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GIT versus Baily-Borel compactification for $K3$'s which are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ ☆

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A B S T R A C T

In previous work, we have introduced a program aimed at studying the birational geometry of locally symmetric varieties of Type IV associated to moduli of certain projective varieties of $K3$ type. In particular, a concrete goal of our program is to understand the relationship between GIT and Baily-Borel compactifications for quartic $K3$ surfaces, $K3$'s which are double covers of a smooth quadric surface, and double EPW sextics. In our first paper [36], based on arithmetic considerations, we have given conjectural decompositions into simple birational transformations of the period maps from the GIT moduli spaces mentioned above to the corresponding Baily-Borel compactifications. In our second paper [35] we studied the case of quartic $K3$'s; we have given geometric meaning to this decomposition and we have partially verified our conjectures. Here, we give a full proof of the conjectures in [36] for the moduli space of $K3$'s which are double covers of a smooth quadric surface. The main new tool here is VGIT for $(2, 4)$ complete intersection curves.

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1. Introduction

1.1. Background and motivation

In the context of the search for geometric compactifications for the moduli spaces of polarized $K3$ surfaces (see [18]; and [19], [3], [45], [33], [2] for some recent references), almost forty years ago, J. Shah [50] analyzed the GIT moduli space $\mathfrak{M}_{6,2}$ of plane sextic curves and then compared it to the Baily-Borel compactification $\mathscr{F}_2^*$ of the period space of degree 2 polarized $K3$ surfaces. The natural period map $\mathfrak{p}\colon \mathfrak{M}_{6,2} \dashrightarrow \mathscr{F}_2^*$ is birational, not regular, with exactly one point of indeterminacy ω. Shah [50,49] proved that the blow up $\epsilon : \widehat{\mathfrak{M}}_{6,2} \to \mathfrak{M}_{6,2}$ of a scheme supported on ω resolves the indeterminacy of $\mathfrak{p}$, giving a regular extended period map $\widehat{\mathfrak{p}}\colon \widehat{\mathfrak{M}}_{6,2} \to \mathscr{F}_2^*$. A few years later, Looijenga [37] revisited Shah's work from a different point of view. Looijenga started from the "other end"(i.e. $\mathscr{F}_2^*$) and noted that $\widehat{\mathfrak{p}}\colon \widehat{\mathfrak{M}}_{6,2} \to \mathscr{F}_2^*$ is a small contraction, that can be interpreted as the $\mathbb{Q}$-factorialization of $\mathscr{F}_2^*$. From this perspective, the blow-up $\epsilon : \widehat{\mathfrak{M}}_{6,2} \to \mathfrak{M}_{6,2}$ can be interpreted as the contraction of (the strict transform of) a certain Heegner divisor H_u in $\mathscr{F}_2$ (here, H_u is the unigonal divisor, which parameterizes elliptic $K3$s). Later Looijenga [38,39] developed analogous ideas for pairs $(\mathscr{F}, H)$, where $\mathscr{F}$ is a locally symmetric variety of ball type or of Type IV, and H is an effective Heegner divisor. Briefly, Looijenga's theory constructs an arithmetic birational modification $\overline{\mathscr{F}}_H$ of the Baily-Borel compactification $\mathscr{F}^*$ which (birationally) contracts the divisor H. In a number of significant geometric examples (e.g. degree 2 $K3$ surfaces, Enriques surfaces [52]), it turns out that, for appropriate choice of H, $\overline{\mathscr{F}}_H$ is isomorphic to a natural GIT model.

Another important case where Looijenga's theory works perfectly is that of cubic fourfolds. As an application of the comparison between GIT and Baily-Borel models for the moduli of cubic fourfolds, Laza [30,31] and Looijenga [40] proved a surjectivity statement for the period map for cubic fourfolds, which complements Voisin's Torelli Theorem [54]. The complexity, both geometric and arithmetic, of the case of cubic fourfolds is similar to that of degree 2 $K3$ surfaces.

A case that stands in stark contrast with the examples discussed above is that of degree 4 $K3$ surfaces and their siblings (e.g. double EPW sextics [44]). Specifically, despite the apparent geometric similarity between the degree 2 and 4 cases, it is not possible to understand the precise relationship between the natural GIT quotient $\mathfrak{M}_{4,3}$ and Baily-Borel compactification $\mathscr{F}_4^*$ for quartic $K3$ surfaces by either Shah's geometric approach (see [51]), or Looijenga's theory (see [39, §8.2]). Looijenga's work hints to an arithmetic explanation for this paradox. Namely, the complexity of $\overline{\mathscr{F}}_H$ associated to a pair $(\mathscr{F}, H)$ as above is related to the codimension of the intersection loci in the hyperplane arrangement $\mathscr{H}$ defining the divisor H (recall, $\mathscr{F} = \mathscr{D}/\Gamma$ and $H = \mathscr{H}/\Gamma$ for some Γ-invariant hyperplane arrangement $\mathscr{H} \subset \mathscr{D}$). The extreme simplicity of the period map of plane sextics is explained by the fact that no two irreducible components of the corresponding arrangement $\mathscr{H}$ meet; similarly, for cubic 4-folds at most two irreducible components of $\mathscr{H}$ meet. By way of contrast, the hyperplane arrangement naturally associated to the period map of quartic surfaces is as complex as it possibly can be, i.e. there are linearly independent hyperplanes with non-empty intersection of any cardinality up to $19(=\dim \mathscr{F}_4)$.

In recent work [36,35], we have set out to revisit the case of quartic $K3$ surfaces. By refining the work of Looijenga [39], we obtained in [36] a conjectural decomposition of the period map $\mathfrak{M}_{4,3} \dashrightarrow \mathscr{F}_4^*$ for quartic surfaces into elementary birational transformations. This (conjectural) wall crossing decomposition is determined, up to a certain "depth", by following the approach of Looijenga, and for higher depths is regulated by subtle second order phenomena (of arithmetic nature) that previously had been overlooked. Our main new tools in [36] are on one hand a variational approach inspired by the so called Hassett–Keel program for the moduli of curves (e.g. see [21,22]), and on the other hand the heavy use of so called Borcherds' relations (with origins in [10,8]). Acknowledging the major influences guiding us, we have baptized ***the Hassett–Keel–Looijenga (HKL) program*** the study of the birational geometry of locally symmetric varieties of Type IV (such as moduli of polarized $K3$s). The guiding principle of this study is that *any natural/tautological model (such as GIT; see [42]) for the moduli of polarized $K3$ surfaces should be obtained via arithmetic modifications from the Baily-Borel compactification $\mathscr{F}^*$*. We regard [20] and [9] as the starting points for the investigation of the birational geometry of the moduli of $K3$ surfaces, and manifestations of this modularity principle.

In [35] we gave strong evidence (of geometric nature) in favor of the correctness of our conjectures regarding this decomposition (but significant parts of [36] still remained conjectural).

In the present paper we completely verify our conjectures on the behavior of the period map for double covers of $\mathbb{P}^1 \times \mathbb{P}^1$. Our main result, **Theorem 1.1** below, gives a highly non-trivial illustration of our Hassett-Keel-Looijenga program. We emphasize that the complexity of the period map for double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ is comparable to that of quartic surfaces (see **Remark 1.2**), and an order of magnitude higher than that for plane sextic curves [37] or cubic fourfolds [31,40]. Similar considerations to those in this paper apply to the study of the moduli space of degree 6 $K3$ surfaces (and the associated GIT models for $(2,3)$ complete intersections in $\mathbb{P}^4$); this is currently under investigation by the first author together with François Greer and Zhiyuan Li. We also expect that a version of the Hassett–Keel–Looijenga program for elliptic $K3$ surfaces (and similarly rational elliptic surfaces) to be tightly connected to the recent work of Ascher–Bejleri [1,2]. Finally, for another potential application of our results, we refer to **Remark 1.3** below.

1.2. The main result

We start by introducing the main actors. Let

$$\mathfrak{M} := |\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)| /\!\!/ \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \tag{1.2.1}$$

be the GIT moduli space of $(4,4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Let C be a $(4,4)$ curve with simple singularities (it is GIT stable by Shah [51, Sect. 4]), and let $\pi\colon X_C \to \mathbb{P}^1 \times \mathbb{P}^1$ be the double cover with branch curve C. Then X_C is a $K3$ surface (eventually with canonical singularities), and $\pi^* \operatorname{NS}(\mathbb{P}^1 \times \mathbb{P}^1)$ is a saturated copy of $U(2)$ in $\operatorname{NS}(X_C)$. Thus X_C is a $U(2)$-hyperelliptic $K3$ surface. The corresponding period space, which we denote by $\mathscr{F}$ (see **Subsection 2.2**) is an 18 dimensional locally symmetric variety. We let $\mathscr{F} \subset \mathscr{F}^*$ be the Baily-Borel compactification. By associating to a generic $[C] \in \mathfrak{M}$ the primitive Hodge structure on $H^2(X_C)$, we get the rational period map

$$\mathfrak{p}\colon \mathfrak{M} \dashrightarrow \mathscr{F}^*. \tag{1.2.2}$$

By Global Torelli, $\mathfrak{p}$ is birational. By Baily-Borel, $\mathscr{F}^*$ is identified with $\operatorname{Proj} R(\mathscr{F}, \lambda)$, where λ is the Hodge ($\mathbb{Q}$-)line bundle on $\mathscr{F}$.

In [36], we proved that also $\mathfrak{M}$ is identified with Proj of a ring of sections of a $\mathbb{Q}$-Cartier divisor on $\mathscr{F}$. Namely, let $H_h \subset \mathscr{F}$ be the Heegner divisor parametrizing periods of $K3$ surfaces which are double covers of a *quadric cone*. Let $\operatorname{Reg}(\mathfrak{p}) \subset \mathfrak{M}$ be the regular locus of $\mathfrak{p}$. Then $\mathfrak{p}(\operatorname{Reg}(\mathfrak{p})) \cap \mathscr{F}$ contains $(\mathscr{F} \setminus H_h)$ (in fact, a posteriori they are equal). We define the *boundary divisor* $\Delta := H_h/2$ (parameterizing the "missing periods"). With this notation, we proved that $\mathfrak{M}$ is identified with $\operatorname{Proj} R(\mathscr{F}, \lambda + \Delta)$ (cf. [36, Prop. 4.0.20]). The main content of [36] is to predict the behavior of the graded $\mathbb{C}$-algebra $R(\mathscr{F}, \lambda + \beta\Delta)$ for $\beta \in [0,1] \cap \mathbb{Q}$ (interpolating between the algebras associated to the Baily-Borel and GIT models). First, we conjecture that it is finitely generated. Secondly we predict the critical values of β, i.e. the Mori chamber decomposition of the sector $\{\lambda + \beta\Delta\}_{\beta \in [0,1] \cap \mathbb{Q}}$. Lastly, we describe the centers of the corresponding flips or contractions. This last part of the conjecture is formulated in terms of towers of closed subvarieties

$$Z^8 \subset Z^7 \subset Z^6 \subset Z^4 \subset Z^3 \subset Z^2 \subset Z^1 \subset \mathscr{F} \tag{1.2.3}$$

and

$$W_0 \subset W_1 \subset W_2 \subset W_3 \subset W_5 \subset W_6 \subset W_7 = \mathfrak{M}^{IV} \subset \mathfrak{M}. \tag{1.2.4}$$

(The "missing" indexes are not misprints.) The superscripts in (1.2.3) denote codimension (in $\mathscr{F}$), while those in (1.2.4) denote dimension. The definition of the stratification (1.2.3) is recalled in **Subsection 2.2**, the definition of (1.2.4) is in **Definition 4.10**. Regarding $Z^\bullet$, it will suffice to recall that $Z^1 = H_h$, and that Z^k are Shimura subvarieties of $\mathscr{F}$ corresponding to certain codimension k intersection loci of the hyperplane arrangement $\mathscr{H}_h$ (where as before, $\mathscr{H}_h$ is associated to the divisor H_h). Regarding $W_\bullet$, it will suffice to recall that $\mathfrak{M}^{IV}$ is the locus parametrizing polystable $(4,4)$ curves C such that the corresponding double cover $X_C \to \mathbb{P}^1 \times \mathbb{P}^1$ has non-slc singularities (i.e. significant limit singularities in the sense of Mumford and Shah [49]). In particular $\mathfrak{M} \setminus \mathfrak{M}^{IV}$ is contained in the regular locus of $\mathfrak{p}$ (in fact, a posteriori equal).

We are ready to state our main result.

Theorem 1.1. *With notation as above, the following hold:*

i) *Let $\beta \in [0,1] \cap \mathbb{Q}$. The ring of sections $R(\mathscr{F}, \lambda + \beta\Delta)$ is a finitely generated $\mathbb{C}$-algebra, and hence $\mathscr{F}(\beta) := \mathrm{Proj} R(\mathscr{F}, \lambda + \beta\Delta)$ is a projective variety interpolating between $\mathscr{F}^* = \mathscr{F}(0)$ and $\mathfrak{M} = \mathscr{F}(1)$.*

ii) *The variation of models $\mathscr{F}(\beta)$ on the interval $[0,1] \cap \mathbb{Q}$ has a Mori chamber decomposition whose set of critical values is*

$$\left\{0, \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}.$$

Hence for consecutive critical values $\beta' > \beta''$ it makes sense to let $\mathscr{F}(\beta', \beta'') := \mathscr{F}(\beta)$, where $\beta' > \beta > \beta''$ is arbitary.

iii) *The period map $\mathfrak{p} \colon \mathfrak{M} = \mathscr{F}(1) \dashrightarrow \mathscr{F}(0) = \mathscr{F}^*$ is the composition of the elementary birational maps in (1.2.5).*

$$\begin{tikzcd}
 & \mathscr{F}(1,\frac{1}{2}) \arrow[dl] \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & \cdots \arrow[dl] & \mathscr{F}(\beta_{k-1},\beta_k) \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & \mathscr{F}(\beta_k,\beta_{k+1}) \arrow[dl] & \cdots \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & \mathscr{F}(\frac{1}{8},0) \arrow[dl] \arrow[dr, "\Phi"] & \\
\mathfrak{M}=\mathscr{F}(1) & & \mathscr{F}(\frac{1}{2}) & & & \mathscr{F}(\beta_k) & & & \mathscr{F}(\frac{1}{8}) & & \mathscr{F}(0)=\mathscr{F}^*
\end{tikzcd} \tag{1.2.5}$$

Here the critical values of β are indexed as in (1.2.6).

k	0	1	2	3	4	5	6	7	8
β_k	1	1/2	1/3	1/4	1/4	1/5	1/6	1/8	0

(1.2.6)

(The equality $\beta_3 = \beta_4$ is not a misprint.) Let $\Omega_-(\beta_k) \subset \mathscr{F}(\beta_{k-1}, \beta_k)$ and $\Omega_+(\beta_k) \subset \mathscr{F}(\beta_k, \beta_{k+1})$ be the exceptional loci of $\mathscr{F}(\beta_{k-1}, \beta_k) \to \mathscr{F}(\beta_k)$ and $\mathscr{F}(\beta_k, \beta_{k+1}) \to \mathscr{F}(\beta_k)$ respectively. Then $\Omega_-(\beta_k)$ is the strict transform of $W_k \subset \mathscr{F}(1) = \mathfrak{M}$ for the birational map $\mathfrak{M} \dashrightarrow \mathscr{F}(\beta_{k-1}, \beta_k)$ and $\Omega_+(\beta_k)$ is the strict transform of $Z^{k+1} \subset \mathscr{F}(0) = \mathscr{F}^*$ if $k \neq 4$, and of $Z^4 \subset \mathscr{F}^*$ if $k = 4$, for the birational map $\mathscr{F}^* \dashrightarrow \mathscr{F}(\beta_k, \beta_{k+1})$.

iv) *The map $\mathscr{F}(1/8, 0) \to \mathscr{F}^*$ is the $\mathbb{Q}$-Cartierization associated to H_h. Moreover $\mathscr{F}(1/8, 0)$ is a moduli space of double covers of quadrics (possibly singular) in $\mathbb{P}^3$* with slc singularities.

Summarizing: Items (i), (ii), (iii) and the first part of Item (iv) prove that our conjectures in [36] hold for the period space of $U(2)$-hyperelliptic $K3$ surfaces, while the second part of Item (iv) is a "bonus" result which we find very interesting. Namely, it says roughly that $\mathscr{F}(1/8, 0)$ is a KSBA-like compactification for the moduli space of $U(2)$-hyperelliptic $K3$ surfaces, and that this compactification is nothing but a small partial resolution of the Baily-Borel compactification $\mathscr{F}^*$. This is an analogue of Shah's main result from [50] (i.e. $\mathscr{F}(1/8,0)$ is the analogue of $\widehat{\mathfrak{M}}_{6,2}$). (We refer to [49] and [25] for some discussion of KSBA versus Hodge theoretic degenerations.)

In addition to the results and techniques of our previous work ([36,35]), the main new tool that allows us to prove the theorem above is VGIT for $(2,4)$ complete intersection curves C in $\mathbb{P}^3$ (the double cover $X_C \to Q$ of the quadric Q containing C ramified over C is a $U(2)$-hyperelliptic $K3$ surface or a degeneration of such surfaces). A similar case, namely $(2,3)$ complete intersections, was analyzed by the first named author and his collaborators in [14,13] in the context of the Hassett-Keel program for genus 4 curves. Here, we follow in rough outline the strategy from [14,13], but as the complexity increases, some streamlining and new ideas (such as the basin of attraction arguments in **Section 6**) are necessary.

Remark 1.2. Arguably, the case of quartic $K3$ surfaces would have been of greater geometric interest, but from the perspective of the HKL program it has almost the same complexity. More precisely, in [36], we have introduced an inductive structure (a certain D-tower of locally symmetric varieties, see [36, Sect. 1]) on the moduli spaces considered here ($U(2)$-hyperelliptic $K3$ surfaces, polarized $K3$ surfaces of degree 4, etc.). We expect then (both for geometric and arithmetic reasons) that **Theorem 1.1** implies the analogous result for quartics. Furthermore, we expect to be able to bootstrap these results and understand the relationship between GIT and periods for EPW sextics ([43,44]); this is the next step in our inductive structure. The study of the moduli of EPW sextics was our original motivation for this investigation, which in turn led us to the development of a general HKL program. We further remark that a non-inductive proof for an analogue of **Theorem 1.1** for quartic $K3$s should be possible by using GIT/VGIT techniques similar to those used in this paper. (We thank O. Benoist for a suggestion that allows to reduce the HKL program for quartics to a feasible GIT computation.)

Remark 1.3. One possible application of **Theorem 1.1** is the computation of the cohomology of the Baily-Borel compactification $\mathscr{F}^*$. Specifically, Kirwan's techniques allow one to compute the cohomology of $\mathfrak{M}$ (see [24] for the case of quartics in $\mathbb{P}^3$), while (1.2.5) gives a simple wall crossing decomposition which allows one to compute (in a standard way by now) the cohomology of $\mathscr{F}^*$. The considerably simpler case of degree 2 $K3$ surfaces (where no flip occurs) was studied by Kirwan-Lee [26,27] (see also [12] for another similar example).

1.3. Structure of the paper

We start our paper with a review of the basic facts about the moduli space of $U(2)$-hyperelliptic $K3$ surfaces. In **Section 2**, we discuss the period space $\mathscr{F} = \mathscr{D}/\Gamma$ for such surfaces. We then recall the definition of the (Shimura type) loci $Z^k \subset \mathscr{F}$ that were identified in [36] as conjectural centers of the birational transformations occurring in the variation of models $\mathscr{F}(\beta)$. We also describe the loci Z^k as the periods of double covers

$X \to Q$ such that Q is a quadric cone, and the branch curve has a certain singularity at the vertex of Q (see **Proposition 2.2**) – roughly, the higher the k (corresponding to the codimension), the worse the singularity of C at the vertex of Q. We end the section with a brief discussion of the boundary components of the Baily-Borel compactification $\mathscr{F}^*$.

In **Section 3**, we discuss the other end of the period map, namely the GIT quotient $\mathfrak{M}$ for $(4,4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. This was first studied by Shah [51, Sect. 4], but we caution the reader that some of the results of [51] are incomplete. We then discuss the Hodge-theoretic stratification of $\mathfrak{M}$ analogous to that of the GIT moduli space of quartics analyzed in [35, Sect. 3] (building on ideas of Shah [49–51]). In particular, we define the tower $(W_\bullet)$ in (1.2.4).

In order to understand the variation of models $\mathscr{F}(\beta)$, we introduce in **Section 5** a "reverse variation" $\mathfrak{M}(t)$ (for $t \in (1/6-\epsilon, 1/2] \cap \mathbb{Q}$) given by VGIT which interpolates between the GIT quotient $\mathfrak{M}$ for $(4,4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and the GIT quotient $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ of the Chow variety of $(2,4)$ complete intersection curves in $\mathbb{P}^3$. The precise definition of $\mathfrak{M}(t)$ is given in (5.2.4), and is mostly based on ideas in [14], where the analogous case of $(2,3)$ curves was discussed. A key point (**Proposition 5.8**) that comes up in our analysis is that only complete intersections $V(f_2, f_4)$ are relevant for the GIT analysis of $\mathfrak{M}(t)$. We also note that for an infinite set of values of t_m, the moduli space $\mathfrak{M}(t_m)$ can be identified with GIT quotients $\mathrm{Hilb}^m_{(2,4)} /\!\!/ \mathrm{SL}(4)$, where $\mathrm{Hilb}^m_{(2,4)}$ denotes the parameter space for m-th Hilbert point of a $(2,4)$ complete intersection. In particular, one sees that the VGIT map $\mathfrak{M}(\frac{1}{2}-\epsilon) \to \mathfrak{M}(\frac{1}{2}) \cong \mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ is induced by the Hilbert-Chow morphism $\mathrm{Hilb}_{(2,4)} \to \mathrm{Chow}_{(2,4)}$.

The actual GIT analysis for $\mathfrak{M}(t)$ is accomplished in **Section 6**. Although the tools that we use are by now standard (we acknowledge the influence of [14] and [7] that studied a similar set-up to ours, and [22], [4] and [6] that focused on the relationship between GIT and Hassett-Keel program in general), the proof is a somewhat indirect, multi-step argument. First, via the numerical criterion, we can destabilize various "bad" $(2,4)$ complete intersection curves. In particular, curves that are relevant to the change of stability are contained in a quadric cone. Since a wall crossing in VGIT involves orbits with positive dimensional stabilizer, we identify a finite list $\{t_k\} \subset (0, \frac{1}{2})$ of potential critical slopes (or walls), and associated (potential) critical curves $\{C_k\}$ by analyzing $(2,4)$ curves with a positive dimensional automorphism group. We prove that the potential critical slopes/orbits actually occur via a two-step argument. First, the numerical criterion allows us to prove that the generic curve in W_k is destabilized for some $t \le t_k$. Secondly, an analysis of the basin of attraction for the potential semistable curves C_k, shows that the generic curve in W_k can not be destabilized before t_k. Thus, the stratum W_k will change from stable to unstable precisely at t_k. Furthermore, it is not hard to see that the replacement stratum is (birational to) Z^{k+1}. This concludes the GIT analysis of the quotients $\mathfrak{M}(t)$ (we refer the reader to Table 2 for a quick summary). In particular, we note that $\mathfrak{M}(\frac{1}{2}-\epsilon)$ parametrizes double covers of irreducible curves with slc singularities.

In **Section 7** we return to the period space $\mathscr{F}$ and the variation of models $\mathscr{F}(\beta)$. Given the VGIT models $\mathfrak{M}(t)$ studied in the previous sections, we complete the proof of the main **Theorem 1.1** by establishing a natural identification

$$\mathfrak{M}(t(\beta)) \cong \mathscr{F}(\beta) \tag{1.3.1}$$

for $t(\beta) = \frac{1}{4\beta+2}$. As a consequence, we note that specializing (1.3.1) to $\beta = 0$ gives:

Corollary 1.4. *The Baily-Borel compactification $\mathscr{F}^*$ for $U(2)$-hyperelliptic $K3$ surfaces is isomorphic to the GIT quotient $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ for the Chow variety of $(2,4)$ curves in $\mathbb{P}^3$.*

In the last section (**Section 8**), we expand on this isomorphism by looking at the wall map $\mathscr{F}(\epsilon) \to \mathscr{F}(0) \cong \mathscr{F}^*$ (for $\epsilon \in (0, 1/8)$). On the arithmetic side, this corresponds to the $\mathbb{Q}$-Cartierization associated to H_h (**Theorem 1.1**(4)), on the geometric side it corresponds to the map of GIT quotients induced by the Hilbert-Chow $\mathrm{Hilb}_{(2,4)} \to \mathrm{Chow}_{(2,4)}$. We believe that this dual description (arithmetic/geometric) for $\mathscr{F}(\epsilon)$, $\mathscr{F}^*$, and for the morphism relating them, is of independent geometric interest.

2. The period space and its Baily-Borel compactification

2.1. Summary

In this section, we briefly review moduli of $U(2)$-hyperelliptic $K3$ surfaces from the perspective of the period map. A $U(2)$-hyperelliptic $K3$ surface is a $U(2)$-polarized $K3$ surface in the sense of Dolgachev [17]. For a generic such surface X the associated transcendental cohomology lattice is $D_{18} \oplus U^2$. The associated 18-dimensional locally symmetric variety $\mathscr{F}$ of Type IV (see [17]) is the moduli space of $U(2)$-hyperelliptic $K3$ surfaces. Furthermore, $\mathscr{F}$ has a natural projective compactification, the Baily-Borel compactification $\mathscr{F}^*$, which can be described (**Theorem 2.3**) following Scattone [47]. We observed in [36] that $\mathscr{F}(= \mathscr{F}(18))$ fits naturally in a tower of locally symmetric varieties $\mathscr{F}(N)$ (with N indicating the dimension). In particular, $\mathscr{F}(19)$ is the moduli space $\mathscr{F}_4$ of polarized $K3$ surfaces of degree 4, while $\mathscr{F}(17)$ is naturally identified with the Heegner divisor $H_h \subset \mathscr{F}$ parameterizing double covers of the quadric cone. More generally, using this tower structure, we can define inductively the Shimura subvarieties $Z^k \subset \mathscr{F}$ mentioned in (1.2.3). A standard argument then gives geometric meaning to the Z^k's; they parametrize double covers $X \to Q$ of the quadric cone Q, such that branch curve $C \subset Q$ has specified singularity behavior at the vertex of Q (see **Proposition 2.2**).

Since most of the arguments here are either standard or occur in our previous work [36,35], we omit the proofs, and focus instead on introducing the notions and results that are needed for the rest of the paper. The interested reader can consult the preprint version of our paper ([34]) for details on the proofs of **Proposition 2.2** and **Theorem 2.3**, which are new to a certain extent.

