PERIODS OF DOUBLE EPW-SEXTICS

KIERAN G. O'GRADY

"SAPIENZA" UNIVERSITÀ DI ROMA

0. Introduction

Let V be a 6-dimensional complex vector space. Let $\mathbb{LG}(\bigwedge^3 V) \subset \mathrm{Gr}(10, \bigwedge^3 V)$ be the symplectic Grassmannian parametrizing subspaces which are lagrangian for the symplectic form given by wedge-product. Given $A \in \mathbb{LG}(\bigwedge^3 V)$ we let

$$Y_A := \{[v] \in \mathbb{P}(V) \mid A \cap (v \wedge \bigwedge^2 V) \neq \{0\}\}.$$

Then Y_A is a degeneracy locus and hence it is naturally a subscheme of $\mathbb{P}(V)$. For certain pathological choices of A we have $Y_A = \mathbb{P}(V)$: barring those cases Y_A is a sextic hypersurface named *EPW-sextic*. An EPW-sextic comes equipped with a double cover [21]

$$f_A \colon X_A \to Y_A. \tag{0.0.1}$$

X_A is what we call a *double EPW-sextic*. There is an open dense subset $\mathbb{LG}(\bigwedge^3 V)^0 \subset \mathbb{LG}(\bigwedge^3 V)$ parametrizing smooth double EPW-sextics - these 4-folds are hyperkähler (HK) deformations of the Hilbert square of a $K3$ (i.e. the blow-up of the diagonal in the symmetric product of a $K3$ surface), see [18]. By varying $A \in \mathbb{LG}(\bigwedge^3 V)^0$ one gets a locally versal family of HK varieties - one of the five known such families in dimensions greater than 2, see [3, 4, 9, 10, 11] for the construction of the other families. The complement of $\mathbb{LG}(\bigwedge^3 V)^0$ in $\mathbb{LG}(\bigwedge^3 V)$ is the union of two prime divisors, Σ and Δ; the former consists of those A containing a non-zero decomposable tri-vector, the latter is defined in **Subsection 1.5**. If A is generic in Σ then X_A is singular along a $K3$ surface, see **Corollary 3.2**, if A is generic in Δ then X_A is singular at a single point whose tangent cone is isomorphic to the contraction of the zero-section of the cotangent sheaf of $\mathbb{P}^2$, see Prop. 3.10 of [21]. By associating to $A \in \mathbb{LG}(\bigwedge^3 V)^0$ the Hodge structure on $H^2(X_A)$ one gets a regular map of quasi-projective varieties [7]

$$\mathcal{P}^0 \colon \mathbb{LG}(\bigwedge^3 V)^0 \longrightarrow \mathbb{D}_\Lambda \tag{0.0.2}$$

where $\mathbb{D}_\Lambda$ is a quasi-projective period domain, the quotient of a bounded symmetric domain of Type IV by the action of an arithmetic group, see **Subsection 1.6**. Let $\mathbb{D}_\Lambda^{BB}$ be the Baily-Borel compactification of $\mathbb{D}_\Lambda$ and

$$\mathcal{P} \colon \mathbb{LG}(\bigwedge^3 V) \dashrightarrow \mathbb{D}_\Lambda^{BB} \tag{0.0.3}$$

the rational map defined by (0.0.2). The map $\mathcal{P}$ descends to the GIT-quotient of $\mathbb{LG}(\bigwedge^3 V)$ for the natural action of $\mathrm{PGL}(V)$. More precisely: the action of $\mathrm{PGL}(V)$ on $\mathbb{LG}(\bigwedge^3 V)$ is uniquely linearized and hence there is an unambiguous GIT quotient

$$\mathfrak{M} := \mathbb{LG}(\bigwedge^3 V)/\!/ \mathrm{PGL}(V). \tag{0.0.4}$$

Let $\mathbb{LG}(\bigwedge^3 V)^{\mathrm{st}}, \mathbb{LG}(\bigwedge^3 V)^{\mathrm{ss}} \subset \mathbb{LG}(\bigwedge^3 V)$ be the loci of (GIT) stable and semistable points of $\mathbb{LG}(\bigwedge^3 V)$. By [22] the open $\mathrm{PGL}(V)$-invariant subset $\mathbb{LG}(\bigwedge^3 V)^0$ is contained in $\mathbb{LG}(\bigwedge^3 V)^{\mathrm{st}}$: we let

$$\mathfrak{M}^0 := \mathbb{LG}(\bigwedge^3 V)^0/\!/ \mathrm{PGL}(V). \tag{0.0.5}$$

Then $\mathcal{P}$ descends to a rational map

$$\mathfrak{p} \colon \mathfrak{M} \dashrightarrow \mathbb{D}_\Lambda^{BB} \tag{0.0.6}$$

Date: September 28 2013.

Supported by PRIN 2010.

which is regular on $\mathfrak{M}^0$. By Verbitsky's global Torelli Theorem and Markman's monodromy results the restriction of $\mathfrak{p}$ to $\mathfrak{M}^0$ is injective, see Theorem 1.3 and Lemma 9.2 of [15]. Since domain and codomain of the period map have the same dimension it follows that $\mathfrak{p}$ is a birational map. In the present paper we will be mainly concerned with the following problem: what is the indeterminacy locus of $\mathfrak{p}$? In order to state our main results we will go through a few more definitions. Given $A \in \mathbb{LG}(\bigwedge^3 V)$ we let

$$\Theta_A := \{W \in \mathrm{Gr}(3, V) \mid \bigwedge^3 W \subset A\}. \tag{0.0.7}$$

Thus $A \in \Sigma$ if and only if $\Theta_A \neq \emptyset$. Suppose that $W \in \Theta_A$: there is a natural determinantal subscheme $C_{W,A} \subset \mathbb{P}(W)$, see [22], with the property that

$$\operatorname{supp} C_{W,A} = \{[v] \in \mathbb{P}(W) \mid \dim(A \cap (v \wedge \bigwedge^2 V)) \geq 2\}. \tag{0.0.8}$$

$C_{W,A}$ is either a sextic curve or (in pathological cases) $\mathbb{P}(W)$. Let

$$|\mathcal{O}_{\mathbb{P}(W)}(6)| \dashrightarrow \mathbb{D}_{\Phi}^{BB} \tag{0.0.9}$$

be the compactified period map where $\mathbb{D}_{\Phi}^{BB}$ is the Baily-Borel compactification of the period space for $K3$ surfaces of degree 2, see [23].

Definition 0.1. (1) Let $\mathfrak{M}^{ADE} \subset \mathfrak{M}$ be the subset of points represented by $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ for which the following hold:
- (1a) The orbit $\mathrm{PGL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$.
- (1b) For all $W \in \Theta_A$ we have that $C_{W,A}$ is a sextic curve with simple singularities.

(2) Let $\mathfrak{I} \subset \mathfrak{M}$ be the subset of points represented by $A \in \mathbb{LG}(\bigwedge^3 V)^{ss}$ for which the following hold:
- (2a) The orbit $\mathrm{PGL}(V)A$ is closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$.
- (2b) There exists $W \in \Theta_A$ such that $C_{W,A}$ is either $\mathbb{P}(W)$ or a sextic curve in the indeterminacy locus of (0.0.9).

Then $\mathfrak{M}^{ADE}$, $\mathfrak{I}$ are respectively open and closed subsets of $\mathfrak{M}$, and since every point of $\mathfrak{M}$ is represented by a single closed $\mathrm{PGL}(V)$-orbit $\mathfrak{I}$ is in the complement of $\mathfrak{M}^{ADE}$. Below is the main result of the present paper.

Theorem 0.2. *The period map $\mathfrak{p}$ is regular away from $\mathfrak{I}$. Let $x \in (\mathfrak{M} \setminus \mathfrak{I})$: then $\mathfrak{p}(x) \in \mathbb{D}_{\Lambda}$ if and only if $x \in \mathfrak{M}^{ADE}$.*

The above result is a first step towards an understanding of the rational map $\mathfrak{p}\colon \mathfrak{M} \dashrightarrow \mathbb{D}_{\Lambda}^{BB}$. Such an understanding will eventually include a characterization of the image of $(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$. (Notice that if $A \in (\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$ then A is a stable point by [22] and hence $[A] \in \mathfrak{M}^{ADE}$ because Θ_A is empty.) We will give a preliminary result, namely we will prove that $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$ is contained in the complement of the union of four arithmetically defined prime divisors in $\mathbb{D}_{\Lambda}$ named , $\mathbb{S}_2^{\star}$, $\mathbb{S}_2'$, $\mathbb{S}_2''$ and $\mathbb{S}_4$. The union of those divisors may be described as the set of Hodge structures which have a $(1,1)$-class which is a root of negative square (see **Subsection 1.7** for details).

Theorem 0.3. *The period map $\mathcal{P}$ maps $(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$ into $(\mathbb{D}_{\Lambda} \setminus (\mathbb{S}_2^{\star} \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4))$.*

Let us briefly summarize the main intermediate results of the paper and the proofs of **Theorem 0.2** and **Theorem 0.3**. In **Subsection 1.7** we will define the prime divisor $\overline{\mathbb{S}}_2^{\star}$ of $\mathbb{D}_{\Lambda}^{BB}$; later we will prove that the closure of $\mathcal{P}(\Sigma)$ is equal to $\overline{\mathbb{S}}_2^{\star}$. In **Subsection 1.7** we will show that the normalization of $\overline{\mathbb{S}}_2^{\star}$ is equal to the Baily-Borel compactification $\mathbb{D}_{\Gamma}^{BB}$ of the quotient of a bounded symmetric domain of Type IV modulo an arithmetic group and we will define a natural finite map $\rho\colon \mathbb{D}_{\Gamma}^{BB} \to \mathbb{D}_{\Phi}^{BB}$ where $\mathbb{D}_{\Phi}^{BB}$ is as in (0.0.9) - the map ρ will play a key rôle in the proof of **Theorem 0.2**. In **Section 2** we will prove that $\mathcal{P}$ is regular away from a certain closed subset of Σ which has codimension 4 in $\mathbb{LG}(\bigwedge^3 V)$. The idea of the proof is the following. Suppose that X_A is smooth and $\mathbf{L} \subset \mathbb{P}(V)$ is a 3-dimensional linear subspace such that $f_A^{-1}(Y_A \cap \mathbf{L})$ is smooth: by Lefschetz' Hyperplane Theorem the periods of X_A inject into the periods of $f_A^{-1}(Y_A \cap \mathbf{L})$. This together with Griffiths' Removable Singularity Theorem gives that the period map extends regularly over the subset of $\mathbb{LG}(\bigwedge^3 V)$ parametrizing those A for which $f_A^{-1}(Y_A \cap \mathbf{L})$ has at most rational double points for generic $\mathbf{L} \subset \mathbb{P}(V)$ as above. We will prove that the latter condition holds away from the union of the subsets of Σ denoted $\Sigma[2]$ and Σ_{∞}, see **Proposition 2.4**. One gets the stated result because the codimensions in $\mathbb{LG}(\bigwedge^3 V)$ of $\Sigma[2]$ and Σ_{∞} are 4 and 7 respectively, see (1.4.3) and (1.4.7). Let $\widehat{\mathbb{LG}}(\bigwedge^3 V) \subset \mathbb{LG}(\bigwedge^3 V) \times \mathbb{D}_\Lambda^{BB}$ be the closure of the graph of $\mathcal{P}$. Since $\mathbb{LG}(\bigwedge^3 V)$ is smooth the projection $p\colon \widehat{\mathbb{LG}}(\bigwedge^3 V) \to \mathbb{LG}(\bigwedge^3 V)$ is identified with the blow-up of the indeterminacy locus of $\mathcal{P}$ and hence the exceptional set of p is the support of the exceptional Cartier divisor of the blow-up. Let $\widehat{\Sigma} \subset \widehat{\mathbb{LG}}(\bigwedge^3 V)$ be the strict transform of Σ. The results of **Section 2** described above give that if A is in the indeterminacy locus of $\mathcal{P}$ (and hence $A \in \Sigma$) then

$$\dim(p^{-1}(A) \cap \widehat{\Sigma}) \geq 2. \tag{0.0.10}$$

Section 3 starts with an analysis of X_A for generic $A \in \Sigma$: we will prove that it is singular along a $K3$ surface S_A which is a double cover of $\mathbb{P}(W)$ where W is the unique element of Θ_A (unique because A is generic in Σ) and that the blow-up of X_A with center S_A - call it $\widetilde{X}_A$ - is a smooth HK variety deformation equivalent to smooth double EPW-sextics, see **Corollary 3.2** and **Corollary 3.6**. It follows that $\mathcal{P}(A)$ is identified with the weight-2 Hodge structure of $\widetilde{X}_A$. Let ζ_A be the Poincaré dual of the exceptional divisor of the blow-up $\widetilde{X}_A \to X_A$. Then ζ_A is a (-2)-root of divisibility 1 perpendicular to the pull-back of $c_1(\mathcal{O}_{Y_A}(1))$: it follows that $\mathcal{P}(A) \in \overline{\mathbb{S}}_2^\star$ and that $\mathcal{P}(\Sigma)$ is dense in $\overline{\mathbb{S}}_2^\star$, see **Proposition 3.13**. We will also define an index-2 inclusion of integral Hodge structures $\zeta_A^\perp \hookrightarrow H^2(S_A;\mathbb{Z})$, see (3.6.7), and we will show that the inclusion may be identified with the value at $\mathcal{P}(A)$ of the finite map $\rho\colon \mathbb{D}_\Gamma^{BB} \to \mathbb{D}_\Phi^{BB}$ mentioned above (this makes sense because ρ is the map associated to an extension of lattices), see (3.6.6). Let

$$\widetilde{\Sigma} := \{(W, A) \in \mathrm{Gr}(3, V) \times \mathbb{LG}(\bigwedge^3 V) \mid \bigwedge^3 W \subset A\}. \tag{0.0.11}$$

The natural forgetful map $\widetilde{\Sigma} \to \Sigma$ is birational (for general $A \in \Sigma$ there is a unique $W \in \mathrm{Gr}(3, V)$ such that $\bigwedge^3 W \subset A$); since the period map is regular at the generic point of Σ it induces a rational map $\widetilde{\Sigma} \dashrightarrow \overline{\mathbb{S}}_2^\star$ and hence a rational map to its normalization

$$\widetilde{\Sigma} \dashrightarrow \mathbb{D}_\Gamma^{BB}. \tag{0.0.12}$$

Let $(W, A) \in \widetilde{\Sigma}$ and suppose that $C_{W,A}$ is a sextic (i.e. $C_{W,A} \neq \mathbb{P}(W)$) and the period map (0.0.9) is regular at $C_{W,A}$: the relation described above between the Hodge structures on $\zeta_A^\perp$ and $H^2(S_A)$ gives that Map (0.0.12) is regular at (W, A). Now let $x \in (\mathfrak{M} \setminus \mathfrak{I})$ and suppose that x is in the indeterminacy locus of the rational period map $\mathfrak{p}$. One reaches a contradiction arguing as follows. Let $A \in \mathbb{LG}(\bigwedge^3 V)$ be semistable with orbit closed in $\mathbb{LG}(\bigwedge^3 V)^{ss}$ and representing x. Results of [22] and [20] give that $\dim \Theta_A \leq 1$; this result combined with the regularity of (0.0.12) at all (W, A) with $W \in \Theta_A$ gives that $\dim(p^{-1}(A) \cap \widehat{\Sigma}) \leq 1$: that contradicts (0.0.10) and hence proves that $\mathfrak{p}$ is regular at x (it proves also the last clause in the statement of **Theorem 0.2**). In **Section 4** we will prove **Theorem 0.3**. The main ingredients of the proof are Verbitsky's Global Torelli Theorem and our knowledge of degenerate EPW-sextics whose periods fill out open dense substes of the divisors $\mathbb{S}_2^\star$, $\mathbb{S}_2'$, $\mathbb{S}_2''$ and $\mathbb{S}_4$. Here "degenerate"means that we have a hyperkähler deformation of the Hilbert square of a $K3$ and a map $f\colon X \to \mathbb{P}^5$: while X is *not* degenerate, the map is degenerate in the sense that it is not a double cover of its image, either it has (some) positive dimensional fibers (as in the case of $\mathbb{S}_2^\star$ that we discussed above) or it has higher degree onto its image.

Acknowledgments. Thanks go to Olivier Debarre for many suggestions on how to improve the exposition.

1. Preliminaries

1.1. **Local equation of EPW-sextics.** We will recall notation and results from [20]. Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and $[v_0] \in \mathbb{P}(V)$. Choose a direct-sum decomposition

$$V = [v_0] \oplus V_0. \tag{1.1.1}$$

We identify V_0 with the open affine $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ via the isomorphism

$$\begin{array}{ccc} V_0 & \overset{\sim}{\longrightarrow} & \mathbb{P}(V) \setminus \mathbb{P}(V_0) \\ v & \mapsto & [v_0 + v]. \end{array} \tag{1.1.2}$$

Thus $0 \in V_0$ corresponds to $[v_0]$. Then

$$Y_A \cap V_0 = V(f_0 + f_1 + \cdots + f_6), \qquad f_i \in \mathrm{S}^i V_0^\vee. \tag{1.1.3}$$

The following result collects together statements contained in Corollary 2.5 and Proposition 2.9 of [20].

Proposition 1.1. *Keep assumptions and hypotheses as above. Let* $k := \dim(A \cap (v_0 \wedge \bigwedge^2 V))$.

(1) *Suppose that there is no* $W \in \Theta_A$ *containing* v_0. *Then the following hold:*
  (1a) $0 = f_0 = \ldots = f_{k-1}$ *and* $f_k \neq 0$.
  (1b) *Suppose that* $k = 2$ *and hence* $[v_0] \in Y_A(2)$. *Then* $Y_A(2)$ *is smooth two-dimensional at* $[v_0]$.

(2) *Suppose that there exists* $W \in \Theta_A$ *containing* v_0. *Then* $0 = f_0 = f_1$.

Next we recall how one describes $Y_A \cap V_0$ under the following assumption:

$$\bigwedge^3 V_0 \cap A = \{0\}. \tag{1.1.4}$$

Decomposition (1.1.1) determines a direct-sum decomposition $\bigwedge^3 V = [v_0] \wedge \bigwedge^2 V_0 \oplus \bigwedge^3 V_0$. We will identify $\bigwedge^2 V_0$ with $v_0 \wedge \bigwedge^2 V_0$ via

$$\begin{array}{ccc} \bigwedge^2 V_0 & \overset{\sim}{\longrightarrow} & v_0 \wedge \bigwedge^2 V_0 \\ \beta & \mapsto & v_0 \wedge \beta \end{array} \tag{1.1.5}$$

By (1.1.4) the subspace A is the graph of a linear map

$$\widetilde{q}_A \colon \bigwedge^2 V_0 \to \bigwedge^3 V_0. \tag{1.1.6}$$

Choose a volume-form

$$\mathrm{vol}_0 \colon \bigwedge^5 V_0 \overset{\sim}{\longrightarrow} \mathbb{C} \tag{1.1.7}$$

in order to identify $\bigwedge^3 V_0$ with $\bigwedge^2 V_0^\vee$. Then $\widetilde{q}_A$ is symmetric because $A \in \mathbb{LG}(\bigwedge^3 V)$. Explicitly

$$\widetilde{q}_A(\alpha) = \gamma \iff (v_0 \wedge \alpha + \gamma) \in A. \tag{1.1.8}$$

We let q_A be the associated quadratic form on $\bigwedge^2 V_0$. Notice that

$$\ker q_A = \{\alpha \in \bigwedge^2 V_0 \mid v_0 \wedge \alpha \in A \cap (v_0 \wedge \bigwedge^2 V)\} \tag{1.1.9}$$

is identified with $A \cap (v_0 \wedge \bigwedge^2 V)$ via (1.1.5). Let $v \in V_0$ and q_v be the Plücker quadratic form defined by

$$\begin{array}{ccc} \bigwedge^2 V_0 & \overset{q_v}{\longrightarrow} & \mathbb{C} \\ \alpha & \mapsto & \mathrm{vol}_0(v \wedge \alpha \wedge \alpha) \end{array} \tag{1.1.10}$$

Proposition 1.2 (Proposition 2.18 of [20]). *Keep notation and hypotheses as above, in particular* (1.1.4) *holds. Then*

$$Y_A \cap V_0 = V(\det(q_A + q_v)). \tag{1.1.11}$$

Next we will state a hypothesis which ensures existence of a decomposition (1.1.1) such that (1.1.4) holds. First recall [19] that we have an isomorphism

$$\begin{array}{ccc} \mathbb{LG}(\bigwedge^3 V) & \overset{\delta}{\underset{\sim}{\longrightarrow}} & \mathbb{LG}(\bigwedge^3 V^\vee) \\ A & \mapsto & \mathrm{Ann}\, A. \end{array} \tag{1.1.12}$$

Let $E \in \mathrm{Gr}(5, V)$; then

$$E \in Y_{\delta(A)} \text{ if and only if } (\bigwedge^3 E) \cap A \neq \{0\}. \tag{1.1.13}$$

(The EPW-sextic $Y_{\delta(A)}$ is the dual of Y_A.) Thus there exists a decomposition (1.1.1) such that (1.1.4) holds if and only if $Y_{\delta(A)} \neq \mathbb{P}(V^\vee)$. The proposition below follows at once from Claim 2.11 and Equation (2.82) of [20].

