Dual double EPW-sextics and their periods

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1 Introduction

Eisenbud, Popescu and Walter have constructed certain singular sextic hypersurfaces (EPW-sextics) in $\mathbb{P}^5$ (Example (9.3) of [4]) which come provided with a natural double cover: we have shown [13] that the generic such double cover is a deformation of the Hilbert square of a $K3$ and that the family of double EPW-sextics is a locally versal family of projective deformations of $(K3)^{[2]}$. Thus the family of double EPW-sextics is similar to the family of Fano varieties of lines on a cubic 4-fold (see [2]), with the following difference: the Plücker ample divisor on the Fano variety of lines has square 6 for the Beauville-Bogomolov quadratic form (see [1, 2]) while the natural polarization of a double EPW-sextic has square 2 (see [13]). Let $Y \subset \mathbb{P}^5$ be a generic EPW-sextic: we proved in [13] that the dual $Y^\vee \subset (\mathbb{P}^5)^\vee$ is another generic EPW-sextic. Thus we may associate to the natural double cover X of Y a "dual"variety $X^\vee$ namely the natural double cover of $Y^\vee$. This construction defines a (rational) involution on the moduli space of double EPW-sextics. In [13] we showed that a generic EPW-sextic is not self-dual and hence the involution on the moduli space of double EPW-sextics is not the identity; in this paper we determine the relation between the periods of a double EPW-sextic and its dual. Before stating the result we recall the definition of EPW-sextics. Let V be a 6-dimensional $\mathbb{C}$-vector space. Choose an isomorphism $vol\colon \wedge^6 V \xrightarrow{\sim} \mathbb{C}$ and let ω be the symplectic form on $\wedge^3 V$ defined by wedge product followed by vol. Let $\mathbb{P}(V)$ be the projective space of 1-dimensional sub vector spaces of V; then ω gives $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ the structure of a symplectic vector-bundle of rank 20. Let F be the sub-vector-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ whose fiber $F_{[v]}$ over $[v] \in \mathbb{P}(V)$ consists of tensors divisible by v:

$$F_{[v]} := \{v \wedge w \mid w \in \wedge^2 V\}. \tag{1.0.1}$$

As is easily checked F is a Lagrangian sub-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$. Let $\mathbb{LG}(\wedge^3 V)$ be the symplectic Grassmannian parametrizing ω-Lagrangian subspaces of $\wedge^3 V$. For $A \in \mathbb{LG}(\wedge^3 V)$ we let λ_A be the composition

$$F \longrightarrow \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{1.0.2}$$

and $Y_A \subset \mathbb{P}(V)$ be the zero-scheme of $\det(\lambda_A)$; unless $\det(\lambda_A)$ is identically zero[1] Y_A is a sextic hypersurface because $\det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6)$. An *EPW-sextic* is a

[1] If A is the fiber of F over a fixed point then $\det(\lambda_A)$ is identically zero.

hypersurface in $\mathbb{P}(V)$ which is equal to Y_A for some $A \in \mathbb{LG}(\wedge^3 V)$. In [13] we described explicitly the non-empty Zariski-open $\mathbb{LG}(\wedge^3 V)^0 \subset \mathbb{LG}(\wedge^3 V)$ parametrizing A such that Y_A has no singularities other than those forced by its description as a degeneracy locus; for $A \in \mathbb{LG}(\wedge^3 V)^0$ the singular locus of Y_A is a smooth degree-40 irreducible surface and at a singular point Y_A is locally (in the classical topology) isomorphic to the product of $sing(Y_A)$ and an A_1-singularity. If $A \in \mathbb{LG}(\wedge^3 V)^0$ there is a natural double cover $f_A \colon X_A \to Y_A$ ramified only over $sing(Y_A)$, with X_A smooth. As shown in [13] the 4-fold X_A is a deformation of $(K3)^{[2]}$ - the Hilbert square of a $K3$. Let L_A be the line-bundle on X_A defined by

$$L_A := f_A^* \mathcal{O}_{Y_A}(1); \tag{1.0.3}$$

then $c_1(L_A)$ has square 2 for the Beauville-Bogomolov quadratic form. The family of (X_A, L_A) for A varying in $\mathbb{LG}(\wedge^3 V)^0$ is a locally complete family of polarized deformations of $(K3)^{[2]}$. Now we recall the duality map. Let $vol^\vee$ be a trivialization of $\wedge^6 V^\vee$ and $\omega^\vee$ be the symplectic form on $\wedge^3 V^\vee$ given by wedge-product followed by $vol^\vee$: let $\mathbb{LG}(\wedge^3 V^\vee)$ be the symplectic Grassmannian parametrizing $\omega^\vee$-Lagrangian subspaces of $\wedge^3 V^\vee$. For $A \in \mathbb{LG}(\wedge^3 V)$ the annihilator $A^\perp \subset \wedge^3 V^\vee$ belongs to $\mathbb{LG}(\wedge^3 V^\vee)$. Thus we have an isomorphism

$$\begin{array}{rccc} \delta\colon \mathbb{LG}(\wedge^3 V) & \overset{\sim}{\longrightarrow} & \mathbb{LG}(\wedge^3 V^\vee) \\ A & \mapsto & A^\perp. \end{array} \tag{1.0.4}$$

Let

$$\mathbb{LG}(\wedge^3 V)^{00} := \mathbb{LG}(\wedge^3 V)^0 \cap \delta^{-1}\mathbb{LG}(\wedge^3 V^\vee)^0. \tag{1.0.5}$$

Thus $\mathbb{LG}(\wedge^3 V)^{00}$ is open dense in $\mathbb{LG}(\wedge^3 V)$. Assume that $A \in \mathbb{LG}(\wedge^3 V)^{00}$. By Proposition (3.1) of [13] $Y_{A^\perp}$ is $Y_A^\vee$ (the dual of Y_A); we let

$$X_A^\vee \to Y_A^\vee = Y_{A^\perp} \tag{1.0.6}$$

be the natural double cover and we call $X_A^\vee$ the *dual of* X_A. We will show how to obtain the Hodge structure on $H^2(X_A^\vee)$ from the Hodge structure on $H^2(X_A)$. First we describe the relevant period space. Let $\widetilde{\Lambda}$ be the even lattice

$$\widetilde{\Lambda} := U^3 \widehat{\oplus} (-E_8)^2 \widehat{\oplus} (-2) \tag{1.0.7}$$

where U is the hyperbolic plane and (-2) is the lattice generated by a single element of square -2; we will denote by $(\cdot,\cdot)$ the symmetric bilinear form on $\widetilde{\Lambda}$. Let U be one of the hyperbolic planes appearing in Decomposition (1.0.7) and choose

$$u \in U \text{ of square 2.} \tag{1.0.8}$$

Let $e_1 \in U$ be a generator of $u^\perp \cap U$ and e_2 be a generator of the direct summand (-2) appearing in Decomposition (1.0.7). Then

$$\Lambda := u^\perp = U^2 \widehat{\oplus} (-E_8)^2 \widehat{\oplus} \mathbb{Z}e_1 \widehat{\oplus} \mathbb{Z}e_2 \cong U^2 \widehat{\oplus} (-E_8)^2 \widehat{\oplus} (-2)^2. \tag{1.0.9}$$

The period domain of interest to us is

$$\mathcal{D}_2 := \{[v] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) |\ (v,v) = 0, \quad (v,\overline{v}) > 0\}. \tag{1.0.10}$$

Let $Stab(u) < O(\widetilde{\Lambda})$ be the subgroup of isometries fixing u and $\rho\colon Stab(u) \to O(\Lambda)$ be the restriction map - this makes sense because $\Lambda = u^\perp$. Let

$$\Gamma := Im(\rho) < O(\Lambda). \tag{1.0.11}$$

The *period moduli space* is the quotient

$$\mathcal{Q}_2 := \Gamma\backslash\mathcal{D}_2. \tag{1.0.12}$$

We will prove in Subsection (2.2) that

$$[O(\Lambda) : \Gamma] = 2. \tag{1.0.13}$$

An explicit $r \in (O(\Lambda)\backslash\Gamma)$ is the isometry which interchanges e_1 with e_2 and fixes the elements of $\{e_1, e_2\}^{\perp}$, see (2.2.12). By (1.0.13) the involution r descends to a non-trivial involution

$$\overline{r}\colon \mathcal{Q}_2 \to \mathcal{Q}_2. \tag{1.0.14}$$

Let $A \in \mathbb{LG}(\wedge^3 V)^0$: the isomorphism class of the Hodge structure on $c_1(L_A)^{\perp} \subset H^2(X_A)$ is a point of $\mathcal{Q}_2$ (see Section (2)) that we denote by $\mathcal{P}(A)$ - this is the *period point of* X_A. The global period map is defined by

$$\begin{array}{ccc} \mathbb{LG}(\wedge^3 V)^0 & \xrightarrow{\mathcal{P}} & \mathcal{Q}_2 \\ A & \mapsto & \mathcal{P}(A). \end{array} \tag{1.0.15}$$

Of course we also have a global period map $\mathbb{LG}(\wedge^3 V^{\vee})^0 \to \mathcal{Q}_2$; we denote it by the same symbol $\mathcal{P}$. If $A \in \mathbb{LG}(\wedge^3 V)^{00}$ we have two period points namely $\mathcal{P}(A)$ and $\mathcal{P}(A^{\perp})$. The following is the main result of this paper.

Theorem 1.1. *Keep notation as above. If $A \in \mathbb{LG}(\wedge^3 V)^{00}$ then*

$$\mathcal{P}(A^{\perp}) = \overline{r} \circ \mathcal{P}(A). \tag{1.0.16}$$

The theorem may be stated informally as follows: the polarized Hodge structures on $c_1(L_{A^{\perp}})^{\perp}$ and $c_1(L_A)^{\perp}$ are always isomorphic while in general there is no Hodge isometry between $H^2(X_A)$ and $H^2(X_{A^{\perp}})$.

There are three steps in the proof of the above theorem. First of all there exists a rational Hodge isometry

$$H^2(X_A) \supset c_1(L_A)^{\perp} \cong c_1(L_{A^{\perp}})^{\perp} \subset H^2(X_{A^{\perp}}). \tag{1.0.17}$$

In fact let $\widetilde{Y}_A, \widetilde{Y}_{A^{\perp}}$ be the desingularizations of $Y_A, Y_{A^{\perp}}$ respectively - they are obtained by blowing up $sing(Y_A)$ and $sing(Y_{A^{\perp}})$ respectively. From the equality $Y_A^{\vee} = Y_{A^{\perp}}$ one gets that

$$\widetilde{Y}_A \cong \widetilde{Y}_{A^{\perp}}. \tag{1.0.18}$$

Now $H^4(Y_A)$ is a subgroup of $Sym^2 H^2(X_A)$ and similarly for $H^4(Y_{A^{\perp}})$; this fact together with (1.0.18) gives (1.0.17) after some work. It follows from (1.0.17) that locally (in the classical topology) there exists $g \in O(\Lambda \otimes \mathbb{Q})$ relating the periods of X_A and $X_{A^{\perp}}$; this means that we have

$$\mathcal{P}_{\Phi}(A^{\perp}) = g \circ \mathcal{P}_{\Psi}(A) \tag{1.0.19}$$

where $\mathcal{P}_{\Psi}, \mathcal{P}_{\Phi}$ are local liftings of the global period maps (Ψ, Φ are markings of the weight-2 cohomology). The second step consists in showing that (1.0.16) holds for A belonging to a certain locally closed codimension-1 submanifold $\Delta^0_*(V) \subset \mathbb{LG}(\wedge^3 V)^0$. If $A \in \Delta^0_*(V)$ then X_A is isomorphic to a moduli space of sheaves on a $K3$ surface $S \subset \mathbb{P}^6$ of degree 10. On the other hand there is a map

$f\colon S^{[2]} \to Y_{A^\perp}$ of degree 2 which contracts a certain plane $P \subset S^{[2]}$ and is finite over $Y_{A^\perp} \setminus f(P)$. One shows that $mult_{f(P)} Y_{A^\perp} = 3$ and hence $A^\perp \notin \mathbb{LG}(\wedge^3 V^\vee)^0$ i.e. $A \notin \mathbb{LG}(\wedge^3 V)^{00}$. Morally $X_{A^\perp}$ is the singular symplectic 4-fold obtained from $S^{[2]}$ by contracting the plane P. We do not prove a precise version of this; however we do prove that the local period map of $\mathbb{LG}(\wedge^3 V^\vee)^0$ extends across $A^\perp$ and that its value is given by the periods of $S^{[2]}$. This will allow us to check that (1.0.16) holds for $A \in \Delta^0_*(V)$. Since the local period map sends $\Delta^0_*(V)$ to the intersection of $\mathcal{D}_2$ and a hyperplane $\zeta^\perp$ it will follow that the rational isometry g of (1.0.19) is (with suitable markings Ψ, Φ) either r or $r \circ r_\zeta$ where r_ζ is the reflection in the span of ζ. We rule out this second possibility by a monodromy argument: this is the third step in the proof of Theorem (1.1).

The paper is organized as follows. In Section (2) we recall the definition of local and global period maps and we prove (1.0.13). In the next section we prove the results on $\Delta^0_*(V)$ that we described above. In the final section we prove the existence of a rational Hodge isometry (1.0.17) and we give the monodromy argument.

Notation: V will always be a complex vector-space of dimension 6.

2 The period map and the lattice Λ

2.1 The period map

Let X be a deformation of $(K3)^{[2]}$. Let $(\cdot,\cdot)_X$ be the Beauville-Bogomolov symmetric bilinear form on $H^2(X)$. The restriction of $(\cdot,\cdot)_X$ to $H^2(X;\mathbb{Z})$ is a non-degenerate integral symmetric form; thus $H^2(X;\mathbb{Z})$ is a lattice. As is well-known $H^2(X;\mathbb{Z})$ and $\widetilde{\Lambda}$ are isometric. Now assume that we are given a holomorphic line-bundle L on X such that $(c_1(L), c_1(L))_X = 2$. Since $O(\widetilde{\Lambda})$ acts transitively on the set of vectors of square 2 there exists an isometry

$$\psi\colon H^2(X;\mathbb{Z}) \xrightarrow{\sim} \widetilde{\Lambda} \tag{2.1.1}$$

such that $\psi(c_1(L)) = u$ where u is as in (1.0.8); this is a *marking of* (X, L). Let $\psi_{\mathbb{C}}\colon H^2(X;\mathbb{C}) \xrightarrow{\sim} \widetilde{\Lambda} \otimes \mathbb{C}$ be the map obtained from ψ by extension of scalars. Let σ be a generator of $H^{2,0}(X)$; then

$$(c_1(L), \sigma)_X = 0 = (\sigma, \sigma)_X, \quad (\sigma, \overline{\sigma})_X > 0. \tag{2.1.2}$$

Thus

$$\mathcal{P}_\psi(X, L) := \psi_{\mathbb{C}}(H^{2,0}(X)) \in \mathcal{D}_2. \tag{2.1.3}$$

This is the *local period point of* (X, L) *associated to* ψ (or periods of (X, L)). Any other marking of (X, L) is given by $\gamma \circ \psi$ where $\gamma \in \Gamma$, thus the Γ-orbit of $\mathcal{P}_\psi(X, L)$ is a well-defined point of $\mathcal{Q}_2$; this is *the global period point of* (X, L), we denote it by $\mathcal{P}(X, L)$. If $A \in \mathbb{LG}(\wedge^3 V)^0$ we let $\mathcal{P}(A) := \mathcal{P}(X_A, L_A)$.

