Higher-dimensional analogues of $K3$ surfaces

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0 Introduction

$K3$ surfaces were known classically as complex smooth projective surfaces whose generic hyperplane section is a canonically embedded curve; an example is provided by a smooth quartic surface in $\mathbb{P}^3$. One naturally encounters $K3$'s in the Enriques-Kodaira classification of compact complex surfaces: they are defined to be compact Kähler surfaces with trivial canonical bundle and vanishing first

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Betti number. Below we list a few among the wonderful properties of these surfaces:

(1) (Kodaira [33]): Any two $K3$ surfaces are deformation equivalent - thus they are all deformations of a quartic surface.

(2) The Kähler cone of a $K3$ surface X is described as follows. Let $\omega \in H^{1,1}_{\mathbb{R}}(X)$ be one Kähler class and $\mathcal{N}_X$ be the set of nodal classes

$$\mathcal{N}_X := \{\alpha \in H^{1,1}_{\mathbb{Z}}(X) \mid \alpha \cdot \alpha = -2, \quad \alpha \cdot \omega > 0\} \tag{0.0.1}$$

The Kähler cone $\mathcal{K}_X$ is given by

$$\mathcal{K}_X := \{\alpha \in H^{1,1}_{\mathbb{R}}(X) \mid \alpha \cdot \alpha > 0, \quad \alpha \cdot \beta > 0 \quad \forall \beta \in \mathcal{N}_X\}. \tag{0.0.2}$$

(3) (Shafarevich & Piatechki - Shapiro [62], Burns & Rapoport [7], Looijenga & Peters [40]): Weak and strong Global Torelli hold. The weak version states that two $K3$ surfaces X, Y are isomorphic if and only if there exists an integral isomorphism of Hodge structures $f\colon H^2(X) \xrightarrow{\sim} H^2(Y)$ which is an isometry (with respect to the intersection forms), the strong version states that f is induced by an isomorphism $\phi\colon Y \xrightarrow{\sim} X$ if and only if it maps effective divisors to effective divisors[1].

The higher-dimensional compact Kähler manifolds closest to $K3$ surfaces are *hyperkähler manifolds* (HK); they are defined to be simply connected with $H^{2,0}$ spanned by the class of a holomorphic symplectic form. The terminology originates from riemannian geometry: Yau's solution of Calabi's conjecture gives that every Kähler class ω on a HK manifold contains a Kähler metric g with holonomy the compact symplectic group. There is a sphere S^2 (the pure quaternions of norm 1) parametrizing complex structures for which g is a Kähler metric - the *twistor family* associated to g; it plays a key role in the general theory of HK manifolds[2]. Notice that a HK manifold has trivial canonical bundle and is of even dimension. An example of Beauville [1] of dimension $2n$: the Douady space $S^{[n]}$ parametrizing length-n analytic subsets of a $K3$ surface S. (Of course $S^{[n]}$ is a Hilbert scheme if S is projective.) We mention right away two results which suggest that HK manifolds might behave like $K3$'s. Let X be HK:

(a) By a theorem of Bogomolov [3] deformations of X are unobstructed[3] i.e. the deformation space $Def(X)$ is smooth of the expected dimension $H^1(T_X)$.

(b) Since the sheaf map $T_X \to \Omega^1_X$ given by contraction with a holomorphic symplectic form is an isomorphism it follows that the differential of the weight-2 period map

$$H^1(T_X) \longrightarrow \mathrm{Hom}(H^{2,0}(X), H^{1,1}(X)) \tag{0.0.3}$$

is injective i.e. infinitesimal Torelli holds.

[1]Effective divisors have a purely Hodge-theoretic description once we have located one Kähler class.

[2]Hyperkähler manifolds are also known as *irreducible symplectic*

[3]The obstruction space $H^2(T_X)$ might be non-zero, e.g. if X is a generalized Kummer, see Subsection 1.1.

We notice that by (a) the generic deformation of X has $h^{1,1}_{\mathbb{Z}} = 0$ - in particular it is not projective. In fact given $\alpha \in H^1(\Omega^1_X)$ and a first order deformation $\kappa \in H^1(T_X)$ we know by Griffiths that α remains of type $(1,1)$ to first order in the direction κ if and only if $Tr(\kappa \cup \alpha) = 0$, moreover the map

$$\begin{array}{ccc} H^1(T_X) & \longrightarrow & H^2(\mathcal{O}_X) \\ \kappa & \mapsto & Tr(\kappa \cup \alpha) \end{array} \tag{0.0.4}$$

is surjective if $\alpha \neq 0$ by Serre duality. Item (b) i.e. Infinitesimal Torelli suggests that the weight-2 Hodge structure of X might capture much of the geometry of X.

We will review some of the known results regarding higher-dimensional HK's and then we will present a program which aims to prove that numerical $K3^{[2]}$'s behave very much like $K3$'s at least as far as Items (1)-(2) and (3) above are concerned - a HK 4-fold X is a *numerical* $(K3)^{[2]}$ if there exists an isomorphism of abelian groups $\psi\colon H^2(X;\mathbb{Z}) \xrightarrow{\sim} H^2(S^{[2]};\mathbb{Z})$ where S is a $K3$ such that

$$\int_X \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4 \qquad \forall \alpha \in H^2(X;\mathbb{Z}). \tag{0.0.5}$$

In the last section we will discuss Global Torelli for deformations of $K3^{[2]}$.

1 Examples

The surprising topological properties of HK manifolds (see Subsection 2.1) led Bogomolov [3] to state erroneously that no higher-dimensional (i.e. of dim > 2) HK exists. Some time later Fujiki [15] realized that $K3^{[2]}$ is a higher-dimensional HK manifold[4]. Soon after that Beauville [1] showed that $K3^{[n]}$ is a HK manifold and constructed another deformation class of HK manifolds in arbitrary even dimension greater that 2 namely deformations of generalized Kummers. We exhibited [55, 56] two "sporadic"deformation classes, one in dimension 6 the other in dimension 10. No other deformation classes are known other than those mentioned above.

1.1 Beauville

Besides $(K3)^{[n]}$ Beauville discovered another class of $2n$-dimensional HK manifolds - generalized Kummers associated to a 2-dimensional compact complex torus. Before defining generalized Kummers we recall that the Douady space $W^{[n]}$ comes with a cycle (Hilbert-Chow) map

$$\begin{array}{ccc} W^{[n]} & \xrightarrow{\kappa_n} & W^{(n)} \\ [Z] & \mapsto & \sum_{p \in W} \ell(\mathcal{O}_{Z,p})p \end{array} \tag{1.1.1}$$

where $W^{(n)}$ is the symmetric product of W. Now suppose that T is a 2-dimensional compact complex torus. We have the summation map $\sigma_n\colon W^{(n)} \to W$. Composing the two above maps (with $(n+1)$ replacing n) we get a locally

[4]Fujiki described $K3^{[2]}$ not as a Douady space but as the blow-up of the diagonal in the symmetric square of a $K3$ surface

(in the classical topology) trivial fibration $\sigma_{n+1} \circ \kappa_{n+1} \colon W^{[n+1]} \to W$. The $2n$-dimensional *generalized Kummer* associated to T is

$$K^{[n]}T := (\sigma_{n+1} \circ \kappa_{n+1})^{-1}(0). \tag{1.1.2}$$

The name is justified by the observation that if $n = 1$ then $K^{[1]}T$ is the Kummer surface associated to T (and hence a $K3$). Beauville [1] proved that $K^{[n]}(T)$ is a HK manifold. Moreover if $n \geq 2$ then

$$b_2((K3)^{[n]}) = 23 \qquad b_2(K^{[n]}T) = 7. \tag{1.1.3}$$

In particular $(K3)^{[n]}$ and $K^{[n]}T$ are not deformation equivalent as soon as $n \geq 2$. The second cohomology of these manifolds is described as follows. Let W be a compact complex surface. There is a “symmetrization map”

$$\mu_n \colon H^2(W;\mathbb{Z}) \longrightarrow H^2(W^{(n)};\mathbb{Z}) \tag{1.1.4}$$

characterized by the following property. Let $\rho_n \colon W^n \to W^{(n)}$ be the quotient map and $\pi_i \colon W^n \to W$ be the i-th projection: then

$$\rho_n^* \circ \mu_n(\alpha) = \sum_{i=1}^{n} \pi_i^* \alpha, \qquad \alpha \in H^2(W;\mathbb{Z}). \tag{1.1.5}$$

Composing with κ_n^* and extending scalars one gets an injection of integral Hodge structures

$$\widetilde{\mu}_n := \kappa_n^* \circ \mu_n \colon H^2(W;\mathbb{C}) \longrightarrow H^2(W^{[n]};\mathbb{C}). \tag{1.1.6}$$

The above map is not surjective unless $n = 1$; we are missing the Poincaré dual of the exceptional set of κ_n i.e.

$$\Delta_n := \{[Z] \in W^{[n]} \mid Z \text{ is non-reduced}\}. \tag{1.1.7}$$

It is known that Δ_n is a prime divisor and that it is divisible[5] by 2 in $\mathrm{Pic}(W^{[n]})$:

$$\mathcal{O}_{W^{[n]}}(\Delta_n) \cong L_n^{\otimes 2}, \quad L_n \in \mathrm{Pic}(W^{[n]}). \tag{1.1.8}$$

Let $\xi_n := c_1(L_n)$; one has

$$H^2(W^{[n]};\mathbb{Z}) = \widetilde{\mu}_n H^2(W;\mathbb{Z}) \oplus \mathbb{Z}\xi_n, \quad \text{if } H_1(W) = 0. \tag{1.1.9}$$

That describes $H^2((K3)^{[n]})$. Beauville proved that an analogous result holds for generalized Kummers, namely we have an isomorphism

$$\begin{array}{ccc} H^2(T;\mathbb{Z}) \oplus \mathbb{Z} & \xrightarrow{\sim} & H^2(K^{[n]}T;\mathbb{Z}) \\ (\alpha, k) & \mapsto & (\widetilde{\mu}_{n+1}(\alpha) + k\xi_{n+1})|_{K^{[n]}T} \end{array} \tag{1.1.10}$$

The above description of the H^2 gives the following interesting result: if $n \geq 2$ the generic deformation of $S^{[n]}$ where S is a $K3$ is not isomorphic to $T^{[n]}$ for some other $K3$ surface T. In fact every deformation of $S^{[n]}$ obtained by deforming S keeps ξ_n of type $(1,1)$ while as noticed previously the generic deformation of a HK manifold has no non-trivial integral $(1,1)$-classes. (Notice that if S is a surface of general type then every deformation of $S^{[n]}$ is indeed obtained by deforming S, see [14].)

[5] If $n = 2$ Equation (1.1.8) follows from existence of the double cover $Bl_{diag}(S^2) \to S^{[2]}$ ramified over Δ_2.

1.2 Mukai and beyond

Mukai [47, 48, 50] and Tyurin [68] analyzed moduli spaces of semistable sheaves on projective $K3$'s and abelian surfaces and obtained other examples of HK manifolds. Let S be a projective $K3$ and $\mathcal{M}$ the moduli space of $\mathcal{O}_S(1)$-semistable sheaves on S with assigned Chern character - by Gieseker and Maruyama $\mathcal{M}$ has a natural structure of projective scheme. A non-zero canonical form on S induces a holomorphic symplectic 2-form on the open $\mathcal{M}^s \subset \mathcal{M}$ parametrizing stable sheaves (notice that $\mathcal{M}^s$ is smooth by Mukai [47]). If $\mathcal{M}^s = \mathcal{M}$ then $\mathcal{M}$ is a HK variety[6], in general it is not isomorphic (nor birational) to $(K3)^{[n]}$ however it can be deformed to $(K3)^{[n]}$ (here $2n = \dim \mathcal{M}$), see [19, 54, 72]. Notice that $S^{[n]}$ may be viewed as a particular case of Mukai's construction by identifying it with the moduli space of rank-1 semistable sheaves on S with $c_1 = 0$ and $c_2 = n$. Notice also that these moduli spaces give explicit deformations of $(K3)^{[n]}$ which are not $(K3)^{[n]}$. Similarly one may consider moduli spaces of semistable sheaves on an abelian surface A: in the case when $\mathcal{M} = \mathcal{M}^s$ one gets deformations of the generalized Kummer. To be precise it is not $\mathcal{M}$ which is a deformation of a generalized Kummer but rather one of its Beauville-Bogomolov factors. Explicitely we consider the map

$$\begin{array}{ccc} \mathcal{M}(A) & \xrightarrow{\mathfrak{a}} & A \times \widehat{A} \\ [F] & \mapsto & (\mathrm{alb}(c_2(F) - c_2(F_0)), [\det F \otimes (\det F_0)^{-1}]) \end{array} \tag{1.2.1}$$

where $[F_0] \in \mathcal{M}$ is a "reference"point and $\mathrm{alb}\colon CH_0(A) \to A$ is the Albanese map. Then $\mathfrak{a}$ is a locally (classical topology) trivial fibration; Yoshioka [73] proved that the fibers of $\mathfrak{a}$ are deformations of a generalized Kummer. What can we say about moduli spaces such that $\mathcal{M} \neq \mathcal{M}^s$? The locus $(\mathcal{M} \setminus \mathcal{M}^s)$ parametrizing S-equivalence classes of semistable non-stable sheaves is the singular locus of $\mathcal{M}$ except for pathological choices of Chern character which do not give anything particularly interesting; thus we assume that $(\mathcal{M} \setminus \mathcal{M}^s)$ is the singular locus of $\mathcal{M}$. A natural question is the following: does there exist a crepant desingularization $\widetilde{\mathcal{M}} \to \mathcal{M}$? We constructed such a desingularization [55, 56] (see also [37]) for the moduli space $\mathcal{M}_4(S)$ of semi-stable rank-2 sheaves on a $K3$ surface S with $c_1 = 0$ and $c_2 = 4$ and for the moduli space $\mathcal{M}_2(A)$ of semi-stable sheaves on an abelian surface A with $c_1 = 0$ and $c_2 = 2$; the singularities of the moduli spaces are the same in both cases and both moduli spaces have dimension 10. Let M_{10} be our desingularization of $\mathcal{M}_4(S)$ where S is a $K3$. Since the resolution is crepant Mukai's holomorphic symplectic form on $(\mathcal{M}(S) \setminus \mathcal{M}(S)^s)$ extends to a holomorphic symplectic form on M_{10}. We proved [55] that M_{10} is HK i.e. it is simply connected and $h^{2,0}(M_{10}) = 1$. Moreover M_{10} is not a deformation of one of Beauville's examples because $b_2(M_{10}) = 24$. (We proved that $b_2(M_{10}) \geq 24$ later Rapagnetta [64] proved that equality holds.) Next let A be an abelian surface and $\widetilde{\mathcal{M}}_2(A) \to \mathcal{M}_2(A)$ be our desingularization. Composing Map (1.2.1) for $\mathcal{M}(A) = \mathcal{M}_2(A)$ with the desingularization map we get a locally (in the classical topology) trivial fibration $\widetilde{\mathfrak{a}}\colon \widetilde{\mathcal{M}}_2(A) \to A \times \widehat{A}$; let M_6 be any fiber of $\widetilde{\mathfrak{a}}$. We proved [55] that M_6 is HK and that $b_2(M_6) = 8$; thus M_6 is not a deformation of one of Beauville's examples. We would like to point out that while all Betti and Hodge numbers of Beauville's examples are known [18] the same is not true of our examples (Rapagnetta [63] computed

[6] A HK variety is a projective HK manifold.

the Euler characteristic of M_6). Of course there are examples of moduli spaces $\mathcal{M}$ with $\mathcal{M} \neq \mathcal{M}^s$ in any even dimension; one would like to desingularize them and produce many more deformation classes of HK manifolds. Kaledin-Lehn-Sorger [31] have proved that in most cases there is no crepant desingularization and that if there is one then it is a deformation of M_{10} if the surface is a $K3$ while in the case of an abelian surface the fibers of Map (1.2.1) composed with the desingularization map are deformations of M_6[7]. In fact all known examples of HK manifolds are deformations either of Beauville's examples or of ours.

