A NEW SIX DIMENSIONAL IRREDUCIBLE SYMPLECTIC VARIETY.

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1. INTRODUCTION.

There are three types of "building blocks" in the Bogomolov decomposition [B, Th.2] of compact Kählerian manifolds with torsion c_1, namely complex tori, Calabi-Yau varieties, and irreducible symplectic manifolds. We are interested in the last type, i.e. simply-connected compact Kählerian manifolds carrying a holomorphic symplectic form which spans $H^{2,0}$. (The holonomy of a Ricci-flat Kähler metric is equal to $Sp(r)$, hence these manifolds are hyperkähler [B].) The stock of available irreducible symplectic manifolds appears to be quite scarce, expecially if we think of the many examples of CalabiYau's. Every known irreducible symplectic manifold is a deformation of one of the following varieties: the Hilbert scheme parametrizing zero-dimensional subschemes of a $K3$ of fixed length [B], the generalized Kummer variety parametrizing zero-dimensional subschemes of a complex torus of fixed length and whose associated 0-cycle sums up to 0 [B], the (10-dimensional) desingularization of the moduli space of rank-two semistable torsion-free sheaves on a $K3$ with $c_1 = 0$, $c_2 = 4$ constructed by the author [O1]. Briefly: all known examples are deformations of an irreducible factor in the Bogomolov decomposition of a moduli space of semistable sheaves on a surface with trivial canonical bundle or, as in the last case, of a symplectic desingularization of such a moduli space. This paper provides a new example in dimension 6: the manifold in question is an irreducible factor in the Bogomolov decomposition of a symplectic desingularization of a moduli space of sheaves on an abelian surface. To put our result in perspective we recall some results on moduli spaces of sheaves on a surface with trivial canonical bundle. Let X be such a surface and D an ample divisor on it: given a vector $\mathbf{w} \in H^*(X; \mathbb{Z})$, we let $\mathcal{M}_{\mathbf{w}}(X, D)$ be the moduli space of D-semistable torsion-free sheaves F on X with Mukai vector

$$(1.1) \qquad v(F) := ch(F)\sqrt{Td(X)} = \mathbf{w}.$$

Mukai [Muk2] proved that the open subset $\mathcal{M}_{\mathbf{w}}^s(X, D)$ parametrizing stable sheaves, if non empty, is smooth symplectic (i.e. it has a regular symplectic from) of pure dimension equal to

$$(1.2) \qquad \dim \mathcal{M}_{\mathbf{w}}^s(X, D) = 2 + (\mathbf{w}, \mathbf{w}),$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

where $(\cdot, \cdot)$ is the quadratic form on $H^*(X; \mathbb{Z})$ defined by

$$(\mathbf{w}, \mathbf{w}) := \int_X (\mathbf{w}_1 \wedge \mathbf{w}_1 - 2\mathbf{w}_0 \wedge \mathbf{w}_2).$$

(Here $\mathbf{w}_i$ is the component of $\mathbf{w}$ belonging to $H^{2i}(X; \mathbb{Z})$.) Yoshioka [Y1,Y2], extending previous partial results [HG,O2], showed that under a technical condition on $(D, \mathbf{w})$ ensuring that all semistable sheaves are stable, the moduli space $\mathcal{M}_{\mathbf{w}}(X, D)$ is either a deformation of a Hilbert scheme of points on a $K3$, or else its Bogomolov factors are abelian surfaces and a deformation of a generalized Kummer variety. In view of this result we search for a moduli space containing points parametrizing strictly semistable sheaves (i.e. non stable), and singular at these points, admitting a symplectic desingularization, in the hope that one of the Bogomolov factors of the desingularization is a new irreducible symplectic variety. This is what was done to produce the new 10-dimensional example mentioned above, the moduli space being that of certain sheaves on $K3$. In this paper we will carry out successfully this program with a moduli space of sheaves on an abelian surface, described as follows. Let C be a smooth irreducible projective curve of genus two and $J := Pic^0(C)$. We set $\mathbf{v} := 2 - 2\eta_J$, where $\eta_J \in H^4(J; \mathbb{Z})$ is the orientation class of J. Let $\mathcal{M}_{\mathbf{v}}$ be the moduli space $\mathcal{M}_{\mathbf{v}}(J, \Theta)$, where Θ is a Theta divisor. Many of the results proved in [O1] for the moduli space $\mathcal{M}_4$ of torsion-free semistable rank-two sheaves on a $K3$ with $c_1 = 0$, $c_2 = 4$, remain valid for $\mathcal{M}_{\mathbf{v}}$, provided one makes the following technical assumption:

(1.3)   The Néron-Severi group of J is generated by Θ, i.e. $H^{1,1}_{\mathbb{Z}}(J) = \mathbb{Z}c_1(\Theta)$.

(As is well-known the generic Jacobian satisfies (1.3).) This assumption will play the role that (0.2) has in [O1]. Strictly speaking the analogue of (0.2) is the assumption that there is no divisor orthogonal to Θ and with self-intersection -2, and this is all that is needed for the results in this paper to hold. We make the stronger hypothesis (1.3) in order to simplify various proofs. The first result of [O1] that extends to our moduli space is the description of the singular locus of $\mathcal{M}_{\mathbf{v}}$: $sing(\mathcal{M}_{\mathbf{v}})$ is the set of S-equivalence classes of strictly semistable sheaves, i.e. equivalent to $I_{p_1} \otimes \xi_1 \oplus I_{p_2} \otimes \xi_2$, where $p_i \in J$ and $\xi_i \in \widehat{J}$ ($\widehat{J} := Pic^0(J)$). Most importantly, the procedure of [O1] carries over to give a symplectic desingularization $\widetilde{\pi}_{\mathbf{v}} : \widetilde{\mathcal{M}}_{\mathbf{v}} \to \mathcal{M}_{\mathbf{v}}$; we let $\widetilde{\omega}_{\mathbf{v}}$ be the symplectic form on $\widetilde{\mathcal{M}}_{\mathbf{v}}$. The variety $\widetilde{\mathcal{M}}_{\mathbf{v}}$ is of pure dimension 10 (see (2.1.3)). It is not symplectically irreducible: consider the map

$$\begin{array}{ccc} \mathcal{M}_{\mathbf{v}} & \xrightarrow{a_{\mathbf{v}}} & J \times \widehat{J} \\ [F] & \mapsto & (\sum c_2(F), [\det F]), \end{array}$$

where $\sum c_2(F)$ (the Albanese map) is the sum of the points (with multiplicities) of any representative of $c_2(F) \in CH_0(J)$. Set $\widetilde{a}_{\mathbf{v}} := a_{\mathbf{v}} \circ \widetilde{\pi}_{\mathbf{v}}$. As is easily checked $\widetilde{a}_{\mathbf{v}}$ is surjective, hence $\widetilde{\mathcal{M}}_{\mathbf{v}}$ is not symplectically irreducible. To "factor out $J \times \widehat{J}$" we set

$$\widetilde{\mathcal{M}} := \widetilde{a}_{\mathbf{v}}^{-1}(0, \widehat{0}), \qquad \widetilde{\omega} := \widetilde{\omega}_{\mathbf{v}}|_{\widetilde{\mathcal{M}}}.$$

The main result of this paper is the following.

(1.4)Theorem. *Keep assumptions as above: $\widetilde{\mathcal{M}}$ is a 6-dimensional irreducible symplectic variety, i.e. one-connected and with $H^{2,0}(\widetilde{\mathcal{M}})$ spanned by the symplectic form $\widetilde{\omega}$. Furthermore $b_2(\widetilde{\mathcal{M}}) = 8$.*

Since the known 6-dimensional irreducible symplectic varieties have $b_2 = 7$ or $b_2 = 23$ [B], it follows that $\widetilde{\mathcal{M}}$ cannot be deformed into one of the known symplectic varieties, not even up to birational equivalence.

The paper is organized as follows. In §2 we construct $\widetilde{\mathcal{M}}_{\mathbf{v}}$: this is straightforward, all the work was done in [O1]. An important feature of this desingularization is that there is a moduli-theoretic interpretation of the complement of $\widetilde{\pi}_{\mathbf{v}}^{-1}\{[I_p(\xi) \oplus I_p(\xi)]\}$ as the set of $\widetilde{S}$-equivalence classes of simple semistable sheaves with the given Chern character, where $\widetilde{S}$-equivalence is a relation finer than S-equivalence and (slightly) coarser than isomorphism. Finally we check that $\widetilde{\mathcal{M}}$ is 6-dimensional and symplectic. In §3 we notice that the generalized Lefschetz Hyperplane Section Theorem, together with methods of [L1,O1], gives the following topological result. Let $\mathcal{M} := a_{\mathbf{v}}^{-1}(0, \widehat{0})$, and $\widetilde{\pi}: \widetilde{\mathcal{M}} \to \mathcal{M}$ be the restriction of $\widetilde{\pi}_{\mathbf{v}}$. Let $\alpha \in J$, and Θ_α be the translate by α of a symmetric theta-divisor Θ. The inclusion

$$(1.5) \qquad \widetilde{\pi}^{-1}\{[F] \in \mathcal{M}|\ F|_{\Theta_\alpha} \text{ is not locally-free semistable}\} \hookrightarrow \widetilde{\mathcal{M}}$$

induces isomorphisms on π_0, π_1, and a surjection on H_2. This is useful because while $\widetilde{\mathcal{M}}$ is mysterious, the left-hand side can be described in terms of other well-known moduli spaces. In fact if α is generic, the left-hand side of (1.5) decomposes into three irreducible components $\widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha \cup \widetilde{V}_\alpha$, which have the following description. The component $\widetilde{\Sigma}_\alpha$, which is mapped by $\widetilde{\pi}$ to the locus parametrizing strictly semistable sheaves singular at some point of Θ_α, is a $\mathbb{P}^1$-bundle over the quotient of $C \times \widehat{J}$ by the equivalence relation (4.2.4). The component $\widetilde{B}_\alpha$, which is mapped by $\widetilde{\pi}$ to the closure of the locus parametrizing stable sheaves which are singular at some point of Θ_α, is a $\mathbb{P}^1$-bundle over $\overline{C} \times K^{[2]}\widehat{J}$, where $\overline{C}$ is obtained from C by identifying the points q_1, q_2 given by (4.2.2), and $K^{[2]}\widehat{J}$ is the (smooth) Kummer surface of $\widehat{J}$. Finally $\widetilde{V}_\alpha$, which is mapped by $\widetilde{\pi}$ to the closure of the locus parametrizing sheaves whose restriction to Θ_α is a non-semistable locally-free sheaf, is obtained from a $\mathbb{P}^1$-bundle over $\widehat{J}$ by flopping sixteen $(-1, -1)$-curves. These results are contained in Sections (4) through (6). These sections also contain descriptions of the double and triple intersections of the sets $\widetilde{\Sigma}_\alpha$, $\widetilde{B}_\alpha$, $\widetilde{V}_\alpha$, and most of the topological results which are needed in the proof of Theorem (1.4). In §7 we give the proof of Theorem (1.4). We show that the left-hand side of Inclusion (1.5) is simply connected and that its second Betti number is at most equal to 8: this implies that $\widetilde{\mathcal{M}}$ is simply connected and that $b_2(\widetilde{\mathcal{M}}) \le 8$. Then we produce a Hodge substructure of $H^2(\widetilde{\mathcal{M}})$ of dimension 8, with $h^{2,0} = 1$; this finishes the proof of (1.4). This shows that Inclusion (1.5) induces an isomorphism on H_2, while a priori the Lefschetz Hyperplane Theorem only gives that it is a surjection; we comment on this in (3.8).

The reader may understand the logical structure of the proof by going through the statements whose numbering is decorated by a superscript $*$. These are the results which are invoked in Section (7) to prove Theorem (1.4); most of them are contained in the last subsection of Sections (2) through (6) (we list the exceptions in the introduction to each section).

Notation. We choose once and for all a Weierstrass point $p_0 \in C$. Let u be the Abel-Jacobi embedding

$$
(1.6) \qquad
\begin{array}{ccc}
C & \overset{u}{\hookrightarrow} & J \\
p & \mapsto & [p - p_0].
\end{array}
$$

Then $\Theta := u(C)$ is a symmetric theta-divisor. For $\alpha \in J$ let $\Theta_\alpha := \Theta + \alpha$. We have the isomorphism

$$
(1.7) \qquad
\begin{array}{ccc}
J & \overset{\delta}{\longrightarrow} & \widehat{J} \\
x & \mapsto & [\Theta_x - \Theta].
\end{array}
$$

We set $\widehat{x} = \delta(x)$; in general points of $\widehat{J}$ are denoted by $\widehat{x}, \widehat{y}, \widehat{p}$, etc. Let $\widehat{\Theta} := \delta(\Theta)$; for $\widehat{\beta} \in \widehat{J}$ let $\widehat{\Theta}_{\widehat{\beta}} := \widehat{\Theta} + \widehat{\beta}$. We will identify both Θ_α and $\widehat{\Theta}_{\widehat{\beta}}$ with C via the maps

$$
(1.8) \qquad
C \ni p \mapsto
\begin{cases}
i_\alpha(p) := \alpha + u(p) \in \Theta_\alpha \\
i_{\widehat{\beta}}(p) := \widehat{\beta} + \delta \circ u(p) \in \widehat{\Theta}_{\widehat{\beta}}.
\end{cases}
$$

Set $j_\alpha := i_\alpha^{-1}$, $j_{\widehat{\beta}} := i_{\widehat{\beta}}^{-1}$.

We let

$$
(1.9) \qquad \phi \colon J \times \widehat{J} \to J, \qquad \widehat{\phi} \colon J \times \widehat{J} \to \widehat{J}
$$

be the projections. Let $\mathcal{L}$ be the normalized Poincaré line-bundle on $J \times \widehat{J}$, i.e. the tautological line-bundle such that $\mathcal{L}|_{0 \times \widehat{J}} \cong \mathcal{O}_{\widehat{J}}$. If $Z \subset J$, $\widehat{Z} \in \widehat{J}$ are zero-dimensional subschemes we set

$$
\mathcal{L}_Z := \widehat{\phi}_*(\mathcal{L} \otimes \phi^* \mathcal{O}_Z) \quad \mathcal{L}_{\widehat{Z}} := \phi_*(\mathcal{L} \otimes \widehat{\phi}^* \mathcal{O}_{\widehat{Z}}).
$$

In particular if $Z = x$ is a single point, $\mathcal{L}_x = \mathcal{L}|_{x \times \widehat{J}}$, and similarly for $\widehat{Z} = \widehat{x}$. We will often use the following formulae:

$$
(1.10) \qquad \mathcal{L}_{\widehat{x}}|_{\Theta_\alpha} \cong j_\alpha^* \mathcal{O}_C(x) \quad \mathcal{L}_x|_{\widehat{\Theta}_{\widehat{\beta}}} \cong j_{\widehat{\beta}}^* \mathcal{O}_C(x).
$$

Here we denote by x both a divisor on C and its linear equivalence class (an element of J); this will be a habit throughout the paper. We reserve the notation $[x]$ for the line-bundle on C corresponding to the invertible sheaf $\mathcal{O}_C(x)$.

For $\alpha \in J$ let

$$
(1.11) \qquad \widehat{J}[2]_{-\widehat{\alpha}} \text{ be the translation of } \widehat{J}[2] \text{ by } (-\widehat{\alpha}),
$$

where $\widehat{J}[2]$ is the kernel of multiplication by two, and

$$
(1.12) \qquad \nu_\alpha \colon \widehat{I}_\alpha \to \widehat{J} \text{ be the blow up of } \widehat{J}[2]_{-\widehat{\alpha}}.
$$

We let E be the exceptional divisor of ν_α, and $E_1, \dots, E_{16}$ be its irreducible components.

Let X be a projective variety and D an ample divisor on X. A torsion-free sheaf F on X is D-*semistable* if it is Gieseker-Maruyama semistable with respect to D, i.e. if for all proper subsheaves $E \subset F$

$$(1.13) \qquad \mathrm{rk}F \cdot \chi(E(nD)) \leq \mathrm{rk}E \cdot \chi(F(nD)), \quad \text{for all } n \gg 0.$$

If there exists $E \subset F$ such that the inequality is an equality then F is *strictly semistable*, otherwise it is stable. There is also the notion of *slope-(semi)stability*: if for all $E \subset F$ with $0 < \mathrm{rk}E < \mathrm{rk}F$

$$\mu(E) := \frac{1}{\mathrm{rk}E}c_1(E) \cdot D^{k-1} \leq \frac{1}{\mathrm{rk}F}c_1(F) \cdot D^{k-1} =: \mu(F), \quad k := \dim X,$$

F is D-slope semistable. It is D-slope stable if the inequality is always strict. Writing out the polynomials appearing in (1.13) one shows that D-semistability implies D-slope semistability, and D-slope stability implies D-stability. Throughout the paper we fix the ample divisor Θ on J: (semi)stability of a sheaf will be Θ-(semi)stability, and similarly for slope-(semi)stability.

We recall that moduli spaces of semistable torsion-free sheaves parametrize S-*equivalence* classes of such sheaves [G]. To define S-equivalence one associates to a semistable sheaf F a direct sum of stable sheaves $Gr(F)$, and then declares that F_1 is S-equivalent to F_2 if $Gr(F_1) \cong Gr(F_2)$. If $\mathrm{rk}F = 2$ (the only case to be considered in this paper)

$$Gr(F) = \begin{cases} F & \text{if } F \text{ is stable} \\ L \oplus (F/L) & \text{if } F \text{ is strictly semistable, } L \subset F \text{ destabilizes.} \end{cases}$$

If F is a semistable sheaf we let $[F]$ be its S-equivalence class.

A *family of sheaves on X parametrized by B* is a sheaf $\mathcal{F}$ on $X \times B$, flat over $\mathcal{O}_B$. For $b \in B$ we set $\mathcal{F}_b := \mathcal{F}|_{X \times \{b\}}$.

2. CONSTRUCTION OF $\widetilde{\mathcal{M}}$

This subsection must be read with the aid of [O1].

2.1. Symplectic desingularization of $\mathcal{M}_{\mathbf{v}}$. We will show that the construction of a symplectic desingularization of $\mathcal{M}_4$, the moduli space of rank-two torsion-free sheaves on a $K3$ surface X with $c_1 = 0$ and $c_2 = 4$ (the ample divisor defining (semi)stability must satisfy (0.2) of [O1], an assumption analogous to (1.3)), carries over to give a symplectic desingularization of $\mathcal{M}_{\mathbf{v}}$. As in [O1] we start by classifying strictly semistable sheaves. Let $[F] \in \mathcal{M}_{\mathbf{v}}$, and assume

$$(2.1.1) \qquad\qquad 0 \to L_1 \to F \to L_2 \to 0$$

is a destabilizing sequence. The following is an immediate consequence of Assumption (1.3).

(2.1.2)Lemma. *Keep notation as above. Then $L_i \cong I_{x_i} \otimes \xi_i$, where I_{x_i} is the ideal sheaf of a point $x_i \in J$, and $\xi_i \in \widehat{J}$. Conversely, if F fits into Exact Sequence (2.1.1) with L_i of this form, then $[F] \in \mathcal{M}_{\mathbf{v}}$ and F is strictly semistable.*

If in the above lemma we replace $I_{x_i}(\xi_i)$ by I_{Z_i}, where Z_i is a length-two subscheme of the $K3$ surface X, we get Lemma (1.1.5) of [O1], in the case $n = 2$.

Moreover, all the results in Subsections (1.4)-(1.8) of [O1] remain valid if one makes the substitution

$$(I_Z \text{ on } X \text{ with } \ell(Z) = 2) \longmapsto (I_x(\xi) \text{ on } J \text{ where } x \in J \text{ and } \xi \in \widehat{J}).$$

Thus we get, as in (1.8) of [O1], a desingularization $\widehat{\pi}_{\mathbf{v}} \colon \widehat{\mathcal{M}}_{\mathbf{v}} \to \mathcal{M}_{\mathbf{v}}$, by blowing up the locus parametrizing the "worst" strictly semistable sheaves

$$\Omega_{\mathbf{v}} := \{[I_x(\xi) \oplus I_x(\xi)] | \ x \in J, \ \xi \in \widehat{J}\},$$

and then blowing up the strict transform of

$$\Sigma_{\mathbf{v}} := \{[I_{x_1}(\xi_1) \oplus I_{x_2}(\xi_2)] | \ x_i \in J, \ \xi_i \in \widehat{J}\}.$$

Associated to a non-zero two-form ω on J there is a regular Mukai two-form $\widehat{\omega}_{\mathbf{v}}$ on $\widehat{\mathcal{M}}_{\mathbf{v}}$, which degenerates on $\widehat{\Omega}_{\mathbf{v}} := \widehat{\pi}^{-1}(\Omega_{\mathbf{v}})$ (copy the proof of [O1, (2.2.3)]). Thus $\widehat{\mathcal{M}}_{\mathbf{v}}$ is not symplectic. One verifies that the proofs of Propositions (2.0.1)-(2.0.3) of [O1] remain valid if we replace everywhere $\widehat{\mathcal{M}}_4$, $\widehat{\Omega}_4$, $X^{[2]}$, I_Z, by $\widehat{\mathcal{M}}_{\mathbf{v}}$, $\widehat{\Omega}_{\mathbf{v}}$, $(J \times \widehat{J})$, $I_x(\xi)$ respectively. Thus Proposition (2.0.3) tells us that we can blow down $\widehat{\Omega}_{\mathbf{v}}$ and get a projective desingularization $\widetilde{\mathcal{M}}_{\mathbf{v}}$ of $\mathcal{M}_{\mathbf{v}}$; let $\widetilde{\pi}_{\mathbf{v}} \colon \widetilde{\mathcal{M}}_{\mathbf{v}} \to \mathcal{M}_{\mathbf{v}}$ be the desingularization map. The two-form $\widetilde{\omega}_{\mathbf{v}}$ on $\widetilde{\mathcal{M}}_{\mathbf{v}}$ induced by $\widehat{\omega}_{\mathbf{v}}$ is symplectic. We set $\widetilde{\Sigma}_{\mathbf{v}} := \widetilde{\pi}_{\mathbf{v}}^{-1}(\Sigma_{\mathbf{v}})$, $\widetilde{\Omega}_{\mathbf{v}} := \widetilde{\pi}_{\mathbf{v}}^{-1}(\Omega_{\mathbf{v}})$. Since $\widetilde{\Sigma}_{\mathbf{v}}$ is a Cartier divisor in $\widetilde{\mathcal{M}}_{\mathbf{v}}$, and $\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Sigma}_{\mathbf{v}} = \widetilde{\pi}_{\mathbf{v}}^{-1}(\mathcal{M}_{\mathbf{v}}^s)$, Equation (1.2) gives that

$$(2.1.3) \qquad \widetilde{\mathcal{M}}_{\mathbf{v}} \text{ is of pure dimension 10.}$$

For later use we set

$$(2.1.4) \qquad B_{\mathbf{v}} := \text{closure of } \{[F] \in \mathcal{M}_{\mathbf{v}} | \ F \text{ is singular and stable}\},$$

and we let $\widetilde{B}_{\mathbf{v}} \subset \widetilde{\mathcal{M}}_{\mathbf{v}}$ be the strict transform of $B_{\mathbf{v}}$.

2.2. Moduli-theoretic interpretation of $\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}}$. We will show that points of $\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}}$ are in one-to-one correspondence with simple semistable sheaves on J, modulo $\widetilde{\text{S}}$-equivalence, a relation finer than S-equivalence. Let F be a torsion-free simple semistable sheaf F on J with $v(F) = \mathbf{v}$, where $v(F)$ is as in (1.1). If F is stable, then E is $\widetilde{\text{S}}$-equivalent to F if and only if it is isomorphic to it. If F is strictly semistable then by Lemma (2.1.2) there is an exact sequence

$$0 \to L_1 \to F \to L_2 \to 0,$$

where $L_i \cong I_{x_i}(\xi_i)$. We associate to F the extension class of the above exact sequence

$$e_F \in \operatorname{Ext}^1(L_2, L_1).$$

This is non-zero because F is simple, and is well-defined modulo $\mathbf{C}^*$, because the destabilizing surjection $F \to L_2$ is determined up to $\mathbf{C}^*$. Yoneda multiplication

$$(2.2.1) \qquad \Upsilon \colon \operatorname{Ext}^1(L_1, L_2) \times \operatorname{Ext}^1(L_2, L_1) \to \operatorname{Ext}^2(L_1, L_1)$$

can be identified with Serre duality, because the trace $\mathrm{Tr} \colon \mathrm{Ext}^2(L_1, L_1) \to H^2(\mathcal{O}_J)$ is an isomorphism. Thus Υ is a perfect pairing. Since F is simple, $L_1 \not\cong L_2$ and hence Riemann-Roch gives $\dim \mathrm{Ext}^1(L_1, L_2) = 2$. Thus the annihilator (with respect to Υ) of e_F is one-dimensional; let $e_F^\perp \in \mathrm{Ext}^1(L_1, L_2)$ be a generator. A sheaf E is $\widetilde{\mathrm{S}}$-equivalent to F if it is isomorphic to F or to the extension

$$(2.2.2) \qquad 0 \to L_2 \to E \to L_1 \to 0$$

determined by $e_F^\perp$. Let $E_{\mathbf{v}}$ be the set of $\widetilde{S}$-equivalence classes of torsion-free simple semistable sheaves F on J with $v(F) = \mathbf{v}$. We will show that there is a natural bijection

$$(2.2.3) \qquad \widetilde{\psi}_{\mathbf{v}} \colon (\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}}) \xrightarrow{\ \sim\ } E_{\mathbf{v}}.$$

The desingularization map $\widetilde{\pi}_{\mathbf{v}} \colon \widetilde{\mathcal{M}}_{\mathbf{v}} \to \mathcal{M}_{\mathbf{v}}$ identifies $(\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Sigma}_{\mathbf{v}})$ with $\mathcal{M}_{\mathbf{v}}^{st}$. The latter is the set of isomorphism classes of torsion-free stable sheaves F with $v(F) = \mathbf{v}$, which injects into $E_{\mathbf{v}}$. This injection defines $\widetilde{\psi}_{\mathbf{v}}$ outside $\widetilde{\Sigma}_{\mathbf{v}}$. Now we define $\widetilde{\psi}_{\mathbf{v}}$ on $(\widetilde{\Sigma}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}})$. Since the map $\widehat{\mathcal{M}}_{\mathbf{v}} \to \widetilde{\mathcal{M}}_{\mathbf{v}}$ is the contraction of $\widehat{\Omega}_{\mathbf{v}}$, it defines an isomorphism

$$(2.2.4) \qquad (\widehat{\Sigma}_{\mathbf{v}} \setminus \widehat{\Omega}_{\mathbf{v}}) \xrightarrow{\ \sim\ } (\widetilde{\Sigma}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}})$$

commuting with $\widehat{\pi}_{\mathbf{v}}$, $\widetilde{\pi}_{\mathbf{v}}$. We will give an injection of $(\widehat{\Sigma}_{\mathbf{v}} \setminus \widehat{\Omega}_{\mathbf{v}})$ into $E_{\mathbf{v}}$. We use the notation of [O1,§1], adapted to our moduli space. Thus Q is the Quot-scheme of which $\mathcal{M}_{\mathbf{v}}$ is the G.I.T. $\mathrm{PGL}(N)$-quotient. For $x \in Q$ we let

$$\mathcal{O}_J(-k)^N \to F_x$$

be the quotient parametrized by x (see [O1,(1.1)]). Let

$$(2.2.5) \qquad \Sigma_Q^0 := \{x \in Q^{ss} \mid F_x \cong L_1 \oplus L_2, \ L_1 \not\cong L_2\},$$

and Σ_Q be its closure in Q. For $x \in \Sigma_Q^0$ with $F_x \cong L_1 \oplus L_2$ we have by [O1,(1.4.1)] a canonical isomorphism

$$(2.2.6) \qquad (C_{\Sigma_Q} Q)_x \cong \{(\epsilon, \eta) \in \mathrm{Ext}^1(L_1, L_2) \oplus \mathrm{Ext}^1(L_2, L_1) \mid \epsilon \cup \eta = 0\},$$

where $C_{\Sigma_Q} Q$ is the normal cone to Σ_Q in Q. Since $\widehat{\mathcal{M}}_{\mathbf{v}}$ is the $\mathrm{PGL}(N)$-quotient of the variety S obtained from Q by first blowing up

$$\Omega_Q := \text{closure of } \{x \in Q^{ss} \mid F_x \cong L \oplus L\},$$

and then the strict transform of Σ_Q, we have

$$(2.2.7) \qquad \widehat{\pi}^{-1}([L_1 \oplus L_2]) \cong \mathbb{P}N(L_1, L_2)//\mathbf{C}^*, \qquad L_1 \not\cong L_2$$

where $N(L_1, L_2)$ is the right-hand side of (2.2.6), and the action of $\lambda \in \mathbf{C}^*$ is given by $\lambda(\epsilon, \eta) = (\lambda\epsilon, \lambda^{-1}\eta)$ (see [O1,(1.4.1)]). Let

$$C(L_1, L_2) := \{\varphi \in \mathrm{Ext}^1(L_2, L_1)^* \otimes \mathrm{Ext}^1(L_2, L_1) \mid 0 = \mathrm{Tr}\,\varphi = \mathrm{Det}\,\varphi\}.$$

As is easily verified

(2.2.8) $\mathbb{P}N(L_1, L_2)//\mathbf{C}^* \cong \mathbb{P}C(L_1, L_2),$

with quotient map given by

(2.2.9)
$$\begin{array}{ccc} \mathbb{P}N(L_1, L_2)^s & \longrightarrow & \mathbb{P}C(L_1, L_2), \\ [(\epsilon, \eta)] & \mapsto & [\epsilon \otimes \eta]. \end{array}$$

(Here $\epsilon \in \mathrm{Ext}^1(L_1, L_2) \cong \mathrm{Ext}^1(L_2, L_1)^*$, the isomorphism being given by (2.2.1).)
We have an injection

$$\begin{array}{ccc} \mathbb{P}C(L_1, L_2) & \hookrightarrow & E_{\mathbf{v}} \\ [\varphi] & \mapsto & ([(\mathrm{Ker}\varphi)^{\perp}], [\mathrm{Im}\varphi]), \end{array}$$

and this defines $\widetilde{\psi}_{\mathbf{v}}$ on $\widetilde{\pi}_{\mathbf{v}}^{-1}([L_1 \oplus L_2]) \cong \widehat{\pi}_{\mathbf{v}}^{-1}([L_1 \oplus L_2])$. We have defined $\widetilde{\psi}_{\mathbf{v}}$ on
all of $(\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}})$; one checks immediately that it is a bijection. Our next task is to
show that $\widetilde{\psi}_{\mathbf{v}}$ is regular on parameter spaces for simple sheaves. Let $\mathcal{E}$ be a family
of torsion-free simple semistable sheaves on J parametrized by a scheme T, with
$v(\mathcal{E}_t) = \mathbf{v}$ for all $t \in T$, and let

$$\begin{array}{ccc} T & \xrightarrow{\psi_{\mathcal{E}}} & \mathcal{M}_{\mathbf{v}} \\ t & \mapsto & [\mathcal{E}_t] \end{array}$$

be the modular map. Since $\mathcal{E}$ is a family of simple semistable sheaves, the image of
$\psi_{\mathcal{E}}$ is contained in $(\mathcal{M}_{\mathbf{v}} \setminus \Omega_{\mathbf{v}})$.

(2.2.10)Proposition. *Keep notation as above. There exists a lift*

$$\widetilde{\psi}_{\mathcal{E}} \colon T \to (\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}})$$

of $\psi_{\mathcal{E}}$ with the following properties.

