Dual double EPW-sextics and their periods

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1 Introduction

Eisenbud, Popescu and Walter have constructed certain singular sextic hypersurfaces (EPW-sextics) in \( \mathbb{P}^5 \) (Example (9.3) of [4]) which come provided with a natural double cover: we have shown [13] that the generic such double cover is a deformation of the Hilbert square of a \( K3 \) and that the family of double EPW-sextics is a locally versal family of projective deformations of \((K3)[2]\). Thus the family of double EPW-sextics is similar to the family of Fano varieties of lines on a cubic 4-fold (see [2]), with the following difference: the Plücker ample divisor on the Fano variety of lines has square 6 for the Beauville-Bogomolov quadratic form (see [1, 2]) while the natural polarization of a double EPW-sextic has square 2 (see [13]). Let \( Y \subset \mathbb{P}^5 \) be a generic EPW-sextic: we proved in [13] that the dual \( Y^\vee \subset (\mathbb{P}^5)^{\vee} \) is another generic EPW-sextic. Thus we may associate to the natural double cover \( X \) of \( Y \) a “dual” variety \( X^\vee \) namely the natural double cover of \( Y^\vee \). This construction defines a (rational) involution on the moduli space of double EPW-sextics. In [13] we showed that a generic EPW-sextic is not self-dual and hence the involution on the moduli space of double EPW-sextics is not the identity; in this paper we determine the relation between the periods of a double EPW-sextic and its dual. Before stating the result we recall the definition of EPW-sextics. Let \( V \) be a 6-dimensional \( \mathbb{C} \)-vector space. Choose an isomorphism \( \text{vol}: \bigwedge^6 V \cong \mathbb{C} \) and let \( \omega \) be the symplectic form on \( \bigwedge^3 V \) defined by wedge product followed by \( \text{vol} \). Let \( \mathbb{P}(V) \) be the projective space of 1-dimensional sub vector spaces of \( V \); then \( \omega \) gives \( \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \) the structure of a symplectic vector-bundle of rank 20. Let \( F \) be the sub-vector-bundle of \( \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \) whose fiber \( F[v] \) over \( [v] \in \mathbb{P}(V) \) consists of tensors divisible by \( v \):

\[
F[v] := \{ v \wedge w | w \in \bigwedge^2 V \}. \tag{1.0.1}
\]

As is easily checked \( F \) is a Lagrangian sub-bundle of \( \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \). Let \( \mathbb{L}G(\bigwedge^3 V) \) be the symplectic Grassmannian parametrizing \( \omega \)-Lagrangian subspaces of \( \bigwedge^3 V \). For \( A \in \mathbb{L}G(\bigwedge^3 V) \) we let \( \lambda_A \) be the composition

\[
F \longrightarrow \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow (\bigwedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{1.0.2}
\]

and \( Y_A \subset \mathbb{P}(V) \) be the zero-scheme of \( \det(\lambda_A) \); unless \( \det(\lambda_A) \) is identically zero\(^1\) \( Y_A \) is a sextic hypersurface because \( \det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6) \). An EPW-sextic is a

\(^1\)If \( A \) is the fiber of \( F \) over a fixed point then \( \det(\lambda_A) \) is identically zero.
hypersurface in $\mathbb{P}(V)$ which is equal to $Y_A$ for some $A \in LG(\wedge^3V)$. In [13] we described explicitly the non-empty Zariski-open $LG(\wedge^3V)^0 \subset LG(\wedge^3V)$ parametrizing $A$ such that $Y_A$ has no singularities other than those forced by its description as a degeneracy locus; for $A \in LG(\wedge^3V)^0$ the singular locus of $Y_A$ is a smooth degree-40 irreducible surface and at a singular point $Y_A$ is locally (in the classical topology) isomorphic to the product of $\text{sing}(Y_A)$ and an $A_1$-singularity. If $A \in LG(\wedge^3V)^0$ there is a natural double cover $f_A : X_A \to Y_A$ ramified only over $\text{sing}(Y_A)$, with $X_A$ smooth. As shown in [13] the 4-fold $X_A$ is a deformation of $(K3)^[2]$ the Hilbert square of a $K3$. Let $L_A$ be the line-bundle on $X_A$ defined by

$$L_A := f_A^* \mathcal{O}_{Y_A}(1);$$

(1.0.3)

then $c_1(L_A)$ has square 2 for the Beauville-Bogomolov quadratic form. The family of $(X_A, L_A)$ for $A$ varying in $LG(\wedge^3V)^0$ is a locally complete family of polarized deformations of $(K3)^[2]$. Now we recall the duality map. Let $\text{vol}^V$ be a trivialization of $\wedge^6V^\vee$ and $\omega^V$ be the symplectic form on $\wedge^3V^\vee$ given by wedge-product followed by $\text{vol}^V$: let $LG(\wedge^3V^\vee)$ be the symplectic Grassmannian parametrizing $\omega^V$-Lagrangian subspaces of $\wedge^3V^\vee$. For $A \in LG(\wedge^3V^\vee)$ the annihilator $A^\perp \subset \wedge^3V^\vee$ belongs to $LG(\wedge^3V^\vee)$. Thus we have an isomorphism

$$\delta : LG(\wedge^3V) \xrightarrow{\sim} LG(\wedge^3V^\vee)$$

(1.0.4)

Let

$$LG(\wedge^3V)^0 := LG(\wedge^3V)^0 \cap \delta^{-1}LG(\wedge^3V^\vee)^0.$$  

(1.0.5)

Thus $LG(\wedge^3V)^0$ is open dense in $LG(\wedge^3V)$. Assume that $A \in LG(\wedge^3V)^0$. By Proposition (3.1) of [13] $Y_A^\vee$ is $Y_A^\vee$ (the dual of $Y_A$); we let

$$X_A^\vee \to Y_A^\vee = Y_A^\perp$$

(1.0.6)

be the natural double cover and we call $X_A^\vee$ the dual of $X_A$. We will show how to obtain the Hodge structure on $H^2(X_A^\vee)$ from the Hodge structure on $H^2(X_A)$. First we describe the relevant period space. Let $\Lambda$ be the even lattice

$$\tilde{\Lambda} := U^3 \hat{\oplus} (-E_8)^2 \hat{\oplus} (-2)$$

(1.0.7)

where $U$ is the hyperbolic plane and $(-2)$ is the lattice generated by a single element of square $-2$; we will denote by $(\cdot, \cdot)$ the symmetric bilinear form on $\tilde{\Lambda}$. Let $U$ be one of the hyperbolic planes appearing in Decomposition (1.0.7) and choose

$$u \in U \text{ of square } 2.$$  

(1.0.8)

Let $e_1 \in U$ be a generator of $u^\perp \cap U$ and $e_2$ be a generator of the direct summand $(-2)$ appearing in Decomposition (1.0.7). Then

$$\Lambda := u^\perp = U^2 \hat{\oplus} (-E_8)^2 \hat{\oplus} Ze_1 \hat{\oplus} Ze_2 \cong U^2 \hat{\oplus} (-E_8)^2 \hat{\oplus} (-2)^2.$$  

(1.0.9)

The period domain of interest to us is

$$D_2 := \{ [v] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (v, v) = 0, \quad (v, \overline{v}) > 0 \}.$$  

(1.0.10)

Let $\text{Stab}(u) < O(\tilde{\Lambda})$ be the subgroup of isometries fixing $u$ and $\rho : \text{Stab}(u) \to O(\Lambda)$ be the restriction map - this makes sense because $\Lambda = u^\perp$. Let

$$\Gamma := \text{Im}(\rho) < O(\Lambda).$$  

(1.0.11)
The period moduli space is the quotient

\[ Q_2 := \Gamma \backslash \mathcal{D}_2. \] (1.0.12)

We will prove in Subsection (2.2) that

\[ [O(\Lambda) : \Gamma] = 2. \] (1.0.13)

An explicit \( r \in (O(\Lambda) \backslash \Gamma) \) is the isometry which interchanges \( e_1 \) with \( e_2 \) and fixes the elements of \( \{ e_1, e_2 \}^\perp \), see (2.2.12). By (1.0.13) the involution \( r \) descends to a non-trivial involution

\( \tau : Q_2 \to Q_2. \) (1.0.14)

Let \( A \in \mathbb{L}G(\wedge^3 V)^0 \): the isomorphism class of the Hodge structure on \( c_1(L_A)^{\perp} \subset H^2(X_A) \) is a point of \( Q_2 \) (see Section (2)) that we denote by \( \mathcal{P}(A) \) - this is the period point of \( X_A \). The global period map is defined by

\[ LG(\wedge^3 V)^0 \xrightarrow{\mathcal{P}} Q_2 \quad A \mapsto \mathcal{P}(A). \] (1.0.15)

Of course we also have a global period map \( LG(\wedge^3 V^\vee)^0 \to Q_2 \); we denote it by the same symbol \( \mathcal{P} \). If \( A \in LG(\wedge^3 V)^{00} \) we have two period points namely \( \mathcal{P}(A) \) and \( \mathcal{P}(A^{\perp}) \). The following is the main result of this paper.

**Theorem 1.1.** Keep notation as above. If \( A \in \mathbb{L}G(\wedge^3 V)^{00} \) then

\[ \mathcal{P}(A^{\perp}) = \tau \circ \mathcal{P}(A). \] (1.0.16)

The theorem may be stated informally as follows: the polarized Hodge structures on \( c_1(L_A^{\perp})^\perp \subset H^2(X_A^{\perp}) \) are always isomorphic while in general there is no Hodge isometry between \( H^2(X_A) \) and \( H^2(X_A^{\perp}) \).

There are three steps in the proof of the above theorem. First of all there exists a rational Hodge isometry

\[ H^2(X_A) \supset c_1(L_A)^{\perp} \cong c_1(L_A^{\perp})^{\perp} \subset H^2(X_A^{\perp}). \] (1.0.17)

In fact let \( \tilde{Y}_A, \tilde{Y}_A^{\perp} \) be the desingularizations of \( Y_A, Y_A^{\perp} \) respectively - they are obtained by blowing up \( sing(Y_A) \) and \( sing(Y_A^{\perp}) \) respectively. From the equality \( \tilde{Y}_A = Y_A^{\perp} \) one gets that

\[ \tilde{Y}_A \cong \tilde{Y}_A^{\perp}. \] (1.0.18)

Now \( H^4(Y_A) \) is a subgroup of \( Sym^2 H^2(X_A) \) and similarly for \( H^4(Y_A^{\perp}) \); this fact together with (1.0.18) gives (1.0.17) after some work. It follows from (1.0.17) that locally (in the classical topology) there exists \( g \in O(\Lambda \otimes \mathbb{Q}) \) relating the periods of \( X_A \) and \( X_A^{\perp} \); this means that we have

\[ \mathcal{P}_\Psi(A^{\perp}) = g \circ \mathcal{P}_\Phi(A) \] (1.0.19)

where \( \mathcal{P}_\Psi, \mathcal{P}_\Phi \) are local liftings of the global period maps (\( \Psi, \Phi \) are markings of the weight-2 cohomology). The second step consists in showing that (1.0.16) holds for \( A \) belonging to a certain locally closed codimension-1 submanifold \( \Delta_1^0(V) \subset LG(\wedge^3 V)^0 \). If \( A \in \Delta_1^0(V) \) then \( X_A \) is isomorphic to a moduli space of sheaves on a \( K3 \) surface \( S \subset \mathbb{P}^6 \) of degree 10. On the other hand there is a map
$f: S_2 \to A_1$ of degree 2 which contracts a certain plane $P \subset S_2$ and is finite over $Y_{A_1} \setminus f(P)$. One shows that $\text{mult}_f(P) = 3$ and hence $A \not\in \mathcal{LG}(\wedge^3 V^*)^0$ i.e. $A \not\in \mathcal{LG}(\wedge^3 V^*)^0$. Morally $X_{A_1}$ is the singular symplectic 4-fold obtained from $S_2$ by contracting the plane $P$. We do not prove a precise version of this; however we do prove that the local period map of $LG$ from $S_i.e. A / A^0$ that (1.0.16) holds for $A^0$. The results on $\Delta$ local and global period maps and we prove (1.0.13). In the next section we prove $f$ acts transitively on the set of vectors of square 2 there exists an isometry $H$ a non-degenerate integral symmetric form; thus symmetric bilinear form on $H$.

Let $V$ be a generator of $\Lambda$ such that $\psi(X,L)$ is as in (1.0.8); this is a marking of $(X,L)$. If $\sigma$ be a generator of $H^{2,0}(X)$; then

$$
(c_1(L),\sigma)_X = 0 = (\sigma,\sigma)_X, \quad (\sigma,\overline{\sigma})_X > 0.
$$

Thus

$$
\mathcal{P}_\psi(X,L) := \psi(H^{2,0}(X)) \subset D_2.
$$

This is the local period point of $(X,L)$ associated to $\psi$ (or periods of $(X,L)$). Any other marking of $(X,L)$ is given by $\gamma \circ \psi$ where $\gamma \in \Gamma$, thus the $\Gamma$-orbit of $\mathcal{P}_\psi(X,L)$ is a well-defined point of $Q_2$; this is the global period point of $(X,L)$, we denote it by $\mathcal{P}(X,L)$. If $A \in LG(\wedge^3 V)^0$ we let $\mathcal{P}(A) := \mathcal{P}(X_A, L_A)$.

Later on we will study global and local period maps for certain families of double EPW-sextics. Let $\pi: X \to T$ be a proper submersive map between complex manifolds such that each fiber is a deformation of $(K3)^2$. For $t \in T$ we let $X_t := \pi^{-1}(t)$. We assume that we

Notation: V will always be a complex vector-space of dimension 6.

## 2 The period map and the lattice $\Lambda$

### 2.1 The period map

Let $X$ be a deformation of $(K3)^2$. Let $(\cdot,\cdot)_X$ be the Beauville-Bogomolov symmetric bilinear form on $H^2(X;\mathbb{Z})$. The restriction of $(\cdot,\cdot)_X$ to $H^2(X;\mathbb{Z})$ is a non-degenerate integral symmetric form; thus $H^2(X;\mathbb{Z})$ is a lattice. As is well-known $H^2(X;\mathbb{Z})$ and $\Lambda$ are isometric. Now assume that we are given a holomorphic line-bundle $L$ on $X$ such that $(c_1(L),c_1(L))_X = 2$. Since $O(\Lambda)$ acts transitively on the set of vectors of square 2 there exists an isometry

$$
\psi: H^2(X;\mathbb{Z}) \xrightarrow{\sim} \widetilde{\Lambda}
$$

such that $\psi(c_1(L)) = u$ where $u$ is as in (1.0.8); this is a marking of $(X,L)$. Let $\psi:\overline{\mathbb{C}}: H^2(X;\overline{\mathbb{C}}) \xrightarrow{\sim} \widetilde{\Lambda} \otimes \overline{\mathbb{C}}$ be the map obtained from $\psi$ by extension of scalars. Let $\sigma$ be a generator of $H^{2,0}(X)$; then

$$
(c_1(L),\sigma)_X = 0 = (\sigma,\sigma)_X, \quad (\sigma,\overline{\sigma})_X > 0.
$$

Thus

$$
\mathcal{P}_\psi(X,L) := \psi(H^{2,0}(X)) \subset D_2.
$$

This is the local period point of $(X,L)$ associated to $\psi$ (or periods of $(X,L)$).

Any other marking of $(X,L)$ is given by $\gamma \circ \psi$ where $\gamma \in \Gamma$, thus the $\Gamma$-orbit of $\mathcal{P}_\psi(X,L)$ is a well-defined point of $Q_2$; this is the global period point of $(X,L)$, we denote it by $\mathcal{P}(X,L)$. If $A \in LG(\wedge^3 V)^0$ we let $\mathcal{P}(A) := \mathcal{P}(X_A, L_A)$.