Proposition 1.3. *Let* $A \in \mathbb{LG}(\bigwedge^3 V)$ *and suppose that* $\dim \Theta_A \leq 2$. *Then*

$$Y_A \neq \mathbb{P}(V), \qquad Y_{\delta(A)} \neq \mathbb{P}(V^\vee).$$

In particular there exists a decomposition (1.1.1) *such that* (1.1.4) *holds.*

Let $A \in \mathbb{LG}(\bigwedge^3 V)$. We will need to consider higher degeneracy loci attached to A. Let

$$Y_A[k] = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap (v \wedge \bigwedge^2 V)) \geq k\}. \tag{1.1.14}$$

Notice that $Y_A[0] = \mathbb{P}(V)$ and $Y_A[1] = Y_A$. Moreover $A \in \Delta$ if and only if $Y_A[3]$ is not empty. We set

$$Y_A(k) := Y_A[k] \setminus Y_A[k+1]. \tag{1.1.15}$$

1.2. **Explicit description of double EPW-sextics.** Throughout the present subsection we will assume that $A \in \mathbb{LG}(\bigwedge^3 V)$ and $Y_A \neq \mathbb{P}(V)$. Let $f_A \colon X_A \to Y_A$ be the double cover of (0.0.1). The following is an immediate consequence of the definition of f_A, see [21]:

$$f_A \text{ is a topological covering of degree 2 over } Y_A(1). \tag{1.2.1}$$

Let $[v_0] \in Y_A$; we will give explicit equations for a neighborhood of $f_A^{-1}([v_0])$ in X_A. We will assume throughout the subsection that we are given a direct-sum decomposition (1.1.1) such that (1.1.4) holds. We start by introducing some notation. Let $K := \ker q_A$ and let $J \subset \bigwedge^2 V_0$ be a maximal subspace over which q_A is non-degenerate; we have a direct-sum decomposition

$$\bigwedge^2 V_0 = J \oplus K. \tag{1.2.2}$$

Choose a basis of $\bigwedge^2 V_0$ adapted to Decomposition (1.2.2). Let $k := \dim K$. The Gram matrices of q_A and q_v (for $v \in V_0$) relative to the chosen basis are given by

$$M(q_A) = \begin{pmatrix} N_J & 0 \\ 0 & 0_k \end{pmatrix}, \quad M(q_v) = \begin{pmatrix} Q_J(v) & R_J(v)^t \\ R_J(v) & P_J(v) \end{pmatrix}. \tag{1.2.3}$$

(We let 0_k be the $k \times k$ zero matrix.) Notice that N_J is invertible and $q_0 = 0$; thus there exist arbitrarily small open (in the classical topology) neighborhoods V_0' of 0 in V_0 such that $(N_J + Q_J(v))^{-1}$ exists for $v \in V_0'$. We let

$$M_J(v) := P_J(v) - R_J(v) \cdot (N_J + Q_J(v))^{-1} \cdot R_J(v)^t, \qquad v \in V_0'. \tag{1.2.4}$$

If V_0' is sufficiently small we may write $(N_J + Q_J(v)) = S(v) \cdot S(v)^t$ for all $v \in V_0'$ where $S(v)$ is an analytic function of v (for this we need V_0' to be open in the classical topology) and $S(v)$ is invertible for all $v \in V_0'$. Let $j := \dim J$. For later use we record the following equality

$$\begin{pmatrix} 1_j & 0 \\ -R_J(v)S^{-1}(v)^t & 1_k \end{pmatrix} \cdot \begin{pmatrix} S(v)^{-1} & 0 \\ 0 & 1_k \end{pmatrix} \cdot \begin{pmatrix} N_J+Q_J(v) & R_J(v)^t \\ R_J(v) & P_J(v) \end{pmatrix} \cdot \begin{pmatrix} S^{-1}(v)^t & 0 \\ 0 & 1_k \end{pmatrix} \cdot \begin{pmatrix} 1_j & -S^{-1}(v)R_J(v)^t \\ 0 & 1_k \end{pmatrix} = \begin{pmatrix} 1_j & 0 \\ 0 & M_J(v) \end{pmatrix} \tag{1.2.5}$$

Let $\mathbf{X}_J \subset V_0' \times \mathbb{C}^k$ be the closed subscheme whose ideal is generated by the entries of the matrices

$$M_J(v) \cdot \xi, \qquad \xi \cdot \xi^t - M_J(v)^c, \tag{1.2.6}$$

where $\xi \in \mathbb{C}^k$ is a column vector and $M_J(v)^c$ is the matrix of cofactors of $M_J(v)$. We identify V_0' with an open neighborhood of $[v_0] \in \mathbb{P}(V)$ via (1.1.2). Projection defines a map $\phi \colon \mathbf{X}_J \to V(\det M_J)$. By (1.2.5) we have $V(\det M_J) = V_0' \cap Y_A$.

Proposition 1.4. *Keep notation and assumptions as above. There exists a commutative diagram*

$$\begin{array}{ccc} (X_A, f_A^{-1}([v_0])) & \xrightarrow{\quad\zeta\quad} & (\mathbf{X}_J, \phi^{-1}([v_0])) \\ & {\scriptstyle f_A}\searrow \qquad \swarrow{\scriptstyle \phi} & \\ & (Y_A, [v_0]) & \end{array}$$

where the germs are in the analytic topology. Furthermore ζ is an isomorphism.

Proof. Let $[v] \in \mathbb{P}(V)$: there is a canonical identification between $v \wedge \bigwedge^2 V$ and the fiber at $[v]$ of $\Omega^3_{\mathbb{P}(V)}(3)$, see for example Proposition 5.11 of [18]. Thus we have an injection $\Omega^3_{\mathbb{P}(V)}(3) \hookrightarrow \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$. Choose $B \in \mathbb{LG}(\bigwedge^3 V)$ transversal to A. The direct-sum decomposition $\bigwedge^3 V = A \oplus B$ gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \Omega^3_{\mathbb{P}(V)}(3) & \xrightarrow{\lambda_A} & A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_A & \to & 0 \\ & & \downarrow{\scriptstyle \mu_{A,B}} & & \downarrow{\scriptstyle \mu^t_{A,B}} & & \downarrow{\scriptstyle \beta_A} & & \\ 0 & \to & A \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda^t_A} & \bigwedge^3 T_{\mathbb{P}(V)}(-3) & \longrightarrow & Ext^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) & \to & 0 \end{array} \tag{1.2.7}$$

(As suggested by our notation the map β_A does not depend on the choice of B.) Choosing B transverse to $v_0 \wedge \bigwedge^2 V$ we can assume that $\mu_{A,B}(0)$ (recall that $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ is identified with V_0 via (1.1.2) and that $[v_0]$ corresponds to 0) is an isomorphism. Then there exist arbitrarily small open (classical topology) neighborhoods $\mathcal{U}$ of 0 such that $\mu_{A,B}(v)$ is an isomorphism for all $v \in \mathcal{U}$. The map $\lambda_A \circ \mu_{A,B}^{-1}$ is symmetric because A is lagrangian. Choose a basis of A and let $L(v)$ be the Gram matrix of $\lambda_A \circ \mu_{A,B}^{-1}(v)$ with respect to the chosen basis. Let $L(v)^c$ be the matrix of cofactors of $L(v)$. Claim 1.3 of [21] gives an embedding

$$f_A^{-1}(\mathcal{U} \cap Y_A) \hookrightarrow \mathcal{U} \times \mathbb{A}^{10} \tag{1.2.8}$$

with image the closed subscheme whose ideal is generated by the entries of the matrices

$$L(v) \cdot \xi \qquad \xi \cdot \xi^t - L(v)^c. \tag{1.2.9}$$

(Here ξ is a 10×1-matrix whose entries are coordinates on $\mathbb{A}^{10}$.) We will denote the above subscheme by $V(L(v) \cdot \xi,\ \xi \cdot \xi^t - L(v)^c)$. Under this embedding the restriction of f_A to $f_A^{-1}(\mathcal{U} \cap Y_A)$ gets identified with the restriction of the projection $\mathcal{U} \times \mathbb{A}^{10} \to \mathcal{U}$. Let $G\colon \mathcal{U} \to \mathrm{GL}_{10}(\mathbb{C})$ be an analytic map and for $v \in \mathcal{U}$ let $H(v) := G^t(v) \cdot L(v) \cdot G(v)$. The automorphism of $\mathcal{U} \times \mathbb{A}^{10}$ given by $(v, \xi) \mapsto (v, G(v)^{-1}\xi)$ restricts to an isomorphism

$$V(L(v) \cdot \xi,\ \xi \cdot \xi^t - L(v)^c) \overset{\sim}{\longrightarrow} V(H(v) \cdot \xi,\ \xi \cdot \xi^t - H(v)^c). \tag{1.2.10}$$

In other words we are free to replace L by an arbitrary congruent matrix function. Let

$$\begin{array}{ccc} \bigwedge^2 V_0 & \overset{\phi_{v_0+v}}{\longrightarrow} & (v_0 + v) \wedge \bigwedge^2 V \\ \alpha & \mapsto & (v_0 + v) \wedge \alpha \end{array} \tag{1.2.11}$$

A straightforward computation gives that

$$\phi_{v_0+v}^t \circ \mu_{A,B}^t(v) \circ (\lambda_A(v) \circ \mu_{A,B}^{-1}(v)) \circ \mu_{A,B}(v) \circ \phi_{v_0+v} = \widetilde{q}_A + \widetilde{q}_v, \qquad v \in \mathcal{U}. \tag{1.2.12}$$

Thus the Gram matrix $M(q_A + q_v)$ is congruent to $L(v)$ and hence we have an embedding (1.2.8) with image $V(M(q_A + q_v) \cdot \xi,\ \xi \cdot \xi^t - M(q_A + q_v)^c)$. On the other hand (1.2.5) shows that $M(q_A + q_v)$ is congruent to the matrix

$$E(v) := \begin{pmatrix} 1_j & 0 \\ 0 & M_J(v) \end{pmatrix} \tag{1.2.13}$$

Thus we have an embedding (1.2.8) with image $V(E(v) \cdot \xi,\ \xi \cdot \xi^t - E(v)^c)$. A straightforward computation shows that the latter subscheme is isomorphic to $\mathbf{X}_J \cap (\mathcal{U} \times \mathbb{C}^k)$. □

1.3. **The subscheme $C_{W,A}$.** Let $(W, A) \in \widetilde{\Sigma}$. For the definition of the subscheme $C_{W,A} \subset \mathbb{P}(W)$ we refer to Subsection 3.1 of [22].

Definition 1.5. Let $\mathcal{B}(W, A) \subset \mathbb{P}(W)$ be the set of $[w]$ such that one of the following holds:

(1) There exists $W' \in (\Theta_A \setminus \{W\})$ containing w.
(2) $\dim(A \cap (w \wedge \bigwedge^2 V) \cap (\bigwedge^2 W \wedge V)) \geq 2$.

The following result is obtained by pasting together Proposition 3.2.6 and Corollary 3.2.7 of [22].

Proposition 1.6. *Let $(W, A) \in \widetilde{\Sigma}$. Then the following hold:*

(1) *$C_{W,A}$ is a smooth curve at $[v_0]$ if and only if $\dim(A \cap (v_0 \wedge \bigwedge^2 V)) = 2$ and $[v_0] \notin \mathcal{B}(W, A)$.*
(2) *$C_{W,A} = \mathbb{P}(W)$ if and only if $\mathcal{B}(W, A) = \mathbb{P}(W)$.*

1.4. **The divisor Σ.** Given $d \geq 0$ we let $\widetilde{\Sigma}[d] \subset \widetilde{\Sigma}$ be

$$\widetilde{\Sigma}[d] := \{(W, A) \in \widetilde{\Sigma} \mid \dim(A \cap (\bigwedge^2 W \wedge V)) \geq d + 1\}. \tag{1.4.1}$$

Notice that $\widetilde{\Sigma} := \widetilde{\Sigma}[0]$. Let

$$\mathrm{Gr}(3, V) \times \mathbb{LG}(\bigwedge^3 V) \overset{\pi}{\longrightarrow} \mathbb{LG}(\bigwedge^3 V) \tag{1.4.2}$$

be projection and $\Sigma[d] := \pi(\widetilde{\Sigma}[d])$. Notice that $\Sigma := \Sigma[0]$. Proposition 3.1 of [20] gives that

$$\mathrm{cod}(\Sigma[d], \mathbb{LG}(\bigwedge^3 V)) = (d^2 + d + 2)/2. \tag{1.4.3}$$

Let

$$\Sigma_+ := \{A \in \Sigma \mid \mathrm{Card}(\Theta_A) > 1\}. \tag{1.4.4}$$

Proposition 3.1 of [20] gives that Σ_+ is a constructible subset of $\mathbb{LG}(\bigwedge^3 V)$ and

$$\mathrm{cod}(\Sigma_+, \mathbb{LG}(\bigwedge^3 V)) = 2. \tag{1.4.5}$$

We claim that

$$\operatorname{sing}\Sigma = \Sigma_+ \cup \Sigma[1]. \tag{1.4.6}$$

In fact $(\overline{\Sigma}_+ \setminus \Sigma_+) \subset \Sigma[1]$ by Equation (3.19) of [20] and hence (1.4.6) follows from Proposition 3.2 of [20]. We let

$$\Sigma_\infty := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid \dim \Theta_A > 0\}. \tag{1.4.7}$$

Theorem 3.37 and Table 3 of [20] give the following:

$$\operatorname{cod}(\Sigma_\infty, \mathbb{LG}(\bigwedge^3 V)) = 7. \tag{1.4.8}$$

1.5. **The divisor Δ.** Let

$$\Delta := \{A \in \mathbb{LG}(\bigwedge^3 V) \mid \exists [v] \in \mathbb{P}(V) \text{ such that } \dim(A \cap (v \wedge \bigwedge^2 V)) \geq 3\}. \tag{1.5.1}$$

A dimension count gives that Δ is a prime divisor in $\mathbb{LG}(\bigwedge^3 V)$, see [21]. Let

$$\widetilde{\Delta}(0) := \{([v], A) \in \mathbb{P}(V) \times \mathbb{LG}(\bigwedge^3 V) \mid \dim(A \cap (v \wedge \bigwedge^2 V)) = 3\}. \tag{1.5.2}$$

The following result will be handy.

Proposition 1.7. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that $\dim Y_A[3] > 0$. Then $A \in (\Sigma_\infty \cup \Sigma[2])$.*

Proof. By contradiction. Thus we assume that $\dim Y_A[3] > 0$ and $A \notin (\Sigma_\infty \cup \Sigma[2])$. By hypothesis there exists an irreducible component C of $Y_A[3]$ of strictly positive dimension. Let $[v] \in C$ be generic. We claim that one of the following holds:

(a) There exist distinct $W_1([v]), W_2([v]) \in \Theta_A$ containing v.
(b) There exists $W([v]) \in \Theta_A$ containing v and such that

$$\dim A \cap S_{W([v])} \cap (v \wedge \bigwedge^2 V) \geq 2. \tag{1.5.3}$$

In fact assume first that $\dim(A \cap (v \wedge \bigwedge^2 V)) = 3$ for $[v]$ in an open dense $C^0 \subset C$. We may assume that C^0 is smooth; then we have an embedding $\iota\colon C^0 \hookrightarrow \widetilde{\Delta}(0)$ defined by mapping $[v] \in C^0$ to $([v], A)$. Let $[v] \in C^0$: the differential of the projection $\widetilde{\Delta}(0) \to \mathbb{LG}(\bigwedge^3 V)$ at $([v], A)$ is *not* injective because it vanishes on $\operatorname{Im} d\iota([v])$. By Corollary 3.4 and Proposition 3.5 of [21] we get that one of Items (a), (b) above holds. Now assume that $\dim(A \cap (v \wedge \bigwedge^2 V)) > 3$ for generic $[v] \in C$ (and hence for all $[v] \in C$). Let notation be as in the proof of Proposition 3.5 of [21]; then $\mathbf{K} \cap \operatorname{Gr}(2, V_0)$ is a zero-dimensional (if it has strictly positive dimension then $\dim \Theta_A > 0$ and hence $A \in \Sigma_\infty$ against our assumption) scheme of length 5. It follows that either Item (a) holds (if $\mathbf{K} \cap \operatorname{Gr}(2, V_0)$ is not a single point) or Item (b) holds (if $\mathbf{K} \cap \operatorname{Gr}(2, V_0)$ is a single point $\mathbf{p}$ and hence the tangent space of $\mathbf{K} \cap \operatorname{Gr}(2, V_0)$ at $\mathbf{p}$ has dimension at least 1). Now we are ready to reach a contradiction. First suppose that Item (a) holds. Since Θ_A is finite there exist distinct $W_1, W_2 \in \Theta_A$ such that $C \subset (\mathbb{P}(W_1) \cap \mathbb{P}(W_2))$. Thus $\dim(W_1 \cap W_2) = 2$ and hence the line

$$\{W \in \operatorname{Gr}(3, V) \mid (W_1 \cap W_2) \subset W \subset (W_1 + W_2)\} \tag{1.5.4}$$

is contained in Θ_A, that is a contradiction. Now suppose that Item (b) holds. Since Θ_A is finite there exists $W \in \Theta_A$ such that $C \subset \mathbb{P}(W)$ and

$$\dim A \cap S_W \cap (v \wedge \bigwedge^2 V) \geq 2 \qquad \forall [v] \in C. \tag{1.5.5}$$

Since $A \notin \Sigma[2]$ we have $\dim(A \cap (\bigwedge^2 W \wedge V)) = 2$. Let $\{w_1, w_2, w_3\}$ be a basis of W; then

$$A \cap (\bigwedge^2 W \wedge V) = \langle w_1 \wedge w_2 \wedge w_3,\, \beta\rangle. \tag{1.5.6}$$

Let $\overline{\beta}$ be the image of β under the quotient map $(\bigwedge^2 W \wedge V) \to (\bigwedge^2 W \wedge V)/\bigwedge^3 W$. Then

$$\overline{\beta} \in \bigwedge^2 W \wedge (V/W) \cong \operatorname{Hom}(W, V/W). \tag{1.5.7}$$

(We choose a volume form on W in order to define the isomorphism above.) By (1.5.5) the kernel of $\overline{\beta}$ (viewed as a map $W \to (V/W)$) contains all v such that $[v] \in C$. Thus $\overline{\beta}$ has rank 1. It follows that β is decomposable: $\beta \in \bigwedge^3 W'$ where $W' \in \Theta_A$ and $\dim W \cap W' = 2$. Then Θ_A contains the line in $\operatorname{Gr}(3, V)$ joining W and W': that is a contradiction. $\square$

1.6. **Lattices and periods.** Let L be an even lattice: we will denote by $(,)$ the bilinear symmetric form on L and for $v \in L$ we let $v^2 := (v,v)$. For a ring R we let $L_R := L \otimes_{\mathbb{Z}} R$ and we let $(,)_R$ be the R-bilinear symmetric form on L_R obtained from $(,)$ by extension of scalars. Let $L^\vee := \mathrm{Hom}(L,\mathbb{Z})$. The bilinear form defines an embedding $L \hookrightarrow L^\vee$: the quotient $D(L) := L^\vee/L$ is the *discriminant group* of L. Let $0 \neq v \in L$ be primitive i.e. $L/\langle v\rangle$ is torsion-free. The *divisibility* of v is the positive generator of (v,L) and is denoted by $\mathrm{div}(v)$; we let $v^* := v/\mathrm{div}(v) \in D(L)$. The group $O(L)$ of isometries of L acts naturally on $D(L)$. The *stable* orthogonal group is equal to

$$\widetilde{O}(L) := \ker(O(L) \to D(L)). \tag{1.6.1}$$

We let $\mathbf{q}_L \colon D(L) \to \mathbb{Q}/2\mathbb{Z}$ and $\mathbf{b}_L \colon D(L) \times D(L) \to \mathbb{Q}/\mathbb{Z}$ be the discriminant quadratic-form and symmetric bilinear form respectively, see [17]. The following criterion of Eichler will be handy.