   (1) *The induced map of sets is (given Identification (2.2.3)):*

$$\widetilde{\psi}_{\mathcal{E}}(t) = \widetilde{S}\text{-equivalence class of } \mathcal{E}_t.$$

   (2) *Let $0 \in T$. If the map from the germ of T at 0 to the deformation space of
      $\mathcal{E}_0$ is an isomorphism, then $\widetilde{\psi}_{\mathcal{E}}$ is an isomorphism in a neighborhood of 0
      (for the classical topology).*

Proof. Clearly it suffices to prove the proposition for $\mathcal{E}$ satisfying the hypothesis of
Item (2); in this case the germ (T, t) is the deformation space of $\mathcal{E}_t$ (after shrinking
T if necessary), hence it suffices to prove that $\psi_{\mathcal{E}}$ lifts in a neighborhood of 0. Let
$E := \mathcal{E}_0$. Assume E is stable: it is well-known that $\psi_{\mathcal{E}}$ is an isomorphism near 0,
and furthermore since $\widetilde{\pi} \colon \widetilde{\mathcal{M}}_{\mathbf{v}} \to \mathcal{M}_{\mathbf{v}}$ is an isomorphism over $\mathcal{M}_{\mathbf{v}}^{st}$, the map $\widetilde{\psi}_{\mathcal{E}}$ lifts
trivially and it has the required properties. Now assume E is strictly semistable,
and let

(2.2.11) $0 \to L_1 \to E \to L_2 \to 0$

be the destabilizing sequence of E. By Serre duality and simpleness of E,

$$\mathrm{Ext}^2(E,E) \cong \mathrm{Ext}^0(E,E)^* \cong \mathbf{C},$$

hence the deformation space of E is smooth by Mukai [Muk2,(0.1)]; thus T is also smooth (after shrinking T around 0 if necessary). Let

$$\Sigma_S^0 := (\pi_R \circ \pi_S)^{-1}(\Sigma_Q^0),$$

where $\pi_R\colon R \to Q$ is the blow up of Ω_Q, $\pi_S\colon S \to R$ is the blow up of the strict transform of Σ_Q (see [O1,(1.1)]), and Σ_Q^0 is as in (2.2.5). Let $x_0 \in Q^{ss}$ be such that $F_{x_0} \cong L_1 \oplus L_2$, where L_1, L_2 are the sheaves fitting into (2.2.11). Let

(2.2.12) $$y_0 := [e_E^{\perp}, e_E] \in \mathbb{P}(N(L_1, L_2)) = (\pi_R \circ \pi_S)^{-1}(x_0),$$

where e_E is the extension class of (2.2.11) (the second equality follows from (2.2.6)). The point y_0 is stable for the $\mathbf{C}^*$-action on $\mathbb{P}(N(L_1, L_2))$, i.e. it is stable with respect to the $\mathrm{PGL}(N)$-action on S. Let $U \subset S^s$ be "Luna's étale slice" normal to the orbit $\mathrm{PGL}(N)y_0$. Then (see [O1,(1.8.8)])

$$\mathrm{Stab}(y_0) \cong \mathbb{Z}/(2)$$

acts on U. Let $\widehat{\mathcal{F}}$ be the sheaf on $J \times U$ obtained pulling back the tautological family of quotients parametrized by Q via the map $U \to Q$. Shrinking U around y_0 (in the classical topology), we may assume that it satisfies the following properties. Let Σ_S be the closure of Σ_S^0, and $\Sigma_U := \Sigma_S \cap U$; there exist families of rank-one torsion-free sheaves $\mathcal{L}_1$, $\mathcal{L}_2$ on J parametrized by Σ_U and an exact sequence

(2.2.13) $$0 \to \mathcal{L}_2 \to \widehat{\mathcal{F}}|_{J \times \Sigma_U} \to \mathcal{L}_1 \to 0,$$

such that:

   (I) the restriction of (2.2.13) to $J \times \{y\}$

$$0 \to L_2(y) \xrightarrow{f_2(y)} \widehat{\mathcal{F}} \xrightarrow{f_1(y)} L_1(y) \to 0$$

   is a destabilizing sequence of $\widehat{\mathcal{F}}_y$ for all $y \in \Sigma_U$,
   (II) for $y = y_0$ we have $L_2(y_0) = L_2$, $L_1(y_0) = L_1$.

Since Σ_U is a Cartier divisor we can construct the elementary modification of $\widehat{\mathcal{F}}$ associated to (2.2.13), i.e. the sheaf $\mathcal{G}$ on $J \times U$ fitting into the exact sequence

(2.2.14) $$0 \to \mathcal{G} \to \widehat{\mathcal{F}} \to i_*(\mathcal{L}_1) \to 0,$$

where $i\colon J \times \Sigma_U \hookrightarrow J \times U$ is the inclusion. Then $\mathcal{G}$ is flat over U. If $y \in (U \setminus \Sigma_U)$ we have $\mathcal{G}_y \cong \widehat{\mathcal{F}}_y$. Now assume $y \in \Sigma_U$, and let $x := \pi_R \circ \pi_S(y)$. By (2.2.6) the point y is an element of $\mathbb{P}(N(L_1, L_2))$, which we write explicitely as

$$y = [\epsilon(y), \eta(y)] \in \mathbb{P}(\mathrm{Ext}^1(L_1(y), L_2(y)) \oplus \mathrm{Ext}^1(L_2(y), L_1(y))).$$

Tensorizing (2.2.14) by $\mathcal{O}_U/m_y$ (m_y is the maximal ideal of $y \in U$) one gets an exact sequence

$$(2.2.15) \qquad\qquad 0 \to L_1(y) \to \mathcal{G}_y \to L_2(y) \to 0,$$

We claim that

$$(2.2.16) \qquad\qquad \eta(y) \text{ is the extension class of } (2.2.15).$$

To prove this, let

$$\kappa: T_y S \to \mathrm{Ext}^1(\widehat{\mathcal{F}}_y, \widehat{\mathcal{F}}_y)$$

be the Kodaira-Spencer map, and $v(y) \in T_y S$ be a vector normal to Σ_U; according to [O4,(1.11)] the extension class of (2.2.15) is equal to

$$f_1(y) \circ \kappa(v(y)) \circ f_2(y).$$

Clearly this equals $\eta(y)$. Since $y \in U \subset S^s$, we have $\eta(y) \neq 0$, and hence $\mathcal{G}$ is a family of simple sheaves. Furthermore by (2.2.16) and (2.2.12) $\mathcal{G}_{y_0} \cong E$. Hence, shrinking U if necessary, the family $\mathcal{G}$ is the pull back of $\mathcal{E}$ by a map $g: U \to T$ with $g(y_0) = 0$. Since E is simple $g(y_1) = g(y_2)$ if and only if $\mathcal{G}_{y_1} \cong \mathcal{G}_{y_2}$. As is easily verified this is the case if and only if y_1 and y_2 are equivalent for the $\mathbb{Z}/(2)$-action on U. Thus we get an injective map

$$\bar{g}: U/\mathbb{Z}(2) \to T, \qquad \bar{g}(y_0) = 0.$$

Both U and T have the same dimension as $\widehat{\mathcal{M}}_{\mathbf{v}}$, so $\dim U = \dim T$. Since T is smooth we get that $\bar{g}$ is an isomorphism near y_0. On the other hand, restricting the quotient map $S^s \to \widehat{\mathcal{M}}_{\mathbf{v}}$ to U we get

$$\rho: U/\mathbb{Z}(2) \hookrightarrow (\widehat{\mathcal{M}}_{\mathbf{v}} \setminus \widehat{\Omega}_{\mathbf{v}}) \cong (\widetilde{\mathcal{M}}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}}),$$

which is an isomorphism onto an open neighborhood of $\rho(y_0)$ (by Luna's étale slice Theorem). Furthermore $\rho(y_0) \in (\widetilde{\Sigma}_{\mathbf{v}} \setminus \widetilde{\Omega}_{\mathbf{v}})$ corresponds to $\mathcal{G}_{y_0} \cong E$ via (2.2.3). Inverting $\bar{g}$ near 0 and composing with ρ we get a lift of $\psi_{\mathcal{E}}$ with the required properties.   △

2.3. The symplectic variety $\widetilde{\mathcal{M}}$. Recall that

$$\mathcal{M} := a_{\mathbf{v}}^{-1}(0, \widehat{0}), \quad \widetilde{a}_{\mathbf{v}} := a_{\mathbf{v}} \circ \widetilde{\pi}_{\mathbf{v}}, \quad \widetilde{\mathcal{M}} := \widetilde{a}_{\mathbf{v}}^{-1}(0, \widehat{0}), \quad \widetilde{\omega} := \widetilde{\omega}_{\mathbf{v}}|_{\widetilde{\mathcal{M}}}.$$

Let us prove that $\widetilde{\mathcal{M}}$ is smooth of pure dimension 6. A straightforward computation shows that the map

$$\begin{array}{ccc} \mathcal{M} \times J \times \widehat{J} & \xrightarrow{q} & \mathcal{M}_{\mathbf{v}} \\ ([F], x, \xi) & \mapsto & [\xi \otimes t_x^* F], \end{array}$$

where $t_x: J \to J$ is translation by x, is the quotient map for the action of $J[2] \times \widehat{J}[2]$ given by

$$([F], x, \xi)^{(x_0, \xi_0)} := ([\xi_0 \otimes t_{x_0}^* F], x - x_0, \xi \otimes \xi_0^{-1}).$$

We claim that there is a $J[2] \times \widehat{J}[2]$-quotient map

$$(2.3.1) \qquad\qquad \widetilde{\mathcal{M}} \times J \times \widehat{J} \xrightarrow{\widetilde{q}} \widetilde{\mathcal{M}}_{\mathbf{v}}$$

covering q. To prove it let $\widehat{\mathcal{M}} := \widehat{\pi}_{\mathbf{v}}^{-1}(0, \widehat{0})$. There exists

$$\widehat{\mathcal{M}} \times J \times \widehat{J} \xrightarrow{\widehat{q}} \widehat{\mathcal{M}}_{\mathbf{v}}$$

covering q, because $\widehat{\pi}$ is obtained by first blowing up $\Omega_{\mathbf{v}}$ and then the strict transform of $\Sigma_{\mathbf{v}}$, and

$$q^* \Omega_{\mathbf{v}} = \Omega \times J \times \widehat{J} \qquad q^* \Sigma_{\mathbf{v}} = \Sigma \times J \times \widehat{J},$$

where $\Omega := \Omega_{\mathbf{v}} \cap \mathcal{M}$, $\Sigma := \Sigma_{\mathbf{v}} \cap \mathcal{M}$. Furthermore, since the loci we blow up are $J[2] \times \widehat{J}[2]$-stable, the action of $J[2] \times \widehat{J}[2]$ on $\mathcal{M} \times J \times \widehat{J}$ lifts to an action on $\widehat{\mathcal{M}} \times J \times \widehat{J}$, and $\widehat{q}$ is the quotient for this action. Since $\widehat{\mathcal{M}}_{\mathbf{v}} \to \widetilde{\mathcal{M}}_{\mathbf{v}}$ is the contraction of $\widehat{\Omega}_{\mathbf{v}}$, and $\widehat{q}^* \widehat{\Omega}_{\mathbf{v}} = \widehat{\Omega} \times J \times \widehat{J}$ (here $\widehat{\Omega} := \widehat{\Omega}_{\mathbf{v}} \cap \widehat{\mathcal{M}}$), the map $\widehat{q}$ descends to a map $\widetilde{q}$ as in (2.3.1); one verifies that $\widetilde{q}$ is the quotient map for a lift of the $J[2] \times \widehat{J}[2]$-action. Now since $\widetilde{q}$ is étale and by (2.1.3) the moduli space $\widetilde{\mathcal{M}}_{\mathbf{v}}$ is smooth of pure dimension 10,

$$(2.3.2)^* \qquad\qquad \widetilde{\mathcal{M}} \text{ is smooth of pure dimension 6.}$$

(2.3.3)Proposition. *Keep notation as above. Then $\widetilde{\mathcal{M}}$ and $J \times \widehat{J}$ are orthogonal for $\widetilde{q}^* \widetilde{\omega}_{\mathbf{v}}$.*

Proof. It suffices to prove orthogonality on the dense subset $\widetilde{q}^{-1} \mathcal{M}_{\mathbf{v}}^{st}$. The composition

$$\widetilde{q}^{-1}(\mathcal{M}_{\mathbf{v}}^s) \xrightarrow{\widetilde{q}} \widetilde{\mathcal{M}}_{\mathbf{v}} \xrightarrow{\widetilde{\pi}} \mathcal{M}_{\mathbf{v}}^s$$

is étale hence the differential at $z \in \widetilde{q}^{-1} \mathcal{M}_{\mathbf{v}}^{st}$ gives an isomorphism

$$(2.3.4) \qquad d(\widetilde{\pi} \circ \widetilde{q}(z)) \colon T_z(\widetilde{\mathcal{M}}_{\mathbf{v}}^0 \times J \times \widehat{J}) \xrightarrow{\sim} \mathrm{Ext}^1(E, E),$$

where $[E] = \widetilde{\pi} \circ \widetilde{q}(z)$. On the other hand we have isomorphisms

$$\widetilde{q}^{-1}(\mathcal{M}_{\mathbf{v}}^s) = \widetilde{\pi}^{-1} \mathcal{M}^{st} \times J \times \widehat{J} \xrightarrow{\widetilde{\pi} \times \mathrm{id}} \mathcal{M}^{st} \times J \times \widehat{J}.$$

Let $(\widetilde{\pi} \times \mathrm{id})(z) = ([F], x, \xi)$ (thus $E \cong \xi \otimes t_x^* F$); by the above isomorphism we have a decomposition

$$T_z(\widetilde{\mathcal{M}} \times J \times \widehat{J}) \cong T_{[F]} \mathcal{M} \oplus T_x J \oplus T_\xi \widehat{J}.$$

Restricting (2.3.4) to the direct summands we get

$$T_{[F]} \mathcal{M} \oplus T_x J \xrightarrow{\sim} \mathrm{Ext}^1(E, E)^0,$$

$$T_\xi \widehat{J} \xrightarrow{\delta} \mathrm{Ext}^1(E, E),$$

where δ is the "diagonal embedding" $H^1(\mathcal{O}_J) \hookrightarrow \mathrm{Ext}^1(E,E)$. By definition of Mukai's form (see (1.9) of [O1]),

$$\widetilde{q}^*\widetilde{\omega}(z)(\alpha,\beta) = \int_J \omega \wedge \mathrm{Tr}(d\widetilde{q}(z)(\alpha) \wedge d\widetilde{q}(z)(\beta)),$$

where the second wedge denotes Yoneda product. If $d\widetilde{q}(z)(\alpha) \in \mathrm{Ext}^1(E,E)^0$ and $d\widetilde{q}(z)(\beta) \in \delta(H^1(\mathcal{O}_J))$, then

$$\mathrm{Tr}(d\widetilde{q}(z)(\alpha) \wedge d\widetilde{q}(z)(\beta)) = 0.$$

Thus $T_{[F]}\mathcal{M}$ is orthogonal to $T_{[\xi]}\widehat{J}$. Now let's prove orthogonality to $T_x J$. Let $\psi\colon U \to \mathcal{M}_{\mathbf{v}}^{st}$ be a neighborhood of $[E]$ in the étale topology such that there is a tautological sheaf $\mathcal{E}$ on $J \times U$, and let $0 \in U$ be a point mapping to $[E]$. We can represent $\psi^*\widetilde{\omega}$ as the Mumford two-form induced by a representative of $c_2(\mathcal{E}) \in CH^2(J \times U)$. More precisely, let $\mathcal{Z}$ be a cycle representing $c_2(\mathcal{E})$ and intersecting $J \times \{0\}$ transversely. Thus

$$\mathcal{Z}|_{J \times \{0\}} = \sum_i \epsilon_i p_i,$$

where the p_i are pairwise distinct, and $\epsilon_i = \pm 1$. Furthermore, by transversality $\mathcal{Z}$ defines for each i a map $f_i\colon T_0 U \to T_{p_i} J$. By [O3,(2.9)] we have, for $\alpha, \beta \in T_0 U$,

$$(2.3.5) \qquad \psi^*\widetilde{\omega}(\alpha,\beta) = -\frac{1}{4\pi^2} \sum_i \epsilon_i \omega(f_i(\alpha), f_i(\beta)).$$

Let

$$\alpha \in d\psi(0)^{-1}(d\widetilde{q}(z)(T_{[F]}\mathcal{M})),$$
$$\beta \in d\psi(0)^{-1}(d\widetilde{q}(z)(T_x J)).$$

Then $\sum \epsilon_i f_i(\alpha) = 0$ (in a trivialization of the tangent bundle of J), and $f_i(\beta)$ is independent of i. Hence the right-hand side of (2.3.5) vanishes. $\triangle$

(2.3.6)*Corollary.* *Keeping notation as above, $\widetilde{\omega}$ is a symplectic form on $\widetilde{\mathcal{M}}$.*

Proof. Since $\widetilde{\omega}_{\mathbf{v}}$ is non-degenerate and $\widetilde{q}$ is étale, the two-form $\widetilde{q}^*\widetilde{\omega}_{\mathbf{v}}$ is non degenerate. By (2.3.3) we get that $\widetilde{\omega}$ is non-degenerate. $\triangle$

Let

$$\widetilde{\Sigma} := \widetilde{\pi}^{-1}\Sigma = \widetilde{S}_{\mathbf{v}} \cap \widetilde{\mathcal{M}}, \quad \widetilde{\Omega} := \widetilde{\pi}^{-1}\Omega = \Omega_{\mathbf{v}} \cap \widetilde{\mathcal{M}}, \quad B := B_{\mathbf{v}} \cap \mathcal{M}, \quad \widetilde{B} := \widetilde{B}_{\mathbf{v}} \cap \widetilde{\mathcal{M}}.$$

(See (2.1.4) and the following definition for $B_{\mathbf{v}}$ and $\widetilde{B}_{\mathbf{v}}$.)

There is an analogue of Proposition (2.2.10) valid for $\widetilde{\mathcal{M}}$, which follows immediately from (2.2.10). Let $\mathcal{E}$ be a family of torsion-free simple semistable sheaves on J parametrized by a scheme T, with $v(\mathcal{E}_t) = \mathbf{v}$, $\det(\mathcal{E}_t) \cong \mathcal{O}_J$, $\sum c_2(\mathcal{E}_t) = 0$ for all $t \in T$. Let $\psi_{\mathcal{E}}\colon T \to \mathcal{M}$ be the modular map.

(2.3.7)Proposition. *Keep notation as above. There exists a lift*

$$\widetilde{\psi}_{\mathcal{E}} \colon T \to (\widetilde{\mathcal{M}} \setminus \widetilde{\Omega})$$

of $\psi_{\mathcal{E}}$ with the following properties.

    (1) *The induced map of sets is (given Identification (2.2.3)):*

$$\widetilde{\psi}_{\mathcal{E}}(t) = \widetilde{S}\text{-equivalence class of } \mathcal{E}_t.$$

    (2) *Let $0 \in T$. Assume the map from the germ $(T, 0)$ to the locus in $\mathrm{Def}(\mathcal{E}_0)$ parametrizing sheaves with trivial determinant and 0 Albanese map is an isomorphism. Then the map $\widetilde{\psi}_{\mathcal{E}}$ is an isomorphism in a neighborhood of 0 (in the classical topology).*

(2.3.8) *Remark.* Let $y_0 \in (\widetilde{\Sigma} \setminus \widetilde{\Omega})$. The correspondence (2.2.3) associates to y_0 two isomorphism classes of simple sheaves on J, a sheaf E fitting into

$$0 \to L_1 \to E \to L_2 \to 0,$$

with extension class e, and a sheaf G fitting into

$$0 \to L_2 \to G \to L_1 \to 0,$$

with extension class $e^{\perp}$. Let T and $\mathcal{E}$ be as in (2.3.7), and assume $\mathcal{E}_0 \cong E$. Then by (2.3.7) a neighborhood of y_0 can be identified with a neighborhood of $0 \in T$. On the other hand, by the same proposition, a neighborhood of $0 \in T$ must also parametrize a family $\mathcal{G}$ of deformations of G. What is the relation between $\mathcal{G}$ and $\mathcal{E}$? One passes from one to the other by means of an elementary modification. In order to explain this, let

$$\widetilde{\Sigma}(T) := \{t \in T \mid \mathcal{E}_t \text{ is strictly semistable}\}.$$

Then (shrinking T in the classical topology, if necessary) there is an exact sequence

$$0 \to \mathcal{L}_1 \to \mathcal{E}|_{J \times \widetilde{\Sigma}(T)} \to \mathcal{L}_2 \to 0,$$

which for every $t \in \widetilde{\Sigma}(T)$ restricts to the destabilizing sequence for $\mathcal{E}_t$. One verifies that $\mathcal{G}$ is the sheaf on $J \times T$ fitting into the exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to i_* \mathcal{L}_2 \to 0,$$

where $i \colon J \times \widetilde{\Sigma}(T) \hookrightarrow J \times T$ is the inclusion. In other words, let $\mathcal{M}^{simple}$ be the moduli space of simple semistable sheaves F with $v(F) = \mathbf{v}$, $\det(F) \cong \mathcal{O}_J$, $\sum c_2(F) = 0$: then the points $[E], [G] \in \mathcal{M}^{simple}$ are not separated. Since by (2.3.7) the tautological family on $J \times \mathcal{M}^{simple}$ induces a regular map $\mathcal{M}^{simple} \to (\widetilde{\mathcal{M}} \setminus \widetilde{\Omega})$, we see that $(\widetilde{\mathcal{M}} \setminus \widetilde{\Omega})$ can be identified with the "separation" of $\mathcal{M}^{simple}$ (obtained identifying all points which are not separated).

(2.3.9) *Description of $\widetilde{\Sigma}$.* The fibers of $\widetilde{\Sigma} \to \Sigma$ over points of $\Sigma \setminus \Omega$ are $\mathbb{P}^1$'s, and over points of Ω they are smooth 3-dimensional quadrics, by (2.0.1)-(2.0.3) of [O1].

A global description of $\widetilde{\Sigma}$ is as follows. Let $A := J \times \widehat{J}$, and $\mathcal{F}, \mathcal{G}$ be "tautologi-cal" sheaves on $J \times A$ such that

$$\mathcal{F}|_{J \times (p,\xi)} \cong I_p \otimes \xi, \quad \mathcal{G}|_{J \times (p,\xi)} \cong I_{-p} \otimes \xi^{-1}.$$

Let $\pi \colon \widetilde{A} \to A$ be the blow up of $A[2]$, with exceptional divisor $E = \bigcup\limits_{i=1}^{256} E_i$, and $f \colon J \times \widetilde{A} \to \widetilde{A}$ be the projection. The relative Ext-sheaf $Ext^1_f((\mathrm{id}_J \times \pi)^* \mathcal{F}, (\mathrm{id}_J \times \pi)^* \mathcal{G})$ is locally-free of rank two away from E, and locally at a point of E is isomorphic to $\mathcal{O}^2 \oplus \mathcal{O}_E$. Thus its double dual $\mathcal{E}$ is locally-free of rank two, and $\mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^1$-fibration over $\widetilde{A}$. Let $\iota \colon \widetilde{A} \to \widetilde{A}$ be the involution lifting multiplication by -1 on A: there is a lift $\bar{\iota} \colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ of ι, with fixed locus the $\mathbb{P}^1$-bundle over E. Let $Y := \mathbb{P}(\mathcal{E})/\langle \bar{\iota} \rangle$, and $\rho \colon Y \to \widetilde{A}/\langle \bar{\iota} \rangle$ be the "quotient" $\mathbb{P}^1$-fibration. Let $E_i \subset \widetilde{A}$ be a component of the exceptional divisor, and let $(p, \xi) = \pi(E_i) \in A[2]$. Since ι is the identity on E_i, there is an embedding $E_i \subset \widetilde{A}$. Of course we have a canonical isomorphism $E_i \cong \mathbb{P}(T_{(p,\xi)}A)$. On $T_{(p,\xi)}A = T_pJ \oplus T_\xi \widehat{J}$ we have the symplectic form $\varphi := \omega \oplus \hat{\omega}$ ($\hat{\omega}$ is the two-form induced by ω on $\widehat{J}$): let $\mathrm{Gr}^\varphi(2, T_{(p,\xi)}A)$ be the grassmannian of φ-isotropic lines in $\mathbb{P}(T_{(p,\xi)}A)$. Then $\rho^{-1}(E_i)$ is isomorphic to the incidence subvariety of $\mathbb{P}(T_{(p,\xi)}A) \times \mathrm{Gr}^\varphi(2, T_{(p,\xi)}A)$, with projection to the first factor corresponding to the restriction of ρ. Projecting to the second factor we get another $\mathbb{P}^1$-fibration

$$\psi_i \colon \rho^{-1}(E_i) \to \mathrm{Gr}^\varphi(2, T_{(p,\xi)}A) \cong \text{smooth 3-dim'l quadric.}$$

Then $\widetilde{\Sigma}$ is the contraction of all the $\rho^{-1}(E_i)$'s along the fibrations ψ_i.

3. AN APPLICATION OF THE GENERALIZED LEFSCHETZ HYPERPLANE THEOREM.

For $\alpha \in J$ let

$$Z_\alpha := \{[F] \in \mathcal{M} | \ F|_{\Theta_\alpha} \text{ is not locally-free semistable}\},$$

(3.1) $\qquad \widetilde{Z}_\alpha := \widetilde{\pi}^{-1}(Z_\alpha).$

(Notice that $F|_{\Theta_\alpha}$ is locally-free semistable if and only if the same holds for all sheaves in the S-equivalence class of F.) Applying the generalized Lefschetz hyperplane Theorem [GM] to the determinant map of $\widetilde{\mathcal{M}}$, and arguing as in [O1,(3.0.1)] one gets the following result.

(3.2)*Proposition.* *Keep notation as above, and let $i \colon \widetilde{Z}_\alpha \hookrightarrow \widetilde{\mathcal{M}}$ be the inclusion. The map*

$$i_\# \colon \pi_q(\widetilde{Z}_\alpha) \to \pi_q(\widetilde{\mathcal{M}})$$

is an isomorphism for $q \leq 1$ and a surjection for $q = 2$. In particular

$$H_2(\widetilde{Z}_\alpha; \mathbb{Z}) \xrightarrow{i_*} H_2(\widetilde{\mathcal{M}}; \mathbb{Z}) \qquad H^2(\widetilde{\mathcal{M}}; \mathbb{Z}) \xrightarrow{i^*} H_2(\widetilde{Z}_\alpha; \mathbb{Z})$$

are surjective and injective, respectively.

Let

$$(3.3) \qquad \Sigma_\alpha := \Sigma \cap Z_\alpha,$$

$$(3.4) \qquad B_\alpha^0 := \{[F] \in \mathcal{M}| \ F \text{ is stable and } F|_{\Theta_\alpha} \text{ is singular}\},$$

$$(3.5) \qquad V_\alpha^0 := \{[F] \in \mathcal{M}| \ F|_{\Theta_\alpha} \text{ is locally-free, not semistable}\},$$

and let B_α, V_α be the closures of B_α^0 and V_α^0 respectively. Let $\widetilde{\Sigma}_\alpha := \widetilde{\pi}^{-1}(\Sigma_\alpha)$, and let $\widetilde{B}_\alpha, \widetilde{V}_\alpha \subset \widetilde{\mathcal{M}}$ be the strict transforms of B_α and V_α, respectively. Clearly we have a decomposition into closed subsets

$$\widetilde{Z}_\alpha = \widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha \cup \widetilde{V}_\alpha.$$

The (easy) proof of the following claim is left to the reader.

(3.6)Claim. *Keep notation as above. If*

$$(3.7) \qquad\qquad \alpha \notin \bigcup_{x \in J[2]} \Theta_x,$$

then $\widetilde{Z}_\alpha \subset (\widetilde{\mathcal{M}} \setminus \widetilde{\Omega})$.

Thus in analyzing $\widetilde{Z}_\alpha$ we will be able to use the moduli-theoretic interpretation of $(\widetilde{\mathcal{M}} \setminus \widetilde{\Omega})$ given by Proposition (2.3.7).

(3.8)Remark. Let $\widetilde{a}_{\mathbf{v}} \colon \widetilde{\mathcal{M}}_{\mathbf{v}} \to J \times \widehat{J}$ be as in §1 and

$$\widetilde{\mathcal{M}}^J := \widetilde{a}_{\mathbf{v}}^{-1}(J \times \{\hat{0}\})$$
$$\widetilde{Z}_\alpha^J := \widetilde{\pi}_{\mathbf{v}}^{-1}\{[F] \in a_{\mathbf{v}}^{-1}(J \times \{\hat{0}\})| \ F|_{\Theta_\alpha} \text{ is not locally-free semistable}\}.$$

Applying the generalized Lefschetz Hyperplane Theorem to the determinant map on $\widetilde{\mathcal{M}}^J$ we get that the inclusion $\widetilde{Z}_\alpha^J \hookrightarrow \widetilde{\mathcal{M}}^J$ induces isomorphisms on π_q for $q \leq 3$, and a surjection on H_4. The restriction of $\widetilde{a}_{\mathbf{v}}$ to $\widetilde{\mathcal{M}}^J$ is a fibration over J with fiber $\widetilde{\mathcal{M}}$, and one verifies easily that $\widetilde{\mathcal{M}}^J$ is cohomologically the product of J and $\widetilde{\mathcal{M}}$. Hence if $\widetilde{Z}_\alpha^J$ were a cohomological product of $\widetilde{Z}_\alpha$ and J we would know that $\widetilde{Z}_\alpha \hookrightarrow \widetilde{\mathcal{M}}$ induces isomorphims on $H^q \otimes \mathbb{Q}$ for $q \leq 3$. Now $\widetilde{Z}_\alpha^J$ is not a fibration: the generic fiber is homeomorphic to $\widetilde{Z}_\alpha$ but special fibers are not, however the equality $b_2(\widetilde{\mathcal{M}}) = b_2(\widetilde{Z}_\alpha)$ that we will prove, shows that up to H^2 the cohomology of $\widetilde{Z}_\alpha^J$ is the product of that of $\widetilde{Z}_\alpha$ and of J.

4. Analysis of $\widetilde{\Sigma}_\alpha$.

The ∗-red tags not contained in Subsection (4.4) are (4.1.1), (4.3.2) and (4.3.10). We choose α so that (3.7) holds. (Co)Homology is with rational coefficients.

4.1. Low-dimensional topology of $\widetilde{\Sigma}_\alpha$. Let $f := \widetilde{\pi}|_{\widetilde{\Sigma}_\alpha}$. By Claim (3.6) we have $\widetilde{\Sigma}_\alpha \subset (\widetilde{\mathcal{M}} \setminus \widetilde{\Omega})$, hence f is a $\mathbb{P}^1$-fibration over Σ_α:

$$
\begin{array}{ccc}
\mathbb{P}^1 & \longrightarrow & \widetilde{\Sigma}_\alpha \\
& & \downarrow f \\
& & \Sigma_\alpha.
\end{array}
$$
$(4.1.1)^*$

Thus

$$(4.1.2) \qquad\qquad H^q(\widetilde{\Sigma}_\alpha) \cong H^q(\Sigma_\alpha), \quad q = 0, 1$$

$$(4.1.3) \qquad\qquad H^2(\widetilde{\Sigma}_\alpha) \cong H^2(\Sigma_\alpha) \oplus \mathbb{Q} c_1(\omega_f),$$

where ω_f is the relative cotangent bundle of f.

4.2. Low-dimensional topology of Σ_α. The map $\kappa \colon C \times \widehat{J} \to \Sigma_\alpha$ defined by

$$\kappa(p, \widehat{x}) := [I_{i_\alpha(p)} \otimes \mathcal{L}_{\widehat{x}} \oplus I_{-i_\alpha(p)} \otimes \mathcal{L}_{-\widehat{x}}]$$

is surjective. It is not injective, because of points which are both in Θ_α and $-\Theta_\alpha$. We claim that

$$(4.2.1) \qquad \Theta_\alpha \cap (-\Theta_\alpha) = \Theta_\alpha \cap \Theta_{-\alpha} = \{i_\alpha(q_1), i_\alpha(q_2)\} \qquad q_1 \neq q_2.$$

In fact $\Theta_\alpha \neq \Theta_{-\alpha}$ because by (3.7) $2\alpha \neq 0$. Thus $i_\alpha^*(\Theta_{-\alpha}) = q_1 + q_2$, where

$$(4.2.2) \qquad\qquad q_1 + q_2 \sim K_C - 2\alpha.$$

It follows from (3.7) that $q_1 \neq q_2$. Clearly $\kappa(q_1, \widehat{x}) = \kappa(q_2, -\widehat{x})$. One easily shows that

$$(4.2.3) \qquad\qquad \Sigma_\alpha \cong (C \times \widehat{J}) / \equiv$$

where $\equiv$ is the equivalence relation generated by setting

$$(4.2.4) \qquad\qquad (q_1, \widehat{x}) \equiv (q_2, -\widehat{x}).$$

In order to discuss the topology of Σ_α we introduce some notation. Denote by $\overline{C}$ the quotient of C obtained identifying q_1 with q_2, and let

$$
\begin{array}{ccc}
\overline{C} & \xrightarrow{\ \mu\ } & \Sigma_\alpha \\
q & \mapsto & [I_{i_\alpha(q)} \oplus I_{-i_\alpha(q)}].
\end{array}
$$

The following result is obtained by a standard argument.

(4.2.5)Proposition. *Keep notation as above.*
   (1) $H^0(\Sigma_\alpha; \mathbb{Q}) \cong \mathbb{Q}$
   (2) *The map $\mu^* \colon H^1(\Sigma_\alpha; \mathbb{Q}) \xrightarrow{\sim} H^1(\overline{C}; \mathbb{Q})$ is an isomorphism.*
   (3) *The map $\kappa^* \colon H^2(\Sigma_\alpha; \mathbb{Q}) \to H^2(C \times \widehat{J}; \mathbb{Q})$ is an isomorphism.*

4.3. The intersection with $\widetilde{B}_\alpha$. Let $\nu_0 \colon \widehat{I}_0 \to \widehat{J}$ be the blow-up of $\widehat{J}[2]$, i.e. the case $\alpha = 0$ of (1.12). Let $\iota \colon \widehat{I}_0 \to \widehat{I}_0$ be the involution covering (-1). Let $\approx$ be the equivalence relation on $C \times \widehat{I}_0$ generated by

$$(4.3.1) \qquad\qquad (q_1, \widehat{x}) \approx (q_2, \iota(\widehat{x})),$$

where q_1, q_2 are as in (4.2.1). We will prove the following result.

(4.3.2)*Proposition. *Keep notation as above. Then:*
   (1) *Given Bijection (2.2.3), $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$ is the subset of $\widetilde{\Sigma}_\alpha$ parametrizing $\widetilde{S}$-equivalence classes of sheaves F which are singular at two distinct points.*
   (2) *There is a map $\widetilde{\kappa} \colon (C \times \widehat{I}_0) \to \widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$ which identifies $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$ with the quotient of $(C \times \widehat{I}_0)$ modulo $\approx$.*
   (3) *Given (4.2.3) and Item (2) above, the restriction of $\widetilde{\pi}$ to $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$ is identified with the map*
   $$(C \times \widehat{I}_0 / \approx) \longrightarrow (C \times \widehat{J} / \equiv)$$
   induced by $\mathrm{id}_C \times \nu_0$.