Later on we will study global and local period maps for certain families of double EPW-sextics. Let $\pi: X \to T$ be a proper submersive map between complex manifolds such that each fiber is a deformation of $(K3)^2$. For $t \in T$ we let $X_t := \pi^{-1}(t)$. We assume that we
are given a (holomorphic) line-bundle $L$ on $X$ such that $c_1(L|_{X_t})$ has square 2 for every $t \in T$; we let $L_t := L|_{X_t}$. The global period map of $(X, L)$ is given by

$$T \overset{\mathcal{P}}{\longrightarrow} \mathbb{Q}_2 \quad t \mapsto \mathcal{P}(t) := \mathcal{P}(X_t, L_t)$$  \hspace{1cm} (2.1.5)

Griffiths proved that $\mathcal{P}$ is a holomorphic map. Let $\pi: X \to LG(\wedge^3 V)^0$ be the tautological family of double EPW-sextics and $L$ be the tautological relatively ample line-bundle on $X$ (see (1.0.3)) on $X_A$; then $\mathcal{P}$ is the period map of (1.0.15), in particular the map of (1.0.15) is holomorphic. Let’s go back to a general family (2.1.4): it is not always possible to lift the global period map $\mathcal{P}$ to a map $T \rightarrow D_2$, in fact a necessary and sufficient condition is that $R^2\pi_*\mathbb{Z}$ is trivial. Suppose that $R^2\pi_*\mathbb{Z}$ is trivial. Then there exists a trivialization $\Psi: R^2\pi_*\mathbb{Z} \cong T \times \tilde{\Lambda}$ sending the section corresponding to $c_1(L)$ to the section given by $u$ - this is a marking of $(X, L)$. Given such a marking we let $\Psi_t: H^2(X_t; \mathbb{Z}) \cong \tilde{\Lambda}$ be the fiber of $\Psi$ over $t$. The local period map of $(X, L)$ associated to $\Psi$ is given by

$$T \overset{\mathcal{P}_\Psi}{\longrightarrow} D_2 \quad t \mapsto \mathcal{P}_\Psi(t) := \mathcal{P}_\Psi(X_t, L_t).$$  \hspace{1cm} (2.1.6)

Griffiths proved that $\mathcal{P}_\Psi$ is holomorphic.

### 2.2 Proof of (1.0.13)

We start by recalling the definition of discriminant group and discriminant quadratic form of an even lattice $(L, (\cdot, \cdot)_L)$, i.e. a free finitely generated abelian group $L$ equipped with a symmetric integral even non-degenerate bilinear form $(\cdot, \cdot)_L$. We follow [10]. The bilinear form $(\cdot, \cdot)_L$ extends to a $\mathbb{Q}$-valued bilinear form on $L \otimes \mathbb{Q}$; abusing notation we denote by $(\cdot, \cdot)_L$ the extended form.

Let $L^\vee := \text{Hom}(L, \mathbb{Z})$; by non-degeneracy of $(\cdot, \cdot)_L$ we have a natural chain of inclusions

$$L \subset L^\vee \subset L \otimes \mathbb{Q}.$$  \hspace{1cm} (2.2.1)

The discriminant group of $L$ is $A_L := L^\vee/L$; it comes provided with the discriminant bilinear-form

$$A_L \times A_L \overset{b_L}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \quad ([\alpha], [\beta]) \mapsto [(\alpha, \beta)_L]$$  \hspace{1cm} (2.2.2)

and the discriminant quadratic-form

$$A_L \overset{q_L}{\longrightarrow} \mathbb{Q}/2\mathbb{Z} \quad [\alpha] \mapsto [(\alpha, \alpha)_L].$$  \hspace{1cm} (2.2.3)

The formula

$$q_L([\alpha + \beta]) \equiv q_L([\alpha]) + q_L([\beta]) + 2b_L([\alpha], [\beta]) \pmod{2\mathbb{Z}}$$  \hspace{1cm} (2.2.4)

shows that $q_L$ determines uniquely $b_L$. An index-$i$ overlattice of $L$ consists of a lattice $M$ and an inclusion of lattices $L \subset M$ (the restriction of $(\cdot, \cdot)_M$ to $L$ is equal to $(\cdot, \cdot)_L$ of index $i$). Two overlattices $M_1 \supset L \subset M_2$ are equivalent if there
exists an isometry $M_1 \cong M_2$ which is the identity on $L$, i.e. if the inclusions $M_i \hookrightarrow L^\vee$ for $i = 1, 2$ have the same image. Suppose that $L \subseteq M$ is an index-$i$ overlattice of $L$; the inclusion $M \subseteq L^\vee$ defines an inclusion $M/L \subseteq A_L$ with image a subgroup of cardinality $i$ which is $q_L$-isotropic. This construction defines a one-to-one correspondence between the set of equivalence classes of index-$i$ overlattices of $L$ and the set of $q_L$-isotropic subgroups of $A_L$ of cardinality $i$.

The correspondence is equivariant for the natural actions of $O(L)$ on both sets. Now consider the lattice $L = \mathbb{Z}u \oplus \Lambda$ where $u$ and $\Lambda$ are as in Section (1): we will describe the discriminant group and discriminant form of $L$. Let $e_1, e_2 \in \Lambda$ be as in Section (1). A straightforward computation gives that

$$A_L = \mathbb{Z}/(2) \left[ \frac{u}{2} \right] \oplus \mathbb{Z}/(2) \left[ \frac{e_1}{2} \right] \oplus \mathbb{Z}/(2) \left[ \frac{e_2}{2} \right]$$

(2.2.5)

and that

$$q_L \left( x \left[ \frac{u}{2} \right] + y_1 \left[ \frac{e_1}{2} \right] + y_2 \left[ \frac{e_2}{2} \right] \right) \equiv \frac{1}{2}(x^2 - y_1^2 - y_2^2) \pmod{2\mathbb{Z}}.$$  (2.2.6)

The set $I \subset A_L$ of non-zero isotropic vectors is given by

$$I = \left\{ \left[ \frac{u}{2} \right] + \left[ \frac{e_1}{2} \right], \left[ \frac{u}{2} \right] + \left[ \frac{e_2}{2} \right] \right\}.$$  (2.2.7)

The group $O(L)$ acts naturally on $A_L$ and hence also on $I$; thus we have a homomorphism

$$O(L) \xrightarrow{\epsilon} Aut(I) \cong \mathbb{Z}/(2).$$  (2.2.8)

The overlattice $\tilde{\Lambda} \supset L$ is of index 2 because $L$ is the kernel of the surjection

$$\tilde{\Lambda} \xrightarrow{v} \mathbb{Z}/(2) \quad (v, u) \pmod{2}.$$  (2.2.9)

The correspondence described above defines an $O(L)$-equivariant bijective map between $I$ and the set of equivalence classes of index-2 overlattices of $L$; thus

$$Im(O(\tilde{\Lambda}) \rightarrow O(L)) = \ker(\epsilon).$$  (2.2.10)

The subgroup of $O(L)$ consisting of isometries which are the identity on $\mathbb{Z}u$ is naturally identified with $O(\Lambda)$; thus $O(\Lambda) < O(L)$. By (2.2.10) and the definition of $\Gamma$ (see (1.0.11)) we get that

$$\Gamma = \ker(\epsilon|_{O(\Lambda)}).$$  (2.2.11)

Let $r \in O(\Lambda)$ be the involution characterized by the following properties:

$$r(e_1) = e_2, \quad r(e_2) = e_1, \quad r|_{\{e_1, e_2\} \perp} = \text{identity}.$$  (2.2.12)

Then $\epsilon(r)$ is the non-trivial permutation of $I$ and hence $\epsilon|_{O(\Lambda)}$ is surjective. Thus (1.0.13) follows from (2.2.11).

3 Explicit dual couples

Let $F_3^3 \subset \mathbb{P}^6$ be the intersection of $Gr(2, \mathbb{C}^5) \subset \mathbb{P}(\wedge^2 \mathbb{C}^5) \cong \mathbb{P}^9$ with a transversal 6-dimensional linear subspace of $\mathbb{P}^9$. Let $T \subset |O_{\mathbb{P}^6}(2)|$ be the open dense subset
period map

the equivalence class of the semistable sheaf $F$

3.1 The locus $\Delta$

extends and its value at $t$

parametrizing quadrics which are transversal to $F_0^3$. For $t \in T$ let $Q_t$ be the quadric corresponding to $t$ and

$$S_t := F_0^3 \cap Q_t. \quad (3.0.1)$$

Then $S_t$ is a degree-10 linearly normal $K3$ surface; in fact the generic such surface is projectively equivalent to $S_t$ for some $t \in T$ by Mukai [8]. Let $D_t$ be the hyperplane divisor class on $S_t$ and $M_t$ be the moduli space of $D_t$-semistable rank-2 sheaves $F$ on $S_t$ with $c_1(F) = c_1(D_t)$ and $c_2(F) = 5$. Suppose that

for all divisors $E$ on $S_t$ we have $E \cdot D_t \equiv 0 \pmod{10}$. \quad (3.0.2)

Then (see Section (5) of [13]) there exists $A_t \in LG(\wedge^3 V)^0$ such that

$$X_{A_t} \cong M_t. \quad (3.0.3)$$

Furthermore $Y_{A_t}$ is explicitly described as follows. Let $\Sigma_t$ be the divisor on $|I_{S_t}(2)|$ parametrizing singular quadrics, since all quadrics parametrized by $|I_{F_0^3}(2)|$ are singular we have

$$\Sigma_t = |I_{F_0^3}(2)| + \Sigma'_t. \quad (3.0.4)$$

Then

$$Y_{A_t} \cong \Sigma'_t. \quad (3.0.5)$$

(Formally $\Sigma'_t$ is a degree-6 divisor; in the above equation we are implicitly stating that $\Sigma'_t$ is a reduced divisor and hence we may view it as a degree-6 hypersurface.) The set of $t$ for which (3.0.2) holds is the complement of a countable union of proper algebraic subvarieties of $T$ however a straightforward argument shows that there is an open dense $T'' \subset T$ such that for $t \in T''$ there exists $A_t \in LG(\wedge^3 V)^0$ for which both (3.0.3) and (3.0.5) hold; we give the argument in Subsection (3.1). We let $\Delta^0(V) \subset LG(\wedge^3 V)^0$ be the set of $A$ such that $X_A$ is isomorphic to $M_t$ for some $t \in T''$; this is a locally closed subset of $LG(\wedge^3 V)^0$. Computing the periods of $X_A$ for $A \in \Delta^0(V)$ we will show that $\Delta^0(V)$ has codimension 1 in $LG(\wedge^3 V)^0$. Let $A \in \Delta^0(V)$, thus $X_A \cong M_t$ for some $t \in T$: the dual $Y^*_A$ is described as follows. The Hilbert scheme $S_t^{[2]}$ contains a copy of $\mathbb{P}^2$, call it $P_t$, parametrizing $Z \subset S_t$ which span a line contained in $F_0^3$. Let $S_t^{[2]} \to N_t$ be the contraction of $P_t$ - thus $N_t$ is a singular symplectic variety. There is an involution on $N_t$ whose quotient is isomorphic to $Y_A^*$. From this it will follow that $A \notin LG(\wedge^3 V)^0$ and hence apparently it will not make sense to compute $P(A^\perp)$. The main point of this subsection is to prove that the period map extends across $A^\perp$, in fact the local period map extends and its value at $A^\perp$ is given by the periods of $(S_t^{[2]}, D_t)$.

3.1 The locus $\Delta^0(V)$

We recall that the Mukai vector $v(F)$ of a sheaf $F$ on $S_t$ is

$$v(F) := ch(F)\sqrt{Td(S_t)} = ch(F)(1 + \eta_t) \in H^*(S_t; \mathbb{Z}). \quad (3.1.1)$$

where $\eta_t \in H^4(S_t; \mathbb{Z})$ is the orientation class. If $[F] \in M_t$ (we let $[F] \in M_t$ be the equivalence class of the semistable sheaf $F$ - we recall that if $F$ is stable this
is the same as the isomorphism class of $\mathcal{F}$) the class $v(\mathcal{F})$ is independent of $\mathcal{F}$, we denote it by $v_t$; explicitly

$$v_t = 2 + c_1(D_t) + 2\eta_t.$$  \hspace{1cm} (3.1.2)

**Proposition 3.1.** Keep notation as above. There is an open dense subset $T'' \subset T$ such that the following holds. Let $t \in T''$; then there exists $A_t \in LG(V)\setminus \mathcal{C}$ such that both (3.0.3) and (3.0.5) hold. In particular we have a canonical identification $\mathbb{P}(V) \cong |I_{S_t}(2)|$.

**Proof.** By Maruyama [7] there exists a projective map $\rho: \mathcal{M} \to T$ with (schematic) fiber $M_t$ over $t \in T$. Let $t \in T$; we say that $S_t$ is unsuitable if there exists a divisor class $C$ on $S_t$ such that

$$\int_{S_t} c_1(C) \wedge c_1(D_t) = 0, \quad -10 \leq \int_{S_t} c_1(C)^2 < 0.$$ \hspace{1cm} (3.1.3)

The set of unsuitable $t$ is a proper closed subset of $T$, thus the complement $T'$ is an open dense subset of $T$. Let $\mathcal{M}' := \rho^{-1}(T')$ and $\rho': \mathcal{M}' \to T'$ be the restriction of $\rho$. It is known that if $t \in T'$ then every sheaf parametrized by $M_t$ is slope-stable and $M_t$ is a smooth 4-dimensional scheme (Main Theorem (0.1.2) and Proposition (2.1) of [11] - notice that $t \in T_1$ if and only if $D_t$ is $|\nu_t|$-generic).

Let’s show that there is an open dense $T'' \subset T'$ such that for $t \in T''$ the Mukai reflection acts on the derived category of coherent sheaves on $S_t$. In order for $\phi_t$ to be a regular involution on $M_t$ it suffices that for all $[\mathcal{F}] \in M_t$ the following hold:

(a) $h^0(\mathcal{F}) = \chi(\mathcal{F}) = 4$,

(b) $\mathcal{F}$ is globally generated away from (at most) a zero-dimensional subset of $S_t$,

(c) the kernel of the evaluation map $H^0(\mathcal{F}) \otimes \mathcal{O}_{S_t} \to \mathcal{F}$, call it $\mathcal{E}$, is a $D_t$-slope-stable sheaf.

If (a)-(b)-(c) are satisfied for all $[\mathcal{F}] \in M_t$ then the generic sheaf $\mathcal{F}$ parametrized by $M_t$ is globally generated and for such a sheaf $\phi([\mathcal{F}]) = [\mathcal{E}]$ where $\mathcal{E}$ is as in (c) above. Let $T'' \subset T'$ be the set of $t$ such that (a)-(b)-(c) hold for every $[\mathcal{F}] \in M_t$. Let’s prove that $T''$ is open. First we show that the subset $T_a \subset T$ of $t \in T'$ such that (a) holds for every $[\mathcal{F}] \in M_t$ is open. Let $t \in T'$: if $[\mathcal{F}] \in M_t$ then by stability $h^2(\mathcal{F}) = 0$ and hence $h^0(\mathcal{F}) \geq 4$, thus by the upper-semicontinuity Theorem the set of $[\mathcal{F}] \in M_t$ that violate (a) is closed. By properness of the map $\rho': \mathcal{M}' \to T'$ it follows that $(T' \setminus T_a)$ is closed i.e. $T_a$ is open. A similar argument shows that the set of $t \in T_a$ such that (b)-(c) hold for every $[\mathcal{F}] \in M_t$ is open; thus $T''$ is open. Since $T'' \subset T'$ contains the subset of $t$ such that (3.0.2) holds it is dense in $T$. Let $T'' \subset T'$ be the set of $t$ such that

$$S_t \text{ contains no effective non-zero divisor } E \text{ with } E \cdot D_t \leq 5.$$ \hspace{1cm} (3.1.4)

Thus $T''$ is open and dense in $T$. We claim that if $t \in T''$ then

$$M_t / \langle \phi_t \rangle \cong \Sigma_t \subset |I_{S_t}(2)| \cong \mathbb{P}^5.$$ \hspace{1cm} (3.1.5)

More precisely Proposition (5.1) of [12] holds for $M_t = \mathcal{M}(v_t)$. In order to prove this it suffices to show that for every $t \in T''$ the following holds:
(1) If \([F] \in M_t\) and \(\sigma \in H^0(F) = \text{Hom}(\mathcal{O}_{S_t}, F)\) is non-zero then the quotient \(\mathcal{F}/\text{Im}(\sigma)\) is locally-free in codimension 1 (\(\sigma\) has isolated zeroes) - this is used in the proof of Lemma (5.4) of [12].