Proposition 1.8 (Eichler's Criterion, see Prop. 3.3 of [6]). *Let L be an even lattice which contains U^2 (the direct sum of two hyperbolic planes). Let $v_1, v_2 \in L$ be non-zero and primitive. There exists $g \in \widetilde{O}(L)$ such that $gv_1 = v_2$ if and only if $v_1^2 = v_2^2$ and $v_1^* = v_2^*$.*

Now suppose that L is an even lattice of signature $(2,n)$. Let

$$\Omega_L := \{[\sigma] \in \mathbb{P}(L_{\mathbb{C}}) \mid (\sigma,\sigma)_{\mathbb{C}} = 0, \quad (\sigma,\overline{\sigma})_{\mathbb{C}} > 0\}. \tag{1.6.2}$$

(Notice that the isomorphism class of Ω_L depends on n only.) Then Ω_L is the union of two disjoint bounded symmetric domains of Type IV on which $O(L)$ acts. By Baily and Borel's fundamental results the quotient

$$\mathbb{D}_L := \widetilde{O}(L)\backslash\Omega_L. \tag{1.6.3}$$

is quasi-projective.

Remark 1.9. Suppose that $v_0 \in L$ has square 2. The reflection

$$\begin{array}{ccc} L & \overset{R_{v_0}}{\longrightarrow} & L \\ v & \mapsto & v - (v,v_0)v_0 \end{array} \tag{1.6.4}$$

belongs to the stable orthogonal group. We claim that R_{v_0} exchanges the two connected components of Ω_L. In fact let $M \subset L_{\mathbb{R}}$ be a positive definite subspace of maximal dimension (i.e. 2) containing v_0. If $[\sigma] \in \Omega_L \cap (M_{\mathbb{C}})$ then $R_{v_0}([\sigma]) = [\overline{\sigma}]$: this proves our claim because conjugation interchanges the two connected components of Ω_L. It follows that if L contains a vector of square 2 then $\mathbb{D}_L$ is connected.

Let us examine the lattices of interest to us. Let J, M, N be three copies of the hyperbolic plane U, let $E_8(-1)$ be the unique unimodular negative definite even lattice of rank 8 and (-2) the rank-1 lattice with generator of square (-2). Let

$$\widetilde{\Lambda} := J \oplus M \oplus N \oplus E_8(-1)^2 \oplus (-2) \cong U^3 \oplus E_8(-1)^2 \oplus (-2). \tag{1.6.5}$$

If X is a HK manifold deformation equivalent to the Hilbert square of a $K3$ then $H^2(X;\mathbb{Z})$ equipped with the Beauville-Bogomolov quadratic form is isometric to $\widetilde{\Lambda}$. A vector in $\widetilde{\Lambda}$ of square 2 has divisibility 1: it follows from **Proposition 1.8** that any two vectors in $\widetilde{\Lambda}$ of square 2 are $O(\widetilde{\Lambda})$-equivalent and hence the isomorphism class of $v^\perp$ for $v^2 = 2$ is independent of v. We choose $v_1 \in J$ of square 2 and let $\Lambda := v_1^\perp$. Then

$$\Lambda \cong U^2 \oplus E_8(-1)^2 \oplus (-2)^2. \tag{1.6.6}$$

We get an inclusion $\widetilde{O}(\Lambda) < O(\widetilde{\Lambda})$ by associating to $g \in \widetilde{O}(\Lambda)$ the unique $\widetilde{g} \in O(\widetilde{\Lambda})$ which is the identity on $\mathbb{Z}v_1$ and restricts to g on $v_1^\perp$ (such a lift exists because $g \in \widetilde{O}(\Lambda)$). Now suppose that X is a HK manifold deformation equivalent to the Hilbert square of $K3$ and that $h \in H^{1,1}_{\mathbb{Z}}(X)$ has square 2. Since there is a single $O(\widetilde{\Lambda})$-orbit of square-2 vectors there exists an isometry

$$\psi \colon H^2(X;\mathbb{Z}) \xrightarrow{\sim} \widetilde{\Lambda}, \qquad \psi(h) = v_1. \tag{1.6.7}$$

Such an isometry is a *marking* of (X,h). If H is a divisor on X of square 2 a marking of (X,H) is a marking of $(X, c_1(\mathcal{O}_X(H)))$. Let $\psi_{\mathbb{C}} \colon H^2(X;\mathbb{C}) \to \widetilde{\Lambda}_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of ψ. Since h is of type $(1,1)$ we have that $\psi_{\mathbb{C}}(H^{2,0}) \in v_1^\perp$. Well-known properties of the Beauville-Bogomolov quadratic form give that $\psi_{\mathbb{C}}(H^{2,0}) \in \Omega_\Lambda$. Any two markings of (X,h) differ by the action of an element of $\widetilde{O}(\Lambda)$. It follows that the equivalence class

$$\Pi(X,h) := [\psi_{\mathbb{C}} H^{2,0}] \in \mathbb{D}_\Lambda \tag{1.6.8}$$

is well-defined i.e. independent of the marking: that is the *period point* of (X,h). Since the lattice Λ contains vectors of square 2 the quotient $\mathbb{D}_\Lambda$ is irreducible by **Remark 1.9**. The discriminant group and discriminant quadratic form of Λ are described as follows. Let e_1 be a generator of $v_1^\perp \cap J$ and let e_2 be a generator of the last summand of (1.6.5):

$$\mathbb{Z}e_1 = v_1^\perp \cap J, \qquad \mathbb{Z}e_2 = (-2). \tag{1.6.9}$$

Then $-2 = e_1^2 = e_2^2$, $(e_1,e_2) = 0$ and $2 = \mathrm{div}_\Lambda(e_1) = \mathrm{div}_\Lambda(e_2)$: here we denote by $\mathrm{div}_\Lambda(e_i)$ the divisibility of e_i as element of Λ, one should notice that the divisibility of e_1 in $\widetilde{\Lambda}$ is 1 (not 2) while the divisibility of e_2 in $\widetilde{\Lambda}$ is 2 (equal to the divisibility of e_2 in Λ). In particular $e_1/2$ and $e_2/2$ are order-2 elements of $D(\Lambda)$. We have the following:

$$\begin{array}{ccc} \mathbb{Z}/(2)\oplus\mathbb{Z}/(2) & \overset{\sim}{\longrightarrow} & D(\Lambda) \\ ([x],[y]) & \mapsto & x(e_1/2)+y(e_2/2) \end{array} \qquad q_\Lambda(x(e_1/2)+y(e_2/2)) \equiv -\frac{1}{2}x^2-\frac{1}{2}y^2 \pmod{2\mathbb{Z}} \tag{1.6.10}$$

In particular we get that

$$[O(\Lambda):\widetilde{O}(\Lambda)] = 2. \tag{1.6.11}$$

Let $\iota \in O(\Lambda)$ be the involution characterized by

$$\iota(e_1) = e_2, \quad \iota(e_2) = e_1, \quad \iota|_{\{e_1,e_2\}^\perp} = \mathrm{Id}_{\{e_1,e_2\}^\perp}\,. \tag{1.6.12}$$

Then $\iota \notin \widetilde{O}(\Lambda)$. Since $[O(\Lambda):\widetilde{O}(\Lambda)] = 2$ we get that ι induces a non-trivial involution

$$\bar{\iota}\colon \mathbb{D}_\Lambda^{BB} \to \mathbb{D}_\Lambda^{BB}. \tag{1.6.13}$$

The geometric counterpart of $\bar{\iota}$ is given by the involution $\delta\colon \mathfrak{M} \to \mathfrak{M}$ induced by the map

$$\begin{array}{ccc} \mathbb{LG}(\bigwedge^3 V) & \overset{\delta_V}{\underset{\sim}{\longrightarrow}} & \mathbb{LG}(\bigwedge^3 V^\vee) \\ A & \mapsto & \delta_V(A) = \mathrm{Ann}\, A. \end{array} \tag{1.6.14}$$

(The geometric meaning of $\delta_V(A)$: for generic $A \in \mathbb{LG}(\bigwedge^3 V)$ the dual of Y_A is equal to $Y_{\delta_V(A)}$.) In [19] we proved that

$$\bar{\iota} \circ \mathfrak{p} = \mathfrak{p} \circ \delta. \tag{1.6.15}$$

1.7. **Roots of Λ.** Let $v_0 \in \Lambda$ be *primitive* and let $v_0^2 = -2d \neq 0$: then v_0 is a *root* if the reflection

$$\begin{array}{ccc} \Lambda_\mathbb{Q} & \overset{R}{\longrightarrow} & \Lambda_\mathbb{Q} \\ v & \mapsto & v + \frac{(v,v_0)v_0}{d} \end{array} \tag{1.7.1}$$

is integral, i.e. $R(\Lambda) \subset \Lambda$. We record the square of v_0 by stating that v_0 is $(-2d)$-root. Notice that if $v_0^2 = \pm 2$ then v_0 is a root. In particular e_1 and e_2 are (-2)-roots of Λ. Let

$$e_3 \in M, \qquad e_3^2 = -2. \tag{1.7.2}$$

Notice that $e_3 \in \Lambda$ and hence it is a (-2)-root of Λ. Since $(e_1+e_2)^2 = -4$ and $\mathrm{div}(e_1+e_2) = 2$ we get that (e_1+e_2) is a (-4)-root of Λ.

Proposition 1.10. *The set of negative roots of Λ breaks up into 4 orbits for the action of $\widetilde{O}(\Lambda)$, namely the orbits of e_1, e_2, e_3 and (e_1+e_2).*

Proof. First let us prove that the orbits of e_1, e_2, e_3 and (e_1+e_2) are pairwise disjoint. Since $-2 = e_1^2 = e_2^2 = e_3^2$ and $(e_1+e_2)^2 = -4$ the orbits of e_1, e_2 and e_3 are disjoint from that of (e_1+e_2). We have $\mathrm{div}_\Lambda(e_3) = 1$ and hence $e_3^* = 0$. Since e_1^*, e_2^* and e_3^* are pairwise distinct elements of $D(\Lambda)$ it follows that the orbits of e_1, e_2, e_3 are pairwise disjoint. Now let $v_0 \in \Lambda$ be a negative root. Since $D(\Lambda)$ is 2-torsion $\mathrm{div}(v_0) \in \{1,2\}$: it follows that v_0 is either a (-2)-root or a (-4)-root, and in the latter case $\mathrm{div}(v_0) = 2$. Suppose first that v_0 is a (-2)-root. If $\mathrm{div}_\Lambda(v_0) = 1$ then $v_0^* = 0$ and hence v_0 is in the orbit of e_3 by **Proposition 1.8**. If $\mathrm{div}_\Lambda(v_0) = 2$ then $v^* \in \{e_1^*, e_2^*\}$ because $q_\Lambda(e_1^*+e_2^*) \equiv -1 \not\equiv -1/2 \pmod{2\mathbb{Z}}$: it follows from **Proposition 1.8** that v_0 belongs either to the $\widetilde{O}(\Lambda)$-orbit of e_1 or to that of e_2. Lastly suppose that v_0 is a (-4)-root. Since $\mathrm{div}(v_0) = 2$ we have $q_\Lambda(v_0^*) = -1$ and hence $v_0^* = e_1/2 + e_2/2$: it follows from **Proposition 1.8** that v_0 belongs to the $\widetilde{O}(\Lambda)$-orbit of (e_1+e_2). □

Let $\kappa\colon \Omega_\Lambda \to \mathbb{D}_\Lambda$ be the quotient map. Let

$$\mathbb{S}_2' := \kappa(e_1^\perp \cap \Omega_\Lambda), \quad \mathbb{S}_2'' := \kappa(e_2^\perp \cap \Omega_\Lambda), \quad \mathbb{S}_2^\star := \kappa(e_3^\perp \cap \Omega_\Lambda), \quad \mathbb{S}_4 := \kappa((e_1+e_2)^\perp \cap \Omega_\Lambda). \tag{1.7.3}$$

Remark 1.11. Let $i = 1, 2, 3$: then $e_i^\perp \cap \Omega_\Lambda$ has two connected components - see **Remark 1.9**. Let $v_0 \in N$ (we refer to (1.6.5)) of square 2. Then $(v_0, e_i) = 0$ for $i = 1, 2, 3$ and hence Reflection (1.6.4) exchanges the two connected components of $e_i^\perp \cap \Omega_\Lambda$ for $i = 1, 2, 3$ and also the two connected components of $(e_1 + e_2)^\perp \cap \Omega_\Lambda$. It follows that each of $\mathbb{S}_2'$, $\mathbb{S}_2''$, $\mathbb{S}_2^\star$ and $\mathbb{S}_4$ is a prime divisor in $\mathbb{D}_\Lambda$.

Let $\bar{\iota}$ be the involution given by (1.6.13): then

$$\bar{\iota}(\mathbb{S}_2^\star) = \mathbb{S}_2^\star, \quad \bar{\iota}(\mathbb{S}_2') = \mathbb{S}_2'', \quad \bar{\iota}(\mathbb{S}_2'') = \mathbb{S}_2', \quad \bar{\iota}(\mathbb{S}_4) = \mathbb{S}_4. \tag{1.7.4}$$

We will describe the normalization of $\mathbb{S}_2^\star$ and we will show that it is a finite cover of the period space for $K3$ surfaces of degree 2. Let v_3 be a generator of $e_3^\perp \cap M$. Let

$$\widetilde{\Gamma} := e_3^\perp = J \oplus \mathbb{Z}v_3 \oplus N \oplus E_8(-1)^2 \oplus \mathbb{Z}e_2 \cong U \oplus (2) \oplus U \oplus E_8(-1)^2 \oplus (-2) \tag{1.7.5}$$

and

$$\Gamma := e_3^\perp \cap \Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}v_3 \oplus N \oplus E_8(-1)^2 \oplus \mathbb{Z}e_2 \cong (-2) \oplus (2) \oplus U \oplus E_8(-1)^2 \oplus (-2). \tag{1.7.6}$$

We have $\Omega_\Gamma = e_3^\perp \cap \Omega_\Lambda$. Viewing $\widetilde{O}(\Gamma)$ as the subgroup of $\widetilde{O}(\Lambda)$ fixing e_3 we get a natural map

$$\nu\colon \mathbb{D}_\Gamma^{BB} \longrightarrow \overline{\mathbb{S}}_2^\star. \tag{1.7.7}$$

Claim 1.12. *Map* (1.7.7) *is the normalization of* $\mathbb{S}_2^\star$.

Proof. Since $\mathbb{D}_\Gamma^{BB}$ is normal and ν is finite it sufffices to show that ν has degree 1. Let $[\sigma] \in e_3^\perp \cap \Omega_\Lambda$ be generic. Let $g \in \widetilde{O}(\Lambda)$ and $[\tau] = g([\sigma])$. We must show that there exists $g' \in \widetilde{O}(\Gamma)$ such that $[\tau] = g'([\sigma])$. Since $[\sigma]$ is generic we have that

$$\sigma^\perp \cap \Lambda = \mathbb{Z}e_3. \tag{1.7.8}$$

It follows that $g(e_3) = \pm e_3$. If $g(e_3) = e_3$ then $g \in \widetilde{O}(\Gamma)$ and we are done. Suppose that $g(e_3) = -e_3$. Let $g' := (-1_\Lambda) \circ g$. Since multiplication by 2 kills $D(\Lambda)$ we have that $(-1_\Lambda) \in \widetilde{O}(\Lambda)$ and hence $g' \in \widetilde{O}(\Lambda)$: in fact $g' \in \widetilde{O}(\Gamma)$ because $g'(e_3) = e_3$. On the other hand $[\tau] = g'([\sigma])$ because (-1_Λ) acts trivially on Ω_Λ. □

Our next task will be to define a finite map from $\mathbb{D}_\Gamma^{BB}$ to the Baily-Borel compactification of the period space for $K3$ surfaces with a polarization of degree 2. Let

$$\widetilde{\Phi} := J \oplus \langle v_3, (v_3 + e_2)/2 \rangle \oplus N \oplus E_8(-1)^2 \cong U^3 \oplus E_8(-1)^2. \tag{1.7.9}$$

Then $\widetilde{\Phi}$ is isometric to the $K3$ *lattice* i.e. $H^2(K3; \mathbb{Z})$ equipped with the intersection form. Let

$$\Phi := v_1^\perp \cap \widetilde{\Phi} := \mathbb{Z}e_1 \oplus \langle v_3, (v_3 + e_2)/2 \rangle \oplus N \oplus E_8(-1)^2 \oplus \cong (-2) \oplus U^2 \oplus E_8(-1)^2. \tag{1.7.10}$$

Then $\mathbb{D}_\Phi$ is the period space for $K3$ surfaces with a polarization of degree 2.

Claim 1.13. $\widetilde{\Phi}$ *is the unique lattice contained in* $\widetilde{\Lambda}_\mathbb{Q}$ *(with quadratic form equal to the restriction of the quadratic form on* $\widetilde{\Lambda}_\mathbb{Q}$*) and containing* $\widetilde{\Gamma}$ *as a sublattice of index* 2.

Proof. First it is clear that $\widetilde{\Gamma}$ is contained in $\widetilde{\Phi}$ as a sublattice of index 2. Now suppose that L is a lattice contained in $\widetilde{\Lambda}_\mathbb{Q}$ and containing $\widetilde{\Gamma}$ as a sublattice of index 2. Then L must be generated by $\widetilde{\Gamma}$ and an isotropic element of $D(\widetilde{\Gamma})$: since there is a unique such element L is unique. □

By **Claim 1.13** every isometry of $\widetilde{\Lambda}$ induces an isometry of $\widetilde{\Phi}$. It follows that we have well-defined injection $\widetilde{O}(\Lambda) < \widetilde{O}(\Phi)$. Since $\Omega_\Lambda = \Omega_\Phi$ there is an induced finite map

$$\rho\colon \mathbb{D}_\Gamma^{BB} \longrightarrow \mathbb{D}_\Phi^{BB}. \tag{1.7.11}$$

Remark 1.14. Keep notation as above. Then $\deg \rho = 2^{20} - 1$.

1.8. **Determinant of a variable quadratic form.** Let U be a complex vector-space of finite dimension d. We view $\mathrm{S}^2 U^\vee$ as the vector-space of quadratic forms on U. Given $q \in \mathrm{S}^2 U^\vee$ we let $\widetilde{q}\colon U \to U^\vee$ be the associated symmetric map. Let $K := \ker q$; then $\widetilde{q}$ may be viewed as a (symmetric) map $\widetilde{q}\colon (U/K) \to \operatorname{Ann} K$. The *dual* quadratic form $q^\vee$ is the quadratic form associated to the symmetric map $\widetilde{q}^{-1}\colon \operatorname{Ann} K \to (U/K)$; thus $q^\vee \in \mathrm{S}^2(U/K)$. We will denote by $\wedge^i q$ the quadratic form induced by q on $\bigwedge^i U$.

Remark 1.15. If $0 \neq \alpha = v_1 \wedge \ldots \wedge v_i$ is a decomposable vector of $\bigwedge^i U$ then $\wedge^i q(\alpha)$ is equal to the determinant of the Gram matrix of $q|_{\langle v_1, \ldots, v_i \rangle}$ with respect to the basis $\{v_1, \ldots, v_i\}$.

The following exercise in linear algebra will be handy.

Lemma 1.16. *Suppose that $q \in S^2 U^\vee$ is non-degenerate. Let $S \subset U$ be a subspace. Then*

$$\operatorname{cork}(q|_S) = \operatorname{cork}(q^\vee|_{\operatorname{Ann}(S)}). \tag{1.8.1}$$

Let $q_* \in S^2 U^\vee$. Then

$$\det(q_* + q) = \Phi_0(q) + \Phi_1(q) + \ldots + \Phi_d(q), \qquad \Phi_i \in S^i(S^2 U). \tag{1.8.2}$$

Of course $\det(q_* + q)$ is well-defined up to multiplication by a non-zero scalar and hence so are the Φ_i's. The result below is well-known (it follows from a straightforward computation).

Proposition 1.17. *Let $q_* \in S^2 U^\vee$ and*

$$K := \ker(q_*), \qquad k := \dim K. \tag{1.8.3}$$

Let Φ_i be the polynomials appearing in (1.8.2). *Then*

(1) *$\Phi_i = 0$ for $i < k$, and*
(2) *there exists $c \neq 0$ such that $\Phi_k(q) = c \det(q|_K)$.*

Keep notation and hypotheses as in **Proposition 1.17**. Let $\mathcal{V}_K \subset S^2 U^\vee$ be the subspace of quadratic forms whose restriction to K vanishes. Given $q \in \mathcal{V}_K$ we have $\widetilde{q}(K) \subset \operatorname{Ann} K$ and hence it makes sense to consider the restriction of $q_*^\vee$ to $\widetilde{q}(K)$.

