Algebro-geometric analogues of Donaldson's polynomials.
O'Grady, Kieran G.
pp. 351 - 396

Terms and Conditions

The Göttingen State and University Library provides access to digitized documents strictly for noncommercial educational, research and private purposes and makes no warranty with regard to their use for other purposes. Some of our collections are protected by copyright. Publication and/or broadcast in any form (including electronic) requires prior written permission from the Goettingen State- and University Library.
Each copy of any part of this document must contain there Terms and Conditions. With the usage of the library's online system to access or download a digitized document you accept there Terms and Conditions.
Reproductions of material on the web site may not be made for or donated to other repositories, nor may be further reproduced without written permission from the Goettingen State- and University Library
For reproduction requests and permissions, please contact us. If citing materials, please give proper attribution of the source.

Contact:

Niedersächsische Staats- und Universitätsbibliothek
Digitalisierungszentrum
37070 Goettingen
Germany
Email: gdz@www.sub.uni-goettingen.de

Purchase a CD-ROM

The Goettingen State and University Library offers CD-ROMs containing whole volumes / monographs in PDF for Adobe Acrobat. The PDF-version contains the table of contents as bookmarks, which allows easy navigation in the document. For availability and pricing, please contact:
Niedersaechisische Staats- und Universitaetsbibliothek Goettingen - Digitalisierungszentrum
37070 Goettingen, Germany, Email: gdz@www.sub.uni-goettingen.de

Invent. math. 107, 351–395 (1992)

Inventiones mathematicae
© Springer-Verlag 1992

Algebro-geometric analogues of Donaldson's polynomials

Kieran G. O'Grady
Columbia University, New York, NY 10027, USA

Oblatum 1-X-1990 & 23-IV-1991

1 Introduction

Donaldson [5] has introduced polynomials on the second homology of any simply connected closed four-manifold, M, with b_2^+ odd (at least 3) which, up to $\pm$, are invariant under orientation preserving diffeomorphisms. For every $c > \frac{3}{4}b_2^+ + \frac{3}{4}$ there is a polynomial $\gamma_c(M) \in \mathrm{Sym}^{d(c)} H^2(S)$, where $d(c) = 4c - \frac{3}{2}(b_2^+ + 1)$; to define it one considers $\mathcal{M}_c$, the set (which is in a natural way a manifold) of Gauge equivalence classes of connections on the SU(2)-bundle on M with $c_2 = c$, anti-self-dual with respect to a generic metric on M. There is a natural map $\mu\colon H_2(M) \to H^2(\mathcal{M}_c)$, namely slant product with $-\frac{1}{4}c_2(P)$, where P is the universal SO(3) principal bundle on $M \times \mathcal{M}_c$. One way of defining $\gamma_c(M)$ [10] is to construct a compactification of $\mathcal{M}_c$, $\mathcal{X}_c$, carrying a fundamental class $[\mathcal{X}_c]$. One proves that μ extends to a map $\bar{\mu}\colon H_2(M) \to H^2(\mathcal{X}_c)$, then one defines

$$\gamma_c(M)(\alpha) = \int_{[\mathcal{X}_c]} \underbrace{\bar{\mu}(\alpha) \wedge \dots \wedge \bar{\mu}(\alpha)}_{d(c)}$$

where $d(c) = \frac{1}{2}\dim \mathcal{X}_c = 4c - \frac{3}{2}(b_2^+ + 1)$.

If M is the C^∞ manifold underlying a projective surface S (by a projective surface we will always mean a smooth projective surface) and we choose the Kähler metric associated to the hyperplane class H, then, by a Theorem of Donaldson [4], $\mathcal{M}_c$ is isomorphic to M_c, the moduli space of H-slope-stable rank two bundles E on S with $c_1(E) = 0$, $c_2(E) = c$. Donaldson [5] was the first to study the polynomials $\gamma_c(S)$ of a projective surface S. In this way he was able to prove some striking new results on the smooth classification of projective surfaces (with $p_g(S) > 0$). By exploiting the algebro-geometric description of $\mathcal{M}_c$ he proved that $\gamma_c(S)$ is not zero for big enough c. By a theorem of Donaldson [5] this implies that if S is a smooth connected sum of two four-manifolds, then the intersection form on one of the two manifolds is negative definite. Friedman and Morgan [9] have studied the case of surfaces with big monodromy (e.g. complete intersections of general type, minimal elliptic surfaces). For such a surface $\gamma_c(S)$ is a polynomial in q and k, the quadratic form and $c_1(K_S)$, respectively. Thus we have

$$\gamma_c(S) = \sum_{i=0}^{[\frac{d(c)}{2}]} a_i q^i k^{d(c)-2i} . \tag{1}$$

This, together with Donaldson's Theorem on the non-vanishing of $\gamma_c(S)$ implies that if $p_g(S)$ is even (and thus $d(c)$ is odd) then $\pm c_1(K_S)$ is a smooth invariant. Now assume that S has big monodromy and that p_g is odd (so $d(c)$ is even). If it so happens that for some $i < \frac{d(c)}{2}$ we have that $a_i \neq 0$ then again $\pm c_1(K_S)$ is invariant under diffeomorphisms. Now notice the following. Let $\Gamma \in H_2(S)$ be Poincarè dual to a holomorphic two-form ω. Let $C \subset S$ be a divisor such that $C^2 = 0$. Assume that for some i

$$\gamma_c(S)(\underbrace{C, \ldots, C}_{2i}, \Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}) \neq 0 \tag{2}$$

where, by abuse of notation, we use $\gamma_c(S)$ to denote both the polynomial and its polarization (we will continue committing this abuse throughout the paper). Then the coefficient $a_{\frac{d(c)}{2}-i}$ in the expansion (1) is non-zero. Hence if $i > 0$ we conclude that $\pm c_1(K_S)$ is a smooth invariant. More in general (i.e. even if S does not have big monodromy), if (2) holds for some $i > 0$ then we conclude that $\gamma_c(S)$ is not a power of the quadratic-form and thus is truly a new smooth invariant.

This paper grew out of an attempt to show that (2) holds for certain surfaces. Given a projective surface S and a polarization H of S we define, for odd big enough values of c, an algebro-geometric polynomial $\delta_c(S, H) \in \mathrm{Sym}^{d(c)} H^2(S)$. Our two main theorems assert that, if we replace $\gamma_c(S)$ by $\delta_c(S, H)$, then (2) holds with $i = 0$ if $|K_S|$ contains a smooth irreducible curve of genus at least two (and c is big enough) and that if furthermore S contains a base-point-free pencil $|C|$, where C is a smooth curve of genus g then (2) holds for all $i \leqq \min\{g - 3, \frac{1}{2}(g + 1)\}$ (more precisely we will show that the left side is positive). In order to define the polynomial $\delta_c(S, H)$ we consider the projective compactification, $\bar{M}_c$, of M_c, given by the closure of M_c in the moduli space of Gieseker-Maruyama H semistable torsion-free sheaves with $c_1 = 0$, $c_2 = c$. If c is odd there is a universal sheaf on $S \times \bar{M}_c$, hence one can define a map $\nu: H_2(S) \to H^2(\bar{M}_c)$ analogous to $\bar{\mu}$ and thus define a polynomial $\delta_c(S, H) \in \mathrm{Sym}^{d(c)} H^2(S)$ analogous to Donaldson's polynomial $\gamma_c(S)$. The natural question at this point is: are $\gamma_c(S)$ and $\delta_c(S, H)$ equal? It is plausible that the answer is positive if some conditions are imposed on H, but we don't know the answer except in the case of K3 surfaces and some elliptic surfaces with geometric genus equal to one [16, 19]. In the last section we will examine more closely the case of complete intersections of general type in a projective space. Arguing as for $\gamma_c(S)$ we will show that if c is big enough and S is the generic such surface then $\delta_c(S, H)$ is a polynomial in the intersection form q and $k = c_1(K_S)$ (the polarization H is the hyperplane class). Then, if the geometric genus of S is odd, our main theorems show that in the expression

$$\delta_c(S, H) = \sum_{i=0}^{\frac{d(c)}{2}} a_i q^i k^{d(c)-2i}$$

the coefficients $a_{\frac{d(c)}{2}}$ and $a_{\frac{d(c)}{2}-1}$ are positive. In fact as the degree of S increases the number of positive coefficients also increases.