1.3 Mukai flops

Let X be a HK manifold of dimension $2n$ containing a submanifold Z isomorphic to $\mathbb{P}^n$. The *Mukai flop of* Z (introduced in [47]) is a bimeromorphic map $X \dashrightarrow X^\vee$ which is an isomorhism away from Z and replaces Z by the dual plane $Z^\vee := (\mathbb{P}^n)^\vee$. Explicitly let $\tau\colon \widetilde{X} \to X$ be the blow-up of Z and $E \subset \widetilde{X}$ be the exceptional divisor. Since Z is Lagrangian the symplectic form on X defines an isomorphism $N_{Z/X} \cong \Omega_Z = \Omega_{\mathbb{P}^n}$. Thus we have

$$E \cong \mathbb{P}(N_{Z/X}) = \mathbb{P}(\Omega_{\mathbb{P}^n}) \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee. \tag{1.3.1}$$

Hence E is a $\mathbb{P}^{n-1}$-fibration in two different ways: we have $\pi\colon E \to \mathbb{P}^n$ i.e. the restriction of τ to E and $\rho\colon E \to (\mathbb{P}^n)^\vee$. A straightforward computation shows that the restriction of $N_{E/\widetilde{X}}$ to a fiber of ρ is $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. By the Fujiki-Nakano contractibility criterion there exists a proper map $\tau^\vee\colon \widetilde{X} \to X^\vee$ to a complex manifold $X^\vee$ which is an isomorphism outside E and which restricts to ρ on E. Clearly $\tau^\vee(E)$ is naturally identified with $Z^\vee$ and we have a bimeromorphic map $X \dashrightarrow X^\vee$ which defines an isomorphism $(X \setminus Z) \xrightarrow{\sim} (X^\vee \setminus Z^\vee)$. Summarizing: we have the following commutative diagram

$$\begin{array}{ccccc} & & \widehat{X} & & \\ & \swarrow^{\tau} & & \searrow^{\tau^\vee} & \\ X & & \dashrightarrow & & X^\vee \\ & \searrow_{c} & & \swarrow_{c^\vee} & \\ & & W & & \end{array} \tag{1.3.2}$$

where $c\colon X \to W$ and $c^\vee\colon X^\vee \to W$ are the contractions of Z and $Z^\vee$ respectively - see the Introduction of [74]. It follows that $X^\vee$ is simply connected and a holomorphic symplectic form on X gives a holomorphic symplectic form on $X^\vee$ spanning $H^0(\Omega^2_{X^\vee})$; thus $X^\vee$ is HK if it is Kähler. We give an example with X and $X^\vee$ projective. Let $f\colon S \to \mathbb{P}^2$ be a double cover branched over a smooth sextic and $\mathcal{O}_S(1) := f^*\mathcal{O}_{\mathbb{P}^2}(1)$: thus S is a $K3$ of degree 2. Let $X := S^{[2]}$ and $\mathcal{M}$ be the moduli space of pure 1-dimensional $\mathcal{O}_S(1)$-semistable sheaves on S with typical member $\iota_*\mathcal{L}$ where $\iota\colon C \hookrightarrow S$ is the inclusion of $C \in |\mathcal{O}_S(1)|$ and $\mathcal{L}$ is a line-bundle on C of degree 2. We have a natural rational map

$$\phi\colon S^{[2]} \dashrightarrow \mathcal{M} \tag{1.3.3}$$

[7] To be precise their result holds if the polarization of the surface is "generic" relative to the chosen Chern character, with this hypothesis the singular locus of $\mathcal{M}$ is, so to speak, as small as possible

which associates to $[W] \in S^{[2]}$ the sheaf $\iota_* \mathcal{L}$ where C is the unique curve containing W (unicity requires W to be generic !) and $\mathcal{L} := \mathcal{O}_C(W)$. If every divisor in $|\mathcal{O}_S(1)|$ is prime (i.e. the branch curve of f has no tritangents) then $\mathcal{M}$ is smooth (projective) and the rational map ϕ is identified with the flop of

$$Z := \{f^{-1}(p) \mid p \in \mathbb{P}^2\}. \tag{1.3.4}$$

Wierzba and Wiśniewsky [74] have proved that any birational map between HK four-folds is a composition of Mukai flops. In higher dimensions Mukai [47] defined more general flops in which the indeterminacy locus is a fibration in projective spaces. Markman [41] constructed *stratified Mukai flops*.

2 General theory

It is fair to state that there are three main ingredients in the general theory of HK manifolds developed by Bogomolov, Beauville, Fujiki, Huybrechts and others:

(1) Deformations are unobstructed (Bogomolov's Theorem).

(2) The canonical Bogomolov-Beauville quadratic form on H^2 of a HK manifold (see the next subsection).

(3) Existence of the twistor family on a HK manifold equipped with a Kähler class: this is a consequence of Yau's solution of Calabi's conjecture.

2.1 Topology

Let X be a HK-manifold of dimension $2n$. Beauville [1] and Fujiki [16] proved that there exist an integral indivisible quadratic form

$$q_X \colon H^2(X) \to \mathbb{C} \tag{2.1.1}$$

(cohomology is with complex coefficients) and $c_X \in \mathbb{Q}_+$ such that

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n! 2^n} q_X(\alpha)^n, \qquad \alpha \in H^2(X). \tag{2.1.2}$$

The above equation determines c_X and q_X with no ambiguity unless n is even. If n is even then q_X is determined up to ± 1: one singles out one of the two choices by imposing the inequality

$$q_X(\omega) > 0 \text{ for } \omega \in H^{1,1}_{\mathbb{R}}(X) \text{ a Kähler class.} \tag{2.1.3}$$

(Notice that if n is odd the above inequality follows from (2.1.2).) The *Beauville-Bogomolov* form and the *Fujiki constant* of X are q_X and c_X respectively. We notice that (2.1.2) is equivalent (by polarization) to

$$\int_X \alpha_1 \wedge \ldots \wedge \alpha_{2n} = c_X \sum_{\sigma \in \mathcal{R}_{2n}} (\alpha_{\sigma(1)}, \alpha_{\sigma(2)})_X \cdot (\alpha_{\sigma(3)}, \alpha_{\sigma(4)})_X \cdots (\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})_X \tag{2.1.4}$$

where $(\cdot,\cdot)_X$ is the symmetric bilinear form associated to q_X and $\mathcal{R}_{2n}$ is a set of representatives for the left cosets of the subgroup $\mathcal{G}_{2n} < \mathcal{S}_{2n}$ of permutations of $\{1,\ldots,2n\}$ generated by transpositions $(2i-1,2i)$ and by products of transpositions $(2i-1,2j-1)(2i,2j)$ - in other words in the right-hand side of (2.1.4) we avoid repeating addends which are equal[8]. The existence of q_X, c_X is by no means trivial; we sketch a proof. Let $f\colon \mathcal{X} \to T$ be a deformation of X representing $Def(X)$; more precisely letting $X_t := f^{-1}\{t\}$ for $t \in T$, we are given $0 \in T$, an isomorphism $X_0 \xrightarrow{\sim} X$ and the induced map of germs $(T,0) \to Def(X)$ is an isomorphism. In particular T is smooth in 0 and hence we may assume that it is a polydisk. The Gauss-Manin connection defines an integral isomorphism $\phi_t\colon H^2(X) \xrightarrow{\sim} H^2(X_t)$. The *local period map* of X is given by

$$\begin{array}{ccc} T & \xrightarrow{\pi} & \mathbb{P}(H^2(X)) \\ t & \mapsto & \phi_t^{-1} H^{2,0}(X_t) \end{array} \tag{2.1.5}$$

By infintesimal Torelli, see (0.0.3) $Im\pi$ is an analytic hypersurface in an open (classical topology) neighborhood of $\pi(0)$ and hence its Zariski closure $V = \overline{Im\pi}$ is either all of $\mathbb{P}(H^2(X))$ or a hypersurface. One shows that the latter holds by considering the (non-zero) degree-$2n$ homogeneous polynomial

$$\begin{array}{ccc} H^2(X) & \xrightarrow{G} & \mathbb{C} \\ \alpha & \mapsto & \int_X \alpha^{2n} \end{array} \tag{2.1.6}$$

In fact if $\sigma_t \in H^{2,0}(X_t)$ then

$$\int_{X_t} \sigma_t^{2n} = 0 \tag{2.1.7}$$

by type consideration and it follows by Gauss-Manin parallel transport that G vanishes on V. Thus $I(V) = (F)$ where F is an irreducible homogeneous polynomial. By considering the derivative of the period map (0.0.3) one checks easily that V is not a hyperplane and hence $\deg F \geq 2$. On the other hand type consideration gives something stronger than (2.1.7), namely

$$\int_{X_t} \sigma_t^{n+1} \wedge \alpha_1 \cdots \wedge \alpha_{n-1} = 0 \qquad \alpha_1,\ldots,\alpha_{n-1} \in H^2(X_t). \tag{2.1.8}$$

It follows that all the derivatives of G up to order $(n-1)$ included vanish on V. Since $\deg G = 2n$ and $\deg F \geq 2$ it follows that $G = c \cdot F^n$ and $\deg F = 2$. By integrality of G there exists $\lambda \in \mathbb{C}^*$ such that $c_X := \lambda c$ is rational positive, $q_X := \lambda \cdot F$ is integral indivisible and (2.1.2) is satisfied.

Of course if X is a $K3$ then q_X is the intersection form of X (and $c_X = 1$). In general q_X gives $H^2(X;\mathbb{Z})$ a structure of lattice just as in the well-known case of $K3$ surfaces. Suppose that X and Y are deformation equivalent HK-manifolds: it follows from (2.1.2) that $c_X = c_Y$ and the lattices $H^2(X;\mathbb{Z}), H^2(Y;\mathbb{Z})$ are isometric (see the comment following (2.1.2) if n is even). Consider the case when $X = (K3)^{[n]}$; then $\widetilde{\mu}_n$ is an isometry, $\xi_n \bot Im\widetilde{\mu}_n$ and $q_X(\xi_n) = -2(n-1)$ i.e.

$$H^2(S^{[n]};\mathbb{Z}) \cong U^3 \widehat{\oplus} E_8\langle -1\rangle^2 \widehat{\oplus} \langle -2(n-1)\rangle \tag{2.1.9}$$

[8]In defining c_X we have introduced a normalization which is not standard in order to avoid a combinatorial factor in Equation (2.1.4).

where $\widehat{\oplus}$ denotes othogonal direct-sum, U is the hyperbolic plane and $E_8\langle -1\rangle$ is the unique rank-8 negative definite unimodular even lattice. Moreover the Fujiki constant is

$$c_{S^{[n]}} = 1. \tag{2.1.10}$$

In [64] the reader will find the B-B quadratic form and Fujiki constant of the other known deformation classes of HK manifolds.

Remark 2.1. Let X be a HK manifold of dimension $2n$ and $\omega \in H^{1,1}_{\mathbb{R}}(X)$ be a Kähler class.

(1) Equation (2.1.2) gives that with respect to $(,)_X$ we have

$$H^{p,q}(X) \bot H^{p',q'}(X) \text{ unless } (p',q') = (2-p, 2-q). \tag{2.1.11}$$

(2) $q_X(\omega) > 0$. In fact let σ be generator of $H^{2,0}(X)$; by Equation (2.1.4) and Item (1) above we have

$$0 < \int_X \sigma^{n-1} \wedge \overline{\sigma}^{n-1} \wedge \omega^2 = c_X (n-1)!(\sigma, \overline{\sigma})_X q_X(\omega). \tag{2.1.12}$$

Since $c_X > 0$ and $(\sigma, \overline{\sigma})_X > 0$ we get that $q_X(\omega) > 0$ as claimed.

(3) The index of q_X is $(3, b_2(X) - 3)$ (i.e. that is the index of its restriction to $H^2(X;\mathbb{R})$). In fact applying Equation (2.1.4) to $\alpha_1 = \ldots = \alpha_{2n-1} = \omega$ and arbitrary α_{2n} we get that $\omega^{\bot}$ is equal to the primitive cohomology $H^2_{pr}(X)$ (primitive with respect to ω). On the other hand Equation (2.1.4) with $\alpha_1 = \ldots = \alpha_{2n-2} = \omega$ and $\alpha_{2n-1}, \alpha_{2n} \in \omega^{\bot}$ gives that a positive multiple of $q_X|_{\omega^{\bot}}$ is equal to the standard quadratic form on $H^2_{pr}(X)$ (recall Inequality (2.1.3)). By the Hodge index Theorem it follows that the restriction of q_X to $\omega^{\bot} \cap H^2(X;\mathbb{R})$ has index $(2, b_2(X) - 3)$. Since $q_X(\omega) > 0$ it follows that q_X has index $(3, b_2(X) - 3)$.

(4) Let D be an effective divisor on X; then $(\omega, D)_X > 0$. (Of course $(\omega, D)_X$ denotes $(\omega, c_1(\mathcal{O}_X(D)))_X$.) In fact the inequality follows from the inequality $\int_D \omega^{2n-1} > 0$ together with (2.1.4) and Item (2) above.

(5) Let $f\colon X \dashrightarrow Y$ be a birational map where Y is a HK manifold. Since X and Y have trivial canonical bundle f defines an isomorphism $U \xrightarrow{\sim} V$ where $U \subset X$ and $V \subset Y$ are open sets with complements of codimension at least 2. It follows that f induces an isomorphism $f^*\colon H^2(Y;\mathbb{Z}) \xrightarrow{\sim} H^2(X;\mathbb{Z})$; f^* is an isometry of lattices, see Lemma 2.6 of [26].

The proof of existence of q_X and c_X may be adapted to prove the following useful generalization of (2.1.2).

Proposition 2.2. *Let X be a HK manifold of dimension $2n$. Let $\mathcal{X} \to T$ be a representative of the deformation space of X. Suppose that $0 \neq \gamma \in H^{p,p}_{\mathbb{R}}(X)$ is a class which remains of type (p,p) under Gauss-Manin parallel transport (e. g. the Chern class $c_p(X)$). Then p is even and moreover there exists $c_\gamma \in \mathbb{R}$ such that*

$$\int_X \gamma \wedge \alpha^{2n-p} = c_\gamma q_X(\alpha)^{n-p/2}. \tag{2.1.13}$$

Our next topic is Verbitsky's theorem [69] (see also [4]). Let X be a HK-manifold of dimension $2n$. Our (sketch of) proof of (2.1.2) shows that

$$\text{if } \alpha \in H^2(X) \text{ and } q_X(\alpha) = 0 \text{ then } \alpha^{n+1} = 0 \text{ in } H^{2n+2}(X). \tag{2.1.14}$$

In fact adopting the notation introduced in the proof of (2.1.2) we have $0 = \sigma_t^{n+1} \in H^{2n+2}(X_t)$ and hence by Gauss-Manin transport we get that $0 = (\psi_t^{-1}\sigma_t)^{n+1} \in H^{2n+2}(X)$. Since the set $\{\psi_t^{-1}\sigma_t \mid t \in T\}$ is Zariski dense in the zero-set $V(q_X) \subset H^2(X)$ we get (2.1.14). Let $I \subset Sym^\bullet H^2(X)$ be the ideal generated by α^{n+1} where $\alpha \in H^2(X)$ and $q_X(\alpha) = 0$:

$$I := \langle \{\alpha^{n+1} \mid \alpha \in H^2(X), \quad q_X(\alpha) = 0\}\rangle. \tag{2.1.15}$$

By (2.1.14) we have a natural map of $\mathbb{C}$-algebras

$$Sym^\bullet H^2(X)/I \longrightarrow H^\bullet(X). \tag{2.1.16}$$

Theorem 2.3. [Verbitsky] *Map* (2.1.16) *is injective.*

In particular we get that cup-product defines an injection

$$\bigoplus_{q=0}^{n} Sym^q H^2(X) \hookrightarrow H^\bullet(X). \tag{2.1.17}$$

S. M. Salamon [65] proved that there is a non-trivial linear constraint on the Betti numbers of a compact Kähler manifold carrying a holomorphic symplectic form (for example a HK manifold); the proof consists in a clever application of the Hirzebruch-Riemann-Roch formula to the sheaves Ω_X^p and the observation that the symplectic form induces an isomorphism $\Omega_X^p \cong \Omega_X^{2n-p}$ where $2n = \dim X$[9].

Theorem 2.4. [S. M. Salamon] *Let X be a compact Kähler manifold of dimension $2n$ carrying a holomorphic symplectic form. Then*

$$nb_{2n}(X) = 2\sum_{i=1}^{2n}(-1)^i(3i^2 - n)b_{2n-i}(X). \tag{2.1.18}$$

The following corollary of Verbitsky's and Salamon's results was obtained by Beauville (unpublished) and Guan [22].

Corollary 2.5. [Beauville and Guan] *Let X be a HK 4-fold. Then $b_2(X) \le 23$. If equality holds then $b_3(X) = 0$ and moreover the map*

$$Sym^2 H^2(X;\mathbb{Q}) \longrightarrow H^4(X;\mathbb{Q}) \tag{2.1.19}$$

induced by cup-product is an isomorphism.

Proof. Let $b_i := b_i(X)$. Salamon's equation (2.1.18) for X reads

$$b_4 = 46 + 10b_2 - b_3. \tag{2.1.20}$$

[9] A non-zero section of the canonical bundle defines an isomorphism $\Omega_X^{2n-p} \cong (\Omega_X^p)^\vee = \wedge^p T_X$ and the symplectic form defines an isomorphism $T_X \cong \Omega_X$ and hence $\wedge^p T_X \cong \Omega_X^p$

By Verbitsky's **Theorem 2.3** - see (2.1.17) - we have

$$\binom{b_2+1}{2} \le b_4. \tag{2.1.21}$$

Replacing b_4 by the right-hand side of (2.1.20) we get that

$$b_2^2 + b_2 \le 92 + 20b_2 - 2b_3 \le 92 + 20b_2. \tag{2.1.22}$$

It follows that $b_2 \le 23$ and that if equality holds then $b_3 = 0$. Suppose that $b_2 = 23$: then $b_4 = 276$ by (2.1.20) and hence (2.1.19) follows from Verbitsly's **Theorem 2.3**. □

We mention that Guan [22] obtained other restrictions on $b_2(X)$ for a HK four-fold X: for example $8 < b_2(X) < 23$ is "forbidden".

2.2 The Kähler cone

Let X be a HK manifold of dimension $2n$. The convex cone $\mathcal{K}_X \subset H^{1,1}_{\mathbb{R}}(X)$ of Kähler classes is the *Kähler cone of* X. Item (3) of **Remark 2.1** gives that the restriction of q_X to $H^{1,1}_{\mathbb{R}}(X)$ is non-degenerate of signature $(1, b_2(X) - 3)$; it follows that the cone

$$\{\alpha \in H^{1,1}_{\mathbb{R}}(X) \mid q_X(\alpha) > 0\} \tag{2.2.1}$$

has two connected components. By Item (2) of **Remark 2.1** $\mathcal{K}_X$ is contained in (2.2.1). Since $\mathcal{K}_X$ is convex it is contained in a single connected component of (2.2.1); that component is the *positive cone* $\mathcal{C}_X$. The following result is proved in the erratum of [26].