2.2. Periods of $U(2)$-hyperelliptic $K3$ surfaces according to [36]

Let Λ_N be the lattice $U^2 \oplus D_{N-2}$, where U is the hyperbolic plane, D_{N-2} is the negative definite lattice corresponding to the Dynkin diagram D_{N-2} (for $N \geq 3$, and where D_1 is the rank 1 lattice with generator of square (-4)), and $\oplus$ will always mean *orthogonal* direct sum. Let

$$\mathscr{D}(N) := \{[\sigma] \in \mathbb{P}(\Lambda_N \otimes \mathbb{C}) \mid \sigma^2 = 0,\ (\sigma + \overline{\sigma})^2 > 0\}.$$

Then $\mathscr{D}(N)$ is a complex manifold of dimension N, and it has two connected components, interchanged by complex conjugation; let $\mathscr{D}^+(N)$ be one of the two connected components. We note that $\mathscr{D}^+(N)$ is a Type IV bounded symmetric domain. Let $O^+(\Lambda_N) < O(\Lambda_N)$ be the index two subgroup mapping $\mathscr{D}^+(N)$ to itself. In [36] we have studied the locally symmetric variety

$$\mathscr{F}(N) := \Gamma(N)\backslash\mathscr{D}^+(N) \tag{2.2.1}$$

where $\Gamma(N) = O^+(\Lambda_N)$ if $n \not\equiv 6 \pmod 8$, and $\Gamma(N)$ is a subgroup of index 3 in $O^+(\Lambda_N)$ if $n \equiv 6 \pmod 8$, see Prop. 1.2.3 [36]. We let $\mathscr{F}(N) \subset \mathscr{F}(N)^*$ be the Baily-Borel compactification.

Definition 2.1. A $U(2)$-*hyperelliptic* $K3$ surface is a double cover $X \to Q$, where Q is an irreducible quadric surface and X is a $K3$ surface (possibly with canonical singularities). The *associated degree* 4 *polarization* of X is the line-bundle L such that $L^{\otimes 2}$ is isomorphic to the pull-back of the (unique) square root of ω_Q^{-1}.

The reason for the reference to $U(2)$ is the following. Let $\widetilde{X} \to X$ and $\widetilde{Q} \to Q$ be the minimal desingularizations of X and Q respectively. Thus $\widetilde{X}$ is a smooth $K3$ surface, and $\widetilde{Q}$ is either $\mathbb{F}_0$ or $\mathbb{F}_2$. The double cover $\rho\colon X \to Q$ lifts to a double cover $\widetilde{\rho}\colon \widetilde{X} \to \widetilde{Q}$. Then $\widetilde{\rho}^* H^2(\widetilde{Q};\mathbb{Z})$ is a saturated sublattice of $H^2(\widetilde{X};\mathbb{Z})$ isomorphic to $U(2)$.

An isomorphism between $U(2)$-hyperelliptic $K3$ surfaces $X_1 \to Q_1$ and $X_2 \to Q_2$ consists of isomorphisms $f\colon Q_1 \to Q_2$ and $\widetilde{f}\colon X_1 \to X_2$ compatible with the double covers. Notice that the associated polarized $K3$ surface (X, L) of degree 4 determines the double cover $X \to Q$ up to isomorphism: in fact the map $\varphi_L\colon X \to |L|^\vee \cong \mathbb{P}^3$ has image a quadric surface isomorphic to Q, and the map $X \to \mathrm{Im}(\varphi_L)$ is isomorphic to $X \to Q$. We will denote a $U(2)$-hyperelliptic $K3$ surface $X \to Q$ also by (X, L), where L is the associated degree 4 polarization.

The period space for $U(2)$-hyperelliptic $K3$ surfaces is $\mathscr{F}(18)$, see Propositions 2.2.1 and 1.4.5 [36]. The period point of a $U(2)$-hyperelliptic $K3$ surface $X \to Q$ is defined as follows. Let $\widetilde{X} \to X$, $\widetilde{Q} \to Q$, and $\widetilde{\rho}\colon \widetilde{X} \to \widetilde{Q}$ be as above. The period point of $X \to Q$ in $\mathscr{F}(18)$ is given by the isomorphism class of the weight 2 sub Hodge structure $(\rho^* H^2(\widetilde{Q};\mathbb{Z}))^\perp \subset H^2(\widetilde{X})$.

In order to simplify notation, we let

$$\Lambda := \Lambda_{18}, \quad \mathscr{D}^{+} := \mathscr{D}^{+}(18), \quad \mathscr{F} := \mathscr{F}(18), \quad \mathscr{F}^{*} := \mathscr{F}(18)^{*}. \tag{2.2.2}$$

We recall that $w \in \Lambda$ is a *hyperelliptic vector* if $w^2 = -4$, and $\operatorname{div}(w) = 2$ (see Definition 1.3.4 and Remark 1.1.3 [36]), where $\operatorname{div}(w)$ is the *divisibility* of w, i.e. the positive generator of (w, Λ). The Heegner divisor $H_h \subset \mathscr{F}$ is the locus of $O^+(\Lambda)$-equivalence classes of points $[\sigma] \in \mathscr{D}^+_\Lambda$ such that $\sigma^\perp$ contains a hyperelliptic vector. The boundary divisor for $\mathscr{F}$ is given by $\Delta := H_h/2$, see Definition 1.3.7 in [36].

Given a smooth $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_C(4)|$, the double cover $X_C \to \mathbb{P}^1 \times \mathbb{P}^1$ ramified over C is a $U(2)$-hyperelliptic $K3$ surface. Thus we have a period map $\mathfrak{p}\colon \mathfrak{M} \dashrightarrow \mathscr{F}^*$ (Shah proved that smooth $(4,4)$ curves are stable, see Theorem 4.8 in [50], or **Proposition 3.2**). By Global Torelli, $\mathfrak{p}$ is birational. The intersection of $\mathscr{F}$ and the image of the regular locus of $\mathfrak{p}$ contains $\mathscr{F} \setminus H_h$ (a posteriori the image, in $\mathscr{F}$, of the regular locus of $\mathfrak{p}$ is equal to $\mathscr{F} \setminus H_h$). The indeterminacy locus of $\mathfrak{p}$ is a subset of $\mathfrak{M}$ of dimension 7, with an intricate Hodge-theoretic stratification. In [36] we have formulated precise predictions on the decomposition of $\mathfrak{p}$ as a composition of elementary birational maps. In particular we defined a tower of closed subsets

$$Z^8 \subset Z^7 \subset Z^6 \subset Z^4 \subset Z^3 \subset Z^2 \subset Z^1 \subset \mathscr{F} \tag{2.2.3}$$

(the superscript denotes codimension), and we motivated the expectation that the centers of the elementary birational maps are birational to the Z^k's. In agreement with Looijenga's vision, most of the Z^k's are the images in $\mathscr{F}$ of the locus of points in $\mathscr{D}^+_\Lambda$ which are orthogonal to k (at least) hyperelliptic vectors. More precisely, the Z^k's are as follows:

(1) If $1 \le k \le 4$, then $Z^k = \operatorname{Im} f_{18-k,18}$, the locus of $O^+(\Lambda)$-equivalence classes of points $[\sigma] \in \mathscr{D}^+_\Lambda$ such that $\sigma^\perp$ contains k pairwise orthogonal hyperelliptic vectors
(2) $Z^6 = \operatorname{Im}(f_{13,18} \circ q_{13})$, the locus of $O^+(\Lambda)$-equivalence classes of points $[\sigma] \in \mathscr{D}^+_\Lambda$ such that $\sigma^\perp$ contains pairwise orthogonal vectors $v_1, \ldots, v_5, w$, with $v_1, \ldots, v_5$ hyperelliptic, $w^2 = -12$, and the divisibility of w in $\{v_1, \ldots, v_5\}^\perp$ equal to 4 (see Prop. 1.6.1 [36]).
(3) $Z^7 = \operatorname{Im}(f_{12,18} \circ m_{12})$, the locus of $O^+(\Lambda)$-equivalence classes of points $[\sigma] \in \mathscr{D}^+_\Lambda$ such that $\sigma^\perp$ contains pairwise orthogonal vectors $v_1, \ldots, v_6, w$, with $v_1, \ldots, v_6$ hyperelliptic, $w^2 = -2$, and the divisibility of w in $\{v_1, \ldots, v_6\}^\perp$ equal to 2 (see Prop. 1.5.2 [36]).
(4) $Z^8 = \operatorname{Im}(f_{11,18} \circ l_{11})$, where $\operatorname{Im}(f_{11,18} \circ l_{11})$ is the locus of $O^+(\Lambda)$-equivalence classes of points $[\sigma] \in \mathscr{D}^+_\Lambda$ such that $\sigma^\perp$ contains pairwise orthogonal vectors $v_1, \ldots, v_7, w$, with $v_1, \ldots, v_7$ hyperelliptic, $w^2 = -4$, and the divisibility of w in $\{v_1, \ldots, v_7\}^\perp$ equal to 4 (see Prop. 1.5.1 [36]).

Notice that $Z^1 = H_h$.

2.3. $U(2)$-hyperelliptic $K3$ surfaces whose periods are parametrized by Z^k

In the present subsection we will freely use notation and results contained in [36].

Proposition 2.2. *Let $X \to Q$ be a $U(2)$-hyperelliptic $K3$ surface, and let $x \in \mathscr{F}$ be its period point. Let C be the branch curve of $X \to Q$. The following hold:*

(1) *Let $1 \le k \le 15$. Then $x \in \operatorname{Im} f_{18-k,18}$ if and only if Q is a quadric cone, and C has an A_m-singularity at the vertex of Q, where $m \ge (k-1)$. If $k = 8$, in addition C must not contain a line.*
(2) *$x \in \operatorname{Im}(f_{13,18} \circ q_{13})$ if and only if Q is a quadric cone, C has an A_m-singularity at the vertex of Q, where $m \ge 4$, and the support of the tangent cone of C at the vertex of Q is tangent to a line of Q.*
(3) *$x \in \operatorname{Im}(f_{12,18} \circ m_{12})$ if and only if C contains a line, and has an A_m-singularity at the vertex of Q, where $m \ge 5$.*
(4) *$x \in \operatorname{Im}(f_{11,18} \circ l_{11})$ if and only if C contains a line, and has an A_7-singularity at the vertex of Q.* $\square$

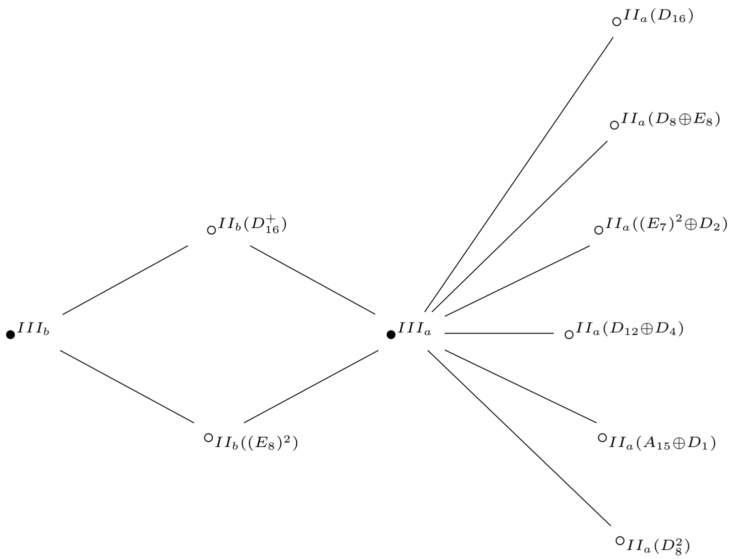
For a detailed proof of **Proposition 2.2**, see [34, §2.2] (part of the proof is already in [48], see Item (iii) of Proposition 5.7 [48]).

2.4. The boundary of the Baily-Borel compactification

The Baily-Borel compactification for quartic $K3$ surfaces (i.e. $\mathscr{F}(19)^*$) was described by Scattone [47, §6.3] (see also [11] for the case $N = 20$). Similar techniques give the following (see [34, Appendix] for details).

Theorem 2.3. *The Baily-Borel compactification $\mathscr{F}^*$ consists of*

i) *Two Type III boundary points, that we label III_a and III_b respectively.*
ii) *Eight Type II boundary components. Six of the Type II boundary components, call them of type a (labeled by D_{16}, $D_8 \oplus E_8$, $(E_7)^2 \oplus D_2$, $D_{12} \oplus D_4$, $A_{15} \oplus D_1$, and D_8^2) are incident only to III_a. The remaining two Type II boundary components (label $(E_8)^2$ and D_{16}^+) are incident to both III_a and III_b. The incidence diagram is given in (2.4.1).*



(2.4.1)

iii) *Furthermore, each of the six Type IIa boundary components are isomorphic to the modular curve* $\mathfrak{h}/\mathrm{SL}(2,\mathbb{Z})$. *Each of the two Type IIb boundary components are isomorphic to the modular curve* $\mathfrak{h}/\Gamma_0(2)$ *where* $\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ c \equiv 0(2) \right\}$. □

Remark 2.4. We recall that the Baily-Borel compactification $\mathscr{F}(19)^*$ for the moduli of degree 4 $K3$ surfaces has 9 Type II boundary components ([47, §6.3]). There is a natural map $f^*_{18,19} : \mathscr{F}^* \to \mathscr{F}(19)^*$, which is the normalization map of the image. The image $f^*_{18,19}(\mathscr{F}^*)$ only meets 8 out of the 9 Type II components of $\mathscr{F}(19)^*$, those are the one showing up in the theorem above. Over the six Type II_a components, $f^*_{18,19}$ is an isomorphism. In contrast, the restriction of $f^*_{18,19}$ to the two Type II_b components, is a 3-to-1 map (this follows from item (iii) in the theorem above).

3. Moduli of $(4,4)$ curves on a smooth quadric

3.1. Summary

The GIT moduli space $\mathfrak{M}$ of $(4,4)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ (see (1.2.1)) was analyzed by Shah [51, Sect. 4]. Furthermore, Shah introduced a Hodge theoretic stratification for the semistable curves C (or rather the associated double covers X_C), which plays a key role for us. The purpose of this section is to recall the results of Shah [51], but we point out that occasionally minor corrections and completions are necessary.

3.2. GIT for $(4,4)$ *curves on* $\mathbb{P}^1 \times \mathbb{P}^1$

Let $q := x_0x_3 - x_1x_2$, and let $Q := V(q) \subset \mathbb{P}^3$. In what follows we will identify $\mathbb{P}^1 \times \mathbb{P}^1$ and Q via the Segre isomorphism

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\sim} & Q \\ ([u_0,u_1],[v_0,v_1]) & \mapsto & [u_0v_0, u_0v_1, u_1v_0, u_1v_1]. \end{array} \tag{3.2.1}$$

Given the above identification, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ can be described either as the group generated by $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ and the involution exchanging the factors, or as the orthogonal group $\mathrm{PO}(q)$. A curve on $\mathbb{P}^1 \times \mathbb{P}^1$ is a *line* if, with the above identification, it is a line in $\mathbb{P}^3$.

As usual, GIT (semi)stability is studied by using the numerical criterion, and analyzing the behavior with respect to 1-parameter subgroups (1-PS). To start, we recall that $\mathrm{diag}(r_0,\ldots,r_n)$ denotes the 1-PS of $\mathrm{SL}(n+1)$ mapping $s \in \mathbb{C}^*$ to the diagonal matrix $(s^{r_0},\ldots,s^{r_n})$ (with $r_0,\ldots,r_n \in \mathbb{Z}$ not all zero, and adding up to 0). We are interested in 1-PS $\widetilde{\lambda}$ of $\mathrm{SL}(2) \times \mathrm{SL}(2)$ acting on $H^0(\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4))$. It turns out that in this case, GIT (semi)stability can be understood by studying the following three 1-PS's:

$$\widetilde{\lambda}_1(s) = ((s^2,s^{-2}),(s,s^{-1})),\ \widetilde{\lambda}_2(s) = ((s,s^{-1}),(s,s^{-1})),\ \widetilde{\lambda}_3(s) = ((s,s^{-1}),(1,1)).$$

Up to isogeny, $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ and $\widetilde{\lambda}_3$ correspond respectively to the 1-PS's of $\mathrm{SO}(q)$

$$\lambda_1 := \mathrm{diag}(3,1,-1,-3),\ \lambda_2 := \mathrm{diag}(1,0,0,-1),\ \lambda_3 := \mathrm{diag}(1,1,-1,-1). \tag{3.2.2}$$

A straightforward argument gives the following results:

(1) If $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$ is unstable, then, up to conjugation, there exists $i \in \{1,2,3\}$ such that C is $\widetilde{\lambda}_i$-unstable.

(2) Let $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$ be properly semistable (i.e. semistable and not stable), and $\sigma \in H^0(\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4))$ be a section whose divisor is C. Then, up to conjugation, there exists $i \in \{1,2,3\}$ such that C is destabilized by $\widetilde{\lambda}_i$, i.e. the limit $\lim\limits_{s\to 0} \widetilde{\lambda}_i(s)^*(\sigma)$ exists.

With these preliminaries, we can state the following two results which summarize the GIT analysis for $(4,4)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 3.1. *Let* $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$. *The following hold:*

(a) *C is desemistabilized by a 1-PS conjugated to $\widetilde{\lambda}_2$ if and only if it has a point of multiplicity at least 5, and it is destabilized by a 1-PS conjugated to $\widetilde{\lambda}_2$ if and only if it has a point of multiplicity at least 4.*

(b) *C is desemistabilized by a 1-PS conjugated to $\widetilde{\lambda}_3$ if and only if $C = 3L + C'$, where L is a line (on the other hand, if $C = 3C' + C''$, where C', C'' are arbitrary, then C has points of multiplicity at least 4, hence we go to Item (a) above). C is destabilized by a 1-PS conjugated to $\widetilde{\lambda}_3$ if and only if $C = 2L + C'$, where L is a line.*

(c) *If C is desemistabilized by a 1-PS conjugated to $\widetilde{\lambda}_1$, then it is destabilized by a 1-PS conjugated to $\widetilde{\lambda}_2$ (hence we go to Item (a) above). C is destabilized by a 1-PS conjugated to $\widetilde{\lambda}_1$ if and only if either it has a singular point p with consecutive triple points and tangent cone equal to $3T_p(L)$ where L is a line, or a point of multiplicity at least 4 (if the latter holds, we go to Item (a) above).* □

Proposition 3.2 *(cf. Proposition 4.5 in [50]). Let $C = V(\sigma) \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$.*

(1) *C is stable if and only if each of its irreducible components has multiplicity at most 2, no component of multiplicity 2 is a line, and each of its points has multiplicity at most 3, with the extra condition, if C is singular at a point p with consecutive triple points, that its tangent cone is* not *equal to $3T_p(L)$ where L is a line.*

(2) *C is properly semistable and polystable (i.e. the $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ orbit of σ is closed) if and only if one of the following holds:*

(a) σ is stabilized by $\widetilde{\lambda}_1$: $\sigma = u_0u_1(a_1u_0v_1^2 + b_1u_1v_0^2)(a_2u_0v_1^2 + b_2u_1v_0^2)$ *in suitable homogeneous coordinates $([u_0, u_1], [v_0, v_1])$, and $(a_1 \cdot b_2, a_2 \cdot b_1) \neq (0, 0)$. Equivalently $C = L_1 + L_2 + T_1 + T_2$, where L_1, L_2 are skew lines, T_1, T_2 are twisted cubics (eventually singular) intersecting each line L_i tangentially at the same point p_i (with p_1, p_2 not belonging to a line), satisfying the condition that no L_i has multiplicity greater than 2. The moduli space of such curves is $\mathbb{P}^1$: map $V(\sigma)$ to $[a_1^2b_2^2 + a_2^2b_1^2, a_1a_2b_1b_2]$.*

(b) σ is stabilized by $\widetilde{\lambda}_2$: $\sigma = \prod_{i=1}^4(a_iu_0v_1 + b_iu_1v_0)$ *in suitable homogeneous coordinates $([u_0, u_1], [v_0, v_1])$, and at most one of the a_i's vanishes, and similarly for the b_j's. Equivalently C is the sum of four members of the pencil of conics through two points not on a line, and no reducible conic appearing in C (if there are any) has multiplicity greater than one. The moduli space of such curves is 3 dimensional. The moduli map is the composition of the map*

$$\begin{array}{ccc} (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)^{ss} & \longrightarrow & \mathbb{P}^5 \\ ([a_1, b_1], \ldots, [a_4, b_4]) & \mapsto & [a_1a_2b_3b_4, a_1a_3b_2b_4, \ldots, a_3a_4b_1b_2] \end{array}$$

and the quotient map for the natural action of $\mathscr{S}_4$ on the image of the above map.

(c) σ is stabilized by $\widetilde{\lambda}_3$: $\sigma = u_0^2u_1^2F(v_0, v_1)$ *in suitable coordinates $([u_0, u_1], [v_0, v_1])$, and $F(v_0, v_1)$ is polystable for the action of* $\mathrm{PGL}(2)$ *on $\mathbb{P}(\mathbb{C}[v_0, v_1]_4)$. Equivalently $C = 2L + 2L' + R_1 + \ldots + R_4$, where L, L' are distinct lines in the same ruling, $R_1, \ldots, R_4$ are lines in the other ruling, and either $R_1, \ldots, R_4$ are dis-*

tinct, or $R_1 = R_2 \neq R_3 = R_4$. The moduli space is identified with that of binary quartics, i.e. $\mathbb{P}^1$ (we map $V(\sigma)$ to the moduli point corresponding to $[F]$).

The remark below will be useful later on.

Remark 3.3. Suppose that $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$ has a point p of multiplicity 4. Then, in the closure of the orbit $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)C$ there exists $C^* = C_1 + \ldots + C_4$ where $C_1, \ldots, C_4$ are conics through distinct points q_1, q_2 not on a line - see Item (a) of **Lemma 3.1**. Moreover the tangent cone $\mathscr{C}_p(C)$ is isomorphic to the tangent cone $\mathscr{C}_{q_i}(C^*)$ at either one of the (multiplicity 4) points q_1, q_2 of C^*.

3.3. Hodge-theoretic stratification of $\mathfrak{M}$

We quickly recall some notions which have been treated in [35, Sect. 3].

Definition 3.4. A reduced (not necessarily irreducible) projective surface X_0 is a *degeneration of $K3$ surfaces* if it is the central fiber of a flat proper family $\mathscr{X}/B$ over a pointed smooth curve $(B, 0)$ such that $\omega_{\mathscr{X}/B} \equiv 0$ and the general fiber X_b is a smooth $K3$ surface. If $p \in X_0$, then X_0 has an *insignificant limit singularity* at p if it has a semi-log-canonical singularity at p.

The above definition ties in with the terminology of Shah. More precisely, the list of singularities baptized as *insignificant limit singularities* by Shah [49] coincides with the list of Gorenstein slc singularities (see [46,29]). For a degeneration of $K3$ surfaces, the Gorenstein assumption is automatic. We recall (see Theorem 4.21 of [29]) that a Gorenstein surface singularity (X, p) with the property that $X \setminus \{p\}$ is semi-smooth (i.e. either smooth, normal crossings with two components, or a pinch point) is semi-log-canonical if and only if it is semi-canonical (see Definition 4.17 [29]), simple elliptic, cusp or a degenerate cusp.

Definition 3.5. Let X_0 be a degeneration of $K3$ surfaces, and let $p \in X_0$ be an insignificant limit singularity. Then

i) X_0 is of Type I at p if (X_0, p) is an ADE singularity (this includes smooth points).
ii) X_0 is of Type II at p if (X_0, p) is simple elliptic, locally normal crossings with exactly two irreducible components containing p, or a pinch point.
iii) X_0 is of Type III at p if (X_0, p) is either a cusp or a degenerate cusp.

Definition 3.6. Let X_0 be a degeneration of $K3$ surfaces.

(1) If X_0 has insignificant limit singularities, then

   (a) X_0 is of Type I if all its points are of Type I.

(b) X_0 is of Type II if all its points are of Type I and II, it does have points of Type II.
(c) X_0 is of Type III, if all its points are of Type I, II or III, and it does have points of Type III.

(2) X_0 has *Type IV* if it has significant limit singularities (i.e. there exists $p \in X_0$ such that (X_0, p) is *not* an insignificant limit singularity).

Remark 3.7. We note that the 1-dimensional components in the singular locus of a Type II degeneration X_0 of $K3$s are either smooth elliptic with no pinch points, or rational with 4 pinch points. Also, recall that the resolution of a simple elliptic singularity is an elliptic curve (of negative self-intersection). Thus, one sees that in all cases, for Type II degeneration X_0 of $K3$ surfaces, there is an associated j-invariant. Typically, X_0 has a single Type II singularity (i.e. simple elliptic, or elliptic double curve, or rational double curve with 4 pinch points), but even if there are multiple Type II singularities, the j-invariant for the various singularities coincides.

The above definitions are of interest to us because they are related to the period map $\mathfrak{p}\colon \mathfrak{M} \dashrightarrow \mathscr{F}^*$. Before explaining this, we introduce one more piece of terminology. Let $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$; we let $X_C \to \mathbb{P}^1 \times \mathbb{P}^1$ be the double cover ramified over C. Let $p \in C$, and let $\widetilde{p} \in X_C$ be the unique point lying over p. We say that C *has an insignificant limit singularity at* p if $(X_C, \widetilde{p})$ is an insignificant limit singularity, and if that holds then C has Type I, II or III at p according to the type of $(X_C, \widetilde{p})$. If $(X_C, \widetilde{p})$ is a significant limit singularity, we say that C has Type IV at p. Similarly C has insignificant limit singularities if all of its points are insignificant limit singularities, and if that is the case the Type of C is that of X_C.

Let $\mathfrak{M}^I, \mathfrak{M}^{II}, \mathfrak{M}^{III}, \mathfrak{M}^{IV} \subset \mathfrak{M}$ be the subsets of points represented by polystable curves $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$ of Type I, II, III and IV respectively. On the other side, let $\mathscr{F}^I := \mathscr{F}$, let $\mathscr{F}^{II}$ be the union of the Type II boundary components of $\mathscr{F}^*$, and let $\mathscr{F}^{III}$ be the union of the Type III boundary components of $\mathscr{F}^*$. The proof Proposition 3.16 of [35] has a straightforward extension to our case, and gives the following result.