Proposition 1.18. *Keep notation and hypotheses as in* **Proposition 1.17**. *The restriction of Φ_i to $\mathcal{V}_K$ vanishes for $I < 2k$. Moreover there exists $c \neq 0$ such that*

$$\Phi_{2k}(q) = c \det(q_*^\vee|_{\widetilde{q}(K)}), \qquad q \in \mathcal{V}_K. \tag{1.8.4}$$

Proof. Choose a basis $\{u_1, \ldots, u_d\}$ of U such that $K = \langle u_1, \ldots, u_k \rangle$ and $\widetilde{q}_*(u_i) = u_i^\vee$ for $k < i \leq d$. Let M be the Gram matrix of q in the chosen basis. Expanding $\det(q_* + tq)$ we get that

$$\det(q_* + tq) \equiv (-1)^k t^{2k} \sum_J (\det M_{\mathbf{k},J})^2 \pmod{t^{2k+1}}$$

where $M_{\mathbf{k},J}$ is the $k \times k$ submatrix of M determined by the first k rows and the columns indicized by $J = (j_1, j_2, \ldots, j_k)$. The claim follows from the equality

$$\sum_J (\det M_{\mathbf{k},J})^2 = \wedge^k (q_*^\vee)(\widetilde{q}(u_1) \wedge \ldots \widetilde{q}(u_k))$$

and **Remark 1.15**. □

Now suppose that

$$\operatorname{cork} \widetilde{q}_* = 1, \qquad \ker \widetilde{q}_* = \langle e \rangle. \tag{1.8.5}$$

We let $\overline{q}_* \in S^2(U/\langle e \rangle)^\vee$ be the non-degenerate quadratic form induced by q_* i.e. $\overline{q}_*(\overline{v}) := q_*(v)$ for $\overline{v} \in U/\langle e \rangle$. Let ..., Φ_i, ... be as in (1.8.2). In particular $\Phi_0 = 0$. Assume that

$$L \subset \ker \Phi_1 = \{q \mid q(e) = 0\} \tag{1.8.6}$$

is a vector subspace. Thus

$$\det(q_* + q)|_L = \Phi_2|_L + \ldots + \Phi_d|_L. \tag{1.8.7}$$

We will compute $\operatorname{rk}(\Phi_2|_L)$. Let $T \subset U$ be defined by

$$T := \operatorname{Ann} \langle \widetilde{q}(e) \rangle_{q \in L} \tag{1.8.8}$$

where L and e are as above. Geometrically: $\mathbb{P}(T)$ is the projective tangent space at $[e]$ of the intersection of the projective quadrics parametrized by $\mathbb{P}(L)$.

Proposition 1.19. *Suppose that $L \subset S^2 U^\vee$ is a vector subspace such that* (1.8.6) *holds. Keep notation as above, in particular T is given by* (1.8.8). *Then*

$$\operatorname{rk}(\Phi_2|_L) = \operatorname{cod}(T, U) - \operatorname{cork}(\overline{q}_*|_{T/\langle e \rangle}). \tag{1.8.9}$$

(The last term on the right-side makes sense because $T \supset \langle e \rangle$.)

Proof. Let

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & (U/\langle e\rangle)^\vee \\ q & \mapsto & \widetilde{q}(e). \end{array} \tag{1.8.10}$$

By **Proposition 1.18** we have

$$\mathrm{rk}(\Phi_2|_L)=\mathrm{rk}(\overline{q}_*^\vee|_{\mathrm{Im}(\alpha)}). \tag{1.8.11}$$

On the other hand **Lemma 1.16** gives that

$$\mathrm{rk}(\overline{q}_*^\vee|_{\mathrm{Im}(\alpha)})=\dim\mathrm{Im}(\alpha)-\mathrm{cork}(\overline{q}_*|_{\mathrm{Ann}(\mathrm{Im}(\alpha))}). \tag{1.8.12}$$

By definition $\mathrm{Ann}(\mathrm{Im}(\alpha))=T/\langle e\rangle$. Since $\dim\mathrm{Im}(\alpha)=\mathrm{cod}(T,U)$ we get the proposition. □

2. First extension of the period map

2.1. **Local structure of Y_A along a singular plane.** Let $(W,A)\in\widetilde{\Sigma}$. Then $\mathbb{P}(W)\subset Y_A$. In this section we will analyze the local structure of Y_A at $v_0\in(\mathbb{P}(W)\setminus C_{W,A})$ under mild hypotheses on A. Let $[v_0]\in\mathbb{P}(W)$ - for the moment being we do not require that $v_0\notin C_{W,A}$. Let $V_0\subset V$ be a subspace transversal to $[v_0]$. We identify V_0 with an open affine neighborhood of $[v_0]$ via (1.1.2); thus $0\in V_0$ corresponds to $[v_0]$. Let $f_i\in \mathrm{S}^i V_0^\vee$ for $i=0,\ldots,6$ be the polynomials of (1.1.3). By Item (2) of **Proposition 1.1** we have

$$Y_A\cap V_0=V(f_2+\ldots+f_6). \tag{2.1.1}$$

Suppose that $Y_A\neq\mathbb{P}(V)$. Then $[v_0]\in\mathrm{sing}\,Y_A$; since $[v_0]$ is an arbitrary point of $\mathbb{P}(W)$ we get that $\mathbb{P}(W)\subset\mathrm{sing}\,Y_A$. It follows that $\mathrm{rk}\,f_2\le 3$.

Proposition 2.1. *Let $(W,A)\in\widetilde{\Sigma}$ and suppose that $Y_{\delta(A)}\neq\mathbb{P}(V^\vee)$. Let $[v_0]\in(\mathbb{P}(W)\setminus C_{W,A})$ and f_2 be the quadratic term of the Taylor expansion of a local equation of Y_A centered at $[v_0]$. Then*

$$\mathrm{rk}\,f_2=4-\dim(A\cap(\bigwedge^2 W\wedge V)). \tag{2.1.2}$$

Proof. By hypothesis there exists a subspace $V_0\subset V$ such that (1.1.1)-(1.1.4) hold. Let $\widetilde{q}_A$ be as in (1.1.6) and q_A be the associated quadratic form on $\bigwedge^2 V_0$. Let $Q_A:=V(q_A)\subset\mathbb{P}(\bigwedge^2 V_0)$. By **Proposition 1.2** we have

$$V(Y_A)|_{V_0}=V(\det(q_A+q_v)) \tag{2.1.3}$$

where q_v is as in (1.1.10). Let $W_0:=W\cap V_0$. Since $[v_0]\notin C_{W,A}$ we have $A\cap(v_0\wedge\bigwedge^2 V)=\bigwedge^3 W$. By (1.1.9) we get that $\mathrm{sing}\,Q_A=\{[\bigwedge^2 W_0]\}$. Thus

$$\det(q_A+q_v)=\Phi_2(v)+\Phi_3(v)+\ldots+\Phi_6(v),\qquad \Phi_i\in\mathrm{S}^i V_0^\vee \tag{2.1.4}$$

and the rank of Φ_2 is given by (1.8.9) with $q_*=q_A$ and $L=V_0$. Let us identify the subspace $T\subset\bigwedge^2 V_0$ given by (1.8.8). Let $U_0\subset V_0$ be a subspace transversal to W_0; since the Plücker quadrics generate the ideal of the Grassmannian we have

$$T=\bigwedge^2 W_0\oplus W_0\wedge U_0. \tag{2.1.5}$$

By **Proposition 1.19** we get that

$$\mathrm{rk}\,\Phi_2=3-\dim\ker(q_A|_{W_0\wedge U}). \tag{2.1.6}$$

We claim that

$$\dim\ker(q_A|_{W_0\wedge U})=\dim(A\cap(\bigwedge^2 W\wedge V)). \tag{2.1.7}$$

In fact let $\alpha\in W_0\wedge U$. Then $\alpha\in\ker(q_A|_{W_0\wedge U})$ if and only if

$$\widetilde{q}_A(\alpha)\in\mathrm{Ann}(W_0\wedge U_0)=\bigwedge^2 W_0\wedge U_0\oplus\bigwedge^3 U_0. \tag{2.1.8}$$

Since $A\subset(\bigwedge^3 W)^\perp$ it follows from (1.1.8) that necessarily $\widetilde{q}_A(\alpha)\in\bigwedge^2 W_0\wedge U_0$. Equation (1.1.8) gives a linear map

$$\begin{array}{ccc} \ker(q_A|_{W_0\wedge U_0}) & \xrightarrow{\varphi} & A\cap(\bigwedge^2 W\wedge U_0) \\ \alpha & \mapsto & v_0\wedge\alpha+\widetilde{q}_A(\alpha). \end{array} \tag{2.1.9}$$

The direct-sum decomposition

$$\bigwedge^2 W\wedge U_0=[v_0]\wedge W_0\wedge U_0\oplus\bigwedge^2 W_0\wedge U_0 \tag{2.1.10}$$

shows that φ is bijective. Since there is an obvious isomorphism $(A\cap(\bigwedge^2 W\wedge U_0))\cong(A\cap(\bigwedge^2 W\wedge V))/\bigwedge^3 W$ we get that (2.1.7) holds. □

Remark 2.2. Suppose that $\dim(A\cap(\bigwedge^2 W\wedge V))>4$. Then Equation (2.1.2) does not make sense. On the other hand $C_{W,A}=\mathbb{P}(W)$ by **Proposition 1.6** and hence there is no $[v_0]\in(\mathbb{P}(W)\setminus C_{W,A})$.

2.2. The extension.

Lemma 2.3. *Let $A_0\in(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma_\infty\setminus\Sigma[2])$. Then $Y_{A_0}[3]$ is finite and C_{W,A_0} is a sextic curve for every $W\in\Theta_{A_0}$.*

Proof. $Y_{A_0}[3]$ is finite by **Proposition 1.7**. Let $W\in\Theta_{A_0}$. Let us show that $\mathcal{B}(W,A_0)\neq\mathbb{P}(W)$. Let $W'\in(\Theta_{A_0}\setminus\{W\})$. Then $\dim(W\cap W')=1$ because otherwise $\bigwedge^3 W$ and $\bigwedge^3 W'$ span a line in $\mathrm{Gr}(3,V)$ which is contained in Θ_{A_0} and that contradicts the assumption that Θ_{A_0} is finite. This proves finiteness of the set of $[w]\in\mathbb{P}(W)$ such that Item (1) of **Definition 1.5** holds. Since $\dim(\bigwedge^2 W\wedge V)\le 2$ a similar argument gives finiteness of the set of $[w]\in\mathbb{P}(W)$ such that Item (2) of **Definition 1.5** holds. This proves that $\mathcal{B}(W,A_0)$ is finite, in particular $\mathcal{B}(W,A_0)\neq\mathbb{P}(W)$. By **Proposition 1.6** it follows that C_{W,A_0} is a sextic curve. □

Proposition 2.4. *Let $A_0\in(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma_\infty\setminus\Sigma[2])$ and $\mathbf{L}\subset\mathbb{P}(V)$ be a generic 3-dimensional linear subspace. Let $\mathcal{U}\subset(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma_\infty\setminus\Sigma[2])$ be a sufficiently small open set containing A_0. Let $A\in\mathcal{U}$. Then the following hold:*

(a) *The scheme-theoretic inverse image $f_A^{-1}\mathbf{L}$ is a reduced surface with DuVal singularities.*
(b) *If in addition $A_0\in\mathbb{LG}(\bigwedge^3 V)^0$ then $f_A^{-1}\mathbf{L}$ is a smooth surface.*

Proof. Let $\mathbf{L}\subset\mathbb{P}(V)$ be a generic 3-dimensional linear subspace. Then

(1) $\mathbf{L}\cap Y_{A_0}[3]=\emptyset$.
(2) $\mathbf{L}\cap C_{W,A_0}=\emptyset$ for every $W\in\Theta_{A_0}$.

In fact $Y_{A_0}[3]$ is finite by **Lemma 2.3** and hence (1) holds. Since Θ_{A_0} is finite and C_{W,A_0} is a sextic curve for every $W\in\Theta_{A_0}$ Item (2) holds as well. We will prove that $f_{A_0}^{-1}\mathbf{L}$ is reduced with DuVal singularities and that it is smooth if $A_0\in\mathbb{LG}(\bigwedge^3 V)^0$. The result will follow because being smooth, reduced or having DuVal singularities is an open property. Write $\Theta_{A_0}=\{W_1,\ldots,W_d\}$. By Item (2) above the intersection $\mathbf{L}\cap\mathbb{P}(W_i)$ is a single point p_i for $i=1,\ldots,d$. Since $p_i\notin C_{W_i,A_0}$ the points $p_1,\ldots,p_d$ are pairwise distinct. By **Proposition 1.1** we know that away from $\bigcup_{W\in\Theta_{A_0}}\mathbb{P}(W)$ the locally closed sets $Y_A(1)$ and $Y_A(2)$ are smooth of dimensions 4 and 2 respectively. By Item (1) it follows that $f_{A_0}^{-1}\mathbf{L}$ is smooth away from

$$f_{A_0}^{-1}\{p_1,\ldots,p_d\}. \tag{2.2.1}$$

It remains to show that $f_{A_0}^{-1}\mathbf{L}$ is DuVal at each point of (2.2.1). Since $p_i\in Y_{A_0}(1)$ the map f_{A_0} is étale of degree 2 over p_i, see (1.2.1). Thus $f_{A_0}^{-1}(p_i)=\{q_i^+,q_i^-\}$ and f_{A_0} defines an isomorphism between the germ $(X_{A_0},q_i^\pm)$ (in the classical topology) and the germ (Y_{A_0},p_i). By **Proposition 2.1** we get that the tangent cone of $f_{A_0}^{-1}\mathbf{L}$ at $q_i^\pm$ is a quadric cone of rank 2 or 3; it follows that $f_{A_0}^{-1}\mathbf{L}$ has a singularity of type A_n at $q_i^\pm$. □

Proposition 2.5. *Let $A_0\in(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma_\infty\setminus\Sigma[2])$. Then $\mathcal{P}$ is regular at A_0 and $\mathcal{P}(A_0)\in\mathbb{D}_\Lambda$.*

Proof. Let $\mathcal{U}$ and $\mathbf{L}$ be as in **Proposition 2.4**. Let $U\subset\mathcal{U}$ be a subset containing A_0, open in the classical topology and contractible. Let $U^0:=U\cap\mathbb{LG}(\bigwedge^3 V)^0$. Let $\overline{A}\in U^0$; thus $X_{\overline{A}}$ is smooth. By **Lemma 2.3** we know that $f_A^{-1}\mathbf{L}$ is a smooth surface for every $A\in U^0$. Thus $\pi_1(U^0,\overline{A})$ acts by monodromy on $H^2(f_{\overline{A}}^{-1}\mathbf{L})$ and by Item (a) of **Proposition 2.4** the image of the monodromy representation is a finite group. On the other hand $H_{\overline{A}}$ is an ample divisor on $X_{\overline{A}}$: by the Lefschetz Hyperplane Theorem the homomorphism

$$H^2(X_{\overline{A}};\mathbb{Z})\longrightarrow H^2(f_{\overline{A}}^{-1}\mathbf{L};\mathbb{Z}) \tag{2.2.2}$$

is injective. The image of (2.2.2) is a subgroup of $H^2(f_{\overline{A}}^{-1}\mathbf{L})$ invariant under the monodromy action of $\pi_1(U^0,\overline{A})$. By injectivity of (2.2.2) the monodromy action of $\pi_1(U^0,\overline{A})$ on $H^2(X_{\overline{A}})$ is finite as well. By Griffith's Removable Singularity Theorem (see p. 41 of [5]) it follows that the restriction of $\mathcal{P}^0$ to U^0 extends to a holomorphic map $U\to\mathbb{D}_\Lambda$. Hence $\mathcal{P}^0$ extends regularly in a neighborhood A_0 and it goes into $\mathbb{D}_\Lambda$. □

Definition 2.6. Let $\widehat{\mathbb{LG}}(\bigwedge^3 V) \subset \mathbb{LG}(\bigwedge^3 V) \times \mathbb{D}_\Lambda^{BB}$ be the closure of the graph of the restriction of $\mathcal{P}$ to the set of its regular points and

$$p\colon \widehat{\mathbb{LG}}(\bigwedge^3 V) \to \mathbb{LG}(\bigwedge^3 V) \tag{2.2.3}$$

the restriction of projection. Let $\widehat{\Sigma} \subset \widehat{\mathbb{LG}}(\bigwedge^3 V)$ be the proper transform of Σ.

Corollary 2.7. *Keep notation as above. Let A be in the indeterminacy locus of $\mathcal{P}$ and p be as in* (2.2.3)*. Then* $\dim(p^{-1}(A) \cap \widehat{\Sigma})$ *has dimension at least* 2.

Proof. Let $\mathrm{Ind}(\mathcal{P})$ be the indeterminacy locus of $\mathcal{P}$. Since $\mathbb{LG}(\bigwedge^3 V)$ is smooth the morphism p identifies $\widehat{\mathbb{LG}}(\bigwedge^3 V)$ with the blow-up of $\mathrm{Ind}(\mathcal{P})$. Hence the exceptional set of p is the support of a Cartier divisor E. By **Proposition 2.5** the indeterminacy locus of $\mathcal{P}$ is contained in Σ and thus $A \in \Sigma$. It follows that $p^{-1}(A) \cap \widehat{\Sigma}$ is not empty. Since $\widehat{\Sigma}$ is a prime divisor in $\widehat{\mathbb{LG}}(\bigwedge^3 V)$ and E is a Cartier divisor every irreducible component of $E \cap \widehat{\Sigma}$ has codimension 2 in $\widehat{\mathbb{LG}}(\bigwedge^3 V)$. On the other hand **Proposition 2.5**, (1.4.3) and (1.4.8) give that $\mathrm{cod}(\mathrm{Ind}(\mathcal{P}), \mathbb{LG}(\bigwedge^3 V)) \geq 4$ and hence every component of a fiber of $E \cap \widehat{\Sigma} \to \mathrm{Ind}(\mathcal{P})$ has dimension at least 2. Since $p^{-1}(A) \cap \widehat{\Sigma}$ is one such fiber we get the corollary. □

3. Second extension of the period map

3.1. X_A **for generic** A **in** Σ**.** Let $A \in (\Sigma \setminus \Sigma[2])$ and $W \in \Theta_A$. Then $\mathcal{B}(W, A) \neq \mathbb{P}(W)$ because by **Lemma 2.3** we know that $C_{W,A} \neq \mathbb{P}(W)$. By the same Lemma $Y_A[3]$ is finite. In particular $(\mathbb{P}(W) \setminus \mathcal{B}(W, A) \setminus Y_A[3])$ is not empty.

Proposition 3.1. *Let $A \in (\Sigma \setminus \Sigma[2])$ and $W \in \Theta_A$. Suppose in addition that* $\dim(A \cap (\bigwedge^2 W \wedge V)) = 1$*. Let*

$$x \in f_A^{-1}(\mathbb{P}(W) \setminus \mathcal{B}(W, A) \setminus Y_A[3]). \tag{3.1.1}$$

The germ (X_A, x) of X_A at x in the classical topology is isomorphic to $(\mathbb{C}^2, 0) \times A_1$ *and* $\mathrm{sing}\, X_A$ *is equal to $f_A^{-1}\mathbb{P}(W)$ in a neighborhood of x.*

Proof. Suppose first that $f_A(x) \notin C_{W,A}$. Then $f_A(x) \in Y_A(1)$ and hence f_A is étale over $f_A(x)$, see (1.2.1). Thus the germ (X_A, x) is isomorphic to the germ $(Y_A, f_A(x))$ and the statement of the proposition follows from **Proposition 2.1** because by hypothesis $B = 0$. It remains to examine the case

$$f_A(x) \in (C_{W,A} \setminus \mathcal{B}(W, A) \setminus Y_A[3]). \tag{3.1.2}$$

Let $f_A(x) = [v_0]$. Since $A \notin \Sigma_\infty$ there exists a subspace $V_0 \subset V$ transversal to $[v_0]$ and such that (1.1.4) holds - see **Proposition 1.3**. Thus we may apply **Proposition 1.4**. We will adopt the notation of that Proposition, in particular we will identify V_0 with $(\mathbb{P}(V) \setminus \mathbb{P}(V_0))$ via (1.1.2). Let $W_0 := W \cap V_0$; thus $\dim W_0 = 2$. Let $K \subset \bigwedge^2 V_0$ be the subspace corresponding to $(v_0 \wedge \bigwedge^2 V) \cap A$ via (1.1.5). By (3.1.2) $\dim K = 2$. Let us prove that there exists a basis $\{w_1, w_2, u_1, u_2, u_3\}$ of V_0 such that $w_1, w_2 \in W_0$ and

$$K = \langle w_1 \wedge w_2,\ w_1 \wedge u_1 + u_2 \wedge u_3 \rangle. \tag{3.1.3}$$

In fact since $[v_0] \notin \mathcal{B}(W, A)$ the following hold:

(1) $\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0) = \{\bigwedge^2 W_0\}$.
(2) $\mathbb{P}(K)$ is *not* tangent to $\mathrm{Gr}(2, V_0)$.