The proof of Proposition (4.3.2) will be given after some preliminary results.

(4.3.3)Lemma. *Let F be a simple torsion-free sheaf on J with $v(F) = \mathbf{v}$, whose $\widetilde{S}$-equivalence class belongs to $\widetilde{B}_\mathbf{v}$ (see the definition following (2.1.4)) via (2.2.3).*
   (1) *The double dual F^{**} fits into an exact sequence*

$$(4.3.4) \qquad\qquad 0 \to \xi_1 \to F^{**} \to \xi_2 \to 0,$$

   where $\xi_i \in \widehat{J}$. In particular $\ell(F^{**}/F) = 2$.
   (2) *Exact sequence (4.3.4) is split if and only if $\xi_1^{-1} \otimes \xi_2 \not\cong \mathcal{O}_J$.*
   (3) *If $[F] \in \widetilde{B}_\alpha$ then $Sing(F) = \{x, -x\}$ where $x \neq -x$.*

Proof. Let $E := F^{**}$. First we prove that

$$(4.3.5) \qquad\qquad c_2^{hom}(E) = 0,$$

where c_q^{hom} is the Chern class in cohomology. Since E is slope-semistable, Bogomolov's inequality tells us that $c_2^{hom}(E) \geq 0$. Thus it suffices to prove that (4.3.5) holds for sheaves F parametrized by an open dense subset of $\widetilde{B}$. We will prove it for $[F] \in \widetilde{B} \setminus \widetilde{\Sigma}$, i.e. we assume F is stable. Since $c_2^{hom}(E) < c_2^{hom}(F)$, because F is singular, $c_2^{hom}(E)$ equals 0 or 1. Let us show that

$$(4.3.6) \qquad\qquad \text{if } c_2^{hom}(E) = 1 \text{ then } E \text{ is not slope-stable.}$$

Assume E is slope-stable. Then

$$(4.3.7) \qquad H^0(E \otimes \mathcal{L}_{\widehat{x}}) = 0 = H^0(E^* \otimes \mathcal{L}_{\widehat{x}}^{-1})^* = H^2(E \otimes \mathcal{L}_{\widehat{x}}).$$

Hence by Riemann-Roch

$$(4.3.8) \qquad\qquad \dim H^1(E \otimes \mathcal{L}_{\widehat{x}}) = 1 \text{ for all } \widehat{x} \in \widehat{J}.$$

From (4.3.7) it follows [Mum,pp.46-55] that $\widehat{\phi}_*(\mathcal{L} \otimes \phi^*E) = R^2\widehat{\phi}_*(\mathcal{L} \otimes \phi^*E) = 0$. Applying Grothendieck-Riemann-Roch we get

$$ch(R^1\widehat{\phi}_*(\mathcal{L} \otimes \phi^*E)) = -\widehat{\phi}_*[ch(\mathcal{L})\phi^*ch(E)] = 1 - 2\eta_{\widehat{J}},$$

where $\eta_{\widehat{J}}$ is the fundamental class of $\widehat{J}$. This is absurd because by (4.3.8) the sheaf $R^1\widehat{\phi}_*(\mathcal{L} \otimes \phi^*E)$ is a line-bundle on $\widehat{J}$. We have proved (4.3.6). Let

$$0 \to \xi_1 \to E \to I_Z \otimes \xi_2 \to 0$$

be a slope-destabilizing sequence, where the ξ_i's are line bundles, and Z is a zero dimensional subscheme of J. Since E is slope-semistable (because F is) we have $\xi_i \cdot \Theta = 0$, so by Assumption (1.3) we have $\xi_i \in \widehat{J}$. If x is the singular point of F the sheaf $I_x \otimes \xi_1$ injects into F. This contradicts our assumption that F is stable, and proves (4.3.5). Item (1) follows because E is slope-semistable and $c_1^{hom}(E) = 0 = c_2^{hom}(E)$. To prove Item (2), first notice that if $\xi_1^{-1} \otimes \xi_2 \not\cong \mathcal{O}_J$ then every extension (4.3.4) is trivial. Thus we may assume that $\xi_1 \cong \xi_2 \cong \xi$, and we must prove that (4.3.4) is non-split. Assume the contrary, i.e. that $F^{**} \cong \xi \oplus \xi$. If x is a singular point of F, one easily checks that $I_x \otimes \xi$ injects into F. Thus, since $[F] \notin \widetilde{\Omega}$ (because F is simple), we must have $-x \neq x$. Since also $I_{-x} \otimes \xi$ injects into F, we get that $F \cong I_x \otimes \xi \oplus I_{-x} \otimes \xi$, contradicting the hypothesis that F is simple. Finally Item (3) follows immediately from Assumption (3.7). $\triangle$

Let $[F] \in \widetilde{\Sigma} \setminus \widetilde{\Omega}$, where F fits into the exact sequence

$$(4.3.9) \qquad\qquad 0 \to I_x \otimes \xi \xrightarrow{\alpha} F \to I_{-x} \otimes \xi^{-1} \xrightarrow{\beta} 0,$$

and is simple. Let $\mathcal{E}$ be a family of torsion-free simple semistable sheaves on J parametrized by T, satisfying the hypotheses of Proposition (2.3.7), with $\mathcal{E}_0 \cong F$. Thus the map $\widetilde{\psi}_\mathcal{E}: T \to \widetilde{\mathcal{M}}$ identifies a neighborhood of $0 \in T$ with a neighborhood of $[F] \in \widetilde{\mathcal{M}}$. Let

$$\Delta(T) := \{t \in T \mid \mathcal{E}_t \text{ is singular}\},$$
$$\widetilde{\Sigma}(T) := \{t \in T \mid \mathcal{E}_t \text{ is strictly semistable}\},$$
$$\widetilde{B}(T) := \text{closure of } \{t \in T \mid \mathcal{E}_t \text{ is singular and stable}\}.$$

Thus $\Delta(T) = \widetilde{\Sigma}(T) \cup \widetilde{B}(T)$ (recall Lemma (2.1.2)).

(4.3.10)*Lemma. *Keep notation as above. Assume that $x \neq -x$. Then (after shrinking T around 0 if necessary)*

$$(4.3.11) \qquad\qquad \widetilde{\Sigma}(T) \cap \widetilde{B}(T) = \{t \in \widetilde{\Sigma}(T) \mid \mathcal{E}_t \text{ is singular at two points}\}.$$

Furthermore $\widetilde{\Sigma}(T)$ and $\widetilde{B}(T)$ intersect transversely.

Proof. Let us prove (4.3.11). Clearly it suffices to show that

$$(4.3.12) \qquad\qquad 0 \in \widetilde{B}(T) \text{ if and only if } F \text{ is singular at } -x.$$

The "only if" follows immediately from Lemma (4.3.3). Now assume F is singular at $(-x)$. From (4.3.9) we get an exact sequence

$$(4.3.13) \qquad\qquad 0 \to \xi \to F^{**} \to \xi^{-1} \to 0.$$

We also have an exact sequence

$$0 \to F \longrightarrow F^{**} \xrightarrow{(p,q)} \mathbf{C}_x \oplus \mathbf{C}_{-x} \to 0,$$

where

$$p(\xi^{\pm 1}(x)) \neq 0, \qquad q(\xi(-x)) = 0.$$

(Here $\xi^{\pm 1}(\pm x)$ is the fiber of $\xi^{\pm 1}$ at $\pm x$.) Choose a surjection

$$F^{**}(-x) \xrightarrow{\ \tau\ } \xi(-x),$$

and let $S := \operatorname{Hom}(\xi(-x), \mathbf{C}_{-x})$. For $s \in S$ let

$$q(s) := q + s \circ \tau,$$

and let F_s be the sheaf on J fitting into the exact sequence

$$0 \to F_s \longrightarrow F^{**} \xrightarrow{(p,q(s))} \mathbf{C}_x \oplus \mathbf{C}_{-x} \to 0.$$

The sheaves F_s form a family of torsion free simple semistable sheaves on J, with $v(F_s) = \mathbf{v}$ for all s. Furthermore F_s is singular for all s, isomorphic to F for $s = 0$, and stable for $s \neq 0$. This shows that $0 \in \widetilde{B}(T)$, and finishes the proof of (4.3.11). Now let's prove that $\widetilde{\Sigma}(T) \cap \widetilde{B}(T)$ intersect transversely. Clearly it suffices to prove that they are transverse at 0. Let $\overline{T}$ be the universal deformation space of F, and $\overline{\mathcal{E}}$ the family of sheaves on J parametrized by $\overline{T}$. Thus $T \subset \overline{T}$, and $\mathcal{E} = \overline{\mathcal{E}}|_{J \times T}$. Set

$$\widetilde{\Sigma}_{\mathbf{v}}(\overline{T}) := \{ t \in \overline{T} | \ \overline{\mathcal{E}}_t \text{ is strictly semistable} \},$$
$$\widetilde{B}_{\mathbf{v}}(\overline{T}) := \text{closure of } \{ t \in \overline{T} | \ \overline{\mathcal{E}}_t \text{ is singular and stable} \}.$$

The map

$$\begin{array}{ccc} \overline{T} & \longrightarrow & J \times \widehat{J} \\ t & \mapsto & (\sum c_2(\overline{\mathcal{E}}_t), \det(\overline{\mathcal{E}}_t)) \end{array}$$

is submersive at 0 and gives fibrations $\widetilde{\Sigma}_{\mathbf{v}}(\overline{T}) \to U$ and $\widetilde{B}_{\mathbf{v}}(\overline{T}) \to U$, where $U \subset J \times \widehat{J}$ is an open subset, with fibers $\widetilde{\Sigma}(T)$, $\widetilde{B}(T)$ respectively. Thus in order to prove that $\widetilde{\Sigma}(T)$ and $\widetilde{B}(T)$ iintersect transversely at 0 it suffices to show that $\widetilde{\Sigma}_{\mathbf{v}}(\overline{T})$ and $\widetilde{B}_{\mathbf{v}}(\overline{T})$ are transverse at 0. By [O4,(1.17)] we have

$$T_0 \widetilde{\Sigma}_{\mathbf{v}}(\overline{T}) = \{ \epsilon \in \operatorname{Ext}^1(F,F) | \ \beta \circ \epsilon \circ \alpha = 0 \},$$

where α, β are as in (4.3.9), and we identify $T_0(\overline{T})$ with $\operatorname{Ext}^1(F,F)$ via the Kodaira-Spencer map. Let S be as above, and $\kappa \colon T_0 S \to \operatorname{Ext}^1(F,F)$ be the Kodaira-Spencer map. It is easy to check that

$$\beta \circ \kappa(\frac{\partial}{\partial s}(0)) \circ \alpha \neq 0.$$

Since $S \subset \widetilde{B}(T) \subset \widetilde{B}_{\mathbf{v}}(\overline{T})$, we see that $T_0\widetilde{\Sigma}_{\mathbf{v}}(\overline{T}) \not\subset T_0\widetilde{B}_{\mathbf{v}}(\overline{T})$. In particular $\widetilde{\Sigma}_{\mathbf{v}}(\overline{T})$ is smooth in 0 (this also follows from the fact that $\widetilde{\Sigma}_{\mathbf{v}}(\overline{T})$ is the pull-back of $\widetilde{\Sigma}_{\mathbf{v}}$ for the map $\overline{T} \to \widetilde{\mathcal{M}}_{\mathbf{v}}$, which is étale in 0.) Hence all that remains to be proved is that $\widetilde{B}_{\mathbf{v}}(\overline{T})$ is smooth at 0. This we do by analyzing the map ρ fitting into the exact sequence

$$\mathrm{Ext}^1(F,F) \xrightarrow{\rho} H^0(Ext^1(F,F)) \to H^2(Hom(F,F)) \xrightarrow{\lambda} \mathrm{Ext}^2(F,F),$$

a piece of the local-to-global Exact Sequence for $\mathrm{Ext}^\bullet(F,F)$. Since

$$H^0(Ext^1(F,F)) \cong Ext^1(F \otimes \mathcal{O}_{J,x}, F \otimes \mathcal{O}_{J,x}) \oplus Ext^1(F \otimes \mathcal{O}_{J,-x}, F \otimes \mathcal{O}_{J,-x}),$$

we can define $\rho_{\pm x}$ as the composition of ρ with projection to the first and second summand respectively. In order to prove that $T_0\widetilde{B}_{\mathbf{v}}(\overline{T}) \neq T_0(\overline{T})$ it suffices to show that ρ_{-x} is surjective. In fact assume this has been proved: by Lemma (4.3.3) first-order deformations belonging to $T_0\widetilde{B}_{\mathbf{v}}(\overline{T})$ do not smooth the singularity of F at $(-x)$, and hence $T_0\widetilde{B}_{\mathbf{v}}(\overline{T})$ is a proper subspace of $\mathrm{Ext}^1(F,F)$. Now let's show that ρ_{-x} is surjective. The transpose of λ is given by

$$\mathrm{Hom}(F,F) \xrightarrow{\lambda^*} \mathrm{Hom}(Hom(F,F), \mathcal{O}_J),$$

From

$$0 \to Hom(F,F) \to Hom(F^{**}, F^{**}) \to \mathbf{C}_x \oplus \mathbf{C}_{-x} \to 0$$

one gets that $\mathrm{Hom}(Hom(F,F), \mathcal{O}_J) \cong \mathrm{Hom}(F^{**}, F^{**})$. With this identification the map λ^* is the canonical one. We claim that

(4.3.14) $\dim \mathrm{Hom}(F,F) = 1, \qquad \dim \mathrm{Hom}(F^{**}, F^{**}) = 2.$

The first equation holds because F is simple. To prove the second equation, look at Exact Sequence (4.3.13). If $\xi \not\cong \xi^{-1}$ the exact sequence is split, and the equation follows. If $\xi \cong \xi^{-1}$, the exact sequence is not split, otherwise we would have $F \cong (I_x \oplus I_{-x}) \otimes \xi$, which is not simple: it follows easily that the second equation holds. From (4.3.14) we get that $\dim \ker \lambda = 1$, and hence

(4.3.15) $\mathrm{cod}(\mathrm{im}\rho) = 1.$

We claim ρ_x is not surjective. Since $\mathrm{Ext}^1(F,F)$ is spanned by $T_0\widetilde{\Sigma}_{\mathbf{v}}(\overline{T})$ and $T_0\widetilde{B}_{\mathbf{v}}(\overline{T})$ (because $T_0\widetilde{\Sigma}_{\mathbf{v}}(\overline{T})$ is of codimension 1 in $\mathrm{Ext}^1(F,F)$ and not equal to $T_0\widetilde{B}_{\mathbf{v}}(\overline{T})$) it suffices to check that every first-order deformation in $T_0\widetilde{\Sigma}_{\mathbf{v}}(\overline{T})$ or in $T_0\widetilde{B}_{\mathbf{v}}(\overline{T})$ does not smooth the singularity in x: this follows from (2.1.2) and (4.3.3). Thus we get from (4.3.15) that ρ_{-x} is surjective. This finishes the proof of the proposition.   △

Proof of Proposition (4.3.2). Item (1) follows at once from Item (3) of (4.3.3) and Lemma (4.3.10). Now we prove Items (2)-(3). Let $U \subset \Sigma$ be the open subset given by

$$U := \{[I_x \otimes \xi \oplus I_{-x} \otimes \xi^{-1}]| \; x \neq -x\},$$

and let $\widetilde{U} := \widetilde{\pi}^{-1}U$. By Proposition (2.3.7) and Lemma (4.3.10) $\widetilde{\Sigma} \cap \widetilde{B} \cap \widetilde{U}$ is smooth. Let $J^0 := (J \setminus J[2])$, where $J[2]$ is the kernel of multiplication by 2, and set

(4.3.16) $\widetilde{X} := (J^0 \times \widehat{J}) \times_U (\widetilde{\Sigma} \cap \widetilde{B} \cap \widetilde{U}),$

where $(J^0 \times \widehat{J}) \to U$ is the étale degree-two map

$$
\begin{array}{ccc}
J^0 \times \widehat{J} & \longrightarrow & U \\
(x, \xi) & \mapsto & [I_x \otimes \xi \oplus I_{-x} \otimes \xi^{-1}].
\end{array}
$$

Since $\widetilde{\Sigma} \cap \widetilde{B} \cap \widetilde{U}$ is smooth and the above map is étale, $\widetilde{X}$ is smooth.

(4.3.17)Claim. *Keep notation as above. The projection $h \colon \widetilde{X} \to (J^0 \times \widehat{J})$ is the blow-up of $(J^0 \times \widehat{J}[2])$.*

Proof of the claim. Since $\widetilde{X}$ is mooth it suffices to show that

$$
(4.3.18) \qquad h^{-1}(x, \xi) \cong \begin{cases} \text{one point} & \text{if } \xi \notin \widehat{J}[2], \\ \mathbb{P}^1 & \text{if } \xi \in \widehat{J}[2]. \end{cases}
$$

The exact sequence

$$
0 \to I_{-x} \otimes \xi^{-1} \to \xi^{-1} \to \xi^{-1}|_{-x} \to 0
$$

gives an exact sequence

$$
0 \to \mathrm{Ext}^1(\xi^{-1}, I_x \otimes \xi) \longrightarrow \mathrm{Ext}^1(I_{-x} \otimes \xi^{-1}, I_x \otimes \xi),
$$

and

$$
(4.3.19) \qquad h^{-1}(x, \xi) = \mathbb{P}(\mathrm{Ext}^1(\xi^{-1}, I_x \otimes \xi)) = \mathbb{P}(H^1(I_x \otimes \xi^{\otimes 2})).
$$

An easy computation gives (4.3.18), and this proves (4.3.17). $\triangle$

Let $g \colon \widetilde{X} \to J^0$ be the projection, and let

$$
\widetilde{X}_\alpha := g^{-1}(\Theta_\alpha) \cong C \times \widehat{I}_0,
$$

where the second isomorphism holds because of (4.3.17). By (4.3.16) we have a natural map

$$
\widetilde{X}_\alpha \to \widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{U} = \widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha,
$$

where the equality follows from Item (3) of (4.3.3). One verifies easily that the map above is the quotient map for $\approx$; proving Item (2). Item (3) is immediate $\triangle$

4.4. Topological results. First notice that by (4.1.1) and (4.2.3)

$$
(4.4.1)^* \qquad\qquad\qquad \widetilde{\Sigma}_\alpha \text{ is irreducible.}
$$

Furthermore (4.1.2) and Item (2) of (4.2.5) give

$$
(4.4.2)^* \qquad\qquad\qquad b^1(\widetilde{\Sigma}_\alpha) = 5.
$$

Now we pass to $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$. Given Item (2) of (4.3.2), the low-dimensional cohomology of $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$ has a description similar to that of Σ_α given in (4.2.5). In

order to give this description, let $E \subset \widehat{I}_0$ be the exceptional divisor of ν_0, and E_i its irreducible components ($i = 1, \ldots, 16$). Choose $z \in E$; the map

$$
\begin{array}{ccc}
C & \longrightarrow & C \times \widehat{I}_0 \\
q & \mapsto & (q, z)
\end{array}
$$

induces a map $\widetilde{\mu} \colon \overline{C} \to \widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$. "Copying" the proof of Item (2) of (4.2.5) one gets that $\widetilde{\mu}^* \colon H^1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha; \mathbb{Q}) \to H^1(\overline{C}; \mathbb{Q})$ is an isomorphism, hence

$$(4.4.3)^* \qquad\qquad\qquad b^1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha) = 5.$$

Furthermore, letting $\tilde{\kappa}$ be as in Item (2) of (4.3.2), one gets that

$$(4.4.4) \qquad \tilde{\kappa}^* \colon H^2(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha; \mathbb{Q}) \to H^2(C \times \widehat{I}_0; \mathbb{Q}) \text{ is an isomorphism}$$

by "copying" the proof of Item (3) of (4.2.5). Thus $\tilde{\kappa}^*$ gives an identification

$$(4.4.5) \qquad H^2(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha; \mathbb{Q}) \cong \bigoplus_{i=1}^{16} \mathbb{Q}[C \times E_i] \bigoplus (\mathrm{id}_C \times \nu_0)^* H^2(C \times \widehat{J}).$$

Next we examine the maps on fundamental groups and cohomology induced by the inclusion $\rho \colon \widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \hookrightarrow \widetilde{\Sigma}_\alpha$.

(4.4.6)*Proposition. *Keeping notation as above $\rho_\# \colon \pi_1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha) \to \pi_1(\widetilde{\Sigma}_\alpha)$ is an isomorphism.*

Proof. The maps

$$\widetilde{\pi}_\# \colon \pi_1(\widetilde{\Sigma}_\alpha) \to \pi_1(\Sigma_\alpha) \qquad \widetilde{\pi}_\# \circ \rho_\# \colon \pi_1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha) \to \pi_1(\Sigma_\alpha)$$

are isomorphisms, and this implies that $\rho_\#$ is an isomorphism. $\triangle$

(4.4.7)Proposition. *The map*

$$H^2(\widetilde{\Sigma}_\alpha; \mathbb{Q}) \xrightarrow{\rho_2^*} H^2(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha; \mathbb{Q})$$

induced by Inclusion ρ is injective. Furthermore, referring to (4.4.5), we have

$$(4.4.6)^* \qquad \mathrm{Im}(\tilde{\kappa}^* \rho_2^*) = \mathbb{Q}[C \times E] \bigoplus (\mathrm{id}_C \times \nu_0)^* H^2(C \times \widehat{J}).$$

Proof. Given that
$$(\mathrm{id}_C \times \kappa) = f|_{\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha},$$

where f is the map of (4.1.1), and given (4.1.3) together with Item (3) of (4.2.5), it suffices to notice that

$$\langle \tilde{\kappa}^* \rho^* c_1(\omega_f), p \times E_i \rangle = -2$$

for every $i = 1, \ldots, 16$ (here $p \in C$). $\triangle$

We will need to know exactly the class of $\tilde{\kappa}^* \rho^*(c_1(\omega_f))$. Let $\mathcal{L}$ be the Poincaré line-bundle on $J \times \widehat{J}$, and i_α as in (1.8). We will prove that

$$(4.4.9)^* \qquad\qquad \tilde{\kappa}^* \rho^*(\omega_f) \cong (i_\alpha \times \nu_0)^*(\mathcal{L}^{\otimes 4}) \otimes [2C \times E].$$

Proof of (4.4.9). Let $\widehat{J}^0 := (\widehat{J} \setminus \widehat{J}[2])$, and $\hat{\iota}: \widehat{J}^0 \hookrightarrow \widehat{J}$ be the inclusion. Notice that we also have a natural inclusion $\widehat{J}^0 \hookrightarrow \widehat{I}_0$. It suffices to prove that

$$(4.4.10) \qquad\qquad \tilde{\kappa}^* \rho^*(\omega_f)|_{C \times \widehat{J}^0} \cong (i_\alpha \times \hat{\iota})^* \mathcal{L}^{\otimes 4}.$$

In fact, since

$$\tilde{\kappa}^* \rho^*(\omega_f)|_{p \times E_i} \cong \mathcal{O}_{E_i}(-2)$$

for $p \in C$ and $i = 1, \dots, 16$, Equality (4.4.10) immediately implies (4.4.9). To prove (4.4.10) we must describe explicitly the inclusion

$$(4.4.11) \qquad\qquad C \times \widehat{J}^0 \hookrightarrow (C \times \widehat{J}^0) \times_{\Sigma_\alpha} \widetilde{\Sigma}_\alpha$$

whose image is dense in $\widetilde{\Sigma}_\alpha \cap \widehat{B}_\alpha$. Let $\Delta_\alpha, -\Delta_\alpha \subset J \times (C \times \widehat{J}^0)$ be given by

$$\Delta_\alpha := \{(i_\alpha(p), p, \xi) |\ p \in C, \quad \xi \in \widehat{J}^0\},$$
$$-\Delta_\alpha := \{(-i_\alpha(p), p, \xi) |\ p \in C, \quad \xi \in \widehat{J}^0\},$$

and let $\varphi: J \times (C \times \widehat{J}^0) \to (C \times \widehat{J}^0)$ be the projection. By (4.3.19) we can identify (4.4.11) with

$$(4.4.12) \qquad \mathbb{P}R^1\varphi_*(I_{\Delta_\alpha} \otimes (\mathcal{L}^0)^{\otimes 2}) \hookrightarrow \mathbb{P}Ext^1_\varphi(I_{-\Delta_\alpha} \otimes \mathcal{L}^0, I_{\Delta_\alpha} \otimes (\mathcal{L}^0)^{-1})),$$

where $\mathcal{L}^0 := (\mathrm{id}_J \times \hat{\iota})^* \mathcal{L}$. (To simplify notation we omit pull-back signs whenever possible.) Thus

$$(4.4.13) \qquad\qquad \tilde{\kappa}^* \rho^*(\omega_f)|_{C \times \widehat{J}^0} \cong R^1\varphi_*(I_{\Delta_\alpha} \otimes (\mathcal{L}^0)^{\otimes 2}) \otimes \mathcal{Q}^{-1},$$

where $\mathcal{Q}$ is the line-bundle fitting into the exact sequence

$$0 \to R^1\varphi_*(\bullet) \to Ext^1_\varphi(\bullet, \bullet) \to \mathcal{Q} \to 0.$$

(Here R^1 and Ext^1 are the sheaves appearing in (4.4.12).) We have an isomorphism

$$\mathcal{Q} \cong Ext^2_\varphi((\mathcal{L}^0)^{-1}|_{-\Delta_\alpha}, I_{\Delta_\alpha} \otimes \mathcal{L}^0),$$

and applying Serre duality we get

$$\mathcal{Q} \cong Hom_\varphi(\mathcal{L}^0, (\mathcal{L}^0)^{-1}|_{-\Delta_\alpha})^{-1} \cong (i_\alpha \times \hat{\iota})^* \mathcal{L}^{-2}.$$

Another easy computation gives

$$R^1\pi_*(I_{\Delta_\alpha} \otimes (\mathcal{L}^0)^{\otimes 2}) \cong (i_\alpha \times \hat{\iota})^* \mathcal{L}^2.$$

Using (4.4.13) we get Equation (4.4.10). $\triangle$

5. Analysis of $\widetilde{B}_\alpha$.

There is a single $*$-red tag not contained in Subsection (5.2), i.e. (5.1.1).

5.1. Realization of $\widetilde{B}_\alpha$ as a $\mathbb{P}^1$-fibration. We will prove that there is a fibration

$$(5.1.1)^* \qquad \begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \widetilde{B}_\alpha \\ & & \downarrow g \\ & & \overline{C} \times K^{[2]}\widehat{J}, \end{array}$$

where $K^{[2]}\widehat{J}$ is the Kummer surface of $\widehat{J}$, i.e. the subset of $\widehat{J}^{[2]}$ parametrizing subschemes whose associated cycle sums up to $\widehat{0}$. We define g by giving its two components. First we define $g_1 \colon \widetilde{B}_\alpha \to \overline{C}$. Let $z \in \widetilde{B}_\alpha$. By Item (3) of (4.3.3) the sheaf F_z has two distinct singularities, say x and $-x$. One at least of $x, -x$ belongs to Θ_α. The set of unordered couples $(x, -x)$ intersecting Θ_α is identified with $\overline{C}$ via the map

$$\begin{array}{ccc} C & \longrightarrow & J \times J \\ p & \mapsto & (i_\alpha(p), -i_\alpha(p)). \end{array}$$

Given the above identification we set

$$(5.1.2) \qquad \begin{array}{ccc} \widetilde{B}_\alpha & \xrightarrow{g_1} & \overline{C} \\ z & \mapsto & sing(F_z). \end{array}$$

In order to define $g_2 \colon \widetilde{B}_\alpha \to K^{[2]}\widehat{J}$ we recall a result of Bryan, Donagi, Leung [BDL]. Let Muk be the moduli space of Mukai-stable rank-two vector-bundles V on J with $\det V \cong \mathcal{O}_J$ and $c_2^{hom}(V) = 0$ (see [BDL] for the definition of Mukai-stability). Let $[V] \in Muk$: as is easily checked either $V \cong \xi \oplus \xi^{-1}$ with $\xi \notin \widehat{J}[2]$, or else V is a non-trivial extension of ξ by itself, where $\xi \in \widehat{J}[2]$. By Theorem (5.6) of [BDL] there is an isomorphism

$$(5.1.3) \qquad \varphi \colon Muk \xrightarrow{\sim} K^{[2]}\widehat{J}.$$

Explicitely, if $V = \xi \oplus \xi^{-1}$ then $\varphi([V]) = \{\xi, \xi^{-1}\}$, and if V is the extension of ξ by itself with extension class $e \neq 0$, then $\varphi([V])$ is the length-two non reduced point supported at ξ, with direction given by $\mathbf{C}e$. Let's define $g_2 \colon \widetilde{B}_\alpha \to K^{[2]}\widehat{J}$. We work locally. Let $z \in \widetilde{B}_\alpha$; by Proposition (2.3.7) there exists a neighborhood $U \subset \widetilde{B}_\alpha$ (in the classical topology) of z parametrizing a family $\mathcal{F}$ of torsion-free simple semistable sheaves on J, with $v(\mathcal{F}_t) = \mathbf{v}$ for all $t \in U$, such that the $\widetilde{S}$-equivalence class of $\mathcal{F}_t$ corresponds to t via (2.2.3). By Proposition (4.3.3) we have $[\mathcal{F}_t^{**}] \in Muk$ for all $t \in U$, hence the family $\mathcal{F}^{**}$ induces a regular map $U \to Muk$, which we can view as a map $U \to \widehat{J}^{[2]}$ by (5.1.3). These maps are independent of the choice of family $\mathcal{F}$ (see Remark (2.3.8)) and they glue together, defining $g_2 \colon \widetilde{B}_\alpha \to K^{[2]}\widehat{J}$.

Claim. *The map $g := (g_1, g_2) \colon \widetilde{B}_\alpha \to \overline{C} \times K^{[2]}\widehat{J}$ is a $\mathbb{P}^1$-fibration.*

Proof. Let $((x, -x), [E]) \in \overline{C} \times K^{[2]}\widehat{J}$. The fiber $g^{-1}((x, -x), [E])$ is isomorphic to the set of $\widetilde{S}$-equivalence classes of simple semistable sheaves F fitting into an exact sequence

$$0 \to F \to E \to \mathbf{C}_x \oplus \mathbf{C}_{-x} \to 0.$$

One verifies easily that this set is isomorphic to $\mathbb{P}^1$. We leave it to the reader to check that g is locally trivial. $\triangle$

5.2. Topological results. From Fibration (5.1.1) we get that

$(5.2.1)^*$ $\widetilde{B}_\alpha$ is irreducible,

and that

$(5.2.2)^*$ $H^1(\widetilde{B}_\alpha) \cong H^1(\overline{C}) \cong \mathbb{Q}^5,$

$(5.2.3)$ $H^2(\widetilde{B}_\alpha) \cong H^2(\overline{C}) \oplus H^2(K^{[2]}\widehat{J}) \oplus \mathbb{Q}c_1(\omega_g),$

where ω_g is the relative cotangent bundle of g.

In proving that $\widetilde{\mathcal{M}}$ is simply connected it will be useful to represent $\pi_1(\widetilde{B}_\alpha)$ as follows. Let $E_1 \subset \widehat{I}_0$ be the component of the exceptional divisor of ν_0 mapping to $\widehat{0}$ and let

$(5.2.4)$ $R := (C \times E_1/\approx) \cong \overline{C} \times E_1,$

where $\approx$ is as in (4.3.1), and $j: R \hookrightarrow \widetilde{B}_\alpha$ be the inclusion given by (4.3.2). Then

$(5.2.5)^*$ $j_\#: \pi_1(R) \to \pi_1(\widetilde{B}_\alpha)$ is an isomorphism

because, letting g_1 be as in (5.1.2), the composition $(g_1 \circ j): R \to \overline{C}$ is the projection to the first factor of (5.2.4), hence $g_{1,\#} \circ j_\#$ is an isomorphism, and $g_{1,\#}: \pi_1(\widetilde{B}_\alpha) \to \pi_1(\overline{C})$ is also an isomorphism.

For the proof of the main Theorem we will need to know the map on 2-cohomology ■ induced by the inclusion $\lambda: (\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha) \hookrightarrow \widetilde{B}_\alpha$.

$(5.2.6)^*$**Proposition.** *Keep notation as above. Then*

$$\lambda^*: H^2(\widetilde{B}_\alpha) \to H^2(\widetilde{B}_\alpha \cap \widetilde{\Sigma}_\alpha)$$

is injective. Furthermore, given Isomorphism (4.4.5), we have
$(5.2.7)^*$

$$\operatorname{Im}(\tilde{\kappa}^*\lambda^*) = \bigoplus_{i=1}^{16} \mathbb{Q}[C \times E_i] \bigoplus (\mathrm{id}_C \times \nu_0)^*(H^2(C) \oplus H^2(\widehat{J}) \oplus \mathbb{Q}c_1((i_\alpha \times \mathrm{id}_{\widehat{j}})^*\mathcal{L})),$$

where $\tilde{\kappa}$ is as in Item (2) of (4.3.2).