Theorem 2.6. [Huybrechts] *Let X be a HK manifold. Let $\mathcal{X} \to T$ be a representative of $Def(X)$ with T irreducible. If $t \in T$ is very general (i.e. outside a countable union of proper analytic subsets of T) then*

$$\mathcal{K}_{X_t} = \mathcal{C}_{X_t}. \tag{2.2.2}$$

Proof. Let $0 \in T$ be the point such that $X_0 \cong X$ and the induced map of germs $(T, 0) \to Def(X)$ is an isomorphism[10]. By shrinking T around 0 if necessary we may assume that T is simply connected and that $\mathcal{X} \to T$ represents $Def(X_t)$ for every $t \in T$. In particular the Gauss-Manin connection gives an isomorphism $P_t \colon H^\bullet(X; \mathbb{Z}) \xrightarrow{\sim} H^\bullet(X_t; \mathbb{Z})$ for every $t \in T$. Given $\gamma \in H^{2p}(X; \mathbb{Z})$ we let

$$T_\gamma := \{t \in T \mid P_t(\gamma) \text{ is of type } (p,p)\}. \tag{2.2.3}$$

Let

$$t \in (T \setminus \bigcup_{T_\gamma \ne T} T_\gamma) \tag{2.2.4}$$

and $Z \subset X_t$ be a closed analytic subset of codimension p; we claim that

$$\int_Z \alpha^{2n-p} > 0 \quad \text{if } q_{X_t}(\alpha) > 0. \tag{2.2.5}$$

[10] The map $(T, 0) \to Def(X)$ depends on the choice of an isomorphism $f \colon X_0 \xrightarrow{\sim} X$ but whether it is an isomorphism or not is independent of f.

In fact let $\gamma \in H^{p,p}_{\mathbb{R}}(X_t)$ be the Poincaré dual of Z. By (2.2.4) γ remains of type (p,p) for every deformation of X_t; by **Proposition 2.2** p is even and moreover there exists $c_\gamma \in \mathbb{R}$ such that

$$\int_Z \alpha^{2n-p} = c_\gamma q_X(\alpha)^{n-p/2} \qquad \forall \alpha \in H^2(X_t). \tag{2.2.6}$$

Let ω be a Kähler class. Since $0 < \int_Z \omega^{2n-p}$ and $0 < q_X(\omega)$ we get that $c_\gamma > 0$; thus (2.2.5) follows from (2.2.6). Now apply Demailly-Paun's version of the Nakai-Moishezon ampleness criterion [11]: $\mathcal{K}_{X_t}$ is a connected component of the set $P(X_t) \subset H^{1,1}_{\mathbb{R}}(X_t)$ of classes α such that $\int_Z \alpha^{2n-p} > 0$ for all closed analytic subsets $Z \subset X_t$ (here $p = \mathrm{cod}(Z, X_t)$). Let t be as in (2.2.4). By (2.2.5) $P(X_t) = \mathcal{C}_{X_t} \coprod (-\mathcal{C}_{X_t})$; since $\mathcal{K}_{X_t} \subset \mathcal{C}_{X_t}$ we get the proposition. □

Huybrechts [26] has proved that **Theorem 2.6** gives the following *projectivity criterion.*

Theorem 2.7. [Huybrechts] *A HK manifold X is projective if and only if there exists a (holomorphic) line-bundle L on X such that $q_X(c_1(L)) > 0$.*

Boucksom [5], elaborating on ideas of Huybrechts, gave the following characterization of $\mathcal{K}_X$ for arbitary X.

Theorem 2.8. [Boucksom] *Let X be a HK manifold. A class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ is Kähler if and only if it belongs to the positive cone $\mathcal{C}_X$ and moreover $\int_C \alpha > 0$ for every rational curve C[11].*

One would like to have a numerical description of the Kähler (or ample) cone as in the 2-dimensional case. Hassett and Tschinkel [24] proved the following result.

Theorem 2.9. [Hassett - Tschinkel] *Let X be a HK variety deformation equivalent to $K3^{[2]}$ and L_0 an ample line-bundle on X. Let L be a line-bundle on X such that $c_1(L) \in \mathcal{C}_X$. Suppose that $(c_1(L), \alpha)_X > 0$ for all $\alpha \in H^{1,1}_{\mathbb{Z}}(X)$ such that $(c_1(L_0), \alpha)_X > 0$ and*

- *(a) $q_X(\alpha) = -2$ or*
- *(b) $q_X(\alpha) = -10$ and $(\alpha, H^2(X; \mathbb{Z}))_X = 2\mathbb{Z}$.*

Then L is ample.

Hassett and Tschinkel [23] conjectured that the converse of the above theorem holds i.e. the above conditions are also necessary for L to be ample. We explain the appearance of the conditions in the above theorem and why one expects that the converse holds. We start with Item (a). Let X be a HK manifold deformation equivalent to $K3^{[2]}$ and L a line-bundle on X: Hirzebruch-Riemann-Roch for X reads

$$\chi(L) = \frac{1}{8}(q(L)+4)(q(L)+6). \tag{2.2.7}$$

(We let $q = q_X$.) It follows that $\chi(L) = 1$ if and only if $q(L) = -2$ or $q(L) = -8$.

[11] A curve is rational if it is irreducible and its normalization is rational

Conjecture 2.10. [Folk?] Let X be a HK manifold deformation equivalent to $K3^{[2]}$. Let L be a line-bundle on X such that $q_X(L) = -2$.

(1) If $(c_1(L), H^2(X;\mathbb{Z}))_X = \mathbb{Z}$ then either L or L^{-1} has a non-zero section.

(2) If $(c_1(L), H^2(X;\mathbb{Z}))_X = 2\mathbb{Z}$ then either L^2 or L^{-2} has a non-zero section. (Notice that $q_X(L^{\pm 2}) = -8$.)

If the above conjecture holds then given $\alpha \in H^{1,1}_{\mathbb{Z}}(X)$ with $q_X(\alpha) = -2$ we have that either $(\alpha, \cdot)_X$ is strictly positive or strictly negative on $\mathcal{K}_X$; in particular the condition corresponding to Item (a) of **Theorem 2.9** is necessary for a line-bundle to be ample. Below are examples of line-bundles satisfying Items (1), (2) above.

Ex. 1 Let S be a $K3$ containing a smooth rational curve C and $X = S^{[2]}$. Let

$$D := \{[Z] \in S^{[2]} \mid Z \cap C \neq \emptyset\}. \tag{2.2.8}$$

Let $L := \mathcal{O}_X(D)$; then $c_1(L) = \widetilde{\mu}_2(c_1(\mathcal{O}_S(C)))$ where $\widetilde{\mu}_2$ is given by (1.1.6). Since $\widetilde{\mu}_2$ is an isometry we have $q_X(L) = C \cdot C = -2$ and moreover $(c_1(L), H^2(X;\mathbb{Z}))_X = \mathbb{Z}$. For another example see Item (5) of **Remark 3.3**

Ex. 2 Let S be a $K3$ and $X = S^{[2]}$. Let L_2 be the square-root of $\mathcal{O}_X(\Delta_2)$ where $\Delta_2 \subset S^{[2]}$ is the divisor parametrizing non-reduced subschemes - thus $c_1(L_2) = \xi_2$. Then $q(L_2) = -2$ and L_2^2 has "the"non-zero section vanishing on Δ_2. Notice that neither L_2 nor L_2^{-1} has a non-zero section.

Summarizing: line-bundles of square -2 on a HK deformation of $K3^{[2]}$ should be similar to (-2)-classes on a K3. (Recall that if L is a line-bundle on a $K3$ with $c_1(L)^2 = -2$ then by Hirzebruch-Riemann-Roch and Serre duality either L or L^{-1} has a non-zero section.) Next we explain Item (b) of **Theorem 2.9**. Suppose that X is a HK deformation of $K3^{[2]}$ and that $Z \subset X$ is a closed submanifold isomorphic to $\mathbb{P}^2$ - see Section 1.3. Let $C \subset Z$ be a line. Since $(,)_X$ is non-degenerate (but not unimodular !) there exists $\beta \in H^2(X;\mathbb{Q})$ such that

$$\int_C \gamma = (\beta, \gamma)_X \qquad \forall \gamma \in H^2(X). \tag{2.2.9}$$

One proves that

$$q_X(\beta) = -\frac{5}{2}. \tag{2.2.10}$$

Equation (2.2.10) follows from Isomorphism (2.1.19) and the good properties of deformations of HK manifolds, see [24], Sect. 4. Since $(\beta, H^2(X;\mathbb{Z}))_X = \mathbb{Z}$ and the discriminant of $(,)_X$ is 2 we have $2\beta \in H^2(X;\mathbb{Z})$; thus $\alpha := 2\beta$ is as in Item (b) of **??**nd if L is ample then $0 < \int_C c_1(L) = \frac{1}{2}(c_1(L), \alpha)_X$.

Hassett and Tschinkel state conjectures that extend **Theorem 2.9** and its converse to general HK varieties, see [25] - in particular they give a conjectural numerical description of the effective cone of a HK variety. The papers [6, 12] contain key results in this circle of ideas.

We close the section by stating a beautiful result of Huybrechts [27] - the proof is based on results on the Kähler cone and uses in an essential way the existence of the twistor family.

Theorem 2.11. *Let X and Y be bimeromorphic HK manifolds. Then X and Y are deformation equivalent.*

3 Complete families of HK varieties

A couple (X, L) where X is a HK variety and L is a primitive[12] ample line-bundle on X with $q_X(L) = d$ is a *HK variety of degree* d; an isomorphism $(X, L) \xrightarrow{\sim} (X', L')$ between HK's of degree d consists of an isomorphism $f : X \xrightarrow{\sim} X'$ such that $f^*L' \cong L$. A family of HK varieties of degree d is a couple

$$(f : \mathcal{X} \to T,\, \mathcal{L}) \tag{3.0.1}$$

where $\mathcal{X} \to T$ is a family of HK varieties deformation equivalent to a fixed HK manifold X and $\mathcal{L}$ is a line-bundle such that (X_t, L_t) is a HK variety of degree d for every $t \in T$ (here $X_t := f^{-1}(t)$ and $L_t := \mathcal{L}|_{X_t}$) - we say that it is a family of HK varieties if we are not intersted in the value of $q_X(L_t)$. The deformation space of (X, L) is a codimension-1 smooth sub-germ $Def(X, L) \subset Def(X)$ with tangent space the kernel of Map (0.0.4) with $\alpha = c_1(L)$. The family (3.0.1) is *locally complete* if given any $t_0 \in T$ the map of germs $(T, t_0) \to Def(X_{t_0}, L_{t_0})$ is surjective, it is *globally complete* if given any HK variety (Y, L) of degree d with Y deformation equivalent to X there exists $t_0 \in T$ such that $(Y, L) \cong (X_{t_0}, L_{t_0})$. In dimension 2 i.e. for $K3$ surfaces one has explicit globally complete families of low degree: If $d = 2$ the family of double covers $S \to \mathbb{P}^2$ branched over a smooth sextic will do[13], if $d = 4$ we may consider the family of smooth quartic surfaces $S \subset \mathbb{P}^3$ with the addition of certain "limit"surfaces (double covers of smooth quadrics and certain elliptic $K3$'s) corresponding to degenerate quartics (double quadrics and the surface swept out by tangents to a rational normal cubic curve respectively). The list goes on for quite a few values of d, see [49, 51] and then it necessarily stops - at least in this form - because moduli spaces of high-degree $K3$'s are not unirational [20]. We remark that in low degree one shows "by hand"that there exists a globally complete family which is irreducible; the same is true in arbitrary degree but I know of no elementary proof, the most direct argument is via Global Torelli. What is the picture in higher (> 2) dimensions ? Four distinct (modulo obvious equivalence) locally complete families of higher-dimensional HK varieties have been constructed - they are all deformations of $K3^{[2]}$. The families are the following:

(1) We constructed [58] the family of double covers of certain special sextic hypersurfaces in $\mathbb{P}^5$ that we named EPW-sextics (they had been introduced by Eisenbud-Popescu-Walter [13]). The polarization is the pull-back of $\mathcal{O}_{\mathbb{P}^5}(1)$; its degree is 2.

(2) Let $Z \subset \mathbb{P}^5$ be a smooth cubic hypersurface; Beauville and Donagi [2] proved that the variety parametrizing lines on Z is a deformation of $K3^{[2]}$. The polarization is given by the Plücker embedding: it has degree 6.

(3) Let σ be a generic 3-form on $\mathbb{C}^{10}$; Debarre and Voisin [10] proved that the set $Y_\sigma \subset Gr(6, \mathbb{C}^{10})$ parametrizing subspaces on which σ vanishes is a

[12] i.e. $c_1(L)$ is indivisibile in $H^2(X; \mathbb{Z})$.

[13] In order to get a global family we must go to a suitable double cover of the parameter space of sextic curves.

deformation of $K3^{[2]}$. The polarization is given by the Plücker embedding: it has degree 22.

(4) Let $Z \subset \mathbb{P}^5$ be a generic cubic hypersurface; Iliev and Ranestad [29, 30] have proved that the variety of sums of powers $VSP(Z, 10)$ [14] is a deformation of $K3^{[2]}$. For the polarization we refer to [30]; the degree is 38 (unpublished computation by Iliev, Ranestad and Van Geemen).

For each of the above families - more precisely for the family obtained by adding "limits"- one might ask whether it is globally complete for HK varieties of the given degree which are deformations of $K3^{[2]}$. As formulated the answer is negative with the possible exception of our family, for a trivial reason: in the lattice $L := H^2(K3^{[2]}; \mathbb{Z})$ the orbit of a primitive vector v under the action of $O(L)$ is determined by the value of the B-B form $q(v)$ plus the extra information on whether

$$(v, L) = \begin{cases} \mathbb{Z} & or \\ 2\mathbb{Z} \end{cases} \tag{3.0.2}$$

In the first case one says that the *divisibility of* v is 1, in the second case that it is 2; if the latter occurs then $q(v) \equiv 6 \pmod 8$. Thus the divisibility of the polarization in Item (1) above equals 1; on the other hand it equals 2 for the families in Item (2)-(4). The correct question regarding global completeness is the following. Let X be a HK deformation of $K3^{[2]}$ with an ample line-bundle L such that either $q(L) = 2$ or $q(L) \in \{6, 22, 38\}$ and the divisibility of $c_1(L)$ is equal to 2: does there exist a variety Y parametrized by one of the above families - or a limit of such - and an isomorphism $(X, L) \cong (Y, \mathcal{O}_Y(1))$? If a "naive"global Torelli holds for HK deformations of $K3^{[2]}$ then the answer is positive, see **Claim 5.4**.

None of the families above is as easy to construct as are the families of low-degree $K3$ surfaces. There is the following Hodge-theoretic explanation. In order to get a locally complete family of varieties one usually constructs complete intersections (or sections of ample vector-bundles) in homogeneous varieties: by Lefschetz' hyperplane Theorem such a construction will never produce a higher-dimensional HK. On the other hand the families of Items (1), (2) and (3) are related to complete intersections as follows (I do not know whether one may view the Iliev-Ranestad family from a similar perspective). First if $f\colon X \to Y$ is a double EPW-sextic (Item (1) above) then f is the quotient map of an involution $X \to X$ which has one-dimensional $(+1)$-eigenspace on $H^2(X)$ - in particular it kills $H^{2,0}$ - and "allows"the quotient to be a hypersurface. Regarding Item (2): let $Z \subset \mathbb{P}^5$ be a smooth cubic hypersurface and X the variety of lines on Z, the incidence correspondence in $Z \times X$ induces an isomorphism of the primitive Hodge structures $H^4(Z)_{pr} \xrightarrow{\sim} H^2(X)_{pr}$. Thus a Tate twist of $H^2(X)_{pr}$ has become the primitive intermediate cohomology of a hypersurface. A similar comment applies to the Debarre-Voisin family (and there is a similar incidence-type construction of double EPW-sextics given by Iliev and Manivel [28]).

In this section we will describe in some detail the family of double EPW-sextics and we will say a few words about analogies with the Beauville-Donagi family.

[14] $VSP(Z, 10)$ parametrizes 9-dimensional linear spaces of $|\mathcal{O}_{\mathbb{P}^5}(3)|$ which contain Z and are 10-secant to the Veronese $\{[L^3] \mid L \in (H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \setminus \{0\})\}$.

3.1 Double EPW-sextics, I

We start by giving the definition of EPW-sextic [13]. Let V be a 6-dimensional complex vector space. We choose a volume-form $vol\colon \wedge^6 V \overset{\sim}{\longrightarrow} \mathbb{C}$ and we equip $\wedge^3 V$ with the symplectic form

$$(\alpha,\beta)_V := \mathrm{vol}(\alpha\wedge\beta). \tag{3.1.1}$$

Let $\mathbb{LG}(\wedge^3 V)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\wedge^3 V$ - notice that $\mathbb{LG}(\wedge^3 V)$ is independent of the chosen volume-form vol. Given a non-zero $v\in V$ we let

$$F_v := \{\alpha\in\wedge^3 V \mid v\wedge\alpha = 0\}. \tag{3.1.2}$$

Notice that $(,)_V$ is zero on F_v and $\dim(F_v)=10$ i.e. $F_v\in\mathbb{LG}(\wedge^3 V)$. Let

$$F\subset \wedge^3 V\otimes\mathcal{O}_{\mathbb{P}(V)} \tag{3.1.3}$$

be the sub-vector-bundle with fiber F_v over $[v]\in\mathbb{P}(V)$. Given $A\in\mathbb{LG}(\wedge^3 V)$ we let

$$Y_A = \{[v]\in\mathbb{P}(V)\mid F_v\cap A\neq\{0\}\}. \tag{3.1.4}$$

Thus Y_A is the degeneracy locus of the map

$$F \overset{\lambda_A}{\longrightarrow} (\wedge^3 V/A)\otimes\mathcal{O}_{\mathbb{P}(V)} \tag{3.1.5}$$

where λ_A is given by Inclusion (3.1.3) followed by the quotient map $\wedge^3 V\otimes\mathcal{O}_{\mathbb{P}(V)}\to(\wedge^3 V/A)\otimes\mathcal{O}_{\mathbb{P}(V)}$. Since the vector-bundles appearing in (3.1.5) have equal rank Y_A is the zero-locus of $\det\lambda_A\in H^0(\det F^\vee)$ - in particular it has a natural structure of closed subscheme of $\mathbb{P}(V)$. A straightforward computation gives that $\det F\cong\mathcal{O}_{\mathbb{P}(V)}(-6)$ and hence Y_A is a sextic hypersurface unless it equals $\mathbb{P}(V)$[15]; if the former holds we say that Y_A is an *EPW-sextic*. What do EPW-sextics look like? The main point is that locally they are the degeneracy locus of a symmetric map of vector-bundles (they were introduced by Eisenbud, Popescu and Walter to give examples of a "quadratic sheaf", namely $\mathrm{coker}(\lambda_A)$, which can not be expressed **globally** as the cokernel of a symmetric map of vector-bundles on $\mathbb{P}^5$). More precisely given $B\in\mathbb{LG}(\wedge^3 V)$ we let $\mathcal{U}_B\subset\mathbb{P}(V)$ be the open subset defined by

$$\mathcal{U}_B := \{[v]\in\mathbb{P}(V)\mid F_v\cap B=\{0\}\}\,. \tag{3.1.6}$$

Now choose B transversal to A. We have a direct-sum decomposition $\wedge^3 V = A\oplus B$; since A is lagrangian the symplectic form $(,)_V$ defines an isomorphism $B\cong A^\vee$. Let $[v]\in\mathcal{U}_B$: since F_v is transversal to B it is the graph of a map

$$\tau_A^B([v])\colon A\to B\cong A^\vee,\qquad [v]\in\mathcal{U}_B\,. \tag{3.1.7}$$

The map $\tau_A^B([v])$ is symmetric because A,B and F_v are lagrangians.

