PAIRWISE INCIDENT PLANES AND HYPERKÄHLER FOUR-FOLDS

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Contents

1. Introduction

A family of pairwise incident lines in a projective space consists of lines through a point or lines contained in a plane. Is there an analogous characterization of families of pairwise incident planes in a complex projective space? A beautiful theorem of Ugo Morin [6] states that an algebraic **irreducible** family of pairwise incident planes is contained in one of the following families:

(1) Planes containing a fixed point.
(2) Planes whose intersection with a fixed plane has dimension at least 1.
(3) Planes contained in a fixed 4-dimensional projective space.
(4) One of the two irreducible components of the set of planes contained in a fixed smooth 4-dimensional quadric.
(5) The planes tangent to a fixed Veronese surface (image of $\mathbb{P}^2 \to |\mathcal{I}_{\mathbb{P}^2}(2)|^\vee$).
(6) The planes intersecting a fixed Veronese surface along a conic.

In the present paper we will address the following question: what are the cardinalities of finite families of pairwise incident planes? As stated the question is not interesting because the families of pairwise incident planes listed above contain sets of arbitrary finite cardinality. In order to formulate a meaningful question we recall the following definition of Morin: a family of pairwise incident planes is *complete* if there exists no plane outside the family which is incident to all planes in the family - in other words if the family is maximal. We ask the following question: what are the cardinalities of finite complete family of pairwise incident planes? Before stating our main result we will describe a finite complete family of pairwise incident planes in $\mathbb{P}^6$. Let $\{v_0, \ldots, v_6\}$ be a basis of $\mathbb{C}^7$. Identify the set $\{[v_0], \ldots, [v_6]\}$ and $\mathbb{P}^2_{\mathbb{F}_2}$ (the projective plane on the field with 2 elements) as follows:

$$[v_0]\mapsto[010],\quad [v_1]\mapsto[011],\quad [v_2]\mapsto[001],\quad [v_3]\mapsto[101],\quad [v_4]\mapsto[100],\quad [v_5]\mapsto[110],\quad [v_6]\mapsto[111].$$

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Given the above identification we let $\Lambda_1,\ldots,\Lambda_7 \in \mathrm{Gr}(2,\mathbb{P}^6)$ be the planes spanned by the points on a line in $\mathbb{P}^2_{\mathbb{F}_2}$. Explicitly

$$\Lambda_1=\mathbb{P}\langle v_0,v_1,v_2\rangle, \quad \Lambda_2=\mathbb{P}\langle v_2,v_3,v_4\rangle, \quad \Lambda_3=\mathbb{P}\langle v_0,v_4,v_5\rangle, \quad \Lambda_4=\mathbb{P}\langle v_1,v_3,v_5,\rangle,$$
$$\Lambda_5=\mathbb{P}\langle v_0,v_3,v_6\rangle, \quad \Lambda_6=\mathbb{P}\langle v_1,v_4,v_6\rangle, \quad \Lambda_7=\mathbb{P}\langle v_2,v_5,v_6\rangle. \tag{1.0.1}$$

As is easily checked the planes $\Lambda_1,\ldots,\Lambda_7$ are pairwise incident: we will show (see **Claim 2.1**) that they form a complete family.

Theorem 1.1. *Let $T\subset \mathrm{Gr}(2,\mathbb{P}^N)$ be a finite complete family of pairwise incident planes. The planes in T span a projective space of dimension 5 or 6. If the span has dimension 6 then T is projectively equivalent to the family $\{\Lambda_1,\ldots,\Lambda_7\}$ described above. If the span has dimension 5 then T has at most 20 elements. For any $10\le k\le 16$ there exists a complete family of k pairwise incident planes: in fact it has at least $(20-k)$ moduli.*

In **Section 2** we will study finite complete families of pairwise incident planes which span a projective space of dimension greater than 5: the proofs are of an elementary nature. In **Section 3** we will make the connection between our question and the geometry of certain Hyperkähler 4-folds which are double covers of special sextic hypersurfaces in $\mathbb{P}^5$ named EPW-sextics. Then we will apply results of Ferretti [4] on degenerations of double EPW-sextics in order to show that there exist finite complete families of pairwise incident planes in $\mathbb{P}^5$ of cardinality between 10 and 16; we will also get the lower bound on the number of moduli given in **Theorem 1.1**. In **Section 4** we will prove that a finite complete family of pairwise incident planes has cardinality at most 20.

A few comments. I suspect that 16 is the maximum cardinality of a finite complete family of pairwise incident planes. Our (we might say Ferretti's) proof that there exist complete families of pairwise incident planes of cardinality between 10 and 16 is a purely existential proof: it does not give explicit families. One may ask for explicit examples. The paper [2] of Dolgachev and Markushevich provides a general framework for the study of this problem. In particular the authors associate to a generic Fano model of an Enriques surface (plus a suitable choice of 10 elliptic curves on the surface) a finite collection of complete families of 10 pairwise incident planes in $\mathbb{P}^5$ - they also study the problem of classifying the irreducible components (there are several such) of the locus parametrizing ordered 10-tuples of pairwise incident planes in $\mathbb{P}^5$. In the same paper Dolgachev and Markushevich give explicit constructions of complete families of 13 pairwise incident planes.

Notation and conventions. We work throughout over $\mathbb{C}$. Let $T\subset \mathrm{Gr}(2,\mathbb{P}^N)$ be a family of planes: the *span* of T is the span of the union of the planes parametrized by T.

2. Families of pairwise incident planes in $\mathbb{P}^N$ for $N>5$

Let $T\subset \mathrm{Gr}(2,\mathbb{P}^N)$ be a finite complete family of pairwise incident planes. If the span of T is contained in a projective space M of dimension at most 4 then T is contained in the infinite family of pairwise incident planes $\mathrm{Gr}(2,M)$, that is a contradiction. Hence the span of T has dimension at least 5. In the present section we will classify finite complete family of pairwise incident planes whose span has dimension greater than 5. We will start by showing that the planes $\Lambda_1,\ldots,\Lambda_7\subset\mathbb{P}^7$ defined by (1.0.1) form a complete family of pairwise incident planes. Let $v_0,\ldots,v_6$ be as in **Section 1**; we let

$$\mathbb{P}^5 := \mathbb{P}\langle v_0,\ldots,v_5\rangle. \tag{2.0.1}$$

The set of lines in $\mathbb{P}^5$ meeting $\Lambda_1, \Lambda_2, \Lambda_3$ has 4 irreducible components, each isomorphic to $\mathbb{P}^2$; more precisely

$$\{L \in \mathrm{Gr}(1,\mathbb{P}^5) \mid L\cap\Lambda_i\neq\emptyset,\ i=1,2,3\} = \mathrm{Gr}(1,\mathbb{P}\langle v_0,v_2,v_4\rangle)\cup \\ \cup\{\mathbb{P}\langle v_0,u\rangle \mid 0\neq u\in\langle v_2,v_3,v_4\rangle\}\cup\{\mathbb{P}\langle v_2,u\rangle \mid 0\neq u\in\langle v_0,v_4,v_5\rangle\}\cup\{\mathbb{P}\langle v_4,u\rangle \mid 0\neq u\in\langle v_0,v_1,v_2\rangle\}. \quad (2.0.2)$$