Now let $\{\alpha, \beta\}$ be a basis of K such that $\bigwedge^2 W_0 = \langle \alpha \rangle$. By (1) we have that $\beta \wedge \beta \neq 0$. Let $S := \mathrm{supp}(\beta \wedge \beta)$; thus $\dim S = 4$. Let us prove that $W_0 \not\subset S$. In fact suppose that $W_0 \subset S$. Then $K \subset \bigwedge^2 S$ and since $\mathrm{Gr}(2, S)$ is a quadric hypersurface in $\mathbb{P}(\bigwedge^2 S)$ it follows that either $\mathbb{P}(K)$ intersects $\mathrm{Gr}(2, U)$ in two points or is tangent to it, that contradicts (1) or (2) above. Let $\{w_1, w_2\}$ be a basis of W_0 such that $w_1 \in W_0 \cap S$; it is clear that there exist $u_1, u_2, u_3 \in S$ linearly independent such that $\beta = w_1 \wedge u_1 + u_2 \wedge u_3$. This proves that (3.1.3) holds. Rescaling u_1, u_3 we may assume that

$$\mathrm{vol}_0(\wedge w_1 \wedge w_2 \wedge u_1 \wedge u_2 \wedge u_3) = 1 \tag{3.1.4}$$

where vol_0 is our chosen volume form, see (1.1.7). Let

$$J := \langle w_1 \wedge u_1,\ w_1 \wedge u_2,\ w_1 \wedge u_3,\ w_2 \wedge u_1,\ w_2 \wedge u_2,\ w_2 \wedge u_3,\ u_1 \wedge u_2,\ u_1 \wedge u_3 \rangle. \tag{3.1.5}$$

Thus J is transversal to K by (3.1.3) and hence we have Decomposition (1.2.2). Given $v \in V_0$ we write

$$v = s_1 w_1 + s_2 w_2 + t_1 u_1 + t_2 u_2 + t_3 u_3. \tag{3.1.6}$$

Thus (s_1,s_2,t_1,t_2,t_3) are affine coordinates on V_0 and hence by (1.1.2) they are also coordinates on an open neighborhood of $[v_0]\in V_0$. Let $N=N_J$, $P=P_J$, $Q=Q_J$, $R=R_J$ be the matrix functions appearing in (1.2.3) . A straightforward computation gives that

$$P(v)=\begin{pmatrix}0 & t_1\\ t_1 & -2s_2\end{pmatrix},\quad R(v)=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & t_3 & -t_2\\ -s_2 & 0 & 0 & s_1 & -t_3 & t_2 & 0 & 0\end{pmatrix}. \tag{3.1.7}$$

The 8×8-matrix $(N+Q(v))$ is invertible for (s,t) in a neighborhood of 0; we set

$$(c_{ij})_{1\le i,j\le 8}=-(N+Q(v))^{-1} \tag{3.1.8}$$

where $c_{ij}\in\mathcal{O}_{V_0,0}$. A straightforward computation gives that

$$P(v)-R(v)\cdot(N+Q(v))^{-1}\cdot R(v)^t=\begin{pmatrix}c_{7,7}t_3^2-2c_{7,8}t_2t_3+c_{8,8}t_2^2 & t_1+\delta\\ t_1+\delta & -2s_2+\epsilon\end{pmatrix} \tag{3.1.9}$$

where $\delta,\epsilon\in\mathfrak{m}_0^2$ (here $\mathfrak{m}_0\subset\mathbb{C}[s_1,s_2,t_1,t_2,t_3]$ is the maximal ideal of $(0,\ldots,0)$). Let us prove that

$$\det\begin{pmatrix}c_{7,7}(0) & -c_{7,8}(0)\\ -c_{8,7}(0) & c_{8,8}(0)\end{pmatrix}\neq 0. \tag{3.1.10}$$

Since $Q(0)=0$ we have $c_{ij}(0)=-(\det N)^{-1}\cdot N^{ij}$ where $N^c=(N^{ij})_{1\le i,j\le 8}$ is the matrix of cofactors of N. Thus (3.1.10) is equivalent to

$$\det\begin{pmatrix}N^{7,7} & N^{7,8}\\ N^{8,7} & N^{8,8}\end{pmatrix}\neq 0. \tag{3.1.11}$$

The quadratic form $q_A|_J$ is non-degenerate and hence we have the dual quadratic form $(q_A|_J)^\vee$ on $J^\vee$. Let $U:=\langle u_1,u_2,u_3\rangle$ where u_1,u_2,u_3 are as in (3.1.3). Applying **Lemma 1.16** to $q_A|_J$ and the subspace $W_0\wedge U\subset J$ we get that

$$\operatorname{cork}(q_A|_{W_0\wedge U})=\operatorname{cork}((q_A|_J)^\vee|_{\operatorname{Ann}(W_0\wedge U)}). \tag{3.1.12}$$

By (2.1.7) $q_A|_{W_0\wedge U}$ is non-degenerate: it follows that $(q_A|_J)^\vee|_{\operatorname{Ann}(W_0\wedge U)}$ is non-degenerate as well. The annihilator of $W_0\wedge U$ in $J^\vee$ is given by

$$\operatorname{Ann}(W_0\wedge U)=\langle u_1^\vee\wedge u_2^\vee,\ u_1^\vee\wedge u_3^\vee\rangle \tag{3.1.13}$$

and the Gram-matrix of $(q_A|_J)^\vee|_{\operatorname{Ann}(W_0\wedge U)}$ with respect to the basis given by (3.1.13) is equal to $(\det N)^{-1}(N^{ij})_{7\le i,j\le 8}$. Hence (3.1.11) holds and this proves that (3.1.10) holds. By (3.1.9) and (3.1.10) there exist new analytic coordinates (x_1,x_2,y_1,y_2,y_3) on an open neighborhood $\mathcal{U}$ of $0\in V_0$ - with $(0,\ldots,0)$ corresponding to $0\in V_0$ - such that

$$P(v)-R(v)\cdot(N+Q(v))^{-1}\cdot R(v)^t=\begin{pmatrix}x_1^2+x_2^2 & y_1\\ y_1 & y_2\end{pmatrix}. \tag{3.1.14}$$

(Recall that $\delta,\epsilon\in\mathfrak{m}_0^2$.) By **Proposition 1.4** we get that

$$f_A^{-1}\mathcal{U}=V(\xi_1^2-y_2,\ \xi_1\xi_2+y_1,\ \xi_2^2-x_1^2-x_2^2,)\subset\mathcal{U}\times\mathbb{C}^2 \tag{3.1.15}$$

where (ξ_1,ξ_2) are coordinates on $\mathbb{C}^2$ and our point $x\in X_A$ has coordinates $(0,\ldots,0)$. (Notice that if $k=2$ then the entries of the first matrix of (1.2.6) belong to the ideal generated by the entries of the second matrix of (1.2.6).) Let $B^3(0,r)\subset\mathbb{C}^3$ be a small open ball centered at the origin and let (x_1,x_2,y_3) be coordinates on $\mathbb{C}^3$; there is an obvious isomorphism between an open neighborhood of $0\in f_A^{-1}\mathcal{U}$ and

$$V(\xi_2^2-x_1^2-x_2^2)\subset B^3(0,r)\times\mathbb{C}^2 \tag{3.1.16}$$

taking $(0,\ldots,0)$ to $(0,\ldots,0)$. This proves that X_A is singular at x with analytic germ as claimed. It follows that $f_A^{-1}(\mathbb{P}(W)\setminus\mathcal{B}(W,A)\setminus Y_A[3])\subset\operatorname{sing}Y_A$. On the other hand an arbitrary point x' in a sufficiently small neighborhood of x is mapped to $Y_A(1)$ and if it does not belong to $f_A^{-1}\mathbb{P}(W)$ the map f_A is étale at x': by **Proposition 1.1** Y_A is smooth at $f(x')$ and therefore X_A is smooth at x'. □

Let Σ^{sm} be the smooth locus of Σ.

Corollary 3.2. *Let $A\in(\Sigma^{\mathrm{sm}}\setminus\Delta)$ and W be the unique element in Θ_A (unique by (1.4.6)). Then*

1. $\operatorname{sing}X_A=f_A^{-1}\mathbb{P}(W)$.
2. *Let $x\in f_A^{-1}\mathbb{P}(W)$. The germ (X_A,x) in the classical topology is isomorphic to $(\mathbb{C}^2,0)\times A_1$.*
3. *$C_{W,A}$ is a smooth sextic curve in $\mathbb{P}(W)$.*

(4) *The map*

$$\begin{array}{ccc} f_A^{-1}\mathbb{P}(W) & \longrightarrow & \mathbb{P}(W) \\ x & \mapsto & f_A(x) \end{array} \tag{3.1.17}$$

is a double cover simply branched over $C_{W,A}$.

Proof. (1)-(2): By (1.4.6) $A \notin (\Sigma_\infty \cup \Sigma[2])$, $\dim(A \cap (\bigwedge^2 W \wedge V)) = 1$ and $\mathcal{B}(W, A) = \emptyset$. Moreover $Y_A[3]$ is empty by definition. By **Proposition 3.1** it follows that $f_A^{-1}\mathbb{P}(W) \subset \operatorname{sing} X_A$ and that the analytic germ at $x \in f_A^{-1}\mathbb{P}(W)$ is as stated. It remains to prove that X_A is smooth at $x \in (X_A \setminus f_A^{-1}\mathbb{P}(W))$. Since $A \notin \Delta$ we have that $f_A(x) \in (Y_A(1) \cup Y_A(2))$. If $f_A(x) \in Y_A(1)$ then f_A is étale over $f_A(x)$ (see (1.2.1)) and Y_A is smooth at $f_A(x)$ by **Proposition 1.1**: it follows that X_A is smooth at x. If $f_A(x) \in Y_A(2)$ then X_A is smooth at x by Lemma 2.5 of [21]. (3): Immediate consequence of **Proposition 1.6**. (4): Map (3.1.17) is an étale cover away from $C_{W,A}$, see (1.2.1), while $f_A^{-1}(y)$ is a single point for $y \in C_{W,A}$ - see (3.1.15). Thus either $f_A^{-1}\mathbb{P}(W)$ is singular or else Map (3.1.17) is simply branched over $C_{W,A}$. Items (1), (2) show that $f_A^{-1}\mathbb{P}(W)$ is smooth: it follows that Item (4) holds. □

Definition 3.3. Suppose that $(W, A) \in \widetilde{\Sigma}$ and that $C_{W,A} \neq \mathbb{P}(W)$. We let

$$S_{W,A} \longrightarrow \mathbb{P}(W) \tag{3.1.18}$$

be the double cover ramified over $C_{W,A}$. If Θ_A has a single element we let $S_A := S_{W,A}$.

Remark 3.4. Let $A \in (\Sigma^{\mathrm{sm}} \setminus \Delta)$ and W be the unique element of Θ_A. By Item (4) of **Corollary 3.2** $f_A^{-1}\mathbb{P}(W)$ is identified with S_A and the restriction of f_A to $f_A^{-1}\mathbb{P}(W)$ is identified with the double cover $S_A \to \mathbb{P}(W)$. In particular $f_A^{-1}\mathbb{P}(W)$ is a polarized $K3$ surface of degree 2.

3.2. **Desingularization of X_A for $A \in (\Sigma^{\mathrm{sm}} \setminus \Delta)$.** Let $A \in (\Sigma^{\mathrm{sm}} \setminus \Delta)$ and W be the unique element of Θ_A. Let

$$\pi_A \colon \widetilde{X}_A \to X_A \tag{3.2.1}$$

be the blow-up of $\operatorname{sing} X_A$. Then $\widetilde{X}_A$ is smooth by **Corollary 3.2**. Let

$$\widetilde{H}_A := \pi_A^* H_A, \quad \widetilde{h}_A := c_1(\mathcal{O}_{\widetilde{X}_A}(\widetilde{H}_A)). \tag{3.2.2}$$

Let

$$\mathcal{U} \subset (\mathbb{LG}(\bigwedge^3 V) \setminus \operatorname{sing}\Sigma \setminus \Delta) \tag{3.2.3}$$

be an open (classical topology) contractible neighborhood of A. We may assume that there exists a tautological family of double EPW-sextics $\mathcal{X} \to \mathcal{U}$, see §2 of [21]. Let $\mathcal{H}$ be the tautological divisor class on $\mathcal{X}$: thus $\mathcal{H}|_{X_A} \sim H_A$. The holomorphic line-bundle $\mathcal{O}_{\mathcal{U}}(\Sigma)$ is trivial and hence there is a well-defined double cover $\phi\colon \mathcal{V} \to \mathcal{U}$ ramified over $\Sigma \cap \mathcal{U}$. Let $\mathcal{X}_2 := \mathcal{V} \times_{\mathcal{U}} \mathcal{X}$ be the base change:

$$\begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{\widetilde{\phi}} & \mathcal{X} \\ \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho} \\ \mathcal{V} & \xrightarrow{\phi} & \mathcal{U}. \end{array} \tag{3.2.4}$$

Given $A' \in \Sigma \cap \mathcal{U}$ we will denote by the same symbol the unique point in $\mathcal{V}$ lying over A'.

Proposition 3.5. *Keep notation and assumptions as above. There is a simultaneous resolution of singularities $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ fitting into a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{\pi} & \mathcal{X}_2 \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \rho_2} & \\ & \mathcal{V} & \end{array} \tag{3.2.5}$$

Moreover π is an isomorphism away from $g^{-1}(\phi^{-1}(\Sigma \cap \mathcal{U}))$ and

$$g^{-1}(A) \cong \widetilde{X}_A, \qquad \pi|_{g^{-1}(A)} = \pi_A, \qquad \pi^*\mathcal{H}|_{g^{-1}(A)} \sim \widetilde{H}_A. \tag{3.2.6}$$

Proof. By Proposition 3.2 of [21] $\mathcal{X}$ is smooth and the map ρ of (3.2.4) is a submersion of smooth manifolds away from points $x \in \mathcal{X}$ such that

$$\rho(x) := A' \in \Sigma \cap \mathcal{U}, \quad x \in S_{A'}. \tag{3.2.7}$$

Let (A', x) be as in (3.2.7). By **Proposition 3.1** and smoothness of $\mathcal{X}$ we get that the map of analytic germs $(\mathcal{X}, x) \to (\mathcal{U}, A')$ is isomorphic to

$$\begin{array}{ccc} (\mathbb{C}^3_\xi \times \mathbb{C}^2_\eta \times \mathbb{C}^{53}_t, \mathbf{0}) & \longrightarrow & (\mathbb{C}^{54}_t, \mathbf{0}) \\ (\xi, \eta, t) & \mapsto & (\xi_1^2 + \xi_2^2 + \xi_3^2, t_2, \ldots, t_{54}) \end{array} \tag{3.2.8}$$

Thus (3.2.5) is obtained by the classical process of simultaneous resolution of ordinary double points of surfaces. More precisely let $\widehat{\mathcal{X}}_2 \to \mathcal{X}_2$ be the blow-up of $\operatorname{sing} \mathcal{X}_2$. Then $\widehat{\mathcal{X}}_2$ is smooth and the exceptional divisor is a fibration over $\operatorname{sing} \mathcal{X}_2$ with fibers isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Since $\operatorname{sing} \mathcal{X}_2$ is simply-connected we get that the exceptional divisor has two rulings by $\mathbb{P}^1$'s. It follows that there are two small resolutions of $\mathcal{X}_2$ obtained by contracting the exceptional divisor along either one of the two rulings. Choose one small resolution and call it $\widetilde{\mathcal{X}}_2$. Then (3.2.6) holds. □

Corollary 3.6. *Let $A \in (\Sigma^{\mathrm{sm}} \setminus \Delta)$ and $A' \in \mathbb{LG}(\bigwedge^3 V)^0$. Then $(\widetilde{X}_A, \widetilde{H}_A)$ is a HK variety deformation equivalent to $(X_{A'}, H_{A'})$. Moreover $\mathcal{P}(A) = \Pi(\widetilde{X}_A, \widetilde{H}_A)$ where $\Pi(\widetilde{X}_A, \widetilde{H}_A)$ is given by* (1.6.8).

Proof. Since $\pi_A \colon \widetilde{X}_A \to X_A$ is a blow-up $\widetilde{X}_A$ is projective. By **Proposition 3.5** $\widetilde{X}_A$ is a (smooth) deformation of $X_{A'}$: it follows that $\widetilde{X}_A$ is a HK variety. The remaining statements are obvious. □

Definition 3.7. Let $A \in (\Sigma^{\mathrm{sm}} \setminus \Delta)$. We let $E_A \subset \widetilde{X}_A$ be the exceptional divisor of $\pi_A \colon \widetilde{X}_A \to X_A$ and $\zeta_A := c_1(\mathcal{O}_{\widetilde{X}_A}(E_A))$.

Given $A \in (\Sigma^{\mathrm{sm}} \setminus \Delta)$ we have a smooth conic bundle[1]

$$p \colon E_A \longrightarrow S_A. \tag{3.2.9}$$

Claim 3.8. *Let $(,)$ be the Beauville-Bogomolov quadratic form of $\widetilde{X}_A$. The following formulae hold:*

$$(\widetilde{h}_A, \zeta_A) = 0, \tag{3.2.10}$$

$$(\zeta_A, \zeta_A) = -2. \tag{3.2.11}$$

Proof. We claim that

$$6(\zeta_A, \widetilde{h}_A) = \int_{\widetilde{X}_A} \zeta_A \wedge \widetilde{h}_A^3 = \int_{S_A} h_A^3 = 0. \tag{3.2.12}$$

In fact the first equality follows from Fujiki's relation

$$\int_X \alpha^4 = 3(\alpha, \alpha)^2, \qquad \alpha \in H^2(X) \tag{3.2.13}$$

valid for any deformation of the Hilbert square of a $K3$ (together with the fact that $(\widetilde{h}_A, \widetilde{h}_A) = 2$) and third equality in (3.2.12) holds because $\dim S_A = 2$. Equation (3.2.10) follows from (3.2.12). In order to prove (3.2.11) we notice that $K_{E_A} \cong \mathcal{O}_E(E_A)$ by adjunction and hence

$$\int_{p^{-1}(s)} \zeta_A = -2, \quad s \in S_A. \tag{3.2.14}$$

Using (3.2.13), (3.2.10) and (3.2.14) one gets that

$$2(\zeta_A, \zeta_A) = (\widetilde{h}_A, \widetilde{h}_A) \cdot (\zeta_A, \zeta_A) = \int_{\widetilde{X}_A} \widetilde{h}_A^2 \wedge \zeta_A^2 = 2 \int_{p^{-1}(s)} \zeta_A = -4. \tag{3.2.15}$$

Equation (3.2.11) follows from the above equality. □

3.3. **Conic bundles in HK fourfolds.** We have shown that if $A \in (\Sigma^{\mathrm{sm}} \setminus \Delta)$ then $\widetilde{X}_A$ contains a divisor which is a smooth conic bundle over a $K3$ surface. In the present section we will discuss HK four-folds containing a smooth conic bundle over a $K3$ surface. (Notice that if a divisor in a HK four-fold is a conic bundle over a smooth base then the base is a holomorphic symplectic surface.)

Proposition 3.9. *Let X be a hyperkähler 4-fold. Suppose that X contains a prime divisor E which carries a conic fibration $p \colon E \longrightarrow S$ over a $K3$ surface S. Let $\zeta := c_1(\mathcal{O}_X(E))$. Then:*

(1) *$h^0(\mathcal{O}_X(E)) = 1$ and $h^p(\mathcal{O}_X(E)) = 0$ for $p > 0$.*
(2) *$q_X(\zeta) < 0$ where q_X is the Beauville-Bogomolov quadratic form of X.*

[1] p is a smooth map and each fiber is isomorphic to $\mathbb{P}^1$.