Proposition 3.8. *The period map $\mathfrak{p}\colon \mathfrak{M} \dashrightarrow \mathscr{F}^*$ is regular away from $\mathfrak{M}^{IV}$, and*

$$\mathfrak{p}(\mathfrak{M}^I) \subset \mathscr{F}, \qquad \mathfrak{p}(\mathfrak{M}^{II}) \subset \mathscr{F}^{II}, \qquad \mathfrak{p}(\mathfrak{M}^{III}) \subset \mathscr{F}^{III}.$$

3.4. The components of $\mathfrak{M}^{II}$

Proposition 3.9. *The irreducible components of $\mathfrak{M}^{II}$ are the following:*

i) $\mathfrak{M}^{II}_{D_8 \oplus E_8}$, *the set parametrizing stable reduced curves C with a singularity of type $\widetilde{E}_8$ at a point p (by* **Proposition 3.2** *the tangent cone at p is $3L$ with L* not *the tangent space to a line through p). The dimension of $\mathfrak{M}^{II}_{D_8 \oplus E_8}$ is* 9.

ii) $\mathfrak{M}^{II}_{D_{12}\oplus D_4}$, *the set parametrizing stable divisors* $C = 2C_0 + D$ *where* C_0 *is a smooth conic, and such that the residual curve* D *intersects* C_0 *transversely. The dimension of* $\mathfrak{M}^{II}_{D_{12}\oplus D_4}$ *is* 5.

iii) $\mathfrak{M}^{II}_{A_{15}\oplus D_1}$, *the set parametrizing stable divisors* $C = 2E$, *where* E *is smooth. The dimension of* $\mathfrak{M}^{II}_{A_{15}\oplus D_1}$ *is* 2.

iv) $\mathfrak{M}^{II}_{D^+_{16}}$, *the set parametrizing stable divisors* $C = 2C_0 + L_1 + L_2$, *where* C_0 *is a twisted cubic, and* L_1, L_2 *are distinct lines intersecting* C_0 *transversally. The dimension of* $\mathfrak{M}^{II}_{D^+_{16}}$ *is* 1.

v) $\mathfrak{M}^{II}_{(E_8)^2}$, *the set parametrizing polystable divisors as in Item (2a) of* **Proposition 3.2** *such that there are two* $\widetilde{E}_8$ *singularities (i.e.* $a_1b_2 \neq 0 \neq a_2b_1$*). The dimension of* $\mathfrak{M}^{II}_{(E_8)^2}$ *is* 1.

vi) $\mathfrak{M}^{II}_{(E_7)^2\oplus D_2}$, *the set parametrizing polystable divisors as in Item (2b) of* **Proposition 3.2** *such that there are two* $\widetilde{E}_7$ *singularities, i.e. the divisor is reduced. The dimension of* $\mathfrak{M}^{II}_{(E_7)^2\oplus D_2}$ *is* 3.

vii) $\mathfrak{M}^{II}_{(D_8)^2}$, *the set parametrizing polystable divisors as in Item (2c) of* **Proposition 3.2** *such that the lines* $R_1, \ldots, R_4$ *are distinct. The dimension of* $\mathfrak{M}^{II}_{(D_8)^2}$ *is* 1.

Proof. This follows from Shah [51, Theorem 4.8, Type II]. The cases (iii) and (iv) are omitted in Shah, but it is clear that they occur. We will discuss in more detail the Type II strata (e.g. normal forms) in **Section 8** (esp. §8.4). In particular, the dimensions will be seen to be as in the statement of the Proposition. For the moment, we only note that the labels are chosen so that they match the labels from **Theorem 2.3**. The justification for this is given by [35, Prop. 7.11, Def. 7.7]. We will revisit the issue in **Section 8**. □

Remark 3.10 *(Type III).* We will not discuss in detail the stratification of $\mathfrak{M}^{III}$, but we note that it corresponds to degenerations of the Type II cases for which the j-invariant associated by **Remark 3.7** becomes ∞ (e.g. simple elliptic singularity degenerate to cusp singularity, or some of the 4 pinch points come together, but at worst with multiplicity 2). There are 5 strata identified by Shah [51, Thm. 4.8] (labeled A-III-i, A-III-ii, B-III-i, B-III-ii, and B-III-iii respectively). We note however, that there is a stratum missing in Shah's analysis, namely, the case of a double twisted cubic together with two tangent lines. This stratum is a specialization of the Type II case labeled D^+_{16} above (again missing from Shah's list). We will label this case III_b; explicitly:

$$(III_b) \;:\; V(x_0x_3 - x_1x_2, x_0x_2^3 + 2x_1^2x_2^2 + x_1^3x_3). \tag{3.4.1}$$

Similarly, the case where C consists of 4 double lines forming a cycle (the case B-III-iii in [50, Thm. 4.8]) will be labeled III_a; explicitly,

$$(III_a) \;:\; V(x_0x_3 - x_1x_2, x_1^2x_2^2). \tag{3.4.2}$$

Clearly, III_a and III_b are isolated points in $\mathfrak{M}^{III}$. It is not hard to see that the closure of any other Type III stratum (or similarly Type II) contains one of those two points. In other words, III_a and III_b are the deepest strata in $\mathfrak{M}^{II} \cup \mathfrak{M}^{III}$. The labels are chosen such that the adjacency of GIT Type II and III strata reflects the adjacency of Type II and III strata in the Baily-Borel compactification (see (2.4.1)). (For a typical picture of the behavior of the GIT vs Baily-Borel Type II and III strata see [33, Figure 2, p. 234].)

4. Stratification of $\mathfrak{M}^{IV}$

4.1. Summary

As discussed above, the indeterminacy locus of the period map $\mathfrak{M} \dashrightarrow \mathscr{F}^*$ is contained in the Type IV locus (in fact, a posteriori the indeterminacy is *equal* to $\mathfrak{M}^{IV}$). The purpose of this section is to define the stratification $W_\bullet$ in (1.2.4). The definition is inspired by our previous analysis [35] for quartics (which in turn is a refinement of Shah [51]). The reader can ignore all this background information, and just regard $W_\bullet$ as a natural stratification of the Type IV stratum in terms of the complexity of the singularities (the lower the index the worse the singularity). Results of Arnold et al. [5] play an essential role here, and will be reviewed below.

4.2. Singularity types (following Arnold)

Let $C \in |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)|$ be polystable, with isolated singularities, and let $X_C \to \mathbb{P}^1 \times \mathbb{P}^1$ be the double cover ramified over C. Suppose that $\operatorname{mult}_p(C) \leq 3$ for all points p (note: this condition holds for all stable C by **Proposition 3.2**). If C does not have singular points with consecutive triple points, then X_C has ADE singularities, and hence $[C] \in \mathfrak{M}^I$, in particular the period map is regular at $[C]$. If C does have consecutive triple points at p, then the initial germ of a defining equation of C at p is equal to x^3, for a suitable local parameter x. The isomorphism classes of such singularities have been classified, see [5, Ch. 16]. We recall the classification, and how to recognize the isomorphism class to which a given singularity belongs. Most of the isomorphism classes of such singularities are of Type IV, i.e. the corresponding double cover has significant limit singularities. They define (together with certain non isolated singularities) our stratification of $\mathfrak{M}^{IV}$.

Theorem 4.1 *(Arnold et al. [5, Ch. 16]). Let $f \in \mathscr{O}_{\mathbb{C}^2,0}$ be the germ of an analytic function of two variables in a neighborhood of the origin. Suppose that f has an isolated singularity at $(0,0)$, of multiplicity 3 with tangent cone a triple line. Then there exist analytic coordinates (x,y) in a neighborhood of 0 (centered at 0) and a decomposition $f(x,y) = u(x,y) \cdot g(x,y)$, where $u(x,y)$ is a unit and $g(x,y)$ is one (and only one) of the functions appearing in the first column of the table below. In the first three rows of the table $k \geq 1$, in the last two rows $k \geq 2$, and in the last row $p > 0$. Moreover*

Table 1
Recognition of singularities (triple points with tangent cone a triple line).

Normal form	Leading Term	$wt(x)$	$wt(y)$	Name	Type
$x^3+y^{3k+1}+\mathbf{a}xy^{2k+1}$	x^3+y^{3k+1}	$\frac{1}{3}$	$\frac{1}{3k+1}$	E_{6k}	I if $k=1$, IV if $k\geq 2$
$x^3+xy^{2k+1}+\mathbf{a}y^{3k+2}$	x^3+xy^{2k+1}	$\frac{1}{3}$	$\frac{2}{3(2k+1)}$	E_{6k+1}	I if $k=1$, IV if $k\geq 2$
$x^3+y^{3k+2}+\mathbf{a}xy^{2k+2}$	x^3+y^{3k+2}	$\frac{1}{3}$	$\frac{1}{3k+2}$	E_{6k+2}	I if $k=1$, IV if $k\geq 2$
$x^3+bx^2y^k+y^{3k}+\mathbf{c}xy^{2k+1}$	$x^3+bx^2y^k+y^{3k}$	$\frac{1}{3}$	$\frac{1}{3k}$	$J_{k,0}$	II if $k=2$, IV if $k\geq 3$
$x^3+x^2y^k+\mathbf{a}x^{3k+p}$	$x^3+x^2y^k$	$\frac{1}{3}$	$\frac{1}{3k}$	$J_{k,p}$	III if $k=2$, IV if $k\geq 3$

$$\mathbf{a} := a_0+\ldots+a_{k-2}y^{k-2}, \quad \mathbf{c} := a_0+\ldots+a_{k-3}y^{k-3}, \quad 4b^3+27\neq 0,$$

($\mathbf{a}=0$ *if* $k=1$*, and* $\mathbf{c}=0$ *if* $k=2$*) and in the last row* $a_0\neq 0$*.*

Proof. This is obtained by putting together Theorems $6_k,\ldots,12_k$ in [5, §16.2]. □

We explain the rôle of the weights appearing in the table above. First notice that the monomials in the leading term have weight 1, and the remaining monomials in the normal form have weight strictly greater than 1. Moreover, with the exception of singularities $J_{k,p}$ with $p>0$, the leading term has an isolated critical point at the origin. Thus, with the exception of singularities $J_{k,p}$ with $p>0$, the singularities in **Theorem 4.1** are semiquasihomogeneous.

Theorem 4.2 *(Arnold et al. [5, Ch. 16]). Let* $f(x,y)\in\mathscr{O}_{\mathbb{C}^2,0}$ *be the germ of an analytic function of two variables in a neighborhood of the origin, and suppose that* f *has an isolated singularity at* $(0,0)$*. Assign weights to* x *and* y *according to a chosen row of the table in* **Theorem 4.1***. If* $f=f_0+f_1$*, where* f_0 *is the leading term of the chosen row and every monomial appearing in* f_1 *has weight strictly greater than* 1*, then there exist analytic coordinates in a neighborhood of* 0 *(which we denote again by* x,y*) such that* $f=u\cdot g$*, where* u *is a unit and* g *is the normal form in the chosen row.*

Remark 4.3. Singularities often have more than one name. Here we note that $J_{2,p}$ is also denoted $T_{2,3,6+p}$ (cusp singularity), and $J_{2,0}=T_{2,3,6}$ is also denoted $\widetilde{E_8}$ (simple elliptic singularity).

We will also make use of the following terminology for certain non isolated singularities.

Definition 4.4. The germ (C,p) of a one dimensional singularity is of *Type* $J_{k,\infty}$ if it is isomorphic to the germ at $(0,0)$ of the planar singularity defined by $x^3+x^2y^k=0$.

4.3. The stratification

The following is obtained by an elementary analysis.

Lemma 4.5. *Let C be a stable $(4,4)$ curve. Then there is at most one Type IV point of C.* $\square$

This allows us to define the following stratification of $\mathfrak{M}^{IV}$.

Definition 4.6.

(1) For each type of isolated singularity T appearing in **Theorem 4.1** (i.e. isolated singularities of multiplicity 3 with tangent cone of multiplicity 3) *of Type IV*, let $\mathfrak{M}^{IV}_T \subset \mathfrak{M}^{IV}$ be the set parametrizing stable curves which have a point of Type T.
(2) Let $\mathfrak{M}^{IV}_{J_{k,\infty}} \subset \mathfrak{M}^{IV}$ be the set parametrizing stable curves which have a point of Type $J_{k,\infty}$.
(3) Let $\mathfrak{M}^{IV}_{J_{k,+}} := \mathfrak{M}^{IV}_{J_{k,\infty}} \sqcup \coprod_{r>0} \mathfrak{M}^{IV}_{J_{k,r}}$.
(4) Let $\mathfrak{M}^{IV}_{(3,1)} \subset \mathfrak{M}^{IV}$ be the set parametrizing curves $3C_0 + C_1$, where $C_0, C_1 \in |\mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(1)|$ are distinct, and C_0 is smooth.
(5) Let $\mathfrak{M}^{IV}_{(4)} \subset \mathfrak{M}^{IV}$ be the singleton whose unique point corresponds to $4C_0$, where $C_0 \in |\mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(1)|$ is smooth.

We have

$$\mathfrak{M}^{IV} = \coprod_T \mathfrak{M}^{IV}_T \sqcup \mathfrak{M}^{IV}_{(3,1)} \sqcup \mathfrak{M}^{IV}_{(4)}. \tag{4.3.1}$$

Our next task is to describe explicitly the curves in the subsets $\mathfrak{M}^{IV}_T$. The following is a slight extension of Lemma 4.6 in [51], we omit the details of the proof.

Lemma 4.7 *(Shah). Let $C \subset \mathbb{P}^3$ be a $(2,4)$ complete intersection curve, let Q be the quadric containing it, and let $p \in C$ be a point* not *belonging to the singular locus of Q. Then the following are equivalent:*

(1) *There exist homogeneous coordinates $[x_0, \ldots, x_3]$ such that $p = [1,0,0,0]$, and*

$$Q = V(x_0x_2 + x_1^2 + ax_3^2), \tag{4.3.2}$$

$$C = V(x_0x_2 + x_1^2 + ax_3^2, x_0x_3^3 + x_1^2g_2(x_2,x_3) + x_1g_3(x_2,x_3) + g_4(x_2,x_3)). \tag{4.3.3}$$

(2) *C has consecutive triple points at p, with tangent cone $3L$, where L is* not *the tangent space to a line of Q.* $\square$

The observation below will be handy when computing the dimensions of the strata $\mathfrak{M}^{IV}_T$.

Proposition 4.8. *Let $C \subset \mathbb{P}^3$ be a $(2,4)$ complete intersection curve, let Q be the quadric containing it, and let $p \in C$ be a point* not *belonging to the singular locus of Q. Suppose that C, Q, p satisfy one of the (equivalent) Items (1) or (2) of* **Lemma 4.7**. *Retain the notation of the quoted lemma, and let $g_d(x_2, x_3) = \sum_{i+j=d} g_d^{i,j} x_2^i x_3^j$. Then the following hold:*

(1) *If $g_2 \neq 0$, then (C, p) is a $J_{2,r}$ singularity, where $r \in \mathbb{N} \cup \{\infty\}$.*
(2) *If $g_2 = 0$, and $g_3^{3,0} \neq 0$, then (C, p) is an E_{12} singularity.*
(3) *If $g_2 = 0$, $g_3^{3,0} = 0$, and $g_3^{2,1} \neq 0$, then (C, p) is an E_{13} singularity.*
(4) *If $g_2 = 0$, $g_3^{3,0} = g_3^{2,1} = 0$, and $g_4^{4,0} \neq 0$, then (C, p) is an E_{14} singularity.*
(5) *If $g_2 = 0$, $g_3^{3,0} = g_3^{2,1} = g_4^{4,0} = 0$, and $g_4^{3,1}((g_3^{1,2})^2 + 4g_4^{3,1}) \neq 0$, then (C, p) is a $J_{3,0}$ singularity.*
(6) *If $g_2 = 0$, $g_3^{3,0} = g_3^{2,1} = g_4^{4,0} = g_4^{3,1}(g_3^{1,2} \cdot g_3^{1,2} + 4g_4^{3,1}) = 0$, and $(g_3^{1,2}, g_4^{3,1}) \neq (0,0)$, then (C, p) is a $J_{3,r}$ singularity, for some $r > 0$ (possibly $r = \infty$).*
(7) *If $g_2 = 0$, $g_3^{3,0} = g_3^{2,1} = g_4^{4,0} = g_3^{1,2} = g_4^{3,1} = 0$, and $g_4^{2,2} \neq 0$, then (C, p) is a $J_{4,\infty}$ singularity.*
(8) *If $g_2 = 0$, $g_3^{3,0} = g_3^{2,1} = g_4^{4,0} = g_3^{1,2} = g_4^{3,1} = g_4^{2,2} = 0$, then $C = 3C_0 + C_1$, where C_0 is a smooth conic.*

Proof. Let (x, y, z) be the affine coordinates, centered at p, given by $x = \frac{x_1}{x_0}$, $y = \frac{x_3}{x_0}$, and $z = \frac{x_2}{x_0}$. The germ of C at p is isomorphic to germ at $(0,0)$ of the affine plane curve with equation given by

$$y^3 + x^2 g_2(-x^2 - ay^2, y) + x g_3(-x^2 - ay^2, y) + g_4(-x^2 - ay^2, y) = 0. \tag{4.3.4}$$

Items (1) - (5) follow from **Theorem 4.2**. In fact, assign weights to x, y according to the table in **Theorem 4.1**, with the rôles of x and y exchanged, i.e. $wt(x) = 1/6$ when proving Item (1), $wt(x) = 1/7$ when proving Item (2), $wt(x) = 2/15$ when proving Item (3), $wt(x) = 1/8$ when proving Item (4), $wt(x) = 1/9$ when proving Item (5), and $wt(y) = 1/3$ in all cases. Then Item (1) holds because the leading term of the equation is $y^3 + x^2 g_2(-x^2, y)$ (in order to recognize the leading term one needs to pass to analytic coordinates $(x, y + \alpha x^2)$ for a suitable choice of α), Item (2) holds because the leading term of the equation is $y^3 - g_3^{3,0} x^7$, Item (3) holds because the leading term of the equation is $y^3 + g_3^{2,1} x^5 y$, Item (4) holds because the leading term of the equation is $y^3 + g_4^{4,0} x^8$, and Item (5) holds because the leading term of the equation is $y^3 - g_3^{1,2} x^3 y^2 - g_4^{3,1} x^6 y$, and there exist non zero distinct α, β such that $y^3 - g_3^{1,2} x^3 y^2 - g_4^{3,1} x^6 y = y(y + \alpha x^3)(y + \beta x^3)$ if and only if $g_4^{3,1}((g_3^{1,2})^2 + 4g_4^{3,1}) \neq 0$.

The proof of (6), (7) and (8) is elementary. □

Corollary 4.9. *Let $T \in \{E_{12}, E_{13}, E_{14}, J_{3,0}, J_{3,+}, J_{4,\infty}, (3,1), (4,0)\}$. Then $\mathfrak{M}_T^{IV}$ is a (non empty) irreducible locally closed subset of $\mathfrak{M}^{IV}$, of dimension given below*

$\mathfrak{M}^{IV}_T$	$\mathfrak{M}^{IV}_{E_{12}}$	$\mathfrak{M}^{IV}_{E_{13}}$	$\mathfrak{M}^{IV}_{E_{14}}$	$\mathfrak{M}^{IV}_{J_{3,0}}$	$\mathfrak{M}^{IV}_{J_{3,+}}$	$\mathfrak{M}^{IV}_{J_{4,\infty}}$	$\mathfrak{M}^{IV}_{(3,1)}$	$\mathfrak{M}^{IV}_{(4)}$
$\dim \mathfrak{M}^{IV}_T$	7	6	5	4	3	2	1	0

(4.3.5)

Moreover

$$\mathfrak{M}^{IV} = \mathfrak{M}^{IV}_{E_{12}} \sqcup \mathfrak{M}^{IV}_{E_{13}} \sqcup \mathfrak{M}^{IV}_{E_{14}} \sqcup \mathfrak{M}^{IV}_{J_{3,0}} \sqcup \mathfrak{M}^{IV}_{J_{3,+}} \sqcup \mathfrak{M}^{IV}_{J_{4,\infty}} \sqcup \mathfrak{M}^{IV}_{(3,1)} \sqcup \mathfrak{M}^{IV}_{(4)}, \tag{4.3.6}$$

and the closure of $\mathfrak{M}^{IV}_T$ is the union of $\mathfrak{M}^{IV}_T$ and the strata $\mathfrak{M}^{IV}_{T'}$ to the right of $\mathfrak{M}^{IV}_T$ in (4.3.6).

Proof. Let $T \in \{E_{12}, E_{13}, E_{14}, J_{3,0}, J_{3,+}, J_{4,\infty}\}$, and let $[C] \in \mathfrak{M}^{IV}_T$, with C polystable. Then C is stable by **Proposition 3.2** and, by the same proposition, the hypotheses of **Proposition 4.8** are satisfied, where Q is the smooth quadric containing C, and $p \in C$ is the unique point of Type IV (see **Lemma 4.5**). By **Proposition 4.8** it follows that $\mathfrak{M}^{IV}_T$ is non empty, irreducible and locally closed. It is elementary that $\mathfrak{M}^{IV}_{(3,1)}$ and $\mathfrak{M}^{IV}_{(4)}$ are irreducible, locally closed. The statement about the closure follows from **Proposition 4.8**, and also the decomposition in (4.3.6).

It remains to prove that the dimensions are given by (4.3.5). In the case of $\mathfrak{M}^{IV}_{(3,1)}$ and $\mathfrak{M}^{IV}_{(4)}$, the computation is straightforward. It remains to deal with $\mathfrak{M}^{IV}_T$ for $T \in \{E_{12}, E_{13}, E_{14}, J_{3,0}, J_{3,+}, J_{4,\infty}\}$. We may assume that C is contained in the smooth quadric $x_0x_2 + x_1^2 - x_3^2$, i.e. we set $a = -1$ in **Lemma 4.7**. By **Lemma 4.7** and **Proposition 4.8**, C is equivalent (under $\mathrm{Aut}(Q)$) to a curve whose equation is given by (4.3.3), with $g_2 = 0$ and such that g_3 and g_4 satisfy the conditions in the Item of **Proposition 4.8** corresponding to the chosen T. Let $\mathscr{S}_T \subset |\mathscr{O}_Q(4)|$ be the subset of such curves. Since C is stable, it follows that the dimension of $\mathfrak{M}^{IV}_T$ is equal to the difference between the dimension of $\mathscr{S}_T$ and the dimension of the subgroup of $G_T < \mathrm{Aut}(Q)$ mapping $\mathscr{S}_T$ to itself. The dimension of $\mathscr{S}_T$ is easily computed: it is 9 if $T = E_{12}$, and it decreases by 1 each time we move one step to the right. The subgroup $G_T < \mathrm{Aut}(Q)$ mapping $\mathscr{S}_T$ to itself is contained in the subgroup of automorphisms ϕ stabilizing $p = [1,0,0,0] \in Q$, and such that the differential $d\phi(p)$ maps to itself the tangent line $V(x_2, x_3)$. Thus $\dim G_T \le 3$. In fact a computation shows that the connected component of the identity of G_T is equal to the subgroup of $\mathrm{PGL}(4)$ given by matrices

$$\begin{pmatrix} 1 & 2\beta & -\beta^2 & 0 \\ 0 & \alpha & -\alpha\beta & 0 \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

with $\alpha \neq 0$. Hence $\dim G_T = 2$, and this gives the dimensions in (4.3.5). □

At this point, we can give the key definition of the W-stratification.

Definition 4.10. For $d \in \{0,1,2,4,5,6,7\}$ (no misprint: $d=3$ *is* missing), we let $W_d \subset \mathfrak{M}^{IV}$ be the union of all the strata $\mathfrak{M}^{IV}_T$ of dimension at most d.

Remark 4.11. The stratum $\mathfrak{M}^{IV}_{J_{3,+}}$ is skipped in the definition of the W-stratification because it is flipped together with $\mathfrak{M}^{IV}_{J_{3,0}}$.

By **Corollary 4.9** we have a ladder of irreducible closed subsets indexed by dimension:

$$W_0 \subset W_1 \subset W_2 \subset W_4 \subset W_5 \subset W_6 \subset W_7 = \mathfrak{M}^{IV} \subset \mathfrak{M}. \tag{4.3.7}$$

This is the counterpart of the stratification Z^k (indexed by codimension) in $\mathscr{F}^*$ (see (2.2.3)).

5. GIT for $(2,4)$ complete intersections in $\mathbb{P}^3$

5.1. Summary

Let U be the parameter space for $(2,4)$ complete intersection curves in $\mathbb{P}^3$, with the natural action of $\mathrm{SL}(4)$. The main tool in this paper is a variation of GIT quotients $\mathfrak{M}(t)$ for U (for $t \in (1/6-\epsilon, 1/2] \cap \mathbb{Q}$). Since U is not projective, we will consider the closure of U in the Hilbert scheme, call it $\mathrm{Hilb}_{(2,4)}$. In order to define a GIT quotient of $\mathrm{Hilb}_{(2,4)}$ modulo the natural action of $\mathrm{SL}(4)$ we must choose an $\mathrm{SL}(4)$-linearized line bundle. For large m, we have the (naturally linearized) ample Plucker line bundle L_m corresponding to the m-th Hilbert point; we let $\mathrm{Hilb}_{(2,4)}/\!\!/_{L_m}\mathrm{SL}(4)$ be the corresponding quotient. There is also the Hilbert-Chow map $c\colon U \to \mathrm{Chow}$ to the Chow variety parametrizing effective 1-cycles on $\mathbb{P}^3$. Let $\mathrm{Chow}_{(2,4)}$ be the closure of the image of c, and let L_∞ be the restriction to $\mathrm{Chow}_{(2,4)}$ of the natural polarization of the Chow variety. As suggested by the notation, for $m \to \infty$ the polarization L_m approaches the pull-back of L_∞, and hence the quotient $\mathrm{Hilb}_{(2,4)}/\!\!/_{L_m}\mathrm{SL}(4)$ approaches the GIT quotient of the Chow variety $\mathrm{Chow}_{(2,4)}/\!\!/\mathrm{SL}(4)$.

In the opposite direction, we may consider m as small as possible, namely $m=4$. The corresponding line bundle L_4 is ample on U but *not* ample on $\mathrm{Hilb}_{(2,4)}$. Here we recall that (semi)stability makes sense with respect to any linearized line bundle, and hence there is a quasi-projective quotient of the open set of semistable points $\mathrm{Hilb}^{ss}_{(2,4)}(L_4)$ (see [41, Thm. 1.10]). On the other hand, we may identify $\mathrm{Hilb}_{(2,4)}/\!\!/_{L_4}\mathrm{SL}(4)$ with the GIT quotient of a space birational to $\mathrm{Hilb}_{(2,4)}$, with a linearization that is *ample*. In fact, let E be the vector-bundle over $|\mathscr{O}_{\mathbb{P}^3}(2)|$ defined by

$$\begin{array}{ccc} H^0(Q,\mathscr{O}_Q(4)) & \subset & E \\ \downarrow & & \downarrow \\ Q & \in & |\mathscr{O}_{\mathbb{P}^3}(2)|. \end{array} \tag{5.1.1}$$

Let $\pi\colon \mathbb{P}E \to |\mathscr{O}_{\mathbb{P}^3}(2)|$ be the structure map. The Picard group of $\mathbb{P}E$ is generated by

$$\eta := \pi^*\mathcal{O}(1), \quad \xi := \mathcal{O}_{\mathbb{P}E}(1). \tag{5.1.2}$$

Proposition 5.1 *([7, Thm 2.7]). Let $t \in \mathbb{Q}$. The $\mathbb{Q}$-Cartier divisor class $\eta + t\xi$ on $\mathbb{P}E$ is ample if and only if $t \in (0, \frac{1}{3}) \cap \mathbb{Q}$.*

Now, $\mathbb{P}E$ contains naturally U, and the complement of U has codimension greater than 1, so that the restriction of L_4 to U extends uniquely to a line-bundle $\overline{L}_4$. A straightforward computation (see the proof of Item (3) of **Theorem 5.6**) shows that $c_1(\overline{L}_4) = 10\eta + \xi$. Thus $\overline{L}_4$ is ample, and from this it follows that the categorical quotient $\mathrm{Hilb}^{ss}_{(2,4)}(L_4)$ is identified with the GIT quotient $\mathbb{P}E/\!\!/_{\overline{L}_4}\mathrm{SL}(4)$. Then, a key fact is that $\mathbb{P}E/\!\!/_{\overline{L}_4}\mathrm{SL}(4)$ is naturally isomorphic to the GIT moduli space $\mathfrak{M}$ of $(4,4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ that was discussed in **Section 3** (see **Theorem 5.6**(2)).