The method we use to prove the theorems is to show that if $\Gamma \in H_2(S)$ is Poincarè dual to the holomorphic two form ω then $\nu(\Gamma)$ is represented by a certain holomorphic two-form $\frac{1}{4\pi^2}\tau_{\bar{M}_c}(\omega)$ (to be precise we first go to a desingularization, $\tilde{M}_c$, of $\bar{M}_c$) which was first studied by Mukai [17] and Tyurin [22]. We prove that if the zero-locus of ω is a smooth connected curve of genus at least two then for c big enough $\tau_{\bar{M}_c}(\omega) + \overline{\tau_{\bar{M}_c}(\omega)}$ is a symplectic form on an open dense subset of $\bar{M}_c$. This implies the first theorem ($i = 0$). The proof of the second result breakes up into two parts. First we show how to represent the Poincarè dual of $\nu(C)^{2i}$ as a sum of algebraic cycles over which the integral of $(\tau_{\bar{M}_c}(\omega) \wedge \overline{\tau_{\bar{M}_c}(\omega)})^{d(c)-2i}$ is either positive or zero. Then to conclude the proof we use the invariance of the two-forms $\tau_M(\omega)$ (which are defined on any moduli space of H slope stable vector bundles on S) under mappings between moduli spaces defined by elementary modifications along a fixed curve (this is analogous to the invariance under translations of the holomorphic one-forms on the Picard group of a curve).

Acknowledgements. It's a pleasure to thank Bob Friedman for many helpful discussions.

2 Preliminaries and statement of results

Sheaves will always assumed to be coherent. Following Gieseker-Maruyama ([12, 15]), if E is a torsion free sheaf of rank r on a smooth polarized projective variety (X, H) we let $p_E(n) = \frac{1}{r}\chi(E(nH))$.

Definition 2.1 Let (X, H) be a smooth polarized projective variety, a torsion-free sheaf, E, on X is *H-stable* (respectively *H-semistable*) if it is stable (respectively semistable) in the sense of Gieseker-Maruyama, i.e. if, for every injective map of sheaves $f\colon F \to E$, $p_F(n) < p_E(n)$ (respectively $p_F(n) \leqq p_E(n)$) for all big n.

Let E be a rank r torsion-free sheaf on (X, H), then we let the slope of E be $\mu(E) = \frac{1}{r}c_1(E)\cdot H$.

Definition 2.2 Let (X, H) be a smooth polarized projective variety, a sheaf, E, on X is *H-slope-stable* (respectively *H-slope-semistable*) if it is stable (respectively semistable) in the sense of Mumford-Takemoto, i.e. if, for every injective map of sheaves $f\colon F \to E$, $\mu(F) < \mu(E)$ (respectively $\mu(F) \leqq \mu(E)$). We say that E is *properly H-slope semistable* if it is H-slope-semistable and there exists a subsheaf $F \subset E$ such that $\mu(F) = \mu(E)$.

Let C be a curve, then a torsion-free sheaf on C is locally free; in this case the polarization H is irrelevant and stability (semistability) is equivalent to slope-stability (respectively slope-semistability). Let L be a line bundle on C, the moduli space of stable rank-two vector bundles, E, on C, with $\det E \cong L$ is a smooth quasi-projective variety, $M(C; L)$, whose tangent space at the point corresponding to E is canonically identified with $H^1(\operatorname{ad} E)$, where $\operatorname{ad} E$ is the sheaf of traceless endomorphisms of E. A natural compactification of $M(C; L)$ is provided by the projective variety $\bar{M}(C; L)$, the moduli space of semistable sheaves, where two semistable bundles are identified if the successive quotients in the Narasimhan-Seshadri filtration are isomorphic. When $L = \mathcal{O}_C$ we let $M(C; \mathcal{O}_C) = M(C; 0)$ and $\bar{M}(C; \mathcal{O}_C) = \bar{M}(C; 0)$.

Remark 2.1 Let S be a surface and let E be a rank-two torsion-free sheaf on S, then, as is easily checked

$$p_E(n) = \tfrac{1}{2}H^2n^2 + \tfrac{1}{2}(\mu(E) - H\cdot K_S)n + \tfrac{1}{2}(\tfrac{1}{2}c_1(E)^2 - \tfrac{1}{2}c_1(E)\cdot K_S - c_2(E)) + \chi(\mathcal{O}_S)\,.$$

If F is of rank one

$$p_F(n) = \tfrac{1}{2}H^2n^2 + (\mu(F) - \tfrac{1}{2}H\cdot K_S)n + \tfrac{1}{2}c_1(F)^2 - \tfrac{1}{2}c_1(F)\cdot K_S - c_2(F) + \chi(\mathcal{O}_S)\,.$$

From these formulas it follows that if E is H-slope-stable then it is also H stable, and that if it is H-semistable then it is H-slope-semistable.

Gieseker and Maruyama [12, 15] have shown that the moduli space of semistable sheaves of a given rank and prescribed Chern classes on a polarized projective surface is a projective variety (as in the case of curves the equivalence relation on properly semistable sheaves is weaker than isomorphism). Let H be a polarization on the surface S and let $c_1 \in H^{1,1}(S, \mathbf{Z})$, $c_2 \in H^4(S, \mathbf{Z})$. We let $M(S, H; c_1, c_2)$ be the moduli space of H-slope-stable rank two vector bundles on S with Chern classes c_1 and c_2. We let $M_{GM}(S, H; c_1, c_2)$ be the moduli space of rank two torsion-free H-semistable sheaves, E, on S such that $c_1(E) = c_1$ and $c_2(E) = c_2$. By the remark above there is a natural embedding $\iota\colon M(S, H; c_1, c_2) \hookrightarrow M_{GM}(S, H; c_1, c_2)$. Since being locally free is an open condition in flat families of sheaves we see that $\iota(M(S, H; c_1, c_2))$ is an open subset of $M_{GM}(S, H; c_1, c_2)$. We let $\bar{M}(S, H; c_1, c_2) \subset M_{GM}(S, H; c_1, c_2)$ be the closure of $\iota(M(S, H; c_1, c_2))$ in $\bar{M}(S, H; c_1, c_2)$. Clearly $\bar{M}(S, H; c_1, c_2)$ is a union of irreducible components of $M_{GM}(S, H; c_1, c_2)$.

The point in a moduli space corresponding to the sheaf E will be denoted by $[E]$. In general we will use $[X]$ to denote the class corresponding to an object X under some equivalence relation or the point in a suitable parameter space corresponding to X. For example if $X \subset S$ is a subvariety $[X]$ might denote the corresponding point in the appropriate Hilbert scheme. If D is a divisor on a variety $[D]$ will denote, as usual, the line bundle corresponding to D.

For the rest of this section we will assume that S is a projective regular surface.