Later on we will study global and local period maps for certain families of double EPW-sextics. Let

$$\pi\colon \mathcal{X} \to T \tag{2.1.4}$$

be a proper submersive map between complex manifolds such that each fiber is a deformation of $(K3)^{[2]}$. For $t \in T$ we let $X_t := \pi^{-1}(t)$. We assume that we are given a (holomorphic) line-bundle $\mathcal{L}$ on $\mathcal{X}$ such that $c_1(\mathcal{L}|_{X_t})$ has square 2 for every $t \in T$; we let $L_t := \mathcal{L}|_{X_t}$. The *global period map of* $(\mathcal{X}, \mathcal{L})$ is given by

$$\begin{array}{ccc} T & \xrightarrow{\mathcal{P}} & \mathcal{Q}_2 \\ t & \mapsto & \mathcal{P}(t) := \mathcal{P}(X_t, L_t) \end{array} \tag{2.1.5}$$

Griffiths proved that $\mathcal{P}$ is a holomorphic map. Let $\pi\colon \mathcal{X} \to \mathbb{LG}(\wedge^3 V)^0$ be the tautological family of double EPW-sextics and $\mathcal{L}$ be the tautological relatively ample line-bundle on $\mathcal{X}$ which restricts to L_A (see (1.0.3)) on X_A; then $\mathcal{P}$ is the period map of (1.0.15), in particular the map of (1.0.15) is holomorphic. Let's go back to a general family (2.1.4): it is not always possible to lift the global period map $\mathcal{P}$ to a map $T \to \mathcal{D}_2$, in fact a necessary and sufficient condition is that $R^2\pi_*\mathbb{Z}$ is trivial. Suppose that $R^2\pi_*\mathbb{Z}$ is trivial. Then there exists a trivialization $\Psi\colon R^2\pi_*\mathbb{Z} \xrightarrow{\sim} T \times \widetilde{\Lambda}$ sending the section corresponding to $c_1(\mathcal{L})$ to the section given by u - this is a *marking of* $(\mathcal{X}, \mathcal{L})$. Given such a marking we let $\Psi_t\colon H^2(X_t; \mathbb{Z}) \xrightarrow{\sim} \widetilde{\Lambda}$ be the fiber of Ψ over t. The *local period map of* $(\mathcal{X}, \mathcal{L})$ *associated to* Ψ is given by

$$\begin{array}{ccc} T & \xrightarrow{\mathcal{P}_\Psi} & \mathcal{D}_2 \\ t & \mapsto & \mathcal{P}_\Psi(t) := \mathcal{P}_{\Psi_t}(X_t, L_t). \end{array} \tag{2.1.6}$$

Griffiths proved that $\mathcal{P}_\Psi$ is holomorphic.

2.2 Proof of (1.0.13)

We start by recalling the definition of discriminant group and discriminant quadratic form of an even lattice $(L, (\cdot,\cdot)_L)$, i.e. a free finitely generated abelian group L equipped with a symmetric integral even non-degenerate bilinear form $(\cdot,\cdot)_L$. We follow [10]. The bilinear form $(\cdot,\cdot)_L$ extends to a $\mathbb{Q}$-valued bilinear form on $L \otimes \mathbb{Q}$; abusing notation we denote by $(\cdot,\cdot)_L$ the extended form. Let $L^\vee := Hom(L, \mathbb{Z})$; by non-degeneracy of $(\cdot,\cdot)_L$ we have a natural chain of inclusions

$$L \subset L^\vee \subset L \otimes \mathbb{Q}. \tag{2.2.1}$$

The *discriminant group of* L is $A_L := L^\vee / L$; it comes provided with the *discriminant bilinear-form*

$$\begin{array}{ccc} A_L \times A_L & \xrightarrow{b_L} & \mathbb{Q}/\mathbb{Z} \\ ([\alpha], [\beta]) & \mapsto & [(\alpha, \beta)_L] \end{array} \tag{2.2.2}$$

and the *discriminant quadratic-form*

$$\begin{array}{ccc} A_L & \xrightarrow{q_L} & \mathbb{Q}/2\mathbb{Z} \\ [\alpha] & \mapsto & [(\alpha, \alpha)_L]. \end{array} \tag{2.2.3}$$

The formula

$$q_L([\alpha + \beta]) \equiv q_L([\alpha]) + q_L([\beta]) + 2b_L([\alpha], [\beta]) \pmod{2\mathbb{Z}} \tag{2.2.4}$$

shows that q_L determines uniquely b_L. An *index-i overlattice of* L consists of a lattice M and an inclusion of lattices $L \subset M$ (the restriction of $(\cdot,\cdot)_M$ to L is equal to $(\cdot,\cdot)_L$) of index i. Two overlattices $M_1 \supset L \subset M_2$ are equivalent if there exists an isometry $M_1 \xrightarrow{\sim} M_2$ which is the identity on L, i.e. if the inclusions $M_i \hookrightarrow L^\vee$ for $i = 1, 2$ have the same image. Suppose that $L \subset M$ is an index-i overlattice of L; the inclusion $M \subset L^\vee$ defines an inclusion $M/L \subset A_L$ with image a subgroup of cardinality i which is q_L-isotropic. This construction defines a one-to-one correspondence between the set of equivalence classes of indexi-i overlattices of L and the set of q_L-isotropic subgroups of A_L of cardinality i. The correspondence is equivariant for the natural actions of $O(L)$ on both sets. Now consider the lattice $L = \mathbb{Z}u \widehat{\oplus} \Lambda$ where u and Λ are as in Section (1): we will describe the discriminant group and discriminant form of L. Let $e_1, e_2 \in \Lambda$ be as in Section (1). A straightforward computation gives that

$$A_L = \mathbb{Z}/(2)\left[\frac{u}{2}\right] \oplus \mathbb{Z}/(2)\left[\frac{e_1}{2}\right] \oplus \mathbb{Z}/(2)\left[\frac{e_2}{2}\right] \tag{2.2.5}$$

and that

$$q_L\left(x\left[\frac{u}{2}\right] + y_1\left[\frac{e_1}{2}\right] + y_2\left[\frac{e_2}{2}\right]\right) \equiv \frac{1}{2}(x^2 - y_1^2 - y_2^2) \pmod{2\mathbb{Z}}. \tag{2.2.6}$$

The set $I \subset A_L$ of non-zero isotropic vectors is given by

$$I = \left\{\left[\frac{u}{2}\right] + \left[\frac{e_1}{2}\right], \left[\frac{u}{2}\right] + \left[\frac{e_2}{2}\right]\right\}. \tag{2.2.7}$$

The group $O(L)$ acts naturally on A_L and hence also on I; thus we have a homomorphism

$$O(L) \xrightarrow{\epsilon} Aut(I) \cong \mathbb{Z}/(2). \tag{2.2.8}$$

The overlattice $\widetilde{\Lambda} \supset L$ is of index 2 because L is the kernel of the surjection

$$\begin{array}{ccc} \widetilde{\Lambda} & \longrightarrow & \mathbb{Z}/(2) \\ v & \mapsto & (v, u) \pmod{2}. \end{array} \tag{2.2.9}$$

The correspondence described above defines an $O(L)$-equivariant bijective map between I and the set of equivalence classes of index-2 overlattices of L; thus

$$Im(O(\widetilde{\Lambda}) \to O(L)) = \ker(\epsilon). \tag{2.2.10}$$

The subgroup of $O(L)$ consisting of isometries which are the identity on $\mathbb{Z}u$ is naturally identified with $O(\Lambda)$; thus $O(\Lambda) < O(L)$. By (2.2.10) and the definition of Γ (see (1.0.11)) we get that

$$\Gamma = \ker(\epsilon|_{O(\Lambda)}). \tag{2.2.11}$$

Let $r \in O(\Lambda)$ be the involution characterized by the following properties:

$$r(e_1) = e_2, \quad r(e_2) = e_1, \quad r|_{\{e_1, e_2\}^\perp} = \text{identity}. \tag{2.2.12}$$

Then $\epsilon(r)$ is the non-trivial permutation of I and hence $\epsilon|_{O(\Lambda)}$ is surjective. Thus (1.0.13) follows from (2.2.11).

3 Explicit dual couples

Let $F_5^3 \subset \mathbb{P}^6$ be the intersection of $Gr(2, \mathbb{C}^5) \subset \mathbb{P}(\wedge^2\mathbb{C}^5) \cong \mathbb{P}^9$ with a transversal 6-dimensional linear subspace of $\mathbb{P}^9$. Let $T \subset |\mathcal{O}_{\mathbb{P}^6}(2)|$ be the open dense subset parametrizing quadrics which are transversal to F_5^3. For $t \in T$ let Q_t be the quadric corresponding to t and

$$S_t := F_5^3 \cap Q_t. \tag{3.0.1}$$

Then S_t is a degree-10 linearly normal $K3$ surface; in fact the generic such surface is projectively equivalent to S_t for some $t \in T$ by Mukai [8]. Let D_t be the hyperplane divisor class on S_t and M_t be the moduli space of D_t-semistable rank-2 sheaves $\mathcal{F}$ on S_t with $c_1(\mathcal{F}) = c_1(D_t)$ and $c_2(\mathcal{F}) = 5$. Suppose that

$$\text{for all divisors } E \text{ on } S_t \text{ we have } E \cdot D_t \equiv 0 \pmod{10}. \tag{3.0.2}$$

Then (see Section (5) of [13]) there exists $A_t \in \mathbb{LG}(\wedge^3 V)^0$ such that

$$X_{A_t} \cong M_t. \tag{3.0.3}$$

Furthermore Y_{A_t} is explicitely described as follows. Let Σ_t be the divisor on $|I_{S_t}(2)|$ parametrizing singular quadrics, since all quadrics parametrized by $|I_{F_5^3}(2)|$ are singular we have

$$\Sigma_t = |I_{F_5^3}(2)| + \Sigma_t'. \tag{3.0.4}$$

Then

$$Y_{A_t} \cong \Sigma_t'. \tag{3.0.5}$$

(Formally Σ_t' is a degree-6 divisor; in the above equation we are implicitely stating that Σ_t' is a reduced divisor and hence we may view it as a degree-6 hypersurface.) The set of t for which (3.0.2) holds is the complement of a countable union of proper algebraic subvarieties of T however a straightforward argument shows that there is an open dense $T''' \subset T$ such that for $t \in T'''$ there exists $A_t \in \mathbb{LG}(\wedge^3 V)^0$ for which both (3.0.3) and (3.0.5) hold; we give the argument in Subsection (3.1). We let $\Delta^0(V) \subset \mathbb{LG}(\wedge^3 V)^0$ be the set of A such that X_A is isomorphic to M_t for some $t \in T'''$; this is a locally closed subset of $\mathbb{LG}(\wedge^3 V)^0$. Computing the periods of X_A for $A \in \Delta^0(V)$ we will show that $\Delta^0(V)$ has codimension 1 in $\mathbb{LG}(\wedge^3 V)^0$. Let $A \in \Delta^0(V)$, thus $X_A \cong M_t$ for some $t \in T$: the dual $Y_A^\vee$ is described as follows. The Hilbert scheme $S_t^{[2]}$ contains a copy of $\mathbb{P}^2$, call it P_t, parametrizing $Z \subset S_t$ which span a line contained in F_5^3. Let $S_t^{[2]} \to N_t$ be the contraction of P_t - thus N_t is a singular symplectic variety. There is an involution on N_t whose quotient is isomorphic to $Y_A^\vee$. From this it will follow that $A \notin \mathbb{LG}(\wedge^3 V)^{00}$ and hence apparently it will not make sense to compute $\mathcal{P}(A^\perp)$. The main point of this subsection is to prove that the period map extends across $A^\perp$, in fact the <u>local</u> period map extends and its value at $A^\perp$ is given by the periods of $(S_t^{[2]}, D_t)$.

3.1 The locus $\Delta^0(V)$

We recall that the *Mukai vector* $v(\mathcal{F})$ of a sheaf $\mathcal{F}$ on S_t is

$$v(\mathcal{F}) := ch(\mathcal{F})\sqrt{Td(S_t)} = ch(\mathcal{F})(1 + \eta_t) \in H^*(S_t; \mathbb{Z}), \tag{3.1.1}$$

where $\eta_t \in H^4(S_t; \mathbb{Z})$ is the orientation class. If $[\mathcal{F}] \in M_t$ (we let $[\mathcal{F}] \in M_t$ be the equivalence class of the semistable sheaf $\mathcal{F}$ - we recall that if $\mathcal{F}$ is stable this is the same as the isomorphism class of $\mathcal{F}$) the class $v(\mathcal{F})$ is independent of $\mathcal{F}$, we denote it by v_t; explicitly

$$v_t = 2 + c_1(D_t) + 2\eta_t. \tag{3.1.2}$$

Proposition 3.1. *Keep notation as above. There is an open dense subset $T''' \subset T$ such that the following holds. Let $t \in T'''$; then there exists $A_t \in \mathbb{LG}(\wedge^3 V)^0$ such that both (3.0.3) and (3.0.5) hold. In particular we have a canonical identification $\mathbb{P}(V) \cong |I_{S_t}(2)|$.*

Proof. By Maruyama [7] there exists a projective map $\rho\colon \mathcal{M} \to T$ with (schematic) fiber M_t over $t \in T$. Let $t \in T$; we say that S_t is *unsuitable* if there exists a divisor class C on S_t such that

$$\int_{S_t} c_1(C) \wedge c_1(D_t) = 0, \quad -10 \le \int_{S_t} c_1(C)^2 < 0. \tag{3.1.3}$$

The set of unsuitable t is a proper closed subset of T, thus the complement T' is an open dense subset of T. Let $\mathcal{M}' := \rho^{-1}(T')$ and $\rho'\colon \mathcal{M}' \to T'$ be the restriction of ρ. It is known that if $t \in T'$ then every sheaf parametrized by M_t is slope-stable and M_t is a smooth 4-dimensional scheme (Main Theorem (0.1.2) and Proposition (2.1) of [11] - notice that $t \in T_1$ if and only if D_t is $|v_t|$-generic). Let's show that there is an open dense $T'' \subset T'$ such that for $t \in T''$ the Mukai reflection (see (4.2.2) of [12]) is a regular involution ϕ_t on M_t - we recall that in general the Mukai reflection acts on the derived category of coherent sheaves on S_t. In order for ϕ_t to be a regular involution on M_t it suffices that for all $[\mathcal{F}] \in M_t$ the following hold:

(a) $h^0(\mathcal{F}) = \chi(\mathcal{F}) = 4$,

(b) $\mathcal{F}$ is globally generated away from (at most) a zero-dimensional subset of S_t,

(c) the kernel of the evaluation map $H^0(\mathcal{F}) \otimes \mathcal{O}_{S_t} \to \mathcal{F}$, call it $\mathcal{E}$, is a D_t-slope-stable sheaf.

If (a)-(b)-(c) are satisfied for all $[\mathcal{F}] \in M_t$ then the generic sheaf $\mathcal{F}$ parametrized by M_t is globally generated and for such a sheaf $\phi_t([\mathcal{F}]) = [\mathcal{E}^\vee]$ where $\mathcal{E}$ is as in (c) above. Let $T'' \subset T'$ be the set of t such that (a)-(b)-(c) hold for every $[\mathcal{F}] \in M_t$. Let's prove that T'' is open. First we show that the subset $T_a \subset T$ of $t \in T'$ such that (a) holds for every $[\mathcal{F}] \in M_t$ is open. Let $t \in T'$: if $[\mathcal{F}] \in M_t$ then by stability $h^2(\mathcal{F}) = 0$ and hence $h^0(\mathcal{F}) \ge 4$, thus by the upper-semicontinuity Theorem the set of $[\mathcal{F}] \in M_t$ that violate (a) is closed. By properness of the map $\rho'\colon \mathcal{M}' \to T'$ it follows that $(T' \setminus T_a)$ is closed i.e. T_a is open. A similar argument shows that the set of $t \in T_a$ such that (b)-(c) hold for every $[\mathcal{F}] \in M_t$ is open; thus T'' is open. Since T'' contains the subset of t such that (3.0.2) holds it is dense in T. Let $T''' \subset T''$ be the set of t such that

$$S_t \text{ contains no effective non-zero divisor } E \text{ with } E \cdot D_t \le 5. \tag{3.1.4}$$

Thus T''' is open and dense in T. We claim that if $t \in T'''$ then

$$M_t/\langle \phi_t \rangle \cong \Sigma_t' \subset |I_{S_t}(2)| \cong \mathbb{P}^5. \tag{3.1.5}$$

More precisely Proposition (5.1) of [12] holds for $M_t = \mathcal{M}(v_t)$. In order to prove this it suffices to show that for every $t \in T'''$ the following holds:

(1) If $[\mathcal{F}] \in M_t$ and $\sigma \in H^0(\mathcal{F}) = \mathrm{Hom}(\mathcal{O}_{S_t}, \mathcal{F})$ is non-zero then the quotient $\mathcal{F}/Im(\sigma)$ is locally-free in codimension 1 (σ has isolated zeroes) - this is used in the proof of Lemma (5.4) of [12].

(2) If $\mathcal{G}$ is a globally generated rank-2 vector bundle on S_t with $\det \mathcal{G} \cong \mathcal{O}_{S_t}(D_t)$ and $c_2(\mathcal{G}) = 5$ then $\mathcal{G}$ is D_t-slope stable - this is used in the proof of Lemma (5.7) of [12].