Proof. By adjunction $K_E \cong \mathcal{O}_E(E)$ and hence

$$\int_{p^{-1}(s)} \zeta = -2, \quad s \in S. \tag{3.3.1}$$

Thus $h^0(\mathcal{O}_E(E)) = 0$ and hence $h^0(\mathcal{O}_X(E)) = 1$. Let us prove that the homomorphism

$$H^q(\mathcal{O}_X) \longrightarrow H^q(\mathcal{O}_E) \tag{3.3.2}$$

induced by restriction is an isomorphism for $q < 4$. It is an isomorphism for $q = 0$ because both X and E are connected. The spectral sequence with E_2 term $H^i(R^j(p|_E)\mathcal{O}_E)$ abutting to $H^q(\mathcal{O}_E)$ gives an isomorphism $H^q(\mathcal{O}_E) \cong H^q(\mathcal{O}_S)$. Since S is a $K3$ surface it follows that $H^q(\mathcal{O}_E) = 0$ for $q = 1, 3$. On the other hand $H^q(\mathcal{O}_X) = 0$ for odd q because X is a HK manifold. Thus (3.3.2) is an isomorphism for $q = 1, 3$. It remains to prove that (3.3.2) is an isomorphism for $q = 2$. By Serre duality it is equivalent to prove that the restriction homomorphism $H^0(\Omega^2_X) \to H^0(\Omega^2_E)$ is an isomorphism. Since $1 = h^0(\Omega^2_X) = h^0(\Omega^2_E)$ it suffices to notice that a holomorphic symplectic form on X cannot vanish on E (the maximum dimension of an isotropic subspace for $\sigma|_{T_xX}$ is equal to 2). This finishes the proof that (3.3.2) is an isomorphism for $q < 4$. The long exact cohomology sequence associated to

$$0 \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0 \tag{3.3.3}$$

gives that $h^q(\mathcal{O}_X(-E)) = 0$ for $q < 4$. By Serre duality we get that Item (1) holds.. Let c_X be the Fujiki constant of X; thus

$$\int_X \alpha^4 = c_X q_X(\alpha)^2, \qquad \alpha \in H^2(X). \tag{3.3.4}$$

Let $\iota\colon E \hookrightarrow X$ be Inclusion. Let σ be a holomorphic symplectic form on X. We proved above that there exists a holomorphic symplectic form τ on S such that $\iota^*\sigma = p^*\tau$. Thus we have

$$\frac{c_X}{3} q_X(\zeta) q_X(\sigma + \overline{\sigma}) = \int_X \zeta^2 \wedge (\sigma + \overline{\sigma})^2 = \int_E \iota^*\zeta \wedge p^*(\tau + \overline{\tau})^2 = -2 \int_S (\tau + \overline{\tau})^2. \tag{3.3.5}$$

(The first equality follows from $(\zeta, \sigma + \overline{\sigma}) = 0$, we used (3.3.1) to get the last equality.) On the other hand $c_X > 0$ and $q_X(\sigma + \overline{\sigma}) > 0$: thus $q_X(\zeta) < 0$. □

Let X and E be as in **Proposition 3.9**. Let $\mathrm{Def}_E(X) \subset \mathrm{Def}(X)$ be the germ representing deformations for which E deforms and $\mathrm{Def}_\zeta \subset \mathrm{Def}(X)$ be the germ representing deformations that keep ζ of type $(1, 1)$. We have an inclusion of germs

$$\mathrm{Def}_E(X) \hookrightarrow \mathrm{Def}_\zeta(X). \tag{3.3.6}$$

Corollary 3.10. *Let X and E be as in* **Proposition 3.9**. *The following hold:*

(1) *Inclusion* (3.3.6) *is an isomorphism.*
(2) *Let C be a fiber of the conic vibration $p\colon E \to S$. Then*

$$\{\alpha \in H^2(X;\mathbb{C}) \mid (\alpha, \zeta) = 0\} = \{\alpha \in H^2(X;\mathbb{C}) \mid \int_C \alpha = 0\}. \tag{3.3.7}$$

(3) *The restriction map $H^2(X;\mathbb{C}) \to H^2(E;\mathbb{C})$ is an isomorphism.*

Proof. Item (1) follows at once from Item (1) of **Proposition 3.9** and upper-semicontinuity of cohomology dimension. Let us prove Item (2). Let X_t be a very generic small deformation of X parametrized by a point of $\mathrm{Def}_\zeta \subset \mathrm{Def}(X)$ and $\zeta_t \in H^{1,1}_{\mathbb{Z}}(X_t)$ be the class deforming ζ. A non-trivial rational Hodge sub-structure of $H^2(X_t)$ is equal to $\zeta_t^\perp$ or to $\mathbb{C}\zeta_t$. On the other hand (3.3.6) is an isomorphism: thus X_t contains a deformation E_t of E and hence also a deformation C_t of C. Clearly $\{\alpha \in H^2(X_t;\mathbb{C}) \mid \int_{C_t} \alpha = 0\}$ is a rational Hodge sub-structure of $H^2(X_t)$ containing $H^{2,0}(X_t)$ and non-trivial by (3.3.1): it follows that

$$\{\alpha \in H^2(X_t;\mathbb{C}) \mid (\alpha, \zeta_t) = 0\} = \{\alpha \in H^2(X_t;\mathbb{C}) \mid \int_{C_t} \alpha = 0\}. \tag{3.3.8}$$

The kernel of the restriction map $H^2(X_t;\mathbb{C}) \to H^2(E_t;\mathbb{C})$ is a rational Hodge sub-structure $V_t \subset H^2(X_t)$. By (3.3.1) we know that $\zeta_t \notin V_t$ and since (3.3.2) is an isomorphism for $q = 2$ we know that $H^{2,0}(X_t) \not\subset V_t$; thus $V_t = 0$. Parallel transport by the Gauss-Manin connection gives Items (2) and (3). □

Let $\iota\colon E\hookrightarrow X$ be Inclusion. By Items (2) and (3) of **Corollary 3.10** we have an isomorphism

$$\begin{array}{ccc} \zeta^{\perp} & \overset{\sim}{\longrightarrow} & \{\beta\in H^2(E;\mathbb{C})\mid \int_C\beta=0\} \\ \alpha & \mapsto & \iota^*\alpha \end{array} \tag{3.3.9}$$

On the other hand $p^*\colon H^2(S;\mathbb{C})\to H^2(E;\mathbb{C})$ defines an isomorphism of $H^2(S;\mathbb{C})$ onto the right-hand side of (3.3.9). Thus (3.3.9) gives an isomorphism

$$r\colon \zeta^{\perp}\overset{\sim}{\longrightarrow} H^2(S;\mathbb{C}). \tag{3.3.10}$$

Claim 3.11. *Let X, E be as in* **Proposition 3.9** *and r be as in* (3.3.10). *Suppose in addition that the Fujiki constant c_X is equal to* 3 *and that $q_X(\zeta)=-2$. Let $\alpha\in\zeta^{\perp}$. Then*

$$q_X(\alpha)=\int_S r(\alpha)^2. \tag{3.3.11}$$

Proof. Equality (3.3.1) gives that

$$-2q_X(\alpha)=\frac{c_X}{3}q_X(\zeta)q_X(\alpha)=\int_X\zeta^2\wedge\alpha^2=\int_E\iota^*\zeta\wedge(\iota^*\alpha)^2=-2\int_S r(\alpha)^2. \tag{3.3.12}$$

□

3.4. **The period map on** $(\Sigma^{\rm sm}\setminus\Delta)$. Let $A_0\in(\Sigma^{\rm sm}\setminus\Delta)$. By (1.4.6) and Theorem 2.4.1 of [22] A_0 belongs to the GIT-stable locus of $\mathbb{LG}(\bigwedge^3 V)$. By Luna's étale slice Theorem [14] it follows that there exists an analytic $PGL(V)$-slice at A_0, call it Z_{A_0}, such that the natural map

$$Z_{A_0}/\operatorname{Stab}(A_0)\to\mathfrak{M} \tag{3.4.1}$$

is an isomorphism onto an open (classical topology) neighborhood of $[A_0]$. We may assume that $Z_{A_0}\subset\mathcal{U}$ where $\mathcal{U}$ is as in (3.2.3). Let $\widetilde{Z}_{A_0}:=\phi^{-1}Z_{A_0}$ where $\phi\colon\mathcal{V}\to\mathcal{U}$ is as in (3.2.4). Then ϕ defines a double cover $\widetilde{Z}_{A_0}\to Z_{A_0}$ ramified over $\Sigma\cap Z_{A_0}$; if $A\in\Sigma\cap Z_{A_0}$ we will denote by the same letter the unique point in $\phi^{-1}(A)$. By **Proposition 3.5** points of $\widetilde{Z}_{A_0}$ parametrize deformations of X_A for $A\in\mathbb{LG}(\bigwedge^3 V)^0$. Since Σ is smooth at A_0 also $\Sigma\cap Z_{A_0}$ is smooth at A_0. Thus $\widetilde{Z}_{A_0}$ is smooth at A_0. Shrinking Z_{A_0} around A_0 if necessary we may assume that $\widetilde{Z}_{A_0}$ is contractible. Hence a marking ψ of $(\widetilde{X}_{A_0},\widetilde{H}_{A_0})$ defines a marking of $(\widetilde{X}_A,\widetilde{H}_A)$ for all $A\in Z_{A_0}$; we will denote it by the same letter ψ. Thus we have a local period map

$$\begin{array}{ccc} \widetilde{Z}_{A_0} & \overset{\widetilde{\mathcal{P}}}{\longrightarrow} & \Omega_\Lambda \\ t & \mapsto & \psi_{\mathbb{C}}(H^{2,0}(g^{-1}t)). \end{array} \tag{3.4.2}$$

Claim 3.12. *The local period map $\widetilde{\mathcal{P}}$ of* (3.4.2) *defines an isomorphism of a sufficiently small open neighborhood of A_0 in $\widetilde{Z}_{A_0}$ onto an open subset of Ω_Λ.*

Proof. Since $\widetilde{Z}_{A_0}$ is smooth and $\dim\widetilde{Z}_{A_0}=\dim\Omega_\Lambda$ it suffices to prove that $d\widetilde{\mathcal{P}}(\widetilde{A}_0)$ is injective. By Luna's étale slice Theorem we have an isomorphism of germs

$$(Z_{A_0},A_0)\overset{\sim}{\longrightarrow}\operatorname{Def}(X_{A_0},H_{A_0}) \tag{3.4.3}$$

induced by the local tautological family of double EPW-sextics parametrized by Z_{A_0}. By **Corollary 3.2** the points of $Z_{A_0}\cap\Sigma$ parametrize deformations of X_{A_0} which are locally trivial at points of S_A. Let $\widetilde{\Sigma}_{A_0}\subset\widetilde{Z}_{A_0}$ be the inverse image of $\Sigma\cap Z_{A_0}$ *with reduced structure*. Let $\operatorname{Def}_{\zeta_{A_0}}(\widetilde{X}_{A_0},\widetilde{H}_{A_0})\subset\operatorname{Def}(\widetilde{X}_{A_0},\widetilde{H}_{A_0})$ be the germ representing deformations that "leave ζ_{A_0} of type $(1,1)$". The natural map of germs

$$(\widetilde{\Sigma}_{A_0},A_0)\longrightarrow\operatorname{Def}_{\zeta_{A_0}}(\widetilde{X}_{A_0},\widetilde{H}_{A_0}) \tag{3.4.4}$$

is an inclusion because Map (3.4.3) is an isomorphism. Notice that $\zeta_{A_0}\in\widetilde{h}_{A_0}^{\perp}$ by (3.2.10); since $\zeta_{A_0}\in H^{1,1}_{\mathbb{Z}}(\widetilde{X}_{A_0})$ we have

$$\widetilde{\mathcal{P}}(\widetilde{\Sigma}_{A_0})\subset\psi(\zeta_{A_0})^{\perp}\cap\Omega_\Lambda. \tag{3.4.5}$$

Notice that $\zeta_{A_0}^{\perp}\cap\Omega_\Lambda$ has codimension 1 and is smooth because $(\zeta_{A_0},\zeta_{A_0})=-2$. By injectivity of the local period map we get injectivity of the period map restricted to $\widetilde{\Sigma}_{A_0}$:

$$\begin{array}{ccc} (\widetilde{\Sigma}_{A_0},A_0) & \hookrightarrow & (\psi(\zeta_{A_0})^{\perp}\cap\Omega_\Lambda,\psi_{\mathbb{C}}H^{2,0}(\widetilde{X}_{A_0})) \\ t & \mapsto & \widetilde{\mathcal{P}}(t) \end{array} \tag{3.4.6}$$

Since domain and codomain have equal dimensions the above map is a local isomorphism. In particular $d\widetilde{\mathcal{P}}(A_0)$ is injective when restricted to the tangent space to $\widetilde{\Sigma}_{A_0}$ at A_0. Thus it will suffice to exhibit a tangent vector $v \in T_{A_0}\widetilde{Z}_{A_0}$ such that $d\widetilde{\mathcal{P}}(v) \notin \psi(\zeta_{A_0})^{\perp}$. By Item (1) of **Corollary 3.10** it suffices to prove that E_{A_0} does not lift to 1-st order in the direction v. Let Δ be the unit complex disc and $\gamma\colon \Delta \hookrightarrow \widetilde{Z}_{A_0}$ be an inclusion with $v := \gamma'(0) \notin \widetilde{\Sigma}_{A_0}$. Let $\widetilde{\mathcal{X}}_\Delta \to \Delta$ be obtained by base-change from $g\colon \widetilde{\mathcal{X}}_2 \to \mathcal{V}$. Let $\mathbb{P}^1$ be an arbitrary fiber of (3.2.9); then $N_{\mathbb{P}^1}\mathcal{X}_\Delta \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It follows that E_{A_0} does not lift to 1-st order in the direction v. This finishes the proof that $d\widetilde{\mathcal{P}}(\widetilde{A}_0)$ is injective. □

Proposition 3.13. *The restriction of* $\mathfrak{p}$ *to* $(\Sigma^{\rm sm} \setminus \Delta)//PGL(V)$ *is a dominant map to* $\mathbb{S}_2^\star$ *with finite fibers. Let* $A \in (\Sigma^{\rm sm} \setminus \Delta)$ *and* ψ *be a marking of* $(\widetilde{X}_A, \widetilde{H}_A)$*: then* $\psi(\zeta_A)$ *is a* (-2)*-root of* Λ *and* $\operatorname{div}(\psi(\zeta_A)) = 1$.

Proof. Let $A \in (\Sigma^{\rm sm} \setminus \Delta)$. By **Claim 3.12** we get that $[A]$ is an isolated point in the fiber $\mathfrak{p}^{-1}(\mathfrak{p}([A]))$. In particular

$$\operatorname{cod}(\mathfrak{p}((\Sigma^{\rm sm} \setminus \Delta)//\operatorname{PGL}(V)), \mathbb{D}_\Lambda) = 1. \tag{3.4.7}$$

By (3.2.10) and (3.2.11) $\psi(\zeta_A)$ is a (-2)-root of Λ. By (3.4.5) and **Proposition 1.10** we get that

$$\mathfrak{p}((\Sigma^{\rm sm} \setminus \Delta)//\operatorname{PGL}(V)) \subset \mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2''. \tag{3.4.8}$$

By (3.4.7) and irreducibility of Σ the left-hand side of (3.4.8) is dense in one of $\mathbb{S}_2^\star$, $\mathbb{S}_2'$, $\mathbb{S}_2''$. Let δ_V be as in (1.6.14) and $\delta\colon \mathfrak{M} \to \mathfrak{M}$ be the induced involution, let $\bar{\iota}\colon \mathbb{D}_\Lambda^{BB} \to \mathbb{D}_\Lambda^{BB}$ be the involution given by (1.6.13)). Then $(\Sigma//\operatorname{PGL}(V))$ is mapped to itself by δ and hence (1.6.15) gives that its image under the period map $\mathfrak{p}$ is mapped to itself by $\bar{\iota}$. By (1.7.4) it follows that $\mathfrak{p}$ maps $(\Sigma^{\rm sm} \setminus \Delta)//PGL(V)$ into $\mathbb{S}_2^\star$ and hence that $\operatorname{div}(\psi(\zeta_A)) = 1$. □

3.5. **Periods of $K3$ surfaces of degree 2.** Let $A \in (\Sigma^{\rm sm} \setminus \Delta)$. We will recall results of Shah on the period map for double covers of $\mathbb{P}^2$ branched over a sextic curve. Let $\mathfrak{C}_6 := |\mathcal{O}_{\mathbb{P}^2}(6)|//PGL_3$ and Φ be the lattice given by (1.7.10). There is a period map

$$\mathfrak{s}\colon \mathfrak{C}_6 \dashrightarrow \mathbb{D}_\Phi^{BB} \tag{3.5.1}$$

whose restriction to the open set parametrizing smooth sextics is defined as follows. Let C be a smooth plane sextic and $f\colon S \to \mathbb{P}^2$ be the double cover branched over C. Then (3.5.1) maps the orbit of C to the period point of the polarized $K3$ surface $(S, f^*\mathcal{O}_{\mathbb{P}^2}(1))$. Shah [23] determined the "boundary"and the indeterminacy locus of the above map. In order to state Shah' results we recall a definition.

Definition 3.14. A curve $C \subset \mathbb{P}^2$ has a *simple singularity* at $p \in C$ if and only if the following hold:

(i) C is reduced in a neighborhood of p.
(ii) $\operatorname{mult}_p(C) \leq 3$ and if equality holds C does not have a consecutive triple point at p.[2]

Remark 3.15. Let $C \subset \mathbb{P}^2$ be a sextic curve. Then C has simple singularities if and only if the double cover $S \to \mathbb{P}^2$ branched over C is a normal surface with DuVal singularities or equivalently the minimal desingularization $\widetilde{S}$ of S is a $K3$ surface (with A-D-E curves lying over the singularities of S), see Theorem 7.1 of [1].

Let $C \subset \mathbb{P}^2$ be a sextic curve with simple singularities. Then C is PGL_3-stable by [23]. We let

$$\mathfrak{C}_6^{ADE} := \{C \in |\mathcal{O}_{\mathbb{P}^2}(6)| \mid C \text{ has simple singularities}\}//PGL_3. \tag{3.5.2}$$

Let C be a plane sextic. If C has simple singularities the period map (3.5.1) is regular at C and takes value in $\mathbb{D}_\Phi$ - see **Remark 3.15**. More generally Shah [23] proved that (3.5.1) is regular at C if and only if C is PGL_3-semistable and the unique closed orbit in $\overline{PGL_3 C} \cap |\mathcal{O}_{\mathbb{P}^2}(6)|^{ss}$ is not that of triple (smooth) conics.

Definition 3.16. Let $\mathbb{LG}(\bigwedge^3 V)^{ADE} \subset \mathbb{LG}(\bigwedge^3 V)$ be the set of A such that $C_{W,A}$ is a curve with simple singularities for every $W \in \Theta_A$. Let $\mathbb{LG}(\bigwedge^3 V)^{ILS} \subset \mathbb{LG}(\bigwedge^3 V)$ be the set of A such that the period map (3.5.1) is regular at $C_{W,A}$ for every $W \in \Theta_A$.

[2] C has a consecutive triple point at p if the strict transform of C in $Bl_p(\mathbb{P}^2)$ has a point of multiplicity 3 lying over p.

Notice that both $\mathbb{LG}(\bigwedge^3 V)^{ADE}$ and $\mathbb{LG}(\bigwedge^3 V)^{ILS}$ are open. We have inclusions

$$(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)\subset\mathbb{LG}(\bigwedge^3 V)^{ADE}\subset\mathbb{LG}(\bigwedge^3 V)^{ILS}. \tag{3.5.3}$$

The reason for the superscript ILS is the following: a curve $C\in|\mathcal{O}_{\mathbb{P}(W)}(6)|$ is in the regular locus of the period map (0.0.9) if and only if the double cover of $\mathbb{P}(W)$ branched over C has *Insignificant Limit Singularities* in the terminology of Mumford, see [24].

Definition 3.17. Let $\Sigma^{ILS}:=\Sigma\cap\mathbb{LG}(\bigwedge^3 V)^{ILS}$. Let $\widetilde{\Sigma}^{ILS}\subset\widetilde{\Sigma}$ be the inverse image of Σ^{ILS} for the natural forgetful map $\widetilde{\Sigma}\to\Sigma$, and $\widehat{\Sigma}^{ILS}\subset\widehat{\Sigma}$

$$\widehat{\Sigma}^{ILS}:=(p|_{\widehat{\Sigma}})^{-1}(\Sigma^{ILS}) \tag{3.5.4}$$

where $p\colon\widehat{\mathbb{LG}(\bigwedge^3 V)}\to\mathbb{LG}(\bigwedge^3 V)$ and $\widehat{\Sigma}$ are as in **Definition 2.6**.

3.6. **The period map on Σ and periods of $K3$ surfaces.** Let

$$\begin{array}{ccc}\widetilde{\Sigma}^{ILS} & \overset{\tau}{\longrightarrow} & \Sigma^{ILS}\\ (W,A) & \mapsto & A\end{array} \tag{3.6.1}$$

be the forgetful map. Let $A\in(\Sigma^{\rm sm}\setminus\Delta)$: then Θ_A is a singleton by (1.4.6) and if W is the unique element of Θ_A then $C_{W,A}$ is smooth sextic by Item (3) of **Corollary 3.2**. It follows that $(\Sigma^{\rm sm}\setminus\Delta)\subset\mathbb{LG}(\bigwedge^3 V)^{ILS}$ and τ defines an isomorphism $\tau^{-1}(\Sigma^{\rm sm}\setminus\Delta)\to(\Sigma^{\rm sm}\setminus\Delta)$. Thus we may regard $(\Sigma^{\rm sm}\setminus\Delta)$ as an (open dense) subset of $\widetilde{\Sigma}^{ILS}$:

$$\iota\colon(\Sigma^{\rm sm}\setminus\Delta)\hookrightarrow\widetilde{\Sigma}^{ILS}. \tag{3.6.2}$$

By definition of $\widetilde{\Sigma}^{ILS}$ we have a regular map

$$\begin{array}{ccc}\widetilde{\Sigma}^{ILS} & \overset{q}{\longrightarrow} & \mathbb{D}^{BB}_{\Phi}\\ (W,A) & \mapsto & \Pi(S_{W,A},D_{W,A})\end{array} \tag{3.6.3}$$

where $D_{W,A}$ is the pull-back to $S_{W,A}$ of $\mathcal{O}_{\mathbb{P}(W)}(1)$ and $\Pi(S_{W,A},D_{W,A})$ is the (extended) period point of $(S_{W,A},D_{W,A})$. Recall that we have defined a finite map $\rho\colon\mathbb{D}^{BB}_{\Gamma}\to\mathbb{D}^{BB}_{\Phi}$, see (1.7.11) and that there is a natural map $\nu\colon\mathbb{D}^{BB}_{\Gamma}\to\overline{\mathbb{S}}^{\star}_2$ which is identified with the normalization of $\overline{\mathbb{S}}^{\star}_2$, see (1.7.7).

Proposition 3.18. *There exists a regular map*

$$Q\colon\widetilde{\Sigma}^{ILS}\to\mathbb{D}^{BB}_{\Gamma} \tag{3.6.4}$$

such that $\rho\circ Q=q$. Moreover the composition $\nu\circ(Q|_{(\Sigma^{\rm sm}\setminus\Delta)})$ is equal to the restriction of the period map $\mathcal{P}$ to $(\Sigma^{\rm sm}\setminus\Delta)$.

Proof. By **Proposition 3.13** the restriction of the period map to $(\Sigma^{\rm sm}\setminus\Delta)$ is a dominant map to $\overline{\mathbb{S}}^{\star}_2$ and therefore it lifts to the normalization of $\overline{\mathbb{S}}^{\star}_2$:

$$\begin{array}{ccc} & & \mathbb{D}^{BB}_{\Gamma}\\ & \overset{Q_0}{\nearrow} & \downarrow\nu\\ (\Sigma^{\rm sm}\setminus\Delta) & \xrightarrow{\mathcal{P}|_{(\Sigma^{\rm sm}\setminus\Delta)}} & \overline{\mathbb{S}}^{\star}_2\end{array} \tag{3.6.5}$$

We claim that

$$\rho\circ Q_0=q|_{(\Sigma^{\rm sm}\setminus\Delta)}. \tag{3.6.6}$$

In fact let $A\in(\Sigma^{\rm sm}\setminus\Delta)$. Let $r\colon\zeta_A^{\perp}\to H^2(S_A;\mathbb{C})$ be the isomorphism given by (3.3.10). Let's prove that

$$[H^2(S_A;\mathbb{Z}):r(\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z}))]=2. \tag{3.6.7}$$

In fact r is a homomorphism of lattices by **Claim 3.11**. Since $H^2(S_A;\mathbb{Z})$ and $\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z})$ have the same rank it follows that $r(\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z}))$ is of finite index in $H^2(S_A;\mathbb{Z})$: let d be the the index. By the last clause of **Proposition 3.13** the lattice $(\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z}))$ is isometric to $\widetilde{\Gamma}$ - see (1.7.5). Hence we have

$$-4=\operatorname{discr}\widetilde{\Gamma}=\operatorname{discr}(\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z}))=d^2\cdot\operatorname{discr}H^2(S_A;\mathbb{Z})=-d^2. \tag{3.6.8}$$

Equation (3.6.7) follows at once. Next let $\psi\colon H^2(\widetilde{X}_A;\mathbb{Z})\xrightarrow{\sim}\widetilde{\Lambda}$ be a marking of $(\widetilde{X}_A,\widetilde{H}_A)$. By the last clause of **Proposition 3.13** we know that $\psi(\zeta_A)$ is a (-2)-root of Λ of divisibility 1. By **Proposition 1.10** there exists $g\in\widetilde{O}(\Lambda)$ such that $g\circ\psi(\zeta_A)=e_3$. Let $\phi:=g\circ\psi$. Then ϕ is a new marking of $(\widetilde{X}_A,\widetilde{H}_A)$ and $\phi(\zeta_A)=e_3$. It follows that $\phi(\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z}))=\widetilde{\Gamma}$. Let $\psi_{\mathbb{Q}}\colon H^2(\widetilde{X}_A;\mathbb{Q})\xrightarrow{\sim}\widetilde{\Lambda}_{\mathbb{Q}}$ be the $\mathbb{Q}$-linear extension of ϕ. By (3.6.7) $H^2(S_A;\mathbb{Z})$ is an overlattice of $\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z})$ and hence it may be emebedded canonically into $H^2(\widetilde{X}_A;\mathbb{Q})$: thus $\phi_{\mathbb{Q}}(H^2(S_A;\mathbb{Z}))$ makes sense. By (3.6.7) we get that $\phi_{\mathbb{Q}}(H^2(S_A;\mathbb{Z}))$ is an overlattice of $\phi(\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z}))$ and that $\phi(\zeta_A^{\perp}\cap H^2(\widetilde{X}_A;\mathbb{Z}))$ has index 2 in $\phi_{\mathbb{Q}}(H^2(S_A;\mathbb{Z}))$. By **Claim 1.13** it follows that $\phi_{\mathbb{Q}}(H^2(S_A;\mathbb{Z}))=\widetilde{\Phi}$. Equation (3.6.6) follows at once from this. By (3.6.6) we have a commutative diagram

$$\begin{array}{ccccc} & & \widetilde{\Sigma}^{ILS}\times_{\mathbb{D}^{BB}_{K_2}}\mathbb{D}^{BB}_{\Gamma} & \longrightarrow & \mathbb{D}^{BB}_{\Gamma} \\ & \overset{(\iota,Q_0)}{\nearrow} & \downarrow & & \downarrow \rho \\ (\Sigma^{\mathrm{sm}}\setminus\Delta) & \xrightarrow{\ \iota\ } & \widetilde{\Sigma}^{ILS} & \xrightarrow{\ q\ } & \mathbb{D}^{BB}_{\Phi}. \end{array} \tag{3.6.9}$$

where ι is the inclusion map (3.6.2). Let $\mathcal{Z}$ be the closure of $\operatorname{Im}(\iota,Q_0)$. Then $\mathcal{Z}$ is an irreducible component of $\widetilde{\Sigma}^{ILS}\times_{\mathbb{D}^{BB}_{K_2}}\mathbb{D}^{BB}_{\Gamma}$ because ι is an open inclusion. The natural projection $\mathcal{Z}\to\widetilde{\Sigma}^{ILS}$ is a finite birational map and hence it is an isomorphism because $\widetilde{\Sigma}^{ILS}$ is smooth. We define the map $Q\colon\widetilde{\Sigma}^{ILS}\to\mathbb{D}^{BB}_{\Gamma}$ as the composition of the inverse $\widetilde{\Sigma}^{ILS}\to\mathcal{Z}$ and the projection $\mathcal{Z}\to\mathbb{D}^{BB}_{\Gamma}$. The properties of Q stated in the proposition hold by commutativity of (3.6.9). □

Corollary 3.19. *The image of the map* $(\tau,\nu\circ Q)\colon\widetilde{\Sigma}^{ILS}\to\Sigma^{ILS}\times\overline{\mathbb{S}}_2^{\star}$ *is equal to* $\widehat{\Sigma}^{ILS}$.

Proof. Let $p\colon\widehat{\mathbb{LG}(\bigwedge^3 V)}\to\mathbb{LG}(\bigwedge^3 V)$ be as in **Definition 2.6**. Since $\mathcal{P}$ is regular on $(\Sigma^{\mathrm{sm}}\setminus\Delta)$ the map $p^{-1}(\Sigma^{\mathrm{sm}}\setminus\Delta)\to(\Sigma^{\mathrm{sm}}\setminus\Delta)$ is an isomorphism and $p^{-1}(\Sigma^{\mathrm{sm}}\setminus\Delta)$ is an open dense subset of $\widehat{\Sigma}^{ILS}$ (recall that Σ is irreducible and hence so is $\widehat{\Sigma}$). By the second clause of **Proposition 3.18** we have that

$$(\tau,\nu\circ Q)(\Sigma^{\mathrm{sm}}\setminus\Delta)=p^{-1}(\Sigma^{\mathrm{sm}}\setminus\Delta). \tag{3.6.10}$$

Since $\widehat{\Sigma}$ is closed in $\widehat{\mathbb{LG}}(\bigwedge^3 V)\times\mathbb{D}^{BB}_{\Lambda}$ it follows that $\operatorname{Im}(\tau,\nu\circ Q)\subset\widehat{\Sigma}$. The commutative diagram

$$\begin{array}{ccc} \widetilde{\Sigma}^{ILS} & \xrightarrow{(\tau,\nu\circ Q)} & \widehat{\Sigma}^{ILS} \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle p|_{\widehat{\Sigma}^{ILS}}} & \\ & \Sigma^{ILS} & \end{array} \tag{3.6.11}$$

gives that $\operatorname{Im}(\tau,\nu\circ Q)\subset\widehat{\Sigma}^{ILS}$. The right-hand side of (3.6.10) is dense in $\widehat{\Sigma}^{ILS}$: thus in order to finish the proof it suffices to show that $\operatorname{Im}(\tau,\nu\circ Q)$ is closed in $\widehat{\Sigma}^{ILS}$. The equality $(p|_{\widehat{\Sigma}^{ILS}})\circ(\tau,\nu\circ Q)=\tau$ and properness of τ give that $(\tau,\nu\circ Q)$ is proper (see Ch. II, Cor. 4.8, Item (e) of [8]) and hence closed: thus $\operatorname{Im}(\tau,\nu\circ Q)$ is closed in $\widehat{\Sigma}^{ILS}$. □

3.7. **Extension of the period map.**

Proposition 3.20. *Let* $A\in\mathbb{LG}(\bigwedge^3 V)^{ILS}$. *If* $\dim\Theta_A\le 1$ *the period map* $\mathcal{P}$ *is regular at* A *and moreover* $\mathcal{P}(A)\in\mathbb{D}_{\Lambda}$ *if and only if* $A\in\mathbb{LG}(\bigwedge^3 V)^{ADE}$.

Proof. If $A\notin\Sigma$ then $\mathcal{P}$ is regular at A by **Proposition 2.5**. Now assume that $A\in\Sigma^{ILS}$. Suppose that $\mathcal{P}$ is not regular at A: we will reach a contradiction. Let $p\colon\widehat{\mathbb{LG}}(\bigwedge^3 V)\to\mathbb{LG}(\bigwedge^3 V)$ be as in **Definition 2.6**. Then $p^{-1}(A)\cap\widehat{\Sigma}$ is a subset of $\{A\}\times\mathbb{D}^{BB}_{\Lambda}$ and hence we may identify it with its projection in $\mathbb{D}^{BB}_{\Lambda}$. This subset is equal to $\nu\circ Q(\tau^{-1}(A))$ by **Corollary 3.19** and Commutative Diagram (3.6.11). On the other hand $\tau^{-1}(A)=\Theta_A$ and hence $\dim\tau^{-1}(A)\le 1$ by hypothesis: it follows that $\dim p^{-1}(A)\le 1$ and this contradicts **Corollary 2.7**. This proves that $\mathcal{P}$ is regular at A. The last clause of the proposition follows from **Corollary 3.19**. □

Proof of **Theorem 0.2**. Let $x\in(\mathfrak{M}\setminus\mathfrak{I})$. There exists a GIT-semistable $A\in\mathbb{LG}(\bigwedge^3 V)$ representing x with $\mathrm{PGL}(V)$-orbit closed in the semistable locus $\mathbb{LG}(\bigwedge^3 V)^{\mathrm{ss}}$, and such A is determined up to the action of $\mathrm{PGL}(V)$. By Luna's étale slice Theorem [14] it suffices to prove that the period map $\mathcal{P}$ is regular at A. If $A\notin\Sigma$ then $\mathcal{P}$ is regular at A and $\mathcal{P}(A)\in\mathbb{D}_{\Lambda}$ by **Proposition 2.5**. Now suppose that $A\in\Sigma$. Then $A\in\Sigma^{ILS}$ because $x\notin\mathfrak{I}$. By **Proposition 3.20** in order to prove that $\mathcal{P}$ is regular at A it will suffice to show that $\dim\Theta_A\le 1$. Suppose that $\dim\Theta_A\ge 2$, we will reach a contradiction. Theorem 3.26 and Theorem 3.36 of [20] give that A belongs to certain subsets of $\mathbb{LG}(\bigwedge^3 V)$ (notice the misprint in the statement of Theorem 3.36: $\mathbb{X}_{\mathcal{D}}$ is to be replaced by $\mathbb{X}_{\mathcal{D},+}$). Since A is semistable the results of [22] give that $A\in(\mathbb{X}_{\mathcal{Y}}\cup\mathbb{X}_{\mathcal{W}}\cup\mathbb{X}_h\cup\mathbb{X}_k\cup\mathbb{X}_+)$. Proposition 4.3.7 of [22] gives that if $A\in\mathbb{X}_{\mathcal{Y}}$ then $A\in\mathrm{PGL}_6 A_+$ (i.e. $A\in\mathbb{X}_+$), thus $A\in(\mathbb{X}_{\mathcal{W}}\cup\mathbb{X}_h\cup\mathbb{X}_k\cup\mathbb{X}_+)$. Then by applying the results of

Sections 4.3 and 4.4 of [22] we get that $A \notin \mathbb{LG}(\bigwedge^3 V)^{ILS}$, that is a contradiction. This shows that $\dim \Theta_A \le 1$ and hence $\mathfrak{p}$ is regular at x. The last clause of **Proposition 3.20** gives that $\mathfrak{p}(x) \in \mathbb{D}_\Lambda$ if and only if $x \in \mathfrak{M}^{ADE}$. □

4. On the image of the period map

We will prove **Theorem 0.3**.

4.1. **Proof that** $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma) \cap \mathbb{S}_2^\star = \emptyset$**.** Let S be a $K3$ surface. We recall [2] the description of $H^2(S^{[2]})$ and the Beauville-Bogomolov form $q_{S^{[2]}}$ in terms of $H^2(S)$. Let $\mu\colon H^2(S) \to H^2(S^{[2]})$ be the composition of the symmetrization map $H^2(S) \to H^2(S^{(2)})$ and the pull-back $H^2(S^{(2)}) \to H^2(S^{[2]})$. There is a direct sum decomposition

$$H^2(S^{[2]}) = \mu(H^2(S;\mathbb{Z})) \oplus \mathbb{Z}\xi \tag{4.1.1}$$

where 2ξ is represented by the locus parametrizing non-reduced subschemes. Moreover if $H^2(S)$ and $H^2(S^{[2]})$ are equipped with the intersection form and Beauville-Bogomolov quadratic form $q_{S^{[2]}}$ respectively, then μ is an isometric embedding, Decomposition (4.1.1) is orthogonal, and $q_{S^{[2]}}(\xi) = -2$. Recall that $\delta_V : \mathbb{LG}(\bigwedge^3 V) \xrightarrow{\sim} \mathbb{LG}(\bigwedge^3 V^\vee)$ is defined by $\delta_V(A) := \operatorname{Ann} A$, see (1.6.14).

Lemma 4.1. $\mathcal{P}(\Delta \setminus \Sigma) \not\subset (\mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4)$ *and* $\mathcal{P}(\delta_V(\Delta) \setminus \Sigma) \not\subset (\mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4)$.

Proof. Let $A \in (\Delta \setminus \Sigma)$ be generic. By Theorem 4.15 of [21] there exist a projective $K3$ surface S_A of genus 6 and a small contraction $S_A^{[2]} \to X_A$. Moreover the period point $\mathcal{P}(A)$ may be identified with the Hodge structure of $S_A^{[2]}$ as follows. The surface S_A comes equipped with an ample divisor D_A of genus 6 i.e. $D_A \cdot D_A = 10$, let d_A be the Poincarè dual of D_A. Then $\mathcal{P}(A)$ is identified with the Hodge structure on $(\mu(D_A) - 2\xi)^\perp$, where ξ is as above. By Proposition 4.14 of [21] we may assume that (S_A, D_A) is a general polarized $K3$ surface of genus 6. It follows that if A is very general in $(\Delta \setminus \Sigma)$ then $H^{1,1}_{\mathbb{Z}}(S_A) = \mathbb{Z}d_A$. Thus for $A \in (\Delta \setminus \Sigma)$ very general we have that

$$H^{1,1}_{\mathbb{Z}}(S_A^{[2]}) \cap (\mu(d_A) - 2\xi)^\perp = \mathbb{Z}(2\mu(d_A) - 5\xi). \tag{4.1.2}$$

Now suppose that $\mathcal{P}(A) \in (\mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4)$: by definition there exists $\alpha \in H^{1,1}_{\mathbb{Z}}(S_A^{[2]}) \cap (\mu(d_A) - 2\xi)^\perp$ of square (-2) or (-4): since $q_{S_A^{[2]}}(2\mu(d_A) - 5\xi) = -10$ that contradicts (4.1.2). This proves that $\mathcal{P}(\Delta \setminus \Sigma) \not\subset (\mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4)$. Next let $\bar{\iota}\colon \mathbb{D}_\Lambda^{BB} \to \mathbb{D}_\Lambda^{BB}$ be the involution defined by δ_V, see (1.6.13). Then $\bar{\iota}$ maps $(\mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4)$ to itself, see (1.7.4), and hence $\mathcal{P}(\delta_V(\Delta) \setminus \Sigma) \not\subset (\mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4)$ because otherwise it would follow that $\mathcal{P}(\Delta \setminus \Sigma) \subset (\mathbb{S}_2^\star \cup \mathbb{S}_2' \cup \mathbb{S}_2'' \cup \mathbb{S}_4)$. □

Suppose that $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma) \cap \mathbb{S}_2^\star \neq \emptyset$. Since $\mathbb{S}_2^\star$ is a $\mathbb{Q}$-Cartier divisor of $\mathbb{D}_\Lambda$ it follows that $\mathcal{P}^{-1}(\mathbb{S}_2^\star) \cap (\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$ has pure codimension 1: let C be one of its irreducible components. Then $C \neq \Delta$ by **Lemma 4.1** and hence $C^0 := C \setminus \Delta$ is a codimension-1 $\mathrm{PGL}(V)$-invariant closed subset of $(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma)$. Since C^0 has pure codimension 1 and is contained in the stable locus of $\mathbb{LG}(\bigwedge^3 V)$ (see [22]) the quotient $C^0/\!/\mathrm{PGL}(V)$ has codimension 1 in $\mathfrak{M}$. If $A \in (C^0 \setminus \Delta)$ then X_A is smooth and hence the local period map $\mathrm{Def}(X_A, H_A) \to \Omega_\Lambda$ is a (local) isomorphism (local Torelli for hyperkähler manifolds): it follows that the restriction of $\mathfrak{p}$ to $C^0/\!/\mathrm{PGL}(V)$ has finite fibers and hence $\mathcal{P}(C^0)$ is dense in $\mathbb{S}_2^\star$. Now consider the period map $\mathfrak{p}\colon (\mathfrak{M} \setminus \mathfrak{I}) \to \mathbb{D}_\Lambda^{BB}$: it is birational by Verbitsky's Global Torelli and Markman's monodromy results, see Theorem 1.3 and Lemma 9.2 of [15]. We have proved that there are (at least) two distinct components in $\mathfrak{p}^{-1}(\overline{\mathbb{S}}_2^\star)$ which are mapped *dominantly* to $\overline{\mathbb{S}}_2^\star$ by $\mathfrak{p}$, namely $((\Sigma/\!/\mathrm{PGL}(V)) \setminus \mathfrak{I})$ and the closure of $C^0/\!/\mathrm{PGL}(V)$: that is a contradiction because $\mathbb{D}_\Lambda^{BB}$ is normal. □