Let $[E] \in \bar{M}(S, H; c_1, c_2)$ correspond to a simple sheaf, E, then the Zariski tangent space to $\bar{M}(S, H; c_1, c_2)$ at $[E]$ is canonically identified with $\operatorname{Ext}^1(E, E)$. If E is locally free then $\operatorname{Ext}^1(E, E) \cong H^1(\operatorname{End} E)$. Let $\operatorname{ad} E$ be the subsheaf of $\operatorname{End} E$ given by traceless endomorphisms. If E is locally free then, since we are assuming S is regular, $H^1(\operatorname{End} E) \cong H^1(\operatorname{ad} E)$. The following criterion for smoothness is due to Mukai ([Theorem 0.3, 17]).

Theorem 2.1 (Mukai) *Let $[E] \in \bar{M}(S, H; c_1, c_2)$ and assume E is simple. If the natural map from $H^0\cdot(K_S)$ to* Hom$(E, E(K_S))$ *is surjective then $\bar{M}(S, H; c_1, c_2)$ is smooth at $[E]$.*

Corollary 2.1 *Assume $h^0(\operatorname{ad} E^{**}(K_S)) = 0$, then $\bar{M}(S, H; c_1, c_2)$ is smooth at $[E]$.*

Definition 2.3 The moduli space $\bar{M}(S, H; c_1, c_2)$ is *locally fine* if every point in $\bar{M}(S, H; c_1, c_2)$ represents a stable sheaf, i.e. if, for rank two sheaves E with $c_1(E) = c_1$, $c_2(E) = c_2$, semistability implies stability.

Definition 2.4 The moduli space $\bar{M}(S, H; c_1, c_2)$ is *globally fine* (for brevity fine) if it is locally fine and there exists a universal sheaf, $\mathscr{E}$, on $S \times \bar{M}(S, H; c_1, c_2)$.

A universal sheaf is not unique, but, since every point of $\bar{M}(S, H; c_1, c_2)$ corresponds to a simple sheaf, if $\mathscr{E}$ and $\mathscr{F}$ are two such sheaves then they differ by a line bundle, L, on $\bar{M}(S, H; c_1, c_2)$. More precisely, let $\pi: S \times \bar{M}(S, H; c_1, c_2) \to \bar{M}(S, H; c_1, c_2)$ be the projection. Then we have that $\mathscr{F} \cong \mathscr{E} \otimes p_{\bar{M}}^*(L)$.

Proposition 2.1 *Let* $\bar{M}(S, H; c_1, c_2)$ *be locally fine, assume that for* $[E] \in \bar{M}(S, H; c_1, c_2)$ *the Euler characteristic* $\chi(E)$ *is odd, then* $\bar{M}(S, H; c_1, c_2)$ *is fine.*

Proof. This follows from Remark A.7 in [18].

Remark 2.2 If in the statement of the proposition we replace $\bar{M}(S, H; c_1, c_2)$ by $M_{GM}(S, H; c_1, c_2)$ then the same conclusion holds. This again follows from Remark A.7 in [18].

Proposition 2.2 *If* $\frac{1}{2}(c_1^2 - c_1 \cdot K_S) - c_2$ *is odd, then* $\bar{M}(S, H; c_1, c_2)$ *is fine.*

Proof. If $[E] \in \bar{M}(S, H; c_1, c_2)$ the constant term of $p_E(n)$ is not an integer, while if L is a rank one sheaf the constant term of $p_L(n)$ is an integer. Hence $p_L(n) = p_E(n)$ cannot hold, so $\bar{M}(S, H; c_1, c_2)$ is locally fine. Since $\chi(E) = \frac{1}{2}(c_1^2 - c_1 \cdot K_S) - c_2 + 2\chi(\mathcal{O}_S)$ Proposition 2.1 applies and we conclude that $\bar{M}(S, H; c_1, c_2)$ is fine.

Definition 2.5 Let D be a divisor on S, we say $M(S, H; c_1, c_2)$ is *D-good* if all its components contain a point, $[E]$, such that $h^2(\operatorname{ad} E(-D)) = 0$.

Let $C \subset S$ be a smooth curve. We let $\mathbf{Def}(E|_C)$ be the versal deformation space of $E|_C$. We let $\mathbf{Def}^0(E|_C) \subset \mathbf{Def}(E|_C)$ be the space parametrizing deformations inducing trivial deformations of $\det(E|_C)$. Let $(M(S, H; c_1, c_2), [E])$ be the germ of $M(S, H; c_1, c_2)$ at $[E]$, then we have a map $\rho_C: (M(S, H; c_1, c_2), [E]) \to \mathbf{Def}^0(E|_C)$ given by restriction of bundles (recall that we are assuming S is regular). Now assume that $h^2(\operatorname{ad} E(-C)) = 0$. Then $h^2(\operatorname{ad} E) = 0$, hence $M(S, H; c_1, c_2)$ is smooth at $[E]$. The tangent space to $M(S, H; c_1, c_2)$ at $[E]$ is identified with $H^1(\operatorname{ad} E)$. On the other hand the cohomology group $H^1(\operatorname{ad} E|_C)$ is canonically identified with the tangent space to $\mathbf{Def}^0(E|_C)$ at the origin (i.e. the point parametrizing $E|_C$). The natural map $f: H^1(\operatorname{ad} E) \to H^1(\operatorname{ad} E|_C)$ is the differential of ρ_C at $[E]$. Hence we conclude that if $M(S, H; c_1, c_2)$ is C-good and $[E]$ is a generic point of $M(S, H; c_1, c_2)$ then the image of ρ_C contains a neighborhood of the origin in $\mathbf{Def}^0(E|_C)$. This gives the geometric meaning of D-good when D is linearly equivalent to a smooth curve C.

An important case is when $D = 0$. Then, following Friedman [7], we simply say that $M(S, H; c_1, c_2)$ is good. In this case, by Theorem 2.1, all the components of $M(S, H; c_1, c_2)$ have the expected dimension $d = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$ and are generically reduced. Viceversa, by the theory of deformations of vector bundles, if all the components of $M(S, H; c_1, c_2)$ are generically reduced and of the expected dimension then $M(S, H; c_1, c_2)$ is good.

We will say that $\bar{M}(S, H; c_1, c_2)$ is D-good if $M(S, H; c_1, c_2)$ is D-good.

We will need the following theorem of Gieseker [12], which shows that in many cases the moduli spaces we are considering are not empty (see Corollary 5.3 for the case $c_1 \neq 0$).

Theorem 2.2 (Gieseker) *Let (S, H) be a polarized surface, let $n \geqq 4[\frac{1}{2}p_g(S)] + 4$, then there exists an H-slope-stable rank two bundle, E, on S with $c_1(E) = 0$ and $c_2(E) = n$.*

We will often use the following theorem, proved by Donaldson [5] and Friedman [8].

Theorem 2.3 (Donaldson) *Let (S, H) be a polarized surface, $c_1 \in H^{1,1}(S, \mathbf{Z})$, D a divisor on S. Let $n(c)$ be the number of moduli of rank two H-slope- semistable vector bundles, E, on S with $c_1(E) = c_1$, $c_2(E) = c$ such that there is a non-zero $\varphi \in H^0(\mathrm{ad}\, E(D + K_S))$, then there exist constants a, b such that*

$$n(c) \leqq 3c + a\sqrt{c} + b\ .$$

Corollary 2.2 *Let $D \subset S$ be a divisor, fix c_1, then for $c_2 \gg 0$ the moduli space $\bar{M}(S, H; c_1, c_2)$ is D-good.*

Corollary 2.3 *Let (S, H) be a polarized surface, fix $c_1 \in H^{1,1}_{\mathbf{Z}}(S)$ and let $\Delta(c)$ be the singular set of $M(S, H; c_1, c)$, then*

$$\dim \Delta(c) < 3c + O(\sqrt{c})\ .$$

Proof. Follows from Theorems 2.1 and 2.3.