On the other hand, the extension of the Chow polarization from U to all of $\mathbb{P}E$ corresponds to $t = 1/2$ in **Proposition 5.1**, and one of our final goals is to prove that the Chow quotient $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ is isomorphic to the Baily-Borel compactification $\mathscr{F}^*$. Thus it is natural to study the one-parameter variation of GIT models $\mathbb{P}E/\!\!/_{\eta+t\xi}\mathrm{SL}(4)$, for $t \in (0, 1/2] \cap \mathbb{Q}$. Since we need to consider values of t for which $\eta + t\xi$ is not ample, i.e. $t \in [1/3, 1/2] \cap \mathbb{Q}$, our approach is not as straightforward as one would like. In **Subsection 5.2** we will define a VGIT on an auxiliary SL(4)-space $\mathscr{P}$, which is somewhat intermediate between $\mathbb{P}E$ and $\mathrm{Chow}_{(2,4)}$. It is natural to expect that the quotient of $\mathscr{P}$ with respect to a suitable polarization N_t which morally corresponds to $(\eta + t\xi)$ is a projective GIT moduli space $\mathfrak{M}(t)$ interpolating between $\mathfrak{M}$ and $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$. The main result of the present section, i.e. **Theorem 5.6**, establishes the expected interpolation.

A similar analysis was carried out in [14] for $(2,3)$ complete intersections. We have modified some of the arguments in [14], in particular we give a streamlined definition of $\mathfrak{M}(t)$, but we point out that a key argument (**Proposition 5.8**) is essentially the same.

5.2. Set up of the VGIT, and statement of the main result

As before, let U be the parameter space for $(2,4)$ complete intersection curves in $\mathbb{P}^3$. Thus we have a regular embedding

$$U \hookrightarrow \mathbb{P}(E) \times \mathrm{Chow}_{(2,4)}\,. \tag{5.2.1}$$

Definition 5.2. Let $\mathscr{P} \subset \mathbb{P}(E) \times \mathrm{Chow}_{(2,4)}$ be the closure of U.

The action of SL(4) on $\mathbb{P}^3$ defines an SL(4)-action on $\mathscr{P}$: this is the space on which we will follow the VGIT. To do this, we need to define a variable SL(4)-linearized ample line bundle on $\mathscr{P}$. Choose

$$0 < \delta < 1/6, \quad \delta \in \mathbb{Q}. \tag{5.2.2}$$

Let p_1, p_2 be the projections of $\mathscr{P}$ onto the first and second factor respectively. For $t \in (\delta, 1/2) \cap \mathbb{Q}$, let

$$N_t := \frac{1-2t}{1-2\delta} \cdot p_1^*(\eta + \delta\xi) + \frac{t-\delta}{2(1-2\delta)} p_2^* L_\infty. \tag{5.2.3}$$

Clearly N_t is ample for $t \in (\delta, 1/2) \cap \mathbb{Q}$, and semi-ample for $t = 1/2$. Thus for each $t \in (\delta, 1/2] \cap \mathbb{Q}$, we have a GIT quotient

$$\mathfrak{M}(t) := \mathscr{P} /\!\!/_{N_t} \mathrm{SL}(4). \tag{5.2.4}$$

Notice that, a priori, $\mathfrak{M}(t)$ depends also on the choice of δ. Formally, we choose one δ, and this justifies the absence of δ from the notation. More substantially, a posteriori we will see that $\mathfrak{M}(t)$ does *not* depend on the choice of δ.

Remark 5.3. For $t \in (\delta, 1/2) \cap \mathbb{Q}$, N_t is ample and thus there is no ambiguity in the definition of $\mathfrak{M}(t) := \mathscr{P} /\!\!/_{N_t} \mathrm{SL}(4)$. For $t = \frac{1}{2}$, N_t is only semi-ample, thus some care should be taken in defining $\mathfrak{M}(\frac{1}{2})$. Specifically, we define

$$\mathfrak{M}\left(\frac{1}{2}\right) := \mathrm{Proj} R\left(\mathscr{P}, N_{\frac{1}{2}}\right)^{\mathrm{SL}(4)}. \tag{5.2.5}$$

By (5.2.3), it is clear that

$$\mathfrak{M}\left(\frac{1}{2}\right) \cong \mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4). \tag{5.2.6}$$

Furthermore, a straightforward application of the functoriality of the numerical function μ ([41, p. 49]) and an application of the numerical criterion, gives that $\mathfrak{M}(t)$ behaves as expected as t attains the value $\frac{1}{2}$. Specifically, the following hold:

i) If $x \in \mathscr{P}^{ss}(t)$ for $t \in \left(\frac{1}{2} - \epsilon, \frac{1}{2}\right)$, then $p_2(x) \in \mathrm{Chow}^{ss}_{(2,4)}$ (where $p_2 : \mathscr{P} \to \mathrm{Chow}_{(2,4)}$ is the projection). Conversely, if $y \in \mathrm{Chow}^{s}_{(2,4)}$, then $p_2^{-1}(y) \in \mathscr{P}^s(t)$. Thus, we can write (by a slight abuse of notation)

$$\mathscr{P}^s\left(\frac{1}{2}\right) \subset \mathscr{P}^s\left(\frac{1}{2} - \epsilon\right) \subset \mathscr{P}^{ss}\left(\frac{1}{2} - \epsilon\right) \subset \mathscr{P}^{ss}\left(\frac{1}{2}\right) \tag{5.2.7}$$

(compare [32, (3.18)] in the standard VGIT set-up).

ii) In particular, there exists a natural birational map

$$\mathfrak{M}\left(\frac{1}{2} - \epsilon\right) \to \mathfrak{M}\left(\frac{1}{2}\right) \tag{5.2.8}$$

which is compatible with the Hilbert-Chow morphism $\mathrm{Hilb}_{(2,4)} \to \mathrm{Chow}_{(2,4)}$.

In order to make the connection with the discussion at the beginning of the present section, and to explain the choice of coefficients in (5.2.3), we prove the following result.

Proposition 5.4. *Let* $\overline{L}_\infty \in \operatorname{Pic}(\mathbb{P}E)_\mathbb{Q}$ *be the unique extension of* c^*L_∞ *(recall that* $c\colon U \to \mathrm{Chow}_{(2,4)}$ *is the restriction of the Hilbert-Chow map). Then*

$$\overline{L}_\infty = 4\eta + 2\xi. \tag{5.2.9}$$

Proof. There exist $x, y \in \mathbb{Q}$ such that $\overline{L}_\infty = x\eta + y\xi$. We compute x and y by computing intersection indices with the following two projective curves in U:

$$\Gamma := \{V(q, \mu_0 f + \mu_1 g) \mid \deg q = 2,\ \deg f = \deg g = 4\}, \tag{5.2.10}$$

$$\Omega := \{V(\mu_0 q + \mu_1 r, f) \mid \deg q = \deg r = 2,\ \deg f = 4\}. \tag{5.2.11}$$

(Here q, r, f, g are chosen generically.) Notice that both Γ and Ω are contained in U. The degree of $L_\infty|_\Gamma$ is equal to the number of curves parametrized by Γ meeting a generic line in $\mathbb{P}^3$, and similarly for $L_\infty|_\Omega$. Thus

$$\deg(L_\infty|_\Gamma) = 2, \qquad \deg(L_\infty|_\Omega) = 4. \tag{5.2.12}$$

On the other hand, we have the following intersection indices

$$\begin{array}{c|c|c} & \eta & \xi \\ \hline \Gamma & 0 & 1 \\ \hline \Omega & 1 & 0 \end{array} \tag{5.2.13}$$

Equation (5.2.9) follows at once from (5.2.12) and (5.2.13). □

Corollary 5.5. *With notation as above,* $\eta + t\xi$ *is the unique divisor class in* $\operatorname{Pic}(\mathbb{P}E)$ *which, restricted to* U*, is equal to* $N_t|_U$*.*

With these preliminaries out of the way, we are almost ready to state the main result of the present section. We will compare GIT quotients of $\mathscr{P}$, $\mathbb{P}E$, $\mathrm{Hilb}_{(2,4)}$ and $\mathrm{Chow}_{(2,4)}$ modulo the natural $\mathrm{SL}(4)$-action. With the exception of the latter, all the listed spaces contain U, the parameter space for $(2,4)$ complete intersections in $\mathbb{P}^3$ (with the natural $\mathrm{SL}(4)$-action), as an open dense subset. Thus U induces birational maps between any $\mathrm{SL}(4)$-quotients of $\mathscr{P}$, $\mathbb{P}E$ or $\mathrm{Hilb}_{(2,4)}$. Similarly, if $V \subset \mathrm{Chow}_{(2,4)}$ is the dense subset of Chow forms of $(2,4)$ intersections, the Hilbert-Chow map induces a birational (regular) map $U \to V$, and hence also a birational map between any $\mathrm{SL}(4)$-quotient of $\mathscr{P}$, $\mathbb{P}E$ or $\mathrm{Hilb}_{(2,4)}$ and any $\mathrm{SL}(4)$-quotient of $\mathrm{Chow}_{(2,4)}$. In the result below, whenever we state that certain moduli spaces are isomorphic, what we really mean is that one of the birational maps discussed above is, in fact, an isomorphism. We will also compare $\mathrm{SL}(4)$-quotients of $\mathbb{P}E$ and the quotient $\mathfrak{M} := |\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)| /\!\!/ \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$; when we state that they are isomorphic, it is understood that the isomorphism is induced by the inclusion $|\mathscr{O}_{\mathbb{P}^1}(4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(4)| \subset U$ determined by the choice of an isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and a smooth quadric in $\mathbb{P}^3$.

Theorem 5.6. *Keep notation as above, in particular δ is as in (5.2.2). The following hold:*

(1) *For $t \in (\delta, 1/3)$, $\mathfrak{M}(t) \cong \mathbb{P}E/\!\!/_{\eta+t\xi}\mathrm{SL}(4)$.*
(2) *For $t \in (\delta, 1/6)$, $\mathfrak{M}(t) \cong \mathfrak{M}$.*
(3) *For $m \geq 4$, we have $\mathrm{Hilb}_{(2,4)}/\!\!/_{L_m}\mathrm{SL}(4) \cong \mathfrak{M}(t(m))$, where*

$$t(m) := \frac{(m-3)^2}{2(m^2-4m+5)}.$$

Theorem 5.6 is inspired by [14], where the analogous results are established for canonical genus 4 curves (or equivalently $(2,3)$ complete intersections in $\mathbb{P}^3$). After a discussion of the GIT stability for $\mathrm{Hilb}_{(2,4)}$ and $\mathbb{P}E$ in **Subsection 5.3** and **Subsection 5.4** respectively, we conclude the proof of the theorem in **Subsection 5.5**

5.3. GIT for $\mathrm{Hilb}_{(2,d)}$

We review the application of the numerical criterion for GIT on $\mathrm{Hilb}_{(2,d)}$ following [23]. We conclude with a proof of **Proposition 5.8**, a key result for what follows.

5.3.1. The m-th Hilbert point

Let $U_{(2,d)}$ be the open subset of the Hilbert scheme of $\mathbb{P}^3$ parametrizing complete intersections of a quadric and a surface of degree d (in short $(2,d)$ c.i.'s), and let $\mathrm{Hilb}_{(2,d)}$ be its closure. Let $P_d \in \mathbb{Z}[m]$ be defined by $P_d = 2dm - d^2 + 2d$. If $C \in U_{(2,d)}$, then

$$\mathrm{rk}(H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m)) \longrightarrow H^0(C, \mathscr{O}_C(m))) = P_d(m), \qquad m \geq d-1.$$

For $m \geq d$, let

$$\begin{array}{ccc} U_{(2,d)} & \xrightarrow{f_{d,m}} & \mathrm{Gr}(H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m)), P_d(m)) \\ C & \mapsto & H^0(\mathbb{P}^3, \mathscr{O}_{|PP^3}(m)) \twoheadrightarrow H^0(C, \mathscr{O}_C(m)) \end{array} \tag{5.3.1}$$

be the natural map, where $\mathrm{Gr}(H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m)), P_d(m))$ is the Grassmannian parametrizing $P_d(m)$-dimensional quotients of $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m))$. Let $\mathrm{Hilb}^m_{(2,d)}$ be the closure of the image of f_m. A point of $\mathrm{Hilb}^m_{(2,d)}$ is determined by a quotient $\varphi\colon H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m)) \twoheadrightarrow V$, where V is a vector space of dimension $P_d(m)$; we denote the corresponding point by $I := \ker\varphi$. Thus $I \subset H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m))$ is a vector subspace of codimension $P_d(m)$. If $m \gg 0$ then $\mathrm{Hilb}^m_{(2,d)}$ is isomorphic to $\mathrm{Hilb}_{(2,d)}$.

By associating to a curve $C \in U_{(2,d)}$ its Chow point (the hypersurface in $(\mathbb{P}^3)^\vee \times (\mathbb{P}^3)^\vee$ parametrizing couples (H_1, H_2) of hyperplanes such that $H_1 \cap H_2 \cap C \neq \varnothing$), we get the

Hilbert-Chow map $c_d\colon U_{(2,d)} \to \mathrm{Chow}(\mathbb{P}^3)$. We let $\mathrm{Chow}_{(2,d)} \subset \mathrm{Chow}(\mathbb{P}^3)$ be the closure of $\mathrm{Im}(c_d)$.

5.3.2. (Semi)stability of points in $\mathrm{Hilb}^m_{(2,4)} \setminus \mathrm{Im}\, f_m$

The group $\mathrm{SL}(4)$ acts on $\mathrm{Hilb}^m_{(2,d)}$, and on the restriction to $\mathrm{Hilb}^m_{(2,d)}$ of the Plücker (ample) line bundle on $\mathrm{Gr}(H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m)), P_d(m))$. Thus we have a notion of (semi)stability of elements of $\mathrm{Hilb}^m_{(2,d)}$. We recall how to determine whether a point of $\mathrm{Hilb}^m_{(2,d)}$ is (semi)stable, following [23].

Let λ be a 1-PS of $\mathrm{SL}(4)$. Let $[x_0, \ldots, x_3]$ be homogeneous coordinates that diagonalize λ, i.e. $\lambda(s) = \mathrm{diag}(s^{r_0}, s^{r_1}, s^{r_2}, s^{r_3})$. Given $A = (A_0, \ldots, A_3) \in \mathbb{N}^4$, we let $x^A := x_0^{A_0} x_1^{A_1} x_2^{A_2} x_3^{A_3}$. The λ-*weight* of x^A is

$$\mathrm{wt}_\lambda(x^A) := \sum_{i=0}^{3} r_i A_i. \tag{5.3.2}$$

Given pairwise distinct monomials $x^{A(1)}, \ldots, x^{A(P_d(m))}$ of degree m, we let

$$\mathrm{wt}_\lambda(x^{A(1)} \wedge \ldots \wedge x^{A(P_d(m))}) := \sum_{j=1}^{P_d(m)} \mathrm{wt}_\lambda(x^{A(j)}).$$

Let $I \in \mathrm{Hilb}^m_{(2,d)}$. The Hilbert-Mumford index of λ with respect to the Plücker linearization is

$$\mu_m(I,\lambda)=\max\{-\mathrm{wt}_\lambda(x^{A(1)}\wedge\ldots\wedge x^{A(P_d(m))}) \mid \{x^{A(1)}, \ldots, x^{A(P_d(m))}\} \pmod{I} \text{ is a basis of } H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(m))/I\}. \tag{5.3.3}$$

By the Hilbert-Mumford Criterion, the point I is semistable if and only if $\mu_m(I, \lambda) \geq 0$ for all λ, and it is stable if and only if strict inequality holds for each λ.

Define a total ordering $\overset{\lambda}{\prec}$ on the set of monomials $\{x^A\}$ by declaring that $x^A \overset{\lambda}{\prec} x^B$ if

(1) $\deg x^A < \deg x^B$, or
(2) $\deg x^A = \deg x^B$, and $\mathrm{wt}_\lambda(x^A) < \mathrm{wt}_\lambda(x^B)$, or
(3) $\deg x^A = \deg x^B$, $\mathrm{wt}_\lambda(x^A) = \mathrm{wt}_\lambda(x^B)$, and x^A precedes x^B in the lexicographical ordering.

Let $f \in \mathbb{C}[x_0, \ldots, x_3]$; we let $\mathrm{in}_{\prec}^{\lambda}(f)$ be the maximum (with respect to $\overset{\lambda}{\prec}$) among monomials appearing with non zero coefficient in the expansion of f. Given a non zero vector subspace $I \subset \mathbb{C}[x_0, \ldots, x_3]$, we let $\mathrm{in}_{\prec}^{\lambda}(I)$ be the set whose elements are the monomials $\mathrm{in}_{\prec}^{\lambda}(f)$ for $f \in I$.

Proposition 5.7. *Let $I \in \mathrm{Hilb}^m_{(2,d)}$, and let λ be a 1-PS of* SL(4). *Then*

$$\mu_m(I,\lambda) = \sum_{x^A \in \mathrm{in}^\lambda_\prec(I)} \mathrm{wt}_\lambda(x^A). \tag{5.3.4}$$

Proof. Following [23], it suffices to show that (in the notation of pp. 43-44 of [23])

$$\mu([X]_m, \rho') = -\sum_{i=1}^{P(m)} \mathrm{wt}_{\rho'}(x^{a(i)}). \tag{5.3.5}$$

The above formula follows from Eqtn (2.5) of [23], and the (easily checked) relation

$$\mathrm{wt}_\rho(x^{a(i)}) = \frac{1}{N+1}(\mathrm{wt}_{\rho'}(x^{a(i)}) + rm).$$

In our case Equation (5.3.5) reads

$$\mu_m(I,\lambda) = -\sum_{x^A \notin \mathrm{in}^\lambda_\prec(I)} \mathrm{wt}_\lambda(x^A). \tag{5.3.6}$$

The above equation is equivalent to (5.3.4), because the sum of $\mathrm{wt}_\lambda(x^A)$ over all degree-m monomials x^A is equal to 0. $\square$

The group SL(4) acts also on $\mathrm{Chow}_{(2,d)}$, and on the restriction to $\mathrm{Chow}_{(2,d)}$ of the Chow polarization. There is a relation between the Hilbert-Mumford index of points in the Hilbert scheme and corresponding points of the Chow variety. First notice that the Hilbert-Chow morphism restricts to a (birational) morphism $\gamma\colon \mathrm{Hilb}_{(2,d)} \to \mathrm{Chow}_{(2,d)}$; then

$$\mu(\gamma(I),\lambda) = \lim_{m\to+\infty} \frac{1}{m^2}\mu_m(I,\lambda). \tag{5.3.7}$$

(The above equation makes sense because $\mathrm{Hilb}^m_{(2,d)} = \mathrm{Hilb}_{(2,d)}$ for $m \gg 0$.)

The result below generalizes Proposition 5.2 in [14].

Proposition 5.8. *Let $d \geq 3$, and $m \geq d$. Then all points of $\mathrm{Hilb}^m_{(2,d)} \setminus U_{(2,d)}$ are* SL(4)-*unstable, and similarly all points of $\mathrm{Chow}_{(2,d)} \setminus U_{(2,d)}$ are* SL(4)-*unstable.*

Proof. Let $I \in \mathrm{Hilb}^m_{(2,d)}$. Then I contains a quadratic form Q, and a degree-d form F which is not a multiple of Q. Now suppose that $I \notin \mathrm{Im}\, f_{d,m}$; then the common zero locus of Q and F is not a curve. It follows that there exist homogeneous coordinates $[x_0, \ldots, x_3]$ on $\mathbb{P}^3$ such that one of the following holds:

(1) $Q = x_0x_1$, and $F = x_0G$, where $G \in \mathbb{C}[x_0, \ldots, x_3]_{d-1}$, and $x_1 \not| G$.

(2) $Q = x_0^2$, and $F = x_0 G$, where $G \in \mathbb{C}[x_0, \ldots, x_3]_{d-1}$, and $x_0 \not| G$.

First assume that Item (1) holds. Let λ be the 1-PS of SL(4) defined by $\lambda = \mathrm{diag}(-3, 1, 1, 1)$ (with respect to the chosen homogeneous coordinates). Let us prove that

$$\mu_m(I, \lambda) \leq -2(a+1)m^2 - (d^2 - 2(2a+3)d + 6(a+1))m + \frac{2}{3}(d-1)(d-2)(d-3(a+1)). \tag{5.3.8}$$

Let

$$U := x_0 x_1 \mathbb{C}[x_0, \ldots, x_3]_{m-2}, \quad V := x_0 G \mathbb{C}[x_0, \ldots, x_3]_{m-d}, \quad T := U + V.$$

Notice that U, V, T are subspaces of I. We claim that

$$\sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(T)} \mathrm{wt}_\lambda(x^A) = -\frac{1}{2}m^3 + \frac{1}{2}(2d-4a-5)m^2 - \frac{1}{6}(9d^2 - 3(8a+13)d + 36(a+1))m + \frac{2}{3}(d-1)(d-2)(d-3(a+1)). \tag{5.3.9}$$

In fact, since $\mathrm{wt}_\lambda(x_0 x_1) = -2$ and the representation of λ on $\dim \mathbb{C}[x_0, \ldots, x_3]_{m-2}$ has determinant 1, we have

$$\sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(U)} \mathrm{wt}_\lambda(x^A) = (-2) \cdot \dim \mathbb{C}[x_0, \ldots, x_3]_{m-2} = (-2) \cdot \binom{m+2}{3}. \tag{5.3.10}$$

Next, let $0 \leq a \leq (d-1)$ be the maximum number such that $x_0^a | G$. Then

$$\sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(V)} \mathrm{wt}_\lambda(x^A) = (d - 4a - 4) \cdot \dim \mathbb{C}[x_0, \ldots, x_3]_{m-d} = (d - 4a - 4) \cdot \binom{m+3-d}{3}. \tag{5.3.11}$$

Lastly, since $U \cap V = x_0 x_1 G \mathbb{C}[x_0, \ldots, x_3]_{m-d-1}$, we have

$$\sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(U \cap V)} \mathrm{wt}_\lambda(x^A) = (d-4a-3) \cdot \dim \mathbb{C}[x_0, \ldots, x_3]_{m-d-1} = (d-4a-3) \cdot \binom{m+2-d}{3}. \tag{5.3.12}$$

Next, we have the Grassmann-like formula

$$\sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(T)} \mathrm{wt}_\lambda(x^A) = \sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(U)} \mathrm{wt}_\lambda(x^A) + \sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(V)} \mathrm{wt}_\lambda(x^A) - \sum_{x^A \in \mathrm{in}_{\prec}^{\lambda}(U \cap V)} \mathrm{wt}_\lambda(x^A).$$

Equation (5.3.9) follows from the above formula, together with (5.3.10), (5.3.11), and (5.3.12).

In order to prove Equation (5.3.8), we notice that $\mathrm{wt}_\lambda(x^A) \leq m$ for every monomial of degree m, and hence

$$\begin{aligned}\mu_m(I,\lambda) &= \sum_{x^A\in \mathrm{in}^{\lambda}_{\prec}(I)} \mathrm{wt}_\lambda(x^A) \le \sum_{x^A\in \mathrm{in}^{\lambda}_{\prec}(T)} \mathrm{wt}_\lambda(x^A) + m\cdot(\dim I - \dim T)\\ &= \sum_{x^A\in \mathrm{in}^{\lambda}_{\prec}(T)} \mathrm{wt}_\lambda(x^A) + \frac{1}{2}(m^3-(2d-1)m^2+d(d-1)m).\end{aligned}$$

Thus Equation (5.3.8) follows from (5.3.9).

Let $P(d,a,m)$ be the right hand side of (5.3.8); we claim that $P(d,a,m)<0$ for $d\ge 3$, $a\in\{0,\dots,d-1\}$, and $m\ge d$. One easily checks that $P(d,a+1,m)<P(d,a,m)$ if $m>0$. Hence it suffices to check that $P(d,0,m)<0$ for $m\ge d$. The function $P(d,0,m)$ is decreasing for $m\ge d$, hence one is reduced to proving that $P(d,0,d)<0$ for $d\ge 3$; this is straightforward.

This proves that if Item (1) holds, then $\mu_m(I,\lambda)<0$, and hence I is unstable. Moreover

$$\begin{aligned}\lim_{m\to+\infty}\frac{1}{m^2}\mu_m(I,\lambda) &\le \lim_{m\to+\infty}\frac{1}{m^2}(-2(a+1)m^2-(d^2-2(2a+3)d+6(a+1))m\\ &\qquad +\frac{2}{3}(d-1)(d-2)(d-3(a+1)))\\ &= -2(a+1)<0.\end{aligned} \tag{5.3.13}$$

By Equation (5.3.7) it follows that $\mu(\gamma(I),\lambda)<0$, i.e. the corresponding point of the Chow variety is unstable.

Next, assume that Item (2) holds. Again, let λ be the 1-PS of $\mathrm{SL}(4)$ defined by $\lambda=\mathrm{diag}(-3,1,1,1)$. We claim that

$$\mu_m(I,\lambda)\le -(2d+6)m^2-(d^2-6d+17)m-(2d^2-6d+10). \tag{5.3.14}$$

We quickly go over the proof of (5.3.14). Let

$$U := x_0^2\mathbb{C}[x_0,\dots,x_3]_{m-2},\quad V := x_0G\mathbb{C}[x_0,\dots,x_3]_{m-d},\quad T := U+V.$$

Then U,V,T are subspaces of I, and computations similar to those performed above give that

$$\sum_{x^A\in \mathrm{in}^{\lambda}_{\prec}(T)} \mathrm{wt}_\lambda(x^A) = -\frac{1}{2}m^3-\frac{2d+13}{2}m^2+\frac{d^2+5d-32}{2}m-2d^2+6d-10. \tag{5.3.15}$$

Since $\mathrm{wt}_\lambda(x^A)\le m$ for every monomial of degree m, it follows that

$$\begin{aligned}\mu_m(I,\lambda) &= \sum_{x^A\in \mathrm{in}^{\lambda}_{\prec}(I)} \mathrm{wt}_\lambda(x^A) \le \sum_{x^A\in \mathrm{in}^{\lambda}_{\prec}(T)} \mathrm{wt}_\lambda(x^A) + m\cdot(\dim I - \dim T)\\ &= \sum_{x^A\in \mathrm{in}^{\lambda}_{\prec}(T)} \mathrm{wt}_\lambda(x^A) + \frac{1}{2}(m^3-(2d-1)m^2-(3d^2-7d+2)m).\end{aligned} \tag{5.3.16}$$

Inequality (5.3.14) follows at once from (5.3.15) and (5.3.16). The right hand side of (5.3.14) is negative for $d \geq 3$ and $m \geq d$; it follows that I is unstable. Proceeding as in the previous case, we also get that the corresponding point of the Chow variety is unstable. $\square$

5.4. On (non-semi)stability of points of $\mathbb{P}E$ *and of* $\mathscr{P}$

We denote points of $\mathbb{P}E$ as follows:

$$\mathbb{P}E = \{([f_2], [\bar{f}_4]) \mid f_d \in \Gamma(\mathcal{O}_{\mathbb{P}^3}(d)), \bar{f}_4 = f_4|_{V(f_2)}\} \tag{5.4.1}$$

The Hilbert-Mumford numerical function for points of $\mathbb{P}E$ relative to $\eta + t\xi$ has been computed by Benoist [7]. First we recall that the Hilbert-Mumford numerical function of a non zero homogeneous polynomial $f = \sum_A f_A x^A$ with respect to a 1-PS $\lambda(s) = \operatorname{diag}(s^{r_0}, \ldots, s^{r_n})$ is

$$\mu(f, \lambda) = \max\{\operatorname{wt}_\lambda(x^A) \mid f_A \neq 0\}. \tag{5.4.2}$$

Thus

$$\mu(f,\lambda) \begin{cases} < 0 & \text{if and only } \lim\limits_{s\to 0} \lambda(s) f = 0, \\ = 0 & \text{if and only } \lim\limits_{s\to 0} \lambda(s) f \neq 0, \\ > 0 & \text{if and only } \lim\limits_{s\to 0} \lambda(s) f \text{ does not exist.} \end{cases}$$

(Recall that $g \in \mathrm{SL}(n+1)$ acts on $f \in \mathbb{C}[x_0, \ldots, x_n]$ by the formula $gf(x) = f(g^{-1}x)$.) Let $([f_2], [\overline{f}_4]) \in \mathbb{P}E$ and let λ be a 1-PS of $\mathrm{SL}(4)$. Then, by Proposition 2.15 of [7]

$$\mu^{\eta+t\xi}(x, \lambda) := \mu(f_2, \lambda) + t \min_{f \in [f_4]} \mu(f, \lambda). \tag{5.4.3}$$

The straightforward proof of the result below is left to the reader.