(2) If \(G\) is a globally generated rank-2 vector bundle on \(S_t\) with \(\det G \cong \mathcal{O}_{S_t}(D_t)\) and \(c_2(G) = 5\) then \(G\) is \(D_t\)-slope stable - this is used in the proof of Lemma (5.7) of [12].

Let’s show that (1) above holds. Suppose that \(\sigma\) does not have isolated zeroes: then it vanishes along a non-zero effective divisor \(E\) and by slope stability of \(\mathcal{F}\) we have \(E \cdot D_t \leq \text{slope}(\mathcal{F}) = 5\) contradicting (3.1.4). Let’s show that (2) above holds. Suppose that \(G\) is not \(D_t\)-slope stable; since \(G\) has rank 2 there is a destabilizing sequence \(G \to I_Z \otimes \mathcal{O}_{S_t}(E)\) where \(Z \subseteq S_t\) is zero-dimensional and \(E\) is a divisor with \(E \cdot D_t \leq \text{slope}(\mathcal{F}) = 5\). Since \(G\) is globally generated there is a non-zero section of \(I_Z \otimes \mathcal{O}_{S_t}(E)\) and hence \(E\) is effective; by (3.1.4) we get that \(E = 0\). Thus we have an exact sequence

\[
0 \to \mathcal{O}_{S_t}(D_t) \to G \to I_Z \to 0. \tag{3.1.6}
\]

Since \(\ell(Z) = c_2(G) = 5\) the zero-dimensional subscheme \(Z \subseteq S_t\) is non-empty and hence \(h^0(I_Z) = 0\); this contradicts the hypothesis that \(G\) is globally generated. We have proved that (3.1.5) holds for \(t \in T''\). By Theorem (1.1) of [13] we get that there exists \(A_t \in \text{LG}(\wedge^3 V)^0\) such that both (3.0.3) and (3.0.5) hold.

□

The proof of the above proposition together with Claim (5.18) of [12] gives the following result.

**Corollary 3.2.** Let \(t \in T''\) and let \(A_t \in \text{LG}(\wedge^3 V)^0\) be such that both (3.0.3) and (3.0.5) hold. The map \(f_{A_t} : X_{A_t} \to Y_{A_t}\) is identified with the quotient map \(M_t \to M_t/\langle \phi_t \rangle\).

Let \(\Delta^0(V) \subset \text{LG}(\wedge^3 V)^0\) be the locus of \(A\) such that \(X_A\) is isomorphic to \(M_t\) for some \(t \in T''\). Our next task is to show that \(\Delta^0(V)\) is a locally closed subset of codimension 1. First we recall how one describes \(H^2(M_t)\) for \(t \in T'\). Let \(u, w \in H^*(S_t)\) and let \(u_q, w_q\) be the degree-\(q\) components of \(u, w\) respectively. One sets \(u' := u_0 - u_2 + u_4\) and

\[
\langle u, w \rangle := \int_{S_t} (u_2 \wedge u_0 - u_0 u_4 - u_4 w_0) = -\int_{S_t} u' \wedge w. \tag{3.1.7}
\]

This is Mukai’s bilinear symmetric form. One defines a positive\(^2\) weight-two Hodge structure on \(H^*(S_t)\) by defining the Hodge filtration as

\[
F^1 H^*(S_t) := H^0(S_t) \oplus F^1 H^2(S_t) \oplus H^4(S_t), \quad F^2 H^*(S_t) := F^2 H^2(S_t). \tag{3.1.8}
\]

Let \(v_t\) be the Mukai vector (3.1.2), then \(v_t\) is integral of type \((1, 1)\) and hence \(v_t^+\) is an integral Hodge substructure of \(H^*(S_t)\) and furthermore the restriction of Mukai’s bilinear symmetric form to \(v_t^+\) is integral. Mukai defined (see [9, 11]) a map

\[
\theta_t : v_t^+ \to H^2(M_t) \tag{3.1.9}
\]

\(^2\)This means that \(h^{p,q} = 0\) if \(p < 0\).
by taking K"unneth components of the Chern character of a tautological sheaf
on $S_1 \times M_t$ (if such a sheaf does not exists one considers a quasi-tautological
sheaf). In [11] we proved that $\theta_t$ is an isomorphism of Hodge structures and an
isometry of lattices - of course the bilinear form on $v^\perp$ is the restriction of
the Mukai pairing. Now assume that $t \in T'''$ and let $A_t$ be as in Proposition (3.1)
and $L_{A_t}$ be as in (1.0.3); then by Corollary (3.2) and Proposition (5.1) of [12]
we have

$$c_1(L_{A_t}) = \theta_t(\eta_t - 1).$$

(3.1.10)

We let $h_t := \theta_t(\eta_t - 1)$.

**Proposition 3.3.** Keep notation as above. Then $\Delta^0(V)$ is a $PGL(V)$-invariant
subset of $L\mathbb{G}(\wedge^3 V)^0$ which is locally (in the classical topology) a codimension 1
submanifold.

**Proof.** The subset $\Delta^0(V)$ is $PGL(V)$-invariant by definition. Let $\pi : X \to
L\mathbb{G}(\wedge^3 V)^0$ be the tautological family of double EPW-sixties and $L$ be the tauto-
logical relatively ample line-bundle on $X$; thus the restriction of $L$ to $A_t$ is
isomorphic to $L_{A_t}$. Let $A_p \in \Delta^0(V)$. Thus there exists $p \in T'''$ such that $X_{A_p} \cong M_p$. Let $U \subset L\mathbb{G}(\wedge^3 V)^0$ be a small open ball containing $A_p$. Let $X_U := \pi^{-1}(U)$
and $L_U := L|_{X_U}$. Since $U$ is contractible there is a marking $\Psi$ of $(X_U, L_U)$;
let $\mathcal{P}_\Psi : U \to D_2$ be the corresponding local period map. We notice that
$(5 + 2c_1(D_p) + 5\eta_p)\in v^\perp_p$ and hence $\theta_p(5 + 2c_1(D_p) + 5\eta_p) \in H^2(M_p)$. Furthermore
since $(\eta_p - 1, 5 + 2c_1(D_p) + 5\eta_p)$ is a flat section of $R^2\rho'' M_p$ and for all $t \in T'''$ we have

$$(\theta_t(5 + 2c_1(D_t) + 5\eta_t), H^2(S_t))_{M_t} = 0$$

(3.1.13)

because $\theta_t(5 + 2c_1(D_t) + 5\eta_t) \in H^{1,1}(M_t)$; the above equality gives (3.1.12).

Next we prove that

$$\mathcal{P}_\Psi(\Delta^0(V) \cap U) \supset \mathcal{P}_\Psi(U) \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)\perp.$$

(3.1.14)

First we prove that

$$\mathcal{P}_\Psi(\Delta^0(V) \cap U) \subset \mathcal{P}_\Psi(U) \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)\perp.$$

(3.1.12)

Let $\mathcal{M}''' := \rho^{-1}(T'''')$ and $\rho''' : \mathcal{M}''' \to T'''$ be the restriction of $\rho$. Then $\theta_t(5 + 2c_1(D_t) + 5\eta_t)$ is a flat section of $R^2\rho''' M_p$ and for all $t \in T'''$ we have

$$\{v_p, \eta_p - 1, 5 + 2c_1(D_p) + 5\eta_p\}^\perp = H^2(S_p)_{prim}$$

(3.1.15)

where the primitive cohomology $H^2(S_p)_{prim} \subset H^2(S)$ is the orthogonal to
c$1(D_p)$. Since $h_p = \theta_p(\eta_p - 1)$ we get that

$$H^2(M_p) \cap \{h_p, \theta_p(5 + 2c_1(D_p) + 5\eta_p)\}^\perp = \theta_p(H^2(S_p)_{prim}).$$

(3.1.16)

Let $K_{10}$ be the period space for $K3$ surfaces with a polarization of degree 10:
Equality (3.1.16) defines an isomorphism

$$D_2 \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)\perp \sim K_{10}$$

(3.1.17)
which is compatible with local period maps defined by the family \( \rho^m : \mathcal{M}^m \to T' \) and the family \( \zeta : S^m \to T^m \) with fiber \( S_t \) over \( t \in T^m \). Let \( S^m_U := \zeta^{-1}(U) \). Since \( S^m \) contains the generic K3 of degree 10 the local period map of the family \( S^m_U \to U \) is a submersive map from \( U \) to an open ball in \( K_{10} \); since (3.1.17) is an isomorphism this proves (3.1.14). We also get that \( \mathcal{P}_\phi \) is submersive and hence in order to show that \( \Delta^0(V) \cap U \) is a codimension 1 submanifold it suffices to prove that
\[
D_2 \cap \Psi_p \circ \theta_p((5 + 2c_1(D_p) + 5\eta_p) \perp)
\]
is smooth. The period domain \( D_2 \) is an open subset of the quadric of isotropic lines for the non-degenerate quadratic form \( \langle \rangle |_{\Lambda \otimes \mathbb{C}} \) and hence if \( \Psi_p \circ \theta_p((5 + 2c_1(D_p) + 5\eta_p)) \) is not isotropic then (3.1.18) is smooth. Since
\[
(\Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p), \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)) = (5 + 2c_1(D_p) + 5\eta_p, 5 + 2c_1(D_p) + 5\eta_p) = -10
\]
the intersection (3.1.18) is indeed smooth. \( \square \)

### 3.2 The dual of \( Y_A \) for \( A \in \Delta^0(V) \)

If \( A \in \mathbb{L}G(\wedge^3 V)^0 \) then \( Y_A^\perp = Y_A^\perp \) by Proposition (3.1) of [13]. As we will see \( \Delta^0(V) \cap \mathbb{L}G(\wedge^3 V)^0 = \emptyset \) and hence in order to show that \( Y_A^\perp = Y_A^\perp \) for \( A \in \Delta^0(V) \) we need to improve on the result of [13].

**Proposition 3.4.** Let \( A \in \mathbb{L}G(\wedge^3 V) \) and \( \mathbb{P}(W) \in \mathbb{P}(V^\perp) \). Then \( \mathbb{P}(W) \in Y_A^\perp \) if and only if
\[
\wedge^3 W \cap A \neq 0.
\]

**Proof.** Let \( \phi \in V^\perp \) be a linear function such that \( W = \ker(\phi) \): then
\[
F_\phi := \{ \phi \wedge \psi \mid \psi \in \wedge^2 V^\perp \} = (\wedge^3 W)^\perp.
\]
By definition \( \mathbb{P}(W) \in Y_A^\perp \) if and only if
\[
\{0\} \neq F_\phi \cap A^\perp = (\wedge^3 W + A)^\perp.
\]
Since \( 10 = \dim(\wedge^3 W) = \dim A \) and \( \dim(\wedge^3 V) = 20 \) we get that (3.2.3) holds if and only if (3.2.1) holds. \( \square \)

**Corollary 3.5.** Let \( A \in \mathbb{L}G(\wedge^3 V)^0 \). Then \( Y_A^\perp \) is a hypersurface and \( Y_A^\perp = Y_A^\perp \).

**Proof.** Let \( \mathbb{P}(W) \in Y_A^\perp \): by Proposition (3.4) this is equivalent to (3.2.1). Let \( 0 \neq \alpha \in (\wedge^3 W \cap A) \): since \( \dim W = 5 \) there exists \( v \in W \) such that \( \alpha \) is divisible by \( v \) and hence \( [v] \in Y_A \). We decompose \( W = C_0 \oplus W_0 \) and write \( \alpha = v \wedge w \) where \( w \in \wedge^2 W_0 \); by Definition (2.5) of [13] the rank of \( w \) is 4, thus \( W = \text{span}(v, w) \). Furthermore if \( \dim(F_{[v]} \cap A) = 1 \) then \( Y_A \) is smooth at \( [v] \) and \( \mathbb{P}(W) \) is the projective tangent space to \( Y_A \) at \( [v] \) - see the proof of Proposition (3.1) of [13]. By Proposition (2.8) of [13] we know that \( \dim(F_{[v]} \cap A) = 1 \) unless \( [v] \in \text{sing}(Y_A) \) and in this case \( \dim(F_{[v]} \cap A) = 2 \). Since \( \text{sing}(Y_A) \) is a surface a straightforward dimension count gives that \( \mathbb{P}(W) \notin Y_A^\perp \) for generic \( \mathbb{P}(W) \in \mathbb{P}(V^\perp) \), thus \( Y_A^\perp \) is a hypersurface. The same dimension count gives that the generic \( \mathbb{P}(W) \in Y_A^\perp \) is tangent to \( Y_A \) at one of its smooth points; this proves the corollary. \( \square \)
We will describe explicitly $Y' = Y_A$, for $A \in \Delta^0(V)$; essentially we will give a refinement of Proposition (5.20) and Corollary (5.21) of [12]. Let $t \in T''$ and $S_t$ be the $K3$ surface corresponding to $t$; in order to simplify notation we temporarily drop the subscript $t$. Let $R$ be the Fano variety of lines on $F_3^3$. If $[\ell] \in R$ then $\ell \not\subset S$ by (3.1.4) and hence $\ell \cap Q$ is a 0-dimensional scheme of length 2 contained in $S$: thus we have a regular map

$$R \longrightarrow S^{[2]} \ell \mapsto \ell \cap Q. \quad (3.2.4)$$

Let $P \subset S^{[2]}_{t}$ be the image of the above map: then (3.2.4) defines a regular map $R \rightarrow P$ with inverse given by

$$P \longrightarrow R [Z] \mapsto \text{span}(Z) \quad (3.2.5)$$

and hence $P$ is isomorphic to $R$. It is known [5] that $R \cong \mathbb{P}^2$, thus $P \cong \mathbb{P}^2$. Since $S^{[2]}_t$ is a symplectic variety it follows that we can contract $P$

$$c: S^{[2]} \rightarrow N. \quad (3.2.6)$$

A priori $N$ is a complex space, we will show soon that it is projective. Let $p := c(P)$; thus $p$ is the unique singular point of $N$. On $S^{[2]}$ there is an interesting map to $|I_{S}(2)| \lor V$, see (4.3) of [12]; we recall the definition. The $K3$ surface $S$ is cut out by quadrics and it contains no lines by (3.1.4); thus we have a regular map

$$S^{[2]} \longrightarrow |I_{S}(2)| \lor \mathbb{P}^5 \{Q_\Lambda \in |I_{S}(2)| \mid \text{span}(Z) \subset Q_\Lambda\}. \quad (3.2.7)$$

Let $W \subset |I_{S}(2)| \lor$ be the image of the above map; thus (3.2.7) defines a map $f: S^{[2]} \rightarrow W$. If $[Z] \in P$ then $f([Z]) = |I_{F_3^3}(2)|$ hence $f$ is constant on $P$; we will see that the point

$$f(P) = |I_{F_3^3}(2)| \in W \quad (3.2.8)$$

is quite special. Since $f$ is constant on $P$ and $N$ is normal the map $f$ descends to a regular map

$$\overline{f}: N \rightarrow W. \quad (3.2.9)$$

**Lemma 3.6.** Keep notation as above. There exist a non trivial involution $\phi: N \rightarrow N$ and a birational morphism $c: N/\langle \phi \rangle \rightarrow W$ with finite fibers such that $\overline{f}$ is the composition

$$N \xrightarrow{\pi} N/\langle \phi \rangle \xrightarrow{c} W. \quad (3.2.10)$$

In particular $N$ is projective and $\deg W = 6$.