Proof. Given (5.2.3) and (4.4.5), in order to prove that λ^* is injective and that (5.2.7) holds, it suffices to show that

$(5.2.8)$ $\tilde{\kappa}^*\lambda^*\omega_g = (\mathrm{id}_C \times \nu_0)^*\mathcal{L}^{\otimes(-4)} \otimes [-2C \times E].$

(Recall (4.3.2).) We will show that

$(5.2.9)$ $\tilde{\kappa}^*\lambda^*(c_1(\omega_g)) = -\tilde{\kappa}^*\rho^*(c_1(\omega_f))$

in the Chow group, and then (5.2.8) will follow from (4.4.9). Let $\widetilde{\Sigma}^* \subset \widetilde{\Sigma}$ and $\widetilde{B}^* \subset \widetilde{B}$ be the open subsets given by

$$\widetilde{\Sigma}^* := \widetilde{\pi}^{-1}\{[I_x \otimes \xi \oplus I_{-x} \otimes \xi^{-1}]|\ x \neq (-x),\quad \xi \not\cong \xi^{-1}\},$$

$$\widetilde{B}^* := \{[F]|\ sing(F) = \{x, -x\},\ x \neq (-x),\quad F^{**} \cong \xi \oplus \xi^{-1},\ \xi \not\cong \xi^{-1}\}.$$

By Lemma (4.3.10) they intersect transversely, and hence by adjunction

$$K_{\widetilde{\Sigma}^* \cap \widetilde{B}^*} \sim [\widetilde{\Sigma} + \widetilde{B}]|_{\widetilde{\Sigma}^* \cap \widetilde{B}^*}.$$

(Because $K_{\widetilde{\mathcal{M}}} \sim 0$.) Restricting $\widetilde{\pi}$ to $\widetilde{\Sigma}^* \cap \widetilde{B}^*$ we get an isomorphism $(\widetilde{\Sigma}^* \cap \widetilde{B}^*) \cong (J^0 \times \widehat{J}^0)$, hence $K_{\widetilde{\Sigma}^* \cap \widetilde{B}^*} \sim 0$. Thus

(5.2.10)
$$[\widetilde{\Sigma}]|_{\widetilde{\Sigma}^* \cap \widetilde{B}^*} \sim -[\widetilde{B}]|_{\widetilde{\Sigma}^* \cap \widetilde{B}^*}.$$

Restricting $\widetilde{\pi}$ to $\widetilde{\Sigma}^*$ we get a $\mathbb{P}^1$-fibration $F: \widetilde{\Sigma}^* \to (J^0 \times \widehat{J}^0)$, and thus

$$[\widetilde{\Sigma}]|_{\widetilde{\Sigma}^*} \sim K_{\widetilde{\Sigma}^*} \sim \omega_F.$$

In fact the first equivalence follows from adjunction, and the second one (where ω_F is the relative cotangent bundle of F) from triviality of the canonical bundle of $(J^0 \times \widehat{J}^0)$. We also have a $\mathbb{P}^1$-fibration

$$\begin{array}{ccc}
\widetilde{B}^* & \xrightarrow{G} & (J^0/(-1)) \times (\widehat{J}^0/(-1)) \\
{}[F] & \mapsto & (sing(F), \{\xi, \xi^{-1}\}),
\end{array}$$

where $F^{**} \cong \xi \oplus \xi^{-1}$. Thus by a similar argument we get

$$[\widetilde{B}]|_{\widetilde{B}^*} \sim K_{\widetilde{B}^*} \sim \omega_G.$$

The last two equivalences together with (5.2.10) prove (5.2.9). $\triangle$

6. ANALYSIS OF $\widetilde{V}_\alpha$.

6.1. Introduction. By definition V_α^0 is the locus parametrizing sheaves F such that there is an exact sequence

(6.1.1)
$$0 \to \lambda \longrightarrow i_\alpha^* F \xrightarrow{f} \xi \to 0$$

with $i_\alpha^* F$ locally-free and ξ a line-bundle of negative degree. Let G be the torsion-free sheaf on J fitting into the exact sequence

(6.1.2)
$$0 \to G \longrightarrow F(\Theta_\alpha) \xrightarrow{\widetilde{f}} i_{\alpha,*}(\xi \otimes K_C) \to 0,$$

where $\widetilde{f}$ is obtained tensorizing f with the identity map of K_C (notice that by adjunction $i_\alpha^*(\Theta_\alpha) \sim K_C$). In other words G is the elementary modification of $F(\Theta_\alpha)$ associated to $\widetilde{f}$. We will show (see Lemma (6.3.22)) that $\deg \xi = -1$, and thus by a straightforward computation

$$v(G) = 2 + \theta =: \mathbf{w},$$

where $\theta := c_1^{hom}(\Theta)$. We will also prove that G is slope-stable (see Lemma (6.3.19)), hence $[G] \in \mathcal{M}_{\mathbf{w}}$. Since (6.1.1) is uniquely determined by F, the above construction

defines a regular map from V_α^0 to $\mathcal{M}_{\mathbf{w}}$. Notice that G comes with extra structure, namely an exact sequence

$$(6.1.3) \qquad\qquad 0 \to \xi \longrightarrow i_\alpha^* G \xrightarrow{\psi} \lambda \otimes K_C \to 0$$

which is part of the long exact sequence of Tor's obtained by applying the functor $\otimes \mathcal{O}_{\Theta_\alpha}$ to (6.1.2). As is easily verified, the sheaf F is the elementary modification of G associated to Exact Sequence (6.1.3). Hence by associating to $[F]$ the couple consisting of $[G]$ and Exact Sequence (6.1.3) we get an isomorphism between V_α^0 and a locally-closed subset of a quot-scheme over $\mathcal{M}_{\mathbf{w}}$. Since V_α^0 is dense in V_α, this will give a way of relating V_α to $\mathcal{M}_{\mathbf{w}}$. The section is organized as follows. In (6.2) following Mukai, Yoshioka [Muk1,Y2] we apply the Mukai transform to prove that $\mathcal{M}_{\mathbf{w}}$ is the product $J \times \widehat{J}$. In (6.3) we show that V_α^0 is isomorphic to P_α^0, a locally closed subset of a Quot-scheme over $\mathcal{M}_{\mathbf{w}}$. We denote by P_α the closure of P_α^0 in the Quot-scheme over $\mathcal{M}_{\mathbf{w}}$; in Subsections(6.4)-(6.5) we give results on P_α and the boundary $P_\alpha \setminus P_\alpha^0$. We will not construct directly P_α (the difficulty lies in analyzing it singular points), instead we construct a $\mathbb{P}^1$-bundle over the blow-up of $\widehat{J}$ at 16 points (we denote it by P_α^-) and then prove that it is a desingularization of P_α: this is done in (6.6). The 3-fold P_α^- has 16 $(-1,-1)$ curves: we let P_α^+ be the manifold obtained by flopping all of these curves. In (6.7) we we show that $\widetilde{V}_\alpha$ is isomorphic to P_α^+. In the last subsection we give the results on the topology of $\widetilde{V}_\alpha$ and its intersections with $\widetilde{\Sigma}_\alpha$, $\widetilde{B}_\alpha$ which are needed in the proof of Theorem (1.4).

6.2. The moduli space $\mathcal{M}_{\mathbf{w}}$. For $(x,\widehat{y}) \in J \times \widehat{J}$, let $E(x,\widehat{y})$ be the sheaf on J defined by

$$E(x,\widehat{y}) := \phi_*(\mathcal{L} \otimes \widehat{\phi}^*(i_{\widehat{y},*}[3p_0 + x])).$$

(Here $\phi, \widehat{\phi}$ are as in (1.9).) We will prove the following result.

(6.2.1)Proposition. *Keep notation as above.*

  (1) $E(x,\widehat{y})^*$ *is a vector-bundle on J, and $v(E(x,\widehat{y})^*) = 2 + \theta$. Furthermore*

$$(6.2.2) \qquad\qquad \det(E(x,\widehat{y})^*) \cong [\Theta] \otimes \mathcal{L}_{-\widehat{x}-2\widehat{y}}$$

$$(6.2.3) \qquad\qquad \sum c_2(E(x,\widehat{y})^*) = -x - y.$$

  (2) $E(x,\widehat{y})^*$ *is slope-stable for every $(x,\widehat{y})$, and the map*

$$\begin{array}{ccc} J \times \widehat{J} & \xrightarrow{\rho} & \mathcal{M}_{\mathbf{w}} \\ (x,\widehat{y}) & \mapsto & [E(x,\widehat{y})^*] \end{array}$$

   is an isomorphism.

(6.2.4) *Remark.* Item (2) of Proposition (6.2.1) is part of a more general result of Yoshioka [Y2] on moduli spaces of sheaves on an abelian surface in the case when semistability implies stability. However, since we will need various results about the moduli space which are not found elsewhere, e.g. (6.2.5), (6.2.11), and which may be used to prove (6.2.1), we have added a proof of (6.2.1).

  Proof of Item (1) of (6.2.1). Since $h^0(C, [3p_0 + x + t]) = 2$ for all $t \in J$, we see [Mum,pp.46-55] that $E(x,\widehat{y})$ is locally-free of rank two. Since the higher direct images vanish, the Chern classes of $E(x,\widehat{y})$ are computed by the Grothendieck-Riemann-Roch formula: one gets $v(E(x,\widehat{y})) = 2 - \theta$. Thus $E(x,\widehat{y})^*$ is a rank-two vector-bundle, and its Mukai vector is as stated. Before going on with the proof of (6.2.1) we determine the restriction of $E(x,\widehat{y})^*$ to Θ_α.

(6.2.5)Proposition. *Keep notation as above. There is an exact sequence*

(6.2.6) $0 \to [p_0 - \alpha - x - y] \to i_\alpha^* E(x, \widehat{y})^* \to [p_0 - y] \to 0.$

The extension class is the (unique up to scalars) non-trivial one if $\alpha + x \neq 0$, and is trivial if $\alpha + x = 0$.

Proof. Let $\Delta \subset C \times C$ be the diagonal, and $\pi_i \colon C \times C \to C$ be the first and second projection respectively, for $i = 1, 2$. Then

(6.2.7) $(i_\alpha \times i_{\widehat{y}})^* \mathcal{L} \cong [\Delta] \otimes \pi_1^*[-p_0 + y] \otimes \pi_2^*[-p_0 + \alpha].$

Thus

(6.2.8) $i_\alpha^* E(x, \widehat{y}) = [-p_0 + y] \otimes \pi_{1,*}([\Delta] \otimes \pi_2^*[2p_0 + \alpha + x]).$

If $\alpha + x = 0$ one gets immediately that

$$i_\alpha^* E(x, \widehat{y})^* \cong \mathcal{O}_C(p_0 - y)^{(2)}.$$

If $\alpha + x \neq 0$, we apply the $\pi_{1,*}$-functor to the exact sequence

$$0 \to \pi_2^*[2p_0 + \alpha + x] \to [\Delta] \otimes \pi_2^*[2p_0 + \alpha + x] \to ([\Delta] \otimes \pi_2^*[2p_0 + \alpha + x])|_\Delta \to 0,$$

and we get an exact sequence

(6.2.9) $0 \to \mathcal{O}_C \to \pi_{1,*}([\Delta] \otimes \pi_2^*[2p_0 + \alpha + x]) \to [\alpha + x] \to 0.$

The extension class is non-trivial, because otherwise we would have a non-zero section of

$$[\Delta] \otimes \pi_2^*[2p_0 + \alpha + x] \otimes \pi_1^*[-\alpha - x],$$

which is absurd since $\mathcal{O}_C(\alpha + x)$ is non-trivial. Tensorizing (6.2.9) by $[-p_0 + y]$ and taking the dual exact sequence we get the proposition. $\triangle$

Let us prove (6.2.2)-(6.2.3). The first formula follows immediately from (6.2.6). In order to prove (6.2.3), choose a canonical divisor $D \in |K_C|$, and consider the exact sequence of sheaves on $J \times \widehat{J}$:

$$0 \to \mathcal{L} \otimes \widehat{\phi}^*(i_{\widehat{y},*}[p_0 + x]) \to \mathcal{L} \otimes \widehat{\phi}^*(i_{\widehat{y},*}[p_0 + x + D]) \to \mathcal{L} \otimes \widehat{\phi}^*(i_{\widehat{y},*}(\mathcal{O}_D)) \to 0.$$

Applying the ϕ_*-functor and recalling that $D \sim 2p_0$, we get an exact sequence of sheaves on J:

(6.2.10) $0 \to E(x, \widehat{y}) \to \mathcal{L}_{\widehat{D}} \to R^1\phi_*(\mathcal{L} \otimes \widehat{\phi}^*(i_{\widehat{y},*}[p_0 + x])) \to 0,$

where $\widehat{D} := i_{\widehat{y},*}(D)$.

Lemma. *Keep notation as above. The sheaf $R^1\phi_*(\mathcal{L}\otimes\widehat{\phi}^*(i_{\widehat{y},*}[p_0+x]))$ is supported on Θ_{-x}. More precisely we have*

(6.2.11)
$$R^1\phi_*(\mathcal{L}\otimes\widehat{\phi}^*(i_{\widehat{y},*}[p_0+x])) = i_{-x,*}[p_0+y].$$

Proof. By Formula (6.2.2) and Exact Sequence (6.2.10) we have (in the Chow ring)

$$-c_1(\Theta) + c_1(\mathcal{L}_{\widehat{x}+2\widehat{y}}) = c_1(E(x,\widehat{y})) = c_1(\mathcal{L}_{\widehat{D}}) - c_1(R^1\phi_*(\mathcal{L}\otimes\widehat{\phi}^*(i_{\widehat{y},*}[p_0+x]))).$$

This gives

$$c_1(R^1\phi_*(\mathcal{L}\otimes\widehat{\phi}^*(i_{\widehat{y},*}[p_0+x]))) = c_1(\Theta_{-x}).$$

One concludes easily that $R^1\phi_*(\mathcal{L}\otimes\widehat{\phi}^*(i_{\widehat{y},*}[p_0+x]))$ is the push-forward by i_{-x} of a line-bundle on C. To compute this line-bundle, pull-back by $(i_{-x}\times i_{\widehat{y}})$ and apply Formula (6.2.7) to get

(6.2.12)
$$R^1\phi_*(\mathcal{L}\otimes\widehat{\phi}^*(i_{\widehat{y},*}[p_0+x])) = i_{-x,*}([-p_0+y]\otimes R^1\pi_{1,*}([\Delta])).$$

Applying the functor $\pi_{1,*}$ to the exact sequence

$$0 \to \mathcal{O}_{C\times C} \to [\Delta] \to [\Delta]|_\Delta \to 0,$$

we get

$$0 \to K_C^{-1} \to H^1(\mathcal{O}_C)\otimes\mathcal{O}_C \to R^1\pi_{1,*}([\Delta]) \to 0.$$

Thus $R^1\pi_{1,*}([\Delta]) \cong K_C$. Plugging this into (6.2.12) we get (6.2.11). $\triangle$

Let us prove (6.2.3). It follows from (6.2.10) and (6.2.11) that

$$c_2(E(x,\widehat{y})) = c_2(\mathcal{L}_{\widehat{D}}) - i_{-x,*}c_1([-p_0+y]).$$

Since $\sum c_2(\mathcal{L}_{\widehat{D}}) = 0$, Formula (6.2.3) follows immediately.

Proof of Item (2) of (6.2.1). By (6.2.5) the restriction of $E(x,\widehat{y})^*$ to Θ_α is semistable; since Θ_α is numerically equivalent to Θ (the ample divisor defining stability) this implies that $E(x,\widehat{y})^*$ is slope-semistable. That $E(x,\widehat{y})^*$ is slope-stable follows from the lemma below, an immediate consequence of Assumption (1.3).

(6.2.13)Lemma. *Let F be a torsion-free slope-semistable sheaf on J with $v(F) = \mathbf{w}$. Then F is slope-stable. In particular every sheaf parametrized by $\mathcal{M}_{\mathbf{w}}$ is slope-stable.*

Now we prove that ρ is an isomorphism. By (6.2.13) all sheaves parametrized by $\mathcal{M}_{\mathbf{w}}$ are stable, hence by Mukai [Muk2,(0.1)] $\mathcal{M}_{\mathbf{w}}$ is smooth of pure dimension 4. The map ρ is injective because of (6.2.2)-(6.2.3), and thus ρ is an isomorphism between $J\times\widehat{J}$ and an irreducible component of $\mathcal{M}_{\mathbf{w}}$ by Zariski's Main Theorem. We finish by showing that ρ is surjective.

(6.2.14)Lemma. *Assume $[F] \in \mathcal{M}_{\mathbf{w}}$. Then the W.I.T. holds for F, with index $i(F) = 1$, i.e. $R^j\widehat{\phi}_*(\mathcal{L} \otimes \phi^*F) = 0$ for $j \neq 1$ (see [Muk1]). Furthermore the Mukai transform of F is given by*

$$\widehat{F} := R^1\widehat{\phi}_*(\mathcal{L} \otimes \phi^*F) = i_{\hat{v},*}(\delta),$$

for a certain $\hat{v} \in \widehat{J}$, where δ is a line-bundle of degree (-1) on C.

Proof. By Serre duality and slope-stability of F,

$$H^2(\mathcal{L}_{\hat{t}} \otimes F) = \mathrm{Hom}(\mathcal{L}_{\hat{t}} \otimes F, \mathcal{O}_J)^* = 0.$$

Applying Grothendieck-Riemann-Roch one gets that

$$\sum_{i=0}^{1}(-1)^j c_1^{hom}(R^j\widehat{\phi}_*(\mathcal{L} \otimes \phi^*F)) = -\theta.$$

Since $\chi(\mathcal{L}_{\hat{t}} \otimes F) = 0$ for all $t \in J$, this implies that there exist $\xi_1, \xi_2 \in \widehat{J}$, with $\xi_1 \not\cong \xi_2$, such that

$$H^0(\xi_1^{-1} \otimes F) \neq 0 \neq H^0(\xi_2^{-1} \otimes F).$$

By slope-stability of F, the map $(\xi_1 \oplus \xi_2) \to F$ is an isomorphism at the generic point, hence we have an exact sequence

$$0 \to \xi_1 \oplus \xi_2 \to F \to i_{u,*}(\eta) \to 0,$$

where η is a line-bundle on C, of degree 1. This shows that $\widehat{\phi}_*(\mathcal{L} \otimes \phi^*F) = 0$, hence the W.I.T. holds for F, with index 1. In order to compute $\widehat{F}$ pull-back the above sequence to $J \times \widehat{J}$, tensorize with $\mathcal{L}$, and apply the functor $\widehat{\phi}_*$: one gets an exact sequence

$$0 \to R^1\widehat{\phi}_*(\mathcal{L} \otimes \phi^*F) \to i_{\hat{v},*}(\lambda) \to \mathbf{C}_{[\xi_1]} \oplus \mathbf{C}_{[\xi_2]} \to 0,$$

where λ is a degree-one line-bundle on C. It follows that $\widehat{F}$ is as claimed.   △

Now let's prove that F is isomorphic to $E(x,\widehat{y})^*$ for some $(x,\widehat{y})$. By a Theorem of Mukai [Muk1, Cor.(2.4)] and (6.2.14) we have

$$F \cong (-1_J)^*\widehat{\hat{F}} := (-1_J)^*R^1\phi_*(\widehat{\phi}^*(\mathcal{L} \otimes i_{\hat{v},*}(\delta))).$$

Applying Serre duality one gets

$$(-1_J)^*R^1\phi_*(\widehat{\phi}^*(\mathcal{L} \otimes i_{\hat{v},*}(\delta))) \cong R^1\phi_*(\widehat{\phi}^*(\mathcal{L}^{-1} \otimes i_{\hat{v},*}(\delta)))$$
$$\cong (\phi_*(\mathcal{L} \otimes \widehat{\phi}^*(i_{\hat{v},*}(2p_0 - \delta))))^* = E(-p_0 - \delta, \hat{v})^*.$$

This finishes the proof of Proposition (6.2.1).

6.3. Description of V_α^0. Let P_α^0 be the variety parametrizing isomorphism classes of couples (G, ψ), where $[G] \in \mathcal{M}_{\mathbf{w}}$, and ψ is a quotient appearing in an exact sequence

$$(6.3.1) \qquad 0 \to \xi \longrightarrow i_\alpha^* G \xrightarrow{\ \psi\ } \zeta \to 0,$$

such that the following holds:

$$(6.3.2) \qquad \zeta \text{ is an invertible sheaf of degree 3,}$$

$$(6.3.3) \qquad \det G \cong [\Theta_\alpha],$$

$$(6.3.4) \qquad \sum c_2(G) - \sum i_{\alpha,*} c_1(\xi) = 0.$$

Clearly P_α^0 is a locally closed subset of a Quot-scheme $\mathcal{Q}$ over the subvariety of $\mathcal{M}_{\mathbf{w}}$ parametrizing sheaves G such that (6.3.3) holds (by [Muk3,(A.6)] there exists a tautological sheaf on $J \times \mathcal{M}_{\mathbf{w}}$); let P_α be the closure of P_α^0 in $\mathcal{Q}$. Thus every point of P_α is represented by a couple (G, ψ) satisfying the conditions above, except for Condition (6.3.2), which is replaced by "ζ is a rank-one sheaf of degree 3". As explained in the introduction to this section, P_α parametrizes sheaves whose moduli belong to V_α, and which are obtained by elementary modification from a sheaf G as above. In order to prove this we introduce some notation.

(6.3.5)Definition. Let $\mathcal{E}^\alpha$ be a rank-two vector-bundle on $J \times \widehat{J}$ such that

$$(6.3.6) \qquad \mathcal{E}_{\hat{y}}^\alpha \cong E(-\alpha - 2y, \hat{y}).$$

(The existence of $\mathcal{E}^\alpha$ is guaranteed by the existence of a tautological sheaf on $J \times \mathcal{M}_{\mathbf{w}}$.) Let $\mathcal{G}^\alpha := (\mathcal{E}^\alpha)^*$.

By (6.2.2) the map

$$\begin{array}{ccc} \widehat{J} & \longrightarrow & \mathcal{M}_{\mathbf{w}} \\ \hat{y} & \mapsto & [\mathcal{G}_{\hat{y}}^\alpha] \end{array}$$

is an isomorphism between $\widehat{J}$ and the subvariety of $\mathcal{M}_{\mathbf{w}}$ parametrizing sheaves with determinant $[\Theta_\alpha]$. Thus P_α is a Quot-scheme over $\widehat{J}$: let

$$(6.3.7) \qquad f : P_\alpha \to \widehat{J}$$

be the natural map.

Thus if $t = (G, \psi) \in P_\alpha$, and $\hat{y} = f(t)$, we have

$$(6.3.8) \qquad G \cong \mathcal{G}_{f(t)}^\alpha \cong E(-\alpha - 2y, \hat{y})^*, \text{ where } \hat{y} = f(t).$$

Let $F = \mathcal{F}_t^\alpha$ be the elementary modification of G associated to (6.3.1), i.e. the sheaf fitting into the exact sequence

$$(6.3.9) \qquad 0 \to F \xrightarrow{\ j\ } G \to i_{\alpha,*}(\zeta) \to 0.$$

Clearly $\mathcal{F}_t^\alpha$ is the restriction to $J \times \{t\}$ of an elementary modification

$$(6.3.10) \qquad 0 \to \mathcal{F}^\alpha \longrightarrow (\mathrm{id}_J \times f)^* \mathcal{G}^\alpha \xrightarrow{\ \Psi^\alpha\ } (i_\alpha \times \mathrm{id}_{P_\alpha})_*(\zeta^\alpha) \to 0.$$

(6.3.11)Lemma. *Keep notation as above. Then $\mathcal{F}^\alpha$ is a family of torsion-free semistable sheaves on J parametrized by P_α, with*

$$(6.3.12) \qquad v(\mathcal{F}_t^\alpha) = \mathbf{v}, \quad \det \mathcal{F}_t^\alpha \cong \mathcal{O}_J, \quad \sum c_2(\mathcal{F}_t^\alpha) = 0$$

for all $t \in P_\alpha$.

Proof. That $\mathcal{F}^\alpha$ is a family of sheaves (i.e. it is flat over $\mathcal{O}_{P_\alpha}$) follows from flatness over $\mathcal{O}_{P_\alpha}$ of the other two sheaves appearing in (6.3.10). Let F and G be as above. Since F is a subsheaf of G it is torsion-free. The formulae of (6.3.12) hold by (6.3.2), (6.3.3) and (6.3.4) respectively. We finish by showing that F is semistable. Suppose F is not semistable, and let

$$(6.3.13) \qquad 0 \to I_Z \otimes L \xrightarrow{f} F \longrightarrow I_W \otimes L^{-1} \to 0$$

be desemistabilizing, where L is a line bundle and Z, W are zero-dimensional sub-schemes. Thus $L \cdot \Theta \geq 0$. Since $j \circ \phi$ is non-zero (here j is as in (6.3.9)), and G is slope-stable by (6.2.13), we get $L \cdot \Theta = 0$. Hence $[L] \in \widehat{J}$ because of Assumption (1.3). From (6.3.13) we get

$$(6.3.14) \qquad 2 = c_2^{hom}(F) = \ell(Z) + \ell(W) - L^2.$$

This shows that Z is empty (or else $I_Z(L)$ would not desemistabilize). The restriction of $j \circ f$ to Θ_α maps $L|_{\Theta_\alpha}$ to ξ: since $\deg(L|_{\Theta_\alpha}) = 0$ and $\deg \xi = -1$, we get that $j \circ f$ vanishes along Θ_α. Thus $j \circ f$ extends to a map $L(\Theta_\alpha) \to G$. This is absurd because G is slope-stable by (6.2.13). $\triangle$

By the above lemma the family $\mathcal{F}^\alpha$ induces a modular map

$$(6.3.15) \qquad \mu_\alpha \colon P_\alpha \to \mathcal{M}.$$

Let μ_α^0 be the restriction of μ_α to P_α^0. The main result of this subsection is the following.

(6.3.16)Proposition. *Keep notation as above. Then μ_α^0 is an isomorphism between P_α^0 and V_α^0. In particular $\mu_\alpha(P_\alpha) = V_\alpha$, and μ_α is a birational morphism from P_α to V_α.*

Proof. Let $t \in P_\alpha$, and set $F = \mathcal{F}_t^\alpha$, $G = \mathcal{G}_{f(t)}^\alpha$. The long exact sequence of Tor's obtained by applying the functor $\otimes \Theta_\alpha$ to (6.3.9) gives an exact sequence

$$(6.3.17) \qquad 0 \to \zeta \otimes K_C^{-1} \to i_\alpha^* F \to \xi \to 0.$$

By (6.2.1) G is locally-free, hence ξ is locally-free. Thus the above exact sequence shows that $i_\alpha^* F$ is singular if and only if ζ is not locally-free. Since F is isomorphic to G outside Θ_α, we get that

$$(6.3.18) \qquad i_\alpha^* \mathcal{F}_t^\alpha \text{ is locally-free iff } \mathcal{F}_t^\alpha \text{ is locally-free iff } t \in P_\alpha^0.$$

Now assume $t \in P_\alpha^0$. Since $\deg \xi = -1$, Exact Sequence (6.3.17) shows that $i_\alpha^* F$ is not semistable, hence $[F] \in V_\alpha^0$. Thus $\mu_\alpha^0(P_\alpha^0) \subset V_\alpha^0$. To finish the proof of the proposition we must define a regular map $V_\alpha^0 \to P_\alpha^0$ inverse to μ_α^0. One proceeds as explained in the introduction to this section, i.e. to $[F] \in V_\alpha^0$ we associate the couple (G, ψ) where G is the sheaf fitting into the exact sequence (6.1.2), and ψ is the map of (6.1.3). Of course we need to prove the following two results.

(6.3.19)Lemma. *Let $[F] \in V_\alpha^0$, and assume (6.1.1) is the desemistabilizing sequence of $i_\alpha^* F$. The sheaf G fitting into Exact Sequence (6.1.2) is locally-free and slope-semistable. If $\deg \xi = -1$ (where ξ is the destabilizing quotient of $i_\alpha^* F$) then G is slope-stable.*

Proof. Since (6.1.1) is the desemistabilizing sequence, both λ and ξ are locally-free: from Exact Sequence (6.1.3) we get that G is locally-free along Θ_α. On the other hand it follows from (2.1.2) and (4.3.3) that F is locally-free: since G is isomorphic to F outside Θ_α, we get that G is locally-free. Now assume that G is not slope-semistable. Let $L \hookrightarrow G$ be a slope-desemistabilizing subsheaf: since G is locally-free we can assume L is locally-free of rank one, and by Assumption (1.3) we have $c_1(L) = x\theta$ with $x \geq 1$. From (6.1.2) we have an injection $L(-\Theta_\alpha) \hookrightarrow F$; since F is slope-semistable the quotient is torsion-free, hence we get an exact sequence

$$0 \to L(-\Theta_\alpha) \to F \to I_Z \otimes L^{-1}(\Theta_\alpha) \to 0,$$

where Z is a zero-dimensional subscheme of J. Since $L(-\Theta_\alpha)$ is topologically trivial, it follows that $c_2^{hom}(F) = \ell(Z)$, and hence $\ell(Z) = 2$. Thus the above exact sequence shows that F is not (Gieseker-Maruyama) semistable, which is absurd. To finish the proof of the lemma we notice that if $\deg \xi = -1$ then $v(G) = \mathbf{w}$, hence G is slope-stable by (6.2.13). $\triangle$

(6.3.20)Lemma. *Let $[F] \in V_\alpha^0$, and assume (6.1.1) is the desemistabilizing sequence. Then $\deg \xi = -1$.*

Proof. Let $d := \deg \lambda$: we know $d \geq 1$. To prove the lemma it suffices to show that $d \leq 1$. From (6.1.2) one gets that

$$c_1^{hom}(G) = \theta, \quad c_2^{hom}(G) = 2 - d.$$

By Lemma (6.3.19) G is slope-semistable, hence by Bogomolov's theorem

$$0 \leq 4c_2^{hom}(G) - c_1^{hom}(G)^2 = 6 - 4d.$$

This gives $d \leq 1$. $\triangle$

Given the above lemmas, the argument described in the introduction to this section shows that the map

$$\begin{array}{ccc} V_\alpha^0 & \longrightarrow & P_\alpha^0 \\ [F] & \mapsto & [(G, \psi)], \end{array}$$

where $\psi \colon i_\alpha^* G \to \lambda \otimes K_C$ is the map appearing in (6.1.3), is the inverse of μ_α^0. This finishes the proof of the proposition. $\triangle$

6.4. First analysis of P_α. Let $\mathcal{Q}_\alpha$ be the scheme parametrizing isomorphism classes of couples (G, ψ), where $[G] \in \mathcal{M}_\mathbf{w}$ and ψ is a quotient appearing in an exact sequence (6.3.1), such that (6.3.3)-(6.3.4) hold, and ζ is assumed only to be a sheaf of degree 3 and rank one. In other words we have weakened Condition (6.3.2) in the definition of P_α^0, hence $P_\alpha^0 \subset \mathcal{Q}_\alpha$. Since $\mathcal{Q}_\alpha$ is projective (it is a closed subscheme of an appropriate Hilbert scheme) we get that $P_\alpha \subset \mathcal{Q}_\alpha$.

(6.4.1)Proposition. *Keeping notation as above, $\mathcal{Q}_\alpha = P_\alpha$. Furthermore the map $f: P_\alpha \to \widehat{J}$ (see (6.3.7)) is a $\mathbb{P}^1$-fibration away from $\widehat{J}[2]_{-\hat{\alpha}}$ (see (1.11)), and the remaining fibers are isomorphic to $\mathbb{P}^2$.*

Proof. The map f extends naturally to a regular map $F: \mathcal{Q}_\alpha \to \widehat{J}$ (see (6.3.7)). Let $\widehat{y} \in \widehat{J}$, and suppose $(G, \psi) \in F^{-1}(\widehat{y})$. By (6.3.8) the isomorphism class of G is determined by $\widehat{y}$. Furthermore, since G is locally-free by (6.2.1), the subsheaf ξ appearing in (6.3.1) is a line-bundle, and its isomorphism class is completely determined by Equation (6.3.4): in fact using (6.2.3) we get

$$(6.4.2) \qquad\qquad \xi \cong [-p_0 + 2\alpha + y].$$

Thus

$$F^{-1}(\widehat{y}) = \mathbb{P}H^0([p_0 - 2\alpha - y] \otimes i_\alpha^* \mathcal{G}_{\widehat{y}}^\alpha),$$
$$F^{-1}(\widehat{y}) \cap P_\alpha^0 = \{[\sigma] \in F^{-1}(\widehat{y}) \mid \sigma \text{ has nozeroes}\}.$$

Exact Sequence (6.2.6) gives

$$(6.4.3) \quad 0 \to H^0([2p_0 - 2\alpha]) \to H^0([p_0 - 2\alpha - y] \otimes i_\alpha^* \mathcal{G}_{\widehat{y}}^\alpha)$$
$$\xrightarrow{\kappa} H^0([2p_0 - 2\alpha - 2y]) \to 0.$$

(Recall that $(2p_0 - 2\alpha) \not\sim K_C$ by (3.7).) This shows that F is a $\mathbb{P}^1$-fibration away from $\widehat{J}[2]_{-\hat{\alpha}}$, and that the remaining fibers are isomorphic to $\mathbb{P}^2$. Let $\epsilon \in H^0([2p_0 - 2\alpha])$ be non-zero, and let $U \subset \widehat{J}$ be the open dense subset of points $\hat{y}$ such that there exists $\tau \in H^0([2p_0 - 2\alpha - 2y])$ whose zero locus is disjoint from (ϵ). We claim that $F^{-1}(U) \subset P_\alpha$. In fact let $\hat{y} \in U$, τ be as above, and $\tilde{\tau}$ be such that $\kappa(\tilde{\tau}) = \tau$: then a suitable linear combination of ϵ and $\tilde{\tau}$ is a section of $[p_0 - 2\alpha - y] \otimes i_\alpha^* \mathcal{G}_{\hat{y}}^\alpha$ with no zeroes. Hence P_α^0 intersects $F^{-1}(\hat{y})$; since P_α^0 is open and $F^{-1}(\hat{y})$ is irreducible we conclude that $F^{-1}(\hat{y}) \subset P_\alpha$.

