Esercizi di Istituzioni di Geometria Superiore (Prof. O'Grady) per il 26/11/2018

Esercizio 1. Let f be an *entire* function, i.e. a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$. Suppose that there exists an integer d such that

$$\lim_{|z| \to +\infty} \frac{|f(z)|}{|z|^{d+1}} = 0. \tag{1}$$

Prove that f is a polynomial of degree at most d, i.e. there exist $a_0, \ldots a_d \in \mathbb{C}$ such that $f(z) = a_0 z^d + \ldots + a_d$. (Hint: prove that $f^{(n)}(0) = 0$ for $n > d$.) In particular one gets *Liouville's Theorem*: a bounded entire function is constant.

Esercizio 2. Let $U \subset \mathbb{C}$ be open, and $a \in U$. Suppose that $f\colon (U \setminus \{a\}) \to \mathbb{C}$ is holomorphic, and that there exists $r > 0$ such that f is bounded on $B(a, r) \cap U$. *Riemann's extension Theorem* states that f extends to a holomorphic function $\widetilde{f}\colon U \to \mathbb{C}$. Prove it as follows. Let $r > 0$ be such that $\overline{B(a, r)} \subset U$. Show that the usual Cauchy integral formula holds for all $z \in (B(a, r) \setminus \{a\})$:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_a(r)} \frac{f(t)}{t - z} dt,$$

and then notice that the right hand side of the above equation extends to a holomorphic function over a as well.

Esercizio 3. Let $U \subset \mathbb{C}$ be open and connected and let $f\colon U \to \mathbb{C}$ be holomorphic non constant. Prove that f is *open*, i.e. it maps open sets to open sets, proceeding as follows. Let $a \in U$, and let

$$f(z) = \sum_{m=0}^{\infty} c_m (z - a)^m$$

be a power series expansion of f in a neighborhhod of a, say $B(a, r)$. Let m_0 be the minimum *strictly positive* natural number such that $c_{m_0} \neq 0$ (since f is not constant on U, such an m_0 exists by the Principle of analytic prolungation). Then, on $B(a, r)$ we have

$$f(z) = c_0 + c_{m_0}(z - a)^{m_0} g(z),$$

where g is holomorphic and $g(a) \neq 0$.

1. Prove that for a sufficiently small positive δ, there exists a homolorphic function $h\colon B(a, \delta)$ such that $g|_{B(a,\delta)} = h^{m_0}$ (use the Inverse function Theorem for holomorphis maps).

2. Let $\varphi\colon B(a, \delta) \to \mathbb{C}$ be the holomorphic function $\varphi(z) = c_{m_0}^{1/m_0}(z - a) \cdot h(z)$. By Item (1), on $B(a, \delta)$ we have $f(z) = c_0 + \varphi(z)^{m_0}$. Check that $\varphi'(a) \neq 0$, and hence $\varphi(B(a, \delta)) \supset B(0, \delta_1)$, for some $\delta_1 > 0$ by the Inverse function Theorem.

3. Conclude that $f(B(a, \delta)) \supset B(c_0, \delta_1^{m_0})$.

Notice that the analogous statement for differentiable (or even analytic) real functions of a real variable is *false*.

Esercizio 4. Prove the *Maximum modulus priciple:* Let $U \subset \mathbb{C}^n$ be open and connected, and let $f\colon U \to \mathbb{C}$ be holomorphic non constant. If $K \subset U$ is compact, any $z_0 \in K$ achieving the maximum of the absolute value function $|f(z)|$ is *not* an interior point of K, i.e $z_0 \in \partial K$. (Hint: if $n = 1$ the result follows at once from **Esercizio ??**. If $n > 1$ reduce to the case $n = 1$ by restricting f to lines in $\mathbb{C}^n$.)

Esercizio 5. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible 2×2 matrix. Then

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{f} & \mathbb{P}^1_{\mathbb{C}} \\ [z_0, z_1] & \mapsto & [cz_1 + dz_0, az_1 + bz_0] \end{array} \tag{2}$$

is an automorphism of $\mathbb{P}^1_{\mathbb{C}}$. (The weird choice of formula in (2) is explained by the formula $f(z) = \frac{az+b}{cz+d}$ valid when using the affine coordinate $z = z_1/z_0$.) Prove that every automorphism of $\mathbb{P}^1_{\mathbb{C}}$ (as complex manifold!) is of the above form, and hence

$$\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}) \cong PGL_2(\mathbb{C}),$$

by arguing as follows.

1. Let $\varphi \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. Composing with a suitable automorphism in (2), we may assume that $\varphi([0,1] = [0,1]$, and hence the restriction of φ to the afffine line $\mathbb{P}^1_{\mathbb{C}} \setminus \{[0,1]\}$ defines a (holomorphic) automorphism of $\mathbb{C}$.

2. Prove that (1) holds for $d = 1$, and conclude that φ is a polynomial function of degree 1 by the first exercise.

Notice that we have also proved that a holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$ is an automorphism if and only if there exists $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ such that $f(z) = az + b$.

Esercizio 6. Prove that the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\}$$

is isomorphic (as complex manifold) to the unit disc $\Delta \subset \mathbb{C}$. (Hint: find an automorphism f of $\mathbb{P}^1_{\mathbb{C}}$ which takes the closure of the real line to the boundary of the unit disc. Either f or $\frac{1}{f}$ will define an isomorphism between $\mathbb{H}$ and Δ.)