Let's show that (1) above holds. Suppose that σ does not have isolated zeroes: then it vanishes along a non-zero effective divisor E and by slope stability of $\mathcal{F}$ we have $E \cdot D_t < slope(\mathcal{F}) = 5$ contradicting (3.1.4). Let's show that (2) above holds. Suppose that $\mathcal{G}$ is not D_t-slope stable; since $\mathcal{G}$ has rank 2 there is a destabilizing sequence $\mathcal{G} \to I_Z \otimes \mathcal{O}_{S_t}(E)$ where $Z \subset S_t$ is zero-dimensional and E is a divisor with $E \cdot D_t \le slope(\mathcal{F}) = 5$. Since $\mathcal{G}$ is globally generated there is a non-zero section of $I_Z \otimes \mathcal{O}_{S_t}(E)$ and hence E is effective; by (3.1.4) we get that $E = 0$. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{S_t}(D_t) \longrightarrow \mathcal{G} \longrightarrow I_Z \longrightarrow 0. \tag{3.1.6}$$

Since $\ell(Z) = c_2(\mathcal{G}) = 5$ the zero-dimensional subscheme $Z \subset S_t$ is non-empty and hence $h^0(I_Z) = 0$; this contradicts the hypothesis that $\mathcal{G}$ is globally generated. We have proved that (3.1.5) holds for $t \in T'''$. By Theorem (1.1) of [13] we get that there exists $A_t \in \mathbb{LG}(\wedge^3 V)^0$ such that both (3.0.3) and (3.0.5) hold. $\square$

The proof of the above proposition together with Claim (5.18) of [12] gives the following result.

Corollary 3.2. *Let $t \in T'''$ and let $A_t \in \mathbb{LG}(\wedge^3 V)^0$ be such that both (3.0.3) and (3.0.5) hold. The map $f_{A_t} \colon X_{A_t} \to Y_{A_t}$ is identified with the quotient map $M_t \to M_t/\langle \phi_t \rangle$.*

Let $\Delta^0(V) \subset \mathbb{LG}(\wedge^3 V)^0$ be the locus of A such that X_A is isomorphic to M_t for some $t \in T'''$. Our next task is to show that $\Delta^0(V)$ is a locally closed subset of codimension 1. First we recall how one describes $H^2(M_t)$ for $t \in T'$. Let $u, w \in H^*(S_t)$ and let u_q, w_q be the degree-q components of u, w respectively. One sets $u^\vee := u_0 - u_2 + u_4$ and

$$\langle u, w \rangle := \int_{S_t} (u_2 \wedge w_2 - u_0 w_4 - u_4 w_0) = -\int_{S_t} u^\vee \wedge w. \tag{3.1.7}$$

This is *Mukai's bilinear symmetric form.* One defines a positive[2] weight-two Hodge structure on $H^*(S_t)$ by defining the Hodge filtration as

$$F^1 H^*(S_t) := H^0(S_t) \oplus F^1 H^2(S_t) \oplus H^4(S_t), \quad F^2 H^*(S_t) := F^2 H^2(S_t). \tag{3.1.8}$$

Let v_t be the Mukai vector (3.1.2), then v_t is integral of type $(1,1)$ and hence $v_t^\perp$ is an integral Hodge substructure of $H^*(S_t)$ and furthermore the restriction of Mukai's bilinear symmetric form to $v_t^\perp$ is integral. Mukai defined (see [9, 11]) a map

$$\theta_t \colon v_t^\perp \longrightarrow H^2(M_t) \tag{3.1.9}$$

[2] This means that $h^{p,q} = 0$ if $p < 0$.

by taking Künneth components of the Chern character of a tautological sheaf on $S_t \times M_t$ (if such a sheaf does not exists one considers a quasi-tautological sheaf). In [11] we proved that θ_t is an isomorphism of Hodge structures and an isometry of lattices - of course the bilinear form on $v_t^{\perp}$ is the restriction of the Mukai pairing. Now assume that $t \in T'''$ and let A_t be as in Proposition (3.1) and L_{A_t} be as in (1.0.3); then by Corollary (3.2) and Proposition (5.1) of [12] we have

$$c_1(L_{A_t}) = \theta_t(\eta_t - 1). \tag{3.1.10}$$

We let $h_t := \theta_t(\eta_t - 1)$.

Proposition 3.3. *Keep notation as above. Then $\Delta^0(V)$ is a $PGL(V)$-invariant subset of $\mathbb{LG}(\wedge^3 V)^0$ which is locally (in the classical topology) a codimension 1 submanifold.*

Proof. The subset $\Delta^0(V)$ is $PGL(V)$-invariant by definition. Let $\pi \colon \mathcal{X} \to \mathbb{LG}(\wedge^3 V)^0$ be the tautological family of double EPW-sextics and $\mathcal{L}$ be the tautological relatively ample line-bundle on $\mathcal{X}$; thus the restriction of $\mathcal{L}$ to X_A is isomorphic to L_A. Let $A_p \in \Delta^0(V)$. Thus there exists $p \in T'''$ such that $X_{A_p} \cong M_p$. Let $U \subset \mathbb{LG}(\wedge^3 V)^0$ be a small open ball contining A_p. Let $\mathcal{X}_U := \pi^{-1}(U)$ and $\mathcal{L}_U := \mathcal{L}|_{\mathcal{X}_U}$. Since U is contractible there is a marking Ψ of $(\mathcal{X}_U, \mathcal{L}_U)$; let $\mathcal{P}_\Psi \colon U \to \mathcal{D}_2$ be the corresponding local period map. We notice that $(5+2c_1(D_p)+5\eta_p) \in v_p^{\perp}$ and hence $\theta_p(5+2c_1(D_p)+5\eta_p) \in H^2(M_p)$. Furthermore since $\langle \eta_p - 1, 5 + 2c_1(D_p) + 5\eta_p \rangle = 0$ we have $(h_p, \theta_p(5+2c_1(D_p)+5\eta_p))_{M_p} = 0$; since $h_p = c_1(L_{A_p})$ we get that $\Psi_p(\theta_p(5+2c_1(D_p)+5\eta_p)) \in u^{\perp}$ where u is our fixed vector of square 2 - see (1.0.8). Let's prove that

$$\mathcal{P}_\Psi(\Delta^0(V) \cap U) = \mathcal{P}_\Psi(U) \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)^{\perp}. \tag{3.1.11}$$

First we prove that

$$\mathcal{P}_\Psi(\Delta^0(V) \cap U) \subset \mathcal{P}_\Psi(U) \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)^{\perp}. \tag{3.1.12}$$

Let $\mathcal{M}''' := \rho^{-1}(T''')$ and $\rho''' \colon \mathcal{M}''' \to T'''$ be the restriction of ρ. Then $\theta_t(5 + 2c_1(D_t) + 5\eta_t)$ is a flat section of $R^2\rho'''_*\mathbb{Z}$ and for all $t \in T'''$ we have

$$(\theta_t(5 + 2c_1(D_t) + 5\eta_t), H^{2,0}(S_t))_{M_t} = 0 \tag{3.1.13}$$

because $\theta_t(5 + 2c_1(D_t) + 5\eta_t) \in H^{1,1}(M_t)$; the above equality gives (3.1.12). Next we prove that

$$\mathcal{P}_\Psi(\Delta^0(V) \cap U) \supset \mathcal{P}_\Psi(U) \cap \Psi_p \circ \theta_p(5 + 2\theta_p(c_1(D_p) + 5\eta_p)^{\perp}. \tag{3.1.14}$$

First we notice that

$$\{v_p, \eta_p - 1, 5 + 2c_1(D_p) + 5\eta_p\}^{\perp} = H^2(S_p)_{prim} \tag{3.1.15}$$

where the primitive cohomology $H^2(S_p)_{prim} \subset H^2(S)$ is the orthogonal to $c_1(D_p)$. Since $h_p = \theta_p(\eta_p - 1)$ we get that

$$H^2(M_p) \supset \{h_p, \theta_p(5 + 2c_1(D_p) + 5\eta_p)\}^{\perp} = \theta_p(H^2(S_p)_{prim}). \tag{3.1.16}$$

Let $\mathcal{K}_{10}$ be the period space for $K3$ surfaces with a polarization of degree 10; Equality (3.1.16) defines an isomorphism

$$\mathcal{D}_2 \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)^{\perp} \xrightarrow{\sim} \mathcal{K}_{10} \tag{3.1.17}$$

which is compatible with local period maps defined by the family $\rho''' \colon \mathcal{M}''' \to T'$ and the family $\zeta \colon \mathcal{S}''' \to T'''$ with fiber S_t over $t \in T'''$. Let $\mathcal{S}'''_U := \zeta^{-1}(U)$. Since $\mathcal{S}'''$ contains the generic $K3$ of degree 10 the local period map of the family $\mathcal{S}'''_U \to U$ is a submersive map from U to an open ball in $\mathcal{K}_{10}$; since (3.1.17) is an isomorphism this proves (3.1.14). We also get that $\mathcal{P}_\Psi$ is submersive and hence in order to show that $\Delta^0(V) \cap U$ is a codimension 1 submanifold it suffices to prove that

$$\mathcal{D}_2 \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)^\perp \tag{3.1.18}$$

is smooth. The period domain $\mathcal{D}_2$ is an open subset of the quadric of isotropic lines for the non-degenerate quadratic form $(,)|_{\Lambda\otimes\mathbb{C}}$ and hence if $\Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)$ is not isotropic then (3.1.18) is smooth. Since

$$\begin{aligned}(\Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p), \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)) = \\ = \langle 5 + 2c_1(D_p) + 5\eta_p, 5 + 2c_1(D_p) + 5\eta_p\rangle = -10\end{aligned} \tag{3.1.19}$$

the intersection (3.1.18) is indeed smooth. □

3.2 The dual of Y_A for $A \in \Delta^0(V)$

If $A \in \mathbb{LG}(\wedge^3 V)^{00}$ then $Y_A^\vee = Y_{A^\perp}$ by Proposition (3.1) of [13]. As we will see $\Delta^0(V) \cap \mathbb{LG}(\wedge^3 V)^{00} = \emptyset$ and hence in order to show that $Y_A^\vee = Y_{A^\perp}$ for $A \in \Delta^0(V)$ we need to improve on the result of [13].

Proposition 3.4. *Let $A \in \mathbb{LG}(\wedge^3 V)$ and $\mathbb{P}(W) \in \mathbb{P}(V^\vee)$. Then $\mathbb{P}(W) \in Y_{A^\perp}$ if and only if*

$$\wedge^3 W \cap A \neq \emptyset. \tag{3.2.1}$$

Proof. Let $\phi \in V^\vee$ be a linear function such that $W = \ker(\phi)$: then

$$F_\phi := \{\phi \wedge \psi |\ \psi \in \wedge^2 V^\vee\} = \left(\wedge^3 W\right)^\perp. \tag{3.2.2}$$

By definition $\mathbb{P}(W) \in Y_{A^\perp}$ if and only if

$$\{0\} \neq F_\phi \cap A^\perp = (\wedge^3 W + A)^\perp. \tag{3.2.3}$$

Since $10 = \dim(\wedge^3 W) = \dim A$ and $\dim(\wedge^3 V) = 20$ we get that (3.2.3) holds if and only if (3.2.1) holds. □

Corollary 3.5. *Let $A \in \mathbb{LG}(\wedge^3 V)^0$. Then $Y_{A^\perp}$ is a hypersurface and $Y_A^\vee = Y_{A^\perp}$.*

Proof. Let $\mathbb{P}(W) \in Y_{A^\perp}$; by Proposition (3.4) this is equivalent to (3.2.1). Let $0 \neq \alpha \in (\wedge^3 W \cap A)$: since $\dim W = 5$ there exists $v \in W$ such that α is divisible by v and hence $[v] \in Y_A$. We decompose $W = \mathbb{C}v \oplus W_0$ and write $\alpha = v \wedge w$ where $w \in \wedge^2 W_0$; by Definition (2.5) of [13] the rank of w is 4, thus $W = span(v, w)$. Furthermore if $\dim(F_{[v]} \cap A) = 1$ then Y_A is smooth at $[v]$ and $\mathbb{P}(W)$ is the projective tangent space to Y_A at $[v]$ - see the proof of Proposition (3.1) of [13]. By Proposition (2.8) of [13] we know that $\dim(F_{[v]} \cap A) = 1$ unless $[v] \in sing(Y_A)$ and in this case $\dim(F_{[v]} \cap A) = 2$. Since $sing(Y_A)$ is a surface a straightforward dimension count gives that $\mathbb{P}(W) \notin Y_{A^\perp}$ for generic $\mathbb{P}(W) \in \mathbb{P}(V^\vee)$, thus $Y_{A^\perp}$ is a hypersurface. The same dimension count gives that the generic $\mathbb{P}(W) \in Y_{A^\perp}$ is tangent to Y_A at one of its smooth points; this proves the corollary. □

We will describe explicitely $Y_A^\vee = Y_{A^\perp}$ for $A \in \Delta^0(V)$; essentially we will give a refinement of Proposition (5.20) and Corollary (5.21) of [12]. Let $t \in T'''$ and S_t be the $K3$ surface corresponding to t; in order to simplify notation we temporarily drop the subscript t. Let R be the Fano variety of lines on F_5^3. If $[\ell] \in R$ then $\ell \not\subset S$ by (3.1.4) and hence $\ell \cap Q$ is a 0-dimensional scheme of length 2 contained in S: thus we have a regular map

$$\begin{array}{ccc} R & \longrightarrow & S^{[2]} \\ \ell & \mapsto & \ell \cap Q. \end{array} \tag{3.2.4}$$

Let $P \subset S_t^{[2]}$ be the image of the above map: then (3.2.4) defines a regular map $R \to P$ with inverse given by

$$\begin{array}{ccc} P & \longrightarrow & R \\ [Z] & \mapsto & span(Z) \end{array} \tag{3.2.5}$$

and hence P is isomorphic to R. It is known [5] that $R \cong \mathbb{P}^2$, thus $P \cong \mathbb{P}^2$. Since $S^{[2]}$ is a symplectic variety it follows that we can contract P:

$$c\colon S^{[2]} \to N. \tag{3.2.6}$$

A priori N is a complex space, we will show soon that it is projective. Let $p := c(P)$; thus p is the unique singular point of N. On $S^{[2]}$ there is an interesting map to $|I_S(2)|^\vee$, see (4.3) of [12]; we recall the definition. The $K3$ surface S is cut out by quadrics and it contains no lines by (3.1.4); thus we have a regular map

$$\begin{array}{ccc} S^{[2]} & \longrightarrow & |I_S(2)|^\vee \cong \mathbb{P}^5 \\ [Z] & \mapsto & \{Q_\lambda \in |I_S(2)| \mid span(Z) \subset Q_\lambda\}. \end{array} \tag{3.2.7}$$

Let $W \subset |I_S(2)|^\vee$ be the image of the above map; thus (3.2.7) defines a map $f\colon S^{[2]} \to W$. If $[Z] \in P$ then $f([Z]) = |I_{F_5^3}(2)|$ hence f is constant on P; we will see that the point

$$f(P) = |I_{F_5^3}(2)| \in W \tag{3.2.8}$$

is quite special. Since f is constant on P and N is normal the map f descends to a regular map

$$\overline{f}\colon N \to W. \tag{3.2.9}$$

Lemma 3.6. *Keep notation as above. There exist a non trivial involution $\overline{\phi}\colon N \to N$ and a birational morphism $\epsilon\colon N/\langle\overline{\phi}\rangle \to W$ with finite fibers such that $\overline{f}$ is the composition*

$$N \xrightarrow{\pi} N/\langle\overline{\phi}\rangle \xrightarrow{\epsilon} W. \tag{3.2.10}$$

In particular N is projective and $\deg W = 6$.