Since $\widehat{J}[2]_{-\hat{\alpha}} \subset U$, and since F is a $\mathbb{P}^1$-fibration away from $\widehat{J}[2]_{-\hat{\alpha}}$, the closure of $F^{-1}(U)$ equals all of $\mathcal{Q}_\alpha$. This proves that $\mathcal{Q}_\alpha = P_\alpha$, and in particular that

$$(6.4.4) \qquad\qquad f^{-1}(\widehat{y}) = \mathbb{P}H^0([p_0 - 2\alpha - y] \otimes i_\alpha^* \mathcal{G}_{\hat{y}}^\alpha).$$

The proposition follows. $\triangle$

(6.4.5) *Remark.* Let $\pi_{\hat{j}}: C \times \widehat{J} \to \widehat{J}$ be the projection. Let $\mathcal{H}$ be a line-bundle on $C \times \widehat{J}$ such that for $\hat{y} \in \widehat{J}$ we have

$$\mathcal{H}_{\hat{y}} \cong [-p_0 + 2\alpha + y].$$

Let $\mathcal{S}^\alpha$ be the sheaf on $\widehat{J}$ defined by

$$(6.4.6) \qquad\qquad \mathcal{S}^\alpha := \pi_{\hat{j},*}(\mathcal{H}^{-1} \otimes (i_\alpha \times \mathrm{id}_{\hat{j}})^* \mathcal{G}^\alpha).$$

In the proof above we have shown that over $(\widehat{J} \setminus \widehat{J}[2]_{-\hat{\alpha}})$ the sheaf $\mathcal{S}^\alpha$ is locally-free of rank two (in fact this is true over all of $\widehat{J}$, see (6.6.1)), and that $f^{-1}(\widehat{J} \setminus \widehat{J}[2]_{-\hat{\alpha}})$ is naturally isomorphic to $\mathbb{P}(\mathcal{S}^\alpha|_{(\widehat{J} \setminus \widehat{J}[2]_{-\hat{\alpha}})})$.

6.5. The boundary of P_α. We analyze $(P_\alpha \setminus P_\alpha^0)$. First notice that by (6.3.18)

$$(6.5.1) \qquad\qquad P_\alpha \setminus P_\alpha^0 = \mu_\alpha^{-1}(B_\alpha \cup \Sigma_\alpha) = \mu_\alpha^{-1}(B \cup \Sigma).$$

For $t = (G, \psi) \in P_\alpha$ let ζ_t be the rank-one sheaf ζ appearing in Exact Sequence (6.3.1) and $Tors(\zeta_t)$ be its torsion subsheaf. Then

$$(6.5.2) \qquad\qquad \ell(Tors(\zeta_t)) \leq 2.$$

In fact since $i_\alpha^* G$ is semistable and we have a surjection

$$i_\alpha^* G \to \zeta_t / Tors(\zeta_t),$$

the line bundle on the right has degree at least 1. This implies immediately (6.5.2). Let

$$W_\alpha := \{t \in P_\alpha \mid \ell(Tors(\zeta_t)) = 2\}, \qquad Y_\alpha^0 := \{t \in P_\alpha \mid \ell(Tors(\zeta_t)) = 1\}.$$

By (6.5.2) W_α is closed. Clearly Y_α^0 is locally closed; we let Y_α be its closure. By (6.3.18) and (6.5.2) we have

$$(6.5.3) \qquad\qquad P_\alpha \setminus P_\alpha^0 = Y_\alpha \cup W_\alpha.$$

(6.5.4)Proposition. *Keep notation as above.*
  (1) *The restriction of f to W_α is identified with the blow-up of $\widehat{J}$ at $\widehat{J}[2]$.*
  (2) *Let $t \in W_\alpha$. Then $\mathcal{F}_t^\alpha$ is singular, with*

$$(6.5.5) \qquad\qquad Sing(\mathcal{F}_t^\alpha) = \{i_\alpha(q_1), i_\alpha(q_2)\},$$

  where q_1, q_2 are given by (4.2.2).
  (3) *If t is generic, $\mathcal{F}_t^\alpha$ is stable. Furthermore $W_\alpha = \mu_\alpha^{-1}(B_\alpha) = \mu_\alpha^{-1}(B)$.*

Proof. Item (1). Let $t = (G, \psi) \in W_\alpha$, and let ζ be the rank-one sheaf appearing in (6.3.1). Then

$$i_\alpha^* G \xrightarrow{\psi} \zeta / Tors(\zeta)$$

is a destabilizing quotient, because $\deg(\zeta / Tors(\zeta)) = 1$. Conversely, assume

$$(6.5.6) \qquad\qquad 0 \to \overline{\xi} \to i_\alpha^* G \to \overline{\zeta} \to 0$$

is destabilizing. Thus $\overline{\xi}, \overline{\zeta}$ are line-bundles of degree 1, because $i_\alpha^* G$ is semistable by (6.2.5). Let ξ be a line-bundle fitting into an exact sequence

$$(6.5.7) \qquad\qquad 0 \to \xi \to \overline{\xi} \to \mathcal{O}_Z \to 0,$$

where Z is a zero-dimensional subscheme of C of length 2 such that

$$(6.5.8) \qquad\qquad \alpha + y - \sum i_{\alpha,*} c_1(\overline{\xi}) + \sum i_{\alpha,*}(Z) = 0,$$

where $\hat{y} = f(t)$. Composing $\xi \to \overline{\xi}$ with $\overline{\xi} \to i_\alpha^* G$ we get an injection $\xi \hookrightarrow i_\alpha^* G$. Let $\zeta := i_\alpha^* G / \xi$. The exact sequence

$$0 \to \xi \longrightarrow i_\alpha^* G \xrightarrow{\psi} \zeta \to 0,$$

satisfies (6.3.3)-(6.3.4), and (6.3.2) except that ζ might be of rank one without being locally-free, i.e. $(G, \psi) \in P_\alpha$. Furthermore, since $\zeta \cong \mathcal{O}_Z \oplus \overline{\zeta}$ we have $(G, \psi) \in W_\alpha$. Thus we have defined a one-to-one correspondence between W_α and the set of triples consisting of $[G] \in \mathcal{M}_\mathbf{w}$ (satisfying (6.3.3)), Exact Sequence (6.5.6) and Exact Sequence (6.5.7) (with Z satisfiyng (6.5.8)). We claim that given (6.5.6) there is one and only one Z such that (6.5.8) holds. In fact from (6.3.8) and (6.2.6) we see that $\overline{\xi} \cong [p_0 + y]$, hence (6.5.8) reads

(6.5.9) $$\sum i_{\alpha,*}(Z) = 0.$$

The map

$$\begin{array}{ccc} C^{(2)} & \longrightarrow & J \\ Z & \mapsto & \sum i_{\alpha,*}(Z) \end{array}$$

is, up to translation, the Abel-Jacobi map, hence it is surjective. Furthermore the fiber over $\sum i_{\alpha,*}(Z)$ is the single cycle Z unless $Z \in |K_C|$. Since $i_{\alpha,*}(K_C) = 2\alpha$, and $2\alpha \neq 0$ by (3.7), we get that Z is uniquely determined by $\overline{\xi}$, as claimed. Thus we have defined a one-to-one correspondence between W_α and the relative Quot-scheme D_α over $\widehat{J}$ parametrizing couples consisting of $[G] \in \mathcal{M}_\mathbf{w}$ (satisfying (6.3.3)) and a destabilizing sequence of $i_\alpha^* G$. In fact this correspondence is functorial, hence it gives an isomorphism

(6.5.10) $$\begin{array}{ccc} W_\alpha & \xrightarrow{\sim} & D_\alpha \\ (G, \psi) & \mapsto & (i_\alpha^* G \to \zeta / Tors(\zeta)). \end{array}$$

By (6.2.5) we get that

$$f^{-1}(\hat{y}) \cap W_\alpha \cong \begin{cases} \text{a single point} & \text{if } 2y \neq 0, \\ \mathbb{P}^1 & \text{if } 2y = 0. \end{cases}$$

Let $\nu_0 \colon \widehat{I}_0 \to \widehat{J}$ be the blow up of $\overline{J}[2]$. To finish the proof of Item (1) it suffices to define a map $\rho \colon \widehat{I}_0 \to W_\alpha$ such that

(6.5.11) $$f \circ \rho = \nu_0$$

(6.5.12) $\rho|_{\nu_0^{-1}(\hat{y}_i)} \colon \nu_0^{-1}(\hat{y}_i) \to f^{-1}(\hat{y}_i)$ is an isomorphism, for $\hat{y}_i \in \widehat{J}[2]$.

In fact it follows immediately from (6.5.11)-(6.5.12) that ρ is bijective, and that its differential is an isomorphism everywhere, hence ρ is an isomorphism. Let $\mathcal{G}^\alpha$ be the tautological vector-bundle on $J \times \widehat{J}$ defined in (6.3.5). By Isomorphism (6.5.10), in order to define ρ it suffices to exhibit a line-bundle $\mathcal{H}$ on $\widehat{I}_0 \times C$ and an injection

$$\mathcal{H} \hookrightarrow (\nu \times i_\alpha)^* \mathcal{G}^\alpha$$

which restricts to a destabilizing subline-bundle of $i_\alpha^* \mathcal{G}_{\nu(u)}^\alpha$ for every $u \in \widehat{I}_0$. Since $\mathcal{G}^\alpha = (\mathcal{E}^\alpha)^*$ (see (6.3.5)) it is equivalent to give an injection $\mathcal{D} \hookrightarrow (\nu \times i_\alpha)^* \mathcal{E}^\alpha$ which restricts to a destabilizing subline-bundle of $i_\alpha^* \mathcal{E}_{\nu(u)}^\alpha$ for every $u \in \widehat{I}_0$. Let

$$\tau, \pi \colon \widehat{I}_0 \times C \times C \to \widehat{I}_0 \times C$$

be the projections which "forget" the third and second factor respectively. By (6.2.8) we have an isomorphism (up to tensoring by a line-bundle on $\widehat{I}_0$)

(6.5.13) $$(\nu_0 \times i_\alpha)^* \mathcal{E}^\alpha \cong \mathcal{H}^1 \otimes \tau_*([\widehat{I}_0 \times \Delta] \otimes \pi^* \mathcal{H}^2),$$

where $\Delta \subset C \times C$ is the diagonal, and $\mathcal{H}^1$, $\mathcal{H}^2$ are line-bundles on $\widehat{I}_0 \times C$ such that for $u \in \widehat{I}_0$ (with $\nu_0(u) = \hat{y}$)

$$\mathcal{H}_u^1 \cong [-p_0 + y], \quad \mathcal{H}_u^2 \cong [2p_0 - 2y].$$

Arguing as in the proof of (6.2.9) we get an exact sequence

$$0 \to \tau_*(\pi^* \mathcal{H}^2) \xrightarrow{g} \tau_*([\widehat{I}_0 \times \Delta] \otimes \pi^* \mathcal{H}^2) \to \mathcal{H}^3 \to \mathcal{O}_{E \times C} \to 0,$$

where $\mathcal{H}^3$ is a line-bundle on $\widehat{I}_0 \times C$ such that $\mathcal{H}_u^3 \cong [2y]$ for $u \in \widehat{I}_0$, and $E \subset \widehat{I}_0$ is the exceptional divisor of ν_0. Since $\tau_*(\pi^* \mathcal{H}^2)$ is a line-bundle, and the map g vanishes to first order along $E \times C$, we get a map

$$\tau_*(\pi^* \mathcal{H}^2) \otimes [E \times C] \to \tau_*([\widehat{I}_0 \times \Delta] \otimes \pi^* \mathcal{H}^2)$$

which is non-zero at every point. Thus by (6.5.13) we get a map

$$\mathcal{D} := \mathcal{H}^1 \otimes \tau_*(\pi^* \mathcal{H}^2) \otimes [E \times C] \xrightarrow{\gamma} (\nu_0 \times i_\alpha)^* \mathcal{E}^\alpha,$$

injective at every point. The restriction of γ to $\{u\} \times C$ is a destabilizing sub line-bundle of $i_\alpha^* \mathcal{E}_{\nu_0(u)}^\alpha$ for every $u \in \widehat{I}_0$. As explained above this defines a map $\rho \colon \widehat{I}_0 \to W_\alpha$. One verifies easily that (6.5.11)-(6.5.12) hold.

Item (2). Let $G = \mathcal{G}_{f(t)}^\alpha$ and $F = \mathcal{F}_t^\alpha$. Exact Sequence (6.3.17) shows that $Sing(F) = i_\alpha(Z)$, and by (6.5.9) we get (6.5.5). Assume F is not stable, and thus by (6.3.11) it is strictly semistable. Let $I_q \otimes L \hookrightarrow F$ be a destabilizing subsheaf. Thus by (2.1.2) $[L] \in \widehat{J}$, and $q \in Sing(F)$. By (6.5.5) we have $q = i_\alpha(q_i)$ for $i = 1$ or $i = 2$.

Item (3). The key result is the following.

(6.5.14)**Claim.** *Keep notation as above. Let $t \in W_\alpha$, and set $\hat{y} := f(t)$. There is an injection*

(6.5.15) $$I_{i_\alpha(q_i)} \otimes L \hookrightarrow \mathcal{F}_t^\alpha$$

if and only if

(6.5.16) $$2y \sim q_{3-i} - r \quad \text{for some } r \in C.$$

Furthermore, if y satisfying (6.5.16) is given, there exists a unique $t \in f^{-1}(\hat{y})$ such that an injection (6.5.15) exists, and this injection is unique up to scalars.

Proof of the claim. Assume (6.5.15) exists: composing with the map j of (6.3.9) we get an injection $h \colon I_{i_\alpha(q_i)} \otimes L \hookrightarrow G$, which comes from a (non-zero) map $\tilde{h} \colon L \to G$, because G is locally-free. Notice that by slope-stability of G and a Chern class computation the map $\tilde{h}$ has a single zero, which is simple, hence we get an exact sequence

$$(6.5.17) \qquad 0 \to L \xrightarrow{\tilde{h}} G \longrightarrow I_p \otimes \Theta_\alpha \otimes L^{-1} \to 0.$$

Since $t \in W_\alpha$, the inclusion $\xi \hookrightarrow i_\alpha^* G$ is the composition

$$\mathcal{O}_C(p_0 + y)(-q_1 - q_2) \hookrightarrow \mathcal{O}_C(p_0 + y) \hookrightarrow i_\alpha^* G,$$

where the second inclusion is a destabilizing sub-line-bundle $i_\alpha^* G$. The restriction to Θ_α of h has image contained in ξ, and hence the restriction to Θ_α of $\tilde{h}$ has image contained in the destabilizing sub-line-bundle of $i_\alpha^* G$, and $\tilde{h}$ must vanish at $i_\alpha(q_{3-i})$. Hence the point p of (6.5.17) is equal to $i_\alpha(q_{3-i})$, and by (6.2.3) we get the equation

$$\alpha + y = \sum c_2(G) = i_{\alpha,*}(q_{3-i} + i_\alpha^* L),$$

which gives

$$(6.5.18) \qquad i_\alpha^* L \cong [p_0 + y - q_{3-i}].$$

There is a further constraint coming from the fact that $h^0(L^{-1} \otimes G) > 0$. Applying the $Hom(\bullet, \mathcal{O}_J)$-functor to (6.2.10) we get the exact sequence

$$0 \to \mathcal{L}_{\hat{D}}^* \to E(x, \hat{y})^* \to i_{-x,*}[p_0 - y] \to 0.$$

(We have used (6.2.11).) Thus we have an exact sequence

$$0 \to H^0(L^{-1} \otimes \mathcal{L}_{\hat{D}}^*) \to H^0(L^{-1} \otimes E(x, \hat{y})^*)$$
$$\to H^0(i_{-x}^* L^{-1} \otimes [p_0 - y]) \to H^1(L^{-1} \otimes \mathcal{L}_{\hat{D}}^*).$$

From this we get that $h^0(L^{-1} \otimes E(x, \hat{y})^*) > 0$ if and only if

$$(6.5.19) \qquad [p_0 - y - r] \cong i_{-x}^* L \text{ for some } r \in C.$$

Since $i_{-x}^* L \cong i_\alpha^* L$, the above isomorphism together with (6.5.18) gives an equation for y which turns out to be (6.5.16). Now let's prove the viceversa. If (6.5.16) is satisfied then the argument just given shows that there exists a map $\tilde{h} \colon L \to G$, which has a single zero at $i_\alpha(q_{3-i})$. Since $\tilde{h}$ vanishes at $i_\alpha(q_{3-i})$, the restriction of $\tilde{h}$ to Θ_α must be contained in a destabilizing sub-line-bundle of $i_\alpha^* G$. This destabilizing sub-line-bundle determines a point $t \in f^{-1}(y) \cap W_\alpha$ such that $\tilde{h}$ "comes" from an injection $I_{i_\alpha(q_i)} \otimes L \hookrightarrow \mathcal{F}_t^\alpha$. This argument also shows that t is unique. Injection (6.5.15) is unique because otherwise we would have $\mathcal{F}_t^\alpha \cong I_{i_\alpha(q_i)} \otimes L \oplus I_{i_\alpha(q_i)} \otimes L$, which is impossible by (3.6). $\triangle$

Let us prove the first statement of Item (3). By the above claim

$$f(W_\alpha \cap \mu_\alpha^{-1}(\Sigma)) = \bigcup_{i=1,2} \{\hat{y} \in \widehat{J} |\ 2y \sim q_{3-i} - r, \quad \text{some } r \in C\}.$$

Since $f(W_\alpha) = \widehat{J}$ we get that $\mathcal{F}_t^\alpha$ is stable for t generic. Let us prove the second statement of Item (3). For $t \in P_\alpha$ the sheaf $F := \mathcal{F}_t^\alpha$ is locally-free outside Θ_α, hence $\mu_\alpha^{-1}(B) = \mu_\alpha^{-1}(B_\alpha)$. Let $t \in \mu_\alpha^{-1}(B)$; by (4.3.3) we have $\ell(F^{**}/F) = 2$, hence (6.3.17) gives that $\ell(Tors(\zeta_t)) = 2$, i.e. $t \in W_\alpha$. This proves that $\mu_\alpha^{-1}(B) \subset W_\alpha$. By Item(2) an open dense subset of W_α is contained in $\mu_\alpha^{-1}(B)$. Since $\mu_\alpha^{-1}(B)$ is closed we get that $W_\alpha \subset \mu_\alpha^{-1}(B)$. This finishes the proof of Item (3). $\triangle$

We finish this subsection by observing that Item (3) of the above proposition together with (6.5.1)-(6.5.3) gives that

$$(6.5.20) \qquad\qquad Y_\alpha = \mu_\alpha^{-1}(\Sigma_\alpha) = \mu_\alpha^{-1}(\Sigma).$$

6.6. A $\mathbb{P}^1$-bundle mapping to P_α. We recall that $f: P_\alpha \to \widehat{J}$ is the natural projection (see (6.3.7)). Remark (6.4.5) identifies $f^{-1}(\widehat{J} \setminus \widehat{J}[2]_{-\hat{\alpha}})$ with an explicit $\mathbb{P}^1$-bundle. Of course by (6.4.1) it is not true that P_α is a $\mathbb{P}^1$-bundle over $\widehat{J}$; notice also that by (6.4.1) there is at least one singular point of P_α lying over each point of $\widehat{J}[2]_{-\hat{\alpha}}$. In this subsection we will construct a $\mathbb{P}^1$-bundle over the blow-up of $\widehat{J}$ at $\widehat{J}[2]_{-\hat{\alpha}}$ which has a birational regular map to P_α, contracting sixteen $(-1, -1)$ curves. Then we will prove that P_α is isomorphic to the contraction of these curves.

(6.6.1)Claim. *The sheaf $\mathcal{S}^\alpha$ given by (6.4.6) is locally-free of rank two.*

Proof. It has been shown in the proof of (6.4.1) that $\mathcal{S}^\alpha$ is locally-free outside $\widehat{J}[2]_{-\hat{\alpha}}$. Let $U := (\widehat{J} \setminus \widehat{J}[2])$; by Assumption (3.7) $\widehat{J}[2]_{-\hat{\alpha}} \subset U$, hence it suffices to prove that $\mathcal{S}^\alpha$ is locally-free of rank two on U. By (6.2.5) there is an exact sequence

$$0 \to \mathcal{L}^+ \to (i_\alpha \times \mathrm{id}_{\hat{j}})^* \mathcal{G}^\alpha|_{C \times U} \to \mathcal{L}^- \to 0,$$

where $\mathcal{L}^+$, $\mathcal{L}^-$ are line-bundles on $C \times U$ such that

$$\mathcal{L}_{\hat{y}}^+ \cong [p_0 + y], \quad \mathcal{L}_{\hat{y}}^- \cong [p_0 - y].$$

Letting $\pi_U: C \times U \to U$ be the projection, and $\mathcal{H}$ be the line-bundle of (6.4.5), we get an exact sequence

$$0 \to \pi_{U,*}(\mathcal{H}^{-1} \otimes \mathcal{L}^+) \to \mathcal{S}^\alpha|_U \to \pi_{U,*}(\mathcal{H}^{-1} \otimes \mathcal{L}^-) \to 0.$$

Since $h^0(\mathcal{H}_{\hat{y}}^{-1} \otimes \mathcal{L}_{\hat{y}}^+) = 1$ for all $\hat{y} \in \widehat{J}$, the sheaf on the left of $\mathcal{S}^\alpha|_U$ is locally-free of rank one. We claim that also the sheaf on the right is locally-free of rank one, although $h^0(\mathcal{H}_{\hat{y}}^{-1} \otimes \mathcal{L}_{\hat{y}}^-)$ jumps: it equals 1 for $\hat{y} \notin \widehat{J}[2]_{-\hat{\alpha}}$ and is 2 for $\hat{y} \in \widehat{J}[2]_{-\hat{\alpha}}$. It is clear that $\pi_{U,*}(\mathcal{H}^{-1} \otimes \mathcal{L}^-)$ is locally-free of rank 1 away from $\widehat{J}[2]_{-\hat{\alpha}}$, we must show the same holds near any point $\hat{y} \in \widehat{J}_{-\hat{\alpha}}$. Applying the $\pi_{U,*}$-functor to the exact sequence

$$0 \to \mathcal{H}^{-1} \otimes \mathcal{L}^- \to \mathcal{H}^{-1} \otimes \mathcal{L}^-(q) \to \mathcal{H}^{-1} \otimes \mathcal{L}^-(q)|_{C \times q} \to 0,$$

($q \in C$ is arbitrary) we get that locally near $\hat{y}$ we have an exact sequence

$$(6.6.2) \qquad \pi_{U,*}(\mathcal{H}^{-1} \otimes \mathcal{L}^{-}) \xrightarrow{h} \mathcal{O}_U^2 \to \mathcal{O}_U \to Q \to 0,$$

where Q is a sheaf supported on $\hat{y}$. It follows immediately that $\pi_{U,*}(\mathcal{H}^{-1} \otimes \mathcal{L}^{-})$ is locally-free of rank 1 near $\hat{y}$. $\triangle$

Let $R_\alpha := \mathbb{P}(\mathcal{S}^\alpha)$, and $g \colon R_\alpha \to \widehat{J}$ be the natural $\mathbb{P}^1$-fibration. Let $\eta \hookrightarrow g^*\mathcal{S}^\alpha$ be the tautological sub-line-bundle. The corresponding section of $\eta^{-1} \otimes g^*(\mathcal{S}^\alpha)$ defines

$$\Phi \in H^0(C \times R_\alpha, \pi_{R_\alpha}^*(\eta^{-1}) \otimes (\mathrm{id}_C \times g)^*\mathcal{H}^{-1} \otimes (i_\alpha \times g)^*(\mathcal{G}^\alpha)),$$

where $\pi_{R_\alpha} \colon C \times R_\alpha \to R_\alpha$ is the projection. We view Φ as an injective map fitting into an exact sequence

$$(6.6.3) \qquad 0 \to \pi_{R_\alpha}^*(\eta) \otimes (\mathrm{id}_C \times g)^*\mathcal{H} \xrightarrow{\Phi} (i_\alpha \times g)^*(\mathcal{G}^\alpha) \longrightarrow \mathcal{Q} \to 0.$$

If $\hat{y} \notin \widehat{J}[2]_{-\hat{\alpha}}$ then for all $t \in g^{-1}\hat{y}$ the restriction of (6.6.3) to $C \times \{t\}$ is an exact sequence of the kind (6.3.1), and hence (6.6.3) induces a map $(R_\alpha \setminus g^{-1}(\widehat{J}[2]_{-\hat{\alpha}})) \to (P_\alpha \setminus f^{-1}(\widehat{J}[2]_{-\hat{\alpha}}))$: this is the isomorphism of (6.4.5). Let ν_α and E_k be as in (1.12) and the subsequent definition: we set

$$(6.6.4) \qquad \hat{y}_k := \nu_\alpha(E_k), \qquad k = 1, \ldots, 16.$$

Thus $\widehat{J}[2]_{-\hat{\alpha}} = \{\hat{y}_1, \ldots, \hat{y}_{16}\}$.

(6.6.5)Claim. *Keep notation as above. There exist $r_1, \ldots, r_{16} \in R_\alpha$, with $g(r_k) = \hat{y}_k$, such that the following holds. The restriction of (6.6.3) to $C \times \{t\}$ is an exact sequence of the kind (6.3.1) for all $t \neq r_1, \ldots, r_{16}$.*

Proof. Let σ^+, σ^- be local generators near $\hat{y}_k$ of the invertible sheaves $\pi_{U,*}(\mathcal{H}^{-1} \otimes \mathcal{L}^+)$ and $\pi_{U,*}(\mathcal{H}^{-1} \otimes \mathcal{L}^{-})$, respectively. Then σ^+ defines a section of $\mathcal{H}^{-1} \otimes \mathcal{L}^+$ (in a neighborhood of $C \times \hat{y}_k$) which restricted to $C \times \{\hat{y}_k\}$ generates $H^0(\mathcal{H}^{-1} \otimes \mathcal{L}^+|_{C \times \{\hat{y}_k\}})$. On the other hand, since the map h of (6.6.2) vanishes at $\hat{y}_k$, the local section of $\mathcal{H}^{-1} \otimes \mathcal{L}^{-}$ defined by σ^- vanishes identically on $C \times \{\hat{y}_k\}$. Hence the Claim holds with $r_k := [\sigma^-]$. $\triangle$

Thus (6.6.3) defines a rational map $R_\alpha \cdots > P_\alpha$ regular outside $\{r_1, \ldots, r_{16}\}$. Let us construct a birational modification of R_α which has a regular map to P_α. The inclusion

$$r_k \times E_k \hookrightarrow \mathbb{P}(\mathcal{S}^\alpha_{\hat{y}_k}) \times E_k = \mathbb{P}(g^*\mathcal{S}^\alpha|_{E_k})$$

gives rise to an exact sequence

$$(6.6.6) \qquad 0 \to \mathcal{O}_{E_k} \longrightarrow \nu_\alpha^*\mathcal{S}^\alpha|_{E_k} \xrightarrow{\rho_k} \mathcal{O}_{E_k} \to 0,$$

where $\mathbb{P}(\ker \rho_k) = r_k \times E_k$. Let $\mathcal{T}^\alpha$ be the rank-two locally-free sheaf on $\widehat{I}_\alpha$ fitting into the exact sequence

$$(6.6.7) \qquad 0 \to \mathcal{T}^\alpha \longrightarrow \nu_\alpha^*\mathcal{S}^\alpha \xrightarrow{\rho} \bigoplus_{k=1}^{16} i_{E_k,*}\mathcal{O}_{E_k} \to 0,$$

where ρ is defined by the ρ_k's, and $i_{E_k}\colon E_k \hookrightarrow \widehat{I}_\alpha$ is the inclusion. We set

$$(6.6.8) \qquad\qquad P_\alpha^- := \mathbb{P}(\mathcal{T}^\alpha).$$

Let us show that P_α^- naturally parametrizes a family of quotients of the kind (6.3.1). Let $f_-\colon P_\alpha^- \to \widehat{I}_\alpha$ be the natural $\mathbb{P}^1$-fibration and $\eta \hookrightarrow f_-^* \mathcal{T}^\alpha$ be the tautological sub-line-bundle. By (6.6.7) we have a map $\eta \to f_-^* \nu_\alpha^*(\mathcal{S}^\alpha)$. The corresponding section of $\eta^{-1} \otimes f_-^* \nu_\alpha^*(\mathcal{S}^\alpha)$ defines a section

$$\Phi^- \in H^0(C \times P_\alpha^-, \pi_{P_\alpha^-}^*(\eta^{-1}) \otimes (\mathrm{id}_C \times (\nu_\alpha \circ f_-))^* \mathcal{H}^{-1} \otimes (i_\alpha \times (\nu_\alpha \circ f_-))^*(\mathcal{G}^\alpha)),$$

where $\pi_{P_\alpha^-}\colon C \times P_\alpha^- \to P_\alpha^-$ is the projection. Let $E_k^- := f_-^{-1}(E_k)$, and $E^- := f_-^{-1}(E)$. The map Φ^- vanishes to order one along $C \times E^-$, hence it gives an exact sequence
(6.6.9)

$$0 \to \pi_{P_\alpha^-}^*(\eta \otimes [E^-]) \otimes (\mathrm{id}_C \times (\nu_\alpha \circ f_-))^* \mathcal{H} \xrightarrow{\Phi^0} (i_\alpha \times (\nu_\alpha \circ f_-))^*(\mathcal{G}^\alpha) \xrightarrow{\Psi} \mathcal{R} \to 0.$$

As is easily checked the restriction of (6.6.9) to $C \times \{u\}$ is an exact sequence of type (6.3.1) for all $u \in P_\alpha^-$. Thus (6.6.9) induces a regular map $\delta\colon P_\alpha^- \to P_\alpha$ such that $f \circ \delta = \nu_\alpha \circ f_-$. An explicit description of δ is as follows. First, $(P_\alpha^- \setminus E^-)$ is canonically isomorphic to $\mathbb{P}(\mathcal{S}^\alpha|_{(\widehat{J} \setminus \widehat{J}[2]_{-\hat{\alpha}})})$, and the latter space is isomorphic to $f^{-1}(\widehat{J} \setminus \widehat{J}[2]_{-\alpha})$ by (6.4.5); the restriction of δ to $(P_\alpha^- \setminus E^-)$ is the composition of these two isomorphisms. It remains to describe δ on E_k^-, for $k = 1, \dots, 16$. Let

$$x \in E_k = \mathbb{P}(H^1(\mathcal{O}_C)) = \mathbb{P}(H^0(K_C))^* = \mathbb{P}(H^0(K_C)),$$

and let $\mathbf{C}\sigma \in \mathbb{P}(H^0(K_C))$ correspond to x under the composition of the above isomorphisms. One easily checks that the restriction of δ to $f_-^{-1}(x)$ is an isomorphism to $\mathbb{P}(\kappa^{-1}(\mathbf{C}\sigma))$, where κ is the map in (6.4.3), and $\mathbb{P}(\kappa^{-1}(\mathbf{C}\sigma)) \subset f^{-1}(\hat{y}_k)$ by (6.4.4). In particular letting $s_k \in f^{-1}(\hat{y}_k)$ be the point corresponding to $\mathbf{C} \ker(\kappa)$, we get a section $\Gamma_k^- := \delta^{-1}(s_k)$ of $f_-|_{E_k^-}$ which is contracted by δ. The curve Γ_k^- corresponds to the natural exact sequence

$$0 \to \mathcal{O}_{E_k}(-E_k) \to \mathcal{T}^\alpha|_{E_k} \to \mathcal{O}_{E_k} \to 0,$$

one gets from (6.6.6) and (6.6.7), i.e. $\Gamma_k^- := \mathbb{P}(\mathcal{O}_{E_k}(-E_k))$. From the above exact sequence we see that E_k^- is isomorphic to $\mathbb{F}_1$ and that the normal bundle of Γ_k^- in E_k^- is isomorphic to $\mathcal{O}_{\Gamma_k^-}(-1)$. Since we also have isomorphisms

$$\mathcal{O}_{\Gamma_k^-}(E_k^-) \overset{df_-}{\cong} \mathcal{O}_{\Gamma_k^-}(f_-^* E_k) \cong \mathcal{O}_{\Gamma_k^-}(-1),$$

we get that

$$(6.6.10) \qquad N_{\Gamma_k^-/P_\alpha^-} \cong \mathcal{O}_{\Gamma_k^-}(-1) \oplus \mathcal{O}_{\Gamma_k^-}(-1), \quad k = 1, \dots, 16$$

i.e. Γ_k^- is a $(-1,-1)$-curve. Let $Contr(P_\alpha^-)$ be the contraction of the sixteen curves Γ_k^-, where $k = 1, \dots, 16$. The map δ induces a regular bijective map $\delta'\colon Contr(P_\alpha^-) \to P_\alpha$.