Proposition 5.9. *Let $t \in (0, 1/2] \cap \mathbb{Q}$. Let $([f_2], [\bar{f}_4]) \in \mathbb{P}E \setminus U$. Then there exist homogeneous coordinates $[x_0, \ldots, x_3]$ on $\mathbb{P}^3$ such that one of the following holds:*

(1) *$f_2 = x_0 x_1$, and $f_4 = x_0 g$, where $g \in \mathbb{C}[x_0, \ldots, x_3]_3$, and $x_1 \not| g$.*
(2) *$f_2 = x_0^2$, and $f_4 = x_0 g$, where $g \in \mathbb{C}[x_0, \ldots, x_3]_3$, and $x_0 \not| g$.*

Moreover, let λ be the 1-PS of $\mathrm{SL}(4)$ given by $\lambda = \operatorname{diag}(-3, 1, 1, 1)$ in the coordinates $[x_0, \ldots, x_3]$ above. Then $\mu^{\eta+t\xi}([f_2], [\bar{f}_4]) \leq -2$.

Corollary 5.10. *Let $t \in (0, 1/3) \cap \mathbb{Q}$. Then $\mathbb{P}E^{ss}(\eta + t\xi) \subset U$.*

Proof. The corollary follows at once from **Proposition 5.9** and the Hilbert-Mumford numerical criterion for semistability, which holds because $\eta+t\xi$ is ample for $t \in (0,1/3)\cap\mathbb{Q}$. $\square$

Proposition 5.11. *Let $t \in (\delta, 1/2]\cap\mathbb{Q}$. Then $\mathscr{P}^{ss}(N_t) \subset U$.*

Proof. Let $z \in (\mathscr{P}\setminus U)$. There exist a curve $\mathscr{C} \subset \mathbb{P}^3$, parametrized by a point of $\mathrm{Hilb}_{(2,4)}$, and a decomposition

$$H^0(\mathbb{P}^3, \mathscr{I}_{\mathscr{C}}(4)) = H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) \cdot f_2 + \mathbb{C} f_4, \quad f_d \in H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(d)),$$

such that

$$z = (([f_2], [\overline{f}_4]), c(\mathscr{C})),$$

(here $c\colon \mathrm{Hilb}_{(2,4)} \to \mathrm{Chow}_{(2,4)}$ is the Hilbert-Chow map) and either Item (a) or Item (b) of **Proposition 5.9** holds. Let $[x_0, \ldots, x_3]$ and λ be the projective coordinates and 1-PS of $\mathrm{SL}(4)$ of **Proposition 5.9**. Then, by linearity of the Hilbert-Mumford numerical function

$$\mu^{N_t}(z,\lambda) = \frac{1-2t}{1-2\delta}\mu^{\eta+\delta\xi}(([f_2],[\overline{f}_4]),\lambda) + \frac{t-\delta}{2(1-2\delta)}\mu^{L_\infty}(c(\mathscr{C}),\lambda). \tag{5.4.4}$$

We claim that both numerical functions in the right hand side of (5.4.4) are strictly negative. In fact $\mu^{\eta+\delta\xi}(([f_2],[\overline{f}_4]),\lambda) < 0$ by **Proposition 5.9**. On the other hand, the proof of **Proposition 5.8** gives that $\mu^{L_\infty}(c(\mathscr{C}),\lambda) < 0$. $\square$

5.5. Proof of the main result (Theorem 5.6)

Item (1): We will apply Lemma 4.17 in [14]. For the reader's convenience, we record below the part of that lemma that we will need.

Lemma 5.12 *([14, Lemma 4.17]). Let X be a projective variety, let G be a reductive group acting on X, and let L be a G-linearized ample line bundle on X. Then the natural map*

$$X^{ss}(L)/\!\!/G \longrightarrow X/\!\!/_L G = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G\right)$$

is an isomorphism.

Applying **Lemma 5.12** to $X = \mathscr{P}$, $G = \mathrm{SL}(4)$, $L = N_t$, and to $X = \mathbb{P}(E)$, $G = \mathrm{SL}(4)$, $L = \eta + t\xi$, we get two isomorphisms

$$\mathscr{P}^{ss}(N_t)/\!\!/\mathrm{SL}(4) \xrightarrow{\sim} \mathfrak{M}(t), \qquad \mathbb{P}E^{ss}(\eta+t\xi)/\!\!/\mathrm{SL}(4) \xrightarrow{\sim} \mathbb{P}E/\!\!/_{\eta+t\xi}\mathrm{SL}(4). \tag{5.5.1}$$

By **Proposition 5.11** we have $\mathscr{P}^{ss}(N_t) \subset U$, and by **Corollary 5.10** we have $\mathbb{P}E^{ss}(\eta+t\xi) \subset U$. On the other hand, $N_t|_U = (\eta+t\xi)|_U$ by **Corollary 5.5**. It follows that both $\mathscr{P}^{ss}(N_t)$ and $\mathbb{P}E^{ss}(\eta+t\xi)$ are equal to the set of points in U which are semistable for the action of $\mathrm{SL}(4)$, with respect to the linearized line bundle $N_t|_U$. Thus $\mathscr{P}^{ss}(N_t) = \mathbb{P}E^{ss}(\eta+t\xi)$ and $\mathscr{P}^{ss}(N_t)/\!\!/\mathrm{SL}(4) = \mathbb{P}E^{ss}(\eta+t\xi)/\!\!/\mathrm{SL}(4)$. Item (1) now follows from (5.5.1).

Item (2): By Item (1) it suffices to prove that

$$\mathbb{P}E/\!\!/_{\eta+t\xi}\mathrm{SL}(4) \cong \mathfrak{M}. \tag{5.5.2}$$

Let $([f_2],[\overline{f}_4]) \in \mathbb{P}E^{ss}(\eta+t\xi)$. We claim that $V(f_2)$ is a smooth quadric. In fact, assume that $V(f_2)$ is singular. Then there exist homogeneous coordinates $[x_0,\ldots,x_3]$ such that $f_2 \in \mathbb{C}[x_1,x_2,x_3]_2$. Let $\lambda = \mathrm{diag}(-3,1,1,1)$. Then $\mu(f_2,\lambda) = -2$, and hence (recall that $t < 1/6$)

$$\mu(f_2,\lambda) + t\min_{f\in[f_4]}\mu(f,\lambda) \le -2+12t < 0. \tag{5.5.3}$$

This is a contradiction. Hence we have proved that if $([f_2],[\overline{f}_4]) \in \mathbb{P}E^{ss}(\eta+t\xi)$, then $V(f_2)$ is a smooth quadric. Now the isomorphism in (5.5.2) follows from the proof of Lemma 4.18 in [14] (which applies verbatim).

Item (3): Arguing as in the proof of Item (1), it suffices to show that

$$L_m|_U = (2(m^2-4m+5)\eta + (m-3)^2\xi)|_U. \tag{5.5.4}$$

Let $\Gamma, \Omega \subset U$ be the projective curves in (5.2.10) and (5.2.11) respectively. By (5.2.13), it suffices to show that

$$\deg(L_m|_\Gamma) = (m-3)^2, \qquad \deg(L_m|_\Omega) = 2(m^2-4m+5). \tag{5.5.5}$$

This is a straightforward computation that we leave to the reader. □

6. The stability analysis for $\mathfrak{M}(t)$

6.1. Summary

In the previous sections, we have defined $\mathfrak{M}(t)$ for $t \in (\delta, \frac{1}{2}]$ (with $0 < \delta < \frac{1}{6}$ fixed) and, for special values of t, we have identified it with natural Hilbert and Chow GIT quotients for $(2,4)$ complete intersection curves. In the present section we analyze the variation of GIT describing $\mathfrak{M}(t)$. To start, we recall that the general theory of variations of GIT quotients [53,15] (see also [32]) says that the interval $(\delta, \frac{1}{2})\cap\mathbb{Q}$ will be partitioned into finitely many (open) intervals, called *chambers*, on which $\mathfrak{M}(t)$ stays constant. The limits of these intervals are called *walls*, or critical values t. At such a critical value, there are birational maps $\mathfrak{M}(t\pm\epsilon) \to \mathfrak{M}(t)$. The composition $\mathfrak{M}(t-\epsilon) \dashrightarrow \mathfrak{M}(t+\epsilon)$

is typically a (generalized) flip, that we will refer to as a *wall crossing*. Since [53,15] a significant number of applications of VGIT to moduli problems have appeared in the literature. Most relevant for us are [22], [14], and [7] from which we borrow a number of techniques and results.

6.2. Main GIT results and structure of the argument

In order to state the main results of the present section, we introduce the following tables.

value of t_k	1/6	1/4	3/10	1/3	1/3	5/14	3/8	2/5	1/2
k	0	1	2	3	4	5	6	7	8

(6.2.1)

(The equality $t_3 = t_4$ is not a misprint.)

type of sing.	$(4,0)$	$(3,1)$	$J_{4,\infty}$	$J_{3,+}$	$J_{3,0}$	E_{14}	E_{13}	E_{12}
tag	0	1	2	3	4	5	6	7

(6.2.2)

Theorem 6.1. *Let $t \in [1/6, \frac{1}{2}) \cap \mathbb{Q}$, and let $C = V(f_2, f_4)$ be a $(2,4)$ curve which is N_t semistable (notation as in* **Subsection 5.2***). Let $X_C \to V(f_2)$ be the double cover ramified over C. Then every non slc singularity of X_C appears in the list in (6.2.2), where notation for singularities is as in* **Section 4***. More precisely, letting $k \in \{0, \ldots, 7\}$, the following holds:*

(1) *If $t = t_k$, then every singularity of X_C has tag at least k.*
(2) *If $t_k < t < t_{k+1}$, then every singularity of X_C has tag at least $k + 1$.*

In particular, if $2/5 < t < 1/2$, then every surface parametrized by $\mathfrak{M}(t)$ has slc singularities, and hence we have a regular *period map $\Phi\colon \mathfrak{M}(t) \to \mathscr{F}^*$.*

Theorem 6.2. *The critical slopes of the VGIT for $\mathfrak{M}(t)$ in the interval $(\delta, \frac{1}{2}) \cap \mathbb{Q}$ are given by the t_k's appearing in (6.2.1), with the exclusion of $t_8 = 1/2$. Hence for $k \in \{0, \ldots, 7\} \setminus \{3\}$ it makes sense to let $\mathfrak{M}(t_k, t_{k+1}) := \mathfrak{M}(t)$ for $t_k < t < t_{k+1}$ (in addition, recall that $\mathfrak{M}(\delta, 1/6) \cong \mathfrak{M}$). The VGIT for $\mathfrak{M}(t)$ gives the sequence of birational maps*

$$
\begin{tikzcd}
\mathfrak{M}\cong\mathfrak{M}(\delta,\frac{1}{6}) \arrow[dr] & & \mathfrak{M}(\frac{1}{6},\frac{1}{4}) \arrow[ll] \arrow[dl] \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & \cdots \arrow[dl] & \mathfrak{M}(t_{k-1},t_k) \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & \mathfrak{M}(t_k,t_{k+1}) \arrow[dl] & \cdots \arrow[dr] \arrow[rr, dashed, leftrightarrow] & & \mathfrak{M}(\frac{2}{5},\frac{1}{2}) \arrow[dl] \arrow[dr, "\Phi"] & \\
& \mathfrak{M}(\frac{1}{6}) & & \mathfrak{M}(\frac{1}{4}) & & & \mathfrak{M}(t_k) & & & \mathfrak{M}(\frac{2}{5}) & & \mathfrak{M}(1/2)
\end{tikzcd}
\tag{6.2.3}
$$

where each dotted arrow denotes a flip, and each solid arrow is a small contraction, with the exception of $\mathfrak{M}(\frac{1}{6}, \frac{1}{4}) \to \mathfrak{M}(\frac{1}{6})$ which is a divisorial contraction, and $\mathfrak{M}(\delta, \frac{1}{6}) \to \mathfrak{M}(\frac{1}{6})$

which is an isomorphism. Furthermore, the following holds. For a critical value t_k, let $\Sigma_-(t_k) \subset \mathfrak{M}(t_{k-1}, t_k)$ (if $k = 0$, we mean $\{[4C]\} \subset \mathfrak{M}(\delta, 1/6)$ where C is a smooth conic) and $\Sigma_+(t_k) \subset \mathfrak{M}(t_k, t_{k+1})$ be the exceptional loci of $\mathfrak{M}(t_{k-1}, t_k) \to \mathfrak{M}(t_k)$ and $\mathfrak{M}(t_k, t_{k+1}) \to \mathfrak{M}(t_k)$ respectively. Then $\Sigma_-(t_k)$ is the strict transform of $W_k \subset \mathfrak{M}$ for the birational map $\mathfrak{M} \dashrightarrow \mathfrak{M}(t_{k-1}, t_k)$ (if $k = 0$ we mean the inverse image of W_0), and $\Sigma_+(t_k)$ is the strict transform of $Z^{k+1} \subset \mathscr{F}^*$ for $k \neq 4$, and of $Z^4 \subset \mathscr{F}^*$ for $k = 4$, for the birational map $\mathscr{F}^* \dashrightarrow \mathfrak{M}(t_k, t_{k+1})$.

Remark 6.3. The value $t = \frac{1}{2}$ is also a critical value, the proof is in **Subsection 7.3**.

Remark 6.4. At $t = \frac{1}{3}$, (the strict transform of) the 4 dimensional locus $W_4 \subset \mathfrak{M}$ is replaced by (the strict transform of) the codimension 4 locus $Z^4 \subset \mathscr{F}^*$. At $t = \frac{1}{3}$, the center of the flip is the curve parametrizing equivalence classes of $(2, 4)$ curves

$$V(x_0x_2 + x_1^2, x_0x_3^3 + 2\alpha x_1x_2x_3^2 - \beta x_2^3x_3),$$

i.e. the set of $[\alpha, \beta] \in W\mathbb{P}(2, 1)$. Moreover W_4 and Z^4 are birationally fibered over it. The special case, $[\alpha, \beta] = [1, 1]$ morally corresponds to (an undefined) W_3, but we have avoided defining W_3 in **Definition 4.10**, because this stratum is flipped together with W_4. Similarly, the undefined Z^5 is "hidden" in Z^4. This behavior is explained by the properties of automorphic forms on $\mathscr{F}(N)$ studied in [36].

Before stating the last main result of the present section, we recall that

$$\mathfrak{p}(t)\colon \mathfrak{M}(t) \dashrightarrow \mathscr{F}^* \tag{6.2.4}$$

is the (birational) period map.

Proposition 6.5. *For $t \in (1/6, \frac{1}{2}] \cap \mathbb{Q}$, the period map $\mathfrak{p}(t)$ is an isomorphism in codimension 1.*

The proof of **Theorem 6.1** is in **Subsection 6.5**, the proof of **Theorem 6.2** and **Proposition 6.5** is in **Subsection 6.7**. Here we outline the main ingredients in the proofs. Similar methods have previously occurred in the GIT analysis associated to the Hassett-Keel program (see esp. [22] and [6]).

Outline of the proof of the main results. Step (1): We compute a set of *potential* critical values t_k for the VGIT $\mathfrak{M}(t)$ and the corresponding *potential* critical curves C_k^*, listed in Table 2. If t is a critical value, then there exists a curve with $\mathbb{C}^*$ stabilizer that is N_t semistable but not $N_{t\pm\epsilon}$ semistable. Based on this observation, we give in **Subsection 6.3** a straightforward algorithm that produces a set of t_k's containing all actual critical values, and corresponding curves C_k^* (there is one curve C_k^* up to projectivities for all $k \neq 3$, while for $k = 3$ we get 1 moduli for the C_k^*'s). The list of potential critical values coincides with the list of critical values in **Theorem 6.2**, but we will be able to prove that C_k^* is N_{t_k}-semistable only at the end of the present section.

Step (2): As always in a GIT analysis, a key rôle is played by the numerical criterion for (semi)stability. In particular it allows us to prove that a curve C with a singularity with tag k in Table (6.2.2) is N_t-unstable for $t_k < t$, see **Proposition 6.11**. The desemistabilizing 1-PS is (conjugated to) the stabilizer of the corresponding curve C_k^* (if $k \in \{0, 1\}$ the stabilizer is not 1 dimensional, one has to choose appropriate 1-PS's of the stabilizers). This is the key step in the proof of **Theorem 6.1**.

Step (3): In order to prove that the t_k's are actual critical values we argue via a basin of attraction argument. This means that for each curve C_k^* we study the curves C such that $\lim_{s\to 0} \lambda(s)C = C_k^*$ for some 1-PS in the stabilizer of C_k^*. Each C_k^* lies on a quadric $V(f_2)$ of rank 3, it has an A_m singularity at the vertex of the quadric, it has a point in the smooth locus of $V(f_2)$ with tag k in (6.2.2), and no other singularities. We show that if C is in the basin of attraction of C_k^* and N_t semistable for $t < t_k$, then it has a point with tag k, while if it is N_t semistable for $t_k < t$, then it lies on a quadric Q of rank 3, it passes through the vertex of Q and near the vertex it is of the same type as C_k^*. **Theorem 6.2** is a straightforward consequence of this result. □

6.3. *Potential critical values and potential critical curves*

By general results on VGIT ([53], [15]; see also [32]) there exists a finite set $\{t_i\} \subset (\delta, 1/2) \cap \mathbb{Q}$ of critical values (or *walls*) for the VGIT $\mathfrak{M}(t)$. (Recall that in **Subsection 5.2** we have chosen a rational $\delta \in (0, 1/6)$ in order to define our VGIT; in the end the choice of δ will make no difference.) A point $t_0 \in (\delta, 1/2) \cap \mathbb{Q}$ is a critical value if for all sufficiently small $\epsilon \in \mathbb{Q}_+$, the following holds (e.g. [32, §3.2.1])

$$\mathscr{P}^{ss}(N_{t_0-\epsilon}) \cap \mathscr{P}^{ss}(N_{t_0+\epsilon}) \subsetneq \mathscr{P}^{ss}(N_{t_0}). \tag{6.3.1}$$

We let $\mathscr{P}^{ss}(N_{t_0})^{\text{new}} \subset \mathscr{P}^{ss}(N_{t_0})$ be the complement of the left hand side of (6.3.1). Notice that $\mathscr{P}^{ss}(N_{t_0})^{\text{new}}$ is a closed PGL(4)-invariant subset of $\mathscr{P}^{ss}(N_{t_0})$, and that all its points are strictly semistable (semistable but not stable) because $\mathscr{P}^{s}(N_{t_0}) \subset \mathscr{P}^{s}(N_{t_0\pm\epsilon})$.

Now let $x \in \mathscr{P}^{ss}(N_{t_0})^{\text{new}}$. It follows from the results recalled above that there is a unique closed PGL(4)-orbit in the closure of the orbit $\mathrm{PGL}(4) \cdot x$ in $\mathscr{P}^{ss}(N_{t_0})^{\text{new}}$, and that if x^* belongs to such a closed orbit, then its stabilizer is a reductive group (cf. Matsushita's criterion) of strictly positive dimension. In particular, x^* is stabilized by a 1-PS λ. In other words the changes of stability are associated to geometric objects stabilized by (at least) a $\mathbb{C}^*$ (N.B. a similar idea occurs in [4]).

Based on this observation, we will write down a finite subset of $(\delta, 1/2)$ containing the set of critical values. The numbers in our list *are* critical values, but before we are in a position to prove that statement, they will be called *potential critical values*. Moreover, for each potential critical value t_i we will give a subset of $\mathscr{P}^{ss}(N_{t_i})$ containing all elements of $\mathscr{P}^{ss}(N_{t_0})^{\text{new}}$ stabilized by a 1-PS - the elements of that subset (or any point in the

Table 2
Potential critical values for the VGIT $\mathfrak{M}(t)$.

k	t_k	Critical curve $C_k^* = V(f_2, f_4)$	1-PS	Sing. at p	Type of C_k^* at v
0	$\frac{1}{6}$	$V(q, x_3^4)$	$(1, \alpha, 2\alpha - 1, -3\alpha)$	$4\times$(conic)	$v \notin C_k^*$
1	$\frac{1}{4}$	$V(q, x_3^3 x_1)$	$(1, \alpha, 2\alpha - 1, -3\alpha)$	$3\times$(conic)	A_1
2	$\frac{3}{10}$	$V(q, x_0 x_3^3 + x_2^2 x_3^2)$	$(7, 3, -1, -9)$	$J_{4,\infty}$	A_2
3	$\frac{1}{3}$	$V(q, x_0 x_3^3 + 2x_1 x_2 x_3^2 - x_2^3 x_3)$	$(3, 1, -1, -3)$	$J_{3,\infty}$	$2\times$(twisted cubic)
4	$\frac{1}{3}$	$V(q, x_0 x_3^3 + 2\alpha x_1 x_2 x_3^2 - \beta x_2^3 x_3)$	$(3, 1, -1, -3)$	$J_{3,0}$	A_3
5	$\frac{5}{14}$	$V(q, x_0 x_3^3 + x_2^4)$	$(17, 5, -7, -15)$	E_{14}	A_4, $\mathscr{C}_v C_k^* = 2T_v(L)$
6	$\frac{3}{8}$	$V(q, x_0 x_3^3 + x_1 x_2^2 x_3)$	$(11, 3, -5, -9)$	E_{13}	A_5, $C_k^* \supset L$
7	$\frac{2}{5}$	$V(q, x_0 x_3^3 + x_1 x_2^3)$	$(4, 1, -2, -3)$	E_{12}	A_7, $C_k^* \supset L$

same PGL(4) orbit) are the *potential critical curves* (notice that by **Proposition 5.11** any element of $\mathscr{P}^{ss}(N_{t_0})$ is in U, i.e. is a $(2,4)$ c.i. curve).

Proposition 6.6. *Keeping notation as above, the set of critical values for the VGIT $\mathfrak{M}(t)$ is included in the set (of potential critical values)*

$$\left\{\frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}, \frac{5}{14}, \frac{3}{8}, \frac{2}{5}\right\}. \tag{6.3.2}$$

For each potential critical value t_k, the corresponding potential critical curve(s) C_k^* appear in the row corresponding to t_k in Table 2. In that table $v = [0,0,0,1] \in \mathbb{P}^3$ is the vertex of the quadric cone $V(q)$ (with $q = x_0x_2 + x_1^2$) containing C_k^*, $p = [1,0,0,0] \in \mathbb{P}^3$ is the unique singular point of C_k^*, $\mathscr{C}_v C_k^*$ is the tangent cone to C_k^* at v, and L is a line of $V(f_2)$ (in fact $L = V(x_0, x_1)$).

(In the row corresponding to $k = 4$, (α, β) and $(1,1)$ are linearly independent. The imprecise notation $2T_v(L)$ means the tangent cone at v of $V(x_0, x_0x_2 + x_1^2) = V(x_0, x_1^2)$.)

Before proving **Proposition 6.6**, we go through a few auxiliary results.

Lemma 6.7. *Let $x = V(f_2, f_4) \in U$. If*

(1) *f_2 has rank at most 2, or*
(2) *there exists a point $p \in V(f_2, f_4)$ which is singular both for $V(f_2)$ and $V(f_4)$,*

then x is t-unstable for all $t \in (\delta, 1/2)$.

Proof. If (1) holds, the proof is similar to that of [14, Prop. 4.6]. If (2) holds, the proof is similar to that of [14, Prop. 4.7]. We omit the details. □

Remark 6.8. We will repeatedly use the function $\mu(f_2, \lambda) + t\mu(f_4, \lambda)$ to destabilize curves $C = V(f_2, f_4)$ at specific values of t. An attentive reader might notice that this is in fact different from the numerical function $\mu^{N_t}((f_2, f_4), \lambda)$ that we should use in the application of the numerical criterion for $\mathscr{P}$ with linearization N_t. In fact, $\mu(f_2, \lambda) + t\mu(f_4, \lambda)$ is the numerical function (for the linearization $\eta + t\xi$) on the $\mathbb{P}E$ model of $\mathscr{P}$. The point is that $\mu^{N_t}((f_2, f_4), \lambda) = \mu(f_2, \lambda) + t\mu(f_4, \lambda)$ if

$$\lim_{t \to 0} \lambda(t)V(f_2, f_4) \in U \tag{6.3.3}$$

(recall that U is a common open subset of $\mathbb{P}E$ and $\mathscr{P}$, and that the linearizations agree over U, cf. **Corollary 5.5**). Equation (6.3.3) always holds in the cases that we will consider.

Lemma 6.9. *Keeping notation as above, let t_0 be a critical value for the VGIT $\mathfrak{M}(t)$. Let $C \in \mathscr{P}^{ss}(N_{t_i})^{\text{new}}$ be a minimal orbit (notice that $\mathscr{P}^{ss}(N_{t_i}) \subset U$ by* **Proposition 5.11***). Then, there exists λ 1-PS of $\mathrm{PGL}(4)$ stabilizing C and equations $C = V(f_2, f_4)$ such that*

$$\mu(f_2, \lambda) \neq 0, \qquad \mu(f_4, \lambda) \neq 0. \tag{6.3.4}$$

(and $\mu(\cdot, \lambda)$ is minimized by f_4 among representatives of $f_4 \pmod{f_2}$).