[15]Given $[v]\in\mathbb{P}(V)$ there exists $A\in\mathbb{LG}(\wedge^3 V)$ such that $A\cap F_v=\{0\}$ and hence $[v]\notin Y_A$; thus Y_A is a sextic hypersurface for generic $A\in\mathbb{LG}(\wedge^3 V)$. On the other hand if $A=F_w$ for some $[w]\in\mathbb{P}(V)$ then $Y_A=\mathbb{P}(V)$.

Remark 3.1. There is one choice of B which produces a "classical" description of Y_A, namely $B = \wedge^3 V_0$ where $V_0 \subset V$ is a codimension-1 subspace[16]. With such a choice of B we have $\mathcal{U}_B = (\mathbb{P}(V) \setminus \mathbb{P}(V_0))$; we identify it with V_0 by choosing $v_0 \in (V \setminus V_0)$ and mapping

$$\begin{array}{ccc} V_0 & \overset{\sim}{\longrightarrow} & \mathbb{P}(V) \setminus \mathbb{P}(V_0) \\ v & \mapsto & [v_0 + v] \end{array} \tag{3.1.8}$$

The direct-sum decomposition $\wedge^3 V = F_{v_0} \oplus \wedge^3 V_0$ and transversality $A \pitchfork \wedge^3 V_0$ allows us to view A as the graph of a (symmetric) map $\widetilde{q}_A \colon F_{v_0} \to \wedge^3 V_0$. Identifying $\wedge^2 V_0$ with F_{v_0} via the isomorphism

$$\begin{array}{ccc} \wedge^2 V_0 & \overset{\sim}{\longrightarrow} & F_{v_0} \\ \alpha & \mapsto & v_0 \wedge \alpha \end{array} \tag{3.1.9}$$

we may view $\widetilde{q}_A$ as a symmetric map

$$\wedge^2 V_0 \longrightarrow \wedge^3 V_0 = \wedge^2 V_0^\vee. \tag{3.1.10}$$

We let $q_A \in Sym^2(\wedge^2 V_0^\vee)$ be the quadratic form corresponding to $\widetilde{q}_A$. Given $v \in V_0$ let $q_v \in Sym^2(\wedge^2 V_0^\vee)$ be the Plücker quadratic form $q_v(\alpha) := vol(v_0 \wedge v \wedge \alpha \wedge \alpha)$. Modulo Identification (3.1.8) we have

$$Y_A \cap (\mathbb{P}(V) \setminus \mathbb{P}(V_0)) = V(\det(q_A + q_v)). \tag{3.1.11}$$

Equivalently let

$$Z_A := V(q_A) \cap \mathbb{G}r(2, V_0) \subset \mathbb{P}(\wedge^2 V_0) \cong \mathbb{P}^9. \tag{3.1.12}$$

Then we have an isomorphism

$$\begin{array}{ccc} \mathbb{P}(V) & \overset{\sim}{\longrightarrow} & |\mathcal{I}_{Z_A}(2)| \\ [\lambda v_0 + \mu v] & \mapsto & V(\lambda q_A + \mu q_v) \end{array} \tag{3.1.13}$$

(Here $\lambda, \mu \in \mathbb{C}$ and $v \in V_0$.) Let $D_A \subset |\mathcal{I}_{Z_A}(2)|$ be the discriminant locus; modulo the above identification we have

$$Y_A \cap (\mathbb{P}(V) \setminus \mathbb{P}(V_0)) = D_A \cap (|\mathcal{I}_{Z_A}(2)| \setminus |\mathcal{I}_{\mathbb{G}r(2,V_0)}(2)|). \tag{3.1.14}$$

Notice that $|\mathcal{I}_{\mathbb{G}r(2,V_0)}(2)|$ is a hyperplane contained in D_A with multiplicity 4; that explains why $\deg Y_A = 6$ while $\deg D_A = 10$.

We go back to general considerations regarding Y_A. The symmetric map τ_A^B of (3.1.7) allows us to give a structure of scheme to the degeneracy locus

$$Y_A[k] = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) \geq k\} \tag{3.1.15}$$

by declaring that $Y_A[k] \cap \mathcal{U}_B = V(\wedge^{(11-k)} \tau_A^B)$. By a standard dimension count we expect that the following holds for generic $A \in \mathbb{LG}(\wedge^3 V)$: $Y_A[3] = \emptyset$, $Y_A[2] = singY_A$ and $Y_A[2]$ is a smooth surface (of degree 40 by (6.7) of [17]), in particular

[16] It might happen that there is no V_0 such that $\wedge^3 V_0$ is transversal to A: in that case A is unstable for the natural $PGL(V)$-action on $\mathbb{LG}(\wedge^3 V)$ and hence we may forget about it.

Y_A should be a very special sextic hypersurface. This is indeed the case; in order to be less "generic"let

$$\begin{aligned} \Delta := & \quad \{A \in \mathbb{LG}(\wedge^3 V) \mid Y_A[3] \neq \emptyset\}, & (3.1.16)\\ \Sigma := & \quad \{A \in \mathbb{LG}(\wedge^3 V) \mid \exists W \in \mathbb{G}r(3,V) \text{ s. t. } \wedge^3 W \subset A\}. & (3.1.17)\end{aligned}$$

A straightforward computation shows that Σ and Δ are distinct closed irreducible codimension-1 subsets of $\mathbb{LG}(\wedge^3 V)$. Let

$$\mathbb{LG}(\wedge^3 V)^0 := \mathbb{LG}(\wedge^3 V) \setminus \Sigma \setminus \Delta. \tag{3.1.18}$$

Then Y_A has the generic behaviour described above if and only if it belongs to $\mathbb{LG}(\wedge^3 V)^0$. Next let $A \in \mathbb{LG}(\wedge^3 V)$ and suppose that $Y_A \neq \mathbb{P}(V)$: then Y_A comes equipped with a natural double cover $f_A\colon X_A \to Y_A$ defined as follows. Let $i\colon Y_A \hookrightarrow \mathbb{P}(V)$ be the inclusion map: since $\mathrm{coker}(\lambda_A)$ is annihilated by a local generator of $\det \lambda_A$ we have $\mathrm{coker}(\lambda_A) = i_*\zeta_A$ for a sheaf ζ_A on Y_A. Choose $B \in \mathbb{LG}(\wedge^3 V)$ transversal to A; the direct-sum decomposition $\wedge^3 V = A \oplus B$ defines a projection map $\wedge^3 V \to A$; thus we get a map $\mu_{A,B}\colon F \to A \otimes \mathcal{O}_{\mathbb{P}(V)}$. We claim that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & F & \xrightarrow{\lambda_A} & A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_A & \to & 0\\ & & \downarrow{\scriptstyle \mu_{A,B}} & & \downarrow{\scriptstyle \mu^t_{A,B}} & & \downarrow{\scriptstyle \beta_A} & & \\ 0 & \to & A \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda^t_A} & F^\vee & \longrightarrow & Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) & \to & 0 \end{array} \tag{3.1.19}$$

(Since A is Lagrangian the symplectic form defines a canonical isomorphism $\big(\wedge^3 V/A\big) \cong A^\vee$; that is why we may write λ_A as above.) In fact the second row is obtained by applying the $Hom(\,\cdot\,, \mathcal{O}_{\mathbb{P}(V)})$-functor to the first row and the equality $\mu^t_{A,B} \circ \lambda_A = \lambda^t_A \circ \mu_{A,B}$ holds because F is a Lagrangian sub-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$. Lastly β_A is defined to be the unique map making the diagram commutative; as suggested by notation it is independent of B. Next by applying the $Hom(i_*\zeta_A, \,\cdot\,)$-functor to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(6) \longrightarrow \mathcal{O}_{Y_A}(6) \longrightarrow 0 \tag{3.1.20}$$

we get the exact sequence

$$0 \longrightarrow i_*Hom(\zeta_A, \mathcal{O}_{Y_A}(6)) \xrightarrow{\partial} Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{n} Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}(6)) \tag{3.1.21}$$

where n is locally equal to multiplication by $\det \lambda_A$. Since the second row of (3.1.19) is exact a local generator of $\det \lambda_A$ annihilates $Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)})$; thus $n = 0$ and hence we get a canonical isomorphism

$$\partial^{-1}\colon Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{\sim} i_*Hom(\zeta_A, \mathcal{O}_{Y_A}(6)). \tag{3.1.22}$$

Let

$$\begin{array}{ccc} \zeta_A \times \zeta_A & \xrightarrow{\widetilde{m}_A} & \mathcal{O}_{Y_A}(6)\\ (\sigma_1, \sigma_2) & \mapsto & (\partial^{-1} \circ \beta_A(\sigma_1))(\sigma_2). \end{array} \tag{3.1.23}$$

Let $\xi_A := \zeta_A(-3)$; tensorizing both sides of (3.1.23) by $\mathcal{O}_{Y_A}(-6)$ we get a multiplication map

$$m_A\colon \xi_A \times \xi_A \to \mathcal{O}_{Y_A}. \tag{3.1.24}$$

The above multiplication map equips $\mathcal{O}_{Y_A}\oplus\xi_A$ with the structure of a commutative and associative $\mathcal{O}_{Y_A}$-algebra. We let

$$X_A := \mathrm{Spec}(\mathcal{O}_{Y_A}\oplus\xi_A), \qquad f_A\colon X_A\to Y_A. \tag{3.1.25}$$

Then X_A is a *double EPW-sextic*. Let $\mathcal{U}_B$ be as in (3.1.6): we may describe $f_A^{-1}(Y_A\cap\mathcal{U}_B)$ as follows. Let M be the symmetric matrix associated to (3.1.7) by a choice of basis of A and M^c be the matrix of cofactors of M. Let $Z=(z_1,\ldots,z_{10})^t$ be the coordinates on A associated to the given basis; then $f_A^{-1}(Y_A\cap\mathcal{U}_B)\subset\mathcal{U}_B\times\mathbb{A}^{10}_Z$ and its ideal is generated by the entries of the matrices

$$M\cdot Z\,,\quad Z\cdot Z^t-M^c\,. \tag{3.1.26}$$

(The "missing"equation $\det M=0$ follows by Cramer's rule.) One may reduce the size of M in a neighborhood of $[v_0]\in\mathcal{U}_B$ as follows. The kernel of the symmetric map $\tau_A^B([v_0])$ equals $A\cap F_{v_0}$; let $J\subset A$ be complementary to $A\cap F_{v_0}$. Diagonalizing the restriction of τ_A^B to J we may assume that

$$M([v])=\begin{pmatrix} M_0([v]) & 0\\ 0 & 1_{10-k}\end{pmatrix} \tag{3.1.27}$$

where $k:=\dim(A\cap F_{v_0})$ and M_0 is a symmetric $k\times k$ matrix. It follows at once that f_A is étale over $(Y_A\setminus Y_A[2])$. We also get the following description of f_A over a point $[v_0]\in(Y_A[2]\setminus Y_A[3])$ under the hypothesis that there is no $0\neq v_0\wedge v_1\wedge v_2\in A$. First $f_A^{-1}([v_0])$ is a single point p_0, secondly X_A is smooth at p_0 and there exists an involution ϕ on (X_A,p_0) with 2-dimensional fixed-point set such that f_A is identified with the quotient map $(X_A,p_0)\to(X_A,p_0)/\langle\phi\rangle$. It follows that X_A is smooth if $A\in\mathbb{LG}(\wedge^3V)^0$. We may fit together all smooth double EPW-sextics by going to a suitable double cover $\rho\colon\mathbb{LG}(\wedge^3V)^\star\to\mathbb{LG}(\wedge^3V)^0$; there exist a family of HK four-folds $\mathcal{X}\to\mathbb{LG}(\wedge^3V)^\star$ and a relatively ample line-bundle $\mathcal{L}$ over $\mathcal{X}$ such that for all $t\in\mathbb{LG}(\wedge^3V)^\star$ we have $(X_t,L_t)\cong(X_{A_t},f_{A_t}^*\mathcal{O}_{Y_{A_t}}(1))$ where

$$X_t:=\rho^{-1}(t),\quad L_t=\mathcal{L}|_{X_t},\quad A_t:=\rho(t). \tag{3.1.28}$$

The following result was proved in [58].

Theorem 3.2. [O'Grady] *Let $A\in\mathbb{LG}(\wedge^3V)^0$. Then X_A is a HK four-fold deformation equivalent to $K3^{[2]}$. Moreover $\mathcal{X}\to\mathbb{LG}(\wedge^3V)^\star$ is a locally complete family of HK varieties of degree* 2.

Sketch of proof following [61]. The main issue is to prove that X_A is a HK deformation of $K3^{[2]}$. In fact once this is known the equality

$$\int_{X_A} f_A^*c_1(\mathcal{O}_{Y_A}(1))^4=2\cdot 6=12 \tag{3.1.29}$$

together with (2.1.2) gives that $q(f_A^*c_1(\mathcal{O}_{Y_A}(1)))=2$ and moreover the family $\mathcal{X}\to\mathbb{LG}(\wedge^3V)^\star$ is locally complete by the following argument. First Kodaira vanishing and Formula (2.2.7) give that

$$h^0(f_A^*\mathcal{O}_{Y_A}(1))=\chi(f_A^*\mathcal{O}_{Y_A}(1))=6 \tag{3.1.30}$$

and hence the map

$$X_A \xrightarrow{f_A} Y_A \hookrightarrow \mathbb{P}(V) \tag{3.1.31}$$

may be identified with the map $X_A \to |f_A^*\mathcal{O}_{Y_A}(1)|^\vee$. From this one gets that the natural map $(\mathbb{LG}(\wedge^3 V)^0//PGL(V), [A]) \to Def(X_A, f_A^*\mathcal{O}_{Y_A}(1))$ is injective. One concludes that $\mathcal{X} \to \mathbb{LG}(\wedge^3 V)^\star$ is locally complete by a dimension count:

$$\dim(\mathbb{LG}(\wedge^3 V)^0//PGL(V)) = 20 = \dim Def(X_A, f_A^*\mathcal{O}_{Y_A}(1)). \tag{3.1.32}$$

Thus we are left with the task of proving that X_A is a HK deformation of $K3^{[2]}$ if $A \in \mathbb{LG}(\wedge^3 V)^0$. We do this by analyzing X_A for

$$A \in (\Delta \setminus \Sigma). \tag{3.1.33}$$

By definition $Y_A[3]$ is non-empty; one shows that it is finite, that $singX_A = f_A^{-1}Y_A[3]$ and that $f_A^{-1}[v_i]$ is a single point for each $[v_i] \in Y_A[3]$. There exists a small resolution

$$\pi_A \colon \widehat{X}_A \longrightarrow X_A, \qquad (f_A \circ \pi_A)^{-1}([v_i]) \cong \mathbb{P}^2 \quad \forall [v_i] \in Y_A. \tag{3.1.34}$$

In fact one gets that locally over the points of $singX_A$ the above resolution is identified with the contraction c (or $c^\vee$) appearing in (1.3.2) - in particular $\widehat{X}_A$ is not unique, in fact there are $2^{|Y_A[3]|}$ choices involved in the construction of $\widehat{X}_A$. The resolution $\widehat{X}_A$ fits into a simultaneous resolution i.e. given a sufficiently small open (in the classical topology) $A \in U \subset (\mathbb{LG}(\wedge^3 V) \setminus \Sigma)$ we have proper maps π, ψ

$$\widehat{\mathcal{X}}_U \xrightarrow{\pi} \mathcal{X}_U \xrightarrow{\psi} U \tag{3.1.35}$$

where ψ is a tautological family of double EPW-sextics over U i.e. $\psi^{-1}A \cong X_A$ and $(\psi \circ \pi)^{-1}A \to \psi^{-1}A = X_A$ is a small resolution as above if $A \in U \cap \Delta$ while $\pi^{-1}A \cong X_A$ if $A \in (U \setminus \Delta)$. Thus it suffices to prove that there exist $A \in (\Delta \setminus \Sigma)$ such that $\widehat{X}_A$ is a HK deformation of $K3^{[2]}$. Let $[v_i] \in Y_A[3]$; we define a $K3$ surface $S_A(v_i)$ as follows. There exists a codimension-1 subspace $V_0 \subset V$ not containing v_i and such that $\wedge^3 V_0$ is transversal to A. Thus Y_A can be described as in **Remark 3.1**: we adopt notation introduced in that remark, in particular we have the quadric $Q_A := V(q_A) \subset \mathbb{P}(\wedge^2 V_0)$. The singular locus of Q_A is $\mathbb{P}(A \cap F_{v_i})$ - we recall Identification (3.1.9). By hypothesis $\mathbb{P}(A \cap F_{v_i}) \cap \mathbb{G}r(2, V_0) = \emptyset$; it follows that $\dim \mathbb{P}(A \cap F_{v_i}) = 2$ (by hypothesis $\dim \mathbb{P}(A \cap F_{v_i}) \geq 2$). Let

$$S_A(v_i) := Q_A^\vee \cap \mathbb{G}r(2, V_0) \subset \mathbb{P}(\wedge^2 V_0^\vee). \tag{3.1.36}$$

Then $S_A(v_i) \subset \mathbb{P}(Ann\,(A \cap F_{v_i})) \cong \mathbb{P}^6$ is the transverse intersection of a smooth quadric and the Fano 3-fold of index 2 and degree 5, i.e. the generic $K3$ of genus 6. There is a natural degree-2 rational map

$$g_i \colon S_A(v_i)^{[2]} \dashrightarrow |\mathcal{I}_{S_A(v_i)}(2)|^\vee \tag{3.1.37}$$

which associates to $[Z]$ the set of quadrics in $|\mathcal{I}_{S_A(v_i)}(2)|$ which contain the line spanned by Z - thus g_i is regular if $S_A(v_i)$ contains no lines. One proves that $Im(g_i)$ may be identified with Y_A; it follows that there exists a birational map

$$h_i \colon S_A(v_i)^{[2]} \dashrightarrow \widehat{X}_A \tag{3.1.38}$$

Moreover if $S_A(v_i)$ contains no lines (that is true for generic $A \in (\Delta \setminus \Sigma)$) there is a choice of small resolution $\widehat{X}_A$ such that h_i is regular and hence an isomorphism - in particular $\widehat{X}_A$ is projective[17]. This proves that X_A is a HK deformation of $K3^{[2]}$ for $A \in \mathbb{LG}(\wedge^3 V)^0$. □

Remark 3.3. The above proof of **Theorem 3.2** provides a description of X_A for $A \in (\Delta \setminus \Sigma)$; what about X_A for $A \in \Sigma$? One proves that if $A \in \Sigma$ is generic - in particular there is a unique $W \in \mathbb{G}r(3, V)$ such that $\wedge^3 W \subset A$ - then the following hold:

(1) $C_{W,A} := \{[v] \in \mathbb{P}(W) \mid \dim(A \cap F_v) \geq 2\}$ is a smooth sextic curve.