**Proof.** Let’s show that $\overline{f}$ has finite fibers of cardinality at most 2 and that the generic fiber has cardinality 2. The fiber of $\overline{f}$ over $|I_{F_3^3}(2)| \in |I_{S}(2)| \lor$ consists of the unique singular point $p$ of $N$. Now let $\Lambda \in (W \setminus \{|I_{F_3^3}(2)|\})$, i.e.

$$\Lambda = f([Z]), \quad [Z] \notin P. \quad (3.2.11)$$

Then $\overline{f}^{-1}(\Lambda) = f^{-1}(\Lambda)$. Let $\Lambda_0 := \Lambda \cap |I_{F_3^3}(2)|$ and choose $\lambda_0 \in (\Lambda \setminus |I_{F_3^3}(2)|)$. One has

$$\bigcap_{\lambda \in \Lambda_0} Q_\lambda = F_3^3 \cup A_\Lambda \quad (3.2.12)$$
where $A_\Lambda$ is a plane such that $A_\Lambda \cap F^3_\lambda = C_\Lambda$ is a conic. (See the proof of Lemma (4.20) of [12].) We claim that $Q_{\lambda_0} \not\supset A_\Lambda$: in fact if $Q_{\lambda_0} \supset A_\Lambda$ then $C_\Lambda \subset S$ contradicting (3.1.4). Thus $Q_{\lambda_0} \cap A_\Lambda$ is a conic $C_\Lambda$. By (3.2.11) the line span$(Z)$ is contained in $C_\Lambda$. Thus $C_\Lambda$ is degenerate and $f^{-1}(\Lambda)$ consists of the set of lines contained in $C_\Lambda$. This shows that $f^{-1}(\Lambda) = \overline{f}^{-1}(\Lambda)$ has cardinality at most 2. It also follows easily that the generic fiber of $\overline{f}$ consists of 2 distinct points. Since $N$ is normal there is a regular covering involution $\overline{\phi}$ such that $\overline{f}$ factors through the quotient map $N \to N/\overline{\phi}$. Since $\overline{f}$ has finite fibers so does $\epsilon$, since the generic fiber of $\overline{f}$ consists of 2 points the map $\epsilon$ is birational. The line-bundle $\overline{f}^*O_W(1)$ is ample because $\overline{f}: N \to W$ has finite fibers, thus $N$ is projective. We know (see (4.3) of [12]) that

\[ \int_{S^{[2]}} c_1(f^*O_W(1)) = 12. \]  

(3.2.13)

Since $f: S^{[2]} \to W$ has generic fiber of cardinality 2 we get that $\deg W = 6$. \hfill \Box

We will show that the map $\epsilon$ of Lemma (3.6) is an isomorphism. Since $(S^{[2]} \setminus P) \cong (N \setminus \{p\})$ the involution $\overline{\phi}$ defines a birational involution

\[ \phi: S^{[2]} \dashrightarrow S^{[2]} \]  

(3.2.14)

(This is the birational involution of Proposition (4.21) of [12].) Let

\[ \beta: Bl_P(S^{[2]}) \to S^{[2]} \]  

(3.2.15)

be the blow-up of $P$: since $P$ is Lagrangian the symplectic form on $S^{[2]}$ induces an isomorphism $N_{P/S^{[2]}} \cong \Omega_P$, and hence the exceptional divisor of $\beta$ is identified with the incidence variety $\Gamma \subset P \times P^\vee$ and the restriction of $\beta$ to the exceptional divisor is identified with the projection $\Gamma \to P$. We abuse notation and view $\Gamma$ as the exceptional divisor in $Bl_P(S^{[2]})$.

**Lemma 3.7.** The map $\phi$ of (3.2.14) is not regular along $P$. There is a regular involution $\phi: Bl_P(S^{[2]}) \to Bl_P(S^{[2]})$ which is equal to $\phi$ on $(S^{[2]} \setminus P) \subset Bl_P(S^{[2]})$. There is an identification $P \cong P^\vee$ such that $\overline{\phi}|_P$ is induced by the involution on $P \times P^\vee$ which interchanges the factors.

**Proof.** The eigenspaces of the isometry $H^2(\phi)$ on $H^2(S^{[2]})$ induced by $\phi$ are given by (see (4.3) of [12])

\[ H^2(\phi)_+ = \mathbb{C}f^*O_W(1), \quad H^2(\phi)_- = f^*O_W(1)^\perp. \]  

(3.2.16)

Suppose $\phi$ is regular along $P$: if $D$ is an ample divisor on $S^{[2]}$ then $c_1(D + \phi^*D)$ is an ample $\phi$-invariant class, this contradicts (3.2.16) because $f^*O_W(1)$ is not ample. Let $\psi: S^{[2]} \dashrightarrow \mathcal{X}$ be the flop of $P$: thus $\psi$ is the inverse of the blowup $Bl_P(S^{[2]}) \to S^{[2]}$ followed by the morphism $Bl_P(S^{[2]}) \to \mathcal{X}$ which contracts $\Gamma$ along the “other” fibration $\Gamma \to P^\vee$. In particular $\mathcal{X}$ contains $P^\vee$. Let $\ell \subset P$ and $\ell^\vee \subset P^\vee$ be lines. The isometry $H^2(\psi)$ identifies $(\ell^\vee)^\perp$ with $\ell^\perp$. In fact we have contractions $c: S^{[2]} \to N$ and $c': \mathcal{X} \to N$ which give identifications $\ell^\perp = H^2(N) = (\ell^\vee)^\perp$. In particular $H^2(\psi)$ sends a nef divisor class in $(\ell^\vee)^\perp$ to a nef divisor class. On the other hand $H^2(\psi)$ maps the half-space $\ell_{=0}$ to the half-space $\ell_{=0}$. By (3.2.16) we get that $(\psi \circ \phi)^*$ maps an ample divisor to an ample
divisor: since $(\psi \circ \phi)$ defines a regular map between the complements of subsets of codimension 2 it follows that $(\psi \circ \phi)$ is regular and hence an isomorphism. It follows also that $\phi$ induces a regular involution $\hat{\phi}: Bl_P(S^{[2]}) \to Bl_P(S^{[2]})$.

Let’s show that the restriction of $\phi$ to $\Gamma$ is as stated. Any automorphism of $\Gamma$ sends the projection $\Gamma \to P$ to itself composed with an automorphism of $P$ or to the projection $\Gamma \to P^V$ composed with an automorphism of $P^V$. Since $\phi$ is not regular the latter holds and it follows that $\hat{\phi}|_{\Gamma}$ is as stated.

**Corollary 3.8.** The map $\epsilon: N/⟨\hat{\phi}⟩ \to W$ of Lemma (3.6) is an isomorphism.

**Proof.** The quotient $Bl_P(S^{[2]})/⟨\hat{\phi}⟩$ is a projective birational model of $N/⟨\hat{\phi}⟩$ and hence it is birational to $W$ by Lemma (3.6). The Kodaira dimension of $Bl_P(S^{[2]})/⟨\hat{\phi}⟩$ is 0 hence also the Kodaira dimension of $W$ is 0. By Lemma (3.6) we know that $\deg W = 6$ and hence by adjunction $W$ is smooth in codimension 1. Thus $W$ is normal: since $\epsilon$ is birational with finite fibers we get that $\epsilon$ is an isomorphism.

Now we reintroduce the subscript $t$; thus we have $S_t, f_t, N_t, W_t$ etc.

**Proposition 3.9.** Keep notation as above. Let $A \in \Delta^0(V)$ and let $t ∈ T^m$ such that $X_{A_t} \cong M_t$. Thus by Proposition (3.1) and Corollary (3.5) we have $Y_{A_t} ⊂ |I_{S_t}(2)|^\vee$. Then

$$Y_{A_t} \cong W_t.$$  \hspace{1cm} (3.2.17)

**Proof.** Proposition (5.20) of [12] gives that the reduced scheme $(Y_{A_t})_{\text{red}}$ is equal to $W_t$ (the hypothesis of that proposition is that (3.0.2) holds, however the same proof goes through because all that is needed is the validity of (3.0.5)). Now $Y_{A_t}$ is a degree-6 hypersurface because $Y_{A_t} \neq \mathbb{P}(V^\vee)$ and on the other hand $W_t$ is a degree-6 hypersurface by Lemma (3.6) and hence from $(Y_{A_t})_{\text{red}} = W_t$ we get that $Y_{A_t}$ is reduced and equal to $W_t$.

Let $A \in \Delta^0(V)$ and let $t ∈ T^m$ such that $X_{A_t} \cong M_t$. Then $|I_{F_t^3}(2)| \in W_t$ - see (3.2.8 - and hence by the above proposition $|I_{F_t^3}(2)| \in Y_{A_t}$; we denote this point by $q_{A_t}$.

**Proposition 3.10.** Let $A \in \Delta^0(V)$ and let $t ∈ T^m$ such that $X_{A_t} \cong M_t$. Then

$$\text{mult}_{q_{A_t}} Y_{A_t} = 3.$$  \hspace{1cm} (3.2.18)

Let $y ∈ (Y_{A_t} \setminus \{q_{A_t}\})$ and hence $\overline{f_t}^{-1}(y) = f_t^{-1}(y)$ consists of two points or of one point.

(a) If $f_t^{-1}(y)$ consists of two points then $Y_{A_t}$ is smooth at $y$.

(b) If $f_t^{-1}(y)$ consists of a single point then the analytic germ $(Y_{A_t}, y)$ is isomorphic to the product of a smooth $2$-dimensional germ and the germ of an $A_1$-singularity.

**Proof.** By Proposition (6.2) of [13] $Y_{A_t}$ has multiplicity $3$ in $|I_{F_t^3}(2)|$; this proves (3.2.18). In order to prove (a) we notice that $Y_{A_t} = W_t$ and by Proposition (3.8) the map $\overline{f_t}: N_t \to W_t$ is identified with the quotient map $N_t \to N_t/⟨\hat{\phi}⟩$. Since $p_t = \overline{f_t}^{-1}(q_A)$ is the unique singular point of $N_t$ this gives Item (a).

In order to prove Item (b) we notice that $(N_t \setminus \{p_t\}) = (S_t^{[2]} \setminus P_t)$ and that the
quotient map \((N_i \setminus \{pt_i\}) \to (N_i \setminus \{pt_i\})/\langle \phi \rangle\) is identified with the quotient map \((S_i^{[2]} \setminus P_i) \to (S_i^{[2]} \setminus P_i)/\langle \phi \rangle\). By Proposition (4.21) of [12] the restriction of \(\phi\) to \((S_i^{[2]} \setminus P_i)\) is an anti-symplectic involution and hence its fixed point set is a Lagrangian surface; this proves (b).

Let \(A \in \Delta^0(V)\); then by (3.2.18) the sextic \(Y_{A^1}\) has a point of multiplicity 3 and hence \(A^1 \not\in \mathbb{L}_G(\wedge^3 V^\vee)^0\) because if \(B \in \mathbb{L}_G(\wedge^3 V^\vee)^0\) then \(Y_B\) has points of multiplicity at most 2. Thus

\[
\Delta^0(V) \cap \mathbb{L}_G(\wedge^3 V^\vee)^0 = \emptyset. \tag{3.2.19}
\]

Let \(\Delta^0_\ast(V) \subset \Delta^0(V)\) be the set of points which are smooth points of the projective variety \(\mathbb{L}_G(\wedge^3 V^\vee)^0\). By Proposition (3.3) we know that \(\Delta^0(V)\) is locally (in the classical topology) a codimension 1 submanifold of \(\mathbb{L}_G(\wedge^3 V^\vee)\); since \(\Delta^0(V) \cap \mathbb{L}_G(\wedge^3 V^\vee)^0 = \emptyset\) it follows that \(\Delta^0_\ast(V)\) is open dense in \(\Delta^0(V)\). We let \(T_\ast \subset T''\) be the set of \(t\) such that \(M_t \cong X_A\) for some \(A \in \Delta^0_\ast(V)\); since \(\Delta^0_\ast(V)\) is open dense in \(\Delta^0(V)\) also \(T_\ast\) is open dense in \(T''\). Let

\[
\mathbb{L}_G(\wedge^3 V^\vee)^0_\ast := \mathbb{L}_G(\wedge^3 V^\vee)^0 \cup \Delta^0_\ast(V). \tag{3.2.20}
\]

**Proposition 3.11.** Keep notation as above. Then \(\mathbb{L}_G(\wedge^3 V^\vee)^0_\ast\) is open in \(\mathbb{L}_G(\wedge^3 V^\vee)^0\) (for the classical topology) and \(\Delta^0_\ast(V)\) is a non-empty codimension-1 submanifold of \(\mathbb{L}_G(\wedge^3 V^\vee)^0_\ast\).

**Proof.** Let \(A \in \Delta^0_\ast(V)\). By definition of \(\Delta^0_\ast(V)\) there exists an open \(U \subset \mathbb{L}_G(\wedge^3 V^\vee)^0\) such that \(U \cap \Delta^0_\ast(V) = U \cap (\mathbb{L}_G(\wedge^3 V^\vee)^0 \setminus \mathbb{L}_G(\wedge^3 V^\vee)^0_\ast)\) and hence \(U \subset \mathbb{L}_G(\wedge^3 V^\vee)^0_\ast\); this proves that \(\mathbb{L}_G(\wedge^3 V^\vee)^0_\ast\) is open. We have already proved that \(\Delta^0_\ast(V)\) is a non-empty locally closed codimension-1 submanifold of \(\mathbb{L}_G(\wedge^3 V^\vee)^0\); this gives the second statement of the proposition. \(\square\)

We let

\[
\Delta^\infty(V^\vee) := \delta(\Delta^0_\ast(V)), \tag{3.2.21}
\]

\[
\mathbb{L}_G(\wedge^3 V^\vee)^\infty := \delta(\mathbb{L}_G(\wedge^3 V^\vee)^0_\ast). \tag{3.2.22}
\]

\[
(3.2.23)
\]

Of course every definition above has a “dual” definition obtained by substituting \(V^\vee\) to \(V\); thus we have \(\Delta^0_\ast(V^\vee) \subset \mathbb{L}_G(\wedge^3 V^\vee)^0_\ast\), \(\Delta^\infty(V^\vee) \subset \mathbb{L}_G(\wedge^3 V^\vee)^\infty\) etc. Let

\[
\mathbb{L}_G(\wedge^3 V^\vee)^2 := \mathbb{L}_G(\wedge^3 V^\vee)^0_\ast \cup \mathbb{L}_G(\wedge^3 V^\vee)^\infty = \mathbb{L}_G(\wedge^3 V^\vee)^0_\ast \cup \Delta^\infty(V^\vee) \cup \Delta^\infty(V^\vee). \tag{3.2.24}
\]

Let \(\mathcal{Y}(V^\vee) \subset \mathbb{L}_G(\wedge^3 V^\vee)^2 \times \mathbb{P}(V^\vee)\) be the tautological EPW-sextic; thus \(\mathcal{Y}(V^\vee)\cap \{B\} \times \mathbb{P}(V^\vee) = Y_B\). If \(B \in \Delta^\infty(V^\vee)\) then by Proposition (3.10) there is a unique point \(q_B \in Y_B\) of multiplicity strictly greater than 2; let

\[
\mathcal{Q}(V^\vee) := \{(B, q_B)\mid B \in \Delta^\infty(V^\vee)\}, \tag{3.2.25}
\]

\[
\mathcal{Y}(V^\vee)^2 := \mathcal{Y}(V^\vee) \setminus \mathcal{Q}(V^\vee). \tag{3.2.26}
\]

**Proposition 3.12.** There exists a double cover \(f : \mathcal{X}(V^\vee) \to \mathcal{Y}(V^\vee)^2\) with the following properties.
Thus if \( \dim(\mathcal{V}) \) replacing \( \mathcal{V} \) where \( \alpha \), \( \q \), \( \zeta \), and \( \in \) in particular we see that there exists a sheaf by Proposition (3.10) and hence \( \dim(H) \) is isomorphic to the natural double cover \( Y_B \rightarrow Y_B \).