Proof. Let $t_0 \in (\delta, 1/2)$ be a critical value, and C an associated critical polystable orbit. Since, the stability of C changes at t_0, it is clear that we can choose a 1-PS λ and equations $C = V(f_2, f_4)$ such that $\mu^t((f_2, f_4), \lambda) = \mu(f_2, \lambda) + t\mu(f_4, \lambda)$ and that $\mu^t((f_2, f_4), \lambda)$ changes sign at $t_0 \neq 0$. It follows that conditions (6.3.4) are satisfied. Finally, the condition $\mu^{t_0}((f_2, f_4), \lambda) = 0$ means that

$$\lim_{s \to 0} \lambda(s) \cdot (f_2)^{\otimes n} \otimes (f_4)^{\otimes m}$$

(for some integers n, m with $t_0 = \frac{m}{n}$) exists and it is non-zero (compare (6.6.4) below). Replacing (f_2, f_4) by the limit $(\overline{f}_2, \overline{f}_4)$, we get that λ stabilizes $V(\overline{f}_2, \overline{f}_4)$ and that (f_2, f_4) and $(\overline{f}_2, \overline{f}_4)$ are in the same SL(4)-orbit (since the orbit is closed). Finally, $\overline{f}_2 = \lim_{s\to 0} s^{cm} \cdot \lambda(s) \cdot f_2$ and $\overline{f}_4 = \lim_{s\to 0} s^{-cn} \cdot \lambda(s) \cdot f_4$ for appropriate n and m as before (and a constant c). We get $\mu(\overline{f}_k, \lambda) = \mu(f_k, \lambda)$ for $k = 2, 4$ (i.e. the monomial computing the λ-weight agree for $\overline{f}_k$ and f_k). $\square$

Proof of Proposition 6.6. Let $t_0 \in (\delta, 1/2) \cap \mathbb{Q}$ be a critical value for the VGIT $\mathfrak{M}(t)$, and let $V(f_2, f_4) \in U(N_{t_0})^{ss}$ be a critical curve for t_0. As noted above, there exists a 1-PS λ of SL(4) stabilizing x, i.e. $\lambda(s)f_2 = s^m f_2$ and $\lambda(s)f_4 \equiv s^n f_4 \pmod{f_2}$ for some m, n. Replacing f_4 by a suitable multiple of f_2, we may assume that $\lambda(s)f_4 = s^n f_4$. Since x is N_{t_0}-semistable, λ acts trivially on the fiber of $\mathscr{O}_{\mathscr{P}}(N_{t_0})$ over x, i.e.

$$\mu(f_2, \lambda) + t_0\mu(f_4, \lambda) = 0. \tag{6.3.5}$$

(Notice that $\mu(f_2, \lambda) = m$ and $\mu(f_4, \lambda) = n$, where m, n are as above.) By **Lemma 6.9** we know that (6.3.4) holds. Since a smooth quadric is semistable, it follows that f_2 is degenerate. On the other hand, we may suppose f_2 has rank at least 3 by **Lemma 6.7**, and hence it has rank equal to 3. A straightforward argument shows that there exist coordinates $(x_0, \dots, x_3)$ on $\mathbb{C}^4$ such that

$$\lambda(s) = \mathrm{diag}(s^{r_0}, \dots, s^{r_3}), \qquad f_2 = x_0 x_2 + x_1^2.$$

Since $\lambda(s) f_2 = s^m f_2$, we have

$$2r_1 = r_0 + r_2.$$

It follows that

$$3r_1 + r_3 = 0$$

because $r_0 + \ldots + r_3 = 0$. By interchanging λ and λ^{-1}, we can assume $r_1 \geq 0 \geq r_3$. Interchanging the variables x_0 and x_2, we can assume $r_0 \geq r_1 = \frac{r_0+r_2}{2} \geq r_2$ (in particular, $r_0 > 0$). At this point we may rescale the r_i's so that $r_0 = 1$ (we will get a virtual 1-PS, it makes no difference as far as our proof is concerned), and we get $r_0 = 1$, $r_1 = \alpha$, $r_2 = 2\alpha - 1$, $r_3 = -3\alpha$, where $\alpha \in [0, 1] \cap \mathbb{Q}$. Thus we let λ_α be the virtual 1-PS

$$\lambda_\alpha(s) := \mathrm{diag}(s, s^\alpha, s^{2\alpha-1}, s^{-3\alpha}), \qquad \alpha \in [0, 1] \cap \mathbb{Q}. \tag{6.3.6}$$

Consider separately the two cases:

(1) f_4 is a multiple of a monomial.
(2) f_4 is not a multiple of a monomial.

Suppose that Item (1) holds. The numerical function $\mu(f_4, \lambda_\alpha)$ is a polynomial in α of degree 1:

$$\mu(f_4, \lambda_\alpha) = c\alpha + d, \qquad c, d \in \mathbb{Q}, \quad c \neq 0. \tag{6.3.7}$$

We are assuming that (6.3.5) holds for a certain $t_0 \in [0, 1/2] \cap \mathbb{Q}$ and $\lambda = \lambda_{\alpha_0}$. Since $\mu(f_2, \lambda_\alpha) = -2\alpha$, it follows that $c > 0$ and $d = 0$. In fact, if $d = 0$ and $c \leq 0$, then clearly (6.3.5) cannot hold for $t_0 > 0$, and if $d \neq 0$, then there exist (many!) values of $\alpha \in [0, 1] \cap \mathbb{Q}$ with the property that $\mu(f_2, \lambda_\alpha) + t_0 \mu(f_4, \lambda) < 0$ or $\mu(f_2, \lambda_\alpha) + t_0 \mu(f_4, \lambda) > 0$, i.e. (after reparametrization, if λ_α is a virtual 1-PS) λ_α fixes $V(f_2, f_4)$ and acts non trivially on the fiber of $\mathscr{O}_{\mathscr{P}}(N_{t_0})$ over $V(f_2, f_4)$. That is a contradiction because $V(f_2, f_4)$ is assumed to be N_{t_0}-semistable. This proves that $c > 0$ and $d = 0$. A straightforward computation then shows that, after rescaling, $f_4 \in \{x_3^4, x_1 x_3^4, x_1^2 x_3^2, x_0 x_2 x_3^2\}$. The critical value for $f_4 = x_3^4$ is $t_0 = 1/6$, the critical value for $f_4 = x_1 x_3^3$ is $t_0 = 1/4$, while $f_4 \in \{x_1^2 x_3^2, x_0 x_2 x_3^2\}$ is impossible, because of **Lemma 6.7**.

Now suppose that Item (2) holds. Thus in the expansion of f_4 there are two (at least) monomials $x_0^{i_0}\dots x_3^{i_3}$ and $x_0^{j_0}\dots x_3^{j_3}$ with non-zero coefficients. Let $k_l = i_l - j_l$ for $l = 0,\dots,3$. Since $\lambda(s)f_4 = s^n f_4$ for some n, we have

$$k_0 + \alpha k_1 + (2\alpha - 1)k_2 - 3\alpha k_3 = 0. \tag{6.3.8}$$

The above equation determines α, and hence we get t_0 upon replacing λ by λ_α in (6.3.5), and solving for t_0. We get the potential critical values in (6.3.2) other than 1/6 by listing all couples of degree 4 monomials and going through the steps described above (or programming a computer to do it in our place). Once we have the potential critical values, it is clear how to compute the potential critical curves associated to each (potential) critical value. $\square$

6.4. *Relations between singularities of C and N_t-(semi)stability*

The following proposition is an easy adaptation to the case of singular quadrics of some of the content of **Proposition 3.2**.

Proposition 6.10. *Let $C \in U$, and assume that C has consecutive triple points at p, and that it has a significant limit singularity at p. Let Q be the unique quadric containing C. Suppose that Q is a quadric cone, and that it is smooth at p. Lastly, let $t \in (\delta, 1/2)$, and suppose that C is N_t-semistable. Then the tangent cone to C at p is* not *equal to $3T_p(L)$, for L the unique line in Q through p.* $\square$

We are now able to complete Step (2) of the proof of **Theorem 6.2**.

Proposition 6.11. *Let $C \in U$, and let $p \in C$ be a point contained in the smooth locus of the unique quadric containing C. Suppose that C has a singularity at p appearing in (6.2.2), with tag k. Then C is N_t-unstable for $t \in (t_k, 1/2]$, and it is N_t-desemistabilized by a 1-PS conjugated to the one appearing in the row of Table 2 with index k.*

Proof. Let us assume that $\mathrm{mult}_p(C) = 4$, i.e. we are in one of the first two cases in Table 2. We may choose homogeneous coordinates $[x_0,\dots,x_4]$ so that $p = [1,0,0,0]$ and $C = V(f_2, f_4)$ where

$$f_2 = x_0x_2 + x_1^2 + ax_3^2, \qquad a \in \mathbb{C}, \quad f_4 \in \mathbb{C}[x_1, x_2, x_3]_4. \tag{6.4.1}$$

In fact, choose coordinates so that f_2 is as above, and let $C = V(f_2, \widetilde{f}_4)$. Then, since $\mathrm{mult}_p(C) = 4$, we can add a suitable multiple of f_2 to $\widetilde{f}_4$ so that we get a quartic polynomial in x_1, x_2, x_3. Choose affine coordinates x_i/x_0 around p, i.e. set $x_0 = 1$. Then (x_1, x_3) are local coordinates on $V(f_2)$ centered at p, and we have an embedding $\mathscr{C}_p(C) \subset \mathbb{A}^2$ as the cone $V(f_4(x_1, 0, x_3))$. It follows that

(1) if $\mathscr{C}_p(C) = 4A$ then *in the generic case* we may make another change of coordinates so that

$$f_4 = x_3^4 + x_2 g_3, \qquad g_3 \in \mathbb{C}[x_1, x_2, x_3]_3,$$

(2) if $\mathscr{C}_p(C) = 3A + B$ (with $A \neq B$) then *in the generic case* we may make another change of coordinates so that

$$f_4 = x_3^3(cx_3 + x_1) + x_2 g_3, \qquad c \in \mathbb{C}, \quad g_3 \in \mathbb{C}[x_1, x_2, x_3]_3.$$

Since the locus of N_t-unstable points is closed, it will suffice to prove N_t-unstability if (1) or (2) above holds. Let λ_α be the virtual 1-PS in (6.3.6) with $\alpha \leq 1/5$. Then the exponents are ordered as follows:

$$1 \geq \alpha \geq -3\alpha \geq 2\alpha - 1. \tag{6.4.2}$$

Now suppose that (1) holds. Then

$$\mu(f_2, \lambda_\alpha) + t\mu(f_4, \lambda_\alpha) = 2\alpha + t \max\{-12\alpha, 5\alpha - 1\}. \tag{6.4.3}$$

Let $0 < \alpha \leq 1/17$. Then $\mu(f_2, \lambda_\alpha) + t\mu(f_4, \lambda_\alpha) = 2\alpha(1 - 6t)$, and hence it is strictly negative for $t > 1/6$. Since $\lambda_\alpha(s) f_2 = s^{2\alpha} f_2$, this proves that C is N_t-unstable if $t \in (1/6, 1/2] \cap \mathbb{Q}$, see **Remark 6.8**.

Next suppose that (2) holds. Then

$$\mu(f_2, \lambda_\alpha) + t\mu(f_4, \lambda_\alpha) = 2\alpha + t \max\{-8\alpha, 5\alpha - 1\}. \tag{6.4.4}$$

Let $0 < \alpha \leq 1/13$. Then $\mu(f_2, \lambda_\alpha) + t\mu(f_4, \lambda_\alpha) = 2\alpha(1 - 4t)$, and hence it is strictly negative for $t > 1/4$. Since $\lambda_\alpha(s) f_2 = s^{2\alpha} f_2$, this proves that C is N_t-unstable if $t \in (1/4, 1/2] \cap \mathbb{Q}$.

Now we suppose that the singularity of C at p appears in one of the remaining rows of Table 2 (the fifth column). By **Lemma 4.7**, we may choose homogeneous coordinates $[x_0, \ldots, x_4]$ so that $p = [1, 0, 0, 0]$ and $C = V(f_2, f_4)$ where

$$\begin{aligned} f_2 &= x_0 x_2 + x_1^2 + a x_3^2 \\ f_4 &= x_0 x_3^3 + x_1 g_3(x_2, x_3) + g_4(x_2, x_3) \end{aligned} \tag{6.4.5}$$

Let λ be the 1-PS appearing in Table 2 on the corresponding row. An elementary computation shows that if $t > t_k$, then $\mu(f_2, \lambda) + t\mu(f_2, \lambda) < 0$, and hence C is N_t-unstable because $\lambda(s) f_2 = s^{2r_1} f_2$, where $\lambda = \mathrm{diag}(r_0, r_1, r_2, r_3)$. $\square$

6.5. *Proof of the first main result*

In the present subsection we prove **Theorem 6.1**. The following key remark (which follows from Arnold's results in **Subsection 4.2**) will be useful.

Remark 6.12. A non slc singularity which is an arbitrary small deformation of a singularity appearing in (6.2.2) is again a singularity appearing in (6.2.2).

Proof of Theorem 6.1. Let us prove that Item (1) holds for $k = 0$. Let C be $N_{1/6}$-semistable. By the classification of potential critical values of the VGIT $\mathfrak{M}(t)$, i.e. **Proposition 6.6**, either C is $N_{1/6-\epsilon}$-semistable, or $C \in \mathscr{P}^{ss}(N_{1/6})^{\text{new}}$. In the former case C sits on a smooth quadric, and defines a semistable point of $\mathfrak{M}$ by **Theorem 5.6**, hence Item (1) holds by **Corollary 4.9**. In the latter case, the closure of the orbit of C contains the curve C_0^* in Table 2. Since C is a quadruple conic, Item (1) follows from **Remark 6.12**.

Let us prove that Item (2) holds for $k = 0$. Let C be N_t-semistable, where $1/6 < t < 1/4$. By **Proposition 6.6**, either C is $N_{1/6-\epsilon}$-semistable, or $C \in \mathscr{P}^{ss}(N_{1/6})^{\text{new}}$. Thus, every non slc singularity of C appears in (6.2.2). Moreover, by **Proposition 6.11** C is not a quadruple conic. This proves that Item (2) holds for $k = 0$.

Let $k_0 \in \{1, \ldots, 7\}$, and assume that Items (1) and (2) hold for all $0 \le k < k_0$. We prove that Items (1) and (2) hold for $k = k_0$.

(1): Let C be $N_{t_{k_0}}$-semistable. Then either C is $N_{t_{k_0}-\epsilon}$-semistable, or $C \in \mathscr{P}^{ss}(N_{t_{k_0}})^{\text{new}}$. In the former case, Item (1) holds for $k = k_0$ because Item (2) holds for $k = k_0 - 1$. In the latter case, the closure of the orbit of C contains the curve $C_{k_0}^*$ in Table 2 (if $k_0 = 3$ there is more than one choice for C_3^*). Since the unique non slc singularity of the curve in Table 2 has tag k_0, Item (1) holds by **Remark 6.12**.

(2): Let C be N_t-semistable, where $t_{k_0} < t < t_{k_0+1}$. By **Proposition 6.6**, either C is $N_{t_{k_0}-\epsilon}$-semistable, or $C \in \mathscr{P}^{ss}(N_{t_{k_0}})^{\text{new}}$. In the former case, Item (2) holds for $k = k_0$ because Item (2) holds for $k = k_0 - 1$. In the latter case, arguing as above, we get that the non slc singularities of C have tag at least k_0. On the other hand C does not have singularities with tag k_0 by **Proposition 6.11**. □

6.6. *Basin of attraction for the potential semistable orbits*

We recall the following general VGIT behavior: assume that x changes stability (say goes from t-semistable to t-unstable) at some critical slope t (or wall). Then there exist some x^* which gives a minimal orbit at t such that $\overline{G \cdot x} \supset G \cdot x^*$. As always, the orbit closures can be tested by 1-PS subgroups. This leads to the notion of basin of attraction, which plays an important role in the GIT analyses related to the Hassett-Keel program (e.g. [22]).

Let $x^* \in \mathscr{P}$ and $t \in (\delta, 1/2) \cap \mathbb{Q}$; we set

$$G_{N_t}(x^*) := \{g \in \mathrm{SL}(4) \mid g(x^*) = x^* \text{ and } g \text{ acts trivially on the fiber of } N_t \text{ at } x^*\}. \tag{6.6.1}$$

Definition 6.13. Let $x^* \in \mathscr{P}$, and let $t \in (\delta, 1/2) \cap \mathbb{Q}$. Suppose that λ is a 1-PS of SL(4) stabilizing x^* and acting trivially on the fiber of N_t at x^* (the last hypothesis is satisfied if x^* is N_t-semistable). The λ-*basin of attraction of* x^* is equal to

$$A_\lambda(x^*) = \{x \in \mathscr{P} \mid \lim_{s\to 0} \lambda(s) \cdot x = x^*\}.$$

The *basin of attraction of* x^* is equal to

$$A(x^*) = \{x \in \mathscr{P} \mid \lim_{s\to 0} \lambda(s) \cdot x = x^* \text{ for some 1-PS } \lambda \text{ of } G_{N_t}(x^*)\}.$$

Remark 6.14. Suppose that $x \in A_\lambda(x^*)$, and let $\widetilde{x}$ be a *non zero* element of the fiber of N_t at x. Then, since the action of λ on the fiber of N_t at x^* is trivial, $\lim_{s\to 0} \lambda(s)\widetilde{x}$ exists and is a non zero element of the fiber of N_t at x^*.

6.6.1. Transition at $t = 1/6$

Let $C^* = V(f_2, f_4)$, where

$$f_2 = x_0x_2 + x_1^2, \qquad f_4 = x_3^4. \tag{6.6.2}$$

Proposition 6.15. *Keep notation as above, and let $C \in \mathscr{P}^{ss}(N_t)$ be in the basin of attraction of C^*. The following hold:*

(1) *If $t \in (\delta, 1/6) \cap \mathbb{Q}$, then C has a point p of multiplicity 4, belonging to the smooth locus of the unique quadric containing C, and such that $\mathscr{C}_p(C) = 4A$, i.e. the singularity of C at p is as in the first row of Table 2.*
(2) *If $t \in (1/6, 1/4) \cap \mathbb{Q}$ then $C = Q \cap S$, where Q is an irreducible quadric, and S is a quartic surface not containing singular points of Q.*

Proof. For $\alpha \in [0, 1] \cap \mathbb{Q}$, let λ_α be the virtual 1-PS of SL(4) given by

$$\lambda_\alpha(s) = (s, s^\alpha, s^{2\alpha-1}, s^{-3\alpha}). \tag{6.6.3}$$

Every virtual 1-PS fixing C^* is equal to λ_α for some $\alpha \in [0, 1] \cap \mathbb{Q}$. Thus $C \in A_{\lambda_\alpha^{\pm 1}}(C^*)$ for some $\alpha \in [0, 1] \cap \mathbb{Q}$.

λ_α-*basin of attraction of* C^*: We determine which (2, 4)-curves $C = V(f_2 + f_2', f_4 + f_4')$ (where $f_d' \in \mathbb{C}[x_0, \dots, x_3]_d$) are in the λ_α-basin of attraction of C^* (N_t-semistable points of $\mathscr{P}$ are actual (2, 4) curves by **Proposition 5.11**). The fiber of $\mathscr{O}_{\mathscr{P}}(N_{1/6})^{\otimes -6}$ at C is identified with $(f_2 + f_2')^{\otimes 6} \otimes (f_4 + f_4')$, and we must determine for which (f_2', f_4') we have

$$\lim_{s\to 0} \lambda_\alpha(s) \left((f_2 + f_2')^{\otimes 6} \otimes (f_4 + f_4')\right) = f_2^{\otimes 6} \otimes f_4. \tag{6.6.4}$$

(See **Remark 6.14**.) Now

$$\lambda_\alpha(s)\left((f_2+f_2')^{\otimes 6}\otimes(f_4+f_4')\right)=(f_2+s^{2\alpha}\lambda_\alpha(s)f_2')^{\otimes 6}\otimes(f_4+s^{-12\alpha}\lambda_\alpha(s)f_4'),\quad (6.6.5)$$

and hence C is in the basin of attraction of C^* for λ_α if and only if

$$\lim_{s\to 0} s^{2\alpha}\lambda_\alpha(s)f_2'=0,\qquad \lim_{s\to 0} s^{-12\alpha}\lambda_\alpha(s)f_4'=0. \quad (6.6.6)$$

Now notice that the ordering of the weights of λ_α changes as we cross the value $\alpha=1/5$. In fact

$$1\geq\alpha\geq 2\alpha-1\geq -3\alpha \quad \text{if } 1/5\leq\alpha, \quad (6.6.7)$$

$$1\geq\alpha\geq -3\alpha\geq 2\alpha-1 \text{ if } 0\leq\alpha\leq 1/5. \quad (6.6.8)$$

It follows that

$$f_2'\in\begin{cases}\langle x_0x_3,x_1x_2,x_1x_3,x_2^2,x_2x_3,x_3^2\rangle & \text{if } 1/5<\alpha,\\ \langle x_1x_2,x_1x_3,x_2^2,x_2x_3,x_3^2\rangle & \text{if } 0\leq\alpha\leq 1/5.\end{cases} \quad (6.6.9)$$

Thus $p=[1,0,0,0]$ is a smooth point of $Q':=V(f_2+f_2')$, and local parameters on Q' around p are $(x_1|_{Q'},x_3|_{Q'})$. Moreover *in* $\mathscr{O}_{Q',p}$ the following holds:

$$x_2|_{Q'}\equiv\begin{cases} bx_3|_{Q'} \pmod{\mathfrak{m}_p^2},\ b\in\mathbb{C} & \text{if } 1/5<\alpha,\\ 0 \pmod{\mathfrak{m}_p^2} & \text{if } 0\leq\alpha\leq 1/5.\end{cases} \quad (6.6.10)$$

On the other hand, if $1/5\leq\alpha\leq 1$, then the second equation in (6.6.6) holds if and only if $f_4'=0$, while if $0\leq\alpha<1/5$ and it holds for f_4', then

$$f_4'=x_2P_3(x_1,x_2,x_3)+x_0x_2^2P_1(x_1,x_2,x_3), \quad (6.6.11)$$

where $P_d\in\mathbb{C}[x_1,x_2,x_3]_d$. It follows (for all α) that C has multiplicity 4 at $p=[1,0,0,0]$, and that the tangent cone at p is equal to $4V(x_3)$.

λ_α^{-1}*-basin of attraction of* C^**:* Let $C=V(f_2+f_2',f_4+f_4')$. Arguing as above, we see that C is in the basin of attraction of C^* for λ_α^{-1} if and only if

$$\lim_{s\to 0} s^{-2\alpha}\lambda_\alpha(s^{-1})f_2'=0,\qquad \lim_{s\to 0} s^{12\alpha}\lambda_\alpha(s^{-1})f_4'=0. \quad (6.6.12)$$

It follows that

$$f_2'\in\begin{cases}\langle x_0^2,x_0x_1\rangle & \text{if } 1/5\leq\alpha\leq 1,\\ \langle x_0^2,x_0x_1,x_0x_3\rangle & \text{if } 0\leq\alpha<1/5,\end{cases} \quad (6.6.13)$$

and hence $V(f_2+f_2')$ has rank 3. Moreover $p\notin V(f_4+f_4')$; in fact $C=V(f_2+f_2',f_4+f_4')$ is projectively equivalent to curves arbitrarily close to $C^*=V(f_2,f_4)$, and since C^* does not contain the vertex of $V(f_2)$, it follows that C does not contain the vertex of $V(f_2')$.

Let us prove Item (1). If $t \in (\delta, 1/6)$, then $V(f_2)$ is a smooth quadric, and hence C is in the λ_α-basin of attraction of C^*. Then Item (1) holds by **Proposition 6.10**. Now assume that $t = 1/6$. If C is in the λ_α-basin of attraction of C^*, the same argument applies. Thus we may assume that C is in the λ_α^{-1}-basin of attraction of C^*, and hence $V(f_2)$ is singular. By **Lemma 6.7** the rank of f_2 is equal to 3, and hence there exist coordinates $(x_0, \ldots, x_3)$ such that $f_2 = x_0x_2 + x_1^2$. Let $\lambda(s) = \mathrm{diag}(s^{-1}, s^{-1}, s^{-1}, s^3)$. Then $\lim_{s\to 0} \lambda(s) f_2^{\otimes 6} \otimes f_4 = f_2^{\otimes 6} \otimes f_4(0,0,0,x_3)$, and we are done.

Let us prove Item (2). Since curves in the λ_α-basin of attraction of C^* are N_t-semistable for $t \leq 1/6$, it follows from general results that C is in the λ_α^{-1}-basin of attraction of C^*. Thus Item (2) holds by the analysis carried out above. □

6.6.2. Transition at $t = 1/4$

Let $C^* = V(f_2, f_4)$, where

$$f_2 = x_0x_2 + x_1^2, \qquad f_4 = x_3^3 x_1. \tag{6.6.14}$$

Proposition 6.16. *Keep notation as above, and let $C \in \mathscr{P}^{ss}(N_t)$ be in the basin of attraction of C^*. Let Q be the unique quadric containing C. The following hold:*

(1) *If $t \in (1/6, 1/4) \cap \mathbb{Q}$, then there exists a point $p \in C$ of multiplicity 4, with tangent cone $\mathscr{C}_p(C) = 3A + B$ ($A \neq B$), and such Q is smooth point at p, i.e. the singularity of C at p is as in the second row of Table 2.*
(2) *If $t \in (1/4, 3/10) \cap \mathbb{Q}$ then Q is a quadric cone (of rank 3 by* **Lemma 6.7***), C contains the vertex v of Q, and has an A_1 singularity at v.*

Proof. The arguments are similar to those of **Proposition 6.15**, we omit the details. □

6.6.3. Transition at the remaining potential critical values

Proposition 6.17. *Let $k \in \{2, \ldots, 6, 7\}$, and let C_k^* be as in the row of Table 2 corresponding to k. Suppose that $C \in \mathscr{P}^{ss}(N_t)$ is in the basin of attraction of C_k^*. Then the following hold:*

(1) *If $t \in (t_{k-1}, t_k) \cap \mathbb{Q}$, there exists a point $p \in C$ such that the unique quadric containing C is smooth at p and the singularity of C at p is of the same type as the unique singularity of C_k^* away from the vertex of $V(f_2)$.*
(2) *If $t \in (t_k, t_{k+1}) \cap \mathbb{Q}$, the unique quadric containing C has rank 3, and its vertex v is contained in C. Moreover, if $k \neq 3$, then C has the same type at v as C_k^* has at the vertex of $V(f_2)$. If $k = 3$, then C has a singularity at v of type A_l, where $l \geq 3$ (possibly $l = \infty$).*

Proof. Let α_k be as follows:

$$\begin{array}{c|c|c|c|c|c|c} k & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \alpha_k & \frac{3}{7} & \frac{1}{3} & \frac{1}{3} & \frac{5}{17} & \frac{3}{11} & \frac{1}{4} \end{array} \tag{6.6.15}$$

Then λ_{α_k} is the component of the identity in the stabilizer of C_k^* (consult Table 2). Thus $C = V(f_2 + f_2', f_4 + f_4') \in \mathscr{P}^{ss}(N_t)$ is either in the λ_{α_k}-basin of attraction of C_k^* or in the $\lambda_{\alpha_k}^{-1}$-basin of attraction of C_k^*.