(2) $singX_A = f_A^{-1}\mathbb{P}(W)$ and the restriction of f_A to $singX_A$ is the double cover of $\mathbb{P}(W)$ branched over $C_{W,A}$, i.e. a $K3$ surface of degree 2.

(3) If $p \in singX_A$ the germ (X_A, p) (in the classical topology) is isomorphic to the product of a smooth 2-dimensional germ and an A_1 singularity; thus the blow-up $\widetilde{X}_A \to X_A$ resolves the singularities of X_A.

(4) Let $U \subset \mathbb{LG}(\wedge^3 V)$ be a small open (classical topology) subset containing A. After a base change $\widetilde{U} \to U$ of order 2 branched over $U \cap \Sigma$ there is a simultaneous resolution of singularities of the tautological family of double EPW's parametrized by $\widetilde{U}$. It follows that $\widetilde{X}_A$ is a HK deformation of $K3^{[2]}$.

(5) Let E_A be the exceptional divisor of the blow-up $\widetilde{X}_A \to X_A$ and $e_A \in H^2(\widetilde{X}_A; \mathbb{Z})$ be its Poincaré dual; then $q(e_A) = -2$ and $(e_A, H^2(\widetilde{X}_A; \mathbb{Z})) = \mathbb{Z}$.

3.2 The Beauville-Donagi family

Let $\mathcal{D}, \mathcal{P} \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the prime divisors parametrizing singular cubics and cubics containing a plane respectively. We recall that if $Z \in (|\mathcal{O}_{\mathbb{P}^5}(3)| \setminus \mathcal{D})$ then

$$X = F(Z) := \{L \in \mathbb{G}r(1, \mathbb{P}^5) \mid L \subset X\} \tag{3.2.1}$$

is a HK four-fold deformation equivalent to $K3^{[2]}$. Let H be the Plücker ample divisor on X and $h = c_1(\mathcal{O}_X(H))$; then

$$q(h) = 6, \qquad (h, H^2(X; \mathbb{Z}))_X = 2\mathbb{Z}. \tag{3.2.2}$$

These results are proved in [2] by considering the codimension-1 locus of Pfaffian cubics; they show that if Z is a generic such Pfaffian cubic then X is isomorphic to $S^{[2]}$ where S is a $K3$ of genus 8 that one associates to Z, moreover the class h is identified with $2\widetilde{\mu}(D) - 5\xi_2$ where D is the class of the (genus 8) hyperplane class of S. Here we will stress the similarities between the HK four-folds parametrized by $\mathcal{D}, \mathcal{P}$ and those parametrized by the loci $\Delta, \Sigma \subset \mathbb{LG}(\wedge^3 V)$ described in the previous subsection. Let $Z \in \mathcal{D}$ be generic. Then Z has a unique singular point p and it is ordinary quadratic, moreover the set of lines in Z containing p is a

[17] There is no reason a priori why $\widehat{X}_A$ should be Kähler, in fact one should expect it to be non-Kähler for some A and some choice of small resolution

$K3$ surface S of genus 4. The variety $X = F(Z)$ parametrizing lines in Z is birational to $S^{[2]}$; the birational map is given by

$$\begin{array}{ccc} S^{[2]} & \dashrightarrow & F(Z) \\ \{L_1, L_2\} & \mapsto & R \end{array} \tag{3.2.3}$$

where $L_1 + L_2 + R = \langle L_1, L_2 \rangle \cdot Z$. Moreover $F(Z)$ is singular with singular locus equal to S. Thus from this point of view $\mathcal{D}$ is similar to Δ. On the other hand let $Z_0 \in (|\mathcal{O}_{\mathbb{P}^5}(3)| \setminus \mathcal{D})$ be "close"to Z; the monodromy action on $H^2(F(Z_0))$ of a loop in $(|\mathcal{O}_{\mathbb{P}^5}(3)| \setminus \mathcal{D})$ which goes once around $\mathcal{D}$ has order 2 and hence as far as monodromy is concerned $\mathcal{D}$ is similar to Σ. (Let $U \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be a small open (classical topology) set containing Z; it is natural to expect that after a base change $\pi\colon \widetilde{U} \to U$ of order 2 ramified over $\mathcal{D}$ the family of $F(Z_u)$ for $u \in (\widetilde{U} \setminus \pi^{-1}\mathcal{D})$ can be completed over points of $\pi^{-1}\mathcal{D}$ with HK four-folds birational (isomorphic?) to $S^{[2]}$.) Now let $Z \in \mathcal{P}$ be generic, in particular it contains a unique plane P. Let $T \cong \mathbb{P}^2$ parametrize 3-dimensional linear supspaces of $\mathbb{P}^5$ containing P; given $t \in T$ and L_t the corresponding 3-space the intersection $L_t \cdot Z$ decomposes as $P + Q_t$ where Q_t is a quadric surface. Let $E \subset X = F(Z)$ be the set defined by

$$E := \{L \in F(Z) \mid \exists t \in T \text{ such that } L \subset Q_t\}. \tag{3.2.4}$$

For Z generic we have a well-defined map $E \to T$ obtained by associating to L the unique t such that $L \subset Q_t$; the Stein factorization of $E \to T$ is $E \to S \to T$ where $S \to T$ is the double cover ramified over the curve $B \subset T$ parametrizing singular quadrics. The locus B is a smooth sextic curve and hence S is a $K3$ surface of genus 2. The picture is: E is a conic bundle over the $K3$ surface S and we have

$$q(E) = -2, \qquad (e, H^2(X;\mathbb{Z})) = \mathbb{Z}, \quad e := c_1(\mathcal{O}_X(E)). \tag{3.2.5}$$

Thus from this point of view $\mathcal{P}$ is similar to Σ - of course if we look at monodromy the analogy fails.

4 Numerical Hilbert squares

A *numerical Hilbert square* is a HK four-fold X such that c_X is equal to the Fujiki constant of $K3^{[2]}$ and the lattice $H^2(X;\mathbb{Z})$ is isometric to $H^2(K3^{[2]};\mathbb{Z})$; by (2.1.9), (2.1.10) this holds if and only if

$$H^2(X;\mathbb{Z}) \cong U^3 \widehat{\oplus} E_8(-1) \widehat{\oplus} \langle -2 \rangle, \qquad c_X = 1. \tag{4.0.1}$$

We will present a program which aims to prove that a numerical Hilbert square is a deformation of $K3^{[2]}$ i.e. an analogue of Kodaira's theorem that any two $K3$'s are deformation equivalent. First we recall how Kodaira [33] proved that $K3$ surfaces form a single deformation class. Let X_0 be a $K3$. Let $\mathcal{X} \to T$ be a representative of the deformation space $Def(X_0)$. The image of the local period map $\pi\colon T \to \mathbb{P}(H^2(X_0))$ contains an open (classical topology) subset of the quadric $\mathcal{Q} := V(q_{X_0})$. The set $\mathcal{Q}(\mathbb{Q})$ of rational points of $\mathcal{Q}$ is dense (classical topology) in the set of real points $\mathcal{Q}(\mathbb{R})$; it follows that the image $\pi(T)$ contains a point $[\sigma]$ such that $\sigma^{\perp} \cap H^2(X_0;\mathbb{Q})$ is generated by a non-zero

α such that $q_X(\alpha) = 0$. Let $t \in T$ such that $\pi(t) = [\sigma]$ and set $X := X_t$; by the Lefschetz $(1,1)$ Theorem we have

$$H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}c_1(L), \qquad q_X(c_1(L)) = 0 \tag{4.0.2}$$

where L is a holomorphic line-bundle on X. By Hirzebruch-Riemann-Roch and Serre duality we get that $h^0(L) + h^0(L^{-1}) \geq 2$. Thus we may assume that $h^0(L) \geq 2$. It follows that L is globally generated, $h^0(L) = 2$ and the map $\phi_L \colon X \to |L| \cong \mathbb{P}^1$ is an elliptic fibration. Kodaira then proved that any two elliptic $K3$'s are deformation equivalent. J. Sawon [66] has launched a similar program with the goal of classifying deformation classes of higher-dimensional HK manifolds[18] by deforming them to Lagrangian fibrations - we notice that Matsushita [43, 44, 45] has proved quite a few results on HK manifolds which have non-trivial fibrations. The program is quite ambitious; it runs immediately into the problem of proving that if L is a non-trivial line-bundle on a HK manifold X with $q_X(c_1(L)) = 0$ then $h^0(L) + h^0(L^{-1}) > 0$[19] On the other hand Kodaira's theorem on $K3$'s may be proved [38] by deforming X_0 to a $K3$ surface X such that $H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}c_1(L)$ where L is a holomorphic line-bundle such that $q_X(L)$ is a small positive integer, say 2. By Hirzebruch-Riemann-Roch and Serre duality $h^0(L) + h^0(L^{-1}) \geq 3$ and hence we may assume that $h^0(L) \geq 3$; it follows easily that L is globally generated, $h^0(L) = 3$ and the map $\phi_L \colon X \to |L|^\vee \cong \mathbb{P}^2$ is a double cover ramified over a smooth sextic curve. Thus every $K3$ is deformation equivalent to a double cover of $\mathbb{P}^2$ ramified over a sextic; since the parameter space for smooth sextics is connected it follows that any two $K3$ surfaces are deformation equivalent. Our idea is to extend this proof to the case of numerical Hilbert squares. In short the plan is as follows. Let X_0 be a numerical Hilbert square. First we deform X_0 to a HK four-fold X such that

$$H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}c_1(L), \qquad q_X(c_1(L)) = 2 \tag{4.0.3}$$

and the Hodge structure of X is very generic given the constraint (4.0.3), see Section 4.1 for the precise conditions. By Huybrechts' Projectivity Criterion **??** we may assume that L is ample and then Hirzebruch-Riemann-Roch together with Kodaira vanishing gives that $h^0(L) = 6$. Thus we must study the map $f \colon X \dashrightarrow |L|^\vee \cong \mathbb{P}^5$. We prove that either f is the natural double cover of an EPW-sextic or else it is birational onto its image (a hypersurface of degree at most 12). We conjecture that the latter never holds; if the conjecture is true then any numerical Hilbert square is a deformation of a double EPW-sextic and hence is a deformation of $K3^{[2]}$.

4.1 The deformation

We recall Huybrechts' Theorem on surjectivity of the global period map for HK manifolds. Let X_0 be a HK manifold. Let L be a lattice isomorphic to the lattice $H^2(X_0;\mathbb{Z})$; we denote by $(,)_L$ the extension to $L \otimes \mathbb{C}$ of the bilinear symmetric form on L. The period domain $\Omega_L \subset \mathbb{P}(L \otimes \mathbb{C})$ is given by

$$\Omega_L := \{[\sigma] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\sigma,\sigma)_L = 0, \quad (\sigma,\overline{\sigma})_L > 0\}. \tag{4.1.1}$$

[18] One should assume that $b_2 \geq 5$ in order to ensure that the set of rational points in $V(q_X)$ is non-empty (and hence dense in the set of real points).

[19] Let $\dim X = 2n$. Hirzebruch-Riemann-Roch gives that $\chi(L) = n + 1$, one would like to show that $h^q(L) = 0$ for $0 < q < 2n$.

A HK manifold X deformation equivalent to X_0 is *marked* if it is equipped with an isometry of lattices $\psi\colon L \xrightarrow{\sim} H^2(X;\mathbb{Z})$. Couples (X,ψ) and (X',ψ') are equivalent if there exists an isomorphism $f\colon X \to X'$ such that $H^2(f)\circ\psi' = \pm\psi$. The moduli space $\mathcal{M}_{X_0}$ of marked HK manifolds deformation equivalent to X_0 is the set of equivalence classes of couples as above. If $t \in \mathcal{M}_{X_0}$ we let (X_t,ψ_t) be a representative of t. Choosing a representative $\mathcal{X} \to T$ of the deformation space of X_t with T contractible we may put a natural structure of (non-separated) complex analytic manifold on $\mathcal{M}_{X_0}$, see for example Thm.(2.4) of [40]. The period map is given by

$$\begin{array}{ccc} \mathcal{M}_{X_0} & \xrightarrow{\mathcal{P}} & \Omega_L \\ (X,\psi) & \mapsto & \psi^{-1}H^{2,0}(X). \end{array} \tag{4.1.2}$$

(we denote by the same symbol both the isometry $L \xrightarrow{\sim} H^2(X;\mathbb{Z})$ and its linear extension $L\otimes\mathbb{C} \to H^2(X;\mathbb{C})$.) The map $\mathcal{P}$ is locally an isomorphism by infinitesimal Torelli and local surjectivity of the period map. The following result is proved in [26]; the proof is an adaptation of Todorov's proof of surjectivity for $K3$ surfaces [67].

Theorem 4.1. [Todorov, Huybrechts] *Keep notation as above and let $\mathcal{M}^0_{X_0}$ be a connected component of $\mathcal{M}_{X_0}$. The restriction of $\mathcal{P}$ to $\mathcal{M}^0_{X_0}$ is surjective.*

Let

$$\Lambda := U^3 \widehat{\oplus} E_8\langle -1\rangle^2 \widehat{\oplus} \langle -2\rangle \tag{4.1.3}$$

be the Hilbert square lattice, see (2.1.9). Thus Ω_Λ is the period space for numerical Hilbert squares. A straightforward computation gives the following result, see Lemma 3.5 of [59].

Lemma 4.2. *Suppose that $\alpha_1,\alpha_2 \in \Lambda$ satisfy*

$$(\alpha_1,\alpha_1)_\Lambda = (\alpha_2,\alpha_2)_\Lambda = 2, \qquad (\alpha_1,\alpha_2)_\Lambda \equiv 1 \mod 2. \tag{4.1.4}$$

Let X_0 be a numerical Hilbert square. Let $\mathcal{M}^0_{X_0}$ be a connected component of the moduli space of marked HK four-folds deformation equivalent to X_0. There exists $1 \le i \le 2$ such that for every $t \in \mathcal{M}^0_{X_0}$ the class of $\psi_t(\alpha_i)^2$ in $H^4(X_t;\mathbb{Z})/Tors$ is indivisible.

Notice that Λ contains (many) couples α_1,α_2 which satisfy (4.1.4); it follows that there exists $\alpha \in \Lambda$ such that for every $t \in \mathcal{M}^0_{X_0}$ the class of $\psi_t(\alpha)^2$ in $H^4(X_t;\mathbb{Z})/Tors$ is indivisible. There exists $[\sigma] \in \Omega_\Lambda$ such that

$$\sigma^\perp \cap \Lambda = \mathbb{Z}\alpha. \tag{4.1.5}$$

By **Theorem 4.1** there exists $t \in \mathcal{M}_{X_0}$ such that $\mathcal{P}(t) = [\sigma]$; Equality (4.1.6) gives that

$$H^{1,1}_{\mathbb{Z}}(X_t) = \mathbb{Z}\alpha. \tag{4.1.6}$$

Since $q(\psi_t(\alpha)) = 2 > 0$ the HK manifold X_t is projective by **Theorem 2.7**; by (4.1.6) either $\psi_t(\alpha)$ or $\psi_t(-\alpha)$ is ample and hence we may assume that $\psi_t(\alpha)$ is ample. Let $X' := X_t$ and H' be the divisor class such that $c_1(\mathcal{O}_{X'}(H')) = \psi_t(\alpha)$; X' is a first approximation to the deformation of X_0 that we will consider. The reason for requiring that $\psi_t(\alpha)^2$ be indivisible in $H^4(X_t;\mathbb{Z})/Tors$ will become apparent in the sketch of the proof of **Theorem 4.5**.