Corollary 2.4 *Let $D \in \mathrm{Pic}(S)$. Fix $c_1 \in H^{1,1}(S, \mathbf{Z})$. If c_2 is big enough and satisfies*

$$c_2 \equiv \tfrac{1}{2}(c_1^2 - c_1 \cdot K_S) + 1 \mod(2)$$

then $\bar{M}(S, H; c_1, c_2)$ is D-good and fine.

We will also need the following proposition, which is in fact used to prove Theorem 2.3, (in [5] the case where $a = -\frac{1}{2}K_S \cdot H$ is proved, a straightforward extension of the proof gives the result for any value of a).

Proposition 2.3 *Let (S, H) be a polarized surface. Fix $c_1 \in H^{1,1}(S, \mathbf{Z})$ and an integer ℓ. Let $n(c)$ be the number of moduli of H-slope-semistable rank two vector bundles, E, on S, such that $c_1(E) = c_1$ and $c_2(E) = c$ and such that there is a non-zero map $f\colon L \to E$, where L is a line bundle such that $L \cdot H \geqq \ell$. Then there exist constants A, B such that*

$$n(c) \leqq 3c + A\sqrt{c} + B\ .$$

Remark 2.3 A simple application of the above proposition gives an upper bound on the number of moduli of H-properly-semistable rank-two bundles on S. More precisely, let $c_1 \in H^{1,1}(S, \mathbf{Z})$, $c \in H^4(S, \mathbf{Z})$, then there exists a scheme (not necessarily of finite type) $P(c_1, c)$ parametrizing (not necessarily effectively) all H-properly-semistable bundles, E, on S with $c_1(E) = c_1$, $c_2(E) = c$. Then, by Proposition 2.3, we have that if c_1 is fixed there exist constants A, B such that

$$\dim P(c_1, c) \leqq 3c + A\sqrt{c} + B\ .$$

The scheme $P(c_1, c)$ can be assumed to have the following property. There is a bundle $\mathscr{V}$ on $S \times P(c_1, c)$ with the following property. Let B be an algebraic variety, let $\mathscr{E}$ be a bundle on $S \times B$ and assume that for all $b \in B$ the bundle $\mathscr{E}|_{S \times \{b\}}$ is H-properly-semistable. Then there is an open dense subset $U \subset B$ and a morphism $f: U \to P(c_1, c)$ such that $\mathscr{E}|_{S \times U} \cong (\mathrm{id}_S \times f)^* \mathscr{V}$.

Now we introduce the algebro-geometric analogue of Donaldson's μ-map. Let S be a projective surface and let B be a quasi-projective variety. Let $\mathscr{E}$ be a B-flat sheaf on $S \times B$. We define the map

$$\nu_{\mathscr{E}}: H_2(S) \to H^2(B)$$

by setting $\nu(\alpha) = c_2(\mathscr{E})/\alpha$.

Let $c_1 = c_1(\mathscr{E}|_{S \times \{b\}})$, where b is any point of B. Let L be a line bundle on B, and let $\pi: S \times B \to B$ be the projection. Let $\mathscr{F} = \mathscr{E} \otimes \pi^* L$. Then, as is easily checked, if $\int_\alpha c_1 = 0$ we have that $\nu_{\mathscr{F}}(\alpha) = \nu_{\mathscr{E}}(\alpha)$. Let $\bar{M} = \bar{M}(S, H; c_1, c_2)$ and assume it is fine, then this allows us to define unambiguously a map $\nu_{\bar{M}}: c_1^{\perp} \to H^2(\bar{M}(S, H; c_1, c_2))$ in the following way. Let $\mathscr{E}$ be a universal sheaf on $S \times \bar{M}$, then we set

$$\nu_{\bar{M}} = \nu_{\mathscr{E}} .$$

Remark 2.4 If $c_1 = 0$ then $\nu_{\bar{M}}$ is defined on all of $H_2(S)$. Let $\iota: M(S, H; 0, c) \hookrightarrow \bar{M}(S, H; 0, c)$ be the inclusion. The moduli space $M(S, H; 0, c)$ is identified with the moduli space of connections on the SU(2)-bundle with $c_2 = c$, anti-self-dual with respect to the Kähler metric associated to H. Then the composition $\iota^* \circ \nu_{\bar{M}}: H_2(S) \to M(S, H; 0, c)$ is identified with Donaldson's μ-map [10].

We also notice that if the Poincaré dual of $\alpha \in H_2(S)$ is in $H^{2,0}(S) + H^{0,2}(S)$ then α is orthogonal to any class in $H^{1,1}(S)$. Thus $\nu_{\bar{M}}$ is always well-defined on the subspace of $H_2(S)$ consisting of Poincaré duals of elements of $H^{2,0}(S) + H^{0,2}(S)$.

Definition 2.6 Let $\bar{M} = \bar{M}(S, H; c_1, c_2)$ be fine and good. Assume $\bar{M}$ is not empty. Let $d = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$ be its dimension. We define $\delta_{\bar{M}} \in \mathrm{Sym}^d(c_1^{\perp})$ as follows:

$$\delta_{\bar{M}}(\alpha_1, \ldots, \alpha_d) = \int_{[\bar{M}]} \nu_{\bar{M}}(\alpha_1) \wedge \ldots \wedge \nu_{\bar{M}}(\alpha_d) .$$

When $c_1 = 0$ and $c_2 = c$ we denote $\delta_{\bar{M}}$ by $\delta_c(S, H)$. Then $\delta_c(S, H) \in \mathrm{Sym}^{d(c)} H^2(S)$ where $d(c) = 4c - 3\chi(\mathcal{O}_S)$. The polynomial $\delta_c(S, H)$ is defined for all c odd and big enough, by Theorem 2.2 and Corollary 2.4. Clearly it is an algebro-geometric analogue of Donaldson's polynomial γ_c but it is not clear how they compare.

Now we can state our main results

Theorem 2.4 *Let (S, H) be a polarized projective regular surface with a smooth irreducible curve $D \in |K_S|$ of genus at least two. Let $c_1 \in H^{1,1}(S, \mathbf{Z})$, $c_2 \in H^4(S, \mathbf{Z})$. Assume that $\bar{M} = \bar{M}(S, H; c_1, c_2)$ is non-empty, fine, K_S-good and even dimensional. Let $\Gamma \in H_2(S)$ be the Poincaré dual of a holomorphic two form ω such that $(\omega) = D$, then*

$$\delta_{\bar{M}}(\Gamma + \bar{\Gamma}) > 0 .$$

Theorem 2.5 *Let (S, H) be as above. Assume that $p_g(S)$ and c are both odd. If $c \gg 0$ then $\bar{M}(S, H; 0, c)$ is non-empty, fine, K_S-good and we have that*

$$\delta_c(S, H)(\Gamma + \bar{\Gamma}) > 0 .$$

The following theorem can be generalized to include also cases in which $c_1 \neq 0$, but for simplicity we limit ourselves to this case.