4.2. **Proof that** $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma) \cap (\mathbb{S}_2' \cup \mathbb{S}_2'') = \emptyset$**.** First we will prove that $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V) \setminus \Sigma) \cap \mathbb{S}_2'' = \emptyset$. Let U be a 3-dimensional complex vector space and $\pi\colon S \to \mathbb{P}(U)$ a double cover branched over a smooth sextic curve; thus S is a $K3$ surface. Let $D \in |\pi^*\mathcal{O}_S(1)|$ and $d \in H^{1,1}_{\mathbb{Z}}(S;\mathbb{Z})$ be its Poincaré dual. Since $S^{[2]}$ is simply connected there is a unique class $\mu(D) \in \mathrm{Pic}(S^{[2]})$ whose first Chern class is equal to $\mu(d)$. One easily checks the following facts. There is a natural isomorphism

$$\mathbb{P}(\mathrm{S}^2 U^\vee) \xrightarrow{\sim} |\mu(D)| \tag{4.2.1}$$

and the composition of the natural maps

$$S^{[2]} \longrightarrow S^{(2)} \longrightarrow \mathbb{P}(U)^{(2)} \longrightarrow \mathbb{P}(\mathrm{S}^2 U) \tag{4.2.2}$$

is identified with the natural map $f\colon S^{[2]}\to|\mu(D)|^{\vee}$. The image of f is the chordal variety $\mathcal{V}_2$ of the Veronese surface $\{[u^2]\mid 0\neq u\in U\}$, and the map $S^{[2]}\to\mathcal{V}_2$ is finite of degree 4. Since $\mu(d)$ has square 2 we have a well-defined period point $\Pi(S^{[2]},\mu(d))\in\mathbb{D}_{\Lambda}$. The class $\xi\in H^{1,1}_{\mathbb{Z}}(S^{[2]})$ is a (-2)-root of divisibility 2 and it is orthogonal to $\mu(d)$: it follows that $\Pi(S^{[2]},\mu(d))\in(\mathbb{S}_2'\cup\mathbb{S}_2'')$. Actually $\Pi(S^{[2]},\mu(d))\in\mathbb{S}_2''$ because the divisibility of ξ as an element of $H^2(S^{[2]};\mathbb{Z})$ is equal to 2 (and not only as element of $\mu(d)^{\perp}$). The periods $\Pi(S^{[2]},\mu(d))$ with S as above fill-out an open dense subset of $\mathbb{S}_2''$. Now suppose that there exists $A\in(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)$ such that $\mathcal{P}(A)\in\mathbb{S}_2''$. Since $\mathbb{S}_2''$ is a $\mathbb{Q}$-Cartier divisor of $\mathbb{D}_{\Lambda}$ it follows that $\mathcal{P}^{-1}(\mathbb{S}_2'')\cap(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)$ has pure codimension 1: let C be one of its irreducible components. By **Lemma 4.1** $C^0:=(C\setminus\Delta)$ is a codimension-1 subset of $\mathbb{LG}(\bigwedge^3 V)^0$ and hence $\mathcal{P}(C^0)$ contains an open dense subset of $\mathbb{S}_2''$. It follows that there exist $A\in C^0$ and a double cover $\pi\colon S\to\mathbb{P}(U)$ as above *with* $\mathrm{Pic}(S)=\mathbb{Z}\mu(D)$ and such that $\mathcal{P}(A)=\Pi(S^{[2]},\mu(d))$. By Verbitsky's Global Torelli Theorem there exists a birational map $\varphi\colon S^{[2]}\dashrightarrow X_A$. Now φ^*h_A is a $(1,1)$-class of square 2: since $\mathrm{Pic}(S)=\mathbb{Z}\mu(D)$ it follows that $\varphi^*h_A=\pm\mu(d)$, and hence $\varphi^*H_A=\mu(D)$ because $|-\mu(D)|$ is empty. But that is a contradiction because the map $f_A\colon X_A\to|H_A|^{\vee}$ is 2-to-1 onto its image while the map $f\colon S^{[2]}\to|\mu(D)|^{\vee}$ has degree 4 onto its image. This proves that

$$\mathcal{P}(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)\cap\mathbb{S}_2''=\emptyset. \tag{4.2.3}$$

It remains to prove that $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)\cap\mathbb{S}_2'=\emptyset$. Suppose that $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)\cap\mathbb{S}_2'\neq\emptyset$. Let $\Sigma(V^{\vee})$ be the locus of $A\in\mathbb{LG}(\bigwedge^3 V^{\vee})$ containing a non-zero decomposable tri-vector. Since $\delta_V(\Sigma)=\Sigma(V^{\vee})$ we get that $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V^{\vee})\setminus\Sigma(V^{\vee}))\cap\mathbb{S}_2''\neq\emptyset$ by (1.6.15) and (1.7.4): that contradicts (4.2.3). □

Remark 4.2. In the above proof we have noticed that the generic point of $\mathbb{S}_2''$ is equal to $\Pi(S^{[2]},\mu(d))$. One may also identify explicitly polarized hyperkähler varieties whose periods belong to $\mathbb{S}_2'$. In fact let $\pi\colon S\to\mathbb{P}(U)$ and D, d be as above. Let $v\in H^*(S;\mathbb{Z})$ be the Mukai vector $v:=(0,d,0)$ and let $\mathcal{M}_v$ be the corresponding moduli space of D-semistable sheaves on S with Mukai vector v: the generic such sheaf is isomorphic to $\iota_*\eta$ where $\iota\colon C\hookrightarrow S$ is the inclusion of a smooth $C\in|D|$ and η is an invertible sheaf on C of degree 1. As is well-known $\mathcal{M}_v$ is a hyperkähler variety deformation equivalent to $K3^{[2]}$. Moreover $H^2(\mathcal{M}_v)$ with its Hodge structure and B-B form is identified with $v^{\perp}$ with the Hodge structure it inherits from the Hodge structure of $H^*(S)$ and the quadratic form given by the Mukai pairing, see [25]. Let $h\in H^2(\mathcal{M}_v)$ correspond to $\pm(1,0,-1)$. Then h has square 2 and, as is easily checked, the period point of $(\mathcal{M}_v,h)$ belongs to $\mathbb{S}_2'$: more precisely $\Pi(\mathcal{M}_v,h)=\bar{\iota}(\Pi(S^{[2]},\mu(d)))$.

4.3. **Proof that** $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)\cap\mathbb{S}_4=\emptyset$**.** Let $S\subset\mathbb{P}^3$ be a smooth quartic surface, $D\in|\mathcal{O}_S(1)|$ and d be the Poincarè dual of D. We have a natural map

$$\begin{array}{ccc} S^{[2]} & \xrightarrow{f} & \mathrm{Gr}(1,\mathbb{P}^3)\subset\mathbb{P}^5 \\ Z & \mapsto & \langle Z\rangle \end{array} \tag{4.3.1}$$

where $\langle Z\rangle$ is the unique line containing the lenght-2 scheme Z. Let $H\in|f^*\mathcal{O}_{\mathbb{P}^5}(1)|$ and h be its Poincarè dual. One checks easily that $h=(\mu(d)-\xi)$, in particular $q_{S^{[2]}}(h)=2$. Moreover pull-back gives an identification of f with the natural map $S^{[2]}\to|H|^{\vee}$. The equalities

$$(h,\mu(d)-2\xi)_{S^{[2]}}=0,\qquad q_{S^{[2]}}(\mu(d)-2\xi)=-4,\qquad (h^{\perp},\mu(d)-2\xi)_{S^{[2]}}=2\mathbb{Z} \tag{4.3.2}$$

(here $h^{\perp}\subset H^2(S^{[2]};\mathbb{Z})$ is the subgroup of classes orthogonal to h) show that $(\mu(d)-2\xi)$ is a (-4)-root of $h^{\perp}$ and hence $\Pi(S^{[2]},h)\in\mathbb{S}_4$ by **Proposition 1.10**. Moreover the generic point of $\mathbb{S}_4$ is equal to $\Pi(S^{[2]},h)$ for some (S,d) as above: in fact $\mathbb{S}_4$ is irreducible, see **Remark 1.11**, of dimension 19 i.e. the dimension of the set of periods $\Pi(S^{[2]},h)$ for (S,d) as above. Now assume that $\mathcal{P}(\mathbb{LG}(\bigwedge^3 V)\setminus\Sigma)\cap\mathbb{S}_4\neq\emptyset$. Arguing as in the previous cases we get that there exists a closed $\mathrm{PGL}(V)$-invariant codimension-1 subvarietry $C^0\subset(\mathbb{LG}(\bigwedge^3 V)\setminus\Delta\setminus\Sigma)$ such that $\mathcal{P}(C^0)\subset\mathbb{S}_4$. Thus $\mathcal{P}(C^0)$ contains an open dense subset of $\mathbb{S}_4$ and therefore if $A\in C^0$ is very generic $h^{1,1}_{\mathbb{Z}}(X_A)=2$. By the discussion above we get that there exist (S,d) as above such that $\Pi(S^{[2]},h)=\Pi(X_A,h_A)$ with $h^{1,1}_{\mathbb{Z}}(X_A)=2$. By Verbitsky's Global Torelli Theorem there exists a birational map $\varphi\colon S^{[2]}\dashrightarrow X_A$. Since the map $f_A\colon X_A\to|H_A|^{\vee}$ is of degree 2 onto its image, and since φ defines an isomorphism between the complement of a codimension-2 subsets of $S^{[2]}$ and the complement of a codimension-2 subsets of X_A (because both are varieties with trivival canonical bundle) we get that

$$q_{S^{[2]}}(\varphi^*h_A)=2,\ |\varphi^*H_A| \text{ has no base divisor, } S^{[2]}\dashrightarrow|\varphi^*H_A|^{\vee} \text{ is of degree 2 onto its image.} \tag{4.3.3}$$

We will get a contradiction by showing that there exists no divisor of square 2 on $S^{[2]}$ such that (4.3.3) holds. Notice that if H is the divisor on $S^{[2]}$ defined above then the first two conditions of (4.3.3) hold but not the third (the degree of the map is equal to 6). This does not finish the proof because the set of elements of $H^{1,1}_{\mathbb{Z}}(S^{[2]})$ whose square is 2 is infinite.

Lemma 4.3. *There exists $n \in \mathbb{Z}$ such that*

$$\varphi^* h_A = x\mu(d) + y\xi, \qquad y + x\sqrt{2} = (-1+\sqrt{2})(3+2\sqrt{2})^n. \tag{4.3.4}$$

Proof. Since $h^{1,1}_{\mathbb{Z}}(X_A) = 2$ we have $h^{1,1}_{\mathbb{Z}}(S^{[2]}) = 2$ and hence $H^{1,1}_{\mathbb{Z}}(S^{[2]})$ is generated (over $\mathbb{Z}$) by $\mu(d)$ and ξ. Let

$$\begin{array}{ccc} H^{1,1}_{\mathbb{Z}}(S^{[2]}) & \xrightarrow{\psi} & \mathbb{Z}[\sqrt{2}] \\ x\mu(d) + y\xi & \mapsto & y + x\sqrt{2} \end{array} \tag{4.3.5}$$

Then

$$(\alpha, \beta) = -\operatorname{Tr}(\psi(\alpha) \cdot \overline{\psi(\beta)}). \tag{4.3.6}$$

Since $\varphi^* h_A$ is an element of square 2 we will need to solve a (negative) Pell equation. Solving Pell's equation $N(y+x\sqrt{2}) = 1$ (see for example Proposition 17.5.2 of [12]) and noting that $N(-1+\sqrt{2}) = -1$ we get that there exists $n \in \mathbb{Z}$ such that

$$\varphi^* h_A = x\mu(d) + y\xi, \qquad y + x\sqrt{2} = \pm(-1+\sqrt{2})(3+2\sqrt{2})^n. \tag{4.3.7}$$

Next notice that S does not contains lines because $h^{1,1}_{\mathbb{Z}}(S) = 1$: it follows that the map $S^{[2]} \to \mathrm{Gr}(1, \mathbb{P}^3)$ is finite and therefore H is ample. Since $|\varphi^* H_A|$ is not empty and $\varphi^* H_A$ is not equivalent to 0 we get that

$$0 < (\varphi^* h_A, h)_{S^{[2]}} = -\operatorname{Tr}\left(\pm(-1+\sqrt{2})(3+2\sqrt{2})^n(-1-\sqrt{2})\right). \tag{4.3.8}$$

It follows that the $\pm$ is actually $+$. □

Next we will consider the analogue of nodal classes on $K3$ surfaces. For $n \in \mathbb{Z}$ we define $\alpha_n \in H^{1,1}_{\mathbb{Z}}(S^{[2]})$ by requiring that

$$\psi(\alpha_n) = -(3-2\sqrt{2})^n. \tag{4.3.9}$$

Thus $q_{S^{[2]}}(\alpha_n) = -2$ for all n.

Lemma 4.4. *If $n > 0$ then $2\alpha_n$ is effective, if $n \leq 0$ then $-2\alpha_n$ is effective.*

Proof. By Theorem 1.11 of [16] either $2\alpha_n$ or $-2\alpha_n$ is effective (because $q_{S^{[2]}}(\alpha_n) = -2$). Since $(\mu(d) - \xi)$ is ample we decide which of $\pm 2\alpha_n$ is effective by requiring that the product with $(\mu(d) - \xi)$ is strictly positive. The result follows easily from (4.3.6). □

Proposition 4.5. *Suppose that $\varphi^* h_A$ is given by* (4.3.4) *with $n \neq 0$. Then there exists an* effective *$\beta \in H^{1,1}_{\mathbb{Z}}(S^{[2]})$ such that $(\varphi^* h_A, \beta)_{S^{[2]}} < 0$.*

Proof. Identify $H^{1,1}_{\mathbb{Z}}(S^{[2]})$ with $\mathbb{Z}[\sqrt{2}]$ via (4.3.5) and let $g\colon H^{1,1}_{\mathbb{Z}}(S^{[2]}) \to H^{1,1}_{\mathbb{Z}}(S^{[2]})$ correspond to multiplication by $(3-2\sqrt{2})$. Since $N(3-2\sqrt{2}) = 1$ the map g is an isometry. Notice that $\alpha_k = g^k(-\xi)$ and by **Lemma 4.3** we have that $\varphi^* h_A = g^{-n}(\mu(d) - \xi)$. Now suppose that $n > 0$. Then $-2\alpha_{-n+1}$ is effective by **Lemma 4.4** and

$$(\varphi^* h_A, -2\alpha_{-n+1})_{S^{[2]}} = (g^{-n}(\mu(d)-\xi), 2g^{-n+1}(\xi))_{S^{[2]}} = (\mu(d)-\xi, 2g(\xi))_{S^{[2]}} = (\mu(d)-\xi, -4\mu(d)+6\xi)_{S^{[2]}} = -4 < 0. \tag{4.3.10}$$

Lastly suppose that $n < 0$. Then $2\alpha_{-n}$ is effective by **Lemma 4.4** and

$$(\varphi^* h_A, 2\alpha_{-n})_{S^{[2]}} = (g^{-n}(\mu(d) - \xi), 2g^{-n}(-\xi))_{S^{[2]}} = (\mu(d) - \xi, -2\xi)_{S^{[2]}} = -4 < 0. \tag{4.3.11}$$

□

Now we are ready to prove that (4.3.3) cannot hold and hence reach a contradiction. By **Lemma 4.3** we know that $\varphi^* h_A$ is given by (4.3.4) for some $n \in \mathbb{Z}$. We have already noticed that (4.3.3) cannot hold if $n = 0$. Suppose that $n \neq 0$. By **Proposition 4.5** there exists an effective $\beta \in H^{1,1}_{\mathbb{Z}}(S^{[2]})$ such that

$$(\varphi^* h_A, \beta)_{S^{[2]}} < 0. \tag{4.3.12}$$

Let B be an effective divisor representing β and $C \in |\varphi^* h_A|$. Then $C \cap B$ does not have codimension 2 i.e. there exists at least one prime divisor B_i which is both in the support of B and in the support of C. In fact suppose the contrary. Let $c \in H^2(S^{[2]})$ be the Poincarè dual of C and σ be a symplectic form on $S^{[2]}$: then

$$0 \leq \int_{B \cap C} \sigma \wedge \overline{\sigma} = (\beta, c)_{S^{[2]}} (\sigma, \overline{\sigma})_{S^{[2]}} \tag{4.3.13}$$

and since $(\sigma, \overline{\sigma})_{S^{[2]}} > 0$ we get that $(\beta, c) \geq 0$ i.e. $(\varphi^* h_A, \beta)_{S^{[2]}} \geq 0$, that contradicts (4.3.12). The conclusion is that there exists a prime divisor B_i which is both in the support of B and of *any* $C \in |\varphi^* h_A|$, i.e. B_i is a base divisor of the linear system $|\varphi^* h_A|$: that shows that (4.3.3) does not hold. □

References

1. W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Ergebnissse der Mathematik und ihrer Grenzgebiete 4, Springer, 1984.
2. A. Beauville, *Variétes Kähleriennes dont la premiére classe de Chern est nulle*, J. Differential geometry 18, 1983, pp. 755-782.
3. A. Beauville, R. Donagi, *La variétés des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. 301, 1985, pp. 703-706.
4. O. Debarre, C. Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. Reine Angew. Math. 649, 2010, pp. 63-87.
5. *Topics in Transcendental Algebraic Geometry*, edited by P. A. Griffiths, Annals of Mathematics Studies 106, 1984.
6. V. Gritsenko, K. Hulek, G. K. Sankaran, *Abelianisation of orthogonal groups and the fundamental group of modular varieties*, J. Algebra 322, 2010, pp. 463-478.
7. V. Gritsenko, K. Hulek, G.K. Sankaran, *Moduli spaces of irreducible symplectic manifolds*, Compos. Math. 146, 2010, pp. 404-434.
8. R. Hartshorne, *Algebraic Geometry*, Springer GTM 52.
9. A. Iliev, K. Ranestad, *$K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, Trans. Amer. Math. Soc. 353, 2001, pp. 1455-1468.
10. A. Iliev, K. Ranestad, *Addendum to "$K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds"*, C. R. Acad. Bulgare Sci. 60, 2007, pp. 1265-1270.
11. C. Lehn, M. Lehn, C. Sorger, D. van Straten, *Twisted cubics on cubic fourfolds*, Mathematics arXiv:1305.0178.
12. K. Ireland, M. Rosen *A classical introduction to modern Number Theory*, Springer GTM 84, 2nd edition, 1990.
13. E. Looijenga, *The period map for cubic fourfolds*, Invent. Math. 177, 2009, pp. 213-233.
14. D. Luna. *Slices étales*, Mém. Soc. Math. France 3, 1973, pp. 81-105.
15. E. Markman *A survey of Torelli and monodromy results for hyperkähler manifolds*, Complex and differential geometry, Springer Proc. Math. 8, 2011, pp. 257-322.
16. E. Markman *Prime exceptional divisors on holomorphic symplectic varieties and monodromy-reflections*, Kyoto J. Math. 53, 2013, pp. 345-403.
17. V. V. Nikulin *Integer symmetric bilinear forms and some of their geometric applications*, Math. USSR-Izv. 14, 1980, pp. 103167.
18. K. O'Grady, *Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics*, Duke Math. J. 34, 2006, pp. 99-137.
19. K. O'Grady. *Dual double EPW-sextics and their periods*, Pure Appl. Math. Q. 4, 2008, pp. 427-468.
20. K. O'Grady, *EPW-sextics: taxonomy*, Manuscripta Math. 138, 2012, pp. 221-272.
21. K. O'Grady, *Double covers of EPW-sextics*, Michigan Math. J. 62, 2013, pp. 143-184.
22. K. O'Grady, *Moduli of double EPW-sextics*, Mathematics arXiv:1111.1395.
23. J. Shah, *A complete moduli space for $K3$ surfaces of Degree* 2, Ann. of Math. 112, 1980, pp. 485-510.
24. J. Shah, *Insignificant limit singularities of surfaces and their mixed Hodge structure*, Ann. of Math. 109, 1979, pp. 497-536.
25. K. Yoshioka, *Some examples of Mukai's reflections on K3 surfaces*, J. Reine Angew. Math. 515, 1999, pp. 97-123.