Remark 4.3. If X is a deformation of $K3^{[2]}$ and $\alpha \in H^2(X;\mathbb{Z})$ is an arbitrary class such that $q(\alpha) = 2$ then the class of α^2 in $H^4(X;\mathbb{Z})/Tors$ is not divisible, see Proposition 3.6 of [59].

Let $\pi\colon \mathcal{X} \to S$ be a representative of the deformation space $Def(X', H')$. Thus letting $X_s := \pi^{-1}(s)$ there exist $0 \in S$ and a given isomorphism $X_0 \xrightarrow{\sim} X'$ and moreover there is a divisor-class $\mathcal{H}$ on $\mathcal{X}$ which restricts to H' on X_0; we let $H_s := \mathcal{H}|_{X_s}$. We will replace (X', H') by (X_s, H_s) for s very general in S in order to ensure that $H^4(X_s)$ has the simplest possible Hodge structure. First we describe the Hodge substructures of $H^4(X_s)$ that are forced by the Beauville-Bogomolov quadratic form and the integral $(1,1)$ class $\psi_t(\alpha)$. Let X be a HK manifold. The Beaville-Bogomolov quadratic form q_X provides us with a non-trivial class $q_X^\vee \in H^{2,2}_{\mathbb{Q}}(X)$. In fact since q_X is non-degenerate it defines an isomorphism

$$L_X\colon H^2(X) \xrightarrow{\sim} H^2(X)^\vee. \tag{4.1.7}$$

Viewing q_X as a symmetric tensor in $H^2(X)^\vee \otimes H^2(X)^\vee$ and applying L_X^{-1} we get a class $(L_X^{-1} \otimes L_X^{-1})(q_X) \in H^2(X) \otimes H^2(X)$; applying the cup-product map $H^2(X) \otimes H^2(X) \to H^4(X)$ to $(L_X^{-1} \otimes L_X^{-1})(q_X)$ we get an element $q_X^\vee \in H^4(X;\mathbb{Q})$ which is of type $(2,2)$ by Equation (2.1.11). Now we assume that X is a numericla Hilbert square and that H is a divisor class such that $q(H) = 2$. Let $h := c_1(\mathcal{O}_X(H))$. We have an orthogonal (with respect to q_X) direct sum decomposition

$$H^2(X) = \mathbb{C}h \widehat{\oplus} h^\perp \tag{4.1.8}$$

into Hodge substructures of levels 0 and 2 respectively. Since $b_2(X) = 23$ we get by **Corollary 2.5** that cup-product defines an isomorphism

$$Sym^2 H^2(X) \xrightarrow{\sim} H^4(X). \tag{4.1.9}$$

Because of (4.1.9) we will identify $H^4(X)$ with $Sym^2 H^2(X)$. Thus (4.1.8) gives a direct sum decomposition

$$H^4(X) = \mathbb{C}h^2 \oplus \left(\mathbb{C}h \otimes h^\perp\right) \oplus Sym^2(h^\perp) \tag{4.1.10}$$

into Hodge substructures of levels 0, 2 and 4 respectively. As is easily checked $q_X^\vee \in (\mathbb{C}h^2 \oplus Sym^2(h^\perp)$. Let

$$W(h) := (q^\vee)^\perp \cap Sym^2(h^\perp). \tag{4.1.11}$$

(To avoid misunderstandings: the first orthogonality is with respect to the intersection form on $H^4(X)$, the second one is with respect to q_X.) One proves easily (see Claim 3.1 of [59]) that $W(h)$ is a codimension-1 rational sub Hodge structure of $Sym^2(h^\perp)$, and that we have a direct sum decomposition

$$\mathbb{C}h^2 \oplus Sym^2(h^\perp) = \mathbb{C}h^2 \oplus \mathbb{C}q^\vee \oplus W(h). \tag{4.1.12}$$

Thus we have the decomposition

$$H^4(X;\mathbb{C}) = \left(\mathbb{C}h^2 \oplus \mathbb{C}q^\vee\right) \oplus \left(\mathbb{C}h \otimes h^\perp\right) \oplus W(h) \tag{4.1.13}$$

into sub-H.S.'s of levels 0, 2 and 4 respectively. The following result is Proposition 3.2 of [59].

Claim 4.4. *Keep notation as above. Let $s \in S$ be very general i.e. outside a countable union of proper analytic subsets of S. Then the following hold:*

(1) $H^{1,1}_{\mathbb{Z}}(X_s) = \mathbb{Z}h_s$ *where* $h_s = c_1(\mathcal{O}_{X_s}(H_s))$.

(2) Let $\Sigma \in Z_1(X_s)$ *be an integral algebraic 1-cycle on* X_s *and* $cl(\Sigma) \in H^{3,3}_{\mathbb{Q}}(X_3)$ *be its Poincaré dual. Then* $cl(\Sigma) = mh_s^3/6$ *for some* $m \in \mathbb{Z}$.

(3) If $V \subset H^4(X_s)$ *is a rational sub Hodge structure then* $V = V_1 \oplus V_2 \oplus V_3$ *where* $V_1 \subset \left(\mathbb{C}h_s^2 \oplus \mathbb{C}q^\vee_{X_s}\right)$, V_2 *is either* 0 *or equal to* $\mathbb{C}h_s \otimes h_s^\perp$ *and* V_3 *is either* 0 *or equal to* $W(h_s)$.

(4) The image of h_s^2 *in* $H^4(X_s;\mathbb{Z})/Tors$ *is indivisible.*

(5) $H^{2,2}_{\mathbb{Z}}(X_s)/Tors \subset \mathbb{Z}(h_s^2/2) \oplus \mathbb{Z}(q^\vee_{X_s}/5)$.

Let $s \in S$ be such that Items(1) through (5) of **Claim 4.4** hold. Let $X := X_s$, $H := H_s$ and $h := c_1(\mathcal{O}_X(H))$. Since H is in the positive cone and h generates $H^{1,1}_{\mathbb{Z}}(X)$ we get that H is ample. By construction X is a deformation of our given numerical Hilbert square. The goal is to analyze the linear system $|H|$. First we compute its dimension. A computation (see pp. 564-565 of [59]) gives that $c_2(X) = 6q^\vee_X/5$; it follows that Equation (2.2.7) holds for numerical Hilbert squares. Thus $\chi(\mathcal{O}_X(H)) = 6$. By Kodaira vanishing we get that $h^0(\mathcal{O}_X(H)) = 6$. Thus we have the map

$$f\colon X \dashrightarrow |H|^\vee \cong \mathbb{P}^5. \tag{4.1.14}$$

The following is the main result of [59].

Theorem 4.5. [O'Grady] *Let (X, H) be as above. One of the following holds:*

(a) The line-bundle $\mathcal{O}_X(H)$ *is globally generated and there exist an anti-symplectic involution* $\phi\colon X \to X$ *and an inclusion* $X/\langle\phi\rangle \hookrightarrow |H|^\vee$ *such that the map f of* (4.1.14) *is identified with the composition*

$$X \xrightarrow{\rho} X/\langle\phi\rangle \hookrightarrow |H|^\vee \tag{4.1.15}$$

where ρ is the quotient map.

(b) The map f of (4.1.14) *is birational onto its image (a hypersurface of degree between* 6 *and* 12*).*

Sketch of proof. The following result follows from Items (4) and (5) of **Claim 4.4** plus a straightforward computation, see Proposition 4.1 of [59].

Claim 4.6. *If* $D_1, D_2 \in |H|$ *are distinct then* $D_1 \cap D_2$ *is a reduced irreducible surface.*

In fact we chose h such that h^2 is not divisible in $H^4(X;\mathbb{Z})/Tors$ precisely to ensure that the above claim holds. Let $Y \subset \mathbb{P}^5$ be the image of f (to be precise the closure of the image by f of its regular points). Thus (abusing notation) we have $f\colon X \dashrightarrow Y$. Of course $\dim Y \le 4$. Suppose that $\dim Y = 4$ and that $\deg f = 2$. Then there exists a non-trivial rational involution $\phi\colon X \dashrightarrow X$ commuting with f. Since $\mathrm{Pic}(X) = \mathbb{Z}[H]$ we get that $\phi^* H \sim H$; since $K_X \sim 0$ it follows that ϕ is regular; it follows easily that (a) holds. Thus it suffices to reach a contradiction assuming that $\dim Y < 4$ or $\dim Y = 4$ and $\deg f > 2$. One goes through a (painful) case-by-case analysis. In each case, with the exception of Y a quartic 4-fold, one invokes either **Claim 4.6** or Item (3) of **Claim 4.4**. We give two "baby"cases. First suppose that Y is a quadric 4-fold. Let Y_0 be an open dense subset containing the image by f of its regular points. There exists a 3-dimensional linear space $L \subset \mathbb{P}^5$ such that $L \cap Y_0$ is a reducible surface. Now L corresponds to the intersection of two distinct $D_1, D_2 \in |H|$ and since $L \cap Y_0$ is reducible so is $D_1 \cap D_2$ - that contradicts **Claim 4.6**. As second example we suppose that Y is a smooth cubic 4-fold and f is regular. Notice that

$$H \cdot H \cdot H \cdot H = 12 \tag{4.1.16}$$

by (2.1.4) and hence $\deg f = 4$. Let $H^4(Y)_{pr} \subset H^4(Y)$ be the primitive cohomology. By Item (3) of **Claim 4.4** we must have $f^* H^4(Y)_{pr} \subset \mathbb{C}h \otimes h^{\perp}$. The restriction to $f^* H^4(Y;\mathbb{Q})_{pr}$ of the intersection form on $H^4(X)$ equals the intersection form on $H^4(Y;\mathbb{Q})_{pr}$ multiplied by 4 because $\deg f = 4$; one gets a contradiction by comparing discriminants. □

Conjecture 4.7. Item (b) of **Theorem 4.5** does not occur.

As we will explain in the next subsection **Conjecture 4.7** implies that a numerical Hilbert square is in fact a deformation of $K3^{[2]}$. The following question arised in connection with the proof of **Theorem 4.5**.

Question 4.8. Is the following true? Let X be a HK 4-fold and H an ample divisor on X. Then $\mathcal{O}_X(2H)$ is globally generated.

The analogous question in $dim = 2$ has a positive answer, see for example [46]. We notice that if X is a 4-fold with trivial canonical bundle and H is ample on X then $\mathcal{O}_X(5H)$ is globally generated by Kawamata [32]. The relation between **Question 4.8** and **Theorem 4.5** is the following.

Claim 4.9. *Suppose that the answer to* **Question 4.8** *is positive. Let X be a numerical Hilbert square equipped with an ample divisor H such that $q_X(H) = 2$. Let $Y \subset |H|^{\vee}$ be the closure of the image of the set of regular points of the rational map $X \dashrightarrow |H|^{\vee}$. Then one of the following holds:*

(1) $\mathcal{O}_X(H)$ is globally generated.

(2) Y is contained in a quadric.

Proof. Suppose that Item (2) does not hold. Then multiplication of sections defines an injection $Sym^2 H^0(\mathcal{O}_X(H)) \hookrightarrow H^0(\mathcal{O}_X(2H))$; on the other hand we have

$$\dim Sym^2 H^0(\mathcal{O}_X(H)) = 21 = \dim H^0(\mathcal{O}_X(2H)). \tag{4.1.17}$$

(The last equation holds by Equation (2.2.7) - valid for numerical Hilbert squares as noticed above.) Since $\mathcal{O}_X(2H)$ is globally generated it follows that $\mathcal{O}_X(H)$ is globally generated as well i.e. Item (1) holds. □

We remark that Items (1) and (2) of the above claim are not mutually exclusive. In fact let $S \subset \mathbb{P}^3$ be a smooth quartic surface (a $K3$) not containing lines. We have a finite map

$$\begin{array}{ccc} S^{[2]} & \xrightarrow{f} & \mathbb{G}r(1,\mathbb{P}^3) \subset \mathbb{P}^5 \\ [Z] & \mapsto & \langle Z \rangle \end{array} \tag{4.1.18}$$

with image the Plücker quadric in $\mathbb{P}^5$. Let $H := f^*\mathcal{O}_{\mathbb{P}^5}(1)$; since f is finite H is ample. Moreover $q(H) = 2$ because $H \cdot H \cdot H \cdot H = 12$; thus (4.1.18) may be identified with the map associated to the complete linear system $|H|$.

4.2 Double EPW-sextics, II

Let (X, H) be as in Item (a) of **Theorem 4.5**: we proved [58] that there exists $A \in \mathbb{LG}(\wedge^3\mathbb{C}^6)^0$ such that $Y_A = f(X)$ and the double cover $X \to f(X)$ may be identified with the canonical double cover $X_A \to Y_A$. Since X_A is a deformation of $K3^{[2]}$ it follows that if **Conjecture 4.7** holds then numerical Hilbert squares are deformations of $K3^{[2]}$. The precise result proved in [58] is the following.

Theorem 4.10. [O'Grady] *Let X be a numerical Hilbert square. Suppose that H is an ample divisor class on X such that the following hold:*

(1) $q_X(H) = 2$ (and hence $\dim |H| = 5$).

(2) $\mathcal{O}_X(H)$ is globally generated.

(3) There exist an anti-symplectic involution $\phi\colon X \to X$ and an inclusion $X/\langle\phi\rangle \hookrightarrow |H|^\vee$ such that the map $X \to |H|^\vee$ is identified with the composition

$$X \xrightarrow{\rho} X/\langle\phi\rangle \hookrightarrow |H|^\vee \tag{4.2.1}$$

where ρ is the quotient map.

Then there exists $A \in \mathbb{LG}(\wedge^3\mathbb{C}^6)^0$ such that $Y_A = Y$ and the double cover $X \to f(X)$ may be identified with the canonical double cover $X_A \to Y_A$.

The proof of the above result goes as follows.

Step I. Let $Y := f(X)$; abusing notation we let $f\colon X \to Y$ be the double cover which is identified with the quotient map for the action of $\langle\phi\rangle$. We have the decomposition $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \eta$ where η is the (-1)-eigensheaf for the action of ϕ on $\mathcal{O}_X$. One proves that $\zeta := \eta \otimes \mathcal{O}_Y(3)$ is globally generated - an intermediate step is the proof that $3H$ is very ample. Thus we have an exact sequence

$$0 \to G \longrightarrow H^0(\zeta) \otimes \mathcal{O}_{|H|^\vee} \longrightarrow i_*\zeta \to 0. \tag{4.2.2}$$

where $i\colon Y \hookrightarrow |H|^\vee$ is inclusion.

Step II. One computes $h^0(\zeta)$ as follows. First $H^0(\zeta)$ is equal to $H^0(\mathcal{O}_X(3H))^-$ i.e. the space of ϕ-anti-invariant sections of $\mathcal{O}_X(3H)$. Using Equation (2.2.7) one gets that $h^0(\zeta) = 10$. A local computation shows that G is locally-free. By invoking Beilinsons' spectral sequence for vector-bundles on projective spaces one gets that $G \cong \Omega^3_{|H|^\vee}(3)$. On the other hand one checks easily (Euler sequence) that the vector-bundle F of (3.1.3) is isomorphic to $\Omega^3_{\mathbb{P}(V)}(3)$. Hence if we identify $\mathbb{P}(V)$ with $|H|^\vee$ then F is isomorphic to the sheaf G appearing in (4.2.2). In other words (4.2.2) starts looking like the top horizontal sequence of (3.1.19).

Step III. The multiplication map $\eta \otimes \eta \to \mathcal{O}_Y$ defines an isomorphism $\beta\colon i_*\zeta \xrightarrow{\sim} Ext^1(i_*\zeta, \mathcal{O}_{|H|^\vee})$. Applying general results of Eisenbud-Popescu-Walter [13] (alternatively see the proof of Claim (2.1) of [8]) one gets that β fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^3_{|H|^\vee}(3) & \xrightarrow{\kappa} & H^0(\theta)\otimes\mathcal{O}_{|H|^\vee} & \longrightarrow & i_*\zeta & \to & 0 \\ & & \Big\downarrow{\scriptstyle s^t} & & \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & H^0(\theta)^\vee\otimes\mathcal{O}_{|H|^\vee} & \xrightarrow{\kappa^t} & \Theta^3_{|H|^\vee}(-3) & \xrightarrow{\partial} & \mathit{Ext}^1(i_*\zeta,\mathcal{O}_{|H|^\vee}) & \longrightarrow & 0 \end{array} \tag{4.2.3}$$

where the second row is obtained from the first one by applying $\mathit{Hom}(\cdot,\mathcal{O}_{|H|^\vee})$.

Step IV. One checks that

$$\Omega^3_{|H|^\vee}(3)\xrightarrow{(\kappa,s^t)}\big(H^0(\zeta)\oplus H^0(\zeta)^\vee\big)\otimes\mathcal{O}_{|H|^\vee} \tag{4.2.4}$$

is an injection of vector-bundles. The transpose of the above map induces an isomorphism $\big(H^0(\zeta)^\vee\oplus H^0(\zeta)\big)\xrightarrow{\sim}H^0(\Omega^3_{|H|^\vee}(3)^\vee)$. The same argument shows that the transpose of (3.1.3) induces an isomorphism $\wedge^3V^\vee\xrightarrow{\sim}H^0(F^\vee)$. Since F is isomorphic to $\Omega^3_{|H|^\vee}(3)$ we get an isomorphism $\rho\colon H^0(\zeta)\oplus H^0(\zeta)^\vee\xrightarrow{\sim}\wedge^3 V$ such that (abusing notation) $\rho(\Omega^3_{|H|^\vee}(3))=F$. Lastly one checks that the standard symplectic form on $(H^0(\zeta)\oplus H^0(\zeta)^\vee)$ is identified (up to a multiple) via ρ with the symplectic form $(,)_V$ of (3.1.1). Now let $A=\rho(H^0(\zeta)^\vee)$; then (4.2.3) is identified with (3.1.19). This ends the proof of **Theorem 4.10**.