In fact suppose that a line L intersects $\Lambda_1, \Lambda_2, \Lambda_3$ and does not belong to $\Lambda_0 := \mathbb{P}\langle v_0, v_2, v_4\rangle$. It suffices to show that one of $[v_0], [v_2], [v_4]$ belongs to L. Suppose the contrary and let $L \cap \mathbb{P}\langle v_0, v_2, v_4\rangle = \{p\}$. Then there exist at least two planes among $\Lambda_1, \Lambda_2, \Lambda_3$ which do not contain p, call them Λ_i, Λ_j. It follows that L belongs to the intersection $\mathbb{P}\langle\Lambda_0, \Lambda_i\rangle \cap \mathbb{P}\langle\Lambda_0, \Lambda_j\rangle$. The latter is equal to Λ_0, that is a contradiction. Equation (2.0.2) gives that there are exactly 3 lines in $\mathbb{P}^5$ meeting $\Lambda_1, \ldots, \Lambda_4$. More precisely let

$$L_5:=\mathbb{P}\langle v_0,v_3\rangle=\Lambda_5\cap\mathbb{P}^5, \quad L_6:=\mathbb{P}\langle v_1,v_4\rangle=\Lambda_6\cap\mathbb{P}^5, \quad L_7:=\mathbb{P}\langle v_2,v_5\rangle=\Lambda_7\cap\mathbb{P}^5. \quad (2.0.3)$$

Then

$$\{L \in \mathrm{Gr}(1, \mathbb{P}^5) \mid L \cap \Lambda_i \neq \emptyset,\ i = 1, 2, 3, 4\} = \{L_5, L_6, L_7\}. \quad (2.0.4)$$

Claim 2.1. *The collection of planes $\Lambda_1, \ldots, \Lambda_7 \subset \mathbb{P}^6$ defined by* (1.0.1) *is a complete family of pairwise incident planes.*

Proof. We need to show that the family is complete. First we notice that the span of $\Lambda_1, \ldots, \Lambda_4$ is equal to $\mathbb{P}^5$, notation as in (2.0.1). Now let $\Lambda \subset \mathbb{P}^6$ be a plane intersecting $\Lambda_1, \ldots, \Lambda_7$. Since the intersection of $\Lambda_1, \ldots, \Lambda_4$ is empty one of the following holds:

(1) $\Lambda \subset \mathbb{P}^5$,
(2) $\dim(\Lambda \cap \mathbb{P}^5) = 1$.

Suppose that (1) holds. Then Λ meets each of the lines L_5, L_6, L_7 given by (2.0.3). Since L_5, L_6, L_7 generate $\mathbb{P}^5$ it follows that Λ intersects L_i in a single point p_i and that Λ is spanned by p_5, p_6, p_7. Imposing the condition that $\langle p_5, p_6, p_7\rangle$ (for $p_i \in L_i$) meet each of $\Lambda_1, \ldots, \Lambda_4$ we get that $\langle p_5, p_6, p_7\rangle$ is one of $\Lambda_1, \ldots, \Lambda_4$. This proves that if (1) holds then $\Lambda \in \{\Lambda_1, \ldots, \Lambda_4\}$. Next suppose that (2) holds and let $L = \Lambda \cap \mathbb{P}^5$. Then L meets each of $\Lambda_1, \ldots, \Lambda_4$. By (2.0.4) it follows that L equals one of L_5, L_6, L_7. Suppose that $L = L_5$. Then Λ meets Λ_6 and Λ_7 in points outside $\mathbb{P}^5$. Now notice that the span of $\Lambda, \Lambda_6, \Lambda_7$ is all of $\mathbb{P}^6$: it follows that $\Lambda, \Lambda_6, \Lambda_7$ meet in a single point, which is necessarily $[v_6]$. Thus $\Lambda = \Lambda_5$. If L equals one of L_6 or L_7 a similar argument shows that $\Lambda = \Lambda_6$ or $\Lambda = \Lambda_7$ respectively. □

Our next goal is to prove that if T is a finite complete family of pairwise incident planes spanning a projective space of dimension greater than 5 then T is projectively equivalent to $\{\Lambda_1, \ldots, \Lambda_7\}$ where the planes $\Lambda_1, \ldots, \Lambda_7$ are defined by (1.0.1). First we make the following observation.

Proposition 2.2. *Let $T \subset \mathrm{Gr}(2, \mathbb{P}^N)$ be a family of pairwise incident planes. Suppose that there exist $\Lambda, \Lambda' \in T$ such that their intersection is a line. Then T is contained in an infinite family of pairwise incident planes.*

Proof. Let $L := \Lambda \cap \Lambda'$ and $M := \langle\Lambda, \Lambda'\rangle$. Thus L is a line and M is a 3-dimensional projective space. Let $\Lambda'' \in T$: since Λ'' intersects both Λ and Λ' one of the following holds:

(1) $\dim(\Lambda'' \cap M) \geq 1$,
(2) $\Lambda'' \cap L \neq \emptyset$.

Now let $\Lambda_0 \subset M$ be a plane containing L. If (1) holds then Λ_0 intersects $(\Lambda'' \cap M)$, if (2) holds then Λ_0 contains the non-empty intersection $(\Lambda'' \cap L)$: in both cases we get that Λ_0 intersects Λ''. Hence the union of T and the set of planes in M containing L is an infinite family of pairwise incident planes containing T. $\square$

The result below follows immediately from **Proposition 2.2**.

Corollary 2.3. *Let $T \subset \mathrm{Gr}(2, \mathbb{P}^N)$ be a finite complete family of pairwise incident planes. If $\Lambda, \Lambda' \in T$ are distinct their intersection is a single point.*

Proposition 2.4. *Let $T \subset \mathrm{Gr}(2, \mathbb{P}^N)$ be a finite complete family of pairwise incident planes. Suppose that the span of T has dimension greater than 5. Then T is projectively equivalent to $\{\Lambda_1, \ldots, \Lambda_7\}$ where $\Lambda_1, \ldots, \Lambda_7$ are as in (1.0.1).*