Theorem 2.6 *Let (S, H) be a polarized surface as above. Assume S has a base-point-free pencil $|C|$, where C is a smooth connected curve of genus $g > 3$. Let $\Gamma \in H_2(S)$ be as above. Let $i \leqq \min\{\frac{1}{2}(g+1), g-3\}$. Then, if c is odd and big enough we have that*

$$\delta_c(S, H)(\underbrace{C, \ldots, C}_{2i}, \Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}) > 0 .$$

Essential for the proof of Theorems 2.4, 2.5 and 2.6 is the following

Theorem 2.7 *Let (S, H) be a polarized projective regular surface. Let $c_1 \in H^{1,1}(S, \mathbf{Z})$, $c_2 \in H^4(S, \mathbf{Z})$. Let $M = M(S, H; c_1, c_2)$. There exists a map $\tau_M: H^0(\Omega_S^2) \to H^0(\Omega_M^2)$, where Ω_M^2 is the sheaf of Kähler differentials on M, with the following property. Assume that M is K_S-good and that there exists a smooth connected curve $D \in |K_S|$ of genus at least two. Let $\omega \in H^0(\Omega_S^2)$ be such that $(\omega) = D$. Let $n = \left[\frac{\dim M}{2}\right]$. There exists an open dense subset $U \subset M$ of smooth points of M such that*

$$\wedge^{2n}(\tau_M(\omega) + \overline{\tau_M(\omega)})|_U$$

is nowhere zero. Thus if $\dim M$ *is even $\tau_M(\omega) + \overline{\tau_M(\omega)}|_U$ is a symplectic form.*

Unfortunately we are not able to determine what is the relation between Donaldson's polynomials and the algebro-geometric polynomials. Nonetheless in the last section we will prove the following theorem, which is analogous to a theorem of Friedman and Morgan [9] on Donaldson's polynomials.

Theorem 2.8 *Let B be an irreducible component of the parameter space for smooth complete intersections in $\mathbf{P}^{r+2}$ of general type. Let $d(c) = 4c - 3\chi(\mathcal{O}_S)$. For $x \in B$ let S_x be the surface corresponding to x. Let H_x be the hyperplane class on $\dot{S}_x$, let $q_x \in \mathrm{Sym}^2 H^2(S_x)$ be the intersection form on $H_2(S_x)$ and let $k_x \in H^2(S_x)$ be $c_1(K_{S_x})$. Then there exists a number k such that if c is odd and $c > k$ the following holds: there is a dense Zariski open subset $V(c) \subset B$ such that if $x \in V(c)$ then $\bar{M}(S_x, H_x; 0, c)$ is good and there are (rational) coefficients a_i^x so that we have*

$$\delta_c(S_x, H_x; 0, c) = \sum_{i=0}^{[\frac{d(c)}{2}]} a_i^x q_x^i k_x^{d(c)-2i} .$$

Furthermore the coefficients a_x^i are independent of x, hence we can set $a_i = a_i^x$.

By applying Theorems 2.5 and 2.6 we will be able to prove

Theorem 2.9 *Let B be an irreducible component of the parameter space for smooth complete intersections in $\mathbf{P}^{r+2}$ of general type. Assume that the geometric genus of surfaces parametrized by B is odd. Then there exists a number k such that the following holds.*

Let c be odd and suppose that $c > k$. There exists an open non-empty subset $V(c) \subset B$ such that if $x \in V(c)$ then

$$\delta_c(S_x, H_x) = \sum_{i=0}^{\left[\frac{d(c)}{2}\right]} a_i q^i k_{d(c)-2i}$$

with $a_{\frac{d(c)}{2}} > 0$, $a_{\frac{d(c)}{2}-1} > 0$.

In fact our theorems show that the number of positive coefficients (starting from $a_{\frac{d(c)}{2}}$) goes to infinity if the degree of S_x goes to infinity.

We will need the following well-known proposition [8]

Proposition 2.4 *Let F be a rank two vector bundle on a smooth genus g curve C. Let* $\det F = \mathcal{O}_C(D)$. *Let* $\mathbf{U} \subset \mathbf{Def}^0(F)$ *be the locus parametrizing unstable deformations of F, let* $\mathbf{V} \subset \mathbf{Def}^0(F)$ *the locus parametrizing deformations which are not stable (i.e. unstable or properly semistable). Then if* $\deg D$ *is even we have that* $\operatorname{cod}_{[F]}(\mathbf{U}, \mathbf{Def}^0(F)) \geqq g + 1$ *and* $\operatorname{cod}_{[F]}(\mathbf{V}, \mathbf{Def}^0(F)) \geqq g - 1$. *If* $\deg D$ *is odd then* $\operatorname{cod}_{[F]}(\mathbf{V}, \mathbf{Def}^0(F)) \geqq g$. *(in this case* $\mathbf{U} = \mathbf{V}$*).*

Corollary 2.5 *Let F be a rank two vector bundle on a smooth genus g curve C. Assume that $g \geqq 2$, then if $x \in \mathbf{Def}^0(F)$ is the generic point it parametrizes the generic stable bundle F_x with* $\det F_x \cong \det F$.

Proof. The set of $x \in \mathbf{Def}^0(F)$ parametrizing stable bundles is non-empty by Proposition 2.4. Since versality is an open condition we conclude that the generic stable bundle is isomorphic to F_x for some x.

Let S be a projective surface and $C \subset S$ a smooth curve. Let B be an algebraic variety and let $\mathscr{E}$ be a B-flat rank-two sheaf on $S \times B$. Assume that for all $b \in B$ the sheaf $E_{b|C}$ is locally free, semistable and $\det E \cong L$. Then, by abuse of notation, we will denote by $\rho_C \colon B \to \bar{M}(C; L|_C)$ the morphism induced by restriction of bundles.

Let E be a torsion-free sheaf on the surface S, then there is a canonical sequence

$$0 \to E \to E^{**} \to Q(E) \to 0$$

where E^{**} is locally free and $Q(E)$ is a sheaf with finite length $l(Q(E))$. We let $Z(E)$ be the zero-dimensional subscheme of S whose ideal sheaf is $\operatorname{Ann} Q(E)$, and we set $l(Z(E)) = h^0(\mathcal{O}_{Z(E)})$.

Let $D \subset X$ be a Cartier divisor on an algebraic variety X. Let $\mathcal{O}_D(F)$ be a locally free sheaf on D (i.e. a sheaf of locally free $\mathcal{O}_X/\mathcal{O}_X(-D)$-modules). Let $\mathscr{E}_1$ be a locally free sheaf on X and $f \colon \mathscr{E}_1 \to \mathcal{O}_D(F)$ a surjection. As is easily checked the sheaf $\mathscr{E}_2 = \ker f$ is a locally free sheaf on X; by definition we have

$$0 \to \mathscr{E}_2 \to \mathscr{E}_1 \xrightarrow{f} \mathcal{O}_D(F) \to 0 .$$

We say that $\mathscr{E}_2$ is the elementary modification of $\mathscr{E}_1$ defined by f. We will use the notation $\mathscr{E}_2 = T(\mathscr{E}_1, f)$.

3 Holomorphic two-forms on parameter spaces for sheaves on surfaces

The purpose of this section is to give explicit representatives (in de Rham cohomology) of the classes $v_{\mathscr{E}}(\Gamma)$ defined in the previous section, when Γ is the Poincaré dual of a holomorphic two-form on the surface S.

The following definition (except for the sign) is as in Tyurin [22] (and Mukai [17] in the case of a K3 surface).