Proof. Let's show that $\overline{f}$ has finite fibers of cardinality at most 2 and that the generic fiber has cardinality 2. The fiber of $\overline{f}$ over $|I_{F_5^3}(2)| \in |I_S(2)|^\vee$ consists of the unique singular point p of N. Now let $\Lambda \in (W \setminus \{|I_{F_5^3}(2)|\})$, i.e.

$$\Lambda = f([Z]), \quad [Z] \notin P. \tag{3.2.11}$$

Then $\overline{f}^{-1}(\Lambda) = f^{-1}(\Lambda)$. Let $\Lambda_0 := \Lambda \cap |I_{F_5^3}(2)|$ and choose $\lambda_0 \in (\Lambda \setminus |I_{F_5^3}(2)|)$. One has

$$\bigcap_{\lambda \in \Lambda_0} Q_\lambda = F_5^3 \cup A_\Lambda \tag{3.2.12}$$

where A_Λ is a plane such that $A_\Lambda \cap F_5^3 = C_\Lambda$ is a conic. (See the proof of Lemma (4.20) of [12].) We claim that $Q_{\lambda_0} \not\supset A_\Lambda$: in fact if $Q_{\lambda_0} \supset A_\Lambda$ then $C_\Lambda \subset S$ contradicting (3.1.4). Thus $Q_{\lambda_0} \cap A_\Lambda$ is a conic C'_Λ. By (3.2.11) the line $span(Z)$ is contained in C'_Λ. Thus C'_Λ is degenerate and $f^{-1}(\Lambda)$ consists of the set of lines contained in C'_Λ. This shows that $f^{-1}(\Lambda) = \overline{f}^{-1}(\Lambda)$ has cardinality at most 2. It also follows easily that the generic fiber of $\overline{f}$ consists of 2 distinct points. Since N is normal there is a regular covering involution $\overline{\phi}$ such that $\overline{f}$ factors through the quotient map $N \to N/\langle\overline{\phi}\rangle$. Since $\overline{f}$ has finite fibers so does ϵ, since the generic fiber of $\overline{f}$ consists of 2 points the map ϵ is birational. The line-bundle $\overline{f}^*\mathcal{O}_W(1)$ is ample because $\overline{f}\colon N \to W$ has finite fibers, thus N is projective. We know (see (4.3) of [12]) that

$$\int_{S^{[2]}} c_1(f^*\mathcal{O}_W(1)) = 12. \tag{3.2.13}$$

Since $f\colon S^{[2]} \to W$ has generic fiber of cardinality 2 we get that $\deg W = 6$. $\square$

We will show that the map ϵ of Lemma (3.6) is an isomorphism. Since $(S^{[2]} \setminus P) \cong (N \setminus \{p\})$ the involution $\overline{\phi}$ defines a birational involution

$$\phi\colon S^{[2]} \dashrightarrow S^{[2]}. \tag{3.2.14}$$

(This is the birational involution of Proposition (4.21) of [12].) Let

$$\beta\colon Bl_P(S^{[2]}) \to S^{[2]} \tag{3.2.15}$$

be the blow-up of P: since P is Lagrangian the symplectic form on $S^{[2]}$ induces an isomorphism $N_{P/S^{[2]}} \cong \Omega^1_P$ and hence the exceptional divisor of β is identified with the incidence variety $\Gamma \subset P \times P^\vee$ and the restriction of β to the exceptional divisor is identified with the projection $\Gamma \to P$. We abuse notation and view Γ as the exceptional divisor in $Bl_P(S^{[2]})$

Lemma 3.7. *The map ϕ of (3.2.14) is not regular along P. There is a regular involution $\widetilde{\phi}\colon Bl_P(S^{[2]}) \to Bl_P(S^{[2]})$ which is equal to ϕ on $(S^{[2]} \setminus P) \subset Bl_P(S^{[2]})$. There is an identification $P \cong P^\vee$ such that $\widetilde{\phi}|_\Gamma$ is induced by the involution on $P \times P^\vee$ which interchanges the factors.*

Proof. The eigenspaces of the isometry $H^2(\phi)$ on $H^2(S^{[2]})$ induced by ϕ are given by (see (4.3) of [12])

$$H^2(\phi)_+ = \mathbb{C} f^*\mathcal{O}_W(1), \quad H^2(\phi)_- = f^*\mathcal{O}_W(1)^\perp. \tag{3.2.16}$$

Suppose ϕ is regular along P: if D is an ample divisor on $S^{[2]}$ then $c_1(D + \phi^*D)$ is an ample ϕ-invariant class, this contradicts (3.2.16) because $f^*\mathcal{O}_W(1)$ is not ample. Let $\psi\colon S^{[2]} \dashrightarrow \mathcal{X}$ be the flop of P: thus ψ is the inverse of the blowup $Bl_P(S^{[2]}) \to S^{[2]}$ followed by the morphism $Bl_P(S^{[2]}) \to \mathcal{X}$ which contracts Γ along the "other"fibration $\Gamma \to P^\vee$. In particular $\mathcal{X}$ contains $P^\vee$. Let $\ell \subset P$ and $\ell^\vee \subset P^\vee$ be lines. The isometry $H^2(\psi)$ identifies $(\ell^\vee)^\perp$ with $\ell^\perp$. In fact we have contractions $c\colon S^{[2]} \to N$ and $c^\vee\colon \mathcal{X} \to N$ which give identifications $\ell^\perp = H^2(N) = (\ell^\vee)^\perp$. In particular $H^2(\psi)$ sends a nef divisor class in $(\ell^\vee)^\perp$ to a nef divisor class. On the other hand $H^2(\psi)$ maps the half-space $\ell^\vee_{>0}$ to the half-space $\ell_{<0}$. By (3.2.16) we get that $(\psi \circ \phi)^*$ maps an ample divisor to an ample divisor: since $(\psi\circ\phi)$ defines a regular map between the complements of subsets of codimension 2 it follows that $(\psi\circ\phi)$ is regular and hence an isomorphism. It follows also that ϕ induces a regular involution $\widetilde{\phi}\colon Bl_P(S^{[2]})\to Bl_P(S^{[2]})$. Let's show that the restriction of $\widetilde{\phi}$ to Γ is as stated. Any automorphism of Γ sends the projection $\Gamma\to P$ to itself composed with an automorphism of P or to the projection $\Gamma\to P^\vee$ composed with an automorphism of $P^\vee$. Since ϕ is not regular the latter holds and it follows that $\widetilde{\phi}|_\Gamma$ is as stated. □

Corollary 3.8. *The map $\epsilon\colon N/\langle\overline{\phi}\rangle\to W$ of Lemma (3.6) is an isomorphism.*

Proof. The quotient $Bl_P(S^{[2]})/\langle\widetilde{\phi}\rangle$ is a projective birational model of $N/\langle\overline{\phi}\rangle$ and hence it is birational to W by Lemma (3.6). The Kodaira dimension of $Bl_P(S^{[2]})/\langle\widetilde{\phi}\rangle$ is 0 hence also the Kodaira dimension of W is 0. By Lemma (3.6) we know that $\deg W=6$ and hence by adjunction W is smooth in codimension 1. Thus W is normal: since ϵ is birational with finite fibers we get that ϵ is an isomorphism. □

Now we reintroduce the subscript t; thus we have S_t, f_t, N_t, W_t etc.

Proposition 3.9. *Keep notation as above. Let $A\in\Delta^0(V)$ and let $t\in T'''$ such that $X_{A_t}\cong M_t$. Thus by Proposition (3.1) and Corollary (3.5) we have $Y_{A^\perp}\subset|I_{S_t}(2)|^\vee$. Then*

$$Y_{A^\perp}=W_t. \tag{3.2.17}$$

Proof. Proposition (5.20) of [12] gives that the reduced scheme $(Y_{A^\perp})_{red}$ is equal to W_t (the hypothesis of that Proposition is that (3.0.2) holds, however the same proof goes through because all that is needed is the validity of (3.0.5)). Now $Y_{A^\perp}$ is a degree-6 hypersurface because $Y_{A^\perp}\neq\mathbb{P}(V^\vee)$ and on the other hand W_t is a degree-6 hypersurface by Lemma (3.6) and hence from $(Y_{A^\perp})_{red}=W_t$ we get that $Y_{A^\perp}$ is reduced and equal to W_t. □

Let $A\in\Delta^0(V)$ and let $t\in T'''$ such that $X_{A_t}\cong M_t$. Then $|I_{F_5^3}(2)|\in W_t$ - see (3.2.8 - and hence by the above proposition $|I_{F_5^3}(2)|\in Y_{A^\perp}$; we denote this point by $q_{A^\perp}$.

Proposition 3.10. *Let $A\in\Delta^0(V)$ and let $t\in T'''$ such that $X_{A_t}\cong M_t$. Then*

$$\operatorname{mult}_{q_{A^\perp}}Y_{A^\perp}=3. \tag{3.2.18}$$

Let $y\in(Y_{A^\perp}\setminus\{q_{A^\perp}\})$ and hence $\overline{f_t}^{-1}(y)=f_t^{-1}(y)$ consists of two points or of one point.

(a) If $f_t^{-1}(y)$ consists of two points then $Y_{A^\perp}$ is smooth at y.

(b) If $f_t^{-1}(y)$ consists of a single point then the analytic germ $(Y_{A^\perp},y)$ is isomorphic to the product of a smooth 2-dimensional germ and the germ of an A_1-singularity.

Proof. By Proposition (6.2) of [13] $Y_A^\vee$ has multiplicity 3 in $|I_{F_5^3}(2)|$; this proves (3.2.18). In order to prove (a)-(b) we notice that $Y_{A^\perp}=W_t$ and by Proposition (3.8) the map $\overline{f}_t\colon N_t\to W_t$ is identified with the quotient map $N_t\to N_t/\langle\overline{\phi}\rangle$. Since $p_t=\overline{f}^{-1}(q_A)$ is the unique singular point of N_t this gives Item (a). In order to prove Item (b) we notice that $(N_t\setminus\{p_t\})=(S_t^{[2]}\setminus P_t)$ and that the quotient map $(N_t \setminus \{p_t\}) \to (N_t \setminus \{p_t\})/\langle\overline{\phi}\rangle$ is identified with the quotient map $(S_t^{[2]} \setminus P_t) \to (S_t^{[2]} \setminus P_t)/\langle\phi\rangle$. By Proposition (4.21) of [12] the restriction of ϕ to $(S_t^{[2]} \setminus P_t)$ is an anti-symplectic involution and hence its fixed point set is a Lagrangian surface; this proves (b). □

Let $A \in \Delta^0(V)$; then by (3.2.18) the sextic $Y_{A^\perp}$ has a point of multiplicity 3 and hence $A^\perp \notin \mathbb{LG}(\wedge^3 V^\vee)^0$ because if $B \in \mathbb{LG}(\wedge^3 V^\vee)^0$ then Y_B has points of multiplicity at most 2. Thus

$$\Delta^0(V) \cap \mathbb{LG}(\wedge^3 V)^{00} = \emptyset. \tag{3.2.19}$$

Let $\Delta^0_*(V) \subset \Delta^0(V)$ be the set of points which are smooth points of the projective variety $\left(\mathbb{LG}(\wedge^3 V) \setminus \mathbb{LG}(\wedge^3 V)^{00}\right)$. By Proposition (3.3) we know that $\Delta^0(V)$ is locally (in the classical topology) a codimension 1 submanifold of $\mathbb{LG}(\wedge^3 V)$; since $\Delta^0(V) \cap \mathbb{LG}(\wedge^3 V)^{00} = \emptyset$ it follows that $\Delta^0_*(V)$ is open dense in $\Delta^0(V)$. We let $T_* \subset T'''$ be the set of t such that $M_t \cong X_A$ for some $A \in \Delta^0_*(V)$; since $\Delta^0_*(V)$ is open dense in $\Delta^0(V)$ also T_* is open dense in T'''. Let

$$\mathbb{LG}(\wedge^3 V)^0_* := \mathbb{LG}(\wedge^3 V)^{00} \cup \Delta^0_*(V). \tag{3.2.20}$$

Proposition 3.11. *Keep notation as above. Then $\mathbb{LG}(\wedge^3 V)^0_*$ is open in $\mathbb{LG}(\wedge^3 V)$ (for the classical topology) and $\Delta^0_*(V)$ is a non-empty codimension-1 submanifold of $\mathbb{LG}(\wedge^3 V)^0_*$.*

Proof. Let $A \in \Delta^0_*(V)$. By definition of $\Delta^0_*(V)$ there exists an open $U \subset \mathbb{LG}(\wedge^3 V)$ such that $U \cap \Delta^0_*(V) = U \cap \left(\mathbb{LG}(\wedge^3 V) \setminus \mathbb{LG}(\wedge^3 V)^{00}\right)$ and hence $U \subset \mathbb{LG}(\wedge^3 V)^0_*$; this proves that $\mathbb{LG}(\wedge^3 V)^0_*$ is open. We have already proved that $\Delta^0_*(V)$ is a non-empty locally closed codimension-1 submanifold of $\mathbb{LG}(\wedge^3 V)$; this gives the second statement of the proposition. □

We let

$$\Delta^\infty_*(V^\vee) := \delta(\Delta^0_*(V)), \tag{3.2.21}$$
$$\mathbb{LG}(\wedge^3 V^\vee)^\infty_* := \delta(\mathbb{LG}(\wedge^3 V)^0_*). \tag{3.2.22}$$
$$\tag{3.2.23}$$

Of course every definition above has a "dual "definition obtained by substituting $V^\vee$ to V; thus we have $\Delta^0_*(V^\vee) \subset \mathbb{LG}(\wedge^3 V^\vee)^0$, $\Delta^\infty_*(V) \subset \mathbb{LG}(\wedge^3 V)$ etc. Let

$$\begin{aligned}\mathbb{LG}(\wedge^3 V^\vee)^\sharp := \mathbb{LG}(\wedge^3 V^\vee)^0_* \cup \mathbb{LG}(\wedge^3 V^\vee)^\infty_* = \\ = \mathbb{LG}(\wedge^3 V^\vee)^{00} \cup \Delta^0_*(V^\vee) \cup \Delta^\infty_*(V^\vee).\end{aligned} \tag{3.2.24}$$

Let $\mathcal{Y}(V^\vee) \subset \mathbb{LG}(\wedge^3 V^\vee)^\sharp \times \mathbb{P}(V^\vee)$ be the tautological EPW-sextic; thus $\mathcal{Y}(V^\vee) \cap \{B\} \times \mathbb{P}(V^\vee) = Y_B$. If $B \in \Delta^\infty_*(V^\vee)$ then by Proposition (3.10) there is a unique point $q_B \in Y_B$ of multiplicity strictly greater than 2; let

$$\mathcal{Q}(V^\vee) := \{(B, q_B) |\ B \in \Delta^\infty_*(V^\vee)\}, \tag{3.2.25}$$
$$\mathcal{Y}(V^\vee)^\sharp := \mathcal{Y}(V^\vee) \setminus \mathcal{Q}(V^\vee). \tag{3.2.26}$$

Proposition 3.12. *There exists a double cover $f\colon \mathcal{X}(V^\vee) \to \mathcal{Y}(V^\vee)^\sharp$ with the following properties.*

(1) *Let* $\pi\colon \mathcal{X}(V^\vee) \to \mathbb{LG}(\wedge^3 V^\vee)^\sharp$ *be the composition of* f *and the projection* $\mathcal{Y}(V^\vee)^\sharp \to \mathbb{LG}(\wedge^3 V^\vee)^\sharp$*; then* π *is a submersion of smooth manifolds.*

(2) *Let* $B \in \mathbb{LG}(\wedge^3 V^\vee)^0_*$*; then* $\pi^{-1}(B) \cong X_B$ *and the map* $\pi^{-1}(B) \to Y_B$ *defined by* f *is isomorphic to the natural double cover* $X_B \to Y_B$.

(3) *Let* $B \in \Delta^\infty_*(V^\vee)$ *and* $A := B^\perp$. *Since* $A \in \Delta^0_*(V)$ *there exists* $t \in T_*$ *such that* $X_A \cong M_t$ *and* $Y_A \cong \Sigma'_t$. *Then* $\pi^{-1}(B) \cong (S^{[2]}_t \setminus \{P_t\})$ *and the map* $\pi^{-1}(B) \to (Y_B \setminus \{q_B\})$ *defined by* f *is isomorphic to the double cover* $(S^{[2]}_t \setminus \{P_t\}) \to (W_t \setminus |I_{F^3_5}(2)|)$ *given by the restriction of (3.2.7). (Recall that* $(Y_B \setminus \{q_B\}) \cong (W_t \setminus |I_{F^3_5}(2)|)$ *by Proposition (3.9)).*

Proof. Let $B \in \mathbb{LG}(\wedge^3 V^\vee)^\sharp$. Then $Y_B \neq \mathbb{P}(V^\vee)$ by Corollary (3.5); thus the map λ_B defined in Section (1) (with $V^\vee$ replacing V) is non-zero and Y_B is the zero-scheme of $\det(\lambda_B)$. Since Y_B is a Lagrangian degeneracy locus there exists locally in $\mathbb{P}(V^\vee)$ a symmetric map of vector-bundles giving a resolution of $coker(\lambda_B)$, i.e. we can cover $\mathbb{P}(V^\vee)$ by open sets U such that on each U we have a locally-free resolution

$$0 \to E_U \xrightarrow{\alpha_U} E_U^\vee \longrightarrow coker(\lambda_B)|_U \to 0 \tag{3.2.27}$$

where α_U is a symmetric map of vector-bundles. Furthermore if $F \hookrightarrow \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}$ is the Lagrangian sub-vector-bundle defined in Section (1) (with $V^\vee$ replacing V) then there is an isomorphism $F_p \cap B \cong ker(\alpha_p)$ for all $p \in U$. Thus if $\dim(F_p \cap B) \geq r$ then $mult_p(Y_B) \geq r$. Let $p \neq q_B$. Then $mult_p(Y_B) \leq 2$ by Proposition (3.10) and hence $\dim(F_p \cap B) \leq 2$. Furthermore one of the following holds:

(1) If $\dim(F_p \cap B) = 1$ then locally around p we have $coker(\lambda_B) \cong i_*\mathcal{O}_{Y_B}$ where $i\colon Y_B \hookrightarrow \mathbb{P}(V^\vee)$ is the inclusion.