Proof. Let $\Lambda_1, \Lambda_2 \in T$ be distinct: by **Corollary 2.3** they intersect in a single point p and hence they span a 4-dimensional projective space M. We claim that there does exist $\Lambda_3 \in T$ which is not contained in M and which intersects Λ_1, Λ_2 in distinct points. In fact suppose the contrary. Then we get an infinite family of pairwise incident planes by adding to T the planes $\Lambda \in \mathrm{Gr}(2, M)$ containing p: that contradicts the hypothesis that T is a finite complete family of pairwise incident planes. Since the planes $\Lambda_1, \Lambda_2, \Lambda_3$ have distinct pairwise intersections and they span a 5-dimensional projective space there exists linearly independent $v_0, \ldots, v_5 \in \mathbb{C}^6$ such that $\Lambda_1, \Lambda_2, \Lambda_3$ are as in (1.0.1). We claim that there exists $\Lambda_4 \in T$ which is contained in $\mathbb{P}^5$ (notation as in (2.0.1)) and does not intersect $\mathbb{P}\langle v_0, v_2, v_4 \rangle$. In fact if no such Λ_4 exists then the plane $\mathbb{P}\langle v_0, v_2, v_4 \rangle$ is incident to all planes in T and intersects each of $\Lambda_1, \Lambda_2, \Lambda_3$ along a line: that is a contradiction because of **Proposition 2.2**. We may rename v_1, v_3, v_5 so that Λ_4 is as in (1.0.1). Now let $\Lambda \in T$: since Λ intersects $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ one of the following holds:

(1) $\Lambda \subset \mathbb{P}^5$.
(2) $\dim(\Lambda \cap \mathbb{P}^5) = 1$ and the line $\Lambda \cap \mathbb{P}^5$ is one of L_5, L_6, L_7, see (2.0.3).

Since the span of T has dimension greater than 5 there does exist $\Lambda \in T$ such that Item (2) holds. By **Corollary 2.3** we have an injection

$$\begin{array}{ccc} T \setminus \mathrm{Gr}(2, \mathbb{P}^5) & \hookrightarrow & \{L_5, L_6, L_7\} \\ \Lambda & \mapsto & \Lambda \cap \mathbb{P}^5 \end{array} \tag{2.0.5}$$

We claim that Map (2.0.5) is surjective. In fact suppose that the image consists of a single line L_i: then every plane containing L_i is incident to every plane in T, that contradicts the hypothesis that T is a finite complete family of pairwise incident planes. Now suppose that the image consists of 2 lines: without loss of generality we may assume that they are L_5, L_6. A straightforward computation gives that

$$\{\Lambda \in \mathrm{Gr}(2,\mathbb{P}^5) | \Lambda \text{ is incident to } L_5,\ L_6,\ \Lambda_1,\ \Lambda_2,\ \Lambda_3 \text{ and } \Lambda_4\} = \mathbb{P}\langle v_0, v_1, v_3, v_4 \rangle \cup \{\mathbb{P}\langle v_0, v_1, av_2+bv_3+cv_4 \rangle\} \cup \\ \cup \{\mathbb{P}\langle v_1, v_3, av_0+bv_4+cv_5 \rangle\} \cup \{\mathbb{P}\langle v_3, v_4, av_0+bv_1+cv_2 \rangle\} \cup \{\mathbb{P}\langle v_0, v_4, av_1+bv_3+cv_5 \rangle\}. \tag{2.0.6}$$

Now notice that the right-hand side of (2.0.6) is an infinite family of pairwise incident planes: that contradicts the hypothesis that T is a finite complete family of pairwise incident planes. We have proved that Map (2.0.5) is surjective. Now let $\Lambda \in T$ be such that Item (1) holds: then Λ is incident to $\Lambda_1, \ldots, \Lambda_4$ and to L_1, L_2, L_3: it follows that $\Lambda \in \{\Lambda_1, \ldots, \Lambda_4\}$ - see the proof of **Claim 2.1**. The set of $\Lambda \in T$ such that Item (2) holds consists of 3 elements, say $\{\Lambda_5, \Lambda_6, \Lambda_7\}$ where $\Lambda_i \cap \mathbb{P}^5 = L_i$. Since L_5, L_6, L_7 span $\mathbb{P}^5$ the planes $\Lambda_5, \Lambda_6, \Lambda_7$ intersect in a single point which lies outside $\mathbb{P}^5$: thus we may complete $v_0, \ldots, v_5$ to a basis of $\mathbb{C}^7$ by adding a vector v_6 such that $\Lambda_5 \cap \Lambda_6 \cap \Lambda_7 = \{[v_6]\}$. Then it is clear that T is projectively equivalent to $\{\Lambda_1, \ldots, \Lambda_7\}$. $\square$

3. Complete finite families of pairwise incident planes in $\mathbb{P}^5$

In the present section we will associate to a finite complete family of pairwise incident planes in $\mathbb{P}^5$ an EPW-sextic - a special sextic hypersurface in $\mathbb{P}^5$ which comes equipped with a double cover. The double cover of a generic EPW-sextic is a Hyperkähler 4-fold deformation equivalent to the Hilbert square of a $K3$. There is a divisor Σ in the space of EPW-sextics whose generic point corresponds to a double cover X whose singular locus is a $K3$-surface of degree 2: it is obtained from a HK 4-fold $\widetilde{X}$ by contracting a divisor E which is a conic bundle over the $K3$, see [12]. Let Y be the EPW-sextic corresponding to X: the covering map $X \to Y$ takes the singular locus of X to a plane. There are more special EPW-sextics parametrized by points of Σ which correspond to a HK 4-fold $\widetilde{X}$ containing more than one of the divisors E: the images of these divisors under the composition $\widetilde{X} \to X \to Y$ are pairwise incident planes. We will show that certain of these EPW-sextics (introduced by Ferretti [4]) provide examples of complete families of k pairwise incident planes in $\mathbb{P}^5$ for $10 \le k \le 16$. Choose a volume-form vol: $\bigwedge^6 \mathbb{C}^6 \xrightarrow{\sim} \mathbb{C}$ and equip $\bigwedge^3 \mathbb{C}^6$ with the symplectic form

$$(\alpha, \beta) := \mathrm{vol}(\alpha \wedge \beta). \tag{3.0.1}$$

Let $A \subset \bigwedge^3 \mathbb{C}^6$ be a subspace: we let

$$\Theta_A := \{W \in \mathrm{Gr}(3, \mathbb{C}^6) \mid \bigwedge^3 W \subset A\}, \tag{3.0.2}$$

$$\boldsymbol{\Theta}_A := \{\Lambda \in \mathrm{Gr}(2, \mathbb{P}^5) \mid \Lambda = \mathbb{P}(W) \text{ where } W \in \Theta_A\}. \tag{3.0.3}$$

The following simple observation will be our starting point.

Remark 3.1. Let $A \subset \bigwedge^3 \mathbb{C}^6$ be **isotropic** for the symplectic form $(,)$. Then $\boldsymbol{\Theta}_A$ is a family of pairwise incident planes. Conversely let $T \subset \mathrm{Gr}(2, \mathbb{P}^5)$ be a family of pairwise incident planes and $B \subset \bigwedge^3 \mathbb{C}^6$ be the subspace spanned by the vectors $\bigwedge^3 W$ for $W \in \mathrm{Gr}(3, \mathbb{C}^6)$ such that $\mathbb{P}(W) \in T$: then B is isotropic for $(,)$.

Let $\mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\bigwedge^3 \mathbb{C}^6$ - of course $\mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ does not depend on the choice of volume-form. Notice that $\dim \bigwedge^3 \mathbb{C}^6 = 20$ and hence elements of $\mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ have dimension 10.