5 Global Torelli and deformations of $K3^{[2]}$

The following question is motivated by the celebrated Global Torelli Theorem for $K3$ surfaces.

Question 5.1. Let $\mathcal{C}$ be a deformation class of HK manifolds. Is the following true? Let X,Y be HK manifolds whose deformation class is $\mathcal{C}$: then X is bimeromorphic to Y if and only if there exists an integral Hodge isometry $H^2(X)\cong H^2(Y)$.

If the answer to the above question is affirmative we say that *Naive Global Torelli* holds for HK manifolds whose deformation class is $\mathcal{C}$. The reason we do not ask for a biregular Global Torelli is that bimeromorphic HK manifolds have isomorphic H^2 cohomologies by Item (5) of **Remark 2.1** and in dimension greater than 2 there do exist examples of bimeromorphic HK manifolds which are not isomorphic, see for example [9] or the domain and codomain of the birational Map (1.3.3). Notice also that bimeromorphic HK manifolds are deformation equivalent by Huybrechts' **Theorem 2.11**. It is known that Naive Global Torelli does not hold for arbitrary deformation classes of HK manifolds. Namikawa [52] proved that it is false for the deformation class of $K^{[n]}T$ as soon as $n\ge 2$: in fact $K^{[n]}T$ and $K^{[n]}\widehat{T}$ have isomorphic H^2's but Namikawa proved that in general they are not bimeromorphic. Markman [42] proved that if $(n-1)$ is not a prime power[20] then Naive Global Torelli fails for deformations of $(K3)^{[n]}$. A refined Global Torelli Question for deformations of $K3^{[n]}$ (based on work of Markman [42]) has been formulated by Gritsenko, Hulek and Sankaran [21]; if $(n-1)$ is a prime power the refined and naive questions coincide. The recent

[20]Here we assume that $n>1$ of course.

preprint [70] by Verbitsky presents a proof of a result which in particular gives an affirmative answer to **Question 5.1** for deformations of $K3^{[n]}$ and $n-1$ a prime power. Here we will not discuss Verbitsky's paper, instead we will concentrate on the deformation class of $K3^{[2]}$. In the first subsection we will assume that naive Global Torelli holds for deformations of $K3^{[2]}$ and we will derive a few easy (but interesting!) geometric consequences. In the second subsection we will give an outline of our work in progress on moduli and periods of double EPW-sextics.

5.1 Torelli and geometry

In this subsection we make the following

Assumption 5.2. Naive Global Torelli holds for deformations of $K3^{[2]}$.

The first consequence that we will derive from the above assumption is about moduli spaces of polarized deformations of $K3^{[2]}$. (For a more general discussion see [21].) First we recall a few results on lattices. Let Λ be a lattice i.e. a free finitely generated abelian group equipped with an integral bilinear symmetric form - we denote by $(,)_\Lambda$ the bilinear form and by q_Λ the associated quadratic form. We recall that

$$H^2(K3^{[2]};\mathbb{Z}) \cong U^3 \widehat{\oplus} E_8\langle -1\rangle^2 \widehat{\oplus} \langle -2\rangle =: \Theta \tag{5.1.1}$$

The *divisibility* of $v \in \Theta$ is

$$div(v) := |\mathbb{Z}/\{(v,w) \mid \ w \in \Theta\}|\,. \tag{5.1.2}$$

Let v be primitive: then $div(v)$ is either 1 or 2 and if it equals 2 then $q_\Theta(v) \equiv 6 \pmod 8$. The following result is a corollary of Nikulin's general results on lattices [53].

Claim 5.3. *Let $v, w \in \Theta$ be primitive. There exists an isometry $\phi \in O(\Theta)$ such that $\phi(v) = w$ if and only if*

$$q_\Theta(v) = q_\Theta(w), \qquad div(v) = div(w) \tag{5.1.3}$$

Let d be a strictly positive integer and $\epsilon \in \{1, 2\}$. We let $\mathfrak{M}^\epsilon_{2d}$ be the coarse moduli space for deformations X of $K3^{[2]}$ equipped with a primitive ample divisor H such that

$$q_X(H) = 2d, \qquad div(\mathcal{O}_X(H)) = \epsilon. \tag{5.1.4}$$

(See [21] for details.) The period moduli space for such couples (X, H) is defined as follows. Let $v \in \Theta$ be primitive such that $q_\Theta(v) = 2d$ and $div(v) = \epsilon$. Let

$$\Omega_{v^\perp} := \quad \{[\sigma] \in \mathbb{P}(v^\perp \otimes_\mathbb{Z} \mathbb{C}) \mid q_\Theta(\sigma) = 0, \quad (\sigma, \overline{\sigma})_\Theta > 0\}, \tag{5.1.5}$$

$$O(\Theta)_v := \{\phi \in O(\Theta) \mid \phi(v) = v\}. \tag{5.1.6}$$

Then $O(\Theta)_v$ acts properly discontinuously on $\Omega_{v^\perp}$; thus the quotient $\mathbb{D}^\epsilon_{2d} := \Omega_{v^\perp}/O(\Theta)_v$ is an analytic space, in fact a quasi-projective variety by a classical result of Baily and Borel. One may define a period map

$$\mathfrak{M}^\epsilon_{2d} \xrightarrow{\mathfrak{p}^\epsilon_{2d}} \mathbb{D}^\epsilon_{2d} \tag{5.1.7}$$

proceeding as in the definition of (4.1.2), with the extra constraint that $\psi(v) = c_1(\mathcal{O}_X(H))$.

Claim 5.4. *If* **Assumption 5.2** *holds then* $\mathfrak{p}^{\epsilon}_{2d}$ *is an isomorphism onto an open dense subset. In particular* $\mathfrak{M}^{\epsilon}_{2d}$ *is irreducible.*

Proof. The period map $\mathfrak{p}^{\epsilon}_{2d}$ has finite fibers and it has open image because the local period map is surjective and $\mathbb{D}^{\epsilon}_{2d}$ is normal. Thus it suffices to prove the following:

(1) $\mathfrak{M}^{\epsilon}_{2d}$ is not empty.

(2) $\deg \mathfrak{p}^{\epsilon}_{2d} = 1$.

Let

$$\mathbb{D}^{\epsilon}_{2d}(1) := \{[\sigma] \in \mathbb{D}^{\epsilon}_{2d} \mid \sigma^{\perp} \cap \Theta = \mathbb{Z}v\}. \tag{5.1.8}$$

(An element of $\mathbb{D}^{\epsilon}_{2d}$ is a $O(\Theta)_v$-orbit in $\Omega_{v^{\perp}}$; to simplify notation we denote it by a representative $[\sigma]$.) Since $\mathbb{D}^{\epsilon}_{2d}(1)$ is dense in $\mathbb{D}^{\epsilon}_{2d}$ it suffices to prove that

$$|(\mathfrak{p}^{\epsilon}_{2d})^{-1}([\sigma])| = 1 \qquad \forall [\sigma] \in \mathbb{D}^{\epsilon}_{2d}(1). \tag{5.1.9}$$

Let $[\sigma] \in \mathbb{D}^{\epsilon}_{2d}(1)$. By **Theorem 4.1** there exist a deformation X of $K3^{[2]}$ and a marking $\psi\colon \Theta \xrightarrow{\sim} H^2(X;\mathbb{Z})$ such that $\psi([\sigma]) = H^{2,0}(X)$. Since $v \perp \sigma$ we have $\psi(v) \in H^{1,1}_{\mathbb{Z}}(X)$ and since $q_X(\psi(v)) = q_{\Theta}(v) = 2$ we get that X is projective by Huybrechts' projectivity criterion **??**. Moreover since $[\sigma] \in \mathbb{D}^{\epsilon}_{2d}(1)$ we know that $\psi(v)$ generates $H^{1,1}_{\mathbb{Z}}(X)$ and hence $\pm\psi(v)$ is ample. Multiplying ψ by (-1) if necessary we may assume that $\psi(v)$ is ample. Let H be a divisor class H on X such that $c_1(\mathcal{O}_X(H)) = \psi(v)$; then $\mathfrak{p}^{\epsilon}_{2d}(X,H) = [\sigma]$. This proves that $(\mathfrak{p}^{\epsilon}_{2d})^{-1}([\sigma])$ is not empty. Let $[(X,H)], [(X',H')] \in \mathfrak{M}^{\epsilon}_{2d}$ be such that

$$\mathfrak{p}^{\epsilon}_{2d}(X,H) = \mathfrak{p}^{\epsilon}_{2d}(X',H') \in \mathbb{D}^{\epsilon}_{2d}(1). \tag{5.1.10}$$

Let's prove that $[(X,H)] = [(X',H')]$. By **Assumption 5.2** there exists a birational map $\phi\colon X \dashrightarrow X'$. Since $\mathfrak{p}^{\epsilon}_{2d}(X',H') \in \mathbb{D}^{\epsilon}_{2d}(1)$ we have $H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}c_1(\mathcal{O}_X(H))$ and $H^{1,1}_{\mathbb{Z}}(X') = \mathbb{Z}c_1(\mathcal{O}_{X'}(H'))$; it follows that $\phi^*H' \sim H$ and hence ϕ is a regular isomorphism because H, H' are ample and X, X' have trivial canonical bundle. □

Next we show that **Assumption 5.2** is linked to the conjectural converse of **Theorem 2.9**.

Claim 5.5. *Suppose that* **Assumption 5.2** *holds. Then Item (2) of* **Conjecture 2.10** *holds.*

Proof. Let X, L be as in Item (2) of **Conjecture 2.10**. Suppose first that

$$H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}c_1(L). \tag{5.1.11}$$

The subspace $c_1(L)^{\perp} \subset H^2(X)$ is a sub Hodge structure because $c_1(L) \in H^{1,1}_{\mathbb{Z}}(X)$. Let $c_1(L)^{\perp}_{\mathbb{Z}} := c_1(L)^{\perp} \cap H^2(X;\mathbb{Z})$; then

$$H^2(X;\mathbb{Z}) = c_1(L)^{\perp}_{\mathbb{Z}} \oplus \mathbb{Z}c_1(L) \tag{5.1.12}$$

because $q_X(L) = -2$ and $(c_1(L), H^2(X;\mathbb{Z}))_X = 2\mathbb{Z}$. By (5.1.12) the lattice $c_1(L)^{\perp}_{\mathbb{Z}}$ is even, unimodular of signature $(3,19)$; it follows that it is isometric to

$U^3\widehat{\oplus}E_8\langle -1\rangle^2$ i.e. the $K3$ lattice. By surjectivity of the period map for $K3$ surfaces (i.e. **Theorem 4.1**) there exist a $K3$ surface S and an isomorphism of integral Hodge structures $\phi_0\colon H^2(S)\xrightarrow{\sim} c_1(L)^{\perp}$ which is an isometry. By (1.1.9) and (2.1.9) ϕ_0 extends to an isomorphism of integral Hodge structures

$$\phi\colon H^2(S^{[2]})\xrightarrow{\sim} H^2(X) \tag{5.1.13}$$

which is an isometry. By **Assumption 5.2** there exists a bimeromorphic map $f\colon X \dashrightarrow S^{[2]}$. Since $H^{1,1}_{\mathbb{Z}}(X)=\mathbb{Z}c_1(L)$ it follows that $f^*\xi_2=\pm c_1(L)$. Let $\Delta_2\subset S^{[2]}$ be the effective divisor parametrizing non-reduced analytic subsets; then $f_*^{-1}\Delta_2$ is an effective divisor and by (1.1.8) we have $c_1(\mathcal{O}_X(f_*^{-1}\Delta_2))=c_1(L^{\pm 2})$. This proves that Item (2) of **Conjecture 2.10** holds if we make the extra assumption (5.1.11). In general one may proceed as follows. Let $\pi\colon\mathcal{X}\to T$ be a representative of $Def(X,L)$. We let $X_t:=\pi^{-1}(t)$ and $0\in T$ such that $X_0\cong X$. Of course we have a line-bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_{X_0}\cong L$; we let $L_t:=\mathcal{L}|_{X_t}$. The set

$$T_{gen}:=\{t\in T\mid H^{1,1}_{\mathbb{Z}}(X_t)=\mathbb{Z}c_1(\mathcal{L}_t)\} \tag{5.1.14}$$

is dense in T. By what we have proved one gets that either $h^0(L_t^2)>0$ for all $t\in T_{gen}$ or else $h^0(L_t^{-2})>0$ for all $t\in T_{gen}$. We may assume that the former holds; by upper semi-continuity of cohomology dimension it follows that $h^0(L_t^2)>0$ for all $t\in T$, in particular $h^0(L_0^2)>0$. □

Remark 5.6. It should be possible to prove that **Assumption 5.2** implies that Item (2) of **Conjecture 2.10** holds and moreover that the conjectural converse of **Theorem 2.9** is true. The proof will be somewhat less elementary because the generic deformation of $K3^{[2]}$ satsifying the hypothesis of Item (2) of **Conjecture 2.10** or Item (b) of **Theorem 2.9** is not bimeromorphic to a $K3^{[2]}$. The natural idea is to start from one 4-fold satisfying the hypothesis and the conclusion and argue by a deformation argument and stability of the divisor in the first case and the lagrangian surface in the second case.

5.2 Double EPW-sextics and Torelli

Let V be a complex vector space of dimension 6. The action of $PGL(V)$ on $\mathbb{LG}(\wedge^3 V)$ lifts (uniquely) to an action on the Plücker line-bundle i.e. it is linearized. Thus there is a GIT quotient

$$\mathfrak{M}:=\mathbb{LG}(\wedge^3 V)//PGL(V)\,. \tag{5.2.1}$$

Given a semistable $A\in\mathbb{LG}(\wedge^3 V)$ we let $[A]\in\mathfrak{M}$ be the corresponding point. Let $A\in\mathbb{LG}(\wedge^3 V)$ and assume that $Y_A\neq\mathbb{P}(V)$; we let $H_A\in|f_A^*\mathcal{O}_{\mathbb{P}(V)}(1)$, thus (X_A,H_A) is a polarized 4-dimensional scheme, if X_A is smooth i.e. $A\in\mathbb{LG}(\wedge^3 V)^0$ then it is a HK deformation of $K3^{[2]}$ of degree 2. We note that if A is semistable then $Y_A\neq\mathbb{P}(V)$, that is proved in [61]. The open dense $\mathbb{LG}(\wedge^3 V)^0\subset\mathbb{LG}(\wedge^3 V)$ (see (3.1.18)) is contained in the stable locus of $\mathbb{LG}(\wedge^3 V)$ (this follows easily from Proposition 6.1 of [58]). Let

$$\mathfrak{M}^0:=\mathbb{LG}(\wedge^3 V)^0//PGL(V). \tag{5.2.2}$$

One proves that points of $\mathfrak{M}^0$ are in one-to-one correspondence with isomorphism classes of double EPW-sextics i.e. $[A]=[B]$ if and only if the polarized

HK 4-folds (X_A, H_A) and (X_B, H_B) are isomorphic. Let $\mathbb{D} := \mathbb{D}_2^1$, notation as in Subsection 5.1. Let $A \in \mathbb{LG}(\wedge^3 V)^0$; then $q(H_A) = 2$ and hence the period map for double EPW-sextics is a regular map of quasi-projective varieties $\mathfrak{p}^0 \colon \mathfrak{M}^0 \to \mathbb{D}$. Let $\mathbb{D}^{BB} \supset \mathbb{D}$ be the Baily-Borel compactification; then $\mathfrak{p}^0$ extends to a rational map

$$\mathfrak{p} \colon \mathfrak{M} \dashrightarrow \mathbb{D}^{BB}. \tag{5.2.3}$$

By **Theorem 3.2** we know that locally in the classic topology $\mathbb{LG}(\wedge^3 V)^0$ parametrizes a locally versal family of HK deformations of $K3^{[2]}$ of degree 2; it follows that $\mathfrak{p}$ is dominant of finite degree. The following claim gives one motivation for studying the period map $\mathfrak{p}$.

Claim 5.7. *Suppose that*

(a) **Conjecture 4.7** *holds and*

(b) $\deg \mathfrak{p} = 1$.

Then Naive Global Torelli holds for deformations of $K3^{[2]}$.

Before proving the claim we discuss a few density results. Let X be a deformation of $K3^{[2]}$. Let $\pi \colon \mathcal{X} \to T$ be a representative of $Def(X)$. As usual we let $X_t := \pi^{-1}(t)$ and $X_0 \cong X$. We will assume that T is small enough; that means that T is simply connected and that the local period map (2.1.5) is an isomorphism onto an open (classical topology) subset of $V(q_X) \subset \mathbb{P}(H^2(X))$. Since T is simply connected the Gauss-Manin connection gives an identification

$$H^2(X) \xrightarrow{\sim} H^2(X_t) \qquad \forall t \in T. \tag{5.2.4}$$

Given $d \in \mathbb{Z}$ we let $T_{2d} \subset T$ be defined by

$$T_{2d} := \{t \in T \mid H^{1,1}_{\mathbb{Z}}(X_t) \ni \gamma, \quad q(\gamma) = 2d, \quad \gamma \text{ primitive}\}. \tag{5.2.5}$$

The following result is proved by copying the proof of Proposition 2 of Le Potier's paper [38].

Proposition 5.8. *Keep notation as above. Then T_{2d} is dense (classical topology) in T.*

We are interested in the case $d = 1$ and we wish to show that a certain subset of T_2 is dense in T as well. First we define complex multiplication HK manifolds. Let X be a HK manifold such that the restriction of q_X to $H^{1,1}_{\mathbb{Z}}(X)$ is non-degenerate, for example a projective one; the *transcendental lattice of* X is the sublattice $T(X) \subset H^2(X; \mathbb{Z})$ perpendicular to $H^{1,1}_{\mathbb{Z}}(X)$. Let $T(X)_{\mathbb{C}} := T(X) \otimes_{\mathbb{Z}} \mathbb{C}$; then $T(X)_{\mathbb{C}}$ is a Hodge substructure of $H^2(X)$ and it is simple[21] because q_X is non-degenerate on $H^{1,1}_{\mathbb{Z}}(X)$. We say that X *has complex multiplication (CM)* if there exists an endomorphism of the Hodge structure $T(X)_{\mathbb{C}}$ which is not a homothety. Now let

$$\mathcal{V} := \{\alpha \in H^2(X; \mathbb{Z}) \mid q_X(\alpha) = 2\}. \tag{5.2.6}$$

Given $\alpha \in \mathcal{V}$ we let

$$T_\alpha := \{t \in T \mid \alpha \in H^{1,1}_{\mathbb{Z}}(X_t)\}. \tag{5.2.7}$$

[21] It contains no non-trivial sub-H.S.

The above definition makes sense because Gauss-Manin gives Identification (5.2.4). Let $t \in T_\alpha$; since $q_{X_t}(\alpha) > 0$ either $(\alpha, \cdot)_{X_t}$ is strictly positive or strictly negative on $\mathcal{C}_{X_t}$, moreover the sign is independent of t by continuity. Thus we have a disjoint union $\mathcal{V} = \mathcal{V}^+ \coprod \mathcal{V}^-$ where

$$\mathcal{V}^+ := \{\alpha \in \mathcal{V} \mid (\alpha,\ \beta)_{X_t} > 0 \ \ \forall t \in T_\alpha \ \ \forall \beta \in \mathcal{C}_{X_t}\} \tag{5.2.8}$$

$$\mathcal{V}^- := \{\alpha \in \mathcal{V} \mid (\alpha,\ \beta)_{X_t} < 0 \ \ \forall t \in T_\alpha \ \ \forall \beta \in \mathcal{C}_{X_t}\}. \tag{5.2.9}$$

Definition 5.9. Let $\alpha \in \mathcal{V}^+$. We let $T_\alpha^{gen} \subset T_\alpha$ be the set of t such that X_t is not CM (this makes sense: q_{X_t} is non degenerate on $H^{1,1}_{\mathbb{Z}}(X_t)$ because $q_{X_t}(\alpha) > 0$) and moreover Items (1) through (5) of **Claim 4.4** hold with $s = t$ and $h_s = \alpha$.

Proposition 5.10. *Keep notation as above. Then T_α^{gen} is dense (classical topology) in T_α.*

Proof. Let $T_\alpha(1) \subset T_\alpha$ be the set of t such that $H^{1,1}_{\mathbb{Z}}(X_t) = \mathbb{Z}\alpha$. A standard argument gives that $T_\alpha(1)$ is the complement of a countable union of proper analytic subsets of T_α. Let $T_\alpha(2) \subset T_\alpha(1)$ be the set of t such that Item (3) of **Claim 4.4** holds with $X_s = X_t$ and $h_s = \alpha$. Again by a standard argument $T_\alpha(2)$ is the complement of a countable union of proper analytic subsets of T_α - see Lemma 3.3 of [59]. Let $t \in T_\alpha(2)$; then Items (1) through (5) of **Claim 4.4** hold with $s = t$ and $h_s = \alpha$. In fact (1) and (3) hold by definition, (4) holds by **Remark 4.3**; Item (2) follows from Item (1) and Item (5) follows from (3) and (4) - see the proof of Proposition 3.2 of [59]. Let $T_\alpha^{CM} \subset T_\alpha$ be the set of t such that X_t has complex multiplication and $T_\alpha^{CM}(1) := T_\alpha^{CM} \cap T_\alpha(1)$. We claim that

$$T_\alpha^{CM} \text{ is contained in a countable union of proper analytic subsets of } T_\alpha. \tag{5.2.10}$$

Since $(T_\alpha \backslash T_\alpha(1))$ is a countable union of proper analytic subsets of T_α it suffices to prove that $T_\alpha^{CM}(1)$ is contained in a countable union of proper analytic subsets of T_α. Let $t \in T_\alpha^{CM}(1)$. Then $H^{1,1}(X_t) = \alpha^\perp$ and hence there exists an integral homomorphism of groups $\phi\colon \alpha^\perp \to \alpha^\perp$ which is not a homothety and such that $H^{2,0}(X_t)$ is an eigenspace of ϕ, say with eigenvalue λ. Since ϕ is not a homothety the λ-eigenspace $V_\lambda \subset \alpha^\perp$ is not all of $\alpha^\perp$; it follows that

$$\{t \in T_\alpha \mid H^{2,0}(X_t) \subset V_\lambda\} \tag{5.2.11}$$

is a proper analytic subset of T_α. The set of integral ϕ as above is countable; it follows that $T_\alpha^{CM}(1)$ is contained in a countable union of proper analytic subsets of T_α; this proves (5.2.10). Since $T_\alpha^{gen} = T_\alpha(2) \setminus T_\alpha^{CM}$ we get that the complement of T_α^{gen} in T_α is contained in a countable union of proper analytic subsets of T_α, in particular T_α^{gen} is dense in T_α. □

Corollary 5.11. *Keep notation as above. Then*

$$\bigcup_{\alpha \in \mathcal{V}^+} T_\alpha^{gen} \tag{5.2.12}$$

is dense in T.

Proof. We have $T_2 = \cup_{\alpha\in\mathcal{V}^+} T_\alpha$. By **Proposition 5.10** we get that the closure of (5.2.12) equals the closure of T_2. Thus the Corollary follows from **Proposition 5.8**. $\square$

Proof of **Claim 5.7**. Suppose that Z, Z' are HK deformations of $K3^{[2]}$ and that there exists an integral isomorphism of Hodge structures

$$\mu\colon H^2(Z) \xrightarrow{\sim} H^2(Z') \tag{5.2.13}$$

which is an isometry with respect to the B-B forms. Composing μ with $-Id_{H^2(Z')}$ we may assume that

$$\mu(\mathcal{C}_Z) = \mathcal{C}_{Z'}. \tag{5.2.14}$$

By the existence of μ we may choose markings ψ, ψ' of Z and Z' repectively such that $\mathcal{P}(Z,\psi) = \mathcal{P}(Z',\psi')$. Let $\pi\colon \mathcal{Z} \to T$ and $\pi'\colon \mathcal{Z}' \to T'$ be representatives of $Def(Z)$ and $Def(Z')$ with T, T' small. (As usual $Z_t = \pi^{-1}(t)$, $Z_0 \cong Z$ and $Z'_t = (\pi')^{-1}(t)$, $Z'_0 \cong Z'$.) By infinitesimal Torelli and local surjectivity of the period map we may shrink T and T' so that there exists an isomorphism $g\colon T \to T'$ such that

$$\mathcal{P}(Z_t,\psi) = \mathcal{P}(Z'_{g(t)},\psi'). \tag{5.2.15}$$

(Here ψ defines a marking of Z_t by Gauss-Manin and similarly for ψ'.) Let $\mathcal{V}^+_Z \subset H^2(Z;\mathbb{Z})$ and $\mathcal{V}^+_{Z'} \subset H^2(Z';\mathbb{Z})$ be defined as in (5.2.8). By (5.2.14) we have $\mu(\mathcal{V}^+_Z) = \mathcal{V}^+_{Z'}$. Let $\alpha \in \mathcal{V}^+_Z$; by (5.2.15) we have $g(T^{gen}_\alpha) = T^{gen}_{\mu(\alpha)}$. Let $t \in T^{gen}_\alpha$. By **Conjecture 4.7** and **Theorem 4.10** there exist $A, A' \in \mathbb{LG}(\wedge^3 V)^0$ such that Z_t, $Z'_{g(t)}$ are isomorphic to the double EPW-sextics X_A, $X_{A'}$ respectively. Moreover α and α' are the natural ample classes on X_A and $X_{A'}$ respectively because they belong to $\mathcal{V}^+_Z$ and $\mathcal{V}^+_{Z'}$ respectively. We have $\mathfrak{p}([A]) = \mathfrak{p}([A'])$ by (5.2.15); since we are assuming that $\deg\mathfrak{p} = 1$ it follows that $X_A \cong X_{A'}$. Let $f\colon X_{A'} \xrightarrow{\sim} X_A$ be an isomorphism. The integral isomorphism of Hodge structures $H^2(f)\colon H^2(X_A) \xrightarrow{\sim} H^2(X_{A'})$ is an isometry with respect to the B-B forms; it sends α to α' and hence $\alpha^\perp$ to $(\alpha')^\perp$. Since $X_A, X_{A'}$ do <u>not</u> have complex multiplication the restriction of $H^2(f)$ to $\alpha^\perp$ is either equal to the restriction of μ or to the restriction of $-\mu$. If the latter occurs we replace f with its composition with the covering involution of $X_A \to Y_A$ and we get that we may assume that $H^2(f) = \mu$. By **Corollary 5.11** there exists a sequence $\{t_i\}$ converging to 0 with $t_i \in \cup_{\alpha\in\mathcal{V}^+} T^{gen}_\alpha$ for all i. For each t_i we have an isomorphism $f_i\colon Z'_{g(t_i)} \xrightarrow{\sim} Z_{t_i}$ such that $H^2(f_i) = \mu$; under these hypotheses Huybrechts (Theorem 4.3 of [26]) proved that the "Main Lemma"of Burns-Rapoport [7] extends to higher-dimensional HK's i.e. a subsequence of the graphs of the f_i converges to the graph of a bimeromorphic map $Z' \dashrightarrow Z$. $\square$

5.3 Periods of double EPW-sextics

The period maps for double EPW-sextics and for cubic hypesurfaces in $\mathbb{P}^5$ have many common features. We will state the main results on periods of double EPW-sextics and then we will point out the analogies with the case of cubic 4-folds. Let $\Delta, \Sigma \subset \mathbb{LG}(\wedge^3 V)$ be defined by (3.1.16), (3.1.17). One shows that $(\Delta \setminus \Sigma)$ is contained in the stable locus and that the generic point of Σ is stable; it follows that

$$\mathfrak{T} := \Delta//PGL(V), \qquad \mathfrak{N} := \Sigma//PGL(V) \tag{5.3.1}$$

are prime divisors in $\mathfrak{M}$. The period map $\mathfrak{p}$ is not regular; one of the main issues is to determine the locus of regular points of $\mathfrak{p}$. One first proves that $\mathfrak{p}$ is regular away from $\mathfrak{N}$. In order to analyze $\mathfrak{p}$ at a point $x \in \mathfrak{N}$ we assume that A belongs to the unique closed orbit[22] representing x. Suppose that $W \in \mathbb{G}r(3,V)$ and that $\wedge^3 W \subset A$. One defines a subscheme $C_{W,A} \subset \mathbb{P}(W)$ as in Item (1) of **Remark 3.3**; it is either a sextic curve or all of $\mathbb{P}(W)$ (pathological case). We let $\mathfrak{M}^\flat \subset \mathfrak{M}$ be the locus of $[A]$ such that the following holds: for all $W \in \mathbb{G}r(3,V)$ such that $\wedge^3 W \subset A$ the scheme $C_{W,A}$ is a $PGL(W)$-semistable sextic which does not contain a triple conic in the closure of its orbit[23]. The map $\mathfrak{p}$ extends regularly over $\mathfrak{M}^\flat$:

$$\begin{array}{ccc} \mathfrak{M}^\flat & \longrightarrow & \mathbb{D}^{BB} \\ [A] & \mapsto & \mathfrak{p}([A]) \end{array} \tag{5.3.2}$$

(In fact we guess that $\mathfrak{M}^\flat$ is equal to the set of regular points of $\mathfrak{p}$.) Let $\mathfrak{M}^{ADE} \subset \mathfrak{M}^\flat$ be the locus of $[A]$ such that the following holds: for all (or equivalently one) $W \in \mathbb{G}r(3,V)$ such that $\wedge^3 W \subset A$ the scheme $C_{W,A}$ is a reduced sextic with ADE singularities i.e. the double cover $S \to \mathbb{P}(W)$ ramified over $C_{W,A}$ has at most DuVal singularities. One has $\mathfrak{M}^{ADE} = \mathfrak{p}^{-1}\mathbb{D} \cap \mathfrak{M}^\flat$ and hence we have

$$\begin{array}{ccc} \mathfrak{M}^{ADE} & \longrightarrow & \mathbb{D} \\ [A] & \mapsto & \mathfrak{p}([A]) \end{array} \tag{5.3.3}$$

Moreover Map (5.3.3) has finite fibers. Next we analyze the restriction of $\mathfrak{p}$ to $\mathfrak{T}^{ADE} := \mathfrak{T} \cap \mathfrak{M}^{ADE}$ and to $\mathfrak{N}^{ADE} := \mathfrak{N} \cap \mathfrak{M}^{ADE}$. The double EPW-sextic parametrized by $[A] \in (\mathfrak{T} \setminus \mathfrak{N})$ is birational to $S_A(v_i)^{[2]}$ where $S_A(v_i)$ is the $K3$ surface described in the proof of **Theorem 3.2**. Similarly let $[A] \in \mathfrak{N}$ be generic as in **Remark 3.3**; then the double cover $S_{W,A} \to \mathbb{P}(W)$ ramified over the smooth sextic $C_{W,A}$ is a $K3$ surface. It is not the case that X_A is birational to a Hilbert square but $e_A^\perp \subset H^2(X_A)$ is a sub-Hodge structure of $H^2(S_{W,A})$ of index 2. (Here e_A is as in Item (5) of **Remark 3.3**.) In both cases Global Torelli for $K3$'s and Riemann-Roch for $K3$ surfaces allow us to analyze the restriction of $\mathfrak{p}$ to $\mathfrak{T}^{ADE}$ and to $\mathfrak{N}^{ADE}$. The closure of $\mathfrak{p}(\mathfrak{N}^{ADE})$ in $\mathbb{D}$ is an irreducible component $\mathbb{S}_2^\star$ of the divisor

$$\{[\sigma] \in \mathbb{D} \mid \exists \gamma \in \{v,\sigma\}^\perp \cap \Theta \text{ such that } q_\Theta(\gamma) = -2\}. \tag{5.3.4}$$

(Here $v \in \Theta$ is a fixed vector such that $q_\Theta(v) = 2$ - see Subsection 5.1.) Moreover the following hold:

(a) The restriction of $\mathfrak{p}$ to $\mathfrak{N}^{ADE}$ is injective.

(b) $\mathfrak{p}$ is not ramified along $\mathfrak{N}$.

(c) $\mathfrak{p}^{-1}(\mathbb{S}_2^\star) \cap \mathfrak{M}^\flat = \mathfrak{N}^{ADE}$.

Similar results hold for the period map on $\mathfrak{T}$. Let's pretend for a moment that $\mathfrak{p}$ is regular; then Items (a)-(c) give that $\deg \mathfrak{p} = 1$ because $(\mathfrak{M} \setminus \mathfrak{M}^\flat)$ contains no divisor - in fact it has relatively high codimension. Going back to the "real" world (i.e. $\mathfrak{p}$ is not regular): if the dimension of $(\mathfrak{M} \setminus \mathfrak{M}^\flat)$ is at

[22] That is closed in the semistable locus $\mathbb{LG}(\wedge^3 V)^{ss}$.

[23] The GIT-quotient of the space of plane sextics is a compactification of the moduli space of degree-2 $K3$ surfaces; the point corresponding to triple conics is the indeterminacy locus of the period map to the Baily-Borel compactification of the relevant period moduli space.

most 6 then one may adapt an argument of Voisin [71] (see the Erratum) and derive $\deg \mathfrak{p} = 1$ from (a) through (c) above. We do not yet know whether the required upper bound holds - what is missing is a complete (or detailed enough) analysis of GIT (semi)stability for the $PGL(V)$ action on $\mathbb{LG}(\wedge^3 V)$. Now we go over the analysis of the period map for cubic hypersurfaces in $\mathbb{P}^5$ according to Voisin [71] and Laza [35, 36] (see also [39]). Let $|\mathcal{O}_{\mathbb{P}^5}(3)|^{spl} \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the open set parametrizing cubics with simple singularities. Then $|\mathcal{O}_{\mathbb{P}^5}(3)|^{spl}$ is $PGL(6)$-invariant and by Laza [35] it is contained in the stable locus of $|\mathcal{O}_{\mathbb{P}^5}(3)|$. Let

$$\mathcal{M}^{spl} := |\mathcal{O}_{\mathbb{P}^5}(3)|^{spl}//PGL(6), \qquad \mathcal{M}_{cbc} := |\mathcal{O}_{\mathbb{P}^5}(3)|//PGL(6). \tag{5.3.5}$$

We have the (rational) period map is $\mathfrak{p}\colon \mathcal{M}_{cbc} \dashrightarrow (\mathbb{D}_6^2)^{BB}$ where $(\mathbb{D}_6^2)^{BB}$ is the Baily-Borel compactification of the period moduli space described in Subsection 5.1. Then $\mathcal{M}^{spl}$ is the analogue of the open $\mathcal{M}^{ADE}$ in the moduli space of double EPW-sextics. In fact $\mathcal{M}^{spl} = \mathfrak{p}^{-1}(\mathbb{D}_6^2) \cap Reg(\mathfrak{p})$ and moreover the restriction of $\mathfrak{p}$ to $\mathcal{M}^{spl}$ has finite fibers (of cardinality 1 by Voisin's Global Torelli for cubics). Next let $\mathcal{D}, \mathcal{P}$ be the prime divisors of $|\mathcal{O}_{\mathbb{P}^5}(3)|$ defined in Subsection 3.2: as shown in that subsection the varieties of lines on cubics parametrized by points of $\mathcal{D}$ are similar to double EPW-sextics parametrized by points of Δ and there is also an analogy between $\mathcal{P}$ and Σ. Voisin [71] proved that analogues of Items (a)-(c) above hold for $\mathcal{P}//PGL(6)$ and from that derived Global Torelli for cubics.

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