Definition 3.1 Let S be a projective surface. Let $\mathscr{E}$ be a B-flat sheaf on $S \times B$, where B is an algebraic variety. Let ω be a holomorphic two-form on S, we associate to it a bilinear form $\tau_{\mathscr{E}}(\omega)$ on the Zariski tangent sheaf of B, Θ_B, as follows. Let Ext^1 be the relative Ext-sheaf $\mathrm{Ext}^1_{\mathscr{O}_B}(\mathscr{E}, \mathscr{E})$ and let $f\colon \Theta_B \to \mathrm{Ext}^1$ be the "Kodaira-Spencer" map. Let $b \in B$ and let $v, w \in \Theta_b$, then we set

$$\tau_{\mathscr{E}}(\omega)(v, w) = -\operatorname{tr}(f(v) \cup f(w) \cup \omega)\,.$$

The $\cup$ stands for Yoneda pairing and we are viewing ω as a section of K_S. The definition makes sense because the product (at a point $b \in B$ such that $\mathscr{E}|_{S \times \{b\}} \cong E$) is in $\mathrm{Ext}^2(E, E \otimes K_S)$.

Proposition 3.1 *The Mukai-Tyurin bilinear form $\tau_{\mathscr{E}}(\omega)$ is skew-symmetric, hence we get a map*

$$\tau_{\mathscr{E}}\colon H^0(\Omega^2_S) \to H^0(\Omega^2_B)\,,$$

where Ω^2_B is the sheaf of Kähler two-differentials on B.

Proof. We must check that $\tau_{\mathscr{E}}(\omega)(v, v) = 0$. Clearly $\tau_{\mathscr{E}}(\omega)(v, v) = -\operatorname{tr}(f(v) \cup f(v)) \cup \omega$. It is shown in [12] that $\operatorname{tr}(f(v) \cup f(v)) \in H^2(\mathscr{O}_S)$ is the obstruction to lifting to second-order the first-order deformation of $\det E$ associated to the first order deformation of E represented by $f(v)$. Since the Picard variety of S is smooth $\operatorname{tr}(f(v) \cup f(v)) = 0$, hence $\tau_{\mathscr{E}}(\omega)(v, v) = 0$.

Remark 3.1 The map $\tau_{\mathscr{E}}$ has the following obvious functorial property: let $f\colon B' \to B$ be a morphism, let $\mathscr{E}' = (\mathrm{id}_S \times f)^* \mathscr{E}$, then $\tau_{\mathscr{E}'}(\omega) = f^* \tau_{\mathscr{E}}(\omega)$.

Remark 3.2 Assume $E \cong \mathscr{E}|_{S \times \{b\}}$ is locally free. Then $f(v) \in H^1(\operatorname{End} E)$ and clearly $\langle \tau_{\mathscr{E}}(\omega), v \wedge w \rangle = -\int_S \operatorname{tr}(f(v) \wedge f(w)) \wedge \omega$. In this case we can check very easily that $\tau_{\mathscr{E}}(\omega)$ is skew-symmetric at b. Locally $f(v)$ is represented by a 2×2 matrix, m_v, with entries $\bar{\partial}$-closed $(0, 1)$-forms. Then $\operatorname{tr}(f(v) \cup f(w)) \in H^{0,2}(S)$ is given, locally, by $\operatorname{tr}(m_v \wedge m_w)$, where $m_v \wedge m_w$ is the usual (i.e. rows by columns) product of m_v and m_w. As is easily checked, $\operatorname{tr}(m_v \wedge m_v) = 0$, hence $\tau_{\mathscr{E}}(\omega)$ is skew-symmetric at b.

Lemma 3.1 *Let S be a projective surface and let B be a smooth algebraic variety. Let p_S, p_B be the projections of $S \times B$ on S and B, respectively. Let $\omega \in H^0(\Omega^2_S)$. Assume that $\mathscr{E}$ is a locally free sheaf on $S \times B$. Let $F \in H^{1,1}(S \times B, \operatorname{End} \mathscr{E})$ be the curvature of a hermitian connection on $\mathscr{E}$ compatible with the complex structure. Then*

$$\tau_{\mathscr{E}}(\omega) = \tfrac{1}{2}(p_B)_*(\operatorname{tr}(F \wedge F) \wedge p_S^*(\omega))\,.$$

Proof. For $b \in B$ let $S_b = S \times \{b\}$. Let $X = S \times B$. Let $v \in \Theta_b$ be a tangent vector to B at b and let $\tilde{v}$ the corresponding section of $\Theta_X | S_b \cong \Theta_S \oplus p_B^* \Theta_B$. We claim that

$f(v) \in H^{(0,1)}(\mathrm{End}\, E)$ (where f is the Kodaira-Spencer map) is represented by the contraction $\tilde{v} \lrcorner (F|_{S_b})$. To prove this we recall that Atiyah [2] has defined a canonical sequence

$$0 \to \mathscr{E} \otimes \Omega^1_X \to D(\mathscr{E}) \to \mathscr{E} \to 0 \tag{1}$$

and has proved that the obstruction $e(D(\mathscr{E})) \in H^1(\mathrm{End}\, \mathscr{E} \otimes \Omega^1_X)$ to the splitting of (1) corresponds, via Dolbeault's isomorphism, to the curvature F. One can also define, as in [17], a canonical exact sequence

$$0 \to \mathscr{E} \otimes_{\mathcal{O}_B} \Omega^1_B \to D_B(\mathscr{E}) \to \mathscr{E} \to 0$$

with associated extension class $e(D_B(\mathscr{E})) \in H^1(\mathrm{End}\, \mathscr{E} \otimes \Omega^1_B)$ and Mukai [17] has proven that

$$f(v) = v \lrcorner\, e(D_B(\mathscr{E}))\ .$$

Since $\Omega^1_X \cong p_S^*(\Omega^1_S) \oplus p_B^*(\Omega^1_B)$ there is a natural map $r: H^1(\mathrm{End}\, \mathscr{E} \otimes \Omega^1_X) \to H^1(\mathrm{End}\, \mathscr{E} \otimes \Omega^1_B)$. As is easily verified $e(D_B(\mathscr{E})) = r(e(D(\mathscr{E})))$, hence $f(v) = \tilde{v} \lrcorner (F|_{S_b})$. The form $\mathrm{tr}(F \wedge F) \wedge \omega$ is of type (4, 2), hence $(p_B)_*(\mathrm{tr}(F \wedge F) \wedge \omega)$ is of type (2, 0). Let $v, w \in \Theta_{B,b}$ be holomorphic tangent vectors. By definition

$$\langle (p_B)_*(\tfrac{1}{2}\mathrm{tr}(F \wedge F) \wedge \omega), v \wedge w \rangle = \int_{S_b} \tilde{v} \wedge \tilde{w} \lrcorner (\tfrac{1}{2}\mathrm{tr}(F \wedge F) \wedge \omega)\ .$$

Type consideration and an easy computation show that $\tilde{v} \wedge \tilde{w} \lrcorner (\frac{1}{2}\mathrm{tr}(F \wedge F) \wedge \omega) = -\,\mathrm{tr}((\tilde{v} \lrcorner F) \wedge (\tilde{w} \lrcorner F)) \wedge \omega$, hence

$$\begin{aligned} \langle (p_B)_*(\tfrac{1}{2}\mathrm{tr}(F \wedge F) \wedge \omega), v \wedge w \rangle &= -\int_{S_b} \mathrm{tr}((\tilde{v} \lrcorner F) \wedge (\tilde{w} \lrcorner F)) \wedge \omega \\ &= \langle \tau_{\mathscr{E}}(\omega), v \wedge w \rangle\ . \end{aligned}$$

Corollary 3.1 *Let S, B, $\mathscr{E}$ and ω be as above. Then the form $\tau_{\mathscr{E}}(\omega)$ is closed.*

Proof. This is because $\mathrm{tr}(F \wedge F) \wedge p_S^*(\omega)$ is closed.