(2) If $\dim(F_p \cap B) = 2$ there exist an open (in the classical topology) $U \subset \mathbb{P}(V^\vee)$ containing p, functions $x, y, z \in Hol(U)$ vanishing at p with linearly independent differentials and an exact sequence

$$0 \to \mathcal{O}_U^2 \xrightarrow{M} \mathcal{O}_U^2 \longrightarrow coker(\lambda_B)|_U \to 0 \tag{3.2.28}$$

where M is the map defined by the matrix

$$\begin{pmatrix} x & y \\ y & z \end{pmatrix}. \tag{3.2.29}$$

In particular we see that there exists a sheaf ζ_B on $(Y_B \setminus \{q_B\})$ such that outside q_B we have $coker(\lambda_B) = i^0_*\zeta_B$ where $i^0\colon (Y_B \setminus \{q_B\}) \hookrightarrow (\mathbb{P}(V^\vee) \setminus \{q_B\})$ is the inclusion. From the local description of $coker(\lambda_B)$ given above we also get a canonical isomorphism of sheaves on $(\mathbb{P}(V^\vee) \setminus \{q_B\})$:

$$Ext^1(coker(\lambda_B), \mathcal{O}_{\mathbb{P}(V^\vee)})|_{\mathbb{P}(V^\vee)\setminus\{q_B\}} \cong i^0_*(\zeta_B^\vee \otimes N_{Y_B/\mathbb{P}(V^\vee)}). \tag{3.2.30}$$

(See Proposition (4.3) of [13].) Let $B^\vee \subset \wedge^3 V^\vee$ be a Lagrangian subspace complementary to B; thus we have a direct sum decomposition

$$\wedge^3 V = B \oplus B^\vee. \tag{3.2.31}$$

Then λ_B can be identified with the map of vector-bundles $F \to B^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}$ associated to Decomposition (3.2.31). Let $\mu_B \colon F \to B \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}$ be the "other" map associated to Decomposition (3.2.31). The diagram

$$\begin{array}{ccc} F & \xrightarrow{\lambda_B} & B^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \\ \downarrow{\scriptstyle \mu_B} & & \downarrow{\scriptstyle \mu_B^\vee} \\ B \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} & \xrightarrow{\lambda_B^\vee} & F^\vee \end{array} \tag{3.2.32}$$

is commutative because $F \overset{(\mu_B,\lambda_B)}{\longrightarrow} (B \oplus B^\vee) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}$ is a Lagrangian embedding. The map λ_B is an injection of sheaves because $Y_B \neq \mathbb{P}(V^\vee)$ and hence also $\lambda_B^\vee$ is an injection of sheaves. Thus there is a unique $\beta_B \colon coker(\lambda_B) \longrightarrow Ext^1(coker(\lambda_B), \mathcal{O}_{\mathbb{P}(V^\vee)})$ making the following diagram commutative with exact horizontal sequences:

$$\begin{array}{ccccccccc} 0 & \to & F & \xrightarrow{\lambda_B} & B^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} & \longrightarrow & coker(\lambda_B) & \to & 0 \\ & & \downarrow{\scriptstyle \mu_B} & & \downarrow{\scriptstyle \mu_B^\vee} & & \downarrow{\scriptstyle \beta_B} & & \\ 0 & \to & B \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} & \xrightarrow{\lambda_B^\vee} & F^\vee & \longrightarrow & Ext^1(coker(\lambda_B), \mathcal{O}_{\mathbb{P}(V^\vee)}) & \to & 0 \end{array} \tag{3.2.33}$$

By Isomorphism (3.2.30) we get that the restriction of β_B to $(\mathbb{P}(V^\vee) \setminus \{q_B\})$ defines a map of sheaves on $(Y_B \setminus \{q_B\})$

$$\zeta_B \to \zeta_B^\vee(6). \tag{3.2.34}$$

Since $F \overset{(\mu_B,\lambda_B)}{\longrightarrow} (B \oplus B^\vee) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}$ is an injection of vector-bundles the above map is an isomorphism - this follows from Claim (4.5) of [13]. Let $\xi_B := \zeta_B(-3)$; then (3.2.34) defines an isomorphism $\xi_B \xrightarrow{\sim} \xi_B^\vee$ which is symmetric and hence it gives $\mathcal{O}_{(Y_B \setminus \{q_B\})} \oplus \xi_B$ the structure of a commutative finite $\mathcal{O}_{(Y_B \setminus \{q_B\})}$-algebra. Let $X_B := Spec(\mathcal{O}_{(Y_B \setminus \{q_B\})} \oplus \xi_B)$ and $f_B \colon X_B \to (Y_B \setminus \{q_B\})$ be the structure map: clearly f_B is finite of degree 2. The above construction is the analogue of the construction of the natural double cover $X_B \to Y_B$ for $B \in \mathbb{LG}(\wedge^3 V^\vee)^0$; thus we have a double cover $f \colon \mathcal{X}(V^\vee) \to \mathcal{Y}^\sharp(V^\vee)$ such that Item (2) holds and such that for $B \in \Delta_*^\infty(V^\vee)$ we have $\pi^{-1}(B) = X_B$ and the map $\pi^{-1}(B) \to (Y_B \setminus \{q_B\})$ is the structure map f_B defined above. It remains to prove that Items (1) and (3) hold. Let

$$\begin{aligned} C_B :=& \{p \in (singY_B \setminus q_B) | \ \dim(F_p \cap B) = 1\} & (3.2.35) \\ D_B :=& \{p \in (singY_B \setminus q_B) | \ \dim(F_p \cap B) = 2\}. & (3.2.36) \end{aligned}$$

Thus $(singY_B \setminus q_B) = C_B \coprod D_B$. The local description given above of $coker(\lambda_B)$ near $p \in (Y_B \setminus q_B)$ shows that both C_B and D_B are smooth and closed (in $(Y_B \setminus q_B)$). Let $p \in (Y_B \setminus q_B)$; the map f_B behaves differently depending on whether $p \in C_B$ or $p \in D_B$: in fact (see [13])

(α) If $p \in C_B$ then f_B is unramified over p.

(β) If $p \in D_B$ then f_B is ramified over p and X_B is smooth at $f_B^{-1}(p)$.

Let's prove that

$$X_B \cong (S_t^{[2]} \setminus P_t). \tag{3.2.37}$$

Let $B' \in \mathbb{LG}(\wedge^3 V^\vee)^0_*$ and define $D_{B'}$ as above. Then $D_{B'}$ is not empty in fact it is a surface; it follows that $D_B \neq \emptyset$ and by Item (β) above we get that the étale covering

$$\big(X_B \setminus f_B^{-1}(singY_B)\big) \longrightarrow (Y_B \setminus singY_B) \tag{3.2.38}$$

is not trivial. On the other hand let $t \in T_*$ such that $X_{A^\perp} \cong M_t$ and $Y_{A^\perp} \cong \Sigma'_t$. Let $f_t \colon S_t^{[2]} \to Y_B$ be the map defined by (3.2.7); then f_t defines an étale double cover

$$(S_t^{[2]} \setminus f_t^{-1}(singY_B)) \longrightarrow (Y_B \setminus singY_B). \tag{3.2.39}$$

Now $S_t^{[2]}$ is simply connected and $f_t^{-1}(singY_B)$ has codimension 2 in $S_t^{[2]}$ hence $(S_t^{[2]} \setminus f_t^{-1}(singY_B))$ is simply connected. Thus $\pi_1(Y_B \setminus singY_B) \cong \mathbb{Z}/(2)$ and hence there is a unique non-trivial double cover of $(Y_B \setminus singY_B)$ and it is given by (3.2.39). Since (3.2.38) is a non-trivial double cover it follows that $C_B = \emptyset$ and that (3.2.37) holds. This proves Items (1)-(3) of the proposition. $\square$

3.3 Extension of the local period map across $\Delta^\infty_*(V^\vee)$

We will prove that the local period map extends across $\Delta^\infty_*(V^\vee)$. Let $\pi^0 \colon \mathcal{X}(V^\vee)^0_* \to \mathbb{LG}(\wedge^3 V^\vee)^0_*$ be the tautological family of double EPW-sextics i.e. the restriction of the map π of Proposition (3.12) to $\pi^{-1}(\mathbb{LG}(\wedge^3 V^\vee)^0_*)$. Since π^0 is proper it defines the variation of Hodge structures

$$(R^2\pi^0_*\mathbb{Z}, F^p) \tag{3.3.1}$$

where $F^0 \supset F^1 \supset F^2$ is the Hodge filtration of $(R^2\pi^0_*\mathbb{Z}) \otimes \mathcal{O}_{\mathbb{LG}(\wedge^3 V^\vee)^0_*}$ i.e. the fiber of F^p over A is

$$F^p_A = F^p H^2(X_A) := \bigoplus_{p' \geq p} H^{p', 2-p'}(X_A). \tag{3.3.2}$$

Furthermore we have the symmetric section

$$\mathbb{B} \in H^0((R^2\pi^0_*\mathbb{Q}) \otimes (R^2\pi^0_*\mathbb{Q})) \tag{3.3.3}$$

which gives the Beauville-Bogomolov bilinear form on $H^2(X_A)$ for every $A \in \mathbb{LG}(\wedge^3 V^\vee)^0_*$.

Proposition 3.13. *There exist*

(1) a local system $\mathbb{H}(V^\vee)$ on $\mathbb{LG}(\wedge^3 V^\vee)^\sharp$,

(2) a decreasing filtration of $\mathbb{H}(V^\vee) \otimes \mathcal{O}_{\mathbb{LG}(\wedge^3 V^\vee)^\sharp}$ by holomorphic sub-bundles $\widetilde{F}^0 \supset \widetilde{F}^1 \supset \widetilde{F}^2$,

(3) a symmetric section $\widetilde{\mathbb{B}} \in H^0((\mathbb{H}(V^\vee) \otimes \mathbb{H}(V^\vee)) \otimes \mathbb{Q}))$

such that the following hold:

(a) $(\mathbb{H}(V^\vee), \widetilde{F}^p)$ extends the variation of Hodge structures (3.3.1).

(b) Let $B \in \Delta^\infty_*(V^\vee)$ and $t \in T_*$ such that $M_t \cong X_{B^\perp}$; then there exists an isomorphism of Hodge structures

$$(\mathbb{H}(V^\vee)_B, \widetilde{F}^p_B) \cong H^2(S_t^{[2]}). \tag{3.3.4}$$

(c) *The restriction of* $\widetilde{\mathbb{B}}$ *to* $\mathbb{LG}(\wedge^3 V^\vee)^0_*$ *is equal to* $\mathbb{B}$.

(d) *Let* $B \in \Delta^\infty_*(V^\vee)$ *and* $t \in T_*$ *such that* $M_t \cong X_{B^\perp}$*; then Isomorphism (3.3.4) is an isometry between* $(\mathbb{H}(V^\vee)_B, \widetilde{\mathbb{B}}_B)$ *and* $H^2(S_t^{[2]})$ *equipped with the Beauville-Bogomolov symmetric bilinear form.*

Proof. Let $\mathcal{U} \subset \mathbb{LG}(\wedge^3 V^\vee)^\sharp$ be an open ball. We assume that $\mathcal{U}$ is small: then there exists a hyperplane $H \subset \mathbb{P}(V^\vee)$ such that $q_B \notin H$ for all $B \in \mathcal{U} \cap \Delta^\infty_*(V^\vee)$. Let π and f be as in Proposition (3.12); we let

$$\mathcal{X}(V^\vee)_{\mathcal{U}} := \pi^{-1}(\mathcal{U}), \tag{3.3.5}$$

$$Z(\mathcal{U}, H) := \mathcal{X}(V^\vee)_{\mathcal{U}} \cap f^{-1}H. \tag{3.3.6}$$

Let $\rho\colon Z(\mathcal{U}, H) \to \mathcal{U}$ be given by the restriction of f. By our choice of H the map ρ is proper submersive with fibers smooth 3-folds. Thus we have a variation of Hodge structures $(R^2\rho_*\mathbb{Z}, F^p)$; we denote it by $(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))$. Let $B \in (\mathcal{U} \setminus \Delta^\infty_*)$; then $rho^{-1}(B)$ is an ample divisor on X_B and hence we have a canonical isomorphism

$$(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))|_{(\mathcal{U}\setminus\Delta^\infty_*)} \cong (R^2\rho_*\mathbb{Z}, F^p)|_{(\mathcal{U}\setminus\Delta^\infty_*)}. \tag{3.3.7}$$

This shows that $(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))$ does not depend on the choice of H and that the collection of $(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))$ gives an extension $(\mathbb{H}(V^\vee), \widetilde{F}^p)$ of the variation of Hodge structures (3.3.1). Now let's prove Item (b). Let $B \in (\mathcal{U} \cap \Delta^\infty_*(V^\vee))$ and let t be as in Item (b). Let $f_t\colon S_t^{[2]} \to W_t = Y_B$ be the map given by (3.2.7). Then $\rho^{-1}(B) = f_t^{-1}H$ and since f_t is semi-small the restriction map $H^2(S_t^{[2]}) \to H^2(f_t^{-1}H)$ is an isomorphism of (integral) Hodge structures; this proves Item (b) because $(\mathbb{H}(V^\vee)_B, \widetilde{F^p})$ is isomorphic to the Hodge structure on $H^2(f_t^{-1}H)$ by definition. We define $\widetilde{\mathbb{B}}$ as follows. Let $\mathcal{U}$ and $\rho\colon Z(\mathcal{U}, H) \to \mathcal{U}$ be as above; let $Z_B := \rho^{-1}(B)$. For $B \in \mathcal{U}$ we have the Lefschetz decomposition

$$\mathbb{H}(V^\vee)_B = H^2(Z_B; \mathbb{Q}) = \mathbb{Q}c_1(L_B)|_{Z_B} \oplus H^2(Z_B; \mathbb{Q})_{prim} \tag{3.3.8}$$

where L_B is the tautological ample line-bundle on X_B. We let $\widetilde{\mathbb{B}}(\mathcal{U}, H)_B$ be the symmetric bilinear form on $\mathbb{H}(V^\vee)_B$ characterized by the following requirements:

(α) Decomposition (3.3.8) is orthogonal for $\widetilde{\mathbb{B}}(\mathcal{U}, H)_B$.

(β) $\widetilde{\mathbb{B}}_B(c_1(L_B)|_{Z_B}, c_1(L_B)|_{Z_B}) = 2$.

(γ) If $x, y \in H^2(Z_B; \mathbb{Q})_{prim}$ then $\widetilde{\mathbb{B}}(\mathcal{U}, H)_B(x, y) = \frac{1}{2}\int_{Z_B} c_1(L_B) \wedge x \wedge y$.

Since the Lefscethz decomposition is flat for the Gauss-Manin connection we have a well-defined section $\widetilde{\mathbb{B}}(\mathcal{U}, H) \in H^0(\mathbb{H}(V^\vee)|_{\mathcal{U}})$ with value $\widetilde{\mathbb{B}}(\mathcal{U}, H)_B$ at $B \in \mathcal{U}$. Let $B \in (\mathcal{U} \setminus \Delta^\infty_*(V^\vee)$ and let $\iota\colon Z_B \hookrightarrow X_B$ be the inclusion. If $\xi_1, \xi_2, \xi_3 \in H^2(X_B)$ then

$$\int_{Z_B} \iota^*\xi_1 \wedge \iota^*\xi_2 \wedge \iota^*\xi_3 = \int_{X_B} c_1(L_B) \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 = \frac{1}{2}\sum_{\sigma \in \mathcal{S}_3} (c_1(L_B), \xi_{\sigma(1)}) \cdot (\xi_{\sigma(2)}, \xi_{\sigma(3)}). \tag{3.3.9}$$

It follows from this that $\widetilde{\mathbb{B}}(\mathcal{U}, H)$ does not depend on H and that the collection of $\widetilde{\mathbb{B}}(\mathcal{U}, H)$'s defines an extension of $\mathbb{B}$. Item (d) holds because Formula (3.3.9) holds if we replace X_B by $S_t^{[2]}$. □

The map δ defines an isomorphism

$$\begin{array}{ccc} \mathbb{LG}(\wedge^3 V)^\sharp & \xrightarrow{\sim} & \mathbb{LG}(\wedge^3 V)^\sharp \\ A & \mapsto & \delta(A) = A^\perp. \end{array} \tag{3.3.10}$$

From now on we will denote by δ what is actually the restriction of δ to $\mathbb{LG}(\wedge^3 V)^\sharp$. Let $\mathcal{U} \subset \mathbb{LG}(\wedge^3 V^\vee)^\sharp$ be an open set: then we have two local systems, namely

$$\mathbb{H}(V)|_{\mathcal{U}}, \qquad \mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}. \tag{3.3.11}$$

Definition 3.14. *Assume that both local systems (3.3.11) are trivial. A marking of $\mathbb{H}(V)|_{\mathcal{U}}$ is an isomorphism $\Psi\colon \mathbb{H}(V)|_{\mathcal{U}} \xrightarrow{\sim} \mathcal{U} \times \widetilde{\Lambda}$ such that:*

(1) $\widetilde{\mathbb{B}}_A(x,y) = (\Psi(x), \Psi(y))$ for every $A \in \mathcal{U}$ and $x, y \in \mathbb{H}(V)_A$, and

(2) Ψ sends the flat section $A \mapsto c_1(L_A)|_{Z(\mathcal{U},H)}$ (notation as in the proof of Proposition (3.13)) to $\mathcal{U} \times u$ where u is given by (1.0.8).

We define similarly a marking Ψ of $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$.

Of course if $\mathcal{U}$ is a small open ball then both local systems (3.3.11) are trivial. Let Φ and Ψ be markings of $\mathbb{H}(V)|_{\mathcal{U}}$ and $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$ respectively. Then we have holomorphic local period maps

$$\begin{array}{ccccccc} \mathcal{U} & \xrightarrow{\mathcal{P}_\Psi} & \mathcal{D}_2 & \qquad & \delta(\mathcal{U}) & \xrightarrow{\mathcal{P}_\Phi} & \mathcal{D}_2 \\ A & \mapsto & \Phi_{\mathbb{C}}(F^2_A) & & B & \mapsto & \Phi_{\mathbb{C}}(F^2_B). \end{array} \tag{3.3.12}$$

The restrictions of $\mathcal{P}_\Psi$ and $\mathcal{P}_\Phi$ to $\mathcal{U} \setminus \Delta^\infty_*(V)$ and $\delta(\mathcal{U}) \setminus \Delta^\infty_*(V^\vee)$ respectively are local period maps for the families of double EPW-sextics parametrized by $\mathbb{LG}(\wedge^3 V)^0$ and $\mathbb{LG}(\wedge^3 V^\vee)^0$ respectively. We will be interested in comparing $\mathcal{P}_\Psi(A)$ and $\mathcal{P}_\Phi(A^\perp)$ for $A \in \Delta^0_*(V)$ - thus $A^\perp \notin \mathbb{LG}(\wedge^3 V^\vee)^0$. Let $t \in T_*$ be such that $M_t \cong X_A$. By (3.3.4) the marking Φ defines a marking $\Phi_{A^\perp}$ of $(S_t^{[2]}, f_t^* \mathcal{O}_{Y_B}(1))$ and

$$\mathcal{P}_\Phi(A^\perp) = \mathcal{P}_{\Phi_{A^\perp}}(S_t^{[2]}, f_t^* \mathcal{O}_{Y_B}(1)). \tag{3.3.13}$$

4 Proof of Theorem (1.1)

If $A \in \mathbb{LG}(\wedge^3 V)^{00}$ we have smooth double covers $f_A\colon X_A \to Y_A$ and $f_{A^\perp}\colon X_{A^\perp} \to Y_{A^\perp}$: we will show that X_A and $X_{A^\perp}$ are "isogenous". Given a small open $\mathcal{U} \subset \mathbb{LG}(\wedge^3 V)^\sharp$ we may consider markings Ψ and Φ of $\mathbb{H}(V)|_{\mathcal{U}}$ and $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$ respectively and the associated local period maps $\mathcal{P}_\Psi$ and $\mathcal{P}_\Phi$. We will show that locally near $\Delta^0_*(V)$ we may choose Ψ and Φ so that $\mathcal{P}_\Phi \circ \delta$ is either $r \circ \mathcal{P}$ or the composition of $r \circ \mathcal{P}$ with a certain specific reflection. In the final subsection we will rule out the latter case by considering the monodromy action; by analytic continuation this will prove Theorem (1.1).

4.1 Isogeny between X_A and $X_{A^\perp}$

Let $A \in \mathbb{LG}(\wedge^3 V)^{00}$ and set $L_A := f_A^* \mathcal{O}_{Y_A}(1)$, $L_{A^\perp} := f_{A^\perp}^* \mathcal{O}_{Y_{A^\perp}}(1)$. Let

$$H^2(X_A)_{prim} := c_1(L_A)^\perp \subset H^2(X_A), \tag{4.1.1}$$

$$H^2(X_{A^\perp})_{prim} := c_1(L_{A^\perp})^\perp \subset H^2(X_{A^\perp}). \tag{4.1.2}$$

Proposition 4.1. *Let $A \in \mathbb{LG}(\wedge^3 V)^{00}$. There exists an isomorphism of rational Hodge structures*

$$g_A \colon H^2(X_A)_{prim} \overset{\sim}{\longrightarrow} H^2(X_{A^\perp})_{prim} \tag{4.1.3}$$

well-defined up to ± 1 and such that for $\gamma \in H^2(X_A)_{prim}$

$$(\gamma, \gamma)_{X_A} = (g_A(\gamma), g_A(\gamma))_{X_{A^\perp}}. \tag{4.1.4}$$

Proof. We recall that $Y_{A^\perp} = Y_A^\vee$, see Corollary (3.5). Let $\Gamma_A \subset Y_A \times Y_{A^\perp}$ be the closure of the Gauss maps:

$$\Gamma_A := \overline{\{(p, T_p Y_A) | \; p \in Y_A^{sm}\}} = \overline{\{(T_q Y_{A^\perp}, q) | \; q \in Y_{A^\perp}^{sm}\}}. \tag{4.1.5}$$

Since $A \in \mathbb{LG}(\wedge^3 V)^0$ the germ of Y_A at each of its singular points is isomorphic to the product of $(\mathbb{C}^2, 0)$ and an A_1-singularity (see Proposition (2.8) of [13]) and similarly for $Y_{A^\perp}$. Thus the projection $\Gamma_A \to Y_A$ is identified with the blow-up $\widetilde{Y}_A \to Y_A$ of $sing(Y_A)$. Similarly the projection $\Gamma_A \to Y_{A^\perp}$ is identified with the blow-up $\widetilde{Y}_{A^\perp} \to Y_{A^\perp}$ of $sing(Y_{A^\perp})$. Thus Γ_A defines an isomorphism $\widetilde{Y}_A \overset{\sim}{\longrightarrow} \widetilde{Y}_{A^\perp}$ and hence it gives an isomorphism of integral Hodge structures

$$H^4(\widetilde{Y}_A) \overset{\sim}{\longrightarrow} H^4(\widetilde{Y}_{A^\perp}). \tag{4.1.6}$$

The cohomology groups $H^4(\widetilde{Y}_A)$ and $H^4(X_A)$ are related as follows. Let $\phi_A \colon X_A \to X_A$ be the involution covering $f_A \colon X_A \to Y_A$. Let $F_A \subset X_A$ be the fixed locus of ϕ_A: this is a Lagrangian smooth surface in X_A because ϕ_A is anti-symplectic. Let $\widetilde{X}_A \to X_A$ be the blow-up of F_A; we have an isomorphism of integral Hodge structures

$$H^4(\widetilde{X}_A) \cong H^4(X_A) \oplus H^2(F_A)(-1). \tag{4.1.7}$$

The involution ϕ_A lifts to an involution $\widetilde{\phi}_A \colon \widetilde{X}_A \to \widetilde{X}_A$ and $\widetilde{Y}_A \cong \widetilde{X}_A / \langle \widetilde{\phi}_A \rangle$. Thus we have an isomorphism of rational Hodge structures

$$H^4(\widetilde{Y}_A) \cong H^4(\widetilde{X}_A)^{\langle \widetilde{\phi}_A \rangle} \cong H^4(X_A)^{\langle \phi_A \rangle} \oplus H^2(F_A)(-1). \tag{4.1.8}$$

Since X_A is a deformation of $(K3)^{[2]}$ we have an isomorphism of rational Hodge Structures $Sym^2 H^2(X_A) \overset{\sim}{\longrightarrow} H^4(X_A)$ defined by cup-product. The action of ϕ_A on $H^2(X_A)$ has $(+1)$-eigenspace generated by $c_1(L_A)$ and (-1)-eigenspace equal to $H^2(X_A)_{prim}$ thus we get an isomorphism of rational Hodge structures

$$H^4(X_A)^{\langle \phi_A \rangle} = \mathbb{C} c_1(L_A)^2 \oplus Sym^2 H^2(X_A)_{prim}. \tag{4.1.9}$$

The right-hand side of the above equality contains a rational $(2,2)$ class $q_A^\vee$ defined by "inverting"the Beauville-Bogomolov bilinear form (see Section (3) of [14]); let $W_A := H^2(X_A)_{prim} \cap (q_A^\vee)^\perp$. One has a decomposition of rational Hodge Structures (Claim (3.1) of [14])

$$\mathbb{C} c_1(L_A)^2 \oplus \mathbb{C} q_A^\vee \oplus W_A. \tag{4.1.10}$$

Of course we have analogous notions for $X_{A^\perp}$ and hence (4.1.6) defines an isomorphism of rational H.S.'s

$$\begin{split} \mathbb{C} c_1(L_A)^2 \oplus \mathbb{C} q_A^\vee \oplus W_A \oplus H^2(F_A)(-1) \cong \\ \mathbb{C} c_1(L_{A^\perp})^2 \oplus \mathbb{C} q_{A^\perp}^\vee \oplus W_{A^\perp} \oplus H^2(F_{A^\perp})(-1). \end{split} \tag{4.1.11}$$

If A is very general (outside a countable union of proper analytic subsets of $\mathbb{LG}(\wedge^3 V)^0 \cap \delta^{-1}\mathbb{LG}(\wedge^3 V^\vee)^0$) then W_A and $W_{A^\perp}$ are both indecomposable rational Hodge Structures - see Section (3) of [14]: since they contain the non-zero components $H^{4,0}(X_A)$ and $H^{4,0}(X_{A^\perp})$ the above isomorphism defines an isomorphism of rational H.S.'s

$$h_A \colon W_A \xrightarrow{\sim} W_{A^\perp}. \tag{4.1.12}$$

Let $Q_A \subset H^2(X_A)_{prim}$ and $Q_{A^\perp} \subset H^2(X_{A^\perp})_{prim}$ be the cones of isotropic classes (with respect to the Beauville-Bogomolov bilinear form). Let

$$\begin{array}{ccc} H^2(X_A)_{prim} & \xrightarrow{\nu_A} & Sym^2 H^2(X_A)_{prim} \\ \alpha & \mapsto & \alpha^2 \end{array} \tag{4.1.13}$$

be the (affine) Veronese map; we define similarly $\nu_{A^\perp}$. Then

$$W_A = \mathrm{span}\nu_A(Q_A), \quad W_{A^\perp} = \mathrm{span}\nu_{A^\perp}(Q_{A^\perp}). \tag{4.1.14}$$

We claim that

$$h_A(\nu_A(Q_A)) = \nu_{A^\perp}(Q_{A^\perp}). \tag{4.1.15}$$

In fact let $\mathcal{U} \subset \mathbb{LG}(\wedge^3 V)^0 \cap \delta^{-1}\mathbb{LG}(\wedge^3 V^\vee)^0$ be an open ball containing A. The Gauss-Manin connection gives identifications

$$H^2(X_B) \cong H^2(X_A), \quad H^2(X_{B^\perp}) \cong H^2(X_{A^\perp}) \tag{4.1.16}$$

for every $B \in \mathcal{U}$. Gauss-Manin gives also identifications $W_B \cong W_A$ and $W_{B^\perp} \cong W_{A^\perp} \cong$ sending $\nu_B(Q_B)$ to $\nu_A(Q_A)$ and $\nu_{B^\perp}(Q_{B^\perp})$ to $\nu_{A^\perp}(Q_{A^\perp})$ respectively. The isomorphism h_B is flat for the Gauss-Manin connection hence it is identified with a (constant) map $h \colon W_A \to W_{A^\perp}$. For $B \in \mathcal{U}$ let $\sigma_B, \sigma_{B^\perp}$ be symplectic forms on X_B and $X_{B^\perp}$ respectively. Since h_B is an isomorphism of H.S.'s we have

$$h[\sigma_B^2] = [\sigma_{B^\perp}^2]. \tag{4.1.17}$$

Now $\sigma_B \in Q_A$ and $\sigma_{B^\perp} \in Q_{A^\perp}$ - here we make the identifications (4.1.16) - and as B varies in $\mathcal{U}$ both $[\sigma_B]$ and $[\sigma_{B^\perp}]$ fill out non-empty open (in the classical topology) subsets of $\mathbb{P}(Q_A)$ and $\mathbb{P}(Q_{A^\perp})$ respectively. Since $\mathbb{P}(Q_A)$ and $\mathbb{P}(Q_{A^\perp})$ are non singular quadrics any non-empty open subset is Zariski-dense and hence (4.1.17) proves (4.1.15). It follows from (4.1.15) that there exists a linear map (4.1.3) well-defined up to ± 1 such that for $\alpha \in Q_A$ we have $h_A(\alpha^2) = g_A(\alpha)^2$. By (4.1.15) we have

$$g_A(\alpha) \in Q_{A^\perp} \text{ if and only if } \alpha \in Q_A. \tag{4.1.18}$$

The rationality of g_A follows from the fact that h_A is defined over $\mathbb{Q}$, and $\nu_A, \nu_{A^\perp}$ give bijective maps between $\mathbb{P}(Q_A)(\mathbb{Q})$, $\nu_A(\mathbb{P}(Q_A))(\mathbb{Q})$ and $\mathbb{P}(Q_{A^\perp})(\mathbb{Q})$, $\nu_{A^\perp}(\mathbb{P}(Q_{A^\perp}))(\mathbb{Q})$ respectively. Finally let's prove that Equation (4.1.4) holds. First we show that

$$(\alpha, \beta)_{X_A} = (g_A(\alpha), g_A(\beta))_{X_{A^\perp}}, \quad \alpha, \beta \in Q_A. \tag{4.1.19}$$

Since (4.1.11) respects the intersection forms we have

$$\int_{X_A} \alpha^2 \wedge \beta^2 = \int_{X_{A^\perp}} h_A(\alpha^2) \wedge h_A(\beta^2) = \int_{X_{A^\perp}} g_A(\alpha)^2 \wedge g_A(\beta)^2. \tag{4.1.20}$$

On the other hand α, β are isotropic and by (4.1.15) $g_A(\alpha), g_A(\beta)$ are isotropic as well; thus

$$\int_{X_A} \alpha^2 \wedge \beta^2 = 2(\alpha,\beta)^2_{X_A} \tag{4.1.21}$$

$$\int_{X_{A^\perp}} g_A(\alpha)^2 \wedge g_A(\beta)^2 = 2(g_A(\alpha), g_A(\beta))^2_{X_{A^\perp}} \tag{4.1.22}$$

$$\tag{4.1.23}$$

(See Section (2) of [14].) Equations (4.1.20), (4.1.21) and (4.1.22) prove that either (4.1.19) holds or else

$$(\alpha,\beta)_{X_A} = -(g_A(\alpha), g_A(\beta))_{X_{A^\perp}}, \quad \alpha, \beta \in Q_A. \tag{4.1.24}$$

Assume that (4.1.24) holds. Let $\sigma_A \in H^{2,0}(X_A)$ be the class of a symplectic forms on X_A; since g_A is an isomorphism of Hodge structures $g_A(\sigma_A)$ is represenyted by a symplectic form on $X_{A^\perp}$. Then

$$(\sigma_A, \overline{\sigma_A})_{X_A} = -(g_A(\sigma_A), \overline{g_A(\sigma_A)})_{X_{A^\perp}}. \tag{4.1.25}$$

This is absurd because the Beaville-Bogomolov $(,)_X$ of an irreducible symplectic manifold X has the property that $(\sigma,\overline{\sigma})_X > 0$ for every symplectic form σ on X. This finishes the proof of Equation (4.1.19). Now let $\gamma \in H^2(X_A)_{prim}$, then $\gamma = \alpha + \beta$ for certain $\alpha, \beta \in Q_A$; by Equations (4.1.18) and (4.1.19) we get

$$(\gamma,\gamma)_{X_A} = 2(\alpha,\beta)_{X_A} = 2(g_A(\alpha), g_A(\beta))_{X_{A^\perp}} = (g_A(\gamma), g_A(\gamma))_{X_{A^\perp}}. \tag{4.1.26}$$

This proves (4.1.4). □

Proposition (4.1) implies that locally there exists $g \in O(\Lambda \otimes \mathbb{Q})$ which relates the periods of $X_{A^\perp}$ to those of X_A and furthermore g normalizes the subgoup of monodromy operators in $O(\Lambda)$. We introduce some notation to formalize this observation. Let $\mathcal{U} \subset \mathbb{LG}(\wedge^3 V)^\sharp$ be a small open ball and let Ψ, Φ be markings of $\mathbb{H}(V)|_{\mathcal{U}}$ and $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$ respectively - see Definition (3.14). Let $\overline{A} \in \mathcal{U}$ be a reference point: the monodromy representation of $\pi_1(\mathbb{LG}(\wedge^3 V)^\sharp, \overline{A})$ on $\mathbb{H}(V)$ determines via $\Psi_{\overline{A}}$ a monodromy representation $\pi_1(\mathbb{LG}(\wedge^3 V)^\sharp, \overline{A}) \to Stab(u)$ where $Stab(u) < O(\widetilde{\Lambda})$ is the subgroup fixing the element u given by (1.0.8). An element of the image of the monodromy representation is a $\Psi_{\overline{A}}$-*monodromy operator*. Similarly we have a monodromy representation of $\pi_1(\mathbb{LG}(\wedge^3 V^\vee), \overline{A})$ on $\mathbb{H}(V^\vee)$; this determines via $\Phi_{\overline{A}^\perp}$ a monodromy representation $\pi_1(\mathbb{LG}(\wedge^3 V^\vee)^\sharp, \overline{A}^\perp) \to Stab(u)$. An element of the image of this second monodromy representation is a $\Phi_{\overline{A}^\perp}$-*monodromy operator*. Here and in the rest of the paper we will adopt the following conventions. First we view both $O(\Lambda)$ and $O(\Lambda \otimes \mathbb{Q})$ as subgroups of $O(\Lambda \otimes \mathbb{C})$. Secondly if $\gamma \in O(\Lambda \otimes \mathbb{C})$ we denote by $\widetilde{\gamma} \in O(\widetilde{\Lambda} \otimes \mathbb{C})$ the isometry which fixes u and equals γ on Λ. As a rule letters decorated by a tilde denote elements of $O(\widetilde{\Lambda} \otimes \mathbb{C})$, letters with no tilde denote elements of $O(\Lambda \otimes \mathbb{C})$.