(2) Let \( B \in \mathcal{L}(\mathcal{V}) \); then \( \pi^{-1}(B) \cong X_B \) and the map \( \pi^{-1}(B) \rightarrow Y_B \) defined by \( f \) is isomorphic to the natural double cover \( Y_B \rightarrow Y_B \).

(3) Let \( B \in \Delta^\infty_1(\mathcal{V}) \) and \( A := B^\perp \). Since \( A \in \Delta^\infty_1(\mathcal{V}) \) there exists \( t \in T \) such that \( X_A \cong M_t \) and \( Y_A \cong \Sigma_t \). Then \( \pi^{-1}(B) \cong (S^1_t \setminus \{P_t\}) \) and the map \( \pi^{-1}(B) \rightarrow (Y_B \setminus \{q_B\}) \) defined by \( f \) is isomorphic to the double cover \( (S^1_t \setminus \{P_t\}) \rightarrow (W_t \setminus |\mathcal{F}_3(2)|) \) given by the restriction of \( (3.2.7) \). (Recall that \( (Y_B \setminus \{q_B\}) \cong (W_t \setminus |\mathcal{F}_3(2)|) \) by Proposition (3.9)).

Proof. Let \( B \in \mathcal{L}(\mathcal{V}) \). Then \( Y_B \neq \mathbb{P}(\mathcal{V}) \) by Corollary (3.5); thus the map \( \lambda_B \) defined in Section (1) (with \( \mathcal{V} \) replacing \( \mathcal{V} \)) is non-zero and \( Y_B \) is the zero-scheme of \( \det(\lambda_B) \). Since \( Y_B \) is a Lagrangian degeneracy locus there exists locally in \( \mathbb{P}(\mathcal{V}) \) a symmetric map of vector-bundles giving a resolution of \( \text{coker}(\lambda_B) \), i.e. we can cover \( \mathbb{P}(\mathcal{V}) \) by open sets \( U \) such that on each \( U \) we have a locally-free resolution

\[
0 \rightarrow E_U \xrightarrow{\alpha_U} E_U' \xrightarrow{\text{coker}(\lambda_B)|_U} 0 \quad (3.2.27)
\]

where \( \alpha_U \) is a symmetric map of vector-bundles. Furthermore if \( F \hookrightarrow \mathcal{V} \) and \( \mathcal{O}_{\mathcal{P}(\mathcal{V})} \) is the Lagrangian sub-vector-bundle defined in Section (1) (with \( \mathcal{V} \) replacing \( \mathcal{V} \)) then there is an isomorphism \( F_p \cap B \cong \ker(\alpha_p) \) for all \( p \in U \). Thus if \( \dim(F_p \cap B) \geq r \) then \( \text{mult}_p(Y_B) \geq r \). Let \( p \neq q_B \). Then \( \text{mult}_p(Y_B) \leq 2 \) by Proposition (3.10) and hence \( \dim(F_p \cap B) \leq 2 \). Furthermore one of the following holds:

1. If \( \dim(F_p \cap B) = 1 \) then locally around \( p \) we have \( \text{coker}(\lambda_B) \cong i_*\mathcal{O}_{Y_B} \) where \( i : Y_B \hookrightarrow \mathbb{P}(\mathcal{V}) \) is the inclusion.

2. If \( \dim(F_p \cap B) = 2 \) there exist an open (in the classical topology) \( U \subset \mathbb{P}(\mathcal{V}) \) containing \( p \), functions \( x, y, z \in \text{Hol}(U) \) vanishing at \( p \) with linearly independent differentials and an exact sequence

\[
0 \rightarrow \mathcal{O}_U^2 \xrightarrow{M} \mathcal{O}_U \xrightarrow{\text{coker}(\lambda_B)|_U} 0 \quad (3.2.28)
\]

where \( M \) is the map defined by the matrix

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}.
\quad (3.2.29)
\]

In particular we see that there exists a sheaf \( \zeta_B \) on \( (Y_B \setminus \{q_B\}) \) such that outside \( q_B \) we have \( \text{coker}(\lambda_B) \cong i^*\zeta_B \) where \( i^* : (Y_B \setminus \{q_B\}) \hookrightarrow (\mathbb{P}(\mathcal{V}) \setminus \{q_B\}) \) is the inclusion. From the local description of \( \text{coker}(\lambda_B) \) given above we also get a canonical isomorphism of sheaves on \( (\mathbb{P}(\mathcal{V}) \setminus \{q_B\}) \):

\[
\text{Ext}^1(\text{coker}(\lambda_B), \mathcal{O}_{\mathbb{P}(\mathcal{V})}|_{\mathbb{P}(\mathcal{V}) \setminus \{q_B\}}) \cong i^* \mathcal{O}_B \otimes N_{Y_B/\mathbb{P}(\mathcal{V})}).
\quad (3.2.30)
\]

(See Proposition (4.3) of [13].) Let \( B' \subset \mathcal{V} \) be a Lagrangian subspace complementary to \( B \); thus we have a direct sum decomposition

\[
\mathcal{V} = B \oplus B'.
\quad (3.2.31)
\]
Then $\lambda_B$ can be identified with the map of vector-bundles $F \to B^\vee \otimes \mathcal{O}_{P(V^\vee)}$ associated to Decomposition (3.2.31). Let $\mu_B : F \to B \otimes \mathcal{O}_{P(V^\vee)}$ be the “other” map associated to Decomposition (3.2.31). The diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\lambda_B} & B^\vee \otimes \mathcal{O}_{P(V^\vee)} \\
\downarrow{\mu_B} & & \downarrow{\mu_B^\vee} \\
B \otimes \mathcal{O}_{P(V^\vee)} & \xrightarrow{\lambda_B^\vee} & F^\vee
\end{array}
$$

(3.2.32)

is commutative because $F \xrightarrow{(\mu_B, \lambda_B)} (B \oplus B^\vee) \otimes \mathcal{O}_{P(V^\vee)}$ is a Lagrangian embedding. The map $\lambda_B$ is an injection of sheaves because $Y_B \neq P(V^\vee)$ and hence also $\lambda_B^\vee$ is an injection of sheaves. Thus there is a unique $\beta_B : coker(\lambda_B) \to Ext^1(coker(\lambda_B), \mathcal{O}_{P(V^\vee)})$ making the following diagram commutative with exact horizontal sequences:

$$
\begin{array}{cccc}
0 & \to & F & \xrightarrow{\lambda_B} & B^\vee \otimes \mathcal{O}_{P(V^\vee)} & \to & coker(\lambda_B) & \to & 0 \\
\downarrow{\mu_B} & & \downarrow{\mu_B^\vee} & & \downarrow{\beta_B} & & \downarrow{\beta_B} & & 0 \\
0 & \to & B \otimes \mathcal{O}_{P(V^\vee)} & \xrightarrow{\lambda_B^\vee} & F^\vee & \to & Ext^1(coker(\lambda_B), \mathcal{O}_{P(V^\vee)}) & \to & 0
\end{array}
$$

(3.2.33)

By Isomorphism (3.2.30) we get that the restriction of $\beta_B$ to $(P(V^\vee) \setminus \{q_B\})$ defines a map of sheaves on $(Y_B \setminus \{q_B\})$

$$\zeta_B \to \zeta_B^{(6)}.$$

(3.2.34)

Since $F \xrightarrow{(\mu_B, \lambda_B)} (B \oplus B^\vee) \otimes \mathcal{O}_{P(V^\vee)}$ is an injection of vector-bundles the above map is an isomorphism - this follows from Claim (4.5) of [13]. Let $\xi_B := \zeta_B(-3)$; then (3.2.34) defines an isomorphism $\xi_B \xrightarrow{\sim} \xi_B'$ which is symmetric and hence it gives $\mathcal{O}_{Y_B \setminus \{q_B\}} \oplus \xi_B$ the structure of a commutative finite $\mathcal{O}_{Y_B \setminus \{q_B\}}$-algebra. Let $X_B := Spec(\mathcal{O}_{Y_B \setminus \{q_B\}})$ and $f_B : X_B \to (Y_B \setminus \{q_B\})$ be the structure map: clearly $f_B$ is finite of degree 2. The above construction is the analogue of the construction of the natural double cover $X_B \to Y_B$ for $B \in \mathbb{L}^3(V^\vee, 0)$, thus we have a double cover $f : X(V^\vee) \to Y_B$ such that Item (2) holds and such that for $B \in \Delta^\infty(V^\vee)$ we have $\pi^{-1}(B) = X_B$ and the map $\pi^{-1}(B) \to (Y_B \setminus \{q_B\})$ is the structure map $f_B$ defined above. It remains to prove that Items (1) and (3) hold. Let

$$C_B := \{p \in (singY_B \setminus \{q_B\}) \mid \dim(F_p \cap B) = 1\}$$

(3.2.35)

$$D_B := \{p \in (singY_B \setminus \{q_B\}) \mid \dim(F_p \cap B) = 2\}.$$  

(3.2.36)

Thus $(singY_B \setminus \{q_B\}) = C_B \bigsqcup D_B$. The local description given above of $coker(\lambda_B)$ near $p \in (Y_B \setminus \{q_B\})$ shows that both $C_B$ and $D_B$ are smooth and closed (in $(Y_B \setminus \{q_B\})$). Let $p \in (Y_B \setminus \{q_B\})$; the map $f_B$ behaves differently depending on whether $p \in C_B$ or $p \in D_B$: in fact (see [13])

(a) If $p \in C_B$ then $f_B$ is unramified over $p$.

(b) If $p \in D_B$ then $f_B$ is ramified over $p$ and $X_B$ is smooth at $f_B^{-1}(p)$.

Let’s prove that

$$X_B \cong (S_t^{[2]} \setminus P_t).$$

(3.2.37)
etale covering

(3.3 Extension of the local period map across $\Delta_\infty^V$)

We will prove that the local period map extends across $\Delta_\infty^V$. It is a surface; it follows that $D_B \neq \emptyset$ and by Item $\beta$ above we get that the etale covering

$$\left( X_B \setminus f_B^{-1}(\text{sing} Y_B) \right) \longrightarrow \left( Y_B \setminus \text{sing} Y_B \right)$$

is not trivial. On the other hand let $t \in T_s$ such that $X_{A_t} \cong M_t$ and $Y_{A_t} \cong \Sigma_t$. Let $f_t : S_{t}^{[2]} \rightarrow Y_B$ be the map defined by (3.2.7); then $f_t$ defines an etale double cover

$$\left( S_{t}^{[2]} \setminus f_t^{-1}(\text{sing} Y_B) \right) \longrightarrow \left( Y_B \setminus \text{sing} Y_B \right).$$

Now $S_{t}^{[2]}$ is simply connected and $f_t^{-1}(\text{sing} Y_B)$ has codimension 2 in $S_{t}^{[2]}$ hence $(S_{t}^{[2]} \setminus f_t^{-1}(\text{sing} Y_B))$ is simply connected. Thus $\pi_1(Y_B \setminus \text{sing} Y_B) \cong \mathbb{Z}/(2)$ and hence there is a unique non-trivial double cover of $(Y_B \setminus \text{sing} Y_B)$ and it is given by (3.2.39). Since (3.2.38) is a non-trivial double cover it follows that $C_B = \emptyset$ and that (3.2.37) holds. This proves Items (1)-(3) of the proposition.

3.3 Extension of the local period map across $\Delta_\infty^V$

We will prove that the local period map extends across $\Delta_\infty^V$. Let $\pi^0 : \mathcal{X}(V^*)_0 \rightarrow L\mathcal{G}(\wedge^3 V^*)_0$ be the tautological family of double EPW-sextics i.e. the restriction of the map $\pi$ of Proposition (3.12) to $\pi^{-1}(L\mathcal{G}(\wedge^3 V^*)_0)$. Since $\pi^0$ is proper it defines the variation of Hodge structures

$$(R^2\pi^0_*\mathbb{Z}, F^p)$$

where $F^0 \supset F^1 \supset F^2$ is the Hodge filtration of $(R^2\pi^0_*\mathbb{Z}) \otimes \mathcal{O}_{L\mathcal{G}(\wedge^3 V^*)_0}$ i.e. the fiber of $F^p$ over $A$ is

$$F^p_A = F^p H^2(X_A) := \bigoplus_{\nu' \geq p} H^{2-\nu'}(X_A).$$

Furthermore we have the symmetric section

$$B \in H^0((R^2\pi^0_*\mathbb{Q}) \otimes (R^2\pi^0_*\mathbb{Q}))$$

which gives the Beauville-Bogomolov bilinear form on $H^2(X_A)$ for every $A \in L\mathcal{G}(\wedge^3 V^*)_0$.

Proposition 3.13. There exist

1. a local system $\mathcal{H}(V^*)$ on $L\mathcal{G}(\wedge^3 V^*)_0$,
2. a decreasing filtration of $\mathcal{H}(V^*) \otimes \mathcal{O}_{L\mathcal{G}(\wedge^3 V^*)_0}$ by holomorphic sub-bundles $\mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2$,
3. a symmetric section $\mathcal{B} \in H^0(\mathcal{H}(V^*) \otimes \mathcal{H}(V^*) \otimes \mathbb{Q})$

such that the following hold:

(a) $(\mathcal{H}(V^*), \mathcal{F}^p)$ extends the variation of Hodge structures (3.3.1).

(b) Let $B \in \Delta_\infty^V$ and $t \in T_s$ such that $M_t \cong X_{B_t}$; then there exists an isomorphism of Hodge structures

$$(\mathcal{H}(V^*)_B, \mathcal{F}^p_B) \cong H^2(S_{t}^{[2]}).$$
(c) The restriction of $\tilde{B}$ to $\text{LG}(\wedge^3 V^\vee)^0$ is equal to $B$.

(d) Let $B \in \Delta^\infty(V^\vee)$ and $t \in T$ such that $M_t \cong X_{B^t}$; then Isomorphism (3.3.4) is an isometry between $(\mathbb{H}(V^\vee)_B, \tilde{B}_B)$ and $H^2(S^2_t)$ equipped with the Beauville-Bogomolov symmetric bilinear form.

Proof. Let $\mathcal{U} \subset \text{LG}(\wedge^3 V^\vee)^\mathbb{R}$ be an open ball. We assume that $\mathcal{U}$ is small: then there exists a hyperplane $H \subset \mathbb{P}(V^\vee)$ such that $q_B \notin H$ for all $B \in \mathcal{U} \cap \Delta^\infty(V^\vee)$. Let $\pi$ and $f$ be as in Proposition (3.12); we let

$$X(V^\vee)_\mathcal{U} := \pi^{-1}(\mathcal{U}),$$

$$Z(\mathcal{U}, H) := X(V^\vee)_\mathcal{U} \cap f^{-1}H. \quad (3.3.5)$$

Let $\rho: Z(\mathcal{U}, H) \to \mathcal{U}$ be given by the restriction of $f$. By our choice of $H$ the map $\rho$ is proper submersive with fibers smooth 3-folds. Thus we have a variation of Hodge structures $(R^2\rho_*\mathbb{Z}, F^p)$; we denote it by $(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))$. Let $B \in (\mathcal{U} \cap \Delta^\infty)$; then $\rho^{-1}(B)$ is an ample divisor on $X_B$ and hence we have a canonical isomorphism

$$(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))|_{(\mathcal{U} \cap \Delta^\infty)} \cong (R^2\rho_*\mathbb{Z}, F^p)|_{(\mathcal{U} \cap \Delta^\infty)}. \quad (3.3.6)$$

This shows that $(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))$ does not depend on the choice of $H$ and that the collection of $(\mathbb{H}(\mathcal{U}, H), F^p(\mathcal{U}, H))$ gives an extension $(\mathbb{H}(V^\vee), F^p)$ of the variation of Hodge structures (3.3.1). Now let’s prove Item (b). Let $B \in (\mathcal{U} \cap \Delta^\infty(V^\vee))$ and let $t$ be as in Item (b). Let $f_t: S^2_t \to W_t = Y_B$ be the map given by (3.2.7). Then $\rho^{-1}(B) = f_t^{-1}H$ and since $f_t$ is semi-small the restriction map $H^2(S^2_t) \to H^2(f_t^{-1}H)$ is an isomorphism of (integral) Hodge structures; this proves Item (b) because $(\mathbb{H}(V^\vee)_B, F^p_B)$ is isomorphic to the Hodge structure on $H^2(f_t^{-1}H)$ by definition. We define $\tilde{B}$ as follows. Let $\mathcal{U}$ and $\rho: Z(\mathcal{U}, H) \to \mathcal{U}$ be as above; let $Z_B := \rho^{-1}(B)$. For $B \in \mathcal{U}$ we have the Lefschetz decomposition

$$\mathbb{H}(V^\vee)_B = H^2(Z_B; \mathbb{Q}) = Qc_1(L_B)|_{Z_B} \oplus H^2(Z_B; \mathbb{Q})_{\text{prim}} \quad (3.3.8)$$

where $L_B$ is the tautological ample line-bundle on $X_B$. We let $\tilde{B}(\mathcal{U}, H)_B$ be the symmetric bilinear form on $\mathbb{H}(V^\vee)_B$ characterized by the following requirements:

(a) Decomposition (3.3.8) is orthogonal for $\tilde{B}(\mathcal{U}, H)_B$.