Remark 3.3 Let $\mathscr{E}$ be a B-flat sheaf on $S \times B$. Let p_S, p_B be the projections of $S \times B$ on S and B, respectively. Let L, M be line bundles on S and B, respectively. Let $\mathscr{F} = \mathscr{E} \otimes p_S^*(L) \otimes p_B^*(M)$. Then, as is easily checked we have that

$$\tau_{\mathscr{F}}(\omega) = \tau_{\mathscr{E}}(\omega)\ .$$

Let $M = M(S, H; c_1, c_2)$. The above remark allows us to define unambiguously a map $\tau_M: H^0(\Omega^2_S) \to H^0(\Omega^2_{M_{\mathrm{red}}})$, where $\Omega^2_{M_{\mathrm{red}}}$ is the sheaf of Kähler two-differentials on the reduced scheme associated to M. In fact we can cover M by open (either in the analytic or in the étale topology) sets U_i so that there is a universal sheaf $\mathscr{E}_i$ on each product $S \times U_i$. Let $\omega \in H^0(\Omega^2_S)$. On each open U_i we can define $\tau_{\mathscr{E}_i}(\omega) \in H^0(\Omega^2_{U_i})$. By Remark 3.3 the two-forms $\tau_{\mathscr{E}_i}(\omega)$ agree on overlaps $U_i \cap U_j$, hence they glue together to define a global section $\tau_M(\omega) \in H^0(\Omega^2_{M_{\mathrm{red}}})$. Now let $\bar{M} = \bar{M}(S, H; c_1, c_2)$ and assume that it is locally fine. Then, by the same procedure, we can define a map $\tau_{\bar{M}}: H^0(\Omega^2_S) \to H^0(\Omega^2_{\bar{M}_{\mathrm{red}}})$, where $\bar{M}_{\mathrm{red}}$ is the reduced scheme associated to $\bar{M}$. In particular if $\bar{M}(S, H; c_1, c_2)$ is fine and $\mathscr{E}$ is a universal sheaf on $S \times \bar{M}_{\mathrm{red}}$ then we can define $\tau_{\bar{M}}$ by setting $\tau_{\bar{M}} = \tau_{\mathscr{E}}$. Now let $\varphi: \tilde{M} \to \bar{M}_{\mathrm{red}}$ be a desingularization. Let $\tilde{\mathscr{E}} = (\mathrm{id}_S \times \varphi)^* \mathscr{E}$, then we set $\tau_{\tilde{M}} = \tau_{\tilde{\mathscr{E}}}$.

Theorem 3.1 *Let S be a regular projective surface. Let B be a smooth algebraic variety. Let $\mathscr{E}$ be a locally free sheaf on $S \times B$. Let $\omega \in H^0(\Omega_S^2)$ and let $\Gamma \in H_2(S)$ be the Poincaré dual of ω. Then $v_{\mathscr{E}}(\Gamma)$ is represented in de Rham cohomology by $\frac{1}{4\pi^2}\tau_{\mathscr{E}}(\omega)$.*

Proof. Let p_S, p_B be the projections of $S \times B$ to S and B, respectively. By definition $v_{\mathscr{E}}(\Gamma) = c_2(\mathscr{E})/\Gamma$. Let F be the curvature of a hermitian connection on $\mathscr{E}$ compatible with the complex structure, then

$$c_2(\mathscr{E}) = \frac{1}{8\pi^2}[\operatorname{tr}(F \wedge F)] - \frac{1}{8\pi^2}[\operatorname{tr} F \wedge \operatorname{tr} F]\ ,$$

hence

$$v_{\mathscr{E}}(\Gamma) = \left[(p_B)_*\left(\frac{1}{8\pi^2}\operatorname{tr}(F \wedge F) \wedge p_S^*\omega - \frac{1}{8\pi^2}\operatorname{tr} F \wedge \operatorname{tr} F \wedge p_S^*\omega\right)\right]. \tag{2}$$

We claim that the de Rham cohomology class $[(p_B)_*(\operatorname{tr} F \wedge \operatorname{tr} F \wedge p_S^*\omega)]$ is zero. In fact, since S is regular, there exist line bundles L, L' on S and B, respectively, such that $\det \mathscr{E} \cong p_S^*(L) \otimes p_B^*(L')$. Let f and f' be the curvatures of holomorphic hermitian connections on L, L', then $[\operatorname{tr} F] = [p_S^* f + p_B^* f']$. Since f is of type (1, 1) we have $p_S^* f \wedge p_S^*\omega = 0$. Clearly $(p_B)_*(p_B^* f' \wedge p_B^* f' \wedge p_S^*\omega) = 0$. This shows that

$$(p_B)_*((p_S^* f + p_B^* f') \wedge (p_S^* f + p_B^* f') \wedge p_S^*\omega) = 0$$

hence $[(p_B)_*(\operatorname{tr} F \wedge \operatorname{tr} F \wedge p_S^*\omega)] = 0$.

From (2) we conclude that $v_{\mathscr{E}}(\Gamma) = \left[(p_B)_*\left(\frac{1}{8\pi^2}\operatorname{tr}(F \wedge F) \wedge \omega\right)\right]$. Then by Lemma 3.1 $v_{\mathscr{E}}(\Gamma) = \frac{1}{4\pi^2}[\tau_{\mathscr{E}}(\omega)]$.

Lemma 3.2 *Let V be a smooth projective variety. Let $D \subset V$ be a proper subvariety, then the restriction map*

$$\rho\colon H^0(\Omega_V^2) \to H_{DR}^2(V \backslash D)$$

is injective.

Proof. Let $\omega \in H^0(\Omega_V^2)$ and assume that $\rho(\omega) = 0$. We will show that this implies that for any two-cycle $\gamma \subset V$, $\int_\gamma \omega = 0$. Thus ω is cohomologous to zero on V. The result follows because on a projective variety a holomorphic form which is cohomologous to zero is necessarily zero.

Let $D_1, \ldots, D_n$ be a set of codimension one components of D such that the $c_1(D_i)$'s are a basis of the span of the Chern classes of all the divisors contained in D. Let $H_1, \ldots, H_r$ be hyperplane sections of V such that $X = H_1 \cap \ldots \cap H_r \cap V$ is a smooth surface. Let $\iota\colon X \hookrightarrow V$ be the inclusion map. By the Lefschetz Hyperplane Section Theorem the classes $\iota^* c_1(D_i) \in H^{1,1}(X, \mathbf{Z})$ are independent. Let $\mathrm{NS}(X)$ be the Néron-Severi group of X, and let $D^\perp \subset H_2(V, \mathbf{Z})$ be the sublattice perpendicular to the $c_1(D_i)$. The pairing between $H^{1,1}(X, \mathbf{Z})$ and $\mathrm{NS}(X)$ given by integration is non-degenerate by the Hodge index theorem. Since the classes $\iota^* c_1(D_i)$ are independent we can decompose $H_2(V, \mathbf{Q})$ as

$$H_2(V, \mathbf{Q}) = \iota_*(\mathrm{NS}(X) \otimes \mathbf{Q}) + (D^\perp \otimes \mathbf{Q})\ . \tag{3}$$

Let $\gamma \in \imath_* \mathrm{NS}(X)$, then clearly $\int_\gamma \omega = 0$. If $\gamma \in D^\perp$ then by a generic position argument there exists a cycle $\gamma' \subset V$ homologous to γ such that $\gamma' \subset V \backslash D$, hence $\int_\gamma \omega = 0$. By the decomposition (3) this implies that $\int_\gamma \omega = 0$ for any two cycle on V, q.e.d.