Corollary 4.2. *Let $\mathcal{U} \subset \mathbb{LG}(\wedge^3 V)^\sharp$ be a small open ball and let Ψ, Φ be markings of $\mathbb{H}(V)|_{\mathcal{U}}$ and $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$ respectively. Let $\mathcal{P}_\Psi, \mathcal{P}_\Phi$ be the local period maps (3.3.12). There exists $g \in O(\Lambda \otimes \mathbb{Q})$ well-determined up to ± 1 such that*

$$\mathcal{P}_\Phi(A^\perp) = g \circ \mathcal{P}_\Psi(A) \tag{4.1.27}$$

for all $A \in \mathcal{U}$. Let $\widetilde{\gamma} \in Stab(u)$ be a $\Psi_{\overline{A}}$-monodromy operator; then $\widetilde{g} \circ \widetilde{\gamma} \circ \widetilde{g}^{-1}$ is a $\Phi_{\overline{A}^{\perp}}$-monodromy operator, in particular $g \circ \gamma \circ g^{-1} \in O(\Lambda)$.

Proof. Equation (4.1.27) holds on $\mathcal{U} \cap \mathbb{LG}(\wedge^3 V)^{00}$ by Proposition (4.1) and flatness of g_A; by continuity Equation (4.1.27) holds on all of $\mathcal{U}$. The statement about monodromy operators holds by flatness of g_A. □

4.2 Restriction to $\Delta^0_*(V)$ of the local period maps

Our next task is to analyze the restriction to $\Delta^0_*(V)$ of the local period maps $\mathcal{P}_\Psi$ and $\mathcal{P}_\Phi$.

Proposition 4.3. *Let $\mathcal{U} \subset \mathbb{LG}(\wedge^3 V)^\sharp$ be a small open ball. There is a choice of markings Ψ and Φ of $\mathbb{H}(V)|_{\mathcal{U}}$ and $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$ respectively such that*

$$\mathcal{P}_\Psi(\Delta^0_*(V) \cap \mathcal{U}) = (e_1 + 2e_2)^\perp \cap \mathcal{P}_\Psi(\mathcal{U}) \tag{4.2.1}$$

where $e_1, e_2 \in \Lambda$ are as in Section (1). Furthermore

$$\mathcal{P}_\Phi(A^\perp) = r \circ \mathcal{P}_\Psi(A), \quad A \in \Delta^0_*(V) \tag{4.2.2}$$

where r is the involution defined by (2.2.12).

Proof. First we embed the lattice $\widetilde{\Lambda}$ in a unimodular lattice as follows. Let $\widehat{\Lambda} := U^4 \widehat{\oplus} (-E_8)^2$. Let $U_1 < \widehat{\Lambda}$ be one of the hyperbolic lattices, let $z \in U_1$ be a vector of square 2 and e_2 be a generator of $z^\perp \cap U_1$. Then we have an isomorphism

$$z^\perp \cong \widetilde{\Lambda} \tag{4.2.3}$$

and we can choose it so that it matches the present e_2 with the vector e_2 appearing in (1.0.9): we fix such an isomorphism once for all. Let $u, e_1 \in \widetilde{\Lambda} = z^\perp$ be as in Section (1); then $\{(u \pm e_1)/2, (z \pm e_2)/2\} \subset \widehat{\Lambda}$. Furthermore the sublattices $\langle (u+e_1)/2, (u-e_1)/2 \rangle$ and $\langle (z+e_2)/2, (z-e_2)/2 \rangle$ are orthogonal hyperbolic planes. Thus we have an orthogonal decomposition

$$\widehat{\Lambda} = \langle (u+e_1)/2, (u-e_1)/2 \rangle \widehat{\oplus} \langle (z+e_2)/2, (z-e_2)/2 \rangle \widehat{\oplus} U^2 \widehat{\oplus} (-E_8)^2. \tag{4.2.4}$$

Now we pass to the geometry. Let T_* be as in Subsection (3.2). Let $A \in \Delta^0_*(V) \cap \mathcal{U}$; by definition there exists $t \in T_*$ such that

$$M_t \cong X_A. \tag{4.2.5}$$

Since $\mathcal{U}$ is a small open ball there exists a small open ball $\mathcal{V} \subset T_*$ such that if $t \in \mathcal{V}$ then (4.2.5) holds for some $A \in \Delta^0_*(V) \cap \mathcal{U}$ and conversely if $A \in \Delta^0_*(V) \cap \mathcal{U}$ then there exists $t \in \mathcal{V}$ such that (4.2.5) holds. Let $\kappa\colon \mathcal{S} \to T_*$ be the tautological family of $K3$ surfaces parametrized by T_*. Let $t \in T_*$. Let $S_t = \kappa^{-1}(t)$, D_t etc. be as in Subsection (3.1) and $A_t \in \Delta^0_*(V) \cap \mathcal{U}$ such that $M_t \cong X_{A_t}$. Let $f_t\colon M_t \to Y_{A_t}$ be the double cover. Let $v_t, w_t \in H^*(S_t)$ be given by

$$v_t := 2 + c_1(D_t) + 2\eta_t, \quad w_t := 1 - \eta_t \tag{4.2.6}$$

where $\eta_t \in H^4(S_t; \mathbb{Z})$ is the orientation class. The Mukai map

$$\theta_{v_t}\colon v_t^\perp \to H^2(M_t) \tag{4.2.7}$$

is an isometry of Hodge structures - see Subsection (3.1). Furthermore one has

$$c_1(f_t^* L_{A_t}) = \theta_{v_t}(\eta_t - 1) = \theta_{v_t}(-w_t^\vee). \tag{4.2.8}$$

(See the line preceding (3.1.7) for the definition of $w_t^\vee$.) The local system $R^2\kappa_*\mathbb{Z}|_{\mathcal{V}}$ is trivial because $\mathcal{V}$ is a small open ball. Thus there exist sections $\alpha, \beta \in \Gamma(R^2\kappa_*\mathbb{Z}|_{\mathcal{V}})$ such that $c_1(D_t) = \alpha_t + 5\beta_t$ for all $t \in \mathcal{V}$. We define a trivialization

$$R\kappa_*\mathbb{Z}|_{\mathcal{V}} = (R^0\kappa_*\mathbb{Z} \oplus R^2\kappa_*\mathbb{Z} \oplus R^4\kappa_*\mathbb{Z})|_{\mathcal{V}} \xrightarrow{\Upsilon} \mathcal{V} \times \widehat{\Lambda} \tag{4.2.9}$$

as follows. For $t \in \mathcal{V}$ let

$$\begin{array}{lcrrrrrrrr} \Upsilon_t(1) & := & - & u/2 & + & e_1/2 & - & z & + & e_2 \\ \Upsilon_t(\eta_t) & := & & u/2 & + & e_1/2 & - & z & + & e_2 \\ \Upsilon_t(\alpha_t) & := & & & - & 2e_1 & + & 5z/2 & - & 3e_2/2 \\ \Upsilon_t(\beta_t) & := & & & & & & z/2 & - & e_2/2 \end{array} \tag{4.2.10}$$

and let

$$\Upsilon_t|_{\{1,\eta_t,\alpha_t,\beta_t\}^\perp} \colon \{1,\eta_t,\alpha_t,\beta_t\}^\perp \xrightarrow{\sim} \{(u \pm e_1)/2, (z \pm e_2)/2\}^\perp \tag{4.2.11}$$

be an arbitrary isometry - notice that $\{1,\eta_t,\alpha_t,\beta_t\}^\perp \cong U^2 \widehat{\oplus} (-E_8)^2$ is isometric to $\{(u \pm e_1)/2, (z \pm e_2)/2\}^\perp$ by (4.2.4). A straightforward computation shows that Υ_t is an isometry. The trivialization (4.2.9) is defined to have value Υ_t at $t \in \mathcal{V}$. Now notice that $\Upsilon_t(v_t) = z$ and hence we have an isometry $\Upsilon_t \circ \theta_{v_t}^{-1} \colon H^2(M_t) \xrightarrow{\sim} z^\perp = \widetilde{\Lambda}$. Since $\Upsilon_t(-w_t^\vee) = u$ Equation (4.2.8) gives that

$$\Upsilon_t \circ \theta_{v_t}^{-1}(c_1(f_t^* L_{A_t})) = u. \tag{4.2.12}$$

Hence $\Upsilon_t \circ \theta_{v_t}^{-1}$ defines a marking of $(M_t, f_t^* L_{A_t})$ for every $t \in \mathcal{V}$; since $\mathbb{H}(V)|_{\mathcal{U}}$ is trivial there exists a marking $\Psi \colon \mathbb{H}(V)|_{\mathcal{U}} \longrightarrow \mathcal{U} \times \widetilde{\Lambda}$ such that

$$\Psi_{A_t} = \Upsilon_t \circ \theta_{v_t}^{-1}, \quad t \in \mathcal{V}. \tag{4.2.13}$$

Equation (4.2.1) follows from Equation (3.1.11) and the equality

$$\Psi_{A_t}(\theta_{v_t}(5 + 2c_1(D_t) + 5\eta_t)) = \Upsilon_t(5 + 2c_1(D_t) + 5\eta_t) = e_1 + 2e_2. \tag{4.2.14}$$

Next we define a marking Φ for $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$. By (3.3.4) this will be equivalent to a marking of $S_t^{[2]}$, hence we first recall the description of $H^2(S_t^{[2]})$. We notice that w_t is the Mukai vector (see (3.1.1)) of any ideal sheaf I_Z where $[Z] \in S_t^{[2]}$; Mukai's map

$$\theta_{w_t} \colon w_t^\perp \to H^2(S_t^{[2]}) \tag{4.2.15}$$

is an isomorphism of polarized Hodge structures and an isometry. Let $g_t \colon S_t^{[2]} \to Y_{A_t^\perp}$ be the map defined by (3.2.7); by Subsection (5.3) of [12]

$$c_1(g_t^* \mathcal{O}_{Y_{A_t^\perp}}(1)) = \theta_{w_t}(-2 + c_1(D_t) - 2\eta_t) = \theta_{w_t}(-v_t^\vee). \tag{4.2.16}$$

We define a trivialization

$$R\kappa_*\mathbb{Z}|_{\mathcal{V}} = (R^0\kappa_*\mathbb{Z} \oplus R^2\kappa_*\mathbb{Z} \oplus R^4\kappa_*\mathbb{Z})|_{\mathcal{V}} \xrightarrow{\Theta} \mathcal{V} \times \widehat{\Lambda} \tag{4.2.17}$$

as follows. For $t \in \mathcal{V}$ let

$$\begin{array}{rclcccccc} \Theta_t(1) & := & u & - & e_1 & + & z/2 & - & e_2/2 \\ \Theta_t(\eta_t) & := & u & - & e_1 & - & z/2 & - & e_2/2 \\ \Theta_t(\alpha_t) & := & 5u/2 & - & 3e_1/2 & & & - & 2e_2 \\ \Theta_t(\beta_t) & := & u/2 & - & e_1/2 & & & & \end{array} \tag{4.2.18}$$

and let the restriction of Θ_t to $\{1, \eta_t, \alpha_t, \beta_t\}^\perp$ be equal to the restriction of Υ_t. A straightforward computation shows that Θ_t is an isometry. The trivialization (4.2.17) is defined to have value Θ_t at $t \in \mathcal{V}$. Now notice that $\Theta_t(w_t) = z$ and hence we have an isometry $\Theta_t \circ \theta_{w_t}^{-1} \colon H^2(S_t^{[2]}) \xrightarrow{\sim} z^\perp = \widetilde{\Lambda}$. Since $\Theta_t(-v_t^\vee) = u$ Equation (4.2.16) gives that

$$\Theta_t \circ \theta_{v_t}^{-1}(c_1(g_t^* \mathcal{O}_{Y_{A_t^\perp}})) = u. \tag{4.2.19}$$

Hence $\Theta_t \circ \theta_{w_t}^{-1}$ defines a marking of $(S_t^{[2]}, g_t^* \mathcal{O}_{Y_{A_t^\perp}}(1))$ for every $t \in \mathcal{V}$; by (3.3.4) and triviality of $\mathbb{H}(V^\vee)|_{\delta(\mathcal{U})}$ there exists a marking $\Phi \colon \mathbb{H}(V^\vee)|_{\delta(\mathcal{U})} \longrightarrow \mathcal{U} \times \widetilde{\Lambda}$ such that

$$\Phi_{A_t} = \Theta_t \circ \theta_{w_t}^{-1}, \quad t \in \mathcal{V}. \tag{4.2.20}$$

Now let's prove (4.2.2). Equation (4.2.2) is equivalent to

$$\mathcal{P}_{\Theta_t \circ \theta_{w_t}^{-1}}(S_t^{[2]}, g_t^* \mathcal{O}_{Y_{A_t^\perp}}(1)) = r \circ \mathcal{P}_{\Upsilon_t \circ \theta_{v_t}^{-1}}(M_t, f_t^* L_A), \quad t \in \mathcal{V} \tag{4.2.21}$$

by Equation (3.3.13). Since θ_{v_t} and θ_{w_t} are isomorphism of Hodge structures the above equation may be rewitten as

$$\Theta_t(H^{2,0}(S_t)) = r \circ \Upsilon_t(H^{2,0}(S_t)), \quad t \in \mathcal{V}. \tag{4.2.22}$$

For $t \in \mathcal{V}$ let

$$\begin{array}{ccc} H^*(S_t; \mathbb{Q}) & \xrightarrow{\Xi_t} & H^*(S_t; \mathbb{Q}) \\ \gamma & \mapsto & -\gamma^\vee + \frac{1}{2}(\gamma^\vee, v_t + w_t)(v_t + w_t) \end{array} \tag{4.2.23}$$

i.e. the composition of the isometry $\gamma \mapsto \gamma^\vee$ and the reflection which is $(+1)$ on $\mathbb{Q}(v_t + w_t)$ and (-1) on $(v_t + w_t)^\perp$; thus Ξ_t is a rational (not integral !) Hodge isometry. A straightforward computation shows that $\Theta_t = r \circ \Upsilon_t \circ \Xi_t$. Thus (4.2.22) follows at once from $\Xi_t(H^{2,0}(S)) = H^{2,0}(S)$. □

If $\gamma \in \Lambda \otimes \mathbb{Q}$ is non-isotropic we let

$$\begin{array}{ccc} \Lambda \otimes \mathbb{Q} & \xrightarrow{r_\gamma} & \Lambda \otimes \mathbb{Q} \\ x & \mapsto & -x + \frac{2}{(\gamma,\gamma)}(x, \gamma)\gamma \end{array} \tag{4.2.24}$$

be the reflection with $(+1)$-eigenspace $\mathbb{Q}\gamma$ and (-1)-eigenspace $\gamma^\perp$. Let

$$\zeta := e_1 + 2e_2. \tag{4.2.25}$$

Corollary 4.4. *Keep notation and assumptions of Proposition (4.3). Let Ψ, Φ be the markings of Proposition (4.3). Then:*

(1) $\mathcal{P}_\Phi(A^\perp) = r \circ \mathcal{P}_\Psi(A)$ for all $A \in \mathcal{U}$ or

(2) $\mathcal{P}_\Phi(A^\perp) = r \circ r_\zeta \circ \mathcal{P}_\Psi(A)$ *for all* $A \in \mathcal{U}$.

Proof. By Corollary (4.2) there exists $g \in O(\Lambda \otimes \mathbb{Q})$ such that $\mathcal{P}_\Phi(A^\perp) = g \circ \mathcal{P}_\Psi(A)$ for all $A \in \mathcal{U}$. By Proposition (4.3) $r^{-1} \circ g$ fixes the points of $\zeta^\perp \cap \mathcal{D}_2$: it follows that $r^{-1} \circ g$ fixes $\zeta^\perp \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$. Since $r \in O(\Lambda)$ we have $r^{-1} \circ g \in O(\Lambda \otimes \mathbb{Q})$: thus we get that that

$$r^{-1} \circ g|_{\zeta^\perp} = \pm Id_{\zeta^\perp}. \tag{4.2.26}$$

Since ζ is non-isotropic (in fact $(\zeta, \zeta) = -10$) and $r^{-1} \circ g \in O(\Lambda \otimes \mathbb{Q})$ we get that $r^{-1} \circ g(\zeta) = \pm\zeta$. It follows that $r^{-1} \circ g = \pm Id$ or $r^{-1} \circ g = \pm r_\zeta$. Since $-Id$ acts trivially on $\mathcal{D}_2 \subset \mathbb{P}(\Lambda)$ the corollary follows. □

4.3 The proof

We will apply the monodromy statement of Corollary (4.2) in order to show that Item (2) of Corollary (4.4) can not hold. We will use the following result.

Claim 4.5. *Let*

$$\xi = a_1 e_1 + a_2 e_2 + \nu \in \Lambda \tag{4.3.1}$$

be a (-2)*-vector i.e.* $(\xi, \xi) = -2$ *and assume that* $r_\zeta \circ r_\xi \circ r_\zeta \in O(\Lambda)$. *Then*

$$(\xi, \zeta) \equiv 0 \pmod 5. \tag{4.3.2}$$

Proof. A tedious straightforward computation gives that

$$\begin{aligned} r_\zeta \circ r_\xi \circ r_\zeta(e_1) = \frac{1}{25}&(18a_1^2 - 48a_1a_2 + 32a_2^2 - 25)e_1 - \\ &- \frac{1}{25}(24a_1^2 - 14a_1a_2 - 24a_2^2)e_2 - \frac{2}{5}(3a_1 - 4a_2)\nu. \end{aligned} \tag{4.3.3}$$

Thus $18a_1^2 - 48a_1a_2 + 32a_2^2 \equiv 0 \pmod{25}$. Since

$$18a_1^2 - 48a_1a_2 + 32a_2^2 = 2(3a_1 - 4a_2)^2 \tag{4.3.4}$$

we get that $3a_1 - 4a_2 \equiv 0 \pmod 5$. This proves (4.3.2) because

$$(\xi, \zeta) = -2a_1 - 4a_2 \equiv 3a_1 - 4a_2 \pmod 5. \tag{4.3.5}$$

□

Proposition 4.6. *Keep notation and assumptions of Proposition (4.3). Let* Ψ, Φ *be the markings of Proposition (4.3) and* $\overline{A} \in \mathcal{U} \cap \Delta^0_*(V)$. *There exists a* (-2)*-vector* $\xi \in \Lambda$ *such that* $-r_\xi$ *is a* $\Psi_{\overline{A}}$*-monodromy operator and*

$$(\xi, \zeta) \not\equiv 0 \pmod 5. \tag{4.3.6}$$

We grant the above proposition for the moment being and we proceed to prove Theorem (1.1). Let notation and assumptions be as in Proposition (4.3) and Ψ, Φ be the markings of Proposition (4.3). Then either (1) or (2) of Corollary (4.4) holds. Suppose that (2) holds; we will arrive at a contradiction. Let ξ be as in Proposition (4.6): by Corollary (4.2) we have $-r \circ r_\zeta \circ r_\xi \circ r_\zeta \circ r \in O(\Lambda)$. Since $r \in O(\Lambda)$ we get that $r_\zeta \circ r_\xi \circ r_\zeta \in O(\Lambda)$: this contradicts Claim (4.5) because of (4.3.6). Thus (1) of Corollary (4.4) holds. Let $\mathcal{U}^{00} := \mathcal{U} \cap \mathbb{LG}(\wedge^3 V)^{00}$; then $\mathcal{U}^{00}$ is an open (in the euclidean topology) non-empty subset of $\mathbb{LG}(\wedge^3 V)^{00}$. Since (1) of Corollary (4.4) holds we have

$$\mathcal{P} \circ \delta|_{\mathcal{U}^{00}} = \overline{r} \circ \mathcal{P}|_{\mathcal{U}^{00}}. \tag{4.3.7}$$

Both $\mathcal{P} \circ \delta$ and $\overline{r} \circ \mathcal{P}$ are holomorphic maps with domain the connected manifold $\mathbb{LG}(\wedge^3 V)^{00}$; by analytic continuation we get that Theorem (1.1) holds.

Proof of Proposition (4.6) Let $F \subset \mathbb{P}^3$ be a smooth quartic, thus F is a $K3$ surface. We have a regular map

$$\begin{array}{ccc} F^{[2]} & \xrightarrow{g} & \mathbb{G}r(1, \mathbb{P}^3) \subset \mathbb{P}^5 \\ [Z] & \mapsto & span(Z) \end{array} \tag{4.3.8}$$

and $c_1(g^* \mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1))$ has square 2 for the Beauville-Bogomolov form. If F does not contain lines the above map is finite and hence $g^* \mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1)$ is an ample line-bundle on $F^{[2]}$, if F contains a line R then $g^* \mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1)$ is big and nef but it restricts to the trivial line-bundle on the $\mathbb{P}^2$ given by $R^{(2)} \subset F^{[2]}$. Assume that F does not contain lines; we proved in Section (6) of [12] that

$$(F^{[2]}, g^* \mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1)) \text{ is deformation equivalent to } (M_t, f_t^* L_{A_t}) \tag{4.3.9}$$

where $t \in T$, i.e. there exists a polarized family of irreducible symplectic 4-folds over a connected basis with one fiber isomorphic to $(F^{[2]}, g^* \mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1))$ and another fiber isomorphic to $(M_t, f_t^* L_{A_t})$. Using this result we will show that the monodromy operator on F given by a suitable (-2)-class orthogonal to $c_1(\mathcal{O}_F(1))$ gives rise to a $\Psi(\overline{A})$-monodromy operator for which (4.3.6) holds. Before proving this we must dive into the details of the proof of (4.3.9). Let $F_0 \subset \mathbb{P}^3$ be a smooth quartic surface containing a line R and with Picard number 2, i.e. $Pic(F_0) = \mathbb{Z}[A_0] \oplus \mathbb{Z}[R]$ where A_0 is the (hyper)plane class. The divisor $(2A_0 - R)$ is very ample and $c_1(2A_0 - R)^2 = 10$; thus we have an embedding

$$F_0 \hookrightarrow |2A_0 - R|^\vee \cong \mathbb{P}^6 \tag{4.3.10}$$

as a linearly normal $K3$ surface of degree 10. Let

$$w_0 := 1 + c_1(A_0) + \eta_0, \quad v_0 := 2 + c_1(2A_0 - R) + 2\eta_0 \tag{4.3.11}$$

where $\eta_0 \in H^4(F_0; \mathbb{Z})$ is the orientation class. Let M_{w_0} be the moduli space of torsion-free sheaves $\mathcal{G}$ on F_0 such that $v(\mathcal{G}) = w_0$; every such sheaf is equal to $I_Z \otimes \mathcal{O}_{F_0}(A_0)$ for a unique $[Z] \in F_0^{[2]}$ and hence

$$M_{w_0} \cong F_0^{[2]}. \tag{4.3.12}$$

We let $L_{w_0} := g_0^* \mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1)$. Let M_{v_0} be the moduli space of $(2A_0 - R)$-semistable sheaves $\mathcal{F}$ on F_0 such that $v(\mathcal{F}) = v_0$. The moduli space M_{v_0} is smooth because $(2A_0 - R)$ is v_0-generic (see Section (6) of [12]) and hence it is a deformation of $(K3)^{[2]}$. Mukai's map gives isometries of Hodge structures

$$\theta_{w_0} \colon w_0^\perp \xrightarrow{\sim} H^2(M_{w_0}), \quad \theta_{v_0} \colon v_0^\perp \xrightarrow{\sim} H^2(M_{v_0}). \tag{4.3.13}$$

One has (see p.1241 of [12])

$$c_1(L_{w_0}) = \theta_{w_0}(\eta_0 - 1). \tag{4.3.14}$$

We let L_{v_0} be the line-bundle on M_{v_0} such that

$$c_1(L_{v_0}) = \theta_{v_0}(\eta_0 - 1). \tag{4.3.15}$$

In Lemma (6.2) of [12] we considered the birational map $M_{w_0} \dashrightarrow M_{v_0}$ whose inverse

$$\varphi\colon M_{v_0} \dashrightarrow M_{w_0} \tag{4.3.16}$$

is the Mukai reflection defined by the (-2)-vector

$$u_0 := (1 + c_1(A_0 - R) + \eta_0) \tag{4.3.17}$$

(notice that $-r_{u_0}(v_0) = w_0$), see [16]. Since M_{v_0} and M_{w_0} are irreducible symplectic manifolds the birational map φ induces an isomorphism of lattices $\varphi^*\colon H^2(M_{w_0}) \cong H^2(M_{v_0})$. By Theorem (2.9) of [16] we have

$$\varphi^*\theta_{w_0}(\alpha) = \theta_{v_0}(-r_{u_0}(v_0)), \tag{4.3.18}$$

in particular by (4.3.14)-(4.3.15) we have

$$\varphi^* L_{w_0} \cong L_{v_0}. \tag{4.3.19}$$

The birational map φ is the flop of

$$\Pi_{w_0} := R^{(2)} \subset F_0^{[2]} = M_{w_0}. \tag{4.3.20}$$

It follows that M_{v_0} contains $\Pi_{v_0} \cong \Pi_{w_0}^{\vee}$ and from (4.3.19) we get that L_{v_0} is big, nef and its restriction to Π_{v_0} is trivial. Let $\mathcal{X} \to B_{v_0}$ be a representative for the deformation space of (M_{v_0}, L_{v_0}), i.e. deformations of M_{v_0} that "keep $c_1(L_{v_0})$ of type $(1,1)$". Similarly let $\mathcal{X}' \to B_{w_0}$ be a representative for the deformation space of (M_{w_0}, L_{w_0}). We let $0 \in B_{v_0}$ and $0 \in B_{w_0}$ be the points corresponding to (M_{v_0}, L_{v_0}) and (M_{w_0}, L_{w_0}) respectively. Thus for each $q \in B_{v_0}$ the fiber X_q of $\mathcal{X} \to B_{v_0}$ over q has a line-bundle L_q which is a deformation of L_{v_0}. Similarly for each $s \in B_{w_0}$ the fiber X'_s of $\mathcal{X}' \to B_{w_0}$ over s has a line-bundle L'_s which is a deformation of L_{w_0}. We may and will assume that B_{v_0}, B_{w_0} are contractible and hence Gauss-Manin gives identifications

$$H^2(X_q) \cong H^2(M_{v_0}), \quad H^2(X'_s) \cong H^2(M_{w_0}), \quad q \in B_{v_0},\ s \in B_{w_0} \tag{4.3.21}$$

which match $c_1(L_q)$ to $c_1(L_{v_0})$ and $c_1(L'_s)$ to $c_1(L_{w_0})$. Let $B(\Pi_{v_0}) \subset B_{v_0}$ be the locus parametrizing deformations X_q which contain a deformation of Π_{v_0}, and similarly let $B(\Pi_{w_0}) \subset B_{w_0}$ be the locus parametrizing deformations of X'_s which contain a deformation of Π_{w_0}. By a Theorem of Voisin [15] each of these loci is smooth of codimension 1. There is a natural isomorphism of germs

$$\mu\colon (B_{w_0}, 0) \xrightarrow{\sim} (B_{v_0}, 0) \tag{4.3.22}$$

such that $\mu(\Pi_{w_0}) = \Pi_{v_0}$ and if $s \notin \Pi_{w_0}$ then $(X_{\mu(s)}, L_{\mu(s)}) \cong (X'_s, L'_s)$. Let $s \in B_{w_0}$ be such that $(X'_s, L'_s) \cong (F^{[2]}, g^*\mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1))$ where F is a quartic containing no lines. Then $s \notin \Pi_{w_0}$ and hence

$$(X_{\mu(s)}, L_{\mu(s)}) \cong (X'_s, L'_s) \cong (F^{[2]}, g^*\mathcal{O}_{\mathbb{G}r(1,\mathbb{P}^3)}(1)). \tag{4.3.23}$$

On the other hand there exists $q \in B_{v_0}$ such that $(X_q, c_1(L_q)) \cong (M_t, \theta_t(\eta_t - 1))$ for $t \in T_*$ because the parameter space for linearly normal $K3$ surfaces of degree 10 (an open subset of the relevant Hilbert scheme) is irreducible - notice that $q \notin \Pi_{v_0}$ because $\theta_t(\eta_t - 1)$ is ample. Thus there exists $\overline{A} \in \Delta^0_*(V)$ such that

$$(X_{\overline{A}}, L_{\overline{A}}) \cong (X_q, L_q). \tag{4.3.24}$$

Let $\gamma \in H^2(F;\mathbb{Z})$ be a (-2)-class orthogonal to $c_1(\mathcal{O}_F(1))$. Then γ determines a monodromy operator on $H^2(F^{[2]})$; by (4.3.23) and (4.3.24) this monodromy operator can be identified with a monodromy operator on $X_{\overline{A}}$ because polarized deformation spaces of irreducible symplectic manifolds are smooth. Given the trivializations (4.3.21) the monodromy operator in question is equal to $-r_{\varphi^*\theta_{w_0}(\gamma_0)}$ - here $\gamma_0 \in H^2(F_0)$ is the class corresponding to γ (the second trivialization of (4.3.21) defines an identification $H^2(F) \cong H^2(F_0)$). The corresponding $\Psi_{\overline{A}}$-monodromy operator is equal to $-r_\xi$ where

$$\xi := \Psi_{\overline{A}}(\varphi^*\theta_{w_0}(\gamma_0)) \in \Lambda \tag{4.3.25}$$

Claim 4.7. *Keep notation as above. Let $\gamma_0 \in c_1(A_0)^\perp \subset H^2(F_0;\mathbb{Z})$ be a (-2)-class such that*

$$\int_{F_0} \gamma_0 \wedge c_1(R) \not\equiv 0 \pmod 5. \tag{4.3.26}$$

Then

$$(\xi, \zeta) \not\equiv 0 \pmod 5. \tag{4.3.27}$$

Proof. By (4.2.14) we must check that

$$(\varphi^*\theta_{w_0}(\gamma_0), \theta_{v_0}(5 + 2c_1(2A_0 - R) + 5\eta_0)) \not\equiv 0 \pmod 5. \tag{4.3.28}$$

By (4.3.18) this is equivalent to

$$(-r_{u_0}(\gamma_0), 5 + 2c_1(2A_0 - R) + 5\eta_0) \not\equiv 0 \pmod 5. \tag{4.3.29}$$

where u_0 is given by (4.3.17). A straightforward computation gives that

$$(-r_{u_0}(\gamma_0), 5 + 2c_1(2A_0 - R) + 5\eta_0) = -8\int_{F_0} \gamma_0 \wedge c_1(R) \tag{4.3.30}$$

and hence we get (4.3.28). □

Certainly there exists a class γ_0 satisfying the hypotheses of Claim (4.7): the vector $\xi \in \Lambda$ given by (4.3.25) satisfies the thesis of Proposition (4.6).

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