Claim 3.2. *Let $T \subset \mathrm{Gr}(2, \mathbb{P}^5)$ be a* **complete** *family of pairwise incident planes. Then there exists $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ such that*

$$\boldsymbol{\Theta}_A = T. \tag{3.0.4}$$

Conversely suppose that $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ is **spanned** *by Θ_A (embedded in $\bigwedge^3 \mathbb{C}^6$ by Plücker). Then $\boldsymbol{\Theta}_A$ is a complete family of pairwise incident planes.*

Proof. Let $B \subset \bigwedge^3 \mathbb{C}^6$ be the subspace spanned by the vectors $\bigwedge^3 W$ for $W \in \mathrm{Gr}(3, \mathbb{C}^6)$ such that $\mathbb{P}(W) \in T$: then B is $(,)$-isotropic, see **Remark 3.1**. Thus there exists $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ containing B. Then $\boldsymbol{\Theta}_A$ is a family of pairwise incident planes, see **Remark 3.1**, and it contains T. Since T is complete we get that (3.0.4) holds. Now suppose that $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ is spanned by $\bigwedge^3 W_1, \ldots, \bigwedge^3 W_{10}$ where $W_1, \ldots, W_{10} \in \Theta_A$. Suppose that $\mathbb{P}(W_*) \in \mathrm{Gr}(2, \mathbb{P}^5)$ is incident to all $\Lambda \in \boldsymbol{\Theta}_A$. Then $\bigwedge^3 W_*$ is orthogonal to $\bigwedge^3 W_1, \ldots, \bigwedge^3 W_{10}$ and hence to all of A. Since A is lagrangian we get that $\mathbb{P}(W_*) \in \boldsymbol{\Theta}_A$. This proves that $\boldsymbol{\Theta}_A$ is a complete family of pairwise incident planes. $\square$

Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$: according to Eisenbud-Popescu-Walter (see the appendix of [3] or [8]) one associates to A a subset of $\mathbb{P}^5$ as follows. Given a non-zero $v \in \mathbb{C}^6$ we let

$$F_v := \{\alpha \in \bigwedge^3 \mathbb{C}^6 \mid v \wedge \alpha = 0\}. \tag{3.0.5}$$

Notice that $F_v \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$. We let

$$Y_A = \{[v] \in \mathbb{P}^5 \mid F_v \cap A \neq \{0\}\}. \tag{3.0.6}$$

The lagrangians F_v are the fibers of a vector-bundle F on $\mathbb{P}^5$ with $\det F \cong \mathcal{O}_{\mathbb{P}^5}(-6)$: it follows that Y_A is the zero-locus of a section of $\mathcal{O}_{\mathbb{P}^5}(6)$. Thus either $Y_A = \mathbb{P}^5$ (this happens for "degenerate" choices of A, for example $A = F_w$) or else Y_A is a sextic hypersurface - an *EPW-sextic*. We emphasize that EPW-sextics are very special hypersurfaces, in particular their singular locus has dimension at least 2. An EPW-sextic Y_A comes equipped with a finite map [10]

$$f_A \colon X_A \to Y_A. \tag{3.0.7}$$

X_A is the *double EPW-sextic* associated to A. The following result [8] motivates the adjective "double". Suppose that

$$\Theta_A = \emptyset \text{ and } \dim(F_v \cap A) \leq 2 \text{ for all } [v] \in \mathbb{P}^5. \tag{3.0.8}$$

(A dimension count shows that (3.0.8) holds for generic $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$.) Then $Y_A \neq \mathbb{P}^5$ and X_A is a Hyperkähler variety deformation equivalent to the Hilbert square of a $K3$ surface[1], moreover (3.0.7) is identified with the quotient map of an anti-symplectic involution on X_A. What if one of the conditions of (3.0.8) are violated? If Θ_A is empty but there do exist $[v] \in \mathbb{P}^5$ such that $\dim(F_v \cap A) > 2$ then necessarily $\dim(F_v \cap A) = 3$ and X_A is obtained from a holomorphic symplectic 4-fold by contracting certain copies of $\mathbb{P}^2$ (one for each point violating the second condition of (3.0.8)): thus X_A is almost as good as a HK variety. On the other hand suppose that $\Lambda \in \mathbf{\Theta}_A$: then $\Lambda \in Y_A$ and Y_A and X_A (assuming that $Y_A \neq \mathbb{P}^5$) may be quite singular along Λ. The following result will be handy.

Proposition 3.3 (Cor. 2.5 of [9] and Prop. 1.11, Claim 1.12 of [10])**.** *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and $[v] \in \mathbb{P}^5$. Then the following hold:*

(1) *If no $\Lambda \in \mathbf{\Theta}_A$ contains $[v]$ then $Y_A \neq \mathbb{P}^5$, $\operatorname{mult}_{[v_0]} Y_A = \dim(A \cap F_{v_0})$ and*
  - (1a) *if $\dim(F_v \cap A) \leq 2$ then X_A is smooth at $f_A^{-1}([v])$,*
  - (1b) *if $\dim(F_v \cap A) > 2$ then the analytic germ of X_A at $f_A^{-1}([v])$ (a single point) is isomorphic to the cone over $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$.*

(2) *If there exists $\Lambda \in \mathbf{\Theta}_A$ containing $[v]$ then either $Y_A = \mathbb{P}^5$ or else X_A is singular at $f_A^{-1}([v])$.*

Next we will define an $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ such that Y_A is a triple quadric: the example will be a key element in the construction of complete families of pairwise incident planes of cardinality between 10 and 16. Choose an isomorphism $\mathbb{C}^6 = \bigwedge^2 U$ where U is a complex vector-space of dimension 4. Thus $\operatorname{Gr}(2, U) \subset \mathbb{P}(\mathbb{C}^6)$ is a smooth quadric hypersurface: we let

$$Q(U) := \operatorname{Gr}(2, U). \tag{3.0.9}$$

We have an embedding

$$\begin{array}{ccc} \mathbb{P}(U) & \overset{i_+}{\hookrightarrow} & \operatorname{Gr}(2, \mathbb{P}^5) \\ [u_0] & \mapsto & \mathbb{P}\{u_0 \wedge u \mid u \in U\} \end{array} \tag{3.0.10}$$

[1]Notice that if A is general then X_A is not isomorphic nor birational to the Hilbert square of a $K3$.

Definition 3.4. Let $A_+(U) \subset \bigwedge^3(\mathbb{C}^6)$ be the subspace spanned by the cone over $\mathrm{Im}(i_+)$ - here we view $\mathrm{Gr}(2,\mathbb{P}^5)$ as embedded in $\mathbb{P}(\wedge^3\mathbb{C}^6)$ by the Plücker map.

Let $\mathcal{L}$ be Plücker line-bundle on $\mathrm{Gr}(2,\mathbb{P}^5)$. Then $i_+^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(U)}(2)$ and the induced map on global sections is surjective: thus $\dim A_+(U) = 10$. On the other hand any two planes in the image of i_+ are incident: thus $A_+(U) \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$, see **Remark 3.1**. One has (see Claim 2.14 of [9])

$$Y_{A_+(U)} = 3Q(U). \tag{3.0.11}$$

Let $\mathbf{K} \subset \mathbb{P}(U)$ be a Kummer quartic surface and let $\mathbf{p}_1,\ldots,\mathbf{p}_{16}$ be its nodes. Choose k nodes $\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_k}$. There exist arbitrarily small deformations of $\mathbf{K}$ which contain exactly k nodes which are small deformations of $\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_k}$ and are smooth elsewhere (it suffices to deform the minimal desingularization of $\mathbf{K}$ keeping the rational curves lying over $\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_k}$ of type $(1,1)$ and not keeping of type $(1,1)$ the rational curves lying over the remaining nodes). Let S_0 be such a small deformation of $\mathbf{K}$ and $p_1,\ldots,p_k$ be its nodes. Let $\widetilde{S}_0 \to S_0$ be the minimal desingularization: thus $\widetilde{S}_0$ is a $K3$ surface containing k smooth rational curves $R_1,\ldots,R_k$ mapping to $p_1,\ldots,p_k$ respectively. The HK 4-fold $\widetilde{S}_0^{[2]}$ contains k disjoint copies of $\mathbb{P}^2$ namely $R_1^{(2)},\ldots,R_k^{(2)}$. We have a regular map

$$\begin{array}{ccc} \widetilde{S}_0^{[2]} \setminus \bigcup_{i=1}^k R_i^{(2)} & \longrightarrow & Q(U) \\ Z & \mapsto & \langle Z\rangle \end{array} \tag{3.0.12}$$

where $\langle Z\rangle$ is the unique line containing the scheme Z. One cannot extend the above map to a regular map over $R_i^{(2)}$. Let $\widetilde{S}_0^{[2]} \dashrightarrow X$ be the flop of $R_1^{(2)},\ldots,R_k^{(2)}$ i.e. the blow-up of each $R_i^{(2)} \cong \mathbb{P}^2$ followed by contraction of the exceptional fiber E_i (which is isomorphic to the incidence variety in $\mathbb{P}^2 \times (\mathbb{P}^2)^\vee$) along the projection $E_i \to (\mathbb{P}^2)^\vee$. Map (3.0.12) extends [4] to a regular degree-6 map

$$X \longrightarrow Q(U). \tag{3.0.13}$$

The following result is due to Ferretti:

Proposition 3.5 (Ferretti, Prop. 4.3 of [4])**.** *Keep notation as above. There exist a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "G"] \arrow[dr, "\pi"'] & & \mathcal{U}\times\mathbb{P}^5 \arrow[dl] \\ & \mathcal{U} & \end{tikzcd} \tag{3.0.14}$$

and maps

$$\mathcal{U} \xrightarrow{A} \mathbb{LG}(\bigwedge^3 \mathbb{C}^6), \qquad \mathcal{U} \xrightarrow{\Lambda_i} \mathrm{Gr}(2,\mathbb{P}^5), \quad i=1,\ldots,k$$

such that the following hold:

1. *$\mathcal{U}$ is a connected contractible manifold of dimension $(20-k)$.*
2. *π is a proper map and a submersion of complex manifolds: for $t \in \mathcal{U}$ we let $X_t := \pi^{-1}(t)$ and $g_t\colon X_t \to \mathbb{P}^5$ be the regular map induced by G.*
3. *There exists $0 \in \mathcal{U}$ and an isomorphism $X_0 \cong X$ such that g_0 gets identified with Map (3.0.13). Moreover $A(0) = A_+(U)$ and $\Lambda_i(0) = i_+(p_i)$.*
4. *There exist a regular map $c_t\colon X_t \to X_{A(t)}$ and prime divisors $E_i(t)$ on X_t for $i=1,\ldots,k$ such that the following hold for all t belonging to an open dense $\mathcal{U}^0 \subset \mathcal{U}$:*
   (4a) *$g_t = f_{A(t)} \circ c_t$.*
   (4b) *$g_t(E_i(t)) = \Lambda_i(t)$ for $i=1,\ldots,k$.*

(4c) *c_t contracts each $E_i(t)$ to a $K3$ surface $S_i(t) \subset X_{A(t)}$ and is an isomorphism of the complement of $\cup_{i=1}^{k} E_i(t)$ onto its image.*

(5) *The period map $\mathcal{U} \to \mathbb{P}(H^2(X_0;\mathbb{C}))$ is an immersion i.e. the family of deformations of X_0 parametrized by $\mathcal{U}$ has $(20-k)$ moduli.*

Given **Proposition 3.5** it is easy to show that there exist complete families of pairwise incident planes of cardinality k for $10 \le k \le 16$. Before stating the relevant result we recall that the $K3$ surface $\widetilde{S}_0$ depends on the choice of nodes $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ and hence so does the variety X.

Proposition 3.6. *Keep notation as in* **Proposition 3.5** *and let $10 \le k \le 16$. Let $t \in \mathcal{U}^0$ be close to 0. One can choose the nodes $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ of $\mathbf{K}$ so that $\mathbf{\Theta}_{A(t)}$ is a complete family of pairwise incident planes of cardinality k.*

Proof. The map i_+ is identified with the map associated to the complete linear system $|\mathcal{O}_{\mathbb{P}(U)}(2)|$. It is well-know that no quadric in $\mathbb{P}(U)$ contains $\mathbf{p}_1, \ldots, \mathbf{p}_{16}$[2]. Thus $i_+(\mathbf{p}_1), \ldots, i_+(\mathbf{p}_{16})$ span a 10-dimensional subspace of $\mathbb{P}(\bigwedge^3 \mathbb{C}^6)$. Since $10 \le k \le 16$ we may choose $\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}$ such that no quadric in $\mathbb{P}(U)$ contains them. It follows that for small enough $t \in \mathcal{U}^0$ the planes $\Lambda_1(t), \ldots, \Lambda_k(y)$ span $\mathbb{P}(A(t))$. By **Claim 3.2** it remains to prove that no other plane is contained in $\mathbf{\Theta}_{A(t)}$. Suppose that $\Lambda \in \mathbf{\Theta}_{A(t)}$ and that $\Lambda \notin \{\Lambda_1(t), \ldots, \Lambda_k(t)\}$. By Item (2) of **Proposition 3.3** we get that $X_{A(t)}$ is singular along $f^{-1}_{A(t)}(\Lambda)$: since $\Lambda \notin \{\Lambda_1(t), \ldots, \Lambda_k(t)\}$ that contradicts Item (4c) of **Proposition 3.5**. □

4. Upper bound

We will prove that a finite complete family of pairwise incident planes in $\mathbb{P}^5$ has at most 20 elements. The key element in the proof is the following construction from [11]: given $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and $W \in \Theta_A$ we consider the locus

$$C_{W,A} := \{[v] \in \mathbb{P}(W) \mid \dim(F_v \cap A) \ge 2\}. \tag{4.0.1}$$

(Notice that $\dim(F_v \cap A) \ge 1$ for $[v] \in \mathbb{P}(W)$ because $\bigwedge^3 W \subset (F_v \cap A)$.) One describes $C_{W,A}$ as the degeneracy locus of a map between vector-bundles of rank 9: the fiber over $[v]$ of the domain is equal to $F_v/\bigwedge^3 W$, the codomain is the trivial vector-bundle with fiber $\bigwedge^3 W^{\perp}/\bigwedge^3 W$ - see [11] for details. It follows that either $C_{W,A} = \mathbb{P}(W)$ or else $C_{W,A}$ is a sextic curve. The link with our problem is the following. Suppose that $C_{W,A} \ne \mathbb{P}(W)$ and that $W' \in \Theta_A$ is distinct from W: then $\mathbb{P}(W \cap W')$ is contained in the singular locus of $C_{W,A}$. In order to state the relevant results from [11] we give a couple of definitions. Let $W \subset V$ be a subspace: we let

$$S_W := (\bigwedge^2 W) \wedge \mathbb{C}^6. \tag{4.0.2}$$

Definition 4.1. Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that $W \in \Theta_A$. We let $\mathcal{B}(W,A) \subset \mathbb{P}(W)$ be the set of $[v]$ such that one of the following holds:

(1) There exists $W' \in (\Theta_A \setminus \{W\})$ such that $[v] \in \mathbb{P}(W')$.
(2) $\dim(A \cap F_v \cap S_W) \ge 2$.

One checks easily that $\mathcal{B}(W,A)$ is closed subset of $\mathbb{P}(W)$.

Proposition 4.2. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that $W \in \Theta_A$. Then $C_{W,A} = \mathbb{P}(W)$ if and only if $\mathcal{B}(W,A) = \mathbb{P}(W)$. Suppose that $C_{W,A} \ne \mathbb{P}(W)$: then every non-reduced component of $C_{W,A}$ is contained in $\mathcal{B}(W,A)$.*

[2] Suppose that the quadric Q_0 contains $p_1, \ldots, p_{16}$. There exist 16 planes $L_1, \ldots, L_{16} \subset \mathbb{P}(U)$ such that each L_j contains 6 of the nodes of $\mathbf{K}$ and moreover $L_j \cdot \mathbf{K} = 2C_j$ where C_j is a smooth conic - see for example Exercise VIII.5 of [1]. It follows that Q_0 contains $C_1, \ldots, C_{16}$ and hence $Q_0 \cap \mathbf{K}$ has degree at least 32: that contradicts Bézout.

Proof. The first statement follows from Corollary 3.2.7 of [11]. The second statement follows from Proposition 3.2.6 of [11]. □

Lemma 4.3. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$. Suppose that Θ_A is finite of cardinality at least 15. Then there exists $W \in \Theta_A$ such that $C_{W,A}$ is a reduced curve.*

Proof. By contradiction. Assume that for every $W \in \Theta_A$ one of the following holds:

(1) $C_{W,A} = \mathbb{P}(W)$.
(2) $C_{W,A}$ is a non-reduced curve.

By **Proposition 4.2** we get that $\dim \mathcal{B}(W,A) \geq 1$. Let $W' \in (\Theta_A \setminus \{W\})$: since Θ_A is finite the planes $\mathbb{P}(W)$ and $\mathbb{P}(W')$ intersect in a single point, see **Corollary 2.3**. It follows that for generic $[v] \in \mathcal{B}(W,A)$ there exists

$$\alpha \in \left((A \cap F_v \cap S_W) \setminus \bigwedge^3 W \right). \tag{4.0.3}$$

Given such α there is a unique $[v] \in \mathbb{P}(W)$ such that (4.0.3) holds. In fact suppose the contrary: then α is a decomposable element whose support is a $W' \in (\Theta_A \setminus \{W\})$ intersecting W in a 2-dimensional subspace, that contradicts the hypothesis that Θ_A is finite (see above). Since $\dim \mathcal{B}(W,A) \geq 1$ it follows that

$$\dim(A \cap S_W) \geq 3. \tag{4.0.4}$$

Thus $\mathbb{P}(A)$ intersects the projective tangent space to $\mathrm{Gr}(2,\mathbb{P}^5)$ (embedded by Plücker) at $\mathbb{P}(W)$ in a linear space of dimension at least 2. Now let $\Omega \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^6)$ be a generic 10-dimensional projective space containing $\mathbb{P}(A)$. Notice that

$$\dim \Omega + \dim \mathrm{Gr}(2,\mathbb{P}^5) = 19 = \dim \mathbb{P}(\bigwedge^3 \mathbb{C}^6).$$

The intersection $\Omega \cap \mathrm{Gr}(2,\mathbb{P}^5)$ is finite because by hypothesis $\boldsymbol{\Theta}_A = \mathbb{P}(A) \cap \mathrm{Gr}(2,\mathbb{P}^5)$ is finite. By (4.0.4) we get that Ω intersects the projective tangent space to $\mathrm{Gr}(2,\mathbb{P}^5)$ at $\mathbb{P}(W)$ in a linear space of dimension at least 2: thus

$$\mathrm{mult}_{\mathbb{P}(W)} \Omega \cdot \mathrm{Gr}(2,\mathbb{P}^5) \geq 3. \tag{4.0.5}$$

Since the cardinality of $\boldsymbol{\Theta}_A$ is at least 15 we get that $\Omega \cdot \mathrm{Gr}(2,\mathbb{P}^5) \geq 45$, that is a contradiction because $\deg \mathrm{Gr}(2,\mathbb{P}^5) = 42$, see p. 247 of [5]. □

Now let T be a finite complete family of pairwise incident planes in $\mathbb{P}^5$. By **Claim 3.2** there exists $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ such that $\boldsymbol{\Theta}_A = T$. Suppose that T has cardinality at least 15: by **Lemma 4.3** there exists $W \in \Theta_A$ such that $C_{W,A}$ is a reduced sextic curves. Let $W' \in (\Theta_A \setminus \{W\})$: by **Corollary 2.3** the intersection $\mathbb{P}(W) \cap \mathbb{P}(W')$ is a point. By **Proposition 4.2** the curve $C_{W,A}$ is singular at $\mathbb{P}(W) \cap \mathbb{P}(W')$. Thus we have a map

$$\begin{array}{ccc} \Theta_A \setminus \{W\} & \xrightarrow{\varphi} & \mathrm{sing}\, C_{W,A} \\ W' & \mapsto & \mathbb{P}(W') \cap \mathbb{P}(W) \end{array} \tag{4.0.6}$$

There are at most 15 singular points of $C_{W,A}$ (the maximum 15 is achieved by sextics which are the union of 6 generic lines): it follows that if φ is injective then $\boldsymbol{\Theta}_A = T$ has at most 16 elements. Since φ is not necessarily injective we will need to answer the following question: what is the relation between the cardinality of $\varphi^{-1}(p)$ and the singularity of $C_{W,A}$ at p? First we will recall how to compute the initial terms in the Taylor expansion of a local equation of $C_{W,A}$ at a given point $[v_0] \in \mathbb{P}(W)$ - here $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and $W \in \Theta_A$ are arbitrary. Let $[w] \in \mathbb{P}(W)$; we let

$$G_w := F_w / \bigwedge^3 W. \tag{4.0.7}$$

Let $W_0 \subset W$ be a subspace complementary to $[v_0]$. We have an isomorphism

$$\begin{array}{ccc} W_0 & \overset{\sim}{\longrightarrow} & \mathbb{P}(W) \setminus \mathbb{P}(W_0) \\ w & \mapsto & [v_0 + w] \end{array} \tag{4.0.8}$$

onto a neighborhood of $[v_0]$; thus $0 \in W_0$ is identified with $[v_0]$. We have

$$C_{W,A} \cap W_0 = V(g_0 + g_1 + \cdots + g_6), \qquad g_i \in \mathrm{S}^i\, W_0^\vee. \tag{4.0.9}$$

Given $w \in W$ we define the Plücker quadratic form $\psi_w^{v_0}$ on G_{v_0} as follows. Let $\overline{\alpha} \in G_{v_0}$ be the equivalence class of $\alpha \in F_{v_0}$. Thus $\alpha = v_0 \wedge \beta$ where $\beta \in \bigwedge^2 V$ is defined modulo $(\bigwedge^2 W + [v_0] \wedge V)$: we let

$$\psi_w^{v_0}(\overline{\alpha}) := \mathrm{vol}(v_0 \wedge w \wedge \beta \wedge \beta). \tag{4.0.10}$$

Proposition 4.4 (Prop. 3.1.2 of [11]). *Keep notation and hypotheses as above. Let $\overline{K} := A \cap F_{v_0}/\bigwedge^3 W$ (notice that $\overline{K} \subset G_{v_0}$) and $\overline{k} := \dim \overline{K} = \dim(A \cap F_{v_0}) - 1$. Then the following hold:*

(1) *$g_i = 0$ for $i < \overline{k}$.*
(2) *There exists $\mu \in \mathbb{C}^*$ such that*

$$g_{\overline{k}}(w) = \mu \det(\psi_w^{v_0}|_{\overline{K}}), \quad w \in W_0. \tag{4.0.11}$$

Next we will give a geometric interpretation of the right-hand side of (4.0.11). Choose a subspace $V_0 \subset \mathbb{C}^6$ complementary to $[v_0]$ and such that $V_0 \cap W = W_0$. Thus have isomorphisms

$$\begin{array}{ccc} \bigwedge^2 V_0 & \overset{\sim}{\longrightarrow} & F_{v_0} \\ \beta & \mapsto & v_0 \wedge \beta \end{array} \tag{4.0.12}$$

and

$$\begin{array}{ccc} \bigwedge^2 V_0/\bigwedge^2 W_0 & \overset{\sim}{\longrightarrow} & G_{v_0} \\ \overline{\beta} & \mapsto & \overline{v_0 \wedge \beta}. \end{array} \tag{4.0.13}$$

Let $\psi_w^{v_0}$ be as in (4.0.10): we will view it as a quadratic form on $\bigwedge^2 V_0/\bigwedge^2 W_0$ via (4.0.13). Let $V(\psi_w^{v_0}) \subset \mathbb{P}(\bigwedge^2 V_0/\bigwedge^2 W_0)$ be the zero-locus of $\psi_w^{v_0}$. Let

$$\widetilde{\rho}\colon \mathbb{P}(\bigwedge^2 V_0) \dashrightarrow \mathbb{P}(\bigwedge^2 V_0/\bigwedge^2 W_0) \tag{4.0.14}$$

be projection with center $\bigwedge^2 W_0$. Let

$$\mathbb{G}r(2, V_0)_{W_0} := \widetilde{\rho}(\mathbb{G}r(2, V_0)). \tag{4.0.15}$$

(The right-hand side is to be interpreted as the closure of $\widetilde{\rho}(\mathbb{G}r(2, V_0) \setminus \{\bigwedge^2 W_0\})$.) Let ρ be the restriction of $\widetilde{\rho}$ to $\mathbb{G}r(2, V_0)$. The rational map

$$\rho\colon \mathbb{G}r(2, V_0) \dashrightarrow \mathbb{G}r(2, V_0)_{W_0} \tag{4.0.16}$$

is birational because $\mathbb{G}r(2, V_0)$ is cut out by quadrics. The following is an easy exercise, see Claim 3.5 of [11].

Claim 4.5. *Keep notation as above. Then*

$$\bigcap_{w \in W_0} V(\psi_w^{v_0}) = \mathbb{G}r(2, V_0)_{W_0} \tag{4.0.17}$$

and the scheme-theoretic intersection on the left is reduced.

Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that $W \in \Theta_A$. Let $p \in \mathbb{P}(W)$. We let

$$n_p := \#\{W' \in (\Theta_A \setminus \{W\}) \mid p \in \mathbb{P}(W')\}. \tag{4.0.18}$$

Notice that if $n_p > 0$ then $p \in C_{W,A}$.

Proposition 4.6. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and suppose that Θ_A is finite. Assume that $W \in \Theta_A$. Let $p \in \mathbb{P}(W)$.*

(1) $n_p \le 4$.
(2) *Assume in addition that* $C_{W,A}$ *is a curve. Then the following hold:*
  (2a) *If* $n_p = 2$ *then either* $C_{W,A}$ *has a cusp*[3] *at* p *or else* $\mathrm{mult}_p C_{W,A} \ge 3$.
  (2b) *If* $n_p = 3$ *or* $n_p = 4$ *then* $\mathrm{mult}_p C_{W,A} \ge 3$

Proof. Throughout the proof we will let $p = [v_0]$. Let $K := A \cap F_{v_0}$: we will view K as a subspace of $\bigwedge^2 V_0$ via Isomorphism (4.0.12).

(1): Suppose that $n_p > 4$. We claim that $\dim K \ge 4$. In fact suppose that $\dim K \le 3$ i.e. $\dim \mathbb{P}(K) \le 2$. Since $n_p \ge 5$ the intersection $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ contains at least 6 points: that is absurd because $\mathrm{Gr}(2, V_0)$ is cut out by quadrics and the intersection $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is finite (recall that Θ_A is finite by hypothesis). This proves that $\dim K \ge 4$. Since $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is finite we get that $\dim \mathbb{P}(K) \le 3$ and hence $\dim \mathbb{P}(K) = 3$. Since the degree of $\mathrm{Gr}(2, V_0)$ is 5 and $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ contains at least 6 points we get that $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is infinite: that is a contradiction.

(2a): If $\dim K \ge 4$ then $\mathrm{mult}_p C_{W,A} \ge 3$ by Item (1) of **Proposition 4.4**. Suppose that $\dim K < 4$ i.e. $\dim \mathbb{P}(K) \le 2$. By hypothesis $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ is finite and contains 3 points. Since $\mathrm{Gr}(2, V_0)$ is cut out by quadrics it follows that $\dim \mathbb{P}(K) = 2$. Let $g_0, \ldots, g_6$ be as in (4.0.9). Then $0 = g_0 = g_1$ because $\dim K = 3$ (see Item (1) of **Proposition 4.4**) and g_2 is given by (4.0.11). Let $\widetilde{\rho}$ be the projection of (4.0.14). The closure of $\widetilde{\rho}(\mathbb{P}(K) \setminus \bigwedge^2 W_0)$ is a line intersecting $\mathrm{Gr}(2, V_0)_{W_0}$ in two distinct points, namely the images under projection of the two points belonging to $(\mathbb{P}(K) \setminus \bigwedge^2 W_0) \cap \mathrm{Gr}(2, V_0)$. By (4.0.9) and **Claim 4.5** we get that $g_2 = l^2$ where $0 \ne l \in W_0^\vee$: thus $C_{W,A}$ has a cusp at p.

(2b): We will prove that $\dim K \ge 4$ - then $\mathrm{mult}_p C_{W,A} \ge 3$ will follow from Item (1) of **Proposition 4.4**. Assume that $\dim K < 4$. Suppose that $n_p = 3$. Then $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ has cardinality 4. Since $\mathrm{Gr}(2, V_0)$ is cut out by quadrics we get that $\dim \mathbb{P}(K) = 2$ and no three among the points of $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ are collinear. Now project $\mathbb{P}(K)$ from $\bigwedge^2 W_0$ - see (4.0.14): we get that $\widetilde{\rho}(\mathbb{P}(K) \setminus \bigwedge^2 W_0)$ is a line intersecting $\mathrm{Gr}(2, V_0)_{W_0}$ in three distinct points, that contradicts **Claim 4.5**. We have proved that if $n_p = 3$ then $\mathrm{mult}_p C_{W,A} \ge 3$. Lastly suppose that $n_p = 4$. Then $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0)$ has cardinality 5 and $\dim \mathbb{P}(K) \le 2$: that is absurd because $\mathrm{Gr}(2, V_0)$ is cut out by quadrics. □

Now let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and assume that Θ_A is finite of cardinality at least 15. By **Lemma 4.3** there exists $W \in \Theta_A$ such that $C_{W,A}$ is a reduced curve. We let

$$L_j := \{p \in \mathbb{P}(W) \mid n_p = j\}, \qquad \ell_j := \# L_j. \tag{4.0.19}$$

By **Proposition 4.6** we have that $\ell_j = 0$ for $j > 4$ and hence

$$\#\Theta_A = 1 + \ell_1 + 2\ell_2 + 3\ell_3 + 4\ell_4. \tag{4.0.20}$$

Lemma 4.7. *Let* $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ *and assume that* Θ_A *is finite of cardinality at least* 15*. Let* $W \in \Theta_A$ *be such that* $C_{W,A}$ *is a reduced curve and keep notation as above. Let* s *be the number of irreducible components of* $C_{W,A}$*. Then*

$$\ell_1 + \ell_2 + 3\ell_3 + 3\ell_4 \le 9 + s. \tag{4.0.21}$$

Proof. Let $C := C_{W,A}$ and $\mu\colon Z \to \mathbb{P}^2$ be a series of blow-ups that desingularize C i.e. such that the strict transform $\widetilde{C} \subset Z$ is smooth. Then

$$-2s \le 2(h^0(K_{\widetilde{C}}) - h^1(K_{\widetilde{C}})) = 2\chi(K_{\widetilde{C}}) = \widetilde{C} \cdot \widetilde{C} + \widetilde{C} \cdot K_Z. \tag{4.0.22}$$

[3] By cusp we mean a plane curve singularity with tangent cone which is quadratic of rank 1.

On the other hand let $p \in \mathbb{P}(W)$: if $n_p \geq 1$ then C is singular at p and if $n_p \geq 3$ then the multiplicity of C at p is at least 3, see **Proposition 4.6**. It follows that

$$\widetilde{C}\cdot\widetilde{C}+\widetilde{C}\cdot K_Z \leq (C\cdot C+C\cdot K_{\mathbb{P}^2})-2(\ell_1+\ell_2)-6(\ell_3+\ell_4) = 18-2(\ell_1+\ell_2)-6(\ell_3+\ell_4). \tag{4.0.23}$$

The proposition follows from (4.0.22) and (4.0.23). □

The result below completes the proof of **Theorem 1.1**.

Proposition 4.8. *Let $A \in \mathbb{LG}(\bigwedge^3 \mathbb{C}^6)$ and assume that Θ_A is finite. Then*

$$\#\Theta_A \leq 20 \tag{4.0.24}$$

Proof. We may assume that $\#\Theta_A > 16$. By **Lemma 4.3** there exists $W \in \Theta_A$ such that $C_{W,A}$ is a reduced curve. Let L_j and ℓ_j be as in (4.0.19) and s be the number of irreducible components of $C_{W,A}$. We recall that $C_{W,A}$ is singular at each point of L_1, it has either a cusp or a point of multiplicity at least 3 at each point of L_2 and it has multiplicity at least 3 at each point of $L_3 \cup L_4$, see **Proposition 4.6**. By (4.0.20) we have that

$$16 \leq \ell_1 + 2\ell_2 + 3\ell_3 + 4\ell_4 \tag{4.0.25}$$

The proof consists of a case-by-case analysis. Suppose first that $s = 1$. Assume that $(\ell_3 + \ell_4) = 0$. Applying Plücker's formulae to $C_{W,A}$ (notice that $\deg C^{\vee}_{W,A} \geq 3$) we get that $(2\ell_1 + 3\ell_2) \leq 27$. It follows that $\#\Theta_A \leq 19$ (recall (4.0.20)) - the "worst" case being $C_{W,A}$ the dual of a smooth cubic i.e. a sextic with 9 cusps. Assume that $(\ell_3 + \ell_4) = 1$. Then $(\ell_1 + \ell_2) \leq 7$ by **Lemma 4.7**: it follows that $\#\Theta_A \leq 19$. If $(\ell_3 + \ell_4) = 2$ then $(\ell_1 + \ell_2) \leq 4$ by **Lemma 4.7**: it follows that $\#\Theta_A = 17$. Next suppose that $s = 2$. A similar analysis shows that necessarily[4] $C_{W,A} = D + L$ where D is an irreducible quintic with 4 cusps (the points of L_2) and 2 nodes (the points of L_4), L is the line through the nodes of D: thus $\#\Theta_A = 17$. Lastly suppose that $s \geq 3$. Then $C_{W,A} = D_1 + D_2 + D_3$ where D_1, D_2 and D_3 are reduced conics (eventually reducible) belonging to the same pencil with reduced base locus (which is equal to $L_3 \cup L_4$). We have $\#\Theta_A \leq (17 + \delta)$ where δ is the number of singular conics among $\{D_1, D_2, D_3\}$. □

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[4]Notice that if an irreducible plane quintic has 5 cusps then it is smooth elsewhere (project the quintic form a hypothetical singular point distinct from the 5 cusps: you will contradict Hurwitz' formula).