(6.6.11)Proposition. *Keep notation as above. The map δ' is an isomorphism.*

Proof. It suffices to prove that P_α is normal. One checks easily that outside $\bigcup_{k=1}^{16} \Gamma_k^-$ the map δ is injective with injective differential, hence P_α is smooth outside $\{s_1, \dots, s_{16}\}$. On the other hand

$$\dim T_{s_k} P_\alpha = 4, \quad k = 1, \dots, 16,$$

because of the exact sequence (see (6.4.1))

$$0 \to T_{s_k} \mathbb{P}^2 \longrightarrow T_{s_k} P_\alpha \xrightarrow{df(s_k)} T_{\hat{y}_k} \widehat{J} \to 0.$$

Thus a neighborhood of s_k is a 3-dimensional hypersurface with an isolated singularity at s_k, hence it is normal. $\triangle$

6.7. A modification of P_α^- isomorphic to $\widetilde{V}_\alpha$. By (6.6.10) each Γ_k^- is a $(-1,-1)$-curve, hence the variety obtained by flopping P_α^- at each of the Γ_k^- is a smooth complex manifold, which we denote by P_α^+. Explicitely, let $\beta_-: \overline{P}_\alpha \to P_\alpha^-$ be the blow-up of Γ^-, let $\overline{\Gamma}$ be the exceptional divisor, and $\overline{\Gamma}_k \subset \overline{\Gamma}$ be the component mapping to Γ_k^-. By (6.6.10) each $\overline{\Gamma}_k$ is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{O}_{\overline{P}_\alpha}(\overline{\Gamma}_k)$ has degree (-1) on the curves of any of its two rulings. Let $\beta_+: \overline{P}_\alpha \to P_\alpha^+$ be the contraction of the $\mathbb{P}^1$'s of $\overline{\Gamma}_k$ (for $k = 1, \dots, 16$) belonging to the ruling opposite to that which is contracted by β_-. The regular map $\mu_\alpha^- := \mu_\alpha \circ \delta$ defines a rational map $\epsilon_\alpha^-: P_\alpha^- \cdots > \widetilde{V}_\alpha$. Since P_α^+ is bimeromorphic to P_α^-, we get a meromorphic map $\epsilon_\alpha^+: P_\alpha^+ \cdots > \widetilde{V}_\alpha$. The main result of this subsection is the following.

(6.7.1)Proposition. *The map ϵ_α^+ is regular and it defines an isomorphism*

$$\epsilon_\alpha^+: P_\alpha^+ \xrightarrow{\sim} \widetilde{V}_\alpha.$$

The proof goes as follows. Let $\mathcal{U}^\alpha := (\mathrm{id}_J \times \delta)^* \mathcal{F}^\alpha$: this is a family of sheaves on J parametrized by P_α^-. Let $\overline{\mathcal{U}}^\alpha := (\mathrm{id}_J \times \beta_-)^* \mathcal{U}^\alpha$. We will show (6.7.2) that the sheaf $\overline{\mathcal{U}}^\alpha$ is simple if and only if $t \notin \overline{\Gamma}$. Starting from $\overline{\mathcal{U}}^\alpha$ we will perform two elementary modifications in order to get simple sheaves. More precisely, the first elementary modification will be a family $\mathcal{V}^\alpha$ of sheaves on J parametrized by $\overline{P}_\alpha$, which coincides with $\overline{\mathcal{U}}^\alpha$ outside $\overline{\Gamma}$, and such that $\mathcal{V}_t^\alpha$ is semistable and simple for all t not belonging to certain fibers of β_-, one for each $\overline{\Gamma}_k$ and denoted by $Z_k(1)$. The second elementary modification is a family $\mathcal{W}^\alpha(\mathcal{N})$ of sheaves on J parametrized by an open neighborhood $\mathcal{N}$ of $\bigcup_{k=1}^{16} Z_k(1)$. The sheaf $\mathcal{W}^\alpha(\mathcal{N})_t$ is semistable and simple for all $t \in \mathcal{N}$, and $\widetilde{S}$-equivalent (see (2.2.2)) to $\mathcal{V}_t^\alpha$ if $t \notin \bigcup_{k=1}^{16} Z_k(1)$. By (2.3.7) the sheaf $\mathcal{V}^\alpha$ defines a regular map $\left(\overline{P}_\alpha \setminus \bigcup_{k=1}^{16} Z_k(1)\right) \to \widetilde{\mathcal{M}}$, and $\mathcal{W}^\alpha(\mathcal{N})$ defines a regular map $\mathcal{N} \to \widetilde{\mathcal{M}}$. By the $\widetilde{S}$-equivalence property stated above these two maps glue together and define a regular map $\bar{\epsilon}_\alpha: \overline{P}_\alpha \to \widetilde{V}_\alpha$. We will verify that $\bar{\epsilon}_\alpha$ descends to a regular map $\epsilon_\alpha^+: P_\alpha^+ \to \widetilde{V}_\alpha$, and finally we will prove that ϵ_α^+ is an isomorphism.

An equivalent description of P_α^+ is the following. By (6.6.11) the space P_α is smooth except at the points $s_1, \ldots, s_{16}$, where it has quadratic singularities. Furthermore, by (6.7.25) the divisor Y_α is not Cartier at each of the s_i, in fact its tangent cone consists of two planes intersecting only at s_i. Hence the blow up of P_α along Y_α is a smooth resolution of P_α: it is isomorphic to P_α^+.

I. The map μ_α^- lifts to $\widetilde{V}_\alpha$ outside Γ^-. This will be an immediate consequence of the following result.

(6.7.2)Proposition. *Keep notation as above. Let $t \in P_\alpha^-$, and set $u := \delta(t)$, $F := \mathcal{F}_u^\alpha$. Then F is simple if and only if $t \notin \Gamma^-$. If $t \in \Gamma_k^-$ then*

$$(6.7.3) \qquad F \cong I_{i_\alpha(q_1)} \otimes L_k(1) \oplus I_{i_\alpha(q_2)} \otimes L_k(2),$$

where $[L_k(i)] = (\hat{y}_k - i_{\hat{0}}(q_{3-i}))$.

Proof. Since $\delta^{-1}(s_k) = \Gamma_k^-$, we must show that F is simple if and only if $u \notin \{s_1, \ldots, s_{16}\}$, and that if $u = s_k$ then (6.7.3) holds. Assume F is not simple. Then F is properly semistable, hence (6.7.3) holds for some $L_k(1), L_k(2) \in \widehat{J}$, and by (6.5.20) we have $u \in Y_\alpha$. Furthermore since F is singular at two points, $u \in W_\alpha$. Thus $u \in Y_\alpha \cap W_\alpha$. Let $\hat{y} := f(u)$; by Claim (6.5.14) we get that

$$2y \in \{q_2 - r \mid r \in C\} \cap \{q_1 - s \mid s \in C\}.$$

The intersection above equals $\widehat{J}[2] \cup \widehat{J}[2]_{\hat{\alpha}}$. According to (6.5.14) there is unique point $u_i \in f^{-1}(\hat{y})$ such that there exists an injection $I_{i_\alpha(q_i)} \otimes L_k(i) \hookrightarrow F$ (for $i = 1, 2$); we must show that $u_1 = u_2$, if and only if $\hat{y} \in \widehat{J}[2]_{-\hat{\alpha}}$. If $u_1 = u_2 = u$ then we have Isomorphism (6.7.3), and in particular $L_k(1) \cong L_k(2)^{-1}$; thus (6.5.18) forces $\hat{y} \in \widehat{J}[2]_{-\hat{\alpha}}$. Now assume $\hat{y} \in \widehat{J}[2]_{-\hat{\alpha}}$; by (6.5.10) the points u_1, u_2 correspond to destabilizing sub-line-bundles of $i_\alpha^* \mathcal{G}_{\hat{y}}^\alpha$, and by (6.2.5) there is only one such sub-line-bundle, hence $u_1 = u_2$. That the isomorphism class of $L_k(i)$ is as claimed follows from (6.5.18). $\triangle$

Let us show that μ_α^- lifts to $\widetilde{V}_\alpha$ outside Γ^-. By the above proposition $\mathcal{U}_t^\alpha$ is simple for all $t \in P_\alpha^- \setminus \Gamma^-$. Furthermore $\mathcal{U}_t^\alpha$ is semistable by (6.3.11) and Equalities (6.3.12) hold. By (2.3.7) $\mathcal{U}^\alpha$ induces a regular map $\epsilon_\alpha^- : (P_\alpha^- \setminus \Gamma^-) \to \widetilde{\mathcal{M}}$ which lifts the restriction of μ_α^- to $(P_\alpha^- \setminus \Gamma^-)$. Since $\mu_\alpha^-(P_\alpha^- \setminus \Gamma^-)$ is dense in V_α by (6.3.16), the image of ϵ_α^- is contained in $\widetilde{V}_\alpha$ (and is dense in it).

II. The elementary modification $\mathcal{V}^\alpha$. As is easily checked from the definition, the restriction of $\mathcal{U}^\alpha$ to $J \times \Gamma_k^-$ is trivial in the Γ_k^--direction, i.e. by (6.7.2) we have an exact sequence

$$0 \to \pi_J^*(I_{i_\alpha(q_1)} \otimes L_k(1)) \longrightarrow \mathcal{U}^\alpha|_{J \times \Gamma_k^-} \xrightarrow{\psi_k} \pi_J^*(I_{i_\alpha(q_2)} \otimes L_k(2)) \to 0,$$

where π_J is projection to J. Pulling back to $J \times \overline{\Gamma}_k$ we get a similar exact sequence for $\overline{\mathcal{U}}^\alpha|_{J \times \overline{\Gamma}_k}$; let $\overline{\psi}_k$ be the analogue of ψ_k. We define $\mathcal{V}^\alpha$ to be the sheaf on $J \times \overline{P}_\alpha$ fitting into the exact sequence

$$(6.7.4) \qquad 0 \to \mathcal{V}^\alpha \longrightarrow \overline{\mathcal{U}}^\alpha \xrightarrow{\overline{\psi}} \bigoplus_{k=1}^{16} i_{k,*}(\pi_J^*(I_{i_\alpha(q_2)} \otimes L_k(2))) \to 0,$$

where $i_k \colon J \times \overline{\Gamma}_k \hookrightarrow J \times \overline{P}_\alpha$ is the inclusion, and $\bar{\psi}$ is defined by the $\bar{\psi}_k$'s. Before stating the main properties of $\mathcal{V}^\alpha$ we introduce some notation. If $p \in C$ we let $p' \in C$ be the point such that

$$(6.7.5) \qquad\qquad p + p' \in |K_C|.$$

Given the identification

$$(6.7.6) \qquad\qquad \Gamma_k^- \cong E_k \cong \mathbb{P}(H^0(K_C))$$

let $z_k(1), z_k(2) \in \Gamma_k^-$ be given by

$$(6.7.7) \qquad\qquad z_k(i) \leftrightarrow q_i + q_i'$$

(where q_1, q_2 are defined by (4.2.2)), and let $Z_k(i) := \beta_-^{-1}(z_k(i))$.

(6.7.8)Proposition. *Keeping notation as above, $\mathcal{V}^\alpha$ is a family of torsion-free semistable sheaves on J parametrized by $\overline{P}_\alpha$, with*

$$(6.7.9) \qquad v(\mathcal{V}_t^\alpha) = \mathbf{v}, \quad \det(\mathcal{V}_t^\alpha) \cong \mathcal{O}_J, \quad \sum c_2(\mathcal{V}_t^\alpha) = 0$$

for all $t \in \overline{P}_\alpha$. Furthermore $\mathcal{V}_t^\alpha$ is simple for all $t \notin \bigcup_{k=1}^{16} Z_k(1)$. If $t \in Z_k(1)$ then

$$\mathcal{V}_t^\alpha \cong I_{i_\alpha(q_1)} \otimes L_k(1) \oplus I_{i_\alpha(q_2)} \otimes L_k(2),$$

where $L_k(i)$ is as in (6.7.2).

Proof. That $\mathcal{V}^\alpha$ is a family of torsion-free sheaves on J follows from an easy local computation. (This computation, away from $h^{-1}(h(z_k(1))$, is essentially written out below in the proof of (6.7.19).) If $t \notin \overline{\Gamma}$ we have

$$(6.7.10) \qquad\qquad \mathcal{V}_t^\alpha \cong \mathcal{U}_{\beta_-(t)}^\alpha,$$

thus (6.7.9) holds for all $t \notin \overline{\Gamma}$, hence by continuity it holds for all $t \in \overline{P}_\alpha$. Furthermore by (6.7.10) and (6.7.2) the sheaf $\mathcal{V}_t^\alpha$ is semistable and simple for all $t \notin \overline{\Gamma}$. Thus we are left with the task of describing $\mathcal{V}_t^\alpha$ for $t \in \overline{\Gamma}$. To simplify notation set

$$\lambda_k(i) := I_{i_\alpha(q_i)} \otimes L_k(i).$$

Applying the $\otimes \mathcal{O}_{J \times \overline{\Gamma}_k}$-functor to (6.7.4) we get an exact sequence

$$(6.7.11) \qquad 0 \to \pi_J^* \lambda_k(2) \otimes \pi_{\overline{\Gamma}_k}^* \mathcal{O}_{\overline{\Gamma}_k}(-\overline{\Gamma}_k) \to \mathcal{V}^\alpha|_{J \times \overline{\Gamma}_k} \to \pi_J^* \lambda_k(1) \to 0,$$

where π_J, $\pi_{\overline{\Gamma}_k}$ are the projections to J and $\overline{\Gamma}_k$ respectively. Thus $\mathcal{V}_t^\alpha$ is semistable also for $t \in \overline{\Gamma}$; it is simple if and only if the restriction of (6.7.11) to $J \times \{t\}$ is non-split (where $k \in \{1, \ldots, 16\}$ is the unique index such that $t \in \overline{\Gamma}_k$). Let e_k be the extension class of (6.7.11): since $\mathrm{Hom}(\lambda_k(1), \lambda_k(2)) = 0$, a simple spectral sequence argument shows that

$$(6.7.12) \qquad e_k \in \mathrm{Ext}^1(\lambda_k(1), \lambda_k(2)) \otimes H^0(\mathcal{O}_{\overline{\Gamma}_k}(-\overline{\Gamma}_k)).$$

The class e_k is equal to a Kodaira-Spencer map. In fact let $t \in \Gamma_k^-$: for $i = 1, 2$ there is an exact sequence

$$(6.7.13) \qquad 0 \to \lambda_k(i) \to \mathcal{U}_t^\alpha \to \lambda_k(3 - i) \to 0.$$

Composing the associated projection

$$\mathrm{Ext}^1(\mathcal{U}_t^\alpha, \mathcal{U}_t^\alpha) \to \mathrm{Ext}^1(\lambda_k(i), \lambda_k(3 - i))$$

with the Kodaira-Spencer map of $\mathcal{U}_t^\alpha$, we get a map of rank-two vector-bundles over Γ_k^-

$$N_{\Gamma_k^-/P_\alpha^-} \xrightarrow{\sigma_k(i)} \mathcal{O}_{\Gamma_k^-} \otimes \mathrm{Ext}^1(\lambda_k(i), \lambda_k(3 - i)).$$

The geometric meaning of $\sigma_k(i)$ is contained in the following observation [O4,(1.17)].

(6.7.14)*Remark.* Let $(x, v) \in N_{\Gamma_k^-/P_\alpha^-}$, i.e. $x \in \Gamma_k^-$ and $v \in T_x P_\alpha^-/T_x \Gamma_k^-$. Then $\sigma_k(i)(x, v) = 0$ if and only if the inclusion $\lambda_k(i) \hookrightarrow \mathcal{U}_x^\alpha$ lifts to first-order in the direction v. (Meaning in the direction of any $\tilde{v} \in T_x P_\alpha^-$ representing v.)

Notice that $\sigma_k(1)$ is an element of the right-hand side of (6.7.12). By [O4,(1.12)] we have

$$(6.7.15) \qquad\qquad\qquad e_k = \sigma_k(1).$$

(6.7.16)Claim. *The map $\sigma_k(i)$ is an isomorphism away from $z_k(i)$.*

Proof of the claim. Let $W_\alpha^- := \delta^{-1}(W_\alpha)$; thus $t \in W_\alpha^-$ if and only if $Tors(\mathcal{R}_t)$ has length 2, where $\mathcal{R}$ is the sheaf appearing in (6.6.9). Arguing as in the proof of Item (1) of (6.5.4) we get that

$$h|_{W_\alpha^-} \colon W_\alpha^- \to \widehat{I}_\alpha$$

is the blow-up of $\nu_\alpha^{-1}(\widehat{J}[2])$. In particular $W_\alpha^- \cap E^- = \Gamma^-$, and the intersection is transverse. Thus we have a direct sum decomposition

$$(6.7.17) \qquad N_{\Gamma_k^-/P_\alpha^-} \cong N_{\Gamma_k^-/W_\alpha^-} \oplus N_{\Gamma_k^-/E_k^-} \cong \mathcal{O}_{\Gamma_k^-}(-1) \oplus \mathcal{O}_{\Gamma_k^-}(-1).$$

On the other hand the E_2-term of the local-to-global spectral sequence abutting to $\mathrm{Ext}^\bullet(\lambda_k(i), \lambda_k(3 - i))$ gives an exact sequence

$$(6.7.18) \quad 0 \to H^1(Hom(\lambda_k(i), \lambda_k(3 - i))) \longrightarrow \mathrm{Ext}^1(\lambda_k(i), \lambda_k(3 - i))$$
$$\xrightarrow{\ell} H^0(Ext^1(\lambda_k(i), \lambda_k(3 - i))) \to 0.$$

Let us show that <u>away from $z_k(i)$</u>

$$(6.7.19)$$
$$\ell \circ \sigma_k(i)(N_{\Gamma_k^-/E_k^-}) \neq 0.$$
$$(6.7.20) \qquad \sigma_k(i)(N_{\Gamma_k^-/W_\alpha^-}) = H^1(Hom(\lambda_k(i), \lambda_k(3 - i))) \otimes \mathcal{O}_{\Gamma_k^-}$$

To this end we recall the geometric meaning of the map ℓ appearing in (6.7.18), i.e. the local analogue of (6.7.14). Let (x, v) be an element of the vector bundle

(6.6.10): then $\ell(\sigma_k(x,v)) = 0$ if and only if the singularity of $\mathcal{U}_x^\alpha$ at $i_\alpha(q_i)$ deforms to first-order in the direction v. Let us prove (6.7.19). Let $G := \mathcal{G}_{\hat{y}_k}^\alpha$, and assume $x \in (\Gamma_k^- \setminus \{z_k(i)\})$. Let $S := h^{-1}(h(x))$: if κ is the map of (6.4.3) then

$$(6.7.21) \qquad\qquad S = \mathbb{P}(\ker(\kappa) \oplus \mathbf{C}\sigma),$$

where $\sigma \in H^0([p_0 - 2\alpha - \hat{y}_k] \otimes i_\alpha^* G)$ is such that $\kappa(\sigma) \in H^0(K_C)$ is non-zero. Since $S \subset E_k^-$, and we have an isomorphism

$$(6.7.22) \qquad\qquad T_x S \xrightarrow{\sim} (N_{\Gamma_k^-/E_k^-})_x,$$

in order to prove (6.7.19) it suffices to analyze the stalk at $(i_\alpha(q_i), x)$ of the sheaf $\mathcal{A} := \overline{\mathcal{U}}^\alpha|_{J \times S}$. By definition we have an exact sequence

$$0 \to \mathcal{A} \longrightarrow G \otimes \mathcal{O}_S \xrightarrow{\Psi} (i_\alpha \times \mathrm{id}_S)_* \mathcal{Q} \to 0$$

(we have dropped the pull-back signs in the middle term). Here $\mathcal{Q}$ is the sheaf on $C \times S$ fitting into the exact sequence

$$0 \to [-p_0 + 2\alpha + \hat{y}_k] \otimes \mathcal{O}_S(-1) \xrightarrow{\Phi} i_\alpha^* G \otimes \mathcal{O}_S \xrightarrow{\Psi} \mathcal{Q} \to 0,$$

where Φ is the tautological map (see (6.7.21)). Let $\{w, z\}$ be a system of local parameters on J centered at $i_\alpha(q_i)$ such that $w = 0$ is a local equation of Θ_α, and s be a linear coordinate on S centered at x. Thus $\{z, s\}$ is a system of local parameters on $\Theta_\alpha \times S$ centered at $(i_\alpha(q_i), x)$: abusing notation we denote by the same symbols their pull-back to $C \times S$ via $(i_\alpha \times \mathrm{id}_S)$. We claim that choosing a suitable local trivialization of $i_\alpha^* G \otimes \mathcal{O}_S$ around (q_i, x) we have

$$(6.7.23) \qquad\qquad \Phi(z, s) = (z, s).$$

To see why let $(\kappa(\sigma)) = p + p'$. Since $x \neq z_k(i)$ we have $q_i \notin \{p, p'\}$. Thus $\sigma(q_i)$ is not contained in the sub line-bundle $[p_0 + \hat{y}_k] \hookrightarrow i_\alpha^* G$ (see (6.2.6)); this implies (6.7.23) because a generator of $\ker(\kappa)$ has a simple zero at q_i and has image contained in the above sub line-bundle. Given (6.7.23) one computes immediately the stalk of $\mathcal{A}$ at $(i_\alpha(q_i), x)$. Explicitly, letting $\mathcal{O}$ be the local ring of $J \times S$ at $(i_\alpha(q_i), x)$ we get an exact sequence

$$
\begin{array}{ccccccc}
0 & \to & \mathcal{A} \otimes \mathcal{O} & \longrightarrow & \mathcal{O}^{(3)} & \longrightarrow & \mathcal{O} & \to & 0. \\
 & & & & (\alpha, \beta, \gamma) & \mapsto & w\alpha - z\beta - s\gamma & &
\end{array}
$$

Let $(\overline{\mathcal{O}}, \overline{m})$ be the local ring of J at $i_\alpha(q_i)$. We may assume that the stalk at $i_\alpha(q_i)$ of (6.7.13) is given by

$$(6.7.24) \qquad\qquad 0 \to \overline{m} \xrightarrow{f} \mathcal{A} \otimes \overline{\mathcal{O}} \xrightarrow{g} \overline{\mathcal{O}} \to 0.$$

Let

$$v := \frac{\partial}{\partial s} \in (N_{\Gamma_k^-/E_k^-})_x.$$

(Yes, we are abusing notation.) By (6.7.22)

$$\ell \circ \sigma_k(i)(x,v) = \epsilon \in Ext^1(\overline{m}, \overline{\mathcal{O}}),$$

where ϵ is obtained as follows. The first-order deformation of $\mathcal{A} \otimes \overline{\mathcal{O}}$ defined by $\mathcal{A} \otimes \mathcal{O}$ gives an exact sequence

$$0 \to \mathcal{A} \otimes \overline{\mathcal{O}} \xrightarrow{\cdot s} \mathcal{A} \otimes \mathcal{O}/(s^2) \longrightarrow \mathcal{A} \otimes \overline{\mathcal{O}} \to 0$$

with extension class $\tilde{\epsilon} \in Ext^1(\mathcal{A} \otimes \overline{\mathcal{O}}, \mathcal{A} \otimes \overline{\mathcal{O}})$; by definition $\epsilon = g \circ \tilde{\epsilon} \circ f$, where f, g are the maps of (6.7.24). An easy computation shows that ϵ is the (non-zero) generator of $Ext^1(\overline{m}, \overline{\mathcal{O}})$, and this proves (6.7.19). Now let's prove (6.7.20). The left-hand side of (6.7.20) is contained in the right-hand side because as x varies in W_α^- the sheaf $\mathcal{U}_x^\alpha$ is equisingular. Hence since both sides of (6.7.20) are line-bundles it suffices to show that $\sigma_k(i)$ is non-zero on $N_{\Gamma_k^-/W_\alpha^-}$, away from $z_k(i)$. Assume $(x,v) \in N_{\Gamma_k^-/W_\alpha^-}$ and $\sigma_k(i)(x,v) = 0$. By (6.7.14) $\delta_*(v)$ is tangent to Y_α at s_k; of course $\delta_*(v)$ is also tangent to W_α at s_k. The proof of the first statement of (6.5.14) works as well for first-order deformations, hence we get that

$$(\nu_\alpha \circ f_-)_*(v) \in T_{\hat{y}_k}\{\hat{y} \in \widehat{J}| \ 2\hat{y} \sim \hat{q}_{3-i} - \hat{r}, \quad \text{some } r \in C.\}.$$

The right-hand side equals $\mathbf{C}\tau$, where $\tau \in H^0(K_C)$ is a section such that $(\tau) = q_i + q_i'$, and we make the canonical identification $\mathbb{P}(T_{\hat{y}_k}) \cong \mathbb{P}(H^0(K_C))$. If $x = p + p'$ then $\mathbf{C}(\nu_\alpha \circ f_-)_*(v) = p + p'$; hence $\sigma_k(i)(x,v) = 0$ implies that $p + p' = q_i + q_i'$, i.e. $x = z_k(i)$. This proves (6.7.20). Claim (6.7.16) follows at once from (6.7.19) and (6.7.20). $\triangle$

Let $Y_\alpha^- \subset P_\alpha^-$ be the closure of the locally-closed subset of points t such that $Tors(\mathcal{R}_t)$ has length 1, where $\mathcal{R}$ is the sheaf appearing in (6.6.9). Thus for $t \in Y_\alpha^-$ the sheaf $\mathcal{U}_t^\alpha$ is strictly semistable (see (6.5.20)).

(6.7.25)Claim. *Keeping notation as above,*

$$Y_\alpha^- \cap \Gamma_k^- = \{z_k(1), z_k(2)\}.$$

Both intersections are transverse.

Proof of the claim. Assume $x \in Y_\alpha^- \cap \Gamma_k^-$. Since $\dim Y_\alpha^- = 2$ there exists $v \in T_x P_\alpha^-$ "normal" to Γ_k^- which is tangent to a curve $D \subset Y_\alpha^-$. Since $\mathcal{U}_t^\alpha$ is properly semistable for all $t \in D$ at least one of the inclusions $\lambda_k(1), \lambda_k(2) \hookrightarrow \mathcal{U}_x^\alpha$ lifts to first-order in the direction determined by v. By (6.7.14) and (6.7.16) we get that $x \in \{z_k(1), z_k(2)\}$ and that if $x = z_k(i)$ then $\lambda_k(3-i) \hookrightarrow \mathcal{U}_x^\alpha$ does not lift. In particular $Y_\alpha^- \cap \Gamma_k^- \subset \{z_k(1), z_k(2)\}$. To prove the reverse inclusion and the transversality of the intersection we consider $Y_\alpha^- \cap E_k^-$. First let's show that the intersection is transverse. Let $x \in E_k$ correspond to a reduced divisor $p + p' \in K_C$ via (6.7.6), and assume also that $x \notin \{h(z_k(1)), h(z_k(2))\}$ (i.e. $p + p' \neq q_i + q_i'$). Arguing as in the proof of (6.7.19) one shows that $f_-^{-1}(x)$ intersects Y_α^- transversely, and hence E_k^- is transverse to Y_α^-. Next, the image

$$\delta(Y_\alpha^- \cap E_k^-) \subset f^{-1}(\hat{y}_k) = \mathbb{P}(H^0([p_0 - 2\alpha - \hat{y}_k] \otimes i_\alpha^* \mathcal{G}_{\hat{y}_k}^\alpha)) \cong \mathbb{P}^2$$

is computed by a Chern class computation. In fact let π_C, $\pi_{\mathbb{P}^2}$ be the projections of $C \times \mathbb{P}^2$ onto C and $\mathbb{P}^2$ respectively. Then

$$[\delta(Y_\alpha^- \cap E_k^-)] = \pi_{\mathbb{P}^2,*}(c_2(\pi_C^*[p_0 - 2\alpha - y_k] \otimes \pi_C^* i_\alpha^* \mathcal{G}_{\hat{y}_k}^\alpha \otimes \pi_{\mathbb{P}^2}^* \mathcal{O}_{\mathbb{P}^2}(1))),$$

where the multiplicity of the left-hand side equals one because Y_α^- is transverse to E_k^-. From the above formula one gets that $\delta(Y_\alpha^- \cap E_k^-)$ is the image of C for the map associated to the linear system $|2K_C - 2\alpha|$. Furthermore $s_k = \delta(\Gamma_k^-)$ is the node of $\delta(Y_\alpha^- \cap E_k^-)$: since the two tangent directions at the node correspond to $z_k(1)$ and $z_k(2)$, we get that $Y_\alpha^- \cap \Gamma_k^- = \{z_k(1), z_k(2)\}$ and that the intersection is transverse at each of these points. $\triangle$

Given claims (6.7.16) and (6.7.25) we can determine $\sigma_k(i)$: let's show that

(6.7.26) $$\sigma_k(i) = \varphi_k(i) \otimes \tau_k(i),$$

where

(6.7.27) $$\varphi_k(i) \in \mathrm{GL}_2(\mathbf{C}), \quad \tau_k(i) \in H^0(\mathcal{O}_{\Gamma_k^-}(1)), \quad \tau_k(i)(z_k(i)) = 0.$$

Since $\mathcal{U}_t^\alpha$ is strictly semistable for every $t \in Y_\alpha^-$, either for $j = i$ or for $j = (3 - i)$ the inclusion $\lambda_k(j) \hookrightarrow \mathcal{U}_{z_k(i)}^\alpha$ lifts for every deformation inside Y_α^-. By (6.7.16) the former inclusion is the one that lifts. By (6.7.25) every direction normal to Γ_k^- at $z_k(i)$ is represented by a vector tangent to Y_α^-, and thus by (6.7.14) we get that $\sigma_k(i)$ vanishes at $z_k(i)$. Hence $\sigma_k(i) = \psi_k(i) \otimes \tau_k(i)$, where $\tau_k(i)$ is as in (6.7.27), and $\psi_k(i) \in \mathrm{End}(\mathcal{O}_{\Gamma_k^-}^{(2)})$. By (6.7.16) $\det \psi_k(i)$ does not vanish outside $z_k(i)$; since $\det \psi_k(i)$ is a section of $\mathcal{O}_{\Gamma_k^-}$ we get that it is nowhere zero, and this proves (6.7.26)-(6.7.27). Now we can finish the proof of (6.7.8). Assume $x \in (\overline{\Gamma}_k \setminus Z_k(1))$ (for $k = 1, \ldots, 16$); we must prove that $\mathcal{V}_x^\alpha$ is simple. This is true because by (6.7.15) and (6.7.26)-(6.7.27) the restriction of (6.7.11) to $J \times \{x\}$ is non-split hence simple. $\triangle$

The elementary modification $\mathcal{W}^\alpha(\mathcal{N})$. Let $\mathcal{N} \subset \overline{P}_\alpha$ be open and such that

(6.7.28) $$Z_k(1) \subset \mathcal{N} \quad k = 1, \ldots, 16.$$

Let $\overline{Y}_\alpha := \beta_-^{-1}(Y_\alpha^-)$. Let $t \in \overline{Y}_\alpha \cap \mathcal{N}$. If $t \notin \bigcup_{k=1}^{16} Z_k(1)$ then $\mathcal{V}_t^\alpha$ is strictly semistable and simple by (6.7.8), hence it has a unique destabilizing sequence. On the other hand if $t \in \bigcup_{k=1}^{16} Z_k(1)$ then Exact Sequence (6.7.13) for $i = 1$ lifts to a destabilizing sequence of $\mathcal{V}_{\tilde{t}}^\alpha$ for every $\tilde{t} \in \overline{Y}_\alpha$ near t, by (6.7.25) and (6.7.16). Thus there is a well-defined exact sequence

(6.7.29) $$0 \to \mathcal{S} \longrightarrow \mathcal{V}^\alpha|_{J \times (\overline{Y}_\alpha \cap \mathcal{N})} \xrightarrow{g} \mathcal{T} \to 0$$

which restricts to a destabilizing sequence of $\mathcal{V}_t^\alpha$ for every $t \in \overline{Y}_\alpha \cap \mathcal{N}$. Let $\mathcal{W}^\alpha(\mathcal{N})$ be the sheaf on $J \times \mathcal{N}$ fitting into the exact sequence

(6.7.30) $$0 \to \mathcal{W}^\alpha(\mathcal{N}) \longrightarrow \mathcal{V}^\alpha|_{J \times \mathcal{N}} \xrightarrow{\tilde{g}} i_* \mathcal{T} \to 0,$$

where $i \colon (\overline{Y}_\alpha \cap \mathcal{N}) \hookrightarrow \mathcal{N}$ is the inclusion and $\tilde{g}$ is the restriction to $J \times (\overline{Y}_\alpha \cap \mathcal{N})$ followed by the map g of (6.7.29).

(6.7.31)Proposition. *With notation as above, $\mathcal{W}^\alpha(\mathcal{N})$ is a family of torsion-free semistable sheaves on J parametrized by $\mathcal{N}$, such that*

$$v(\mathcal{W}^\alpha(\mathcal{N})_t) = \mathbf{v}, \quad \det \mathcal{W}^\alpha(\mathcal{N})_t \cong \mathcal{O}_J, \quad \sum c_2(\mathcal{W}^\alpha(\mathcal{N})_t) = 0$$

for all $t \in \mathcal{N}$. If $t \in (\mathcal{N} \setminus \overline{Y}_\alpha)$ then $\mathcal{W}^\alpha(\mathcal{N})_t \cong \mathcal{V}_t^\alpha$. Finally, if $\mathcal{N}$ is a sufficiently small open set satisfying (6.7.28) then $\mathcal{W}^\alpha(\mathcal{N})_t$ is simple for all $t \in \mathcal{N}$.

Proof. All the statements except the last are standard. Let us prove that if $t \in (\overline{Y}_\alpha \cap \mathcal{N})$ the sheaf $\mathcal{W}^\alpha(\mathcal{N})_t$ is simple, if $\mathcal{N}$ is sufficiently small. Applying the functor $\otimes \mathcal{O}_{J \times (\overline{Y}_\alpha \cap \mathcal{N})}$ to (6.7.30) we get an exact sequence

$$0 \to \mathcal{T} \otimes \pi_{\mathcal{N}}^* \mathcal{O}_{\mathcal{N}}(-\overline{Y}_\alpha) \to \mathcal{W}^\alpha(\mathcal{N})|_{J \times (\overline{Y}_\alpha \cap \mathcal{N})} \to \mathcal{S} \to 0,$$

where $\pi_{\mathcal{N}} \colon J \times \mathcal{N} \to \mathcal{N}$ is the projection. Hence if $t \in (\overline{Y}_\alpha \cap \mathcal{N})$ we get an exact sequence

(6.7.32) $$0 \to \mathcal{T}_t \to \mathcal{W}^\alpha(\mathcal{N})_t \to \mathcal{S}_t \to 0.$$

Thus it suffices to show that the above extension is non-split. Since $\mathcal{N}$ can be chosen to be an arbitrarily small neighborhood of $\bigcup_{k=1}^{16} Z_k(1)$, it is enough to prove that (6.7.32) is non-split for $t = p \in \bigcup_{k=1}^{16} Z_k(1)$. Let $\Lambda \subset \overline{\Gamma}_k$ be the $\mathbb{P}^1$ containing p and belonging to the ruling opposite to that of $Z_k(1)$. Let $\varphi_k(1)(\Lambda) = \mathbf{C}e$ where $\varphi_k(1)$ is as in (6.7.26). Thus e is the class of a non-trivial extension

(6.7.33) $$0 \to \lambda_k(2) \to \mathcal{H} \to \lambda_k(1) \to 0.$$

Let

(6.7.34) $$0 \to \pi_J^* \lambda_k(2) \to \pi_J^* \mathcal{H} \to \pi_J^* \lambda_k(1) \to 0$$

be its pull-back to $J \times \Lambda$, where $\pi_J \colon J \times \Lambda \to J$ is the projection. By (6.7.11), (6.7.15), (6.7.26) and transversality of the intersection $\overline{Y}_\alpha \cap \overline{\Gamma}$ (see (6.7.25)) the restriction $\mathcal{V}^\alpha|_{J \times \Lambda}$ is the pull-back of (6.7.33) by the inclusion $\pi_J^* \lambda_k(1)(-\pi_\Lambda^*(p)) \hookrightarrow \pi_J^* \lambda_k(1)$, where $\pi_\Lambda \colon J \times \Lambda \to \Lambda$ is the projection. In other words $\mathcal{V}^\alpha|_{J \times \Lambda}$ is the elementary modification of $\pi_J^* \mathcal{H}$ fitting into the exact sequence

(6.7.35) $$0 \to \mathcal{V}^\alpha|_{J \times \Lambda} \to \pi_J^* \mathcal{H} \to j_* \lambda_k(1) \to 0,$$

where $j \colon J \times \{p\} \hookrightarrow J \times \Lambda$ is the inclusion. Restricting to $J \times p$ we get a surjection

(6.7.36) $$j^* \mathcal{V}^\alpha \xrightarrow{f} \lambda_k(2).$$

Since $\mathcal{W}^\alpha(\mathcal{N})|_{J \times \Lambda}$ is the elementary modification of $\mathcal{V}^\alpha|_{J \times \Lambda}$ fitting into the exact sequence

(6.7.37) $$0 \to \mathcal{W}^\alpha(\mathcal{N})|_{J \times \Lambda} \to \mathcal{V}^\alpha|_{J \times \Lambda} \xrightarrow{f} j_*(\lambda_k(2)) \to 0,$$

i.e. the modification "inverse" to (6.7.35), we see that

(6.7.38) $$\mathcal{W}^\alpha(\mathcal{N})|_{J \times \Lambda} \cong \pi_J^* \mathcal{H} \otimes \pi_\Lambda^*(-p).$$

In particular $\mathcal{W}^\alpha(\mathcal{N})_p$ is isomorphic to the non-split extension (6.7.33), hence it is simple. This finishes the proof of the proposition. $\triangle$

The map ϵ_α^+. Let $\mathcal{N}$ be sufficiently small, so that (6.7.31) applies. Then the sheaf $\mathcal{W}^\alpha(\mathcal{N})$ defines a regular map $\mu_\alpha(\mathcal{N})\colon\mathcal{N} \to \widetilde{V}_\alpha$. On the other hand by (6.7.8) the restriction of $\mathcal{V}^\alpha$ to $J \times (\overline{P}_\alpha \setminus \bigcup_{k=1}^{16} Z_k(1))$ defines a regular map

$$(6.7.39) \qquad (\overline{P}_\alpha \setminus \bigcup_{k=1}^{16} Z_k(1)) \to \widetilde{V}_\alpha.$$

By (6.7.31) the two maps coincide on $(\mathcal{N} \setminus \overline{Y}_\alpha)$ hence they glue together and they define a regular map $\bar{\epsilon}_\alpha\colon\overline{P}_\alpha \to \widetilde{V}_\alpha$. By (6.7.26)-(6.7.27) $\bar{\epsilon}_\alpha$ is constant on every $\mathbb{P}^1 \subset \overline{\Gamma}_k$ belonging to the ruling "opposite" to that of $Z_k(1)$, hence $\bar{\epsilon}_\alpha$ descends to a regular map $\epsilon_\alpha^+\colon P_\alpha^+ \to \widetilde{V}_\alpha$.

The map ϵ_α^+ away from $(\epsilon_\alpha^+)^{-1}(\widetilde{\Sigma}_\alpha)$. We will prove that

$$(6.7.40) \qquad \text{the restriction of } \epsilon_\alpha^+ \text{ to } (P_\alpha^+ \setminus Y_\alpha^+) \text{ is an isomorphism onto } (\widetilde{V}_\alpha \setminus \widetilde{\Sigma}).$$

Since $Y_\alpha^+ = (\epsilon_\alpha^+)^{-1}(\widetilde{\Sigma})$ and $\widetilde{\pi}\colon\widetilde{\mathcal{M}} \to \mathcal{M}$ is an isomorphism outside $\Sigma = \widetilde{\pi}(\widetilde{\Sigma})$, it is equivalent to show that the restriction of μ_α to $(P_\alpha \setminus Y_\alpha)$ is an isomorphism onto $(V_\alpha \setminus \Sigma)$. One defines an inverse of this map proceeding as in the proof of (6.3.16); what is needed is an analogue of (6.3.20) valid for all $[F] \in (V_\alpha \setminus \Sigma)$. This is the content of the following lemma.

(6.7.41)Lemma. *Let $[F] \in \mathcal{M}$ and assume F is stable. Then there is at most one exact sequence*

$$(6.7.42) \qquad 0 \to \lambda \to i_\alpha^* F \to \xi \to 0$$

such that λ, ξ are rank-one sheaves, with $\deg(\lambda) = 1$, $\deg(\xi) = -1$.

Proof of the lemma. If $i_\alpha^* F$ is locally-free the result follows from (6.3.20), hence we assume $i_\alpha^* F$ is singular. Thus $[F] \in (B_\alpha \setminus \Sigma)$. Consider the canonical exact sequence

$$(6.7.43) \qquad 0 \to F \longrightarrow F^{**} \overset{\phi}{\longrightarrow} Q \to 0.$$

By (4.3.3) we have $Q = \mathbf{C}_{x_1} \oplus \mathbf{C}_{x_2}$, where $x_1 + x_2 = 0$ and $x_1 \neq x_2$. By the same proposition

$$(6.7.44) \qquad F^{**} \cong L \oplus L^{-1}, \quad L^{\otimes 2} \not\cong \mathcal{O}_J$$

or else there is a non-trivial extension

$$(6.7.45) \qquad 0 \to L \to F^{**} \to L \to 0, \quad L^{\otimes 2} \cong \mathcal{O}_J.$$

Let ϕ_{x_i}, L_{x_i} be the fibers of ϕ and L over x_i: we claim that

$$(6.7.46) \qquad \ker(\phi_{x_i}) \cap L_{x_i}^{\pm 1} = \{0\}.$$

In fact assume (6.7.46) does not hold, say $\ker(\phi_{x_i}) = L_{x_i}$, then from (6.7.43) we get an inclusion $I_{x_{3-i}} \otimes L \subset F$, contradicting stablility of F. Applying the functor i_α^* to (6.7.43) we get an exact sequence

$$(6.7.47) \qquad 0 \to i_\alpha^* Q \longrightarrow i_\alpha^* F \longrightarrow i_\alpha^* F^{**} \xrightarrow{i_\alpha^* \phi} i_\alpha^* Q \to 0.$$

Let $V := i_\alpha^* F / i_\alpha^* Q$; we claim that this is a slope-semistable vector bundle on C, i.e. that if η is a line-bundle on C such that $\mathrm{Hom}(\eta, V) \neq 0$ then $\deg(\eta) \leq (-1)$. If (6.7.44) holds this is an immediate consequence of (6.7.46). If (6.7.45) holds, we notice that since the map $H^1(\mathcal{O}_J) \to H^1(\mathcal{O}_C)$ induced by i_α is an isomorphism, Exact Sequence(6.7.45) remains exact when we pull-back by i_α, and this together with (6.7.46) easily implies that V is slope-semistable. Any rank-one subsheaf of $i_\alpha^* F$ is of the form $\eta \oplus T$, where η is a sub-line-bundle of V and $T \subset i_\alpha^* Q$. Assume that $\deg(\eta \oplus T) = 1$: since $\deg \eta \leq (-1)$ we get that $\ell(T) = 2$. Thus $x_1, x_2 \in \Theta_\alpha$ so that $x_i = q_i$ (where q_i is given by (4.2.2)), $T = \mathbf{C}_{x_1} \oplus \mathbf{C}_{x_2}$, and $\deg \eta = (-1)$. This gives a bijection between the set of Exact Sequences (6.7.42) and the set of sub-line-bundles $\eta \subset V$ with $\deg(\eta) = (-1)$. We will prove that if (6.7.44) holds then there is at most one such sub-line-bundle. The proof of the analogous property when (6.7.45) holds is left to the reader. Assume there are two distinct sub-line-bundles $\eta_1 \subset V$, $\eta_2 \subset V$ of degree (-1): since $\deg V = (-2)$ we get that

$$(6.7.48) \qquad V \cong \eta_1 \oplus \eta_2.$$

Let $\ell := i_\alpha^* L$; composing $\eta_j \hookrightarrow V$ with the injection $V \to i_\alpha^* F^{**}$ we get an injection $\eta_j \hookrightarrow (\ell \oplus \ell^{-1})$. It follows from (6.7.46) that the projections $\eta_j \to \ell^{\pm 1}$ are non-zero, hence

$$\eta_j \cong \ell(-p_j) \cong \ell^{-1}(-r_j)$$

for some p_j, r_j. Thus

$$(6.7.49) \qquad \ell^{\otimes 2} \cong [p_j - r_j].$$

Since $\ell^{\otimes 2} \not\cong \mathcal{O}_C$ this implies that

$$(6.7.50) \qquad r_2 = p_1', \quad p_2 = r_1'.$$

On the other hand $\det(V) \cong [-q_1 - q_2]$ and thus (6.7.48) gives

$$\ell^{\otimes 2}(-p_1 - p_2) \cong [-q_1 - q_2].$$

Using (6.7.49) and (6.7.50) we get that $q_1 + q_2 \in |K_C|$, a contradiction. $\triangle$

The restriction of ϵ_α^+ to Y_α^+. We begin by describing $\overline{Y}_\alpha$. Let $\pi_\alpha \colon T_\alpha \to C \times \widehat{J}$ be the $\mathbb{P}^1$-fibration with fiber $\mathbb{P}(\mathrm{Ext}^1(I_{-p} \otimes \mathcal{L}_{\hat{w}}^{-1}, I_p \otimes \mathcal{L}_{\hat{w}}))$ over $(p, \hat{w})$. Thus we have a quotient map

$$(6.7.51) \qquad \zeta_\alpha \colon T_\alpha \to \widetilde{\Sigma}_\alpha,$$

corresponding to the equivalence relation generated by setting $[e_F] \equiv [e_F^\perp]$ for $[e_F] \in \mathbb{P}(\mathrm{Ext}^1(I_{i_\alpha(q_2)} \otimes \mathcal{L}_{\hat{w}}^{-1}, I_{i_\alpha(q_1)} \otimes \mathcal{L}_{\hat{w}}))$, and $e_F^\perp$ generating the annihilator of e_F in the Serre duality pairing. We define a regular map

$$\rho_\alpha \colon \overline{Y}_\alpha \to T_\alpha$$

as follows. If $t \notin \bigcup_{k=1}^{16} Z_k(1)$ then $\mathcal{V}_t^\alpha$ is simple and semistable, hence there exists a unique destabilizing exact sequence

$$0 \to I_{-p} \otimes \mathcal{L}_{\hat{w}}^{-1} \to \mathcal{V}_t^\alpha \to I_p \otimes \mathcal{L}_{\hat{w}} \to 0,$$

and this determines a point of T_α. If $t \in Z_k(1)$ the sheaf $\mathcal{W}^\alpha(\mathcal{N})_t$ is simple and semistable, hence it fits into a unique destabilizing exact sequence which determines an extension class $e_{\mathcal{W}^\alpha(\mathcal{N})_t} \in \mathbb{P}(\mathrm{Ext}^1(I_{i_\alpha(q_1)} \otimes \mathcal{L}_{\hat{w}}^{-1}, I_{i_\alpha(q_2)} \otimes \mathcal{L}_{\hat{w}}))$ for a suitable w. We set $\rho_\alpha(t) := e_{\mathcal{W}^\alpha(\mathcal{N})_t}^\perp$. That ρ_α is regular follows from (6.7.31). Clearly

$$(6.7.52) \qquad\qquad\qquad \bar{\epsilon}_\alpha|_{\overline{Y}_\alpha} = \zeta_\alpha \circ \rho_\alpha.$$

(6.7.53)Lemma. *Keeping notation as above,*

$$(6.7.54) \qquad \mathrm{Im}(\pi_\alpha \circ \rho_\alpha) = \{(p, \hat{w})|\ p - 2w \sim r - 2\alpha \text{ for some } r \in C\} =: \Xi_\alpha.$$

Furthermore $\pi_\alpha \circ \rho_\alpha$ is the blow-up of Ξ_α with center

$$\Omega_\alpha := \{(q_i, \hat{y}_k - i_{\hat{0}}(q_{3-i})) = (q_i, [L_k(i)])\}, \quad i = 1, 2, \quad k = 1, \ldots, 16.$$

Proof of the lemma. First we show that $\mathrm{Im}(\pi_\alpha \circ \rho_\alpha) \subset \Xi_\alpha$. Assume $t \in (\overline{Y}_\alpha \setminus \overline{\Gamma})$, and let $I_p \otimes \mathcal{L}_{\hat{w}} \hookrightarrow \mathcal{V}_t^\alpha$ be destabilizing. Let $u := \delta \circ \beta_-(t)$. Then we have an injection

$$(6.7.55) \qquad\qquad\qquad I_p \otimes \mathcal{L}_{\hat{w}} \hookrightarrow \mathcal{F}_u^\alpha.$$

Composing with $\mathcal{F}_u^\alpha \hookrightarrow \mathcal{G}_{f(u)}^\alpha$ we get that

$$h^0(\mathcal{L}_{\hat{w}}^{-1} \otimes \mathcal{G}_{f(u)}^\alpha) > 0,$$

hence by (6.5.19) we get that

$$(6.7.56) \qquad\qquad [w] \cong i_\alpha^* \mathcal{L}_{\hat{w}} \cong [p_0 - \widehat{f(u)} - r] \text{ for some } r \in C.$$

On the other hand (6.7.55) gives that the line-bundle ξ appearing in (6.3.1) (with $G = \mathcal{G}_{f(u)}^\alpha$) must be equal to $\mathcal{O}_C(w - p)$ and thus from (6.4.2) we get that

$$(6.7.57) \qquad\qquad\qquad w - p \sim -p_0 + 2\alpha + \widehat{f(u)}.$$

From (6.7.56)-(6.7.57) we get that $\mathrm{Im}(\pi_\alpha \circ \rho_\alpha) \subset \Xi_\alpha$. Since Ξ_α is smooth irreducible and $\overline{Y}_\alpha$ is a surface smooth at points of $Z_k(j)$ (for $j = 1, 2$ and $k = 1, \ldots, 16$), in order to finish the proof of the lemma it suffices to show that for $(p, \hat{w}) \in \mathrm{Im}(\pi_\alpha \circ \rho_\alpha)$ we have

$$(\pi_\alpha \circ \rho_\alpha)^{-1}(p, \hat{w}) = \begin{cases} \text{one point} & \text{if } (p, \hat{w}) \notin \Omega_\alpha, \\ Z_k(i) & \text{if } (p, \hat{w}) \in \Omega_\alpha. \end{cases}$$

Let $t_1, t_2 \in (\pi_\alpha \circ \rho_\alpha)^{-1}(p, \hat{w})$, and set $u_j := \delta \circ \beta_-(t_j)$. It suffices to show that $u_1 = u_2$. From (6.7.57) we get that $f(u_1) = f(u_2)$. Thus each of u_1, u_2 corresponds to an Exact Sequence (6.3.1) with $G = \mathcal{G}_{\hat{y}_0}^\alpha$, where $\hat{y}_0 := f(u_1) = f(u_2)$. An easy

stability argument shows that $h^0(\mathcal{L}_{\hat{w}}^{-1} \otimes \mathcal{G}_{\hat{y}_0}^\alpha) = 1$, and thus there is a unique Exact Sequence (6.3.1) (with $G = \mathcal{G}_{f(u)}^\alpha$) such that the elementary modification F fitting into (6.3.9) is strictly semistable. This shows that $u_1 = u_2$. $\triangle$

Now we prove that

(6.7.58) $\epsilon_\alpha^+|_{Y_\alpha^+} \colon Y_\alpha^+ \to \widetilde{V}_\alpha \cap \widetilde{\Sigma}$ is bijective.

The map is surjective by (6.5.20). Let us prove injectivity. First notice that

(6.7.59) $Y_\alpha^+ = \overline{Y}_\alpha / \sim,$

where $\sim$ is the equivalence relation identifying $t \in Z_k(1)$ with the point $t' \in Z_k(2)$ belonging to the same $\mathbb{P}^1$ of the ruling opposite to that of $Z_k(i)$. By (6.7.52) and (6.7.59) injectivity of ϵ_α restricted to Y_α^+ will follow from

(6.7.60) $\zeta_\alpha \circ \rho_\alpha(t_1) = \zeta_\alpha \circ \rho_\alpha(t_2)$ implies $t_1 \sim t_2$, for $t_1, t_2 \in \overline{Y}_\alpha.$

If $\rho_\alpha(t_1) = \rho_\alpha(t_2)$ it follows immediately from (6.7.53) that (6.7.60) holds. Thus we may assume that $\rho_\alpha(t_1) \equiv \rho_\alpha(t_2)$ but $\rho_\alpha(t_1) \neq \rho_\alpha(t_2)$. Without loss of generality we may suppose that

$$\pi_\alpha \circ \rho_\alpha(t_j) = (q_j, (-1)^j \hat{w}), \quad \hat{w} \in \widehat{J}.$$

By (6.7.54) we get that

(6.7.61) $q_j - 2(-1)^j w \sim r_j - 2\alpha, \quad j = 1, 2, \quad r_j \in C,$

and hence $r_1 + r_2 \in |K_C + 2\alpha|$. Thus

(6.7.62) $r_j \sim K_C - q_j, \quad j = 1, 2$ or
(6.7.63) $r_j \sim K_C - q_{3-j} \quad j = 1, 2.$

Let $u_j := \delta \circ \beta_-(t_j)$. If (6.7.62) holds then

$$2f(u_1) = 2f(u_2) = -2\alpha,$$

hence $t_1, t_2 \in \overline{\Gamma}$ and one readily verifies that $t_1 \sim t_2$. Now assume (6.7.63) holds. Then

$$\widehat{f(u_j)} = w + i_{\alpha,*}(q_{3-j} - p_0),$$

and thus $2f(u_j) \neq -2\alpha$. Hence

$$\mathcal{V}_{t_j}^\alpha \cong \mathcal{F}_{\delta \circ \beta_-(t_j)}^\alpha =: F_j,$$

Thus we have a non-split exact sequence

(6.7.64) $0 \to I_{i_\alpha(q_j)} \otimes \mathcal{L}_{\hat{w}} \xrightarrow{\phi_j} F_j \longrightarrow I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}} \to 0.$

(By (6.7.63)-(6.7.61) we have $2w = 0$.) Let $e_j \in \mathrm{Ext}^1(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}, I_{i_\alpha(q_j)} \otimes \mathcal{L}_{\hat{w}})$ be the extension class of (6.7.64): we must prove that

$$(6.7.65) \qquad\qquad \langle e_1, e_2 \rangle \neq 0,$$

where $\langle \ , \ \rangle$ is the Serre duality pairing. The local-to-global spectral sequence abutting to $\mathrm{Ext}^\bullet(\ ,\ )$ gives a natural isomorphism

$$\gamma_j \colon \mathrm{Ext}^1(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}, I_{i_\alpha(q_j)} \otimes \mathcal{L}_{\hat{w}}) \xrightarrow{\ \sim\ } H^1(\mathcal{O}_J).$$

commuting with Serre duality. Since a one-dimensional subspace $\ell \subset H^1(\mathcal{O}_J)$ is the Serre-duality annihilator of itself, we get that (6.7.65) is equivalent to

$$(6.7.66) \qquad\qquad [\gamma_1(e_1)] \neq [\gamma_2(e_2)].$$

We claim that via the composition of canonical isomorphisms

$$\mathbb{P}(H^1(\mathcal{O}_J)) \cong \mathbb{P}(H^1(\mathcal{O}_C)) \cong \mathbb{P}(H^0(K_C)),$$

we have

$$(6.7.67) \qquad\qquad [\gamma_j(e_j)] = q_{3-j} + q'_{3-j}.$$

By (3.7) this will imply (6.7.66). In order to prove (6.7.67) we set $G_j := \mathcal{G}^\alpha_{f(u_j)}$. Composing the map ϕ_j of (6.7.64) with the inclusion $F_j \hookrightarrow G_j$ we get an injection $\mathcal{L}_{\hat{w}} \hookrightarrow G_j$, which gives rise to a (non-split) exact sequence

$$0 \to \mathcal{L}_{\hat{w}} \to G_j \to I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}(\Theta_\alpha) \to 0.$$

The corresponding extension class f_j is a generator of $\mathrm{Ext}^1(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}(\Theta_\alpha), \mathcal{L}_{\hat{w}})$. Applying the $\mathrm{Hom}(\bullet, \mathcal{L}_{\hat{w}})$-functor to the exact sequence

$$0 \to I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}} \to I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}(\Theta_\alpha) \to i_{\alpha,*}(\mathbf{C}_{q3-i} \oplus K_C(-q_{3-j})) \otimes \mathcal{L}_{\hat{w}} \to 0$$

we get an exact sequence

$$(6.7.68) \quad \mathrm{Ext}^1(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}(\Theta_\alpha), \mathcal{L}_{\hat{w}}) \xrightarrow{\ \beta_j\ } \mathrm{Ext}^1(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}, \mathcal{L}_{\hat{w}})$$

$$\longrightarrow \mathrm{Ext}^2(i_{\alpha,*}(\mathbf{C}_{q_{3-i}} \oplus K_C(-q_{3-j})) \otimes \mathcal{L}_{\hat{w}}, \mathcal{L}_{\hat{w}}) \xrightarrow{\ \lambda_j\ } \mathrm{Ext}^2(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}(\Theta_\alpha), \mathcal{L}_{\hat{w}}) \to 0.$$

Given the natural isomorphism

$$\mathrm{Ext}^1(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}, I_{i_\alpha(q_j)} \otimes \mathcal{L}_{\hat{w}}) \cong \mathrm{Ext}^1(I_{i_\alpha(q_{3-j})} \otimes \mathcal{L}_{\hat{w}}, \mathcal{L}_{\hat{w}})$$

we have $e_j = \beta_j(f_j)$. The transpose of λ_j is given by the restriction map

$$H^0(I_{i_\alpha(q_{3-j})}(\Theta_\alpha)) \xrightarrow{\ \lambda_j^*\ } H^0(C; \mathbf{C}_{q_{3-i}} \oplus K_C(-q_{3-j})),$$

which has image equal to $H^0(C; \mathbf{C}_{q_{3-i}})$. By (6.7.68) we get that

$$\mathrm{Ann}(\gamma_j(e_j)) = \mathrm{Im}(H^0(C; K_C(-q_{3-j})) \hookrightarrow H^0(C; K_C)).$$

(We make the identification $H^1(\mathcal{O}_J)^* \cong H^1(\mathcal{O}_C)^* \cong H^0(K_C)$.) This proves (6.7.67) and finishes the proof of (6.7.58).

Proof of Proposition (6.7.1). By (6.7.40)-(6.7.58) we see that ϵ_α^+ is a bijection between Y_α^+ and $\widetilde{V}_\alpha$. Since P_α^+ is smooth, in order to prove the proposition it suffices to show that ϵ_α^+ has injective differential everywhere. This is true away from Y_α^+ by (6.7.40), hence we are left with the task of proving that

$$(6.7.69) \qquad d\epsilon_\alpha^+(t) \text{ is injective for } t \in Y_\alpha^+.$$

First we notice that we have an equality of Cartier divisors

$$(6.7.70) \qquad (\epsilon_\alpha^+)^* \widetilde{\Sigma} = Y_\alpha^+.$$

In fact by (6.7.53) Y_α^+ is irreducible, hence it suffices to show that for some $t \in Y_\alpha^+$ there exists $v \in T_t P_\alpha^+$ such that

$$(6.7.71) \qquad d\epsilon_\alpha^+(v) \notin T_{\epsilon_\alpha^+(t)} \widetilde{\Sigma}.$$

In order to exhibit such a t we let $\Gamma_k^+ := \beta_+(\overline{\Gamma}_k)$ and Γ^+ be the union of the Γ_k^+'s (for $k = 1, \ldots, 16$): thus Γ_k^+ is a $(-1, -1)$-curve of P_α^+. By (6.7.25) and (6.7.59)

$$(6.7.72) \qquad \Gamma^+ \subset Y_\alpha^+ \text{ and is the nodal curve of } Y_\alpha^+.$$

Assume $\mathcal{N}$ is sufficiently small (as in (6.7.31)): then (6.7.71) holds for all $t \in (\beta_+(\mathcal{N}) \setminus \Gamma^+)$ because $\mathcal{W}^\alpha(\mathcal{N})_t$ is simple. Now let $t \in (Y_\alpha^+ \setminus \Gamma^+)$, i.e. in the smooth locus of Y_α^+. By (6.7.70) in order to prove (6.7.69) it suffices to verify that the restriction of $d\epsilon_\alpha^+(t)$ to $T_t Y_\alpha^+$ is injective: this is straightforward. If instead $t \in \Gamma^+$ one verifies by an explicit computation that (6.7.69) holds.

6.8. Topological results. First notice that $\widetilde{V}_\alpha$ is isomorphic to P_α^+ by (6.7.1), and that P_α^+ is birational to P_α^-, hence $\widetilde{V}_\alpha$ is birational to P_α^-. Since P_α^- is a $\mathbb{P}^1$-bundle over $\widehat{I}_\alpha$ (see (6.6.8)) it is irreducible, and thus

$$(6.8.1)^* \qquad\qquad \widetilde{V}_\alpha \text{ is irreducible.}$$

Since both $\widetilde{V}_\alpha$ and P_α^- are smooth we have $H^1(\widetilde{V}_\alpha) \cong H^1(P_\alpha^-)$. Composing the isomorphisms

$$H^1(\widehat{J}) \xrightarrow{\overset{\nu_\alpha^*}{\sim}} H^1(\widehat{I}_\alpha) \xrightarrow{\overset{f_-^*}{\sim}} H^1(P_\alpha^-)$$

we get that $b^1(P_\alpha^-) = 4$, hence

$$(6.8.2)^* \qquad\qquad b^1(\widetilde{V}_\alpha) = 4.$$

Let $\Delta_k := \epsilon_\alpha^+(\Gamma_k^+)$ (for $k = 1, \ldots, 16$) and $\Delta := \epsilon_\alpha^+(\Gamma^+)$. (Thus Δ_k is a $(-1, -1)$ curve of $\widetilde{V}_\alpha$.) Let $\Lambda(\widetilde{V}_\alpha) \subset H^2(\widetilde{V}_\alpha; \mathbb{Z})$ be the subgroup defined by

$$\Lambda(\widetilde{V}_\alpha) := \{ \tau \in H^2(\widetilde{V}_\alpha; \mathbb{Z}) \mid \int_{\Delta_j} \tau = \int_{\Delta_k} \tau, \quad j, k = 1, \ldots, 16 \}.$$

We claim that

$$(6.8.3)^* \qquad \mathrm{Im}(H^2(\widetilde{\mathcal{M}}; \mathbb{Z}) \to H^2(\widetilde{V}_\alpha; \mathbb{Z})) \subset \Lambda(\widetilde{V}_\alpha),$$

where the map between H^2's is given by restriction. To prove (6.8.3) we notice that by (6.7.72) and (6.7.58) each Δ_k belongs to $\widetilde{\Sigma}$. Furthermore each Δ_k is a fiber of $\widetilde{\Sigma} \xrightarrow{\widetilde{\pi}} \Sigma$. Over $\Sigma \setminus \Omega$ (where Ω is the set of equivalence classes of sheaves $(I_p \otimes L) \oplus (I_p \otimes L)$) the map $\widetilde{\pi}$ is a locally-trivial fibration, hence all the fibers represent the same homology class in $H_2(\widetilde{\mathcal{M}}; \mathbb{Z})$. Thus the integral over Δ_k of a class in $H^2(\widetilde{\mathcal{M}}; \mathbb{Z})$ is independent of k, and this proves (6.8.3). We will need to know that

$$(6.8.4)^* \qquad\qquad \Lambda(\widetilde{V}_\alpha) \cong \mathbb{Z}^8.$$

In order to prove it we first notice that there are canonical isomorphisms

$$(6.8.5) \qquad H^2(\widetilde{V}_\alpha; \mathbb{Z}) \cong H^2(\widetilde{V}_\alpha \setminus \Delta; \mathbb{Z}) \cong H^2(P_\alpha^- \setminus \Gamma^-; \mathbb{Z}) \cong H^2(P_\alpha^-; \mathbb{Z}).$$

In fact the first map is an isomorphism because $\widetilde{V}_\alpha$ is smooth and Δ has (complex) codimension 2, the second map is an isomorphism because $(\widetilde{V}_\alpha \setminus \Delta) \cong (P_\alpha^- \setminus \Gamma^-)$, and the third map is an isomorphism because P_α^- is smooth and Γ^- has (complex) codimension 2. The last term on the right of (6.8.5) is isomorphic to $\mathbb{Z}^{23}$ because P_α^- is a $\mathbb{P}^1$-bundle over $\widehat{I}_\alpha$, hence in order to prove (6.8.4) we must show that the classes in $H_2(\widetilde{V}_\alpha; \mathbb{Z})$ represented by $\Delta_1, \ldots, \Delta_{16}$ are independent. Let $\overline{E}_k \subset \overline{P}_\alpha$ be the strict transform of E_k^- (for the blow-up β_-), let $E_k^+ := \beta_+(\overline{E}_k)$, and $D_k := \epsilon_\alpha^+(E_k^+)$. One easily checks that

$$\langle c_1(D_i), \Delta_j \rangle = \delta_{i,j},$$

where $\delta_{i,j}$ is Kronecker's symbol. This shows that the homology classes represented by $\Delta_1, \ldots, \Delta_{16}$ are independent, and proves (6.8.4).

Now we give results on $\widetilde{V}_\alpha \cap \widetilde{\Sigma}_\alpha$. By (6.7.59), (6.7.72), (6.7.58) and (6.7.53) we get the following result.

(6.8.6)*Proposition. *The intersection $\widetilde{V}_\alpha \cap \widetilde{\Sigma}_\alpha$ contains Δ as a double curve, and is smooth away from Δ. The normalization of $\widetilde{V}_\alpha \cap \widetilde{\Sigma}_\alpha$ is isomorphic to the blow up of Ξ_α at the points of Ω_α (see the statement of (6.7.53)).*

Now we prove that

$$(6.8.7)^* \qquad\qquad b^1(\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha) = 24.$$

We identify $\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha$ with Y_α^+ and its normalization with $\overline{Y}_\alpha$, according to (6.7.58)-(6.7.59). Let $E_k(i) \subset \overline{Y}_\alpha$ be the exceptional divisor of $\pi_\alpha \circ \rho_\alpha$ mapping to the point of Ω_α indicized by k, i (see (6.7.53)). By (6.7.59) the smooth locus of Y_α^+ is isomorphic to $(\overline{Y}_\alpha \setminus \bigcup_{k,i} E_k(i))$. We claim that the long exact sequence of the couple $(Y_\alpha^+, sm(Y_\alpha^+))$ (we let $sm(X)$ be the smooth locus of a variety X) gives an exact sequence

$$(6.8.8) \qquad 0 \to H^1(Y_\alpha^+, sm(Y_\alpha^+)) \xrightarrow{\lambda} H^1(Y_\alpha^+) \xrightarrow{\phi} H^1(\overline{Y}_\alpha \setminus \bigcup_{k,i} E_k(i)) \to 0.$$

In fact injectivity of λ follows immediately from connectivity of $\overline{Y}_\alpha$ (see (6.7.53)). To show that ϕ is surjective we must prove that

$$H^2(Y_\alpha^+, sm(Y_\alpha^+)) \to H^2(Y_\alpha^+)$$

is injective. It suffices to verify injectivity of the map

$$\mu \colon H^2(Y_\alpha^+, sm(Y_\alpha^+)) \to H^2(\overline{Y}_\alpha)$$

obtained by composing with the homomorphism $H^2(Y_\alpha^+) \to H^2(\overline{Y}_\alpha)$ induced by normalization. Let $U_1, \ldots, U_{16} \subset Y_\alpha^+$ be pairwise disjoint neighborhoods of $\Gamma_1^+, \ldots, \Gamma_{16}^+$ gotten by gluing toghether tubular neighborhoods of $E_k(1)$ and $E_k(2)$, for $k = 1, \ldots, 16$. Excision and Mayer-Vietoris give

$$H^2(Y_\alpha^+, sm(Y_\alpha^+)) = \bigoplus_{k=1}^{16} \mathbb{Q}\gamma_k,$$

where the restriction of γ_k to Γ_k^+ is a generator of $H^2(\Gamma_k^+)$, say of degree (-1), and the restriction to Γ_i^+ for $i \neq k$ vanishes. Then $\mu(\gamma_k) = c_1(E_k(1)) + c_1(E_k(2))$. Since the classes $\{c_1(E_k(j))| \ k = 1, \ldots, 16, \ j = 1, 2\}$ are linearly independent, μ is injective. This proves that (6.8.8) is exact. We go on with the proof of (6.8.7): we have

$$H^1(\overline{Y}_\alpha \setminus \bigcup_{k,i} E_k(i)) \cong H^1(\Xi_\alpha \setminus \Omega_\alpha) \cong H^1(\Xi_\alpha),$$

where Ξ_α is as in (6.7.53). Below we will prove that

(6.8.9) $$b^1(\Xi_\alpha) = 8.$$

Granting this, $b^1(\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha)$ is computed as follows. The first term appearing in (6.8.8) is computed applying excision and Künneth: the result is that

$$\dim H^1(Y_\alpha^+, Y_\alpha^+ \setminus sing(Y_\alpha^+)) = 16.$$

Thus (6.8.8) gives $b^1(\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha) = 24$.

Proof of (6.8.9). Let $m_2 \colon J \to J$ be "multiplication by two". Let $\widetilde{C} := m_2^{-1}(\Theta)$, and

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{f} & C \\ x & \mapsto & u^{-1}(2x), \end{array}$$

where u is the Abel-Jacobi map of (1.6). Thus f is the Galois cover of C with $J[2]$ as group of deck transformations. The map

(6.8.10) $$\begin{array}{ccc} \widetilde{C} \times \widetilde{C} & \xrightarrow{q_\alpha} & \Xi_\alpha \\ (x,y) & \mapsto & (f(x), \hat{x} - \hat{y} + \hat{\alpha}) \end{array}$$

is the quotient map for the diagonal action of $J[2]$ on $\widetilde{C} \times \widetilde{C}$. Thus $H^1(\Xi_\alpha)$ is isomorphic to the subspace of $H^1(\widetilde{C} \times \widetilde{C})$ invariant for this action. If $\pi_i \colon \widetilde{C} \times \widetilde{C} \to \widetilde{C}$ is the projection to the i-th factor,

$$H^1(\widetilde{C} \times \widetilde{C})^{J[2]} = \pi_1^* f^* H^1(C) \oplus \pi_2^* f^* H^1(C).$$

This proves (6.8.9). $\quad \triangle$

The next result will be used to prove that $\widetilde{\mathcal{M}}$ is simply-connected. Let R be as in (5.2.4): by Item (2) of (4.3.2) we have an inclusion $i \colon R \hookrightarrow \widetilde{\Sigma}_\alpha$.

(6.8.11)Lemma. *Keeping notation as above, we have*

$$\mathrm{Im}(i_{\#}) \subset \mathrm{Im}(\pi_1(\widetilde{V}_\alpha \cap \widetilde{\Sigma}_\alpha) \to \pi_1(\widetilde{\Sigma}_\alpha)).$$

Proof. Referring to (6.7.53) let h_α be given by

$$
\begin{array}{ccc}
C & \xrightarrow{h_\alpha} & \Xi_\alpha \\
p & \mapsto & (p, \hat{\alpha}).
\end{array}
$$

By (6.7.53) the map h_α lifts to a map $\bar{h}_\alpha \colon C \hookrightarrow \overline{Y}_\alpha$. Composing with the quotient map $\overline{Y}_\alpha \to Y_\alpha^+$ (see (6.7.59)) we get an inclusion $h_\alpha^+ \colon C \hookrightarrow Y_\alpha^+$. Let $\ell_\alpha^+ \colon Y_\alpha^+ \hookrightarrow \widetilde{\Sigma}_\alpha$ be the inclusion given by (6.7.58). As is easily checked the image of

$$(6.8.12) \qquad\qquad \ell_{\alpha,\#}^+ \circ h_{\alpha,\#}^+ \colon \pi_1(C) \to \pi_1(\widetilde{\Sigma}_\alpha)$$

is contained in the image of $i_{\#}$. Next let $L_1, L_2 \subset \overline{Y}_\alpha$ be two exceptional divisors of $\pi_\alpha \circ \rho_\alpha$ whose images in Ω_α are indicized by the same k (see (6.7.53)): thus the equivalence relation $\sim$ of (6.7.59) glues together L_1 and L_2. Let $\bar{\gamma} \colon [1, 2] \to \overline{Y}_\alpha$ be a continuous path such that $\bar{\gamma}(i) \in L_i$ and $\gamma(1) \sim \gamma(2)$. Composing $\bar{\gamma}$ with the quotient map $\overline{Y}_\alpha \to Y_\alpha^+$ we get a loop $\gamma^+ \colon [1, 2] \to Y_\alpha^+$ such that $\ell_{\alpha,\#}^+(\gamma^+)$ is in the image of $i_{\#}$. Furthermore one easily checks that $i_{\#}\pi_1(R)$ is generated by the image of (6.8.12) and by $\ell_{\alpha,\#}^+(\gamma^+)$. Since both $\ell_{\alpha,\#}^+ \circ h_{\alpha,\#}^+(C)$ and $\ell_{\alpha,\#}^+(\gamma^+)$ are contained in $\widetilde{V}_\alpha \cap \widetilde{\Sigma}_\alpha$, this implies the lemma. $\triangle$

Next we give results on $\widetilde{V}_\alpha \cap \widetilde{B}_\alpha$. The composition

$$P_\alpha^+ \overset{\beta_+^{-1}}{\dashrightarrow} \overline{P}_\alpha \xrightarrow{\beta_-} P_\alpha^- \xrightarrow{f_-} \widehat{I}_\alpha$$

is a rational map $f_+ \colon P_\alpha^+ \dashrightarrow \widehat{I}_\alpha$, regular away from Γ^+. The following result follows immediately from (6.5.4) and (6.7.17).

(6.8.13)*Proposition. *Keeping notation as above, the intersection of $\widetilde{V}_\alpha \cap \widetilde{B}_\alpha$ and Δ inside $\widetilde{V}_\alpha$ is transverse, and consists of 16 points, one on each Δ_k. The rational map*

$$f_+ \circ (\epsilon_\alpha^+)^{-1}|_{\widetilde{V}_\alpha \cap \widetilde{B}_\alpha} \colon \widetilde{V}_\alpha \cap \widetilde{B}_\alpha \dashrightarrow \widehat{I}_\alpha$$

is obtained by blowing up $\widetilde{B}_\alpha \cap \Delta$, and then contracting 16 disjoint (-1) curves (not intersecting the exceptional divisor of the blow-up of $\widetilde{B}_\alpha \cap \Delta$) to $\nu_\alpha^{-1}\widehat{J}[2]$.

(6.8.14)*Corollary. *The map*

$$(6.8.15) \qquad\qquad \pi_1(\widetilde{V}_\alpha \cap \widetilde{B}_\alpha) \to \pi_1(\widetilde{V}_\alpha)$$

induced by inclusion is an isomorphism. In particular

$$(6.8.16)^* \qquad\qquad b^1(\widetilde{V}_\alpha \cap \widetilde{B}_\alpha) = 4.$$

Proof of the corollary. We have a series of isomorphisms

$$(6.8.17) \qquad \pi_1(\widetilde{V}_\alpha) \cong \pi_1(P_\alpha^+) \cong \pi_1(P_\alpha^+ \setminus \Gamma^+) \xrightarrow{f_{+,\#}} \pi_1(\widehat{I}_\alpha \setminus E) \cong \pi_1(\widehat{I}_\alpha).$$

If we compose with the map of (6.8.15) we get an isomorphism $\pi_1(\widetilde{V}_\alpha \cap \widetilde{B}_\alpha) \xrightarrow{\sim} \pi_1(\widehat{I}_\alpha)$, by (6.8.13). Thus the map of (6.8.15) must be an isomorphism. Formula (6.8.16) follows from the first statement of the corollary and (6.8.2). $\triangle$

Another result on $\widetilde{V}_\alpha \cap \widetilde{B}_\alpha$ that will be useful is the following:

(6.8.18)*   the map $\pi_1(\widetilde{V}_\alpha \cap \widetilde{B}_\alpha) \to \pi_1(\widetilde{B}_\alpha)$ induced by inclusion is trivial.

In fact by (6.5.4) we have

$$\widetilde{V}_\alpha \cap \widetilde{B}_\alpha \subset g_1^{-1}(\text{node of } \overline{C}),$$

where g_1 is the map of (5.1.2). Since the right-hand side of the above formula is a $\mathbb{P}^1$-bundle over the simply-connected surface $K^{[2]}\widehat{J}$ we get (6.8.18).

We give a description of the triple intersection $\widetilde{V}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{\Sigma}_\alpha$. Let $D_i \subset \widehat{J}$ (for $i = 1, 2$) be the smooth irreducible curve defined by

$$D_i := \{\hat{y} \in \widehat{J} |\ 2y \sim q_{3-i} - r, \quad \text{some } r \in C\}.$$

The last statement of (6.5.14) gives an inclusion $D_i \hookrightarrow P_\alpha$. Composing with the birational map $P_\alpha \cdots > \widetilde{V}_\alpha$ we get an inclusion $h_i \colon D_i \hookrightarrow \widetilde{V}_\alpha$. It follows from (6.5.14) that

(6.8.19) $h_1(D_1) \cap h_2(D_2) = \widetilde{B}_\alpha \cap \Delta.$

The following proposition also follows easily from (6.5.14).

(6.8.20)***Proposition.** *Keeping notation as above*

$$\widetilde{V}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{\Sigma}_\alpha = h_1(D_1) \cup h_2(D_2).$$

In particular by (6.8.19) the curve $\widetilde{V}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{\Sigma}_\alpha$ is connected.

Finally we give some results on restriction maps. Let $\Pi(\widetilde{\Sigma}_\alpha) \subset H^2(\widetilde{\Sigma}_\alpha)$ be given by

(6.8.21)* $\Pi(\widetilde{\Sigma}_\alpha) := \mathbb{Q}c_1(\omega_f) \oplus H^2(C) \oplus H^2(\widehat{J}) \oplus \mathbb{Q}c_1((i_\alpha \times \mathrm{id}_{\widehat{j}})^* \mathcal{L}),$

where the right-hand side is viewed as a subspace of $H^2(\widetilde{\Sigma}_\alpha)$ thanks to (4.1.3) and Item (3) of (4.2.5). Let ζ_α be as in (6.7.51): we claim that

(6.8.22)* $\Pi(\widetilde{\Sigma}_\alpha) \to H^2(\zeta_\alpha^{-1}(\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha))$ is injective.

(The map is the restriction to $\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha$ followed by pull-back for ζ_α.) By (6.7.58) and (6.7.59) we have $\zeta_\alpha^{-1}(\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha) \cong \overline{Y}_\alpha$, and the restriction to $\overline{Y}_\alpha$ of the map $f \circ \zeta_\alpha$ (f is as in (4.1.1)) is identified with the blow-down map $\overline{Y}_\alpha \to \Xi_\alpha$, where Ξ_α is as in (6.7.53). Thus ω_f has degree (-2) on each of the exceptional divisors of $\overline{Y}_\alpha \to \Xi_\alpha$, and all the elements of the other direct summands of (6.8.21) have

degree zero on the exceptional divisors. Hence the kernel of the map of (6.8.22) is contained in

$$(6.8.23) \qquad\qquad H^2(C) \oplus H^2(\widehat{J}) \oplus \mathbb{Q}c_1((i_\alpha \times \mathrm{id}_{\widehat{j}})^* \mathcal{L}).$$

Let $q_\alpha: \widetilde{C} \times \widetilde{C} \to \Xi_\alpha$ be as in (6.8.10). In order to prove (6.8.22) it suffices to show that the natural map from (6.8.23) to $H^2(\widetilde{C} \times \widetilde{C})$ is injective. This is an easy exercise given the explicit formula for q_α of (6.8.10).

Let

$$H^1(\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha) \xrightarrow{r_1} H^1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha) \qquad H^1(\widetilde{B}_\alpha \cap \widetilde{V}_\alpha) \xrightarrow{r_2} H^1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha)$$

be the maps induced by restriction. We claim that

$$(6.8.24)^* \qquad\qquad \dim \mathrm{Im}(r_1) + \dim \mathrm{Im}(r_2) \geq 19.$$

In fact consider the exact sequence

$$(6.8.25) \quad 0 \to \mathbb{Q} \longrightarrow H^1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha, sm(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha))$$
$$\longrightarrow H^1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha) \xrightarrow{\psi} H^1(sm(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha))$$

We claim that the image of the map λ of (6.8.8) (recall that $\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha$ is identified with Y_α^+) maps surjectively to the kernel of the map ψ of (6.8.25), and hence

$$(6.8.26) \qquad\qquad \ker(\psi) \subset \mathrm{Im}(r_1).$$

In fact, adopting the notation used in the proof of (6.8.8),

$$H^1(Y_\alpha^+, sm(Y_\alpha^+)) \cong \bigoplus_{k=1}^{16} \widetilde{H}^0(U_k, sm(U_k)),$$

where $\widetilde{H}$ is reduced cohomology. Similarly

$$H^1(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha, sm(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha)) \cong \bigoplus_{k=1}^{16} \widetilde{H}^0(U_k \cap \widetilde{B}_\alpha, sm(U_k \cap \widetilde{B}_\alpha)).$$

The map

$$\bigoplus_{k=1}^{16} \widetilde{H}^0(U_k, sm(U_k)) \to \bigoplus_{k=1}^{16} \widetilde{H}^0(U_k \cap \widetilde{B}_\alpha, sm(U_k \cap \widetilde{B}_\alpha))$$

is an isomorphism because by (6.8.19) the nodes of $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha$ belong to the nodal curve of $\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha$, with each irreducible component of the nodal curve containing exactly one node. This proves (6.8.26). On the other hand the composition

$$H^1(\widetilde{B}_\alpha \cap \widetilde{V}_\alpha) \xrightarrow{l} H^1(D_1) \xrightarrow{m} H^1(sm(\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha))$$

is injective because both l and m are injective, and hence (6.8.16) gives that $\dim \psi(\mathrm{Im}(r_2)) = 4$. By (6.8.26) we conclude that (6.8.24) holds.

7. Proof of Theorem (1.4)

By (2.3.2) and (2.3.6) we know that $\widetilde{\mathcal{M}}$ is of pure dimension 6 and that $\widetilde{\omega}$ is symplectic.

7.1. Proof that $\widetilde{\mathcal{M}}$ is connected and simply connected.
By (3.2) it suffices to show that

$$(7.1.1) \qquad\qquad \widetilde{Z}_\alpha \text{ is connected,}$$

$$(7.1.2) \qquad\qquad \pi_1(\widetilde{Z}_\alpha) = \{1\},$$

where

$$(7.1.3) \qquad\qquad \widetilde{Z}_\alpha = \widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha \cup \widetilde{V}_\alpha.$$

Let us prove (7.1.1). By (4.4.1), (5.2.1), (6.8.1) each of $\widetilde{\Sigma}_\alpha$, $\widetilde{B}_\alpha$, $\widetilde{V}_\alpha$ is irreducible, and by (4.3.2), (6.8.6) and (6.8.13) every pairwise intersection is non-empty. By (7.1.3) we get that $\widetilde{Z}_\alpha$ is path-connected. Now we prove (7.1.2). First let's show that $\pi_1(\widetilde{Z}_\alpha)$ is generated by the images of $\pi_1(\widetilde{\Sigma}_\alpha)$, $\pi_1(\widetilde{B}_\alpha)$ and $\pi_1(\widetilde{V}_\alpha)$. By (6.8.6)-(6.8.13) both $\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha$ and $\widetilde{B}_\alpha \cap \widetilde{V}_\alpha$ are irreducible, hence connected, and by (6.8.20) the triple intersection $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha$ is non-empty: thus $(\widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha) \cap \widetilde{V}_\alpha$ is connected. Applying Van Kampen to $\widetilde{Z}_\alpha = (\widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha) \cup \widetilde{V}_\alpha$ we get that $\pi_1(\widetilde{Z}_\alpha)$ is generated by $\pi_1(\widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha)$ and $\pi_1(\widetilde{V}_\alpha)$. By (4.3.2) $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha$ is irreducible, hence connected, so that by Van Kampen $\pi_1(\widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha)$ is generated by $\pi_1(\widetilde{\Sigma}_\alpha)$ and $\pi_1(\widetilde{B}_\alpha)$. This proves that $\pi_1(\widetilde{Z}_\alpha)$ is generated by the images of $\pi_1(\widetilde{\Sigma}_\alpha)$, $\pi_1(\widetilde{B}_\alpha)$ and $\pi_1(\widetilde{V}_\alpha)$. By (4.4.6) and (6.8.14) the images of $\pi_1(\widetilde{\Sigma}_\alpha)$ and $\pi_1(\widetilde{V}_\alpha)$ are contained in the image of $\pi_1(\widetilde{B}_\alpha)$, hence $\pi_1(\widetilde{B}_\alpha) \to \pi_1(\widetilde{Z}_\alpha)$ is surjective. By (5.2.5) we get that $\pi_1(R) \to \pi_1(\widetilde{Z}_\alpha)$ is surjective, where R is as in (5.2.4). Thus (6.8.11) gives that $\pi_1(\widetilde{V}_\alpha) \to \pi_1(\widetilde{Z}_\alpha)$ is surjective, and hence by (6.8.14) $\pi_1(\widetilde{V}_\alpha \cap \widetilde{B}_\alpha)$ generates $\pi_1(\widetilde{Z}_\alpha)$. By (6.8.18) we conclude that $\widetilde{Z}_\alpha$ is simply-connected.

7.2. Proof that $b_2(\widetilde{\mathcal{M}}) \leq 8$.
The key is the following result.

(7.2.1) Proposition. *The map $H^2(\widetilde{\mathcal{M}} : \mathbb{Z}) \to H^2(\widetilde{V}_\alpha; \mathbb{Z})$ induced by restriction is injective.*

The proposition implies the claimed bound on $b_2(\widetilde{\mathcal{M}})$ because by (6.8.3) the image of the restriction map is contained in $\Lambda(\widetilde{V}_\alpha)$, and this group has rank 8 by Formula (6.8.4).

The first element in the proof of (7.2.1) consists of a Mayer-Vietoris argument. Let $X_1 := \widetilde{\Sigma}_\alpha$, $X_2 := \widetilde{B}_\alpha$, $X_3 := \widetilde{V}_\alpha$, and

$$C^p(H^q) := \bigoplus_{1 \leq i_0 < \cdots < i_p \leq 3} H^q(X_{i_0} \cap \cdots \cap X_{i_p}; \mathbb{Q}).$$

We let $\delta \colon C^p(H^q) \to C^{p+1}(H^q)$ be the usual Čech cochain map, and $Z^p(H^q) \subset C^p(H^q)$ be the group of δ-cocycles.

(7.2.2)Proposition. *The map*

$$H^2(\widetilde{Z}_\alpha) \to Z^0(H^2)$$

induced by restriction is injective.

Proof. This is proved by considering the Mayer-Vietoris spectral sequence associated to the decomposition $\widetilde{Z}_\alpha = \widetilde{\Sigma}_\alpha \cup \widetilde{B}_\alpha \cup \widetilde{V}_\alpha$, and applying our results on the cohomology maps induced by restriction. More precisely one can triangulate $\widetilde{Z}_\alpha$ so that every intersection of the X_i's is the support of a sub-triangulation. Let

$$E_0^{p,q} := \bigoplus_{1 \le i_0 < \cdots < i_p \le 3} S^q(X_{i_0} \cap \cdots \cap X_{i_p}),$$

where $S^q(X_{i_0} \cap \cdots \cap X_{i_p})$ is the group of cochains supported on $X_{i_0} \cap \cdots \cap X_{i_p}$. Letting $\tilde{\delta}: E_0^{p,q} \to E_0^{p+1,q}$ be the Čech cochain map, and $d: E_0^{p,q} \to E_0^{p,q+1}$ be the map induced by the coboundary of simplicial cochains, we get a double complex which computes the cohomology of $\widetilde{Z}_\alpha$, because $\tilde{\delta}$ is exact except at $E_0^{0,q}$, where its homology is the group of simplicial cochains of $\widetilde{Z}_\alpha$ (see [W, p. 202]). On the other hand the filtration "by p" generates a spectral sequence with E_1-term given by

$$E_1^{p,q} = C^p(H^q),$$

whose differential $E_1^{p,q} \to E_1^{p,q+1}$ is equal to the Čech cochain map δ considered above. Thus $E_2^{0,2} = Z^0(H^2)$, and in order to prove the lemma it suffices to show that

$$E_2^{2,0} = E_2^{1,1} = 0.$$

The first vanishing is trivial: by (6.8.20) the intersection $\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha \cap \widetilde{V}_\alpha$ is connected, hence $\delta: E_1^{1,0} \to E_1^{2,0}$ is surjective. To prove the second vanishing we must show that the complex

$$(7.2.3) \qquad\qquad C^0(H^1) \xrightarrow{\delta^{0,1}} C^1(H^1) \xrightarrow{\delta^{1,1}} C^2(H^1)$$

is exact. By (4.4.2), (5.2.2) and (6.8.2) we have

$$(7.2.4) \qquad\qquad\qquad \dim C^0(H^1) = 14,$$

and Formulae (4.4.3), (6.8.7), (6.8.16) give

$$(7.2.5) \qquad\qquad\qquad \dim C^1(H^1) = 33.$$

We claim that

$$(7.2.6) \qquad\qquad \text{the map } \delta^{0,1} \text{ is injective.}$$

This may be verified directly, or else one may argue that since $E_2^{2,0} = 0$ we have $H^1(\widetilde{Z}_\alpha) = E_2^{0,1}$; by (7.1.2) we get $E_2^{0,1} = 0$, i.e. $\ker(\delta^{0,1}) = 0$. By (6.8.24) we have

$$(7.2.7) \qquad\qquad\qquad \text{rk}(\delta^{1,1}) \ge 19.$$

Exactness of (7.2.3) follows at once from (7.2.4), (7.2.5), (7.2.6) and (7.2.7).   △

We represent elements of $Z^0(H^2)$ as $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_i \in H^2(X_i; \mathbb{Q})$. The second element in the proof of (7.2.1) consists of the following result.

(7.2.8)Proposition. *Keep notation as above. The map*

$$
\begin{array}{ccc}
Z^0(H^2) & \longrightarrow & H^2(\widetilde{V}_\alpha) \\
(\gamma_1, \gamma_2, \gamma_3) & \mapsto & \gamma_3
\end{array}
$$

is injective.

Proof. Since γ is a δ-cocycle

$$
(7.2.9) \qquad \qquad \gamma_1|_{\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha} = \gamma_2|_{\widetilde{\Sigma}_\alpha \cap \widetilde{B}_\alpha}.
$$

Thus (5.2.7) gives that

$$
\tilde{\kappa}^*(\gamma_1) \in \mathbb{Q}[C \times E] \oplus (\mathrm{id}_C \times \nu_0)^*(H^2(C) \oplus H^2(\widehat{J}) \oplus \mathbb{Q}c_1((i_\alpha \times \mathrm{id}_{\widehat{j}})^*\mathcal{L})).
$$

Applying (4.4.9) we get that $\gamma_1 \in \Pi(\widetilde{\Sigma}_\alpha)$, where $\Pi(\widetilde{\Sigma}_\alpha)$ is as in (6.8.21). Now assume $\gamma_3 = 0$. Since γ is a cocycle we get that the restriction of γ_1 to $\widetilde{\Sigma}_\alpha \cap \widetilde{V}_\alpha$ vanishes, hence by injectivity of (6.8.22) we get $\gamma_1 = 0$. By (7.2.9) and (5.2.6) we also get $\gamma_2 = 0$. $\triangle$

Proof of Proposition (7.2.1). Since $\widetilde{\mathcal{M}}$ is simply-connected $H^2(\widetilde{\mathcal{M}}; \mathbb{Z})$ has no torsion, and thus it suffices to prove that the restriction map $H^2(\widetilde{\mathcal{M}}; \mathbb{Q}) \to H^2(\widetilde{V}_\alpha; \mathbb{Q})$ is injective. This map is equal to the composition of three maps:

$$
H^2(\widetilde{\mathcal{M}}) \to H^2(\widetilde{Z}_\alpha) \to Z^0(H^2) \to H^2(\widetilde{V}_\alpha).
$$

The first map is injective by (3.2), the second map is injective by (7.2.2) and the third map is injective by (7.2.8). Hence the composition is injective.

7.3. Generators of $H^2(\widetilde{\mathcal{M}}; \mathbb{Q})$. There exists a compactification of the moduli space of slope-stable vector-bundles on a projective surface which is "smaller" than the moduli space of Gieseker-Maruyama semistable sheaves, namely Uhlenbeck's compactification [L2,Mor]. Let $\mathcal{M}_{\mathbf{v}}^U$ be the Uhlenbeck compactification of the moduli space of slope-stable vector-bundles F on J with $v(F) = \mathbf{v}$: it is a projective variety and there exists a regular map $\varphi \colon \mathcal{M}_{\mathbf{v}} \to \mathcal{M}_{\mathbf{v}}^U$ which is an isomorphism on the subset parametrizing slope-stable vector-bundles. Let $\mathcal{M}^U := \varphi(\mathcal{M})$. We have Donaldson's homomorphism [L2,Mor]

$$
(7.3.1) \qquad \qquad \mu \colon H^2(J; \mathbb{Z}) \to H^2(\mathcal{M}^U; \mathbb{Z}),
$$

characterized by the following property: if $\mathcal{F}$ is a family of semistable sheaves on J parametrized by S, with moduli belonging to $\mathcal{M}$, and $f \colon S \to \mathcal{M}$ is the the modular map, then

$$
(7.3.2) \qquad \qquad f^* \circ \varphi^* \circ \mu(\alpha) = p_{S,*}(c_2(\mathcal{F}) \cup \alpha).
$$

(Here $p_S \colon J \times S \to S$ is the projection.)

(7.3.3)Proposition. *The homomorphism* $\widetilde{\pi}^* \circ \varphi^* \circ \mu \colon H^2(J;\mathbb{Z}) \to H^2(\widetilde{\mathcal{M}};\mathbb{Z})$ *is injective, and*

$$\widetilde{\pi}^* \circ \varphi^* \circ \mu(H^2(J;\mathbb{Z})) \otimes \mathbb{Q}, \quad \mathbb{Q}c_1(\widetilde{\Sigma}), \quad \mathbb{Q}c_1(\widetilde{B})$$

are linearly independent subspaces of $H^2(\widetilde{\mathcal{M}};\mathbb{Q})$.

Proof. Let $x \in (J \setminus J[2])$. The sheaf on $\widehat{J}$ given by

$$(7.3.4) \qquad\qquad Ext^1_{\widehat{\phi}}(\phi^*(I_x) \otimes \mathcal{L}, \phi^*(I_{-x}) \otimes \mathcal{L}^{-1})$$

is locally-free of rank two because $x \neq -x$. (Here $\phi, \widehat{\phi}$ are as in (1.9).) Let T be the projectivization of (7.3.4): it parametrizes semistable simple sheaves on J whose moduli belong to $\mathcal{M}$. In fact, letting $h \colon T \to \widehat{J}$ be the $\mathbb{P}^1$-fibration, π_J, π_T the projections of $J \times T$ to J and T respectively, and ξ the tautological sub-line-bundle on T, we have a tautological exact sequence of sheaves on $J \times T$

$$0 \to \pi_J^*(I_{-x}) \otimes (\pi_J \times h)^*\mathcal{L}^{-1} \to \mathcal{E} \to \pi_J^*(I_x) \otimes (\pi_J \times h)^*\mathcal{L} \otimes \pi_T^*\xi \to 0.$$

Since $\mathcal{E}$ is a family of simple semistable sheaves on J parametrized by T, with $v(\mathcal{E}_t) = \mathbf{v}$ and $\det(\mathcal{E}_t) \cong \mathcal{O}_J$, $\sum c_2(\mathcal{E}_t) = 0$, it induces by (2.3.7) a regular map $\gamma \colon T \to \widetilde{\mathcal{M}}$. Applying (7.3.2) one easily gets that

$$\gamma^* \circ \widetilde{\pi}^* \circ \varphi^* \circ \mu(\alpha) = h^* \circ (\delta^{-1})^*(\alpha),$$

where $\delta \colon J \to \widehat{J}$ is the isomorphism (1.7). This shows that $\widetilde{\pi}^* \circ \varphi^* \circ \mu$ is injective. Now we prove the second statement of the proposition. Let $\Gamma \subset \widetilde{\Sigma}_\alpha$ and $\Lambda \subset \widetilde{B}_\alpha$ be generic fibers of the fibrations f of (4.1.1) and g of (5.1.1), respectively. Let us prove that the intersection numbers of $c_1(\widetilde{\Sigma})$ and $c_1(\widetilde{B})$ with Γ and Λ are given by the entries of the following intersection matrix

(7.3.5)

	$c_1(\widetilde{\Sigma})$	$c_1(\widetilde{B})$
Γ	-2	1
Λ	2	-2

The restriction of $\widetilde{\pi}$ to $(\widetilde{\Sigma} \setminus \widetilde{\Omega})$ is a $\mathbb{P}^1$-fibration over $(\Sigma \setminus \Omega)$, of which Γ is a fiber: this gives the top left entry because by adjunction $K_{\widetilde{\Sigma}} \cong \mathcal{O}_{\widetilde{\Sigma}}(\widetilde{\Sigma})$. To get the other diagonal entry one proceeds similarly: a dense open subset $\widetilde{U} \subset \widetilde{B}$ containing Λ is the total space of a $\mathbb{P}^1$-fibration $G \colon \widetilde{U} \to U$ defined similarly to the map g of (5.1.1), and Λ is a fiber of G. Since by adjunction $K_{\widetilde{B}} \cong \mathcal{O}_{\widetilde{B}}(\widetilde{B})$, we get the bottom right entry. To get the off-diagonal entries first notice that by (4.3.10) $\widetilde{\Sigma}$ and $\widetilde{B}$ intersect transversely outside $\widetilde{\pi}^{-1}(\{[I_x \otimes L \oplus I_x \otimes L^{-1}\})$, hence

$$\langle c_1(\widetilde{\Sigma}), \Lambda \rangle = \#\{\lambda \in \Lambda \mid \mathcal{G}_\lambda \text{ is strictly semistable}\},$$

$$\langle c_1(\widetilde{B}), \Gamma \rangle = \#\{\gamma \in \Gamma \mid \mathcal{F}_\gamma \text{ has two singular points}\}.$$

Here $\mathcal{G}$ and $\mathcal{F}$ are families of simple semistable sheaves on J parametrized by Λ and Γ respectively which induce the inclusion maps of Λ and Γ in $\widetilde{\mathcal{M}}$ respectively.

(We have used (4.3.2) to get the second equation.) Computing the right-hand side of the above equalities we get the off-diagonal entries of (7.3.5). Now assume that

$$xc_1(\widetilde{\Sigma}) + yc_1(\widetilde{B}) + u = 0,$$

where $x, y \in \mathbb{Q}$ and $u \in \widetilde{\pi}^* \circ \varphi^* \circ \mu(H^2(J; \mathbb{Z})) \otimes \mathbb{Q}$. We intersect with Γ and Λ, and notice that

$$\langle u, \Gamma \rangle = \langle u, \Lambda \rangle = 0.$$

In fact the first intersection number vanishes because $\widetilde{\pi}$ contracts Γ, and the second intersection number vanishes because $\widetilde{\pi}(\Lambda)$ is contracted by φ (see [L2]). Since the intersection matrix of (7.3.5) is non-singular we get that $x = y = 0$, and hence also $u = 0$. $\triangle$

Now we can finish the proof of Theorem (1.4). By the previous subsection we have $b_2(\widetilde{\mathcal{M}}) \leq 8$, and by the above proposition $b_2(\widetilde{\mathcal{M}}) \geq 8$, hence $b_2(\widetilde{\mathcal{M}}) = 8$. Finally, since $c_1(\widetilde{\Sigma})$ and $c_1(\widetilde{B})$ are of type $(1,1)$ and the map $\widetilde{\pi}^* \circ \varphi^* \circ \mu$ is a morphism of Hodge structures (this follows from (7.3.2)), we get that

$$h^{2,0}(\widetilde{\mathcal{M}}) = h^{2,0}(J) = 1.$$

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