Lemma 3.3 *Let S be an algebraic surface and let B be an algebraic variety. Let $\mathscr{E}$ be a B-flat sheaf on $S \times B$ such that for all $b \in B$ the sheaf $\mathscr{E}|_{S \times \{b\}}$ is torsion-free. Let X be an algebraic variety and let $\varphi: X \to B$ be a morphism. Set $\mathscr{F} = (\mathrm{id}_S \times \varphi)^* \mathscr{E}$. Assume that either φ is a birational morphism or the inclusion of a subvariety of B. Then for any $\alpha \in H_2(S)$ we have that*

$$\nu_{\mathscr{F}}(\alpha) = \varphi^* \nu_{\mathscr{E}}(\alpha) .$$

Proof. We will show that $c_2(\mathscr{F}) = (\mathrm{id}_S \times \varphi)^* c_2(\mathscr{E})$, this clearly implies the lemma.

We claim that the sheaf $\mathscr{E}$ has a short locally free resolution, i.e. there exist locally free sheaves $\mathscr{E}_0$, $\mathscr{E}_1$ on $S \times B$ such that $\mathscr{E}$ fits into the exact sequence

$$0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{E} \to 0 . \tag{4}$$

This is true on each slice $S \times \{b\}$ because S is a surface and $\mathscr{E}|_{S \times \{b\}}$ is torsion-free. Then there is a parametrized short resolution by the B-flatness of $\mathscr{E}$.

Then if φ satisfies our hypothesis the pull-back of the sequence (4) by $\mathrm{id}_S \times \varphi$ remains exact. Hence we get the exact sequence

$$0 \to (\mathrm{id}_S \times \varphi)^* \mathscr{E}_1 \to (\mathrm{id}_S \times \varphi)^* \mathscr{E}_0 \to (\mathrm{id}_S \times \varphi)^* \mathscr{E} \to 0 .$$

This implies that $c_2(\mathscr{F}) = (\mathrm{id}_S \times \varphi)^* c_2(\mathscr{E})$, q.e.d.

Now we are ready to prove the first main theorem of this section.

Theorem 3.2 *Let (S, H) be a polarized regular projective surface. Let $\bar{M} = \bar{M}(S, H; c_1, c_2)$ be a fine moduli space. Let $\bar{M}_{\mathrm{red}}$ be the reduced scheme associated to $\bar{M}$. Let $\varphi: \tilde{M} \to \bar{M}_{\mathrm{red}}$ be a desingularization. Let $\omega \in H^0(\Omega_S^2)$ and let $\Gamma \in H_2(S)$ be the Poincaré dual of ω. Then the class $\varphi^* \nu(\Gamma) \in H^2(\tilde{M})$ is represented, in de Rham cohomology, by $\frac{1}{4\pi^2} \tau_{\tilde{M}}(\omega)$.*

Proof. Let $\tilde{\mathscr{E}} = (\mathrm{id}_S \times \varphi)^* \mathscr{E}$. By Lemma 3.3 we have that $\varphi^* \nu_{\mathscr{E}}(\Gamma) = \nu_{\tilde{\mathscr{E}}}(\Gamma)$. Since $c_2(\tilde{\mathscr{E}}) \in H^{2,2}(S \times \tilde{M})$ and Γ is Poincaré dual to a $(2, 0)$-form we conclude that $\nu_{\tilde{\mathscr{E}}}(\Gamma) \in H^{2,0}(\tilde{M})$. Let $M_0 = (M(S, H; c_1, c_2))_{\mathrm{red}} \subset (\bar{M}(S, H; c_1, c_2))_{\mathrm{red}}$ be the open dense subset parametrizing slope-stable bundles, and set $\tilde{M}_0 = \varphi^{-1}(M_0)$.

By Theorem 3.1 we have that $\nu_{\tilde{\mathscr{E}}}(\Gamma)|_{\tilde{M}_0} = \frac{1}{4\pi^2} [\tau_{\tilde{\mathscr{E}}}(\omega)|_{\tilde{M}_0}]$. Applying Lemma 3.2 to each irreducible component of $\tilde{M}$ we conclude that $\nu_{\tilde{\mathscr{E}}}(\Gamma) = \frac{1}{4\pi^2} [\tau_{\tilde{\mathscr{E}}}(\omega)]$, q.e.d.

Now we will concentrate on special properties of families of rank-two sheaves on a surface. Up to now the rank of the sheaves on S played no role. In fact the theorem above would also hold if we considered fine moduli spaces of sheaves of any rank. The theorems that we now wish to prove are special to rank two, the reason being that in this case the zero-locus of a section (assuming it is zero-dimensional) represents the second Chern class of the sheaf.

We first need a few definitions. Let ω be a holomorphic two-form on the projective surface S. Let $\mathrm{Hilb}^d(S)$ be the Hilbert scheme of zero-dimensional subschemes of S of length d. Let $\iota: \mathscr{W} \to S \times \mathrm{Hilb}^d(S)$ be the universal subscheme. Let $U \subset \mathrm{Hilb}^d(S)$ be the non-empty open set parametrizing subschemes supported on d distinct points. Beauville [3] has proved that there exists a holomorphic two-form on $\mathrm{Hilb}^d(S)$, which we will denote by $\omega^{(d)}$, such that $\omega^{(d)}|_U = (p_U)_* \iota^* p_S{}^* \omega$, where p_S, p_U are the projections of $S \times U$ on S and U, respectively. In other words $\omega^{(d)}$ extends the form obtained by integrating ω on the cycles $\mathscr{W} \cdot (S \times \{u\})$, for $u \in U$.

Let S be a surface, B be a smooth algebraic variety, $\mathscr{Z} \subset S \times B$ a subscheme. Assume there is an open dense subset $U \subset B$ such that, for all $u \in U$, $\mathscr{Z} \cdot S \times \{u\}$ is a zero-dimensional subscheme. Let the cycle $[\mathscr{Z}]$ break up as $[\mathscr{Z}] = \sum_i n_i [\mathscr{Z}_i]$, where for each i the scheme $\mathscr{Z}_i$ is irreducible and generically reduced. There exists an open dense subset $U_0 \subset U$ such that $p: \mathscr{Z}_i \to U_0$ is flat for all i, where p is the projection. Let d_i be the length of $\mathscr{Z}_i \cdot (S \times \{u\}), u \in U_0$. Each $\mathscr{Z}_i$ defines a morphism $\alpha_i: U_0 \to \mathrm{Hilb}^{d_i}(S)$. Let $\alpha': U_0 \to \times_i \mathrm{Hilb}^{d_i}(S)$ be the product of the maps α_i. Let $\alpha: B \to \times_i \mathrm{Hilb}^{d_i}(S)$ be the rational map extending α'. Consider the holomorphic two-form $\sum_i n_i \omega^{(d_i)}$ on $\times_i \mathrm{Hilb}^{d_i}(S)$. Then $\alpha^*(\sum_i n_i \omega^{(d_i)})$ is a global holomorphic two-form on B, extending $(\alpha')^*(\sum_i n_i \omega^{(d_i)})$. We will denote $\alpha^*(\sum_i n_i \omega^{(d_i)})$ by $\omega_{\mathscr{Z}}$. Loosely speaking $\omega_{\mathscr{Z}}$ is obtained by integrating ω along the fibