There exists an open affine $U \subset \mathbb{P}(V)$ containing ℓ_0 and a symplectic trivialization (2.3) of $\wedge^3 V \otimes \mathcal{O}_U$ such that for every $\ell \in U$ both A and F_ℓ are transversal to $\mathcal{L}_\ell^\vee$.*

[2] The second equation of (1.3) follows from the first equation - see [15].

Proof. The set of Lagrangian subspaces of $(\wedge^3 V)$ which are transversal to a given Lagrangian subspace is an open dense subset of $\mathbb{LG}(\wedge^3 V)$. Thus there exists $C \in \mathbb{LG}(\wedge^3 V)$ which is transversal both to A and to F_{ℓ_0}. Since the condition of being transversal is open there exists an open affine $U \subset \mathbb{P}(V)$ containing ℓ_0 such that F_ℓ is transversal to C for all $\ell \in U$. Let $B \in \mathbb{LG}(\wedge^3 V)$ be transversal to C; thus

$$\wedge^3 V = B \oplus C \tag{2.4}$$

and hence $(B \otimes \mathcal{O}_U, C \otimes \mathcal{O}_U)$ is a symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$. Letting $\mathcal{L} := B \otimes \mathcal{O}_U$ we have an isomorphism $\mathcal{L}^\vee \cong C \otimes \mathcal{O}_U$ induced by σ and we write the chosen symplectic trivialization as (2.3); by construction both A and F_ℓ are transversal to $\mathcal{L}_\ell^\vee$ for every $\ell \in U$. $\square$

Choose $A \in \mathbb{LG}(\wedge^3 V)$. Let $\ell_0 \in \mathbb{P}(V)$. Assume that we have U and a symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$ as in the claim above. We will define a closed degeneracy subscheme $D_i(A, F, U, \mathcal{L}, \mathcal{L}^\vee) \subset U$ - after doing this we will define the subcheme $D_i(A, F) \subset \mathbb{P}(V)$ by gluing together the local degeneracy loci. Via (2.3) we may identify both $A \otimes \mathcal{O}_U$ and $F|_U$ with the graphs of maps

$$q_A, q_F \colon \mathcal{L} \to \mathcal{L}^\vee \tag{2.5}$$

because for every $\ell \in U$ both A and F_ℓ are transversal to $\mathcal{L}_\ell^\vee$. Since both $A \otimes \mathcal{O}_U$ and $F|_U$ are Lagrangian sub-vector-bundles the maps q_A and q_F are symmetric. Choosing a trivialization of $\mathcal{L}$ we view q_A, q_F as symmetric 10×10 matrices with entries in $\mathbb{C}[U]$.

Definition 2.3. *Keep notation as above. We let $D_i(A, F, U, \mathcal{L}, \mathcal{L}^\vee) \subset U$ be the closed subscheme defined by the vanishing of determinants of $(11-i) \times (11-i)$-minors of $(q_A - q_F)$.*

Notice that the definition above makes sense because if we change the trivialization of $\mathcal{L}$ the relevant determinants are multiplied by units of $\mathbb{C}[U]$

Lemma 2.4. *Let $U_1, U_2 \subset \mathbb{P}(V)$ be open affine and $\mathcal{L}_j \oplus \mathcal{L}_j^\vee = \wedge^3 V \otimes \mathcal{O}_{U_j}$ be symplectic trivializations. Then*

$$D_i(A, F, U_1, \mathcal{L}_1, \mathcal{L}_1^\vee) \cap U_1 \cap U_2 = D_i(A, F, U_2, \mathcal{L}_2, \mathcal{L}_2^\vee) \cap U_1 \cap U_2. \tag{2.6}$$

Proof. It suffices to prove the lemma for $U_1 = U_2 = U$. The constructions above can be carried out more generally for a trivialization $\mathcal{V} \oplus \mathcal{W} \cong \wedge^3 V \otimes \mathcal{O}_U$ where $\mathcal{V}, \mathcal{W} \subset \wedge^3 V \otimes \mathcal{O}_U$ are trivial rank-10 sub-vector-bundles (not necessarily Lagrangian) such that for every $\ell \in U$ both A and F_ℓ are transversal to $\mathcal{W}_\ell$. We identify $A \otimes \mathcal{O}_U$ and $F|_U$ with the graphs of maps $q_A \colon \mathcal{V} \to \mathcal{W}$ and $q_F \colon \mathcal{V} \to \mathcal{W}$ respectively - notice that in general it does not make sense to ask whether q_A, q_F are symmetric! Trivializing $\mathcal{V}$ and $\mathcal{W}$ we view q_A, q_F as 10×10 matrices with entries in $\mathbb{C}[U]$. We let $D_i(A, F, U, \mathcal{V}, \mathcal{W}) \subset U$ be the subscheme defined by the vanishing of determinants of $(11-i) \times (11-i)$-minors of $(q_A - q_F)$. One checks easily that if we change $\mathcal{V}$ (leaving $\mathcal{W}$ fixed) or if we change $\mathcal{W}$ (leaving $\mathcal{V}$ fixed) the scheme $D_i(A, F, U, \mathcal{V}, \mathcal{W})$ remains the same. The lemma follows immediately. $\square$

Now we define $D_i(A, F)$. Consider the collection of symplectic trivializations $\mathcal{L}_j \oplus \mathcal{L}_j^\vee = \wedge^3 V \otimes \mathcal{O}_{U_j}$ with $U_j \subset \mathbb{P}(V)$ open affine and the corresponding closed subschemes $D_i(A, F, U_j, \mathcal{L}_j, \mathcal{L}_j^\vee)$. By Claim (2.2) the subsets U_j cover $\mathbb{P}(V)$ and by Lemma (2.4) the $D_i(A, F, U_j, \mathcal{L}_j, \mathcal{L}_j^\vee)$ for different j's match on overlaps; thus they glue together and they define a closed subscheme $D_i(A, F) \subset \mathbb{P}(V)$. Clearly (2.1) holds and furthermore $D_{i+1}(A, F)$ is a subscheme of $D_i(A, F)$. We claim that

$$Y_A := D_1(A, F). \tag{2.7}$$

It is clear from (2.1) that $supp(Y_A) = supp(D_1(A, F))$ and hence we need only check that the scheme structures coincide in a neighborhood of any point $\ell_0 \in D_1(A, F)$. There exists $B \in \mathbb{L}G(\wedge^3 V)$ which is transversal both to F_{ℓ_0} and A. There is an open neighborhood U of ℓ_0 such that B is transversal to F_ℓ for $\ell \in U$. The symplectic form σ defines an isomorphism $B \cong A^\vee$. Consider the symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$ given by $\mathcal{L} := A \otimes \mathcal{O}_U$ and $\mathcal{L}^\vee := B \otimes \mathcal{O}_U$; since $q_A = 0$ we have $D_1(A, F, U, \mathcal{L}, \mathcal{L}^\vee) = Y_A \cap U$ and we are done. To simplify notation we let

$$W_A := D_2(A, F). \tag{2.8}$$

As shown in Section (1) - see (1.7) - Y_A is never empty. We claim that W_A is never empty as well. In fact Formula (6.7) of [5] and Equation (1.8) give that if W_A has the expected dimension i.e. 2 (see Equations (2.13)-(2.14)) or is empty then the cohomology class of the cycle $[W_A]$ is

$$cl([W_A]) = 2c_3(F) - c_1(F)c_2(F) = 40\omega^3. \tag{2.9}$$

Since the right-hand side of the above equation is non-zero it follows that necessarily $W_A \neq \emptyset$.

Definition 2.5. *Let $\mathbb{LG}(\wedge^3 V)^\times \subset \mathbb{LG}(\wedge^3 V)$ be the set of A such that for all $\ell \in \mathbb{P}(V)$ we have*

$$\dim A \cap F_\ell \le 2, \tag{2.10}$$

i.e. $D_3(A, F) = \emptyset$. Let $\mathbb{LG}(\wedge^3 V)^0 \subset \mathbb{LG}(\wedge^3 V)^\times$ be the set of A which do not contain a non-zero completely decomposable element $v_0 \wedge v_1 \wedge v_2$.

A straightforward dimension count gives the following result.

Claim 2.6. *Both $\mathbb{LG}(\wedge^3 V)^\times$ and $\mathbb{LG}(\wedge^3 V)^0$ are open dense subsets of $\mathbb{LG}(\wedge^3 V)$.*

We will show that $\mathbb{LG}(\wedge^3 V)^0$ is the open subset of $\mathbb{LG}(\wedge^3 V)$ parametrizing A such that Y_A and W_A are as nice as possible. First we describe $\Theta_{\ell_0} D_i(A, F)$ at a point $\ell_0 \in D_i(A, F)$. Proceeding as in the proof of Claim (2.2) we consider a symplectic trivialization

$$\wedge^3 V \otimes \mathcal{O}_U = A \otimes \mathcal{O}_U \oplus A^\vee \otimes \mathcal{O}_U \tag{2.11}$$

where $U \subset \mathbb{P}(V)$ is a suitable open affine subset containing ℓ_0 and $A^\vee \in \mathbb{LG}(\wedge^3 V)$ is transversal to A and to F_ℓ for every $\ell \in U$. We view $F \otimes \mathcal{O}_U$ as the graph of a symmetric map $q_F \colon A \otimes \mathcal{O}_U \to A^\vee \otimes \mathcal{O}_U$. Let

$$\begin{array}{ccc} U & \xrightarrow{\psi} & Sym_2 A^\vee \\ \ell & \mapsto & \psi(\ell) := q_F(\ell) \end{array} \tag{2.12}$$

and let $\Sigma_i \subset Sym_2 A^\vee$ be the closed subscheme parametrizing quadratic forms of corank at least i; then

$$D_i(A, F) \cap U = \psi^* \Sigma_i. \tag{2.13}$$

We recall that

$$cod(\Sigma_i, Sym_2 A^\vee) = i(i+1)/2. \tag{2.14}$$

Let $\overline{q} \in (\Sigma_i \setminus \Sigma_{i+1})$. Then Σ_i is smooth at $\overline{q}$ and the tangent space $\Theta_{\overline{q}}\Sigma_i$ is described as follows. Identify $Sym_2 A^\vee$ with its tangent space at $\overline{q}$; then

$$\Theta_{\overline{q}}\Sigma_i = \{q \in Sym_2 A^\vee \mid q|_{\ker(\overline{q})} = 0\}. \tag{2.15}$$

Thus we also get a natural identification

$$(N_{\Sigma_i/Sym_2 A^\vee})_{\overline{q}} = Sym_2 \ker(\overline{q})^\vee. \tag{2.16}$$

Lemma 2.7. *Keep notation as above. Suppose that $A \in \mathbb{LG}(\wedge^3 V)^\times$. Let $\ell_0 \in D_i(A, F)$ and let*

$$A \cap F_{\ell_0} = \ell_0 \otimes W \tag{2.17}$$

where $W \subset \wedge^2(V/\ell_0)$. The composition

$$\Theta_{\ell_0}\mathbb{P}(V) \overset{d\psi(\ell_0)}{\longrightarrow} Sym_2 A^\vee \longrightarrow Sym_2 \ker\psi(\ell_0)^\vee \tag{2.18}$$

is surjective if and only if W contains no non-zero decomposable element.

Proof. Let $\ell_0 = \mathbb{C}v_0$. Choose a codimension 1 subspace $V_0 \subset V$ transversal to ℓ_0. Thus we have

$$V/\ell_0 \cong \oplus V_0, \quad W \subset \wedge^2 V_0. \tag{2.19}$$

The map

$$\begin{array}{ccc} V_0 & \longrightarrow & \mathbb{P}(V) \\ u & \mapsto & [v_0 + u] \end{array} \tag{2.20}$$

gives an isomorphism between V_0 and an open affine subspace of $\mathbb{P}(V)$ containing ℓ_0 - with $0 \in V_0$ corresponding to ℓ_0. Shrinking U if necessary we may assume that $U \subset V_0$ is an open subset containing 0. For $u \in U$ the map $\psi(u)\colon A \to A^\vee$ is characterized by the equation

$$\alpha + \psi(u)(\alpha) = (v_0 + u) \wedge \gamma(u, \alpha), \quad \gamma(u,\alpha) \in \wedge^2 V_0 \tag{2.21}$$

where $\alpha \in A$. Thus when we view $\psi(u)$ as a symmetric bilinear form we have the formula

$$\psi(u)(\alpha, \beta) = vol\left((v_0 + u) \wedge \gamma(u, \alpha) \wedge \beta\right) \tag{2.22}$$

for $\alpha, \beta \in A$. Now assume that $\alpha, \beta \in \ker\psi(\ell_0)$ and hence

$$\alpha = v_0 \wedge \alpha_0, \quad \beta = v_0 \wedge \beta_0, \quad \alpha_0, \beta_0 \in \wedge^2 V_0. \tag{2.23}$$

Let

$$\tau \in \Theta_0 U \cong V_0 \tag{2.24}$$

and let $u(t)$ be a "parametrized curve"in U with $u(0) = 0$ and $\dot{u}(0) = \tau$. Then

$$d\psi(\tau)(v_0 \wedge \alpha_0, v_0 \wedge \beta_0) = \frac{d}{dt}_{|t=0} vol\left((v_0 + u(t)) \wedge \gamma(u(t), v_0 \wedge \alpha_0) \wedge v_0 \wedge \beta_0\right). \tag{2.25}$$

Differentiating and observing that $\gamma(0, v_0 \wedge \alpha_0) = \alpha_0$ we get that

$$d\psi(\tau)(v_0 \wedge \alpha_0, v_0 \wedge \beta_0) = -vol\left(v_0 \wedge \tau \wedge \alpha_0 \wedge \beta_0\right). \tag{2.26}$$

Let's prove the proposition. If $i = 0$ there is nothing to prove. If $i = 1$ let $\ker \psi(\ell_0) = \mathbb{C}v_0 \wedge \alpha_0$. We apply the above formula with $\beta_0 = \alpha_0$; since $\tau \in V_0$ is an arbitrary element we get that Composition (2.18) is surjective if and only if $\alpha_0 \wedge \alpha_0 \neq 0$ i.e. if and only if α_0 is not decomposable. This proves the proposition when $i = 1$. By definition of $\mathbb{LG}(\wedge^3 V)^{\times}$ we are left with the case $i = 2$. First notice that if $\alpha_0 \in W$ is decomposable then $d\psi(\tau)(v_0 \wedge \alpha_0, v_0 \wedge \alpha_0) = 0$ for all τ by Equation (2.26); thus if W contains a non-zero decomposable element then Composition (2.18) is not surjective. Now assume that W does not contain non-zero decomposable elements. Then we have a well-defined regular map

$$\begin{array}{ccc} \mathbb{P}(W) & \xrightarrow{\rho} & \mathbb{P}(\wedge^4 V_0) \\ [\alpha] & \mapsto & [\alpha \wedge \alpha] \end{array} \tag{2.27}$$

We claim that ρ is injective. In fact assume that we have $[\alpha] \neq [\beta]$ and $[\alpha \wedge \alpha] = [\beta \wedge \beta]$. Then $span(\alpha) = span(\beta) = S$ where $S \subset V_0$ is a subspace of dimension 4. Since α, β span W we get that $span(\gamma) = S$ for all $\gamma \in W$. Thus

$$\gamma \wedge \gamma \in Im(\wedge^2 S \to \wedge^2 V_0) \tag{2.28}$$

for all $\gamma \in W$. Since $\mathbb{G}r(2, S) \subset \mathbb{P}(\wedge^2 S)$ is a hypersurface we get that there exists a decomposable non-zero $\gamma \in W$, contradiction. Thus ρ is injective; since ρ is defined by quadratic polynomials we get that $Im(\rho)$ is a conic in $\mathbb{P}(\wedge^4 V_0)$ and hence

$$\wedge^4 V_0^{\vee} \longrightarrow H^0(\mathcal{O}_{\mathbb{P}(W)}(2)) \tag{2.29}$$

is surjective. Given Formula (2.26) this implies surjectivity of Composition (2.18). $\square$

Proposition 2.8. *Keep notation as above. Suppose that $A \in \mathbb{LG}(\wedge^3 V)^{\times}$. The following statements are equivalent.*

(1) $A \in \mathbb{LG}(\wedge^3 V)^0$.

(2) $(D_i(A, F) \setminus D_{i+1}(A, F))$ *is smooth of codimension equal to the expected codimension* $i(i+1)/2$ *for all* i.

(3) $(Y_A \setminus W_A)$ *is smooth and for every* $\ell_0 \in W_A$ *the following holds. There exist* $U \subset \mathbb{P}(V)$ *open in the analytic topology containing* ℓ_0 *and* x, y, z *holomorphic functions on* U *vanishing at* ℓ_0 *with* $dx(\ell_0), dy(\ell_0), dz(\ell_0)$ *linearly independent such that* $Y_A \cap U = V(xz - y^2)$.

Proof. We prove equivalence of (1) and (2) - the proof of equivalence of (1) and (3) is similar, we leave it to the reader. Let $\ell_0 \in (D_i(A, F) \setminus D_{i+1}(A, F))$. Since we have Equation (2.13) and since $(\Sigma_i \setminus \Sigma_{i+1})$ is smooth of codimension $i(i+1)/2$ (see Equation (2.14)) $D_i(A, F)$ is smooth of codimension $i(i+1)/2$ at ℓ_0 if and only if Composition (2.18) is surjective. Let $\ell_0 = \mathbb{C}v_0$. By Lemma (2.7) Composition (2.18) is surjective if and only if A does not contain a non-zero completely decomposable element divisible by v_0, i.e. of the form $v_0 \wedge v_1 \wedge v_3$. The proposition follows immediately. $\square$

Remark 2.9. Let $A \in \mathbb{LG}(\wedge^3 V)^0$.

(1) By Proposition (2.8) Y_A has canonical singularities; since Y_A is a sextic adjunction gives that Y_A has Kodaira dimension 0.

(2) The family of Y_A's (for $A \in \mathbb{LG}(\wedge^3 V)^0$) is topologically locally trivial.

3 The dual of an EPW-sextic

Let $A \in \mathbb{LG}(\wedge^3 V)$ be such that Y_A is a reduced hypersurface. For $\ell \in Y_A^{sm}$ a smooth point of Y_A let $T_\ell Y_A \subset \mathbb{P}(V^\vee)$ be the projective tangent space to Y_A at ℓ; the dual $Y_A^\vee \subset \mathbb{P}(V^\vee)$ is (as usual) the closure of $\bigcup_{\ell \in Y_A^{sm}} T_\ell Y_A$. We will show that if A is generic then $Y_A^\vee$ is isomorphic to an EPW-sextic. The trivialization $vol\colon \wedge^6 V \xrightarrow{\sim} \mathbb{C}$ defines a trivialization $vol^\vee\colon \wedge^6 V^\vee \xrightarrow{\sim} \mathbb{C}$ and hence a symplectic form $\sigma^\vee$ on $\wedge^3 V^\vee$; let $\mathbb{LG}(\wedge^3 V^\vee)$ be the symplectic Grassmannian parametrizing $\sigma^\vee$-Lagrangian subspaces of $\wedge^3 V^\vee$. For $A \in \mathbb{LG}(\wedge^3 V)$ we let

$$A^\perp := \{\phi \in \wedge^3 V^\vee \mid \langle \phi, A\rangle = 0\} \subset \wedge^3 V^\vee. \tag{3.1}$$

As is easily checked $A^\perp \in \mathbb{LG}(\wedge^3 V^\vee)$. Thus we have an isomorphism

$$\begin{array}{rcl} \delta_V\colon \mathbb{LG}(\wedge^3 V) & \xrightarrow{\sim} & \mathbb{LG}(\wedge^3 V^\vee) \\ A & \mapsto & A^\perp. \end{array} \tag{3.2}$$

Proposition 3.1. *Keep notation as above and assume that*

$$A \in \mathbb{LG}(\wedge^3 V)^0 \cap \delta_V^{-1} \mathbb{LG}(\wedge^3 V^\vee)^0. \tag{3.3}$$

Then $Y_A^\vee = Y_{A^\perp}$.

Proof. We claim that it suffices to prove that

$$Y_A^\vee \subset Y_{A^\perp}. \tag{3.4}$$

In fact by Item (1) of Remark (2.9) we know that Y_A is not covered by positive-dimensional linear spaces and hence $Y_A^\vee$ is 4-dimensional; since $A^\perp \in \mathbb{LG}(\wedge^3 V^\vee)^0$ we know that $Y_{A^\perp}$ is irreducible (see Proposition (2.8)) and hence (3.4) implies that $Y_A^\vee = Y_{A^\perp}$. Let's prove (3.4). Let ψ be as in (2.12) and let's adopt the notation introduced in the proof of Lemma (2.7). Let $\ell_0 = \mathbb{C}v_0 \in Y_A^{sm}$. By Proposition (2.8) we know that $\ell_0 \notin W_A$ and hence $\ker \psi(\ell_0) = \mathbb{C}v_0 \wedge \alpha_0$ with $\alpha_0 \in \wedge^2 V_0$ an indecomposable element; let $J_0 \subset V_0$ be the span of α_0, thus $\dim J_0 = 4$ because α_0 is indecomposable. Let $E_0 \subset V$ be the codimension-1 subspace spanned by v_0 and J_0. It follows immediately from (2.26) with $\beta_0 = \alpha_0$ that

$$T_{\ell_0} Y_A = \mathbb{P}(E_0). \tag{3.5}$$

Now notice that $v_0 \wedge \alpha_0 \in \wedge^3 E_0 \cap A$ and hence

$$\{0\} \neq (\wedge^3 V / \wedge^3 E_0 + A)^\vee = (\wedge^3 E_0)^\perp \cap A^\perp. \tag{3.6}$$

Let $E_0^\perp = \mathbb{C}\phi_0$; then

$$(\wedge^3 E_0)^\perp = \mathbb{C}\phi_0 \otimes \wedge^2 (V^\vee / \mathbb{C}\phi_0) = F_{\mathbb{C}\phi_0}. \tag{3.7}$$

By (3.5) and (3.6)-(3.7) we get that $T_{\ell_0} Y_A \in Y_{A^\perp}$; this proves (3.4). $\square$

By the above proposition duality defines a rational involution on the set of projective equivalence classes of EPW-sextics. We will show later on - see Section (6) - that a generic EPW-sextic is not self-dual, i.e. the rational involution defined by duality is not the identity.

4 Double covers of EPW-sextics

We give the details of the following observation: for $A \in \mathbb{LG}(\wedge^3 V)^0$ the variety Y_A supports a quadratic sheaf (see Definition (0.2) of [3]) and if $X_A \to Y_A$ is the associated double cover then X_A is smooth. Let $A \in \mathbb{LG}(\wedge^3 V)$ and let $A^\vee \subset \wedge^3 V$ be a Lagrangian subspace transversal to A - see Section (2). Thus we have

$$\wedge^3 V = A \oplus A^\vee. \tag{4.1}$$

Let

$$\widetilde{\lambda}_A \colon \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \to A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}, \tag{4.2}$$

be the projection corresponding to Decomposition (4.1). Let ν and λ_A be given by (1.6) and (1.7) respectively; then $\lambda_A = \widetilde{\lambda}_A \circ \nu$. We will study the sheaf $coker(\lambda_A)$ fitting into the exact sequence

$$0 \to F \xrightarrow{\lambda_A} A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow coker(\lambda_A) \to 0. \tag{4.3}$$

Proposition 4.1. *Keep notation as above and assume that $A \in \mathbb{LG}(\wedge^3 V)^0$. Let $\ell_0 \in \mathbb{P}(V)$.*

(1) If $\ell_0 \notin Y_A$ then $coker(\lambda_A)$ is zero in a neighborhood of ℓ_0.

(2) If $\ell_0 \in (Y_A \setminus W_A)$ there exist an open affine $U \subset \mathbb{P}(V)$ containing ℓ_0 and trivializations $F|_U \cong \mathcal{O}_U^9 \oplus \mathcal{O}_U$ and $A \otimes \mathcal{O}_U \cong \mathcal{O}_U^9 \oplus \mathcal{O}_U$ such that (4.3) restricted to U reads

$$0 \longrightarrow \mathcal{O}_U^9 \oplus \mathcal{O}_U \stackrel{(Id,x)}{\longrightarrow} \mathcal{O}_U^9 \oplus \mathcal{O}_U \longrightarrow coker(\lambda_A) \otimes \mathcal{O}_U \longrightarrow 0 \tag{4.4}$$

where x is a local generator of $I_{Y_A \cap U}$.

(3) If $\ell_0 \in W_A$ there exist an open affine $U \subset \mathbb{P}(V)$ containing ℓ_0 and trivializations $F|_U \cong \mathcal{O}_U^8 \oplus \mathcal{O}_U^2$ and $A \otimes \mathcal{O}_U \cong \mathcal{O}_U^8 \oplus \mathcal{O}_U^2$ such that (4.3) restricted to U reads

$$0 \longrightarrow \mathcal{O}_U^8 \oplus \mathcal{O}_U^2 \stackrel{(Id,M)}{\longrightarrow} \mathcal{O}_U^8 \oplus \mathcal{O}_U^2 \longrightarrow coker(\lambda_A) \otimes \mathcal{O}_U \longrightarrow 0 \tag{4.5}$$

where

$$M = \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \tag{4.6}$$

with x, y, z generators of $I_{W_A \cap U}$.

Proof. This is a straightforward consequence of the proof of Lemma (2.7); we leave the details to the reader. □

We will need a few results on the sheaf $coker(\lambda_A)$ and sheaves which locally look like $coker(\lambda_A)$.

Definition 4.2. *A coherent sheaf $\mathcal{F}$ on a smooth projective variety Z is a Casnati-Catanese sheaf if for every $p \in Z$ there exists $U \subset Z$ open in the classical topology containing p such that one of the following holds:*

(1) $\mathcal{F}|_U = 0$.

(2) There exist $x \in Hol(U)$ with $x(p) = 0$, $dx(p) \neq 0$ and an exact sequence

$$0 \to \mathcal{O}_U \xrightarrow{\cdot x} \mathcal{O}_U \longrightarrow \mathcal{F} \to 0. \tag{4.7}$$

(3) There exist $x, y, z \in Hol(U)$ vanishing at p with $dx(p), dy(p), dz(p)$ linearly independent and an exact sequence

$$0 \to \mathcal{O}_U^2 \xrightarrow{M} \mathcal{O}_U^2 \longrightarrow \mathcal{F} \to 0 \tag{4.8}$$

where M is the map defined by Matrix (4.6).

Thus $coker(\lambda_A)$ is a typical example of a Casnati-Catanese sheaf. If $\mathcal{F}$ is a Casnati-Catanese sheaf the schematic support of $\mathcal{F}$ is a divisor D on Z; thus letting $i\colon D \hookrightarrow Z$ be the inclusion we have

$$\mathcal{F} = i_* \mathcal{G} \tag{4.9}$$

for a coherent sheaf $\mathcal{G}$ on D. In particular $coker(\lambda_A) = i_* \zeta_A$ for $i\colon Y_A \hookrightarrow \mathbb{P}(V)$ the inclusion map and ζ_A a certain coherent sheaf on Y_A.

Proposition 4.3. *Let $\mathcal{F}$ be a Casnati-Catanese sheaf on a smooth projective variety Z. Let D be the schematic support of $\mathcal{F}$ and hence $\mathcal{F} = i_* \mathcal{G}$ where $i\colon D \hookrightarrow Z$ is the inclusion, see (4.9).*

(1) If $q \geq 2$ then $Tor_q(\mathcal{F}, \mathcal{E}) = 0$ for any coherent sheaf $\mathcal{E}$ on Z.

(2) There is an isomorphism

$$Ext^1(\mathcal{F}, \mathcal{O}_Z) \cong i_* \left(\mathcal{G}^\vee \otimes N_{D/Z} \right). \tag{4.10}$$

(3) The sheaf $\mathcal{G}$ is locally isomorphic to $\mathcal{G}^\vee := Hom(\mathcal{G}, \mathcal{O}_D)$. In particular $\mathcal{G}$ is pure i.e. there does not exist a non-zero subsheaf of $\mathcal{G}$ supported on a proper subscheme of D.

(4) The map of sheaves

$$\mathcal{O}_D \to Hom(\mathcal{G}, \mathcal{G}) \tag{4.11}$$

which associates to $f \in \mathcal{O}_{D,p}$ multiplication by f is an isomorphism.

Proof. (1) follows immediately from the given local resolutions of $\mathcal{F}$. (2): Let $\mathcal{E}_0 \to \mathcal{F}$ be a surjection with $\mathcal{E}_0$ locally-free and let $\mathcal{E}_1$ be the kernel of the surjection. Thus we have an exact sequence

$$0 \to \mathcal{E}_1 \xrightarrow{h} \mathcal{E}_0 \longrightarrow \mathcal{F} \to 0. \tag{4.12}$$

By Item (1) the sheaf $\mathcal{E}_1$ is locally-free. The dual of (4.12) is the exact sequence

$$0 \to \mathcal{E}_0^\vee \xrightarrow{h} \mathcal{E}_1^\vee \xrightarrow{\partial} Ext^1(\mathcal{F}, \mathcal{O}_Z) \to 0. \tag{4.13}$$

Multiplication defines an inclusion

$$\mathcal{F} \otimes \mathcal{O}_Z(-D) = (\mathcal{E}_0/\mathcal{E}_1) \otimes \mathcal{O}_Z(-D) \to \mathcal{E}_0(-D)/\mathcal{E}_1(-D). \tag{4.14}$$

Since $\mathcal{F}$ is supported on D we have an inclusion $\mathcal{E}_0(-D) \hookrightarrow \mathcal{E}_1$ and hence an inclusion

$$\mathcal{E}_0(-D)/\mathcal{E}_1(-D) \hookrightarrow \mathcal{E}_1/\mathcal{E}_1(-D) = \mathcal{E}_1 \otimes \mathcal{O}_D. \tag{4.15}$$

Composing Map (4.14) and Map (4.15) we get an inclusion

$$\mathcal{G} \otimes \mathcal{O}_D(-D) \hookrightarrow \mathcal{E}_1|_D \tag{4.16}$$

whose dual is a surjection

$$\mathcal{E}_1^\vee|_D \to \mathcal{G} \otimes N_{D/Z}. \tag{4.17}$$

Since $\mathcal{F}$ is supported on D the connecting homomorphism map of (4.13) annihilates $\mathcal{E}_1^\vee(-D)$ and hence it may be identified with a quotient map of $\mathcal{E}_1^\vee|_D$: a local computation shows that the quotient map is (4.17). This proves Item (2). Let's prove Item (3): We work in a neighborhood of $p \in Z$. If (1) or (2) of Definition (4.2) holds the result is immediate. Now assume that (3) of Definition (4.2) holds. Use the locally free resolution of $\mathcal{F}$ given by (4.8) to compute $Ext^1(\mathcal{F}, \mathcal{O}_Z)$: since M is symmetric we get that (locally) $Ext^1(\mathcal{F}, \mathcal{O}_Z) \cong \mathcal{F}$. On the other hand $N_{D/Z}$ is locally free of rank one because D is a Cartier divisor in Z and hence (4.10) gives that $Ext^1(\mathcal{F}, \mathcal{O}_Z) \cong i_*(\mathcal{G}^\vee)$. Thus we get that locally

$$i_*\mathcal{G} = \mathcal{F} \cong i_*(\mathcal{G}^\vee) \tag{4.18}$$

and hence $\mathcal{G} \cong \mathcal{G}^\vee$ (locally). Finally let's prove Item (4): We work in a neighborhood of $p \in Z$. The result is trivial if (1) or (2) of Definition (4.2) holds. Thus we may suppose that (3) of Definition (4.2) holds. By GAGA it suffices to prove that (4.11) induces an isomorphism of the corresponding analytic sheaves. Let U be an analytic open neighborohood of p and α be an endomorphism of the restriction of $\mathcal{G}$ to $D \cap U$. To simplify notation we denote by $\mathcal{G}$ the restriction of $\mathcal{G}$ to $D \cap U$. Let $D^{sm} := (D \setminus singD)$; since the restriction of $\mathcal{G}$ to D^{sm} is locally-free of rank one there exists $f^0 \in Hol(D^{sm})$ such that α restricted to D^{sm} equals multiplication by f^0. There exists $f \in Hol(D \cap U)$ extending f^0 because $D \cap U$ is normal (we have $D = V(xz - y^2)$); abusing notation we let $f \colon \mathcal{G} \to \mathcal{G}$ be multiplication by f. Then $(\alpha - f)$ is zero on D^{sm} and thus $Im(\alpha - f)$ is a subsheaf of $\mathcal{G}$ supported on a proper analytic subspace of D. We know that $\mathcal{G}$ is pure (see Item (3)) and hence $Im(\alpha - f) = 0$ i.e. $\alpha = f$. This proves Item (4). □

We set

$$\xi_A := \zeta_A \otimes \mathcal{O}_{Y_A}(-3). \tag{4.19}$$

Proposition 4.4. *Keep notation as above and assume that $A \in \mathbb{LG}(\wedge^3 V)^0$.*

(1) There exists a symmetric isomorphism

$$\alpha_A \colon \xi_A \xrightarrow{\sim} \xi_A^\vee \tag{4.20}$$

defining a commutative multiplication map

$$\overline{\alpha}_A \colon \xi_A \otimes \xi_A \longrightarrow \mathcal{O}_{Y_A}. \tag{4.21}$$

(2) *Multiplication (4.21) is an isomorphism away from W_A and near $\ell_0 \in W_A$ is described as follows. There exist an open affine $U \subset Y_A$ containing ℓ_0, global generators $\{e_1, e_2\}$ of $\xi_A \otimes \mathcal{O}_U$ and $x, y, z \in \mathbb{C}[U]$ generating the ideal of $W_A \cap U$ such that*

$$\overline{\alpha}_A(e_1 \otimes e_1) = x, \quad \overline{\alpha}_A(e_1 \otimes e_2) = \overline{\alpha}_A(e_2 \otimes e_1) = y, \quad \overline{\alpha}_A(e_2 \otimes e_2) = z. \tag{4.22}$$

(3) *Any map $\gamma \colon \xi_A \to \xi_A^\vee$ is a constant multiple of α_A.*

Proof. Let

$$\widetilde{\mu}_A \colon \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \to A \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{4.23}$$

be the projection given by Decomposition (4.1) and let $\mu_A \colon F \to A \otimes \mathcal{O}_{\mathbb{P}(V)}$ be defined by $\mu_A := \widetilde{\mu}_A \circ \nu$ where ν is as in (1.6). The diagram

$$\begin{array}{ccc} F & \xrightarrow{\lambda_A} & A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \\ \big\downarrow{\scriptstyle \mu_A} & & \big\downarrow{\scriptstyle \mu_A^\vee} \\ A \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda_A^\vee} & F^\vee \end{array} \tag{4.24}$$

is commutative because $F \xrightarrow{(\mu_A, \lambda_A)} (A \oplus A^\vee) \otimes \mathcal{O}_{\mathbb{P}(V)}$ is a Lagrangian embedding. The map λ_A is an injection of sheaves because $Y_A \neq \mathbb{P}(V)$ and hence also $\lambda_A^\vee$ is an injection of sheaves. Thus there is a unique $\beta_A \colon i_*\zeta_A \longrightarrow Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)})$ making the following diagram commutative with exact horizontal sequences:

$$\begin{array}{ccccccccc} 0 & \to & F & \xrightarrow{\lambda_A} & A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_A & \to & 0 \\ & & \big\downarrow{\scriptstyle \mu_A} & & \big\downarrow{\scriptstyle \mu_A^\vee} & & \big\downarrow{\scriptstyle \beta_A} & & \\ 0 & \to & A \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda_A^\vee} & F^\vee & \longrightarrow & Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) & \to & 0 \end{array} \tag{4.25}$$

By Isomorphism (4.10) we get that β_A may be viewed as a map $\beta_A \colon \zeta_A \xrightarrow{\sim} \zeta_A^\vee(6)$.

Claim 4.5. *Let Z be a smooth projective variety and $\mathcal{F}$ be a Casnati-Catanese sheaf on Z. Suppose that there exist vector-bundles $\mathcal{E}_0, \mathcal{E}_1$ on Z and an exact sequence*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{E}_1 & \xrightarrow{\lambda} & \mathcal{E}_0 & \longrightarrow & \mathcal{F} & \to & 0 \\ & & \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu^\vee} & & \big\downarrow{\scriptstyle \beta} & & \\ 0 & \to & \mathcal{E}_0^\vee & \xrightarrow{\lambda^\vee} & \mathcal{E}_1^\vee & \longrightarrow & Ext^1(\mathcal{F}, \mathcal{O}_Z) & \to & 0 \end{array} \tag{4.26}$$

Then β is an isomorphism if and only if the map

$$\mathcal{E}_1 \xrightarrow{(\lambda,\mu)} \mathcal{E}_0 \oplus \mathcal{E}_0^\vee \tag{4.27}$$

is an injection of vector-bundles i.e. it is injective on fibers.

Proof. Let $\mathcal{F} = i_*\mathcal{G}$ where $i \colon D \hookrightarrow Z$ is the inclusion. First we notice that β is an isomorphism if and only if it is surjective; in fact by Items (2) and (3) of Proposition (4.3) we have local identifications of the sheaves $Hom(\mathcal{F}, Ext^1(\mathcal{F}, \mathcal{O}_Z))$

with $Hom(\mathcal{G},\mathcal{G})$ and it follows from Item (4) of the same proposition that a map of stalks $\mathcal{G}_p \to \mathcal{G}_p$ is an isomorphism if and only if it is surjective. By Nakayama's Lemma (or by a direct computation) we get that β is an isomorphism if and only if for every $p \in Z$ the map from the fiber of $\mathcal{F}$ at p to the fiber of $Ext^1(\mathcal{F},\mathcal{O}_Z)$ at p is surjective. As is easily checked there exist $U \subset Z$ open in the classical topology containing p, trivial vector-bundles $\mathcal{A}_i, \mathcal{B}_i$ on U for $i = 0, 1$ and isomorphisms $\mathcal{E}_i|_U \cong \mathcal{A}_i \oplus \mathcal{B}_i$ such that the restriction of (4.26) to U reads

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_1 \oplus \mathcal{B}_1 & \xrightarrow{(\phi,\psi)} & \mathcal{A}_0 \oplus \mathcal{B}_0 & \longrightarrow & \mathcal{F}|_U & \to & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu^\vee} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \to & \mathcal{A}_0^\vee \oplus \mathcal{B}_0^\vee & \xrightarrow{(\phi^\vee,\psi^\vee)} & \mathcal{A}_1^\vee \oplus \mathcal{B}_1^\vee & \longrightarrow & Ext^1(\mathcal{F}_U,\mathcal{O}_U) & \to & 0 \end{array} \tag{4.28}$$

where $\phi\colon \mathcal{A}_1 \to \mathcal{A}_0$ is an isomorphism and $\psi\colon \mathcal{B}_1 \to \mathcal{B}_0$ is a standard resolution of $\mathcal{F}_U$ as given in Definition (4.2), i.e. $\mathcal{B}_1 = \mathcal{B}_0 = 0$ if (1) of Definition (4.2) holds, $\mathcal{B}_1 \cong \mathcal{B}_0 \cong \mathcal{O}_U$ and ψ is the map of (4.7) if (2) of Definition (4.2) holds and finally if (3) of Definition (4.2) holds then $\mathcal{B}_1 \cong \mathcal{B}_0 \cong \mathcal{O}_U^2$ and ψ is the map of (4.8). The map β is a surjection from the fiber of $\mathcal{F}$ at p to the fiber of $Ext^1(\mathcal{F}_U,\mathcal{O}_U)$ if and only if the composition

$$\mathcal{B}_0 \xrightarrow{\iota} \mathcal{A}_0 \oplus \mathcal{B}_0 \xrightarrow{\mu^\vee} \mathcal{A}_1^\vee \oplus \mathcal{B}_1^\vee \xrightarrow{\pi} \mathcal{B}_1^\vee \tag{4.29}$$

is surjective at p. On the other hand since ψ is zero at p the map $((\phi,\psi),\mu)$ is injective at p if and only if the composition

$$\mathcal{B}_1 \xrightarrow{\pi^\vee} \mathcal{A}_1 \oplus \mathcal{B}_1 \xrightarrow{\mu} \mathcal{A}_0^\vee \oplus \mathcal{B}_0^\vee \tag{4.30}$$

is injective at p. Since ψ vanishes at p while $\phi^\vee$ is an isomorphism at p the equality $(\phi^\vee,\psi^\vee) \circ \mu = \mu^\vee \circ (\phi,\psi)$ gives that the composition

$$\mathcal{B}_1 \xrightarrow{\pi^\vee} \mathcal{A}_1 \oplus \mathcal{B}_1 \xrightarrow{\mu} \mathcal{A}_0^\vee \oplus \mathcal{B}_0^\vee \longrightarrow \mathcal{A}_0^\vee \tag{4.31}$$

vanishes at p. Thus (4.30) is injective at p if and only if the composition

$$\mathcal{B}_1 \xrightarrow{\pi^\vee} \mathcal{A}_1 \oplus \mathcal{B}_1 \xrightarrow{\mu} \mathcal{A}_0^\vee \oplus \mathcal{B}_0^\vee \xrightarrow{\iota^\vee} \mathcal{B}_0^\vee \tag{4.32}$$

is injective at p. Since Composition (4.32) is the transpose of Composition (4.29) this proves that β is a surjection at p if and only if $((\phi,\psi),\mu)$ is injective at p. $\square$

The above result together with Exact Sequence (4.25) gives that β_A is an isomorphism. We define the map α_A of (4.20) to be the tensor product of β_A times the identity map on $\mathcal{O}_{Y_A}(-3)$. Since β_A is an isomorphism we get that α_A is an isomorphism . The map $\alpha_A^\vee - \alpha_A$ is zero on $(Y_A \setminus W_A)$ because ξ_A is locally-free of rank 1 on $(Y_A \setminus W_A)$. By Item (3) of Proposition (4.3) ζ_A is pure and hence ξ_A is pure; thus $\alpha_A^\vee - \alpha_A$ is zero on Y_A i.e. α_A is symmetric. This proves Item (1). Let's prove Item (2). Since ξ_A is locally-free of rank 1 away from W_A it is clear that (4.21) is an isomorphism away from W_A. Equation (4.22) defines a symmetric isomorphism $\mu\colon \xi_A \otimes \mathcal{O}_U \to \xi_A^\vee \otimes \mathcal{O}_U$. By Item (4) of Proposition (4.3) any other symmetric isomorphism $\xi_A \otimes \mathcal{O}_U \to \xi_A^\vee \otimes \mathcal{O}_U$ is equal to $f \cdot \mu$ where $f \in \mathbb{C}[U]^\times$ is invertible; Item (2) of the proposition follows at once from this. Lastly we prove Item (3). The composition $\alpha_A^{-1} \circ \gamma$ is a global section of $Hom(\xi_A, \xi_A)$ and by Item (4) of Proposition (4.3) we get that $\alpha_A^{-1} \circ \gamma$ is multiplication by a constant c; thus $\gamma = c\alpha_A$. $\square$

Now we are ready to define the double cover of X_A for $A \in \mathbb{LG}(\wedge^3 V)^0$. Map (4.21) gives $\mathcal{O}_{Y_A} \oplus \xi_A$ the structure of a commutative finite $\mathcal{O}_{Y_A}$-algebra. Let

$$X_A := Spec\,(\mathcal{O}_{Y_A} \oplus \xi_A) \tag{4.33}$$

and $f\colon X_A \to Y_A$ be the structure map. Thus f is a finite map of degree 2; let $\phi\colon X_A \to X_A$ be the covering involution - thus ϕ corresponds to the map on $\mathcal{O}_{Y_A} \oplus \xi_A$ which is the identity on $\mathcal{O}_{Y_A}$ and multiplication by (-1) on ξ_A. Equivalently

$$f_*\mathcal{O}_{X_A} = \mathcal{O}_{Y_A} \oplus \xi_A \tag{4.34}$$

with $\mathcal{O}_{Y_A}$ and ξ_A the $+1$ and (-1)-eigensheaf respectively. It follows at once from our lemma that f is ramified exactly over W_A and that X_A is smooth. Let A vary in $\mathbb{LG}(\wedge^3 V)^0$. By Item (2) of Remark (2.9) the family of Y_A's is locally trivial; since $\mathbb{LG}(\wedge^3 V)^0$ is irreducible we get the following result.

Proposition 4.6. *The varieties X_A for A varying in $\mathbb{LG}(\wedge^3 V)^0$ are all deformation equivalent.*

5 Proof of Theorem (1.1)

Throughout this section X is a numerical $(K3)^{[2]}$ with an ample divisor H such that both (1.3) and Item (a) of Section (1) hold. Thus we have an anti-symplectic involution $\phi\colon X \to X$ with quotient map $f\colon X \to X/\langle\phi\rangle =: Y$ and an embedding

$$j\colon Y \hookrightarrow |H|^\vee, \quad \deg Y = 6 \tag{5.1}$$

such that $j \circ f$ is the tautological map $X \to |H|^\vee$. Let X^ϕ be the fixed locus of ϕ; then $singY \cong X^\phi$. Since ϕ is anti-symplectic X^ϕ is a smooth Lagrangian surface - not empty because a smooth hypersurface in $\mathbb{P}^5$ is simply connected. Thus

$$\dim(singY) = 2 \tag{5.2}$$

and f is unramified over $(Y \setminus singY)$. We have a decomposition

$$f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \eta, \tag{5.3}$$

where η is the (-1)-eigensheaf of the involution $\phi^*\colon f_*\mathcal{O}_X \to f_*\mathcal{O}_X$.

Claim 5.1. *Keep notation as above. The sheaf $j_*\eta$ is a Casnati-Catanese sheaf on $|H|^\vee$.*

Proof. Let $p \in (|H|^\vee \setminus Y)$; then $j_*\eta$ is zero in a neighborhood of p. Let $p \in (Y \setminus singY)$; then η is locally-free of rank 1 in a neighborhood of p (in Y) and hence we see that (2) of Definition (4.2) holds. Let $p \in singY$. Let $f^{-1}(p) = \{\widetilde{p}\}$. Since $\dim X^\phi = 2$ there exist a $\widetilde{U} \subset X$ open in the classical topology containing $\widetilde{p}$ with analytic coordinates $\{u_1, u_2, v_1, v_2\}$ centered at $\widetilde{p}$ and an open $U \subset |H|^\vee$ containing p with analytic coordinates $\{x_1, \ldots, x_5\}$ centered at p such that

$$\phi^*(u_1, u_2, v_1, v_2) = (-u_1, -u_2, v_1, v_2) \tag{5.4}$$
$$f^*(x_1, x_2, x_3, x_4, x_5) = (u_1^2, u_2^2, -u_1u_2, v_1, v_2). \tag{5.5}$$

It follows from (5.4) that

$$j_*\eta|_U \text{ is generated by } (u_1, u_2). \tag{5.6}$$

A straightforward computation gives the free presentation

$$0 \to \mathcal{O}_U^2 \xrightarrow{\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}} \mathcal{O}_U^2 \xrightarrow{(u_2,u_1)} (j_*\eta)|_U \to 0. \tag{5.7}$$

Thus (3) of Definition (4.2) holds. $\square$

For future use we notice the following: keeping notation as in the above proof we may assume by shrinking U that $f^{-1}(U) = \widetilde{U}$ and then

$$Y \cap U = V(x_1x_2 - x_3^3). \tag{5.8}$$

Multiplication on $f_*\mathcal{O}_X$ defines a symmetric map

$$\overline{\alpha} \colon \eta \otimes \eta \to \mathcal{O}_Y \tag{5.9}$$

making η a quadratic sheaf in the sense of Casnati-Catanese [3]. A straightforward computation shows that $\overline{\alpha}$ defines a symmetric isomorphism

$$\alpha \colon \eta \xrightarrow{\sim} \eta^\vee := \mathcal{H}om(\eta, \mathcal{O}_Y). \tag{5.10}$$

The symmetric map (5.9) makes $\mathcal{O}_Y \oplus \eta$ a commutative $\mathcal{O}_Y$-algebra and we have the tautological isomorphism

$$X \cong \mathit{Spec}(\mathcal{O}_Y \oplus \eta). \tag{5.11}$$

The main result of this section is the following.

Theorem 5.2. *Keep notation and hypotheses as above. There exists $A \in \mathbb{LG}(\wedge^3 V)^0$ such that $Y = Y_A$ and $\eta \cong \xi_A$.*

The proof of Theorem (5.2) will be given at the end of this section. Let's show that Theorem (1.1) follows from Theorem (5.2). Let's prove (1) of Theorem (1.1) i.e. that $f \colon X \to Y$ is identified with the natural double cover $X_A \to Y_A$. According to Theorem (5.2) we may identify η and ξ_A. We claim that with this identification the map α of (5.10) gets identified with a non-zero constant multiple of α_A; in fact α is non-zero and hence the claim follows from Item (3) of Proposition (4.4). Thus multiplying the isomorphism $\eta \xrightarrow{\sim} \xi_A$ by a suitable constant we may assume that the map α gets identified with α_A; by (5.11) and (4.33) we get that $f \colon X \to Y$ is identified with $X_A \to Y_A$. Let's prove (2) of Theorem (5.2) i.e. that if $A \in \mathbb{LG}(\wedge^3 V)^0$ then $X_A \to Y_A$ is a deformation of $(K3)^{[2]}$. By Proposition (4.6) the X_A for $A \in \mathbb{LG}(\wedge^3 V)^0$ are all deformation equivalent. Since every Kähler deformation of an irreducible symplectic manifold is an irreducible symplectic manifold it suffices to prove that there exists one $A \in \mathbb{LG}(\wedge^3 V)^0$ such that X_A is a symplectic irreducible variety deformation equivalent of $(K3)^{[2]}$. By Item (1) of Theorem (1.1) it suffices to exhibit X, H where X is a deformation of $(K3)^{[2]}$ and H is an ample divisor on X such that both (1.3) and Item (a) of Section (1) hold. Such an example was given by Mukai (Ex.(5.17) of [12]), details are in Subsection (5.4) of [14]. We briefly describe the example. Let $F \subset \mathbb{P}^6$ be "the"Fano 3-fold of index 2 and degree 5 i.e. the transversal intersection of $Gr(2, \mathbb{C}^5) \subset \mathbb{P}(\wedge^2 \mathbb{C}^5)$ and a 6-dimensional linear subspace of $\mathbb{P}(\wedge^2 \mathbb{C}^5)$. Let $\overline{Q} \subset \mathbb{P}^6$ be a quadric hypersurface intersecting transversely F and let

$$S := F \cap \overline{Q}. \tag{5.12}$$

Thus $(S, \mathcal{O}_S(1))$ is a generic polarized $K3$ surface of degree 10. We assume that for all divisors E divisor on S

$$E \cdot c_1(\mathcal{O}_S(1)) \equiv 0 \pmod{10}. \tag{5.13}$$

This is Hypothesis (4.8) of [14] and it holds for $\overline{Q}$ very general (i.e. in the complement of a denumerable union of proper algebraic subvarieties of the parameter space for quadrics). We have $\dim |I_F(2)| = 4$ and $\dim |I_S(2)| = 5$. Let Σ be the degree-7 divisor on $|I_S(2)|$ parametrizing singular quadrics. Every $Q \in |I_F(2)|$ is singular and hence

$$\Sigma = |I_S(2)| + Y. \tag{5.14}$$

Since $\deg \Sigma = 7$ we have $\deg Y = 6$. The generic $Q \in Y$ has corank 1 and hence it has two rulings by 3-dimensional linear spaces. Thus there exists a natural double cover of an open dense subset of Y. It turns out (see Ex.(5.17) of [12] and Subsection (5.4) of [14]) that the double cover extends to a double cover $f \colon X \to Y$, that

$$X = \{F \text{ stable sheaf on } S,\ rk(F) = 2,\ c_1(F) = c_1(\mathcal{O}_S(1),\ c_2(F) = 5\}/\text{isomorphism} \tag{5.15}$$

and that X is a deformation of $S^{[2]}$. Let $H := f^*\mathcal{O}_Y(1)$. As shown in Subsection (5.4) of [14] both (1.3) and Item (a) of Section (1) hold. This proves that (2) of Theorem (1.1) holds.

5.1 A locally free resolution of $j_*(\eta \otimes \mathcal{O}_Y(3))$

Proposition 5.3. *Let X be a numerical $(K3)^{[2]}$ with an ample divisor H such that both (1.3) and Item (a) of Section (1) hold. If $k \geq 3$ then $\mathcal{O}_X(kH)$ is very ample.*

Proof. By hypothesis nH is very ample for some $n \gg 0$ and hence it suffices to prove that the multiplication map

$$H^0(\mathcal{O}_X(kH)) \otimes H^0(\mathcal{O}_X(H)) \longrightarrow H^0(\mathcal{O}_X((k+1)H)) \tag{5.16}$$

is surjective for $k \geq 3$. By (5.2) there exists a plane $\Lambda \subset |H|^\vee$ such that $\Lambda \cap (singY) = \emptyset$ and Λ is transversal to Y. Let $C := \Lambda \cap Y$ and $\widetilde{C} := f^{-1}C$. Then C is a smooth plane sextic by (5.1) and $\pi := f|_C \colon \widetilde{C} \to C$ is an unramified double cover. Let's prove that

$$H^0(\mathcal{O}_{\widetilde{C}}(kH)) \otimes H^0(\mathcal{O}_{\widetilde{C}}(H)) \longrightarrow H^0(\mathcal{O}_{\widetilde{C}}((k+1)H)) \tag{5.17}$$

is surjective for $k \geq 3$. We have

$$\pi_*\mathcal{O}_{\widetilde{C}} = \mathcal{O}_C \oplus \lambda \tag{5.18}$$

where λ is a non-trivial square root of $\mathcal{O}_{\widetilde{C}}$. Thus

$$H^0(\mathcal{O}_{\widetilde{C}}(kH)) = H^0(\pi_*\mathcal{O}_{\widetilde{C}}(kH)) = H^0(\mathcal{O}_C(k)) \oplus H^0(\lambda(k)). \tag{5.19}$$

Since the multiplication map $H^0(\mathcal{O}_C(k)) \otimes H^0(\mathcal{O}_C(1)) \to H^0(\mathcal{O}_C(k+1))$ is surjective it suffices to prove surjectivity of

$$\mu^k \colon H^0(\lambda(k)) \otimes H^0(\mathcal{O}_C(1)) \to H^0(\lambda(k+1)) \tag{5.20}$$

for $k \geq 3$. We will use the following result (well-known to experts).

Claim 5.4. *Let C be a curve and L, M line-bundles on C. Suppose that M is globally generated and that*

$$h^1(L \otimes M^{-1}) = 0. \tag{5.21}$$

Then the multiplication map

$$H^0(L) \otimes H^0(M) \to H^0(L \otimes M) \tag{5.22}$$

is surjective.

Proof. Since M is a globally generated line-bundle on a curve there exist $\epsilon_0, \epsilon_1 \in H^0(M)$ with no common zeroes. Thus we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \ker(\tau) & \to & L \oplus L & \xrightarrow{\tau} & L \otimes M & \to & 0 \\ & & & & (s_0, s_1) & \mapsto & s_0\epsilon_0 + s_1\epsilon_1 & & \end{array} \tag{5.23}$$

The sheaf $\ker(\tau)$ is locally-free of rank one and taking the determinants of the sheaves appearing in (5.23) we get that

$$\ker(\tau) \cong L \otimes M^{-1}. \tag{5.24}$$

Now consider the long exact cohomology sequence associated to (5.23): by (5.24)-(5.21) we get that $H^0(\tau)$ is surjective. Since the image of $H^0(\tau)$ is contained in the image of Map (5.22) this proves the claim. $\square$

Let C be our smooth plane sextic and $L = \lambda(k)$, $M = \mathcal{O}_C(1)$. Certainly $\mathcal{O}_C(1)$ is globally generated hence the hypotheses of Claim (5.4) are satisfied provided

$$h^1(\lambda(k-1)) = 0. \tag{5.25}$$

By adjunction $K_C \cong \mathcal{O}_C(3)$ and thus (5.25) holds if $k \geq 4$; this proves surjectivity of μ^k for $k \geq 4$. It remains to prove surjectivity of μ^3 under the hypothesis that

$$0 < h^1(\lambda(2)) = h^0(\lambda(1)). \tag{5.26}$$

(The equality follows from Serre duality.) First we notice that

$$h^0(\lambda(1)) = 1. \tag{5.27}$$

In fact assume that $h^0(\lambda(1)) > 1$: since $\deg C = 6$ it follows that $\lambda(1) \cong \mathcal{O}_C(p-q)(1)$ (let E be a divisor in the moving part of $|\lambda(1)|$ equal to the sum of $d \leq 6$ pairwise distinct points, by Riemann-Roch E does not impose independent conditions on cubics, as is easily checked this implies that E contains 5 points on a line), since $\lambda^{\otimes 2} \cong \mathcal{O}_C$ we get that $2p \cong 2q$ and hence $p = q$ (C is not hyperelliptic) thus $\lambda \cong \mathcal{O}_C$, contradiction. By (5.27) there exists a unique $D \in |\lambda(1)|$; let $q \in (C \setminus suppD)$ and let

$$H^0(\lambda(3)) \otimes H^0(\mathcal{O}_C(-q)(1)) \xrightarrow{\mu_q^3} H^0(\lambda \otimes \mathcal{O}_C(-q)(4)) \tag{5.28}$$

be the multiplication map. We claim that μ^3_q is surjective i.e. that

$$Im(\mu^3_q) = H^0(\lambda \otimes \mathcal{O}_C(-q)(4)). \tag{5.29}$$

To prove this we apply Claim (5.4). Let C be our plane sextic, $L = \lambda(3)$ and $M = \mathcal{O}_C(-q)(1)$. We check that the hypotheses of Claim (5.4) are satisfied. Clearly $\mathcal{O}_C(-q)(1)$ is globally generated. By Serre duality

$$h^1(L \otimes M^{-1}) = h^1(\mathcal{O}_C(q) \otimes \lambda(2)) = h^0(\mathcal{O}_C(-q) \otimes \lambda(1)) \tag{5.30}$$

and since q is outside the support of the unique (up to scalars) non-vanishing section of $\lambda(1)$ we get that $h^0(\mathcal{O}_C(-q) \otimes \lambda(1)) = 0$; thus $h^1(L \otimes M^{-1}) = 0$. This proves (5.29). Of course $Im(\mu^3_q)$ is contained in the image of μ^3 and thus

$$Im(\mu^3) \supset H^0(\lambda \otimes \mathcal{O}_C(-q)(4)) \quad q \notin suppD. \tag{5.31}$$

(As above D is the unique element of $|\lambda(1)|$.) The line-bundle $\lambda(4)$ is very ample because $\deg(\lambda(4)) > 20 = \deg(K_C) + 2$ and hence $H^0(\lambda \otimes \mathcal{O}_C(-q)(4)) \neq H^0(\lambda \otimes \mathcal{O}_C(-q')(4))$ for $q \neq q'$; since both subspaces have codimension 1 in $H^0(\lambda(4))$ it follows from (5.31) that μ^3 is surjective. This finishes the proof that (5.17) is surjective for $k \geq 3$. Now we apply an argument of Gallego - Purnaprajna (Observation (1.3) of [10]) in order to prove that Map (5.16) is surjective. Let $X = X_4 \supset X_3 \supset X_2 \supset X_1 = \widetilde{C}$ be a chain of smooth linear sections of X, i.e. $X_3 \in |H|$ and $X_2 = D \cap D'$ where $D, D' \in |H|$ intersect transversely. Let

$$\mu^k_{X_i} \colon H^0(\mathcal{O}_{X_i}(kH)) \otimes H^0(\mathcal{O}_{X_i}(H)) \longrightarrow H^0(\mathcal{O}_{X_i}((k+1)H)) \tag{5.32}$$

be the multiplication map.

Claim 5.5. *Keep notation as above. Let $1 \leq i \leq 3$ and $k \geq 3$. If $\mu^k_{X_i}$ is surjective then $\mu^k_{X_{i+1}}$ is surjective as well.*

Proof. The restriction map

$$\rho^n_{X_{i+1}} \colon H^0(\mathcal{O}_{X_{i+1}}(nH)) \to H^0(\mathcal{O}_{X_i}(nH)). \tag{5.33}$$

is surjective for $n = 1$ and $n \geq 3$: in fact

$$h^1(\mathcal{O}_{X_{i+1}}((n-1)H)) = 0, \quad n = 1, \quad n \geq 3, \tag{5.34}$$

by the Lefschetz Hyperplane Section Theorem and Kodaira Vanishing. Now suppose that $k \geq 3$ and let $\sigma \in H^0(\mathcal{O}_{X_{i+1}}((k+1)H))$; by hypothesis $\rho^{k+1}_{X_{i+1}}(\sigma) \in Im(\mu^k_{X_i})$ and hence

$$\rho^{k+1}_{X_{i+1}}(\sigma) = \mu^k_{X_i}(\sum_{j \in J} \tau_j \otimes \nu_j), \quad \tau_j \in H^0(\mathcal{O}_{X_i}(kH)), \quad \nu_j \in H^0(\mathcal{O}_{X_i}(H)). \tag{5.35}$$

Since $\rho^k_{X_{i+1}}$ and $\rho^1_{X_{i+1}}$ are both surjective there exist $\widetilde{\tau}_j \in H^0(\mathcal{O}_{X_{i+1}}(kH))$ and $\widetilde{\nu}_j \in H^0(\mathcal{O}_{X_{i+1}}(H))$ such that $\rho^k_{X_{i+1}}(\widetilde{\tau}_j) = \tau_j$ and $\rho^1_{X_{i+1}}(\widetilde{\nu}_j) = \nu_j$. Then

$$\sigma = \mu^k_{X_{i+1}}(\sum_{j \in J} \widetilde{\tau}_j \otimes \widetilde{\nu}_j) + \epsilon \tag{5.36}$$

where $\epsilon \in H^0(\mathcal{O}_{X_{i+1}}((k+1)H))$ vanishes on X_i. Since X_i is the zero-locus of a section of $\mathcal{O}_{X_{i+1}}(H)$ we have $\epsilon = \mu^k_{X_{i+1}}(\epsilon' \otimes \epsilon'')$ where $\epsilon'' \in H^0(\mathcal{O}_{X_{i+1}}(H))$ is the unique (up to scalars) section whose zero-locus is X_i. Thus $\sigma \in Im(\mu^k_{X_{i+1}})$; since σ was arbitrary this shows that $\mu^k_{X_{i+1}}$ is surjective. $\square$

Now we finish proving the proposition. Since $X_1 = \widetilde{C}$ surjectivity of $\mu^k_{X_1}$ for $k \geq 3$ has been proved above. By Claim (5.5) it follows that $\mu^k_{X_4}$ is surjective for $k \geq 3$; since $\mu^k_{X_4}$ is Map (5.16) this proves the proposition. $\square$

Let

$$\theta := \eta \otimes \mathcal{O}_Y(3). \tag{5.37}$$

Corollary 5.6. *Keep notation and hypotheses as above. Then θ is globally generated.*

Proof. Let $H^0(\mathcal{O}_X(3H))^- \subset H^0(\mathcal{O}_X(3H))$ be the (-1)-eigenspace for the action of ϕ^*. Then

$$H^0(\mathcal{O}_X(3H)) = H^0(\mathcal{O}_Y(3H)) \oplus H^0(\theta), \quad H^0(\theta) = H^0(\mathcal{O}_X(3H))^-. \tag{5.38}$$

Let $p \in Y$ and let θ_p be the fiber of θ at p; we must show that

$$H^0(\mathcal{O}_X(3H))^- \longrightarrow \theta_p \tag{5.39}$$

is surjective. Assume that $p \in (Y \setminus singY)$. Let $f^{-1}(p) = \{p_1, p_2\}$ and let L_{p_i} be the fiber of $\mathcal{O}_X(3H)$ at p_i. Since $\mathcal{O}_X(3H)$ is very ample the evaluation map $H^0(\mathcal{O}_X(3H)) \to (L_{p_1} \oplus L_{p_2})$ is surjective. Since θ_p is identified with the (-1)-eigenspace for the action of ϕ^* on $(L_{p_1} \oplus L_{p_2})$ we get that (5.39) is surjective. Finally assume that $p \in singY$ and let $f^{-1}(p) = \{\widetilde{p}\}$. Then ϕ acts on $\Omega^1_{X,\widetilde{p}}$ and since ϕ is an involution we have a decomposition

$$\Omega^1_{X,\widetilde{p}} = V^+_{\widetilde{p}} \oplus V^-_{\widetilde{p}} \tag{5.40}$$

where $V^{\pm}_{\widetilde{p}}$ is the $(\pm)$-eigenspace for the action of ϕ (since ϕ is anti-symplectic $\dim V^{\pm}_{\widetilde{p}} = 2$). Equation (5.6) gives an identification between θ_p and $V^-_{\widetilde{p}}$; with this identification Map (5.39) gets identified with

$$\begin{array}{ccc} H^0(\mathcal{O}_X(3H))^- & \longrightarrow & V^-_{\widetilde{p}} \\ \sigma & \mapsto & d\sigma(\widetilde{p}). \end{array} \tag{5.41}$$

Since $\mathcal{O}_X(3H)$ is very ample the differential at $\widetilde{p}$ of the map $X \to |3H|^\vee$ is injective and hence

$$\begin{array}{ccc} H^0(I_{\widetilde{p}}(3H)) & \xrightarrow{\Psi} & \Omega^1_{X,\widetilde{p}} \\ \sigma & \mapsto & d\sigma(\widetilde{p}). \end{array} \tag{5.42}$$

is surjective. Since $\Psi(H^0(I_{\widetilde{p}}(3H))^+) \subset V^+_{\widetilde{p}}$ we get that $\Psi(H^0(I_{\widetilde{p}}(3H))^-) = V^-_{\widetilde{p}}$. Of course $H^0(I_{\widetilde{p}}(3H))^- = H^0(\mathcal{O}_X(3H))^-$ and hence we get that (5.41) is surjective. This finishes the proof that (5.39) is a surjection. $\square$

Let $\epsilon \colon H^0(\theta) \otimes \mathcal{O}_{|H|^\vee} \to j_*\theta$ be the evaluation map. By the above corollary ϵ is surjective; let G be the kernel of ϵ. Thus we have an exact sequence

$$0 \to G \longrightarrow H^0(\theta) \otimes \mathcal{O}_{|H|^\vee} \xrightarrow{\epsilon} j_*\theta \to 0. \tag{5.43}$$

Proposition 5.7. *Keep notation and assumptions as above. Then*

$$G \cong \Omega^3_{|H|^\vee}(3). \tag{5.44}$$

Proof. First we prove that G is locally-free. By Claim (5.1) the sheaf $j_*\eta$ is Casnati-Catanese and hence so is $j_*\theta$. By (1) of Proposition (4.3) we get that G is locally-free. Since G is locally-free Beilinson's spectral sequence with

$$E_1^{p,q} = H^q(G(p)) \otimes \Omega^{-p}_{|H|^\vee}(-p) \tag{5.45}$$

converges in degree 0 to the graded sheaf associated to a filtration on G, see [16] p. 240. Thus it suffices to prove that

$$h^q(G(p)) = \begin{cases} 0 & \text{if } -5 \leq p \leq 0 \text{ and } (p,q) \neq (-3,3), \\ 1 & \text{if } (p,q) = (-3,3). \end{cases} \tag{5.46}$$

This follows from a straightforward computation which goes as follows. Tensorizing (5.43) by $\mathcal{O}_{|H|^\vee}(p)$ and taking the associated cohomology exact sequence we get

$$\begin{aligned} \cdots \to H^0(\theta) \otimes H^{q-1}(\mathcal{O}_{|H|^\vee}(p)) \longrightarrow H^{q-1}(\theta(p)) \xrightarrow{\partial} \\ \xrightarrow{\partial} H^q(G(p)) \to H^0(\theta) \otimes H^q(\mathcal{O}_{|H|^\vee}(p)) \longrightarrow \cdots \end{aligned} \tag{5.47}$$

(We let $\theta(p) := \theta \otimes \mathcal{O}_Y(p)$.) From this one easily gets that $h^1(G) = 0$ and that

$$h^{q-1}(\theta(p)) = h^q(G(p)), \quad -5 \leq p \leq 0,\ (p,q) \neq (0,1). \tag{5.48}$$

Thus in order to prove (5.46) it suffices to show that

$$h^r(\theta(p)) = \begin{cases} 0 & \text{if } -5 \leq p \leq 0 \text{ and } (p,r) \notin \{(0,0),(-3,2)\}, \\ 1 & \text{if } (p,r) = (-3,2). \end{cases} \tag{5.49}$$

The map $f\colon X \to Y$ is finite and we have (5.3)-(5.37); thus

$$h^r(\mathcal{O}_X((3+p)H)) = h^r(f_*\mathcal{O}_X((3+p)H)) = h^r(\mathcal{O}_Y(3+p)) + h^r(\theta(p)). \tag{5.50}$$

We will compute $h^r(\theta(p))$ by computing $h^r(\mathcal{O}_X((3+p)H))$ and $h^r(\mathcal{O}_Y(3+p))$. First we claim that

$$h^0(\mathcal{O}_X((3+p)H)) = h^0(\mathcal{O}_Y((3+p)H)), \quad p \leq -1. \tag{5.51}$$

For $p \leq -3$ the equation is trivial and for $p = -2$ it holds by hypothesis. To check equality for $p = -1$ we apply Formula (4.0.4) of [15]:

$$\chi(\mathcal{O}_X(nH)) = \frac{1}{2}n^4 + \frac{5}{2}n^2 + 3. \tag{5.52}$$

Since H is ample Kodaira Vanishing gives that $h^0(\mathcal{O}_X(2H)) = \chi(\mathcal{O}_X(2H))$ and by the above formula we get $h^0(\mathcal{O}_X(2H)) = 21$. On the other hand a straightforward computation gives that $h^0(\mathcal{O}_Y(2)) = 21$ and this finishes the proof of (5.51); from (5.51)-(5.50) we get that

$$h^0(\theta(p)) = 0, \quad p \leq -1. \tag{5.53}$$

Next we show that

$$h^4(\mathcal{O}_X((3+p)H)) = h^4(\mathcal{O}_Y((3+p)H)), \quad -5 \le p. \tag{5.54}$$

Since $K_X \sim 0$ Serre duality gives that

$$h^4(\mathcal{O}_X((3+p)H)) = h^0(\mathcal{O}_X((-3-p)H)). \tag{5.55}$$

Now write $\mathcal{O}_X((-3-p)H) = \mathcal{O}_X(3+(-6-p)H))$; if $-5 \le p$ then $(-6-p) \le -1$ and hence by (5.51)

$$h^0(\mathcal{O}_X((-3-p)H)) = h^0(\mathcal{O}_Y(-3-p)), \quad -5 \le p. \tag{5.56}$$

Since Y is a sextic hypersurface in $\mathbb{P}^5$ the dualizing sheaf of Y is trivial and hence Serre duality gives that

$$h^0(\mathcal{O}_Y(-3-p)) = h^4(\mathcal{O}_Y(3+p)). \tag{5.57}$$

Equation (5.54) follows from (5.55), (5.56) and (5.57). From (5.54) and (5.50) we get that

$$h^4(\theta(p)) = 0, \quad -5 \le p. \tag{5.58}$$

Kodaira Vanishing gives that $h^r(\mathcal{O}_X((3+p)H)) = 0$ for $0 < r < 4$ and $p \ne -3$; by (5.50) we get that

$$h^r(\theta(p)) = 0 \text{ for } 0 < r < 4 \text{ and } p \ne -3. \tag{5.59}$$

Of course $h^r(\theta(p)) = 0$ for $r < 0$ and $h^r(\theta(p)) = 0$ for $4 < r$ because θ is supported on the 4-dimensional variety Y. Thus in order to finish the proof of (5.49) we must show that

$$h^r(\theta(-3)) = \begin{cases} 0 & \text{if } r = 1, 3, \\ 1 & \text{if } r = 2. \end{cases} \tag{5.60}$$

From (5.50) we get that

$$h^r(\mathcal{O}_X) = h^r(\mathcal{O}_Y) + h^r(\theta(-3)). \tag{5.61}$$

We have

$$0 = h^1(\mathcal{O}_X) = h^3(\mathcal{O}_X) = h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = h^3(\mathcal{O}_Y), \quad h^2(\mathcal{O}_X) = 1. \tag{5.62}$$

Equation (5.60) follows from (5.61) and (5.62). $\square$

By the above proposition and by (5.43) we have an exact sequence

$$0 \to \Omega^3_{|H|^\vee}(3) \xrightarrow{\kappa} H^0(\theta) \otimes \mathcal{O}_{|H|^\vee} \xrightarrow{\epsilon} j_*\theta \to 0. \tag{5.63}$$

5.2 Proof of Theorem (5.2)

Claim 5.8. *Keep notation as above. There exists an isomorphism*

$$\beta \colon j_*\theta \xrightarrow{\sim} Ext^1(j_*\theta, \mathcal{O}_{|H|^\vee}). \tag{5.64}$$

Proof. Since $j_*\theta$ is a Casnati-Catanese sheaf we have an isomorphism

$$Ext^1(j_*\theta, \mathcal{O}_{|H|^\vee}) \cong j_*\left(\theta^\vee \otimes N_{Y/|H|^\vee}\right) \tag{5.65}$$

because of Item (2) of Proposition (4.3). By (5.37) and Isomorphism (5.10) we get

$$j_*\left(\theta^\vee \otimes N_{Y/|H|^\vee}\right) = j_*\left(\eta^\vee \otimes \mathcal{O}_Y(-3) \otimes N_{Y/|H|^\vee}\right) \cong j_*\left(\eta \otimes \mathcal{O}_Y(-3) \otimes N_{Y/|H|^\vee}\right). \tag{5.66}$$

By (5.1) we have $N_{Y/|H|^\vee} \cong \mathcal{O}_Y(6)$ and hence we get that

$$j_*\left(\eta \otimes \mathcal{O}_Y(-3) \otimes N_{Y/|H|^\vee}\right) \cong j_*\left(\eta \otimes \mathcal{O}_Y(3)\right) = j_*\theta. \tag{5.67}$$

The above equations prove (5.64). □

Let κ, ϵ be as in (5.63) and β be as in (5.64). We claim that there exists a map

$$s\colon H^0(\theta) \otimes \mathcal{O}_{|H|^\vee} \to \left(\Omega^3_{|H|^\vee}(3)\right)^\vee = \Theta^3_{|H|^\vee}(-3) \tag{5.68}$$

such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \Omega^3_{|H|^\vee}(3) \arrow[r, "\kappa"] \arrow[d, "s^\vee"] & H^0(\theta) \otimes \mathcal{O}_{|H|^\vee} \arrow[r, "\epsilon"] \arrow[d, "s"] & j_*\theta \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & H^0(\theta)^\vee \otimes \mathcal{O}_{|H|^\vee} \arrow[r, "\kappa^\vee"] & \Theta^3_{|H|^\vee}(-3) \arrow[r, "\partial"] & Ext^1(j_*\theta, \mathcal{O}_{|H|^\vee}) \arrow[r] & 0 \end{tikzcd} \tag{5.69}$$

In fact this follows from the results of Casnati-Catanese [3] or of Eisenbud-Popescu-Walter [4]: by the proof of Claim (2.1) of [3] the obstruction to existence of s lies in $H^1(Sym_2\left(H^0(\theta) \otimes \mathcal{O}_{|H|^\vee}\right)^\vee)$ which is zero and hence s exists.

Remark 5.9. Proposition (1.6) of [3] does not hold with $\mathcal{F} = j_*\theta$ because $\chi(j_*\theta(-3))$ is not even, see Theorem (9.1) of [4] - in fact $\chi(j_*\theta(-3)) = 1$. Thus unlike the surfaces considered by Casnati-Catanese the 4-fold Y cannot be presented as the degeneracy locus of a symmetric map of vector-bundles.

Claim 5.10. *Keep notation and assumptions as above. Then*

$$\Omega^3_{|H|^\vee}(3) \xrightarrow{(\kappa, s^\vee)} \left(H^0(\theta) \oplus H^0(\theta)^\vee\right) \otimes \mathcal{O}_{|H|^\vee} \tag{5.70}$$

is an injection of vector-bundles. The image of $(\kappa, s^\vee)$ is Lagrangian for the tautological symplectic form on $H^0(\theta) \oplus H^0(\theta)^\vee$ given by

$$\lambda((\alpha, \psi), (\alpha', \psi')) := \psi(\alpha) - \psi'(\alpha'). \tag{5.71}$$

Proof. The sheaf $j_*\theta$ is a Casnati-Catanese sheaf on $|H|^\vee$; since β is an isomorphism we get by Claim (4.5) that $(\kappa, s^\vee)$ is an injection of vector-bundles. The tautological symplectic form vanishes on $Im(\kappa, s^\vee)$ by commutativity of Diagram (5.69). Since $\Omega^3_{|H|^\vee}(3)$ has rank 10 it follows that $Im(\kappa, s^\vee)$ is Lagrangian. □

We will show that Diagram (5.69) can be identified with Diagram (4.25) for a suitable A. Let V be a 6-dimensional complex vector-space and $F \hookrightarrow \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ be the sub-vector-bundle defined by (1.5).

Proposition 5.11. *Keep notation as above. Then*

$$F \cong \Omega^3_{\mathbb{P}(V)}(3). \tag{5.72}$$

Proof. Let $Q := \Theta_{\mathbb{P}(V)}(-1)$. Thus we have the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow Q \to 0 \tag{5.73}$$

and by definition

$$F \cong (\wedge^2 Q)(-1). \tag{5.74}$$

The perfect pairing $\wedge^2 Q \times \wedge^3 Q \to \wedge^5 Q \cong \mathcal{O}_{\mathbb{P}(V)}(1)$ gives an isomorphism $\wedge^2 Q \cong (\wedge^3 Q^\vee)(1)$ and hence

$$F \cong \wedge^3 Q^\vee \cong \Omega^3_{\mathbb{P}(V)}(3). \tag{5.75}$$

□

In order to identify (5.69) with (4.25) we will need a few properties of the vector-bundle $F \cong \Omega^3_{\mathbb{P}(V)}(3)$. First of all Bott Vanishing gives that

$$h^i(F) = 0 \text{ for all } i. \tag{5.76}$$

Proposition 5.12. *Keep notation as above. Assume that $\mathcal{W}$ is a symplectic vector-bundle and that $\mu\colon F \to \mathcal{W}$ is an injection of vector-bundles such that $\mu(F)$ is a Lagrangian sub-vector-bundle. Then $\mu^\vee\colon \mathcal{W}^\vee \to F^\vee$ induces an isomorphism*

$$H^0(\mathcal{W}^\vee) \xrightarrow{\sim} H^0(F^\vee). \tag{5.77}$$

Proof. Since $\mu(F)$ is Lagrangian the symplectic form on $\mathcal{W}$ induces an isomorphism $\mathcal{W}/\mu(F) \cong F^\vee$. Thus we have an exact sequence

$$0 \to F \xrightarrow{\mu} \mathcal{W} \longrightarrow F^\vee \to 0. \tag{5.78}$$

and its dual

$$0 \to F \longrightarrow \mathcal{W}^\vee \xrightarrow{\mu^\vee} F^\vee \to 0. \tag{5.79}$$

The proposition follows at once from (5.76). □

Applying the above proposition to (1.6) we get an isomorphism

$$\wedge^3 V^\vee \xrightarrow{\sim} H^0(F^\vee). \tag{5.80}$$

Now let $V := H^0(\mathcal{O}_X(H))^\vee$: by Proposition (5.11) we have $\Omega^3_{|H|^\vee}(3) \cong F$. By Claim (5.10), Equation (5.80) and Proposition (5.12) we have a sequence of isomorphisms

$$\wedge^3 V^\vee \xrightarrow{\sim} H^0(F^\vee) \xrightarrow{\sim} H^0(\wedge^3 \Theta_{|H|^\vee}(-3)) \xrightarrow{\sim} H^0(\theta)^\vee \oplus H^0(\theta). \tag{5.81}$$

Let

$$\rho\colon H^0(\theta) \oplus H^0(\theta)^\vee \xrightarrow{\sim} \wedge^3 V \tag{5.82}$$

be the transpose of the composition of the maps in (5.81). Then (abusing notation)

$$\rho(\Omega^3_{|H|^\vee}(3)) = F. \tag{5.83}$$

Thus (5.69) starts looking like (4.25). One missing link: we have not proved that $\rho(H^0(\theta)^\vee)$ is Lagrangian for the symplectic form σ on $\wedge^3 V$ defined in Section (1).

Proposition 5.13. *Let V be a 6-dimensional complex vector-space. Suppose that $\tau\colon \wedge^3 V \times \wedge^3 V \to \mathbb{C}$ is a symplectic form such that $\tau|_{F_\ell} = 0$ for every $\ell \in \mathbb{P}(V)$. Then $\tau = c\sigma$ for a certain $c \in \mathbb{C}^*$.*

Proof. Let $\{e_0, \ldots, e_5\}$ be a basis of V. Let $\mathcal{S} \subset \mathcal{P}(\{0, \ldots, 5\})$ be the family of $I \subset \{0, \ldots, 5\}$ of cardinality 3. For $\{i, j, k\} \in \mathcal{S}$ with $i < j < k$ we let $e_I := e_i \wedge e_j \wedge e_k$; then $\{\ldots, e_I, \ldots\}_{I \in \mathcal{S}}$ is basis of $\wedge^3 V$. Let $\alpha, \beta \in \wedge^2 V$. For $i \in \{0, \ldots, 5\}$ we have $e_i \wedge \alpha, e_i \wedge \beta \in F_{e_i}$; by our hypothesis we get that $\tau(e_i \wedge \alpha, e_i \wedge \beta) = 0$ and hence

$$\tau(e_I, e_J) = 0 \text{ if } I \cap J \neq \emptyset. \tag{5.84}$$

Let $j \in \{0, \ldots, 5\}$; then

$$0 = \tau((e_i + e_j) \wedge \alpha, (e_i + e_j) \wedge \beta) = \tau(e_i \wedge \alpha, e_j \wedge \beta) + \tau(e_j \wedge \alpha, e_i \wedge \beta). \tag{5.85}$$

This implies that

$$sign(I, I^c)\tau(e_I, e_{I^c}) = sign(J, J^c)\tau(e_J, e_{J^c}), \quad I, J \in \mathcal{S} \tag{5.86}$$

where I^c, J^c are the complements of I, J in $\{0, \ldots, 5\}$ respectively. The proposition is an immediate consequence of (5.84) and (5.86). □

Corollary 5.14. *Keep notation and hypotheses as above. Then*

$$\rho^*\sigma = c\lambda \tag{5.87}$$

where ρ is the isomorphism of (5.83), σ is the symplectic form defined in Section (1), c is a non-zero constant and λ is the tautological symplectic form defined in (5.71).

Proof. The corollary is equivalent to the equality

$$(\rho^{-1})^*\lambda = c^{-1}\sigma, \quad c \in \mathbb{C}^*. \tag{5.88}$$

By Claim (5.10) $\rho(\Omega^3_{|H|^\vee}(3))$ (yes, we abuse notation again) is a Lagrangian sub-vector-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ equipped with symplectic form $(\rho^{-1})^*\lambda$. By Equality (5.83) we get that for any $\ell \in \mathbb{P}(V)$ the restriction of $(\rho^{-1})^*\lambda$ to F_ℓ is zero; by Proposition (5.13) we get that (5.88) holds. □

Let $A := \rho(H^0(\theta)^\vee)$. Since $H^0(\theta)^\vee$ is a Lagrangian subspace of $H^0(\theta) \oplus H^0(\theta)^\vee$ equipped with the symplectic form λ the above corollary gives that $A \in \mathbb{LG}(\wedge^3 V)$. It is clear from Diagram (5.69) that we have equality of reduced $Y = (Y_A)_{red}$ where $(Y_A)_{red}$ is the reduced Y_A. Since $\deg Y = 6 = \deg(Y_A)$ and Y, Y_A are both Cartier divisors we get that

$$Y = Y_A. \tag{5.89}$$

Claim 5.15. *Keep notation and assumptions as above. Then $A \in \mathbb{LG}(\wedge^3 V)^0$.*

Proof. First notice that $A \in \mathbb{LG}(\wedge^3 V)^\times$ simply because $j_*\theta$ is a Casnati-Catanese sheaf. We notice also that

$$D_1(A, F) \setminus D_2(A, F) = Y \setminus singY. \tag{5.90}$$

Thus we may apply Proposition (2.8) in order to prove the claim. Item (3) of the proposition is satisfied by (5.90) and (5.8) and hence $A \in \mathbb{LG}(\wedge^3 V)^0$. Alternatively one can check that Item (2) of Proposition (2.8) holds. The only unproved fact is that W_A is smooth. By (2.13)-(2.14) and (5.2) we know that W_A is a local complete intersection; since $singY = (W_A)_{red}$ and $singY$ is smooth we get that if W_A is not smooth then it is not reduced. Thus by Formula (2.9) we get that it suffices to show that $\deg(singY) = 40$; this follows at once form the formulae in Item (1) of Theorem (1.1) of [15]. □

Comparing Diagrams (4.25) and (5.69) we see that $\theta = \zeta_A$; by (4.19) and (5.37) we get that $\eta = \xi_A$. This completes the proof of Theorem (5.2).

6 An involution on a moduli space

We start by proving an elementary GIT result that will give a structure of quasi-projective variety to the set of isomorphism classes of EPW-sextics parametrized by $\mathbb{LG}(\wedge^3 V)^0$. Let W be a finite-dimensional complex vector space with $\dim W = n+1$. Let $Z \subset \mathbb{P}(W)$ be a hypersurface of degree $(n+1)$: we say that Z is *stable* if the corresponding point of $\mathbb{P}(Sym^{n+1} W^\vee)$ is stable for the standard $\mathbb{PGL}(W)$-action on $\mathbb{P}(Sym^{n+1} W^\vee)$ with respect to the unique linearization of the action, see [13].

Proposition 6.1. *Keep notation as above. Assume that $Z \subset \mathbb{P}(W)$ is a hypersurface of degree $(n+1)$ such that if $q \in singV(P)$ then $mult_q V(P) = 2$ and the tangent cone of Z at q is a quadric of rank at least 3. Then Z is stable.*

Proof. We will apply the Hilbert-Mumford numerical criterion for stability. Let $\lambda\colon \mathbb{C}^\times \to SL(W)$ be a an arbitrary one-parameter subgroup: there exist homogeneous coordinates $X_0, \ldots, X_n \in W^\vee$ which diagonalize λ and

$$\lambda(t) = diag(t^{a_0}, \ldots, t^{a_n}), \quad a_0 \le a_1 \le \cdots \le a_n, \quad a_0 < 0 < a_n. \tag{6.1}$$

Let $Z = V(P)$ where

$$P = \sum_{I \in M} c_I X^I, \tag{6.2}$$

M is the set of multiindices $I = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = n+1$ and $X^I := \prod_{s=0}^n X_s^{i_s}$. By the numerical criterion, see p. 50 of [11], it suffices to prove that there exists $I \in M$ such that

$$c_I \ne 0, \quad \langle \lambda, I \rangle := \sum_{s=0}^n a_s i_s < 0. \tag{6.3}$$

If $P(1, 0, \ldots, 0) \ne 0$ then $I = (n+1, 0, \ldots, 0)$ does the job. Now assume that $P(1, 0, \ldots, 0) = 0$ i.e. $q := [1, 0, \ldots, 0] \in V(P)$. We distinguish two cases: q is a smooth point of $V(P)$ and q is a singular point of $V(P)$. In the former case there exists $J = (n, j_1, \ldots, j_n)$ with $j_k = 1$ for some $1 \le k \le n$ such that $c_J \ne 0$. We have

$$\langle \lambda, J \rangle = n a_0 + a_k \le a_0 + a_1 + \cdots a_n = 0. \tag{6.4}$$

If the inequality appearing above is strict we set $I = J$ and we are done. Now assume that the inequality is an equality; by (6.1) we know that $a_0 < a_n$ hence $k = n$ and

$$a_0 = \cdots = a_{n-1}. \tag{6.5}$$

By our hypothesis $V(P)$ is reduced and irreducible thus there exists $I = (i_0, \ldots i_{n-1}, 0)$ such that $c_I \neq 0$; then $\langle \lambda, I \rangle < 0$ by (6.5). Lastly suppose that $q \in singV(P)$. Let Ω be the set of multiindices $J = (n-1, j_1, \ldots, j_{n-1}, 0)$ such that $c_J \neq 0$. The set Ω is not empty because $rk(C_q V(P)) \geq 3$. Suppose that there exists $I \in \Omega$ with $j_{n-1} \leq 1$; an easy computation gives that $\langle \lambda, I \rangle < 0$ and we are done. Thus we may assume that

$$\Omega = \{J\}, \quad J := (n-1, 0, \ldots, 2, 0). \tag{6.6}$$

As is easily checked $\langle \lambda, J \rangle \leq 0$ with equality if and only if

$$a_0 = \cdots = a_{n-2}, \quad a_{n-1} = a_n. \tag{6.7}$$

Thus if (6.7) does not hold we are done. We are left to deal with P such that both (6.6) and (6.7) hold. Since $rk(C_q V(P)) \geq 3$ there exists $I = (n-1, i_1, \ldots, i_n)$ with $i_k > 0$ for some $1 \leq k \leq n-2$ such that $c_I \neq 0$; then $\langle \lambda, I \rangle < 0$ by (6.7). $\square$

Now let V be a complex vector space of dimension 6. We have a regular map

$$\begin{array}{ccc} \mathbb{LG}(\wedge^3 V)^0 & \longrightarrow & \mathbb{P}(Sym^6 V^\vee) \\ A & \mapsto & Y_A \end{array} \tag{6.8}$$

with image a locally closed $\mathbb{PGL}(V)$-invariant subset $\mathcal{EPW}(V)^0$. Thus $\mathbb{PGL}(V)$ acts on $\mathcal{EPW}(V)^0$; let

$$\mathcal{K}_2^0 = \mathcal{EPW}(V)^0 // \mathbb{PGL}(V). \tag{6.9}$$

If $A \in \mathbb{LG}(\wedge^3 V)^0$ then by Proposition (4.1) the sextic Y_A satisfies the hypothesis of Proposition (6.1). Thus all points of $\mathcal{EPW}(V)^0$ are $\mathbb{PGL}(V)$-stable and hence the points of $\mathcal{K}_2^0$ are in one-to-one correspondence with projective equivalence classes of EPW-sextics parametrized by $A \in \mathbb{LG}(\wedge^3 V)^0$. We are really interested in isomorphism classes of double covers of EPW-sextics. Let (X, H) and (X', H') be polarized projective varieties (i.e. H, H' are ample divisor classes on X and X' respectively); then (X, H) *is isomorphic to* (X', H') if there exists an isomorphism $\psi \colon X \xrightarrow{\sim} X'$ such that $\psi^* H' \sim H$. For $A \in \mathbb{LG}(\wedge^3 V)^0$ we let $f_A \colon X_A \to Y_A$ be the natural double cover defined in Section (4) and $H_A := f_A^* \mathcal{O}_{Y_A}(1)$.

Proposition 6.2. *Let $A_i \in \mathbb{LG}(\wedge^3 V)^0$ for $i = 1, 2$. Then (X_{A_1}, H_{A_1}) is isomorphic to (X_{A_2}, H_{A_2}) if and only if Y_{A_1} is projectively equivalent to Y_{A_2}.*

Proof. Assume that (X_{A_1}, H_{A_1}) is isomorphic to (X_{A_2}, H_{A_2}). Since Y_{A_i} is the image of the map $X_{A_i} \to |H_{A_i}|^\vee$ we get that Y_{A_1} is projectively equivalent to Y_{A_2}. Now suppose that Y_{A_1} is projectively equivalent to Y_{A_2}. Since $f_{A_i} \colon X_{A_i} \to Y_{A_i}$ is a double cover ramified over $singY_{A_i}$ it suffices to show that if $A \in \mathbb{LG}(\wedge^3 V)^0$ then there is (up to isomorphism) a unique double cover of Y_A ramified over $singY_A$. Thus it suffices to prove that

$$\pi_1(Y_A^{sm}) \cong \mathbb{Z}_2. \tag{6.10}$$

Let $Z_A := f_A^{-1}(singY_A)$; we have an étale double cover

$$(X_A \setminus Z_A) \to Y_A^{sm}. \tag{6.11}$$

X_A is simply connected because it is a deformation of $(K3)^{[2]}$. Since $\mathrm{cod}(Z_A, X_A) = 2$ we get that $(X_A \setminus Z_A)$ is simply connected. Thus (6.11) is the universal double cover of Y_A^{sm}; since it has degree 2 we get that (6.10) holds. □

By Theorem (1.1) and the above proposition points of $\mathcal{K}_2^0$ are in one-to-one correspondence with isomorphism classes of couples (X, H) where X is a numerical $(K3)^{[2]}$ and H an ample divisor on X such that (1.3) and (a) of Section (1) both hold. We will view $\mathcal{K}_2^0$ as the moduli space of such couples; for $A \in \mathbb{LG}(\wedge^3 V)^0$ we let $[X_A] \in \mathcal{K}_2^0$ be the point corresponding to the isomorphism class of (X_A, H_A). We remark that the second equality of (1.3) follows from the first one. Furthermore the first equality of (1.3) should be thought of as the analogue of self-intersection 2 for an ample divisor on a $K3$ surface, see the observation in parentheses which follows (1.3). We recall also that Condition (a) is an open condition. Thus $\mathcal{K}_2^0$ is an open subset of the (a priori non-separated) moduli space of couples (X, H) where X is a numerical $(K3)^{[2]}$ and H is an ample divisor on X of square 2 for Beauville's quadratic form. Let $\delta_V : \mathbb{LG}(\wedge^3 V) \xrightarrow{\sim} \mathbb{LG}(\wedge^3 V^\vee)$ be the isomorphism of (3.2). The subset of $\mathbb{LG}(\wedge^3 V)^0$ appearing in (3.3) is open dense and $\mathbb{PGL}(V)$-invariant hence

$$\mathcal{U} := \left(\mathbb{LG}(\wedge^3 V)^0 \cap \delta_V^{-1}\mathbb{LG}(\wedge^3 V^\vee)^0\right) //\mathbb{PGL}(V). \tag{6.12}$$

is an open and dense subset of $\mathcal{K}_2^0$. If $[X_A] \in \mathcal{U}$ then we have the nautural double cover $X_{A^\perp} \to Y_{A^\perp}$ and furthermore $[X_{A^\perp}] \in \mathcal{K}_2^0$ by Theorem (1.1). Hence duality defines a regular involution

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\delta} & \mathcal{U} \\ [X_A] & \mapsto & [X_{A^\perp}] \end{array} \tag{6.13}$$

Let's prove that δ is not the identity. We consider the example of Mukai that we presented in the proof that Theorem (5.2) implies Theorem (1.1). Thus S is a $K3$ given by (5.12) satisfying (5.13), and Y, X are given by (5.14) and (5.15) respectively. Then $Y \subset |I_S(2)|$ corresponds to a certain $M \in \mathbb{LG}(\wedge^3 H^0(I_S(2)))^0$ i.e. $Y = Y_M$. According to Subsubsection (5.4.2) of [14] the dual $Y_M^\vee \subset |I_S(2)|^\vee$ is described as follows. By (5.13) S is cut out by quadrics and contains no lines hence we have a well-defined regular map

$$\begin{array}{ccc} S^{[2]} & \xrightarrow{g} & |I_S(2)|^\vee \\ [Z] & \mapsto & \{Q \mid Q \supset \langle Z\rangle\} \end{array} \tag{6.14}$$

where $\langle Z\rangle \subset \mathbb{P}^6$ is the unique line containing Z. Then $Y_M^\vee = Im(g)$. One has $|I_F(2)| \in Im(g)$ - in fact $|I_F(2)|$ is the image by g of

$$B_S := \{[Z] \in S^{[2]} \mid \langle Z\rangle \subset F\}. \tag{6.15}$$

Iskovskih (Cor. (6.6) of [9]) proved that $B_S \cong \mathbb{P}^2$. We proved that g has degree 2 onto its image and that

$$\deg Y_M^\vee = 6. \tag{6.16}$$

For $[Z] \in (S^{[2]} \setminus B_S)$ there exists a unique conic $C \subset F$ containing Z. Then $C \cap S$ is a scheme of length 4 containing Z and there is a well-defined residual scheme Z' of Z, see Subsection (4.3) of [14]. Let $\phi([Z]) := [Z']$; then

$$g^{-1}(g([Z])) = \{[Z], \phi([Z])\}. \tag{6.17}$$

For $[Z]$ generic Z' can be characterized as the unique $Z' \subset S$ of length 2 such that

$$Z' \cap Z = \emptyset, \quad \langle Z' \rangle \cap \langle Z \rangle \neq \emptyset \tag{6.18}$$

Proposition 6.3. *Keep notation and assumptions as above. Then* $|I_F(2)| \in Y_M^\vee$ *is a point of multiplicity* 3.

Proof. Let $\Lambda \subset |I_F(2)|$ be a generic linear subspace with $\dim \Lambda = 3$. Thus

$$\Lambda^\perp := \{\Omega \in |I_S(2)|^\vee \mid \Lambda \subset \Omega\} \tag{6.19}$$

is a generic line in $|I_S(2)|^\vee$ containing $|I_F(2)|$. By (6.16) we must prove that

$$\sharp \left(\Lambda^\perp \cap Y_M^\vee \setminus \{|I_F(2)|\}\right) = 3. \tag{6.20}$$

Let $[Z] \in S^{[2]}$; then $g([Z]) \in \left(\Lambda^\perp \cap Y_M^\vee \setminus \{|I_F(2)|\}\right)$ if and only if

$$\langle Z \rangle \subset \bigcap_{t \in \Lambda} Q_t \quad \langle Z \rangle \not\subset F. \tag{6.21}$$

(Here $Q_t \subset \mathbb{P}^6$ is the quadric corresponding to t.) In the proof of Item (1) of Lemma (4.20) of [14] we showed that

$$\bigcap_{t \in \Lambda} Q_t = F \cup \langle C \rangle \tag{6.22}$$

where $C \subset F$ is a certain conic with span $\langle C \rangle$ such that $F \cap \langle C \rangle = C$. Let $\overline{Q} \subset \mathbb{P}^6$ be a quadric such that $S = F \cap \overline{Q}$, see (5.12). By (5.13) the surface S does not contain conics and hence $\overline{Q} \cap C$ is a finite set of length 4. Since Λ is chosen generically $\overline{Q} \cap C$ consists of 4 distinct points $p_1, \ldots, p_4$ and hence

$$\langle C \rangle \cap S = \{p_1, \ldots, p_4\}. \tag{6.23}$$

Thus $[Z] \in S^{[2]}$ satisfies (6.21) if and only if Z is a subset of $\{p_1, \ldots, p_4\}$; since there are 6 such Z's and since g has degree 2 onto its image (see (6.17)) we get that (6.20) holds. $\square$

The above proposition shows that the involution δ is not the identity. In fact $M \in \mathbb{LG}(\wedge^3 H^0(I_S(2)))^0$ and hence every singular point of Y_M has multiplicity 2. Since $Y_M^\vee$ has a point of multiplicity 3 we get that $Y_M^\vee$ is not projectively isomorphic to Y_M.

References

[1] A. Beauville, *Variétes Kähleriennes dont la premiére classe de Chern est nulle*, J. Differential geometry 18, 1983, pp. 755-782.

[2] A. Beauville - R. Donagi, *La variétés des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. 301, 1985, pp. 703-706.

[3] G. Casnati - F. Catanese, *Even sets of nodes are bundle symmetric*, J. Diff. Geom. 47, 1997, pp. 237-256.

[4] D. Eisenbud - S. Popescu - C. Walter, *Lagrangian subbundles and codimension 3 subcanonical subschemes*, Duke Math. J. 107, 2001, pp. 427-467.

[5] W. Fulton - P. Pragacz, *Schubert Varieties and Degeneracy Loci*, LNM 1689.

[6] D. Huybrechts, *Compact hyper-Khler manifolds: basic results*, Invent. Math. 135, 1999, pp. 63-113.

[7] D. Huybrechts, *Erratum: "Compact hyper-Khler manifolds: basic results"[Invent. Math. 135 (1999), no. 1, 63–113]*, Invent. Math. 152, 2003, pp. 209-212.

[8] A. Iliev - K. Ranestad, *Addendum to "K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds"*, preprint http://folk.uio.no/ranestad/papers.html

[9] V. A. Iskovskih, *Fano 3-folds, I*, Math. USSR Izvestija, Vol. 11, 1977, pp. 485-527.

[10] F. J. Gallego - B. P. Purnaprajna, *Very ampleness and higher syzigies for Calabi-Yau threefolds*, Math. Ann. 312 (1998), pp. 133-149.

[11] D. Gieseker, *Geometric invariant theory and applications to moduli problems*, Invariant theory (Montecatini, 1982), Springer LNM 996(1983), pp. 45-73.

[12] S. Mukai, *Moduli of vector bundles on $K3$ surfaces and symplectic manifolds*, Sugaku Expos. 1, 1988, pp. 139-174.

[13] D. Mumford - J. Fogarty - F. Kirwan, *Geometric invariant theory*, 3rd edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34. Springer (1994).

[14] K. G. O'Grady, *Involutions and linear systems on holomorphic symplectic manifolds*, math.AG/0403519, to appear on GAFA.

[15] K. G. O'Grady, *Irreducible symplectic 4-folds numerically equivalent to* $(K3)^{[2]}$, math.AG/0504434.

[16] C. Okonek - M. Schneider - H. Spindler, *Vector bundles on Projective spaces*, Progress in Mathematics 3, Birkhähuser (1980).

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