(b) $\tilde{B}_B(c_1(L_B)|_{Z_B}, c_1(L_B)|_{Z_B}) = 2$.

(c) $\tilde{B}_B(c_1(L_B)|_{Z_B}, c_1(L_B)|_{Z_B}) = 2$.

(d) $\tilde{B}_B(c_1(L_B)|_{Z_B}, c_1(L_B)|_{Z_B}) = 2$.

Since the Lefschetz decomposition is flat for the Gauss-Manin connection we have a well-defined section $\tilde{B}(\mathcal{U}, H) \in H^0(\mathbb{H}(V^\vee)|_{\mathcal{U}})$ with value $\tilde{B}(\mathcal{U}, H)_B$ at $B \in \mathcal{U}$. Let $B \in (\mathcal{U} \setminus \Delta^\infty(V^\vee))$ and let $c: Z_B \hookrightarrow X_B$ be the inclusion. If $\xi_1, \xi_2, \xi_3 \in H^2(X_B)$ then

$$\int_{Z_B} c_1(L_B)\wedge \xi_1 \wedge \xi_2 \wedge \xi_3 = \int_{X_B} c_1(L_B)\wedge \xi_1 \wedge \xi_2 \wedge \xi_3 = \frac{1}{2} \sum_{\sigma \in S_3} (c_1(L_B), \xi_{\sigma(1)}) (\xi_{\sigma(2)}, \xi_{\sigma(3)}). \quad (3.3.9)$$

It follows from this that $\tilde{B}(\mathcal{U}, H)$ does not depend on $H$ and that the collection of $\tilde{B}(\mathcal{U}, H)$’s defines an extension of $B$. Item (d) holds because Formula (3.3.9) holds if we replace $X_B$ by $S^2_t$.
The map $\delta$ defines an isomorphism
\[
\Lambda \xrightarrow{\delta} \Lambda
\]
(3.3.10)
From now on we will denote by $\delta$ what is actually the restriction of $\delta$ to $\Lambda$. Let $U \subset \Lambda$ be an open set: then we have two local systems, namely
\[
\mathbb{H}(V)|_U, \quad \mathbb{H}(V')|_{\delta(U)}.
\]
(3.3.11)
Definition 3.14. Assume that both local systems (3.3.11) are trivial. A marking of $H$ is an isomorphism $\Psi: \mathbb{H}(V)|_U \sim \mu U \times \mu$ such that:

1. $\tilde{\Phi}(x, y) = (\Psi(x), \Psi(y))$ for every $A \in U$ and $x, y \in \mathbb{H}(V)$, and
2. $\Psi$ sends the flat section $A \mapsto c_1(L_A)_\mathcal{Y}(U, H)$ (notation as in the proof of Proposition (3.13)) to $U \times u$ where $u$ is given by (1.0.8).

We define similarly a marking $\Psi$ of $\mathbb{H}(V')|_{\delta(U)}$.

Of course if $U$ is a small open ball then both local systems (3.3.11) are trivial. Let $\Phi$ and $\Psi$ be markings of $\mathbb{H}(V)|_U$ and $\mathbb{H}(V')|_{\delta(U)}$ respectively. Then we have holomorphic local period maps
\[
U \xrightarrow{\mathcal{P}_\Phi} \mathcal{D}_2 \quad \delta(U) \xrightarrow{\mathcal{P}_\Phi} \mathcal{D}_2.
\]
(3.3.12)
The restrictions of $\mathcal{P}_\Phi$ and $\mathcal{P}_\Psi$ to $U \setminus \Delta^\infty_1(V)$ and $\delta(U) \setminus \Delta^\infty_1(V')$ respectively are local period maps for the families of double EPW-sextics parametrized by $\mathbb{L}(\Lambda^3 V)^0$ and $\mathbb{L}(\Lambda^3 V')^0$ respectively. We will be interested in comparing $\mathcal{P}_\Phi(A)$ and $\mathcal{P}_\Psi(A)$ for $A \in \Delta^\infty_1(V)$ - thus $A^\perp \notin \mathbb{L}(\Lambda^3 V)^0$. Let $t \in T_*$ be such that $M_t \cong X_A$. By (3.3.4) the marking $\Phi$ defines a marking $\Phi(A^\perp)$ of $(S_1^2, f_t(\mathcal{O}_{Y_t}(1)))$ and
\[
\mathcal{P}_\Phi(A^\perp) = \mathcal{P}_\Phi(A^\perp)(S_1^2, f_t(\mathcal{O}_{Y_t}(1))).
\]
(3.3.13)

4 Proof of Theorem (1.1)

If $A \in \mathbb{L}(\Lambda^3 \mathcal{V})^{00}$ we have smooth double covers $f_A: X_A \to Y_A$ and $f_A^\perp: X_{A^\perp} \to Y_{A^\perp}$: we will show that $X_A$ and $X_{A^\perp}$ are “isogenous”. Given a small open $U \subset \mathbb{L}(\Lambda^3 V)^0$ we may consider markings $\Phi$ and $\Psi$ of $\mathbb{H}(V)|_U$ and $\mathbb{H}(V')|_{\delta(U)}$ respectively and the associated local period maps $\mathcal{P}_\Phi$ and $\mathcal{P}_\Psi$. We will show that locally near $\Delta^\infty_1(V)$ we may choose $\Psi$ and $\Phi$ so that $\mathcal{P}_\Phi \circ \delta$ is either $r \circ \mathcal{P}$ or the composition of $r \circ \mathcal{P}$ with a certain specific reflection. In the final subsection we will rule out the latter case by considering the monodromy action; by analytic continuation this will prove Theorem (1.1).

4.1 Isogeny between $X_A$ and $X_{A^\perp}$

Let $A \in \mathbb{L}(\Lambda^3 \mathcal{V})^{00}$ and set $L_A := f_A^* \mathcal{O}_{Y_A}(1)$, $L_{A^\perp} := f_{A^\perp}^* \mathcal{O}_{Y_{A^\perp}}(1)$. Let
\[
H^2(X_A)_{\text{prim}} := c_1(L_A)^\perp \subset H^2(X_A),
\]
(4.1.1)
\[
H^2(X_{A^\perp})_{\text{prim}} := c_1(L_{A^\perp})^\perp \subset H^2(X_{A^\perp}).
\]
(4.1.2)
Proposition 4.1. Let $A \in \operatorname{LG}(\wedge^3 V)^{\text{prim}}$. There exists an isomorphism of rational Hodge structures
\begin{equation}
\varphi_A : H^2(X_A)_{\text{prim}} \sim \to H^2(X_{A^+})_{\text{prim}}
\end{equation}
well-defined up to $\pm 1$ and such that for $\gamma \in H^2(X_A)_{\text{prim}}$
\begin{equation}
(\gamma, \gamma)_{X_A} = (\varphi_A(\gamma), \varphi_A(\gamma))_{X_{A^+}}.
\end{equation}

Proof. We recall that $Y_{A^+} = Y_{A^+}'$, see Corollary (3.5). Let $\Gamma_A \subset Y_A \times Y_{A^+}$ be the closure of the Gauss maps:
\begin{equation}
\Gamma_A := \{(p, T_p Y_A) | p \in Y_A^{\text{prim}}\} = \{(T_q Y_{A^+}, q) | q \in Y_{A^+}^{\text{prim}}\}.
\end{equation}
Since $A \in \operatorname{LG}(\wedge^3 V)^0$ the germ of $Y_A$ at each of its singular points is isomorphic to the product of $(\mathbb{C}^2, 0)$ and an $A_1$-singularity (see Proposition (2.8) of [13]) and similarly for $Y_{A^+}$. Thus the projection $\Gamma_A \to Y_A$ is identified with the blow-up $\tilde{Y}_A \to Y_A$ of $\text{sing}(Y_A)$. Similarly the projection $\Gamma_A \to Y_{A^+}$ is identified with the blow-up $\tilde{Y}_{A^+} \to Y_{A^+}$ of $\text{sing}(Y_{A^+})$. Thus $\Gamma_A$ defines an isomorphism $\tilde{Y}_A \sim \to \tilde{Y}_{A^+}$ and hence it gives an isomorphism of integral Hodge structures
\begin{equation}
H^4(\tilde{Y}_A) \sim \to H^4(\tilde{Y}_{A^+}).
\end{equation}
The cohomology groups $H^4(\tilde{Y}_A)$ and $H^4(X_A)$ are related as follows. Let $\phi_A : X_A \to X_A$ be the involution covering $f_A : X_A \to Y_A$. Let $F_A \subset X_A$ be the fixed locus of $\phi_A$; this is a Lagrangian smooth surface in $X_A$ because $\phi_A$ is anti-symplectic. Let $X_A \to X_A$ be the blow-up of $F_A$; we have an isomorphism of integral Hodge structures
\begin{equation}
H^4(X_A) \cong H^4(X_A) \oplus H^2(F_A)(-1).
\end{equation}
The involution $\phi_A$ lifts to an involution $\tilde{\phi}_A : \tilde{X}_A \to \tilde{X}_A$ and $\tilde{Y}_A \cong \tilde{X}_A / \langle \tilde{\phi}_A \rangle$. Thus we have an isomorphism of rational Hodge structures
\begin{equation}
H^4(\tilde{Y}_A) \cong H^4(\tilde{X}_A)^{(\tilde{\phi}_A)} \cong H^4(X_A)^{(\phi_A)} \oplus H^2(F_A)(-1).
\end{equation}
Since $X_A$ is a deformation of $(K3)^{[2]}$ we have an isomorphism of rational Hodge Structures $\operatorname{Sym}^2 H^2(X_A) \sim \to H^4(X_A)$ defined by cup-product. The action of $\phi_A$ on $H^2(X_A)$ has $(+1)$-eigenspace generated by $c_1(L_A)$ and $(-1)$-eigenspace equal to $H^2(X_A)_{\text{prim}}$ thus we get an isomorphism of rational Hodge structures
\begin{equation}
H^4(X_A)^{(\phi_A)} = \mathbb{C} c_1(L_A)^2 \oplus \operatorname{Sym}^2 H^2(X_A)_{\text{prim}}.
\end{equation}
The right-hand side of the above equality contains a rational $(2, 2)$ class $q_A^\vee$ defined by “inverting”the Beauville-Bogomolov bilinear form (see Section (3) of [14]); let $W_A := H^2(X_A)_{\text{prim}} \cap (q_A^\vee)^\perp$. One has a decomposition of rational Hodge Structures (Claim (3.1) of [14])
\begin{equation}
\mathbb{C} c_1(L_A)^2 \oplus \mathbb{C} q_A^\vee \oplus W_A.
\end{equation}
Of course we have analogous notions for $X_{A^+}$ and hence (4.6) defines an isomorphism of rational H.S.’s
\begin{equation}
\mathbb{C} c_1(L_A)^2 \oplus \mathbb{C} q_A^\vee \oplus W_A \oplus H^2(F_A)(-1) \cong \\
\mathbb{C} c_1(L_{A^+})^2 \oplus \mathbb{C} q_{A^+}^\vee \oplus W_{A^+} \oplus H^2(F_{A^+})(-1).
\end{equation}
If $A$ is very general (outside a countable union of proper analytic subsets of $\mathbb{LG}(\wedge^3 V)^0 \cap \delta^{-1}\mathbb{LG}(\wedge^3 V)^0$) then $W_A$ and $W_{A^+}$ are both indecomposable rational Hodge Structures - see Section (3) of [14]; since they contain the non-zero components $H^{4,0}(X_A)$ and $H^{4,0}(X_{A^+})$ the above isomorphism defines an isomorphism of rational H.S.’s

$$h_A: W_A \xrightarrow{\sim} W_{A^+}. \quad (4.1.12)$$

Let $Q_A \subset H^2(X_A)_{prim}$ and $Q_{A^+} \subset H^2(X_{A^+})_{prim}$ be the cones of isotropic classes (with respect to the Beauville-Bogomolov bilinear form). Let

$$H^2(X_A)_{prim} \xrightarrow{\nu_A} \text{Sym}^2 H^2(X_A)_{prim} \quad (4.1.13)$$

be the (affine) Veronese map; we define similarly $\nu_{A^+}$. Then

$$W_A = \text{span}_{\nu_A}(Q_A), \quad W_{A^+} = \text{span}_{\nu_{A^+}}(Q_{A^+}). \quad (4.1.14)$$

We claim that

$$h_A(\nu_A(Q_A)) = \nu_{A^+}(Q_{A^+}). \quad (4.1.15)$$

In fact let $U \subset \mathbb{LG}(\wedge^3 V)^0 \cap \delta^{-1}\mathbb{LG}(\wedge^3 V)^0$ be an open ball containing $A$. The Gauss-Manin connection gives identifications

$$H^2(X_B) \cong H^2(X_A), \quad H^2(X_{B^+}) \cong H^2(X_{A^+}) \quad (4.1.16)$$

for every $B \in U$. Gauss-Manin gives also identifications $W_B \cong W_A$ and $W_{B^+} \cong W_{A^+}$ respectively. The isomorphism $h_B$ is flat for the Gauss-Manin connection hence it is identified with a (constant) map $h: W_A \rightarrow W_{A^+}$. For $B \in U$ let $\sigma_B, \sigma_{B^+}$ be symplectic forms on $X_B$ and $X_{B^+}$ respectively. Since $h_B$ is an isomorphism of H.S.’s we have

$$h(\sigma_B^2) = [\sigma_{B^+}^2]. \quad (4.1.17)$$

Now $\sigma_B \in Q_A$ and $\sigma_{B^+} \in Q_{A^+}$ - here we make the identifications (4.1.16) - and as $B$ varies in $U$ both $[\sigma_B]$ and $[\sigma_{B^+}]$ fill out non-empty open (in the classical topology) subsets of $\mathbb{P}(Q_A)$ and $\mathbb{P}(Q_{A^+})$ respectively. Since $\mathbb{P}(Q_A)$ and $\mathbb{P}(Q_{A^+})$ are non singular quadrics any non-empty open subset is Zariski-dense and hence (4.1.17) proves (4.1.15). It follows from (4.1.15) that there exists a linear map (4.1.3) well-defined up to $\pm 1$ such that for $\alpha \in Q_A$ we have $h_A(\alpha^2) = g_A(\alpha)^2$. By (4.1.15) we have

$$g_A(\alpha) \in Q_{A^+} \text{ if and only if } \alpha \in Q_A. \quad (4.1.18)$$

The rationality of $g_A$ follows from the fact that $h_A$ is defined over $\mathbb{Q}$ and $\nu_A, \nu_{A^+}$ give bijective maps between $\mathbb{P}(Q_A)(\mathbb{Q}), \nu_A(\mathbb{P}(Q_A))(\mathbb{Q})$ and $\mathbb{P}(Q_{A^+})(\mathbb{Q}), \nu_{A^+}(\mathbb{P}(Q_{A^+}))(\mathbb{Q})$ respectively. Finally let’s prove that Equation (4.1.4) holds. First we show that

$$(\alpha, \beta)_{X_A} = (g_A(\alpha), g_A(\beta))_{X_{A^+}}, \quad \alpha, \beta \in Q_A. \quad (4.1.19)$$

Since (4.1.11) respects the intersection forms we have

$$\int_{X_A} \alpha^2 \wedge \beta^2 = \int_{X_{A^+}} h_A(\alpha^2) \wedge h_A(\beta^2) = \int_{X_{A^+}} g_A(\alpha)^2 \wedge g_A(\beta)^2. \quad (4.1.20)$$
Corollary 4.2. Let $U \subset \mathcal{LG}(\Lambda^3 V)\sharp$ be a small open ball and let $\Psi, \Phi$ be markings of $H(V|_U)$ and $H(V^\vee)|_{\delta(U)}$ respectively - see Definition (3.14). Let $\mathcal{A} \in \mathcal{U}$ be a reference point: the monodromy representation of $\pi_1(\mathcal{LG}(\Lambda^3 V), \mathcal{A})$ on $H(V)$ determines via $\Psi_\mathcal{A}$ a monodromy representation $\pi_1(\mathcal{LG}(\Lambda^3 V), \mathcal{A}) \to \text{Stab}(u)$ where $\text{Stab}(u) < O(\Lambda)$ is the subgroup fixing the element $u$ given by (1.0.8). An element of the image of the monodromy representation is a $\Psi_\mathcal{A}$-monodromy operator. Similarly we have a monodromy representation of $\pi_1(\mathcal{LG}(\Lambda^3 V^\vee), \overline{\mathcal{A}})$ on $H(V^\vee)$; this determines via $\Phi_{\overline{\mathcal{A}}}$ a monodromy representation $\pi_1(\mathcal{LG}(\Lambda^3 V^\vee), \overline{\mathcal{A}}) \to \text{Stab}(\overline{u})$. An element of the image of this second monodromy representation is a $\Phi_{\overline{\mathcal{A}}}$-monodromy operator. Here and in the rest of the paper we will adopt the following conventions. First we view both $O(\Lambda)$ and $O(\Lambda \otimes \mathbb{Q})$ as subgroups of $O(\Lambda \otimes \mathbb{C})$. Secondly if $\gamma \in O(\Lambda \otimes \mathbb{C})$ we denote by $\tilde{\gamma} \in O(\Lambda \otimes \mathbb{C})$ the isometry which fixes $u$ and equals $\gamma$ on $\Lambda$. As a rule letters decorated by a tilde denote elements of $O(\Lambda \otimes \mathbb{C})$, letters with no tilde denote elements of $O(\Lambda \otimes \mathbb{C})$.

**Corollary 4.2.** Let $U \subset \mathcal{LG}(\Lambda^3 V)\sharp$ be a small open ball and let $\Psi, \Phi$ be markings of $H(V|_U)$ and $H(V^\vee)|_{\delta(U)}$ respectively. Let $\mathcal{P}_\Psi, \mathcal{P}_\Phi$ be the local period maps (3.3.12). There exists $g \in O(\Lambda \otimes \mathbb{Q})$ well-determined up to $\pm 1$ such that

$$\mathcal{P}_\Psi(A) = g \circ \mathcal{P}_\Psi(A)$$

(4.1.27)
for all $A \in \mathcal{U}$. Let $\bar{\gamma} \in \text{Stab}(u)$ be a $\Psi_{\Lambda}$-monodromy operator; then $\bar{g} \circ \bar{\gamma} \circ \bar{g}^{-1}$ is a $\Phi_{\Lambda}$-monodromy operator, in particular $g \circ \gamma \circ g^{-1} \in O(\Lambda)$.

**Proof.** Equation (4.1.27) holds on $\mathcal{U} \cap LG(\wedge^3 V)^00$ by Proposition (4.1) and flatness of $g_A$; by continuity Equation (4.1.27) holds on all of $\mathcal{U}$. The statement about monodromy operators holds by flatness of $g_A$. \hfill \Box

### 4.2 Restriction to $\Delta^0(V)$ of the local period maps

Our next task is to analyze the restriction to $\Delta^0(V)$ of the local period maps $\mathcal{P}_\Psi$ and $\mathcal{P}_\Phi$.

**Proposition 4.3.** Let $\mathcal{U} \subset LG(\wedge^3 V)^{1}$ be a small open ball. There is a choice of markings $\Psi$ and $\Phi$ of $H(V)|_\mathcal{U}$ and $H(V^*)|_\mathcal{U}$ respectively such that

$$\mathcal{P}_\Psi(\Delta^0(V) \cap \mathcal{U}) = (e_1 + 2e_2)^{\perp} \cap \mathcal{P}_\Phi(\mathcal{U}) \quad (4.2.1)$$

where $e_1, e_2 \in \Lambda$ are as in Section (1). Furthermore

$$\mathcal{P}_\Phi(A^{\perp}) = r \circ \mathcal{P}_\Phi(A), \quad A \in \Delta^0(V) \quad (4.2.2)$$

where $r$ is the involution defined by (2.2.12).

**Proof.** First we embed the lattice $\tilde{\Lambda}$ in a unimodular lattice as follows. Let $\tilde{\Lambda} := U^+ \oplus (-E_8)^2$. Let $U_1 < \tilde{\Lambda}$ be one of the hyperbolic lattices, let $z \in U_1$ be a vector of square 2 and $e_2$ be a generator of $z^{\perp} \cap U_1$. Then we have an isomorphism

$$z^{\perp} \cong \tilde{\Lambda} \quad (4.2.3)$$

and we can choose it so that it matches the present $e_2$ with the vector $e_2$ appearing in (1.0.9): we fix such an isomorphism once and for all. Let $u, e_1 \in \Lambda = z^{\perp}$ as in Section (1); then \{$(u \pm e_1)/2, (z \pm e_2)/2$\} $\subset \tilde{\Lambda}$. Furthermore the sublattices \{(u + e_1)/2, (u - e_1)/2\} and \{(z + e_2)/2, (z - e_2)/2\} are orthogonal hyperbolic planes. Thus we have an orthogonal decomposition

$$\tilde{\Lambda} \equiv \{(u + e_1)/2, (u - e_1)/2\} \oplus \{(z + e_2)/2, (z - e_2)/2\} \oplus U^2 \oplus (-E_8)^2. \quad (4.2.4)$$

Now we pass to the geometry. Let $T_s$ be as in Subsection (3.2). Let $A \in \Delta^0(V) \cap \mathcal{U}$; by definition there exists $t \in T_s$ such that

$$M_t \cong X_A. \quad (4.2.5)$$

Since $\mathcal{U}$ is a small open ball there exists a small open ball $\mathcal{V} \subset T_s$ such that if $t \in \mathcal{V}$ then (4.2.5) holds for some $A \in \Delta^0(V) \cap \mathcal{U}$ and conversely if $A \in \Delta^0(V) \cap \mathcal{U}$ then there exists $t \in \mathcal{V}$ such that (4.2.5) holds. Let $\kappa: S \to T_s$ be the tautological family of K3 surfaces parametrized by $T_s$. Let $t \in T_s$. Let $S_t = \kappa^{-1}(t), D_t$ etc. be as in Subsection (3.1) and $A_t \in \Delta^0(V) \cap \mathcal{U}$ such that $M_t \cong X_{A_t}$. Let $f_t: M_t \to Y_{A_t}$ be the double cover. Let $v_t, w_t \in H^*(S_t)$ be given by

$$v_t := 2 + c_1(D_t) + 2\eta_t, \quad w_t := 1 - \eta_t \quad (4.2.6)$$

where $\eta_t \in H^4(S_t; \mathbb{Z})$ is the orientation class. The Mukai map

$$\theta_{v_t}: v_t^{\perp} \to H^2(M_t) \quad (4.2.7)$$

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is an isometry of Hodge structures - see Subsection (3.1). Furthermore one has
\[ c_1(f_t^* L_{A_t}) = \theta_{v_t}(\eta_t - 1) = \theta_{v_t}(-w_t^\vee). \] (4.2.8)

(See the line preceding (3.1.7) for the definition of \( w_t^\vee \).) The local system \( R^2 \kappa_* Z|_V \) is trivial because \( V \) is a small open ball. Thus there exist sections \( \alpha, \beta \in \Gamma(R^2 \kappa_* Z|_V) \) such that \( c_1(D_t) = \alpha_t + 5\beta_t \) for all \( t \in V \). We define a trivialization
\[ R\kappa_* Z|_V = (R^0 \kappa_* Z \oplus R^2 \kappa_* Z \oplus R^4 \kappa_* Z)|_V \xrightarrow{\theta} V \times \Lambda \] (4.2.9)
as follows. For \( t \in V \) let
\[ \Upsilon_t(1) := -\frac{u}{2} + e_1/2 - z + e_2, \]
\[ \Upsilon_t(\eta_t) := \frac{u}{2} + e_1/2 - z + e_2, \]
\[ \Upsilon_t(\alpha_t) := -2e_1 + 5z/2 - 3e_2/2, \]
\[ \Upsilon_t(\beta_t) := \frac{z}{2} - e_2/2 \] (4.2.10)
and let
\[ \Psi_t = \{ (u, \eta, \alpha, \beta, 1) : (u \pm e_1)/2, (z \pm e_2)/2 \} \]
be an arbitrary isometry - notice that \( \Upsilon_t \) is an isometry. The trivialization (4.2.9) is defined to have value \( \Upsilon_t \) at \( t \in V \).

Now notice that \( \Upsilon_t(v_t) = z \) and hence we have an isometry \( \Upsilon_t \circ \theta_{v_t}^{-1} : H^2(M_t) \cong \mathbb{P} \) for all \( t \in V \). Since \( \Upsilon_t(-w_t^\vee) = u \) Equation (4.2.8) gives that
\[ \Upsilon_t \circ \theta_{v_t}^{-1}(c_1(f_t^* L_{A_t})) = u. \] (4.2.12)

Hence \( \Upsilon_t \circ \theta_{v_t}^{-1} \) defines a marking of \( (M_t, f_t^* L_{A_t}) \) for every \( t \in V \); since \( \mathbb{H}(V)|_U \) is trivial there exists a marking \( \Psi : \mathbb{H}(V)|_U \longrightarrow U \times \Lambda \) such that
\[ \Psi_{A_t} = \Upsilon_t \circ \theta_{v_t}^{-1}, \quad t \in V. \] (4.2.13)

Equation (4.2.1) follows from Equation (3.1.11) and the equality
\[ \Psi_{A_t}(\theta_{v_t}(5 + 2c_1(D_t) + 5\eta_t)) = \Upsilon_t(5 + 2c_1(D_t) + 5\eta_t) = c_1 + 2e_2. \] (4.2.14)

Next we define a marking \( \Phi \) for \( \mathbb{H}(V)|_U \). By (3.3.4) this will be equivalent to a marking of \( S^{[2]}_1 \), hence we first recall the description of \( H^2(S^{[2]}_1) \). We notice that \( w_t \) is the Mukai vector (see (3.1.1)) of any ideal sheaf \( I_Z \) where \( [Z] \in S^{[2]}_1 \); Mukai’s map
\[ \theta_{w_t} : w_t^\vee \rightarrow H^2(S^{[2]}_1) \] (4.2.15)
is an isomorphism of polarized Hodge structures and an isometry. Let \( g_t : S^{[2]}_1 \rightarrow Y_{A_t} \) be the map defined by (3.2.7); by Subsection (5.3) of [12]
\[ c_1(g_t^* O_{Y_{A_t}}(1)) = \theta_{w_t}(-2 + c_1(D_t) - 2\eta_t) = \theta_{w_t}(-v_t^\vee). \] (4.2.16)

We define a trivialization
\[ R\kappa_* Z|_V = (R^0 \kappa_* Z \oplus R^2 \kappa_* Z \oplus R^4 \kappa_* Z)|_V \xrightarrow{\Theta} V \times \Lambda \] (4.2.17)
as follows. For \( t \in \mathcal{V} \) let

\[
\begin{align*}
\Theta_t(1) & := u - e_1 + z/2 - e_2/2 \\
\Theta_t(q_t) & := u - e_1 - z/2 - e_2/2 \\
\Theta_t(\alpha_t) & := 5u/2 - 3e_1/2 - 2e_2 \\
\Theta_t(\beta_t) & := u/2 - e_1/2
\end{align*}
\] (4.2.18)

and let the restriction of \( \Theta_t \) to \( \{ 1, q_t, \alpha_t, \beta_t \} \) be equal to the restriction of \( \Upsilon_t \). A straightforward computation shows that \( \Theta_t \) is an isometry. The trivialization (4.2.17) is defined to have value \( \Theta_t \) at \( t \in \mathcal{V} \). Now notice that \( \Theta_t(w_t) = z \) and hence we have an isometry \( \Theta_t \circ \theta_w^{-1} : H^2(S_t^{[2]}) \rightarrow z\perp = \Lambda \).

Since \( \Theta_t(-v_t^J) = u \) Equation (4.2.16) gives that

\[
\Theta_t \circ \theta_{v_t}^{-1}(c_1(g_\gamma^* \mathcal{O}_{\mathcal{Y}_{\Lambda_t}^\gamma})) = u. \] (4.2.19)

Hence \( \Theta_t \circ \theta_{v_t}^{-1} \) defines a marking of \((S_t^{[2]}, g_\gamma^* \mathcal{O}_{\mathcal{Y}_{\Lambda_t}^\gamma}(1)) \) for every \( t \in \mathcal{V} \); by (3.3.4) and triviality of \( \mathcal{H}(V^\vee)_{\delta(U)} \) there exists a marking \( \Phi : \mathcal{H}(V^\vee)_{\delta(U)} \rightarrow \mathcal{U} \times \Lambda \) such that

\[
\Phi_{\Lambda_t} = \Theta_t \circ \theta_{v_t}^{-1}, \quad t \in \mathcal{V}. \] (4.2.20)

Now let's prove (4.2.2). Equation (4.2.2) is equivalent to

\[
\mathcal{P}_{\Theta_t \circ \theta_{v_t}^{-1}}((S_t^{[2]}, g_\gamma^* \mathcal{O}_{\mathcal{Y}_{\Lambda_t}^\gamma}(1))) = r \circ \mathcal{P}_{\Theta_t \circ \theta_{w_t}^{-1}}((M_t, f_t^* L_A)), \quad t \in \mathcal{V} \] (4.2.21)

by Equation (3.3.13). Since \( \theta_{v_t} \) and \( \theta_{w_t} \) are isomorphism of Hodge structures the above equation may be rewritten as

\[
\Theta_t(H^{2,0}(S_t)) = r \circ \Upsilon_t(H^{2,0}(S_t)), \quad t \in \mathcal{V}. \] (4.2.22)

For \( t \in \mathcal{V} \) let

\[
\begin{array}{c}
H^*(S_t; \mathbb{Q}) \\
\gamma
\end{array} \xrightarrow{\Xi_t} \begin{array}{c}
H^*(S_t; \mathbb{Q}) \\
-\gamma^\vee + \frac{1}{2}(\gamma^\vee, v_t + w_t)(v_t + w_t)
\end{array} \] (4.2.23)

i.e. the composition of the isometry \( \gamma \mapsto \gamma^\vee \) and the reflection which is (+1) on \( \mathbb{Q}(v_t + w_t) \) and (-1) on \( (v_t + w_t)^\perp \); thus \( \Xi_t \) is a rational (not integral !) Hodge isometry. A straightforward computation shows that \( \Theta_t = r \circ \Upsilon_t \circ \Xi_t \). Thus (4.2.22) follows at once from \( \Xi_t(H^{2,0}(S)) = H^{2,0}(S) \).

If \( \gamma \in \Lambda \otimes \mathbb{Q} \) is non-isotropic we let

\[
\begin{array}{c}
\Lambda \otimes \mathbb{Q} \\
x
\end{array} \xrightarrow{r_{\gamma}} \begin{array}{c}
\Lambda \otimes \mathbb{Q} \\
x + \frac{1}{2}(x, \gamma)(x, \gamma)
\end{array} \] (4.2.24)

be the reflection with (+1)-eigenspace \( \mathbb{Q}\gamma \) and (-1)-eigenspace \( \gamma^\perp \). Let

\[
\zeta := e_1 + 2e_2. \] (4.2.25)

**Corollary 4.4.** Keep notation and assumptions of Proposition (4.3). Let \( \Psi, \Phi \) be the markings of Proposition (4.3). Then:

1. \( \mathcal{P}_{\Psi}(A^\perp) = r \circ \mathcal{P}_{\Phi}(A) \) for all \( A \in \mathcal{U} \) or
(2) $P_\Phi(A^\perp) = r \circ r_\zeta \circ P_\Phi(A)$ for all $A \in \mathcal{U}$.

**Proof.** By Corollary (4.2) there exists $g \in O(\Lambda \otimes \mathbb{Q})$ such that $P_\Phi(A^\perp) = g \circ P_\Phi(A)$ for all $A \in \mathcal{U}$. By Proposition (4.3) $r^{-1} \circ g$ fixes the points of $\zeta^\perp \cap D_2$: it follows that $r^{-1} \circ g$ fixes $\zeta^\perp \subset P(\Lambda \otimes \mathbb{C})$. Since $r \in O(\Lambda)$ we have $r^{-1} \circ g \in O(\Lambda \otimes \mathbb{Q})$: thus we get that $r^{-1} \circ g|_{\zeta^\perp} = \pm Id_{\zeta^\perp}$. (4.2.26)

Since $\zeta$ is non-isotropic (in fact $(\zeta, \zeta) = -10$) and $r^{-1} \circ g \in O(\Lambda \otimes \mathbb{Q})$ we get that $r^{-1} \circ g(\zeta) = \pm \zeta$. It follows that $r^{-1} \circ g = \pm Id$ or $r^{-1} \circ g = \pm r_\zeta$. Since $-Id$ acts trivially on $D_2 \subset P(\Lambda)$ the corollary follows. \qed

### 4.3 The proof

We will apply the monodromy statement of Corollary (4.2) in order to show that Item (2) of Corollary (4.4) can not hold. We will use the following result.

**Claim 4.5.** Let

$$\xi = a_1 e_1 + a_2 e_2 + \nu \in \Lambda$$  

be a $(-2)$-vector i.e. $(\xi, \xi) = -2$ and assume that $r_\zeta \circ r_\xi \circ r_\zeta \in O(\Lambda)$. Then

$$(\xi, \zeta) \equiv 0 \pmod{5}.$$  

**Proof.** A tedious straightforward computation gives that

$$r_\zeta \circ r_\xi \circ r_\zeta(e_1) = \frac{1}{25}(18a_1^2 - 48a_1a_2 + 32a_2^2 - 25)e_1 - \frac{1}{25}(24a_1^2 - 14a_1a_2 - 24a_2^2)e_2 - \frac{2}{5}(3a_1 - 4a_2)\nu.$$  

Thus $18a_1^2 - 48a_1a_2 + 32a_2^2 \equiv 0 \pmod{25}$. Since

$$18a_1^2 - 48a_1a_2 + 32a_2^2 = 2(3a_1 - 4a_2)^2$$  

we get that $3a_1 - 4a_2 \equiv 0 \pmod{5}$. This proves (4.3.2) because

$$(\xi, \zeta) = -2a_1 - 4a_2 \equiv 3a_1 - 4a_2 \pmod{5}.$$  

\qed

**Proposition 4.6.** Keep notation and assumptions of Proposition (4.3). Let $\Psi$, $\Phi$ be the markings of Proposition (4.3) and $\overrightarrow{A} \in \mathcal{U} \cap \Delta_0^\perp(V)$. There exists a $(-2)$-vector $\xi \in \Lambda$ such that $-r_\xi$ is a $\Psi$-$\pi$-monodromy operator and

$$(\xi, \zeta) \not\equiv 0 \pmod{5}.$$  

We grant the above proposition for the moment being and we proceed to prove Theorem (1.1). Let notation and assumptions be as in Proposition (4.3) and $\Psi$, $\Phi$ be the markings of Proposition (4.3). Then either (1) or (2) of Corollary (4.4) holds. Suppose that (2) holds; we will arrive at a contradiction. Let $\xi$ be as in Proposition (4.6): by Corollary (4.2) we have $-r_\zeta \circ r_\xi \circ r_\zeta \circ r \in O(\Lambda)$. Since $r \in O(\Lambda)$ we get that $r_\zeta \circ r_\xi \circ r_\zeta \in O(\Lambda)$: this contradicts Claim (4.5)
because of (4.3.6). Thus (1) of Corollary (4.4) holds. Let \( U^0 := U \cap LG(\Lambda^3 V)^{\text{iso}} \), then \( U^0 \) is an open (in the euclidean topology) non-empty subset of \( LG(\Lambda^3 V)^{\text{iso}} \). Since (1) of Corollary (4.4) holds we have

\[
\mathcal{P} \circ \delta|_{U^0} = \pi \circ \mathcal{P}|_{U^0}, \tag{4.3.7}
\]

Both \( \mathcal{P} \circ \delta \) and \( \pi \circ \mathcal{P} \) are holomorphic maps with domain the connected manifold \( LG(\Lambda^3 V)^{\text{iso}} \); by analytic continuation we get that Theorem (1.1) holds.

**Proof of Proposition (4.6)** Let \( F \subset \mathbb{P}^3 \) be a smooth quartic, thus \( F \) is a K3 surface. We have a regular map

\[
\begin{align*}
F^{[2]} & \xrightarrow{\varphi} \text{Gr}(1, \mathbb{P}^3) \subset \mathbb{P}^5 \\
[Z] & \mapsto \text{span}(Z)
\end{align*}
\]

and \( c_1(g^*O_{Gr(1,\mathbb{P}^3)}(1)) \) has square 2 for the Beauville-Bogomolov form. If \( F \) does not contain lines the above map is finite and hence \( g^*O_{Gr(1,\mathbb{P}^3)}(1) \) is an ample line-bundle on \( F^{[2]} \). If \( F \) contains a line \( R \) then \( g^*O_{Gr(1,\mathbb{P}^3)}(1) \) is big and nef but it restricts to the trivial line-bundle on the \( \mathbb{P}^2 \) given by \( R^{(2)} \subset F^{[2]} \). Assume that \( F \) does not contain lines; we proved in Section (6) of [12] that

\[
(F^{[2]}, g^*O_{Gr(1,\mathbb{P}^3)}(1))
\]

is deformation equivalent to \( (M_t, f^*L_A) \); (4.3.9)

where \( t \in T \), i.e. there exists a polarized family of irreducible symplectic 4-folds over a connected basis with one fiber isomorphic to \( (F^{[2]}, g^*O_{Gr(1,\mathbb{P}^3)}(1)) \) and another fiber isomorphic to \( (M_t, f^*L_A) \). Using this result we will show that the monodromy operator on \( F \) given by a suitable \((-2)\)-class orthogonal to \( c_1(O_F(1)) \) gives rise to a \( \Psi(\text{A}) \)-monodromy operator for which (4.3.6) holds. Before proving this we must dive into the details of the proof of (4.3.9). Let \( F_0 \subset \mathbb{P}^3 \) be a smooth quartic surface containing a line \( R \) and with Picard number 2, i.e. \( \text{Pic}(F_0) = \mathbb{Z}[A_0] \oplus \mathbb{Z}[R] \) where \( A_0 \) is the (hyper)plane class. The divisor \( (2A_0 - R) \) is very ample and \( c_1(2A_0 - R)^2 = 10 \); thus we have an embedding

\[
F_0 \hookrightarrow |2A_0 - R| \cong \mathbb{P}^6
\]

(4.3.10)
as a linearly normal K3 surface of degree 10. Let

\[
w_0 := 1 + c_1(A_0) + \eta_0, \quad v_0 := 2 + c_1(2A_0 - R) + 2\eta_0
\]

(4.3.11)

where \( \eta_0 \in H^4(F_0; \mathbb{Z}) \) is the orientation class. Let \( M_{w_0} \) be the moduli space of torsion-free sheaves \( \mathcal{G} \) on \( F_0 \) such that \( v(\mathcal{G}) = w_0 \); every such sheaf is equal to \( I_Z \otimes O_{F_0}(A_0) \) for a unique \( |Z| \in F_0^{[2]} \) and hence

\[
M_{w_0} \cong F_0^{[2]}.
\]

(4.3.12)

We let \( L_{w_0} := g_0^*O_{Gr(1,\mathbb{P}^3)}(1) \). Let \( M_{v_0} \) be the moduli space of \( (2A_0 - R) \)-semistable sheaves \( \mathcal{F} \) on \( F_0 \) such that \( v(\mathcal{F}) = v_0 \). The moduli space \( M_{v_0} \) is smooth because \( (2A_0 - R) \) is \( v_0 \)-generic (see Section (6) of [12]) and hence it is a deformation of \((K3)^{[2]}\). Mukai’s maps give isometries of Hodge structures

\[
\theta_{w_0} : w_0^1 \sim H^2(M_{w_0}), \quad \theta_{v_0} : v_0^1 \sim H^2(M_{v_0}).
\]

(4.3.13)

One has (see p.1241 of [12])

\[
c_1(L_{w_0}) = \theta_{w_0}(\eta_0 - 1).
\]

(4.3.14)
We let $L_{v_0}$ be the line-bundle on $M_{v_0}$ such that

$$c_1(L_{v_0}) = \theta_{v_0}(\eta_0 - 1). \tag{4.3.15}$$

In Lemma (6.2) of [12] we considered the birational map $M_{w_0} \dashrightarrow M_{v_0}$ whose inverse

$$\varphi: M_{v_0} \dashrightarrow M_{w_0} \tag{4.3.16}$$
is the Mukai reflection defined by the $(-2)$-vector

$$w_0 := (1 + c_1(A_0 - R) + \eta_0) \tag{4.3.17}$$

(notice that $-r_{w_0}(v_0) = w_0), see [16]$. Since $M_{v_0}$ and $M_{w_0}$ are irreducible symplectic manifolds the birational map $\varphi$ induces an isomorphism of lattices $\varphi^*: H^2(M_{v_0}) \cong H^2(M_{w_0})$. By Theorem (2.9) of [16] we have

$$\varphi^*\theta_{v_0}(\alpha) = \theta_{v_0}(-r_{w_0}(v_0)), \tag{4.3.18}$$
in particular by (4.3.14)-(4.3.15) we have

$$\varphi^*L_{w_0} \cong L_{v_0}. \tag{4.3.19}$$

The birational map $\varphi$ is the flop of

$$\Pi_{w_0} := R(2) \subset F_{0}^{[2]} = M_{w_0}. \tag{4.3.20}$$

It follows that $M_{v_0}$ contains $\Pi_{v_0} \cong \Pi_{v_0}'$ and from (4.3.19) we get that $L_{v_0}$ is big, nef and its restriction to $\Pi_{v_0}$ is trivial. Let $X \to B_{v_0}$ be a representative for the deformation space of $(M_{v_0}, L_{v_0})$, i.e. deformations of $M_{v_0}$ that “keep $c_1(L_{v_0})$ of type $(1,1)$”. Similarly let $X' \to B_{w_0}$ be a representative for the deformation space of $(M_{w_0}, L_{w_0})$. We let $0 \in B_{v_0}$ and $0 \in B_{w_0}$ be the points corresponding to $(M_{v_0}, L_{v_0})$ and $(M_{w_0}, L_{w_0})$ respectively. Thus for each $q \in B_{v_0}$ the fiber $X_q$ of $X \to B_{v_0}$ over $q$ has a line-bundle $L_q$ which is a deformation of $L_{v_0}$. Similarly for each $s \in B_{w_0}$ the fiber $X'_s$ of $X' \to B_{w_0}$ over $s$ has a line-bundle $L'_s$ which is a deformation of $L_{w_0}$. We may and will assume that $B_{v_0}, B_{w_0}$ are contractible and hence Gauss-Manin gives identifications

$$H^2(X_q) \cong H^2(M_{v_0}), \quad H^2(X'_s) \cong H^2(M_{w_0}), \quad q \in B_{v_0}, \quad s \in B_{w_0} \tag{4.3.21}$$

which match $c_1(L_q)$ to $c_1(L_{v_0})$ and $c_1(L'_s)$ to $c_1(L_{w_0})$. Let $B(\Pi_{v_0}) \subset B_{v_0}$ be the locus parametrizing deformations $X_q$ which contain a deformation of $\Pi_{v_0}$, and similarly let $B(\Pi_{w_0}) \subset B_{w_0}$ be the locus parametrizing deformations of $X'_s$ which contain a deformation of $\Pi_{w_0}$. By a Theorem of Voisin [15] each of these loci is smooth of codimension 1. There is a natural isomorphism of germs

$$\mu: (B_{w_0}, 0) \simto (B_{v_0}, 0) \tag{4.3.22}$$
such that $\mu(\Pi_{w_0}) = \Pi_{v_0}$ and if $s \notin \Pi_{w_0}$ then $(X_{\mu(s)}, L_{\mu(s)}) \cong (X'_s, L'_s)$. Let $s \in B_{w_0}$ be such that $(X'_s, L'_s) \cong (F^{[2]}, g^*O_{Gr(1,\mathbb{P}^3)}(1))$ where $F$ is a quartic containing no lines. Then $s \notin \Pi_{w_0}$ and hence

$$(X_{\mu(s)}, L_{\mu(s)}) \cong (X'_s, L'_s) \cong (F^{[2]}, g^*O_{Gr(1,\mathbb{P}^3)}(1)). \tag{4.3.23}$$
On the other hand there exists \( q \in B_{v_0} \) such that \((X_q, c_1(L_q)) \cong (M_t, \theta_t(\eta_t - 1))\) for \( t \in T_* \) because the parameter space for linearly normal \( K3 \) surfaces of degree 10 (an open subset of the relevant Hilbert scheme) is irreducible - notice that \( q \notin \Pi_{v_0} \) because \( \theta_t(\eta_t - 1) \) is ample. Thus there exists \( \mathfrak{F} \in \Delta^0_s(V) \) such that

\[
(X_{\mathfrak{F}}, L_{\mathfrak{F}}) \cong (X_q, L_q).
\] (4.3.24)

Let \( \gamma \in H^2(F; \mathbb{Z}) \) be a \((-2)\)-class orthogonal to \( c_1(O_F(1)) \). Then \( \gamma \) determines a monodromy operator on \( H^2(F^{[2]}) \); by (4.3.23) and (4.3.24) this monodromy operator can be identified with a monodromy operator on \( X_{\mathfrak{F}} \) because polarized deformation spaces of irreducible symplectic manifolds are smooth. Given the trivializations (4.3.21) the monodromy operator in question is equal to \(-r_{\nu_0}^*\theta_{u_0}(\gamma_0)\) where \( u_0 \) is given by (4.3.17). A straightforward computation gives that

\[
(-r_{\nu_0}^*\theta_{u_0}(\gamma_0), 5 + 2c_1(2A_0 - R) + 5\eta_0) \neq 0 \quad (\text{mod } 5).
\] (4.3.28)

By (4.3.18) this is equivalent to

\[
(-r_{\nu_0}(\gamma_0), 5 + 2c_1(2A_0 - R) + 5\eta_0) \neq 0 \quad (\text{mod } 5).
\] (4.3.29)

where \( u_0 \) is given by (4.3.17). A straightforward computation gives that

\[
(-r_{\nu_0}(\gamma_0), 5 + 2c_1(2A_0 - R) + 5\eta_0) = -8 \int_{F_0} \gamma_0 \wedge c_1(R) \quad (\text{mod } 5).
\] (4.3.30)

and hence we get (4.3.28). \( \square \)

Certainly there exists a class \( \gamma_0 \) satisfying the hypotheses of Claim (4.7): the vector \( \xi \in \Lambda \) given by (4.3.25) satisfies the thesis of Proposition (4.6).

References