λ_{α_k}-basin of attraction of C_k^*: We will prove that

$$\text{if } C \text{ is in the } \lambda_{\alpha_k}\text{-basin of attraction of } C_k^*, \text{ then Item (1) holds.} \tag{6.6.16}$$

Noting that $t_k = 2\alpha_k/(9\alpha_k - 1)$, we get that C is in the λ_{α_k}-basin of C_k^* if and only if

$$\lim_{s\to 0} s^{2\alpha_k}\lambda_{\alpha_k}(s)f_2' = 0, \qquad \lim_{s\to 0} s^{-(9\alpha_k-1)}\lambda_{\alpha_k}(s)f_4' = 0. \tag{6.6.17}$$

Since $1/5 < \alpha_k$, the first equation of (6.6.17) gives that the first alternative in (6.6.9) holds. In particular $p := [1, 0, 0, 0]$ is a smooth point of the quadric $V(f_2 + f_2')$, local coordinates on Q' centered at p are

$$x := x_3|Q', \quad y := x_1|Q'.$$

Moreover $x_2|_{Q'} = \varphi$, where φ is an analytic function such that $\varphi \equiv bx \pmod{\mathfrak{m}_p^2}$ for some $b \in \mathbb{C}$ (see (6.6.10)). Thus a local equation of $C \subset Q'$ centered at p is given by

$$f_4(1, y, \varphi(x, y), x) + f_4'(1, y, \varphi(x, y), x) = 0.$$

Let us prove that the singularity of C at p is as in the row of Table 2 corresponding to k.

We assign weights to x and y as follows:

$$\mathrm{wt}_k(x) := \frac{1}{3}, \qquad \mathrm{wt}_k(y) := \frac{1 - \alpha_k}{3(3\alpha_k + 1)}. \tag{6.6.18}$$

Notice the following:

(1) The weights of x and y in (6.6.18) are equal to the weights of x and y in Table 1 corresponding to the singularity type listed in Table 2 (on the row with index k) - this is a straightforward computation. (Warning: the index k in Table 1 has no relation to the index k in Table 2.)

(2) $\varphi(x, y) = y^2 + \psi(x, y)$ (see (6.6.10)) where all monomials appearing in $\psi(x, y)$ have weight greater than $\mathrm{wt}_k(y^2)$ - this because $2\,\mathrm{wt}_k(y) < \mathrm{wt}_k(x)$.

One easily checks that $f_4(1,y,\varphi(x,y),x) = h(x,y)+g(x,y)$ where $h(x,y)$ is homogeneous of weight 1 and all monomials appearing in $g(x,y)$ have weight strictly greater than 1, and moreover $h(x,y)$ is equal to the leading term appearing in Table 1 on the row with index k (with the exception of the case $k=3$ in Table 2, where it is equal to the leading (and "unique") term of the $J_{3,\infty}$ singularity).

Thus, by **Theorem 4.2**, in order to prove (6.6.16), it suffices to check that all monomials appearing in $f_4'(1,y,\varphi(x,y),x)$ have weight strictly greater than 1. Every such monomial is obtained from a monomial $x_0^{i_0}x_1^{i_1}x_2^{i_2}x_3^{i_3}$ appearing in f_4' by setting $x_0=1$, $x_1=y$, $x_2=\varphi(x,y)$ and $x_3=x$. By item (2) above it suffices to check that the weight of $x^{i_3}y^{i_1+2i_2}$ is strictly greater than 1, i.e. that

$$(i_1+2i_2)(1-\alpha_k)+i_3(3\alpha_k+1) > 3(3\alpha_k+1).$$

The above inequality follows from the second equation in (6.6.17).

$\lambda_{\alpha_k}^{-1}$-basin of attraction of C_k^*: We will prove that

$$\text{if } C \text{ is in the } \lambda_{\alpha_k}^{-1}\text{-basin of attraction of } C_k^*, \text{ then Item (2) holds.} \tag{6.6.19}$$

First C is in the $\lambda_{\alpha_k}^{-1}$-basin of C_k^* if and only if

$$\lim_{s\to 0} s^{-2\alpha_k}\lambda_{\alpha_k}(s^{-1})f_2' = 0, \qquad \lim_{s\to 0} s^{9\alpha_k-1}\lambda_{\alpha_k}(s^{-1})f_4' = 0. \tag{6.6.20}$$

Since $1/5 < \alpha_k$ it follows from (6.6.13) that there exist $c, d \in \mathbb{C}$ such that

$$f_2 + f_2' = x_0(cx_0+dx_1+x_2)+x_1^2. \tag{6.6.21}$$

Thus $V(f_2+f_2')$ has rank 3, and its singular point is $v=[0,0,0,1]$. In order to examine the consequences of the second equation in (6.6.20), we introduce some notation. Given a 1-PS $\lambda(s)=\operatorname{diag}(s^{r_0},\ldots,s^{r_3})$ we define $\operatorname{wt}_\lambda(g)$ for $0\neq g\in\mathbb{C}[x_0,\ldots,x_3]$ as

$$\operatorname{wt}_\lambda(g) := \min\{\operatorname{wt}_\lambda(\text{monomial in } g)\}, \tag{6.6.22}$$

where $\operatorname{wt}_\lambda$ of a monomial is given by (5.3.2). Then the second equation in (6.6.20) is equivalent to

$$\operatorname{wt}_{\lambda_{\alpha_k}}(f_4') > 1-9\alpha_k. \tag{6.6.23}$$

Inequality (6.6.23) gives that v is a singular point of $V(f_4')$. Since v is a smooth point of $V(f_4)$, with tangent plane $V(x_0)$, it follows that v is a smooth point of $V(f_4+f_4')$, with tangent plane $V(x_0)$. Now set $x_3=1$, and hence (x_0,x_1,x_2) are local coordinates in a neighborhood of v in $\mathbb{P}^3$; by the Implicit Function Theorem, the restrictions of x_1,x_2 to in $V(f_4+f_4')$ are local coordinates in a neighborhood of v in $V(f_4+f_4')$. Abusing notation, we use the same symbol for x_1,x_2 and their restrictions. There exists an analytic function φ of two variables defined in a neighborhood of $(0,0)$ such that $x_0 = \varphi(x_1,x_2)$ *on* $V(f_4+f_4')$. By (6.6.21), a local equation of the plane singularity (C,v) is given by

$$x_1^2 + \varphi(x_1,x_2)(c\varphi(x_1,x_2) + dx_1 + x_2) = 0. \tag{6.6.24}$$

Define

$$\mathrm{wt}_m(x_1) := 1/2, \quad \mathrm{wt}_m(x_2) := 1/(m+1), \tag{6.6.25}$$

and extend it to a weight function on non zero elements of $\mathbb{C}[[x_1,x_2]]$ by defining $\mathrm{wt}_m(g)$ as the minimum of weights of monomials appearing in g. In order to prove (6.6.19) it suffices to check that

$$\mathrm{wt}_m(\varphi) \geq \frac{m}{m+1}, \tag{6.6.26}$$

and the extra condition involving a line $L \subset V(f_2+f_2')$ if $k \in \{5,6,7\}$.

Since $f_4(\varphi(x_1,x_2),x_1,x_2,1) + f_4'(\varphi(x_1,x_2),x_1,x_2,1) = 0$, we get that

$$\mathrm{wt}_m(\varphi) = \mathrm{wt}_m(f_4(0,x_1,x_2,1) + f_4'(0,x_1,x_2,1)). \tag{6.6.27}$$

A straightforward computation shows that $\mathrm{wt}_m(f_4(0,x_1,x_2,1)) = m/(m+1)$. Thus it suffices to check that

$$\mathrm{wt}_m(f_4'(0,x_1,x_2,1)) \geq \frac{m}{m+1}. \tag{6.6.28}$$

This follows from the values in Table (6.6.29) - notice that it suffices to consider monomials $x_3^{i_3}x_2^{i_2}x_1^{i_1}$ with $i_1 \in \{0,1\}$, because if $i_1 \geq 2$ the associated weight is at least 1. (It helps to notice that α_k's are decreasing.)

monomial x^I	$x_3^2x_2^2$	$x_3^2x_2x_1$	$x_3x_2^3$	$x_3x_2^2x_1$	x_2^4	$x_2^3x_1$
$\mathrm{wt}_{\lambda_{\alpha_k}}(x^I)$	$-2\alpha_k-2$	$-3\alpha_k-1$	$3\alpha_k-3$	$2\alpha_k-2$	$8\alpha_k-4$	$7\alpha_k-3$
$\mathrm{wt}_{\lambda_{\alpha_k}}(x^I) > 1-9\alpha_k$ iff	no k	$k=2$	$k=2$	$k\in\{2,\dots,5\}$	$k\in\{2,3,4\}$	$k\in\{2,\dots,6\}$

(6.6.29)

Lastly we prove that, if $k \in \{5,6,7\}$, the extra condition involving a line $L \subset V(f_2+f_2')$ holds. First notice that $L := V(x_0,x_1)$ is a line contained in $V(f_2+f_2')$. If $k=5$ the condition $\mathscr{C}_vC = 2T_v(L)$ holds by the local equation of C, see (6.6.24). If $k \in \{6,7\}$ the line L belongs to $V(f_4)$ by the computations above.

Let us finish the proof of the proposition. Suppose that $t \in (t_k,t_{k+1})$. By the analysis above, either the thesis of Item (1) or the thesis of Item (2) holds. The former is excluded by **Proposition 6.11**.

Now suppose that $t \in (t_{k-1}, t_k)$. By the analysis above, either the thesis of Item (1) or the thesis of Item (2) holds. Assume that the latter holds. By the analysis of the $\lambda^{\pm}_{\alpha_k}$-basin of attraction of C^*_k, it follows that there exist N_t semistable curves C, with $t_k < t$, which are in the basin of attraction of C^*_k for which the thesis of Item (1) holds. This contradicts **Proposition 6.11**. Hence the thesis of Item (1) holds. □

6.7. Proof of the remaining main results of the section

Proof of Theorem 6.2. Let us prove that the critical values for the VGIT $\mathfrak{M}(t)$ in the interval $(\delta, 1/2) \cap \mathbb{Q}$ are given by the t_k's appearing in (6.2.1), with the exclusion of $t_8 = 1/2$. By **Proposition 6.6**, every critical value is equal to one of the t_k's. Hence it remains to prove that t_k is a critical value, for each $k \in \{0, \ldots, 7\}$. There exists an $N_{1/6-\epsilon}$-semistable $(4, 4)$ curve C_k on a smooth quadric Q, such that the double cover $X_{C_k} \to Q$ ramified over C_k has a non slc singularity with tag k and no non slc singularity with tag strictly less than k. This is immediate for $k \in \{0, 1\}$, and it follows from **Lemma 4.5** for $k \in \{2, \ldots, 7\}$. (The argument would work equally well if we knew that the curve C^*_k in Table 2 is N_{t_k}-semistable, but at this stage we have not yet proved this.) By **Proposition 6.11** C_k is N_t-unstable for $t_k < t$. Thus it will suffice to show that C_k is N_t-semistable for $t < t_k$. Suppose the contrary. Since C_k is $N_{1/6-\epsilon}$-semistable, there exists $0 \leq k_0 < k$ such that C_k is in the basin of attraction (form the left) of a curve projectively equivalent to $C^*_{k_0}$. By the results of **Subsection 6.6** this implies that C_k has a point with tag k_0, contradicting our choice of C_k. This proves that C_k is N_t-semistable for $t < t_k$, and hence proves that t_k is critical value.

It remains to prove that the exceptional loci of $\mathfrak{M}(t_{k-1}, t_k) \to \mathfrak{M}(t_k)$ and $\mathfrak{M}(t_k, t_{k+1}) \to \mathfrak{M}(t_k)$ are as claimed. We have already established the fact that the exceptional loci of $\mathfrak{M}(t_{k-1}, t_k) \to \mathfrak{M}(t_k)$ are naturally birational to W_k. Specifically, the generic point ζ_k in $W_k \subset \mathfrak{M}$ is N_t stable for $t < \frac{1}{6}$, it becomes unstable for $t > t_k$ (cf. **Proposition 6.11**), but via the basin of attraction argument, ζ_k can not become unstable before t_k. For the exceptional loci of $\mathfrak{M}(t_k, t_{k+1}) \to \mathfrak{M}(t_k)$, we note first that the cases $k = 0, 1$ are discussed in detail in **Proposition 6.15** and **Proposition 6.16**. For the cases $k = 2, 4$, the minimal orbit C^*_k at t_k has a singularity of type A_2 and respectively A_3 at the vertex v of the quadric cone containing C^*_k. For the cases $k = 5, 6, 7$, there is a singularity at v of type A_4, A_5, and A_7 respectively, and additionally a line L in special position with respect to the curve C^*_k and the singularity at v (see Table 2). As previously discussed, the exceptional locus $\Sigma_+(t_k)$ of $\mathfrak{M}(t_k, t_{k+1}) \to \mathfrak{M}(t_k)$ is obtained via the basin of attraction of C^*_k. By the arguments given in the previous subsection, it follows that the generic point ξ_k of $\Sigma_+(t_k)$ will correspond to a curve having the same type of singularity at v (and position of L) as C^*_k. Furthermore, this curve will have at worst some additional nodes (imposed by the special position of the line L). The resulting conditions are exactly the conditions that have been used to define the loci Z^{k+1} in $\mathscr{F}^*$ (see **Proposition 2.2**). In other words, there are natural rational maps $\Sigma_+(t_k) \dashrightarrow Z^{k+1}$ (and clearly one-to-one onto the image). To conclude that the two spaces are birational, we note that Z^{k+1} are irreducible and that $\Sigma_+(t_k)$ and Z^{k+1} have the same dimension. As discussed, the index of Z corresponds to the codimension. On the other hand, the dimension of $\Sigma_+(t_k)$ can be computed as being complementary to that of $\Sigma_-(t_k)$, or equivalently W_k. (See **Remark 6.4** for a discussion of the dimensions in the special case $t = \frac{1}{3}$.) □

Proof of Proposition 6.5.

Claim 6.18. *Let $C = V(f_2, f_4)$ be a $(2,4)$ c.i. curve such that the associated double cover is a $K3$ surface with canonical singularities. Then the following hold:*

(1) *C is N_t-stable for $t \in (2/5, 1/2] \cap \mathbb{Q}$ (and hence asymptotic GIT stable).*
(2) *If in addition $V(f_2)$ is smooth along C, then C is N_t-semistable for $t \in (1/6, 2/5] \cap \mathbb{Q}$.*

Proof. (1): Let $t \in (2/5, 1/2) \cap \mathbb{Q}$. By **Theorem 6.1**, every point of $\mathfrak{M}(2/5, 1/2)$ parametrizes a (polystable) surface with slc singularities, and the regular period map $\Phi\colon \mathfrak{M}(2/5, 1/2) \to \mathscr{F}^*$ is dominant, hence surjective. Thus the fiber of Φ over a point of $\mathscr{F}$ is a surface with slc singularities, and moreover, if the double cover $X \to V(f_2)$ ramified over $C = V(f_2, f_4)$ has slc singularities, then C must be N_t-semistable. On the other hand, if the double cover $X \to V(f_2)$ ramified over $C = V(f_2, f_4)$ has slc singularities, then the automorphism group of C is finite; it follows that such a curve is necessarily N_t-stable.

The same argument applies for $t = 1/2$. In fact, since the map $\mathfrak{M}(2/5, 1/2) \to \mathfrak{M}(1/2)$ induced by the Hilbert-Chow morphism is surjective, every point of $\mathscr{P}^{ss}(N_{1/2})$ is represented by a curve $V(f_2, f_4)$ which is N_t-semistable for $t \in (2/5, 1/2) \cap \mathbb{Q}$.

(2): By the results of **Subsection 6.6**, more precisely by Item (2) of **Proposition 6.15**, Item (2) of **Proposition 6.16**, and Item (2) of **Proposition 6.17**, C is not in the basin of attraction (from the right) of any critical value t_k with $k \in \{1, \dots, 7\}$. It follows that C remains semistable for all $t \in (1/6, 2/5] \cap \mathbb{Q}$. □

Motivated by **Claim 6.18**, we give a definition that will be useful here and also later on.

Definition 6.19. Let $U_0 \subset U$ be the (open) subset parametrizing curves $C = V(f_2, f_4)$ such that the associated double cover is a $K3$ surface with canonical singularities, and $V(f_2)$ is smooth along C.

Let

$$\mathfrak{M}(t)_0 := U_0 /\!\!/_{N_t} \mathrm{SL}(4). \tag{6.7.1}$$

Thus $\mathfrak{M}(t)_0$ is an open subset of $\mathfrak{M}(t)$. A dimension count shows that

$$\operatorname{cod}(\mathfrak{M}(t) \setminus \mathfrak{M}(t)_0, \mathfrak{M}(t)) \geq 2. \tag{6.7.2}$$

Since $\mathfrak{M}(t)_0$ is contained in the regular locus of the period map $\mathfrak{p}(t)$, all that remains to prove is that the complement of $\mathfrak{p}(t)(\mathfrak{M}(t)_0)$ in $\mathscr{F}^*$ has codimension at least 2. By **Proposition 2.2**, the complement of $\mathfrak{p}(t)(\mathfrak{M}(t)_0)$ in $\mathscr{F}$ is equal to Z^2, which has codimension 2. Since the boundary $\mathscr{F}^* \setminus \mathscr{F}$ has codimension 17, we are done. □

7. Proof of the main result

7.1. Summary

In the present section, we prove **Theorem 1.1**. The proofs of Items (i) and (ii) involve our GIT models $\mathfrak{M}(t)$ for $t \in (1/6, 1/2] \cap \mathbb{Q}$. Let

$$\mathfrak{p}(t) \colon \mathfrak{M}(t) \dashrightarrow \mathscr{F} \tag{7.1.1}$$

be the period map: if $C = V(f_2, f_4)$ represents a generic stable point $x \in \mathfrak{M}(t)$, then $\mathfrak{p}(t)(x)$ is the period point of the double cover of $V(f_2)$ branched over C (a $U(2)$-hyperelliptic $K3$ surface). The key ingredients in our proofs of Items (i)-(ii) are **Proposition 6.5** and results about the Picard groups of $\mathscr{F}$ and $\mathfrak{M}(t)$ that we discuss in the following subsection.

7.2. Divisor classes on the locally symmetric and GIT models

The locally symmetric variety $\mathscr{F} = \mathscr{D}/\Gamma$ is a $\mathbb{Q}$-factorial quasi-projective variety. Let H_n and H_h be the nodal and hyperelliptic divisors of $\mathscr{F}$, respectively (see Definition 1.3.4 in [36]). Informally, H_n is the closure of the locus of periods of $U(2)$-hyperelliptic $K3$'s which are double covers $X \to V(f_2)$ ramified over a $(2,4)$ curve $V(f_2, f_4)$ which is smooth except for a node at a smooth point of $V(f_2)$, while H_h is the locus of periods of $U(2)$-hyperelliptic $K3$'s which are double covers $X \to V(f_2)$ of a quadric cone ramified over a $(2,4)$ curve with ADE singularities. We recall that λ is the Hodge (or automorphic) $\mathbb{Q}$ divisor class.

If Z is an algebraic variety, we let $\operatorname{Pic}(Z)_{\mathbb{Q}} := \operatorname{Pic}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$. A $\mathbb{Q}$-Cartier divisor D on Z determines an element $[D]$ of $\operatorname{Pic}(Z)_{\mathbb{Q}}$.

Proposition 7.1 *(cf. [36, Sect. 3]). Let $\mathscr{F}$ be the period space of $U(2)$-hyperelliptic $K3$'s. Then*

(1) $\operatorname{Pic}(\mathscr{F})_{\mathbb{Q}} = \mathbb{Q}[H_n] \oplus \mathbb{Q}[H_h]$, *and*
(2) $136\lambda \equiv H_n + 16H_h$.

Next, we come to the Picard group of $\mathfrak{M}(t)$ for $t \in (1/6, 1/2] \cap \mathbb{Q}$. Let $\mathscr{P}^{ss}(N_t) \subset \mathscr{P}$ be the locus of N_t-semistable points. Let U be the parameter space for $(2,4)$ complete intersection curves in $\mathbb{P}^3$. Then $\mathscr{P}^{ss}(N_t) \subset U$ by **Proposition 5.11**. Since $U \subset \mathbb{P}E$, it makes sense to restrict η and ξ (see (5.1.2)) to $\mathscr{P}^{ss}(N_t)$. The restriction of the SL(4)-linearized ample divisor class N_t to U is isomorphic to $(\eta + t\xi)|_U$. Hence the following definition makes sense because of **Corollary 5.5**.

Definition 7.2. For $t \in [1/6, 1/2] \cap \mathbb{Q}$, let $D(t)$ be the $\mathbb{Q}$-Cartier divisor class on $\mathfrak{M}(t)$ obtained by descent from the divisor class $(\eta + t\xi)|_{\mathscr{P}^{ss}(N_t)}$.

Remark 7.3. Let $t \in [1/6, 1/2] \cap \mathbb{Q}$. Then $D(t)$ is an ample $\mathbb{Q}$-Cartier divisor class, because N_t is ample on $\mathscr{P}$ for $t \in [1/6, 1/2) \cap \mathbb{Q}$, and by (5.2.5) if $t = 1/2$.

Proposition 7.4. *Let $t \in (1/6, 1/2) \cap \mathbb{Q}$, and assume that t is* not *one of the critical slopes for the VGIT $\mathfrak{M}(t)$ (see* **Theorem 6.2***). Then both $\eta|_{\mathscr{P}^{ss}(N_t)}$ and $\xi|_{\mathscr{P}^{ss}(N_t)}$ descend to $\mathbb{Q}$-Cartier divisor classes $\overline{\eta}(t)$ and $\overline{\xi}(t)$ on $\mathfrak{M}(t)$, and* $\mathrm{Pic}(\mathfrak{M}(t))_{\mathbb{Q}} = \mathbb{Q}\overline{\eta}(t) \oplus \mathbb{Q}\overline{\xi}(t)$.

Proof. One checks easily that $\mathrm{Pic}^G(\mathscr{P}^{ss}(N_t))_{\mathbb{Q}} = \mathbb{Q}[\eta|_{\mathscr{P}^{ss}(N_t)}] \oplus \mathbb{Q}[\xi|_{\mathscr{P}^{ss}(N_t)}]$. In order to prove that both $\eta|_{\mathscr{P}^{ss}(N_t)}$ and $\xi|_{\mathscr{P}^{ss}(N_t)}$ descend to $\mathbb{Q}$-Cartier divisor classes on $\mathfrak{M}(t)$ we apply Theorem 2.3 in [16]. One has to check that if $C = V(f_2, f_4)$ is N_t-polystable, then the stabilizer $\mathrm{Stab}(C)$ acts trivially on the fiber of η or ξ at C. Since G is a linearly reductive group, it suffices to check that any 1-PS λ contained in $\mathrm{Stab}(C)$ acts trivially on the fiber of η or ξ at C. Now, t is not one of the critical slopes for the VGIT $\mathfrak{M}(t)$, hence $\mu(f_2, \lambda) = 0$ and $\mu(f_4, \lambda) = 0$. The fiber of η at C is identified with $(\mathbb{C}f_2)^\vee$, and the fiber of ξ at C is identified with $(\mathbb{C}f_4)^\vee \pmod{f_2}$; it follows that λ acts trivially both on the fiber of η and of ξ at C. □

Our next task is to compare $\mathbb{Q}$-Cartier divisors on $\mathfrak{M}(t)$ with the pull-back via $\mathfrak{p}(t)$ of $\mathbb{Q}$-Cartier divisors on $\mathscr{F}$. Let $U_0 \subset U$ be as in **Definition 6.19**, and let $\widetilde{\mathfrak{p}}_0 \colon U_0 \to \mathscr{F}$ be the (regular) period map.

Lemma 7.5. *Let $t \in (1/6, 1/2] \cap \mathbb{Q}$. In $\mathrm{Pic}(U_0)_{\mathbb{Q}}$, we have*

$$\widetilde{\mathfrak{p}}_0^* H_h = 4\eta_{|U_0}, \qquad \widetilde{\mathfrak{p}}_0^* H_n = (72\eta + 68\xi)_{|U_0}. \tag{7.2.1}$$

Proof. We have

$$\widetilde{\mathfrak{p}}_0^* H_h = \{([f_2], [\bar{f}_4]) \in U_0 \mid V(f_2) \text{ is singular}\},$$

and

$$\widetilde{\mathfrak{p}}_0^* H_n = \text{closure of } \{([f_2], [\bar{f}_4]) \in U_0 \mid C = V(f_2, f_4) \text{is singular at a smooth point of } V(f_2)\}.$$

Since the locus of singular quadrics is a degree 4 hypersurface in $|\mathscr{O}_{\mathbb{P}^3(2)}|$, the first equality in (7.2.1) is clear. The second equality in (7.2.1) is proved by a computation analogous to the one done in [13, Prop. 1.1] for $(2,3)$ complete intersections in $\mathbb{P}^3$. We omit the details. □

For $t \neq 0$, let $\beta(t) = \frac{1-2t}{4t}$.

Proposition 7.6. *Let $t \in (1/6, 1/2] \cap \mathbb{Q}$. Then $\mathfrak{p}(t)^*(\lambda + \beta(t)\Delta)$ is a $\mathbb{Q}$-Cartier divisor on $\mathfrak{M}(t)$, and we have the relation*

$$D(t) = 2t\mathfrak{p}(t)^*(\lambda + \beta(t)\Delta). \tag{7.2.2}$$

Proof. We recall that, since SL(4) has no non trivial characters, $\mathrm{Pic}(\mathfrak{M}(t))$ injects into the group of SL(4)-linearized line bundles on $\mathscr{P}^{ss}(N_t)$. Since $\mathscr{P}^{ss}(N_t) \setminus U_0$ has codimension at least 2 in $\mathscr{P}^{ss}(N_t)$, and U_0 is smooth, $\widetilde{\mathfrak{p}}^*(\lambda + \beta(t)\Delta)$ extends uniquely to a $\mathbb{Q}$-Cartier divisor on $\mathscr{P}^{ss}(N_t)$, and moreover it suffices to prove that

$$(\eta + t\xi)_{|U_0} = 2t\widetilde{\mathfrak{p}}^*(\lambda + \beta(t)\Delta)_{|U_0}. \tag{7.2.3}$$

Since $\Delta = \frac{1}{2}H_h$, we have $\widetilde{\mathfrak{p}}^*\Delta_{|U_0} = 2\eta_{|U_0}$ by **Lemma 7.5**. On the other hand, by **Proposition 7.1** and by **Lemma 7.5**,

$$\widetilde{\mathfrak{p}}^*\lambda_{|U_0} = \frac{1}{136}\widetilde{\mathfrak{p}}^*(H_n + 16H_h)_{|U_0} = \frac{1}{136}(72\eta + 68\xi + 64\eta)_{|U_0} = (\eta + \frac{1}{2}\xi)_{|U_0}.$$

Thus,

$$\widetilde{\mathfrak{p}}^*(\lambda + \beta(t)\Delta)_{|U_0} = (\eta + \frac{1}{2}\xi + \left(\frac{1}{2t} - 1\right)\eta)_{|U_0} = \frac{1}{2t}(\eta + t\xi)_{|U_0}.$$

This proves (7.2.3). □

7.3. Proof of Items (i)-(iii) of Theorem 1.1

First we notice that β defines an invertible function $[1/6, 1/2] \cap \mathbb{Q} \to [0,1] \cap \mathbb{Q}$. In fact the inverse is given by

$$t(\beta) := \frac{1}{4\beta + 2}. \tag{7.3.1}$$

Let us prove Item (i). Since $D(t)$ is ample (see **Remark 7.3**), it follows that it will suffice to prove that the period map induces an isomorphism of rings

$$\mathfrak{p}(t)^* \colon R(\mathscr{F}, \lambda + \beta(t)\Delta) \xrightarrow{\sim} R(\mathfrak{M}(t), D(t)). \tag{7.3.2}$$

If $\beta = 1$, then $t = 1/6$, and in that case (7.3.2) has been proved in [36], Prop. 4.0.20. More precisely, in that proposition $D(1/6)$ is replaced by an ample $L(18)$, but $\mathrm{Pic}(\mathfrak{M})_{\mathbb{Q}}$

has rank one because $\mathfrak{M} = |\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)|/\!\!/\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1)$, and hence $D(1/6)$ is a positive multiple of $L(18)$, and (7.3.2) for $t = 1/6$ follows.

Thus we may assume that $t \in (1/6, 1/2] \cap \mathbb{Q}$. Let $\mathfrak{M}(t)_0 \subset \mathfrak{M}(t)$ be as in (6.7.1). The period map is regular on $\mathfrak{M}(t)_0$, and it defines an isomorphism between $\mathfrak{M}(t)_0$ and its image $\mathscr{F}_0 \subset \mathscr{F}$. Moreover

$$\mathrm{cod}(\mathfrak{M}(t) \setminus \mathfrak{M}(t)_0, \mathfrak{M}(t)) \geq 2, \qquad \mathrm{cod}(\mathscr{F} \setminus \mathscr{F}_0, \mathscr{F}) \geq 2. \tag{7.3.3}$$

In fact, the first inequality is (6.7.2), the second one has been proved in the proof of **Proposition 6.5**. Since $\mathscr{F}$ and $\mathfrak{M}(t)$ are normal varieties, Equation (7.3.2) now follows from **Proposition 7.6** and (7.3.3).

We have proved Item (i).

In order to prove Items (ii) and (iii) we take the Proj of both sides of (7.3.2), and we get an isomorphism

$$\mathfrak{p}(t(\beta))^{-1} \colon \mathscr{F}(\beta) \xrightarrow{\sim} \mathfrak{M}(t(\beta)). \tag{7.3.4}$$

Given the above isomorphism, Items (ii) and (iii) follow from **Theorem 6.2**, except that we do not know yet whether $\beta = 1$ is a critical value. For this we must show that for $\epsilon \in \mathbb{Q}_+$ small the period map $\mathfrak{p}(t(\epsilon)) \colon \mathscr{F}(t(\epsilon)) \dashrightarrow \mathscr{F}^*$ is not an isomorphism. Suppose that it is an isomorphism. Then $\lambda + t(\epsilon)\Delta$ is a $\mathbb{Q}$-Cartier divisor, and since λ is $\mathbb{Q}$-Cartier, so is Δ. Thus H_h is $\mathbb{Q}$-Cartier, and this is a contradiction, one knows that H_h is not $\mathbb{Q}$-Cartier (e.g. it follows from [39, Cor. 3.5]).

7.4. Proof of Item (iv) of Theorem 1.1

By the discussion above, we have $\mathscr{F}(\epsilon) \cong \mathfrak{M}\left(\frac{1}{2} - \epsilon'\right) \cong \mathrm{Hilb}^{\gg 0}_{(2,4)}/\!\!/\mathrm{SL}(4)$ (see **Theorem 5.6**) (with $\frac{1}{2} - \epsilon' = t(\epsilon)$, and $0 < \epsilon, \epsilon' \ll 1$). Similarly, $\mathscr{F}^* \cong \mathscr{F}(0) \cong \mathfrak{M}\left(\frac{1}{2}\right) \cong \mathrm{Chow}_{(2,4)}/\!\!/\mathrm{SL}(4)$. Furthermore, with these identifications, $\mathscr{F}(\epsilon) \to \mathscr{F}^*$ is compatible with the natural Hilbert-Chow map (see **Remark 5.3**).

By **Proposition 7.4**, it follows that $\mathscr{F}(\epsilon)$ is $\mathbb{Q}$-factorial with Picard number 2. As already noted, $\mathscr{F}^*$ is not $\mathbb{Q}$-factorial with (the closure of) H_h being a Weil divisor, which is not $\mathbb{Q}$-Cartier. By the GIT description, it is clear that $\mathscr{F}(\epsilon) \to \mathscr{F}^*$ is a small map (e.g. **Proposition 6.5**). It follows then that $\mathscr{F}(\epsilon)$ is isomorphic to the $\mathbb{Q}$-factorialization of Looijenga associated to the divisor H_h (see [28, Lemma 6.2]).

8. The structure of the Chow and (asymptotic) Hilbert GIT quotients

8.1. Summary

As previously discussed, the period map induces an isomorphism $\mathfrak{M}(t(\beta)) \cong \mathscr{F}(\beta)$ for $\beta \in [0,1] \cap \mathbb{Q}$. The purpose of this section is to discuss the geometric meaning of this isomorphism for β close to 0 (or equivalently t close to $\frac{1}{2}$). Specifically, we are interested in the following diagram:

$$\begin{array}{ccc} \mathfrak{M}(\frac{1}{2}-\epsilon) \cong \mathrm{Hilb}^{m\gg 0}_{(2,4)} /\!\!/ \mathrm{SL}(4) & \xrightarrow{\cong} & \widehat{\mathscr{F}} \cong \mathscr{F}(\epsilon) \\ \Big\downarrow \Psi & & \Big\downarrow \Pi \\ \mathfrak{M}(\frac{1}{2}) \cong \mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4) & \xrightarrow{\cong} & \mathscr{F}^* \cong \mathscr{F}(0) \end{array} \tag{8.1.1}$$

where

i) $\mathscr{F}^*$ is the Baily-Borel compactification of $\mathscr{F}$, $\widehat{\mathscr{F}}$ is Looijenga's $\mathbb{Q}$-factorialization of $\mathscr{F}^*$, and $\Pi : \widehat{\mathscr{F}} \to \mathscr{F}^*$ is the structure morphism constructed by Looijenga [39],
ii) $\mathrm{Hilb}^{m\gg 0}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ is the GIT quotient of the Hilbert scheme for $(2,4)$ complete intersections (see **Subsubsection 5.3.1** and **Theorem 5.6**), and similarly $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ is the Chow quotient. The map Ψ is induced by the Hilbert-Chow morphism,
iii) the horizontal isomorphisms are those of (7.3.4) (i.e. induced by the period map).

8.2. Structure of Looijenga's $\mathbb{Q}$-factorization $\widehat{\mathscr{F}}$

We have already discussed the structure of the Baily-Borel compactification $\mathscr{F}^*$ in **Subsection 2.4**. We recall that while $\mathscr{F}$ is $\mathbb{Q}$-factorial, its compactification $\mathscr{F}^*$ is not. For this reason, Looijenga [39] has introduced the semitoric compactifications (that offer common generalization of both Baily-Borel and toroidal compactifications) that for appropriate choices give the $\mathbb{Q}$-Cartierizations of the closures of Heegner divisors in $\mathscr{F}$. As already used elsewhere in the paper, we denote by $\widehat{\mathscr{F}}$ the $\mathbb{Q}$-Cartierization associated to the divisor Δ; we have $\widehat{\mathscr{F}} \cong \mathscr{F}(\epsilon)$.

By definition, $\widehat{\mathscr{F}} \to \mathscr{F}^*$ is a small map, which is an isomorphism over $\mathscr{F}$ (recall $\mathscr{F}$ is $\mathbb{Q}$-factorial). The structure of the Baily-Borel compactification was discussed in **Theorem 2.3**. Then, following Looijenga [39], the fibers of $\widehat{\mathscr{F}} \to \mathscr{F}^*$ reflect the arithmetic structure of the hyperplane arrangement associated to the divisor Δ (or equivalently $H_h \subset \mathscr{F}$) at the boundary of the period domain. Explicitly, in our situation the following holds.

Proposition 8.1. *Let $\widehat{\mathscr{F}} \to \mathscr{F}^*$ be the Looijenga $\mathbb{Q}$-factorialization associated to the divisor H_h. Then*

i) *The dimensions of the pre-images in $\widehat{\mathscr{F}}$ of the eight Type II components in $\mathscr{F}^*$ are as given in Table 3.*
ii) *With the exception of the component labeled D_{16}, the remaining 7 Type II boundary components in $\widehat{\mathscr{F}}$ are naturally birational to 7 Type II boundary components in the GIT quotient for $(4,4)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ (see* **Proposition 3.9** *- the labeling is chosen compatibly). The geometric meaning is given in Table 3.*

Table 3
The Type II boundary components of $\widehat{\mathscr{F}} \to \mathscr{F}^*$.

Label	Type	Dim. in $\widehat{\mathscr{F}}$	Geometric Meaning	Quartic Case
D_{16}	a	1	double twisted cubic (on the quadric cone)	D_{17}
$D_8 \oplus E_8$	a	9	$\widetilde{E}_8$, double line	$D_9 \oplus E_8$
$D_{12} \oplus D_4$	a	5	double conic	$D_{12} \oplus D_5$
$(E_7)^2 \oplus D_2$	a	3	two $\widetilde{E}_7$ singularities	$(E_7)^2 \oplus D_3$
$A_{15} \oplus D_1$	a	2	double elliptic quartic	$A_{15} \oplus D_2$
$(D_8)^2$	a	1	two skew double lines; smooth quadric	$(D_8)^2 \oplus D_1$
$(E_8)^2$	b	1	two $\widetilde{E}_8$ singularities; smooth quadric	$(E_8)^2 \oplus D_1$
$(D_{16})^+$	b	1	double twisted cubic; smooth quadric	$D_{16} \oplus D_1$

Proof. In [35, Sect. 7], we have analyzed the $\mathbb{Q}$-factorialization $\widehat{\mathscr{F}}(19) \to \mathscr{F}(19)^*$ for quartic surfaces. In particular, we refer to [35, Prop. 7.6] for the computation of the dimensions of the boundary components, and to [35, Def. 7.7] and [35, Prop. 7.11] for the geometric meaning. The proof for the case of $U(2)$-hyperelliptic $K3$'s is essentially verbatim. For the reader's convenience, the last column of Table 3 indicates the analogous case for quartic $K3$ surfaces. As previously indicated (see **Remark 2.4**), one of the cases occurring for quartics does not occur for hyperelliptic quartics. $\square$

8.3. Arithmetic dictates the structure of the Chow and Hilbert quotients

In conclusion, the isomorphisms of (8.1.1), the structure of Baily-Borel compactification (**Theorem 2.3**), and the structure of the $\mathbb{Q}$-factorialization (**Proposition 8.1**), give the following information on the structure of the Chow GIT quotient $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ and asymptotic Hilbert quotients $\mathrm{Hilb}^{\gg 0}_{(2,4)} /\!\!/ \mathrm{SL}(4)$:

1. (1) There are 8 one-dimensional Type II strata in the Chow GIT $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$. Additionally, there are two Type III points in $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$. The union of the Type II and III boundary strata is the complement of the ADE locus (as in **Claim 6.18**) in $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$.
2. (2) In the Hilbert GIT $\mathrm{Hilb}^m_{(2,4)} /\!\!/ \mathrm{SL}(4)$ ($m \gg 0$), there are 8 Type II boundary components of dimensions between 1 and 9 according to the third column of Table 3.
3. (3) Seven of the 8 Type boundary II components in $\mathrm{Hilb}^m_{(2,4)} /\!\!/ \mathrm{SL}(4)$ are birational to the seven Type II boundary components of $\mathfrak{M}$ identified by **Proposition 3.9**. More precisely, the VGIT $\mathfrak{M}(t)$ for $t \in (\delta, 1/2)$ affects these seven components only birationally. In particular, they have the same dimension in $\mathfrak{M}$ as in $\mathrm{Hilb}^m_{(2,4)} /\!\!/ \mathrm{SL}(4)$ (given by Table 3).
4. (4) Finally, the eighth Type II component (label D_{16}) only exists in the range $t \in (1/3, 1/2]$. (It appears in the exceptional locus of the flip at $t = \frac{1}{3}$, when the locus $W_4 \subset \mathfrak{M}$ is replaced by the stratum $Z^4 \subset \mathscr{F}$; see **Remark 6.4**).

The above results are much more involved and subtle than those present in the existing literature ([50], [39], [40], [31]). In particular Item (4) is completely new, and it offers an elegant explanation to an apparent contradiction to the Shah/Looijenga study of GIT versus Baily-Borel for quartic surfaces. Namely, in the case of degree 2 $K3$ surfaces (and similarly for cubic fourfolds), the number of Type II components on the GIT and Baily-Borel models agree. In contrast, for quartic $K3$ surfaces (and similarly for $U(2)$-hyperelliptic $K3$'s), comparing the number of the Type II components in the GIT model (cf. [51, Thm. 2.4]) to the number of Type II components for the Baily-Borel model (cf. [47, §6.3]) one observes a discrepancy of 1 (i.e. 8 vs. 9). This is somewhat unexpected from Looijenga's theory [39]. Our previous work [36,35] gives a conjectural explanation for this discrepancy (a "second order" arithmetic correction). The present paper establishes that this conjectural behavior is accurate (at least for $U(2)$-hyperelliptic $K3$'s).

8.4. The GIT analysis of the Chow and asymptotic Hilbert quotients

The following result is the geometric counterpart of the discussion from **Subsection 8.3**. The essential new aspect here is the geometric explanation for the drop in dimensions for Type II components as we pass from the Hilbert to the Chow GIT quotient. Somewhat surprisingly, there are four different geometric behaviors (labeled (A)–(D) in the proof below) that occur here.

Theorem 8.2. *The following hold:*

a) *Let C be an irreducible $(2,4)$ complete intersection with only planar singularities of type ADE (equivalently, the associated double cover is a $K3$ surface with canonical singularities). Then C is GIT stable w.r.t. the Chow polarization. Consequently, we can view $\mathscr{F}$ as an open subset in $\mathfrak{M}(\frac{1}{2})$.*
b) *The boundary of $\mathscr{F}$ in the Chow GIT $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4)$ is the union of 8 rational curves (the closure of eight 1-dimensional Type II components listed below), meeting as in diagram (2.4.1). In particular, there are two Type III points (compare* **Remark 3.10***).*

Furthermore,

(II) *The polystable curves parametrized by the 8 Type II boundary components are given by the following equations, where the cases are labeled according to the labels of the Type II components in the Baily-Borel compactification $\mathscr{F}^*$ (cf.* **Theorem 2.3***), using the identification $\mathrm{Chow}_{(2,4)} /\!\!/ \mathrm{SL}(4) \cong \mathscr{F}^*$.*
   i) $(D_8 \oplus E_8)$*:* $V\left(x_1^2 + x_0x_2, x_0x_3^3 + x_1^2x_2^2 + ax_1^2x_2x_3\right)$*;*
   ii) $(D_{12} \oplus D_4)$*:* $V\left(x_1^2 + x_0x_2, x_1(x_0 + ax_2)x_3^2\right)$*;*
   iii) $(A_{15} \oplus D_1)$*:* $V(f_2, g_2^2)$*, where $V(f_2, g_2)$ is an elliptic normal curve;*
   iv) (D_{16}^+)*:* $V\left((u_0 + u_1)(u_0 + au_1)(u_0v_1^2 + u_1v_0^2)^2\right) \subset \mathbb{P}^1 \times \mathbb{P}^1$*;*

v) (E_8^2): $V\left(u_0u_1(u_0v_1^2+u_1v_0^2)(u_0v_1^2+au_1v_0^2)\right)\subset\mathbb{P}^1\times\mathbb{P}^1$;
vi) ($E_7^2\oplus D_2$): $V\left(x_0x_3, x_1x_2(x_1-x_2)(x_1-ax_2)\right)$;
vii) (D_8^2): $V(u_0^2u_1^2v_0v_1(v_0-v_1)(v_0-av_1))\subset\mathbb{P}^1\times\mathbb{P}^1$;
viii) (D_{16}): $V\left(x_1^2+x_0x_2, (x_3+x_1+ax_2)(x_0x_3^2+2x_1x_2x_3-x_2^2x_3)\right)$.

The dimensions of the preimages (via $\mathfrak{M}(1/2-\epsilon)\to\mathfrak{M}(1/2)$) of these strata in $\mathfrak{M}(1/2-\epsilon)(\cong \mathrm{Hilb}^{m\gg 0}_{(2,4)}/\!\!/\mathrm{SL}(4)\cong\widehat{\mathscr{F}})$ are given in the third column of Table 3. Furthermore, the Type II components in $\mathfrak{M}(1/2-\epsilon)$ corresponding to first seven cases (i)–(vii) are birational to the seven Type II components in $\mathfrak{M}$ (via the natural map $\mathfrak{M}\cong\mathfrak{M}(1/6-\epsilon)\dashrightarrow\mathfrak{M}(1/2-\epsilon)$) listed in **Proposition 3.9**. *The eighth stratum (label D_{16}) is visible (as a Type II stratum) in $\mathfrak{M}(t)$ only for $t\in(1/3,1/2]$.*

(III) *The polystable orbits corresponding to the two Type III points have equations:*

$$(III_a): V(x_0x_3, x_1^2x_2^2);$$
$$(III_b): V(x_0x_3-x_1x_2, x_0x_2^3+2x_1^2x_2^2+x_1^3x_3).$$

Proof. Item (a) was established in **Claim 6.18**.

We consider the behavior of the Type II polystable orbits listed in **Proposition 3.9**. According to the analysis of **Subsection 6.6**, there is no change of (semi)stability for a Type II (and similarly for Type III) curve C for $t\in(\delta,1)$ (i.e. if C is stable/semistable as a $(4,4)$ curve or equivalently for $t=\frac{1}{6}-\epsilon$, the same will be true for $t=1-\epsilon$). Thus the seven Type II strata in $\mathfrak{M}\cong\mathfrak{M}(1/6-\epsilon)$ listed in **Proposition 3.9** will survive (birationally) in $\mathrm{Hilb}^{m\gg 0}_{(2,4)}/\!\!/\mathrm{SL}(4)\cong\mathfrak{M}(1/2-\epsilon)\cong\widehat{\mathscr{F}}$. In particular, we note that the dimensions of these strata listed in **Proposition 3.9** match the dimensions of the corresponding Type II components in $\widehat{\mathscr{F}}$ (cf. the third column of Table 3). We know that the Hilbert-Chow morphism induces the VGIT map $\mathfrak{M}(1/2-\epsilon)\to\mathfrak{M}(1/2)$, and then (by our main theorem) this morphism is identified to $\widehat{\mathscr{F}}\to\mathscr{F}^*$. Since the Type II components in $\mathscr{F}^*$ are 1-dimensional, it follows that the Type II components in $\mathfrak{M}(1-\epsilon)$ of dimension larger than 1 will collapse to 1-dimensional components in $\mathfrak{M}(1/2)\cong\mathrm{Chow}_{(2,4)}/\!\!/\mathrm{SL}(4)$. As explained, this is a corollary of our main result (see **Subsection 8.3** above), but we would like to see this behavior purely in GIT terms. In particular, this allows us to identify the minimal orbits for $\mathrm{Chow}_{(2,4)}$.

A Type II stratum in $\mathfrak{M}(1/2-\epsilon)$ might drop dimension in $\mathfrak{M}(1/2)$ for two distinct reasons. The first reason (the typical behavior in VGIT) is the creation of new semistable orbits at $t=\frac{1}{2}$ which absorb some of the $(\frac{1}{2}-\epsilon)$-polystable orbits. The second reason is a contraction is induced by the Hilbert-Mumford morphism. The latter case is only relevant for non-reduced curves, and since we restrict to Type II, it affects only curves in the stratum $A_{15}+D_1$. Returning to the former case, an analysis similar to that of **Section 6** allows us to identify the following new critical orbits at $t=\frac{1}{2}$:

$$V(x_1^2+x_0x_2, x_1^2f_2(x_2,x_3)+x_0f_3(x_2,x_3)) \tag{8.4.1}$$

with stabilizer $\lambda = (5, 1, -3, -3)$. This curve has a double line passing through the vertex v of the cone, and an $\widetilde{E}_8$ singularity at the point $p = [1, 0, 0, 0]$.

$$V(x_0x_3, f_4(x_1, x_2)) \tag{8.4.2}$$

stabilized by $\lambda = (3, 1, 1, -1)$. In this situation, the quadric becomes reducible, and it is cut out by 4 planes (that share an axis).

$$V(x_1^2 + x_0x_2, q(x_0, x_1, x_2)x_3^2) \tag{8.4.3}$$

stabilized by $\lambda = (1, 1, 1, -3)$. In this situation, we have a double conic, together with 4 lines passing through the vertex v. (As limiting cases, we obtain III_a and III_b listed in the theorem.)

In conclusion, we identify the following cases for the behavior of the Type II GIT boundary:

Case A: *The* 1*-dimensional Type II boundary components in* **Proposition 3.9** *(label* $(D_8)^2$*,* $(E_8)^2$*, and* $(D_{16})^+$*).* The stability in these cases does not change in the interval $(0, \frac{1}{2}]$ (two of the cases are strictly semistable for all t, while the third one is stable). In all cases, the relevant curves sit on the smoth quadric.

Case B: *The stratum* $A_{15} \oplus D_1$ *(double elliptic normal curve).* Such curves $C = V(f_2, g_2^2)$ are stable at all time. The difference between this case and case A is that the dimension of the stratum drops by one via the Hilbert-Chow morphism. Geometrically, in the Hilbert scheme, the unique quadric containing C (i.e. $V(f_2)$) is recorded, while in the Chow variety it is not.

Case C: *Strata* $D_8 \oplus E_8$*,* $D_{12} \oplus D_4$ *and* $(E_7)^2 \oplus D_2$. In these cases the stability is not affected in the interval $(0, \frac{1}{2})$, but at $t = \frac{1}{2}$, new orbits become semistable and absorb the polystable orbits (of the given 3 types). The arguments and computations are very similar to those of **Section 6**.

For instance, assume that we are in the situation of the stratum with a single $\widetilde{E}_8$ and no special line. Then, **Lemma 4.7** gives the normal form:

$$\begin{aligned} f_2 &= x_0x_2 + x_1^2 + ax_3^2 \\ f_4 &= bx_0x_3^3 + x_1^2 g_2(x_2, x_3) + x_1 g_3(x_2, x_3) + g_4(x_2, x_3) \end{aligned}$$

As usual, we are interested in singularity at $p = [1, 0, 0, 0]$. In affine coordinates, $x_2 = x_1^2 + ax_3^2$, and once we substitute ($x_0 = 1$, $x_3 = v$, $x_1 = w$, $x_2 = v^2 + aw^3$) the leading term of f_4 becomes

$$v^3 + w^2 g_2(w^2, v)$$

If the leading term defines an isolated singularity, we get the singularity $J_{2,0} = \widetilde{E_8}$. If this is not the case, we get $J_{2,p}$, which is the same as $T_{2,3,6+p}$ (a cusp singularity, still insignificant, but of Type III). Here g_2 is assumed non-vanishing, otherwise we get Type IV case discussed previously. Very similarly to the E_{12} case discussed in **Section 6**, the potential critical orbit (case (8.4.1) above) is

$$V(x_0x_2 + x_1^2, x_0x_3^3 + x_1^2 g_2(x_2, x_3))$$

with stabilizer $\lambda = (5, 1, -3, -3)$. This can be semistable only at $t = \frac{1}{2}$. For $t > \frac{1}{2}$, we see $\nu^t(x, \lambda) < 0$ for all points in the $\widetilde{E}_8$ stratum. At $t = \frac{1}{2}$, the limit of x in this stratum with respect to λ is the orbit with $\mathbb{C}^*$-stabilized as above.

The case of $2\widetilde{E}_7$ is similar. The polystable orbits $V(f_2, f_4(x_1, x_2))$ (i.e. a quadric cut by 4-coaxial planes) will further degenerate to the case of $f_2 = x_0x_3$ (as discussed in **Lemma 6.7**, the reducible quadric case can not be semistable until $t = \frac{1}{2}$). This leads to the equation (8.4.2) above.

In the case of a double conic, the residual curve will be (in general) an elliptic curve E (Type (2,2) in the smooth quadric case) cutting the double conic in 4 points. At $t = \frac{1}{2}$ this will degenerate to the curve of arithmetic genus 1 with a 4-tuple elliptic point (i.e. 4 lines passing through the origin in $\mathbb{A}^3$). This type of degeneration is not allowed until $t = \frac{1}{2}$ (N.B. by **Lemma 6.7**, for $t < \frac{1}{2}$ only planar singularities are allowed). This corresponds to (8.4.3) above.

Case D: *The stratum D_{16} is not visible in GIT quotient $\mathfrak{M}$ for $(4, 4)$ curves, but becomes visible in $\mathfrak{M}(t)$ for $t > \frac{1}{3}$.* Geometrically, the relevant polystable curves sit only on the quadric cone. They consist of a double twisted cubic, together with a residual conic. The minimal orbits at $t = \frac{1}{3}$ are

$$V(x_1^2 + x_0x_2, x_0x_3^3 + 2\alpha x_1x_2x_3^2 - \beta x_2^3x_3).$$

For generic α, β, there will be a singularity of type A_3 at the vertex of the cone $v = [0, 0, 0, 1]$, and a singularity of type $E_{3,0}$ at $p = [1, 0, 0, 0]$. For the special value of $\alpha = \beta = 1$, we obtain the double twisted cubic. The singularity at v will be of type A_∞, while the singularity at p is of type $J_{3,\infty}$. The curves that will have this polystable orbit in their orbit closure at $t = \frac{1}{2}$ will have either a singularity of type $J_{3,k}$ $(k > 0)$ at p or a singularity of type A_k ($k > 3$, and such that the curve doesn't split a line through the vertex). At t increases, we have seen that all $J_{3,k}$ (we allow also $k = 0, \infty$) are destabilized, while the curves with A_k singularity (allow also $k = \infty$, but require that it doesn't split a line) at the vertex are allowed to become stable. In practice, we modify the equation $f_4 = x_0x_3^3 + 2\alpha x_1x_2x_3^2 - \beta x_2^3x_3$ by adding monomials of lower weight with respect to $\lambda = (3, 1, -1, -3)$. We are interested in preserving the double twisted cubic thus

$$f_4 = x_3(x_0x_3^2 + 2x_1x_2x_3 - x_2^2x_3)$$

is modified by moving the conic $V(x_1^2 + x_0x_2, x_3)$, i.e. modify the linear form x_3 to $\ell(x_0, x_1, x_2, x_3)$. It needs to avoid passing through the vertex, which gives the normal form from the theorem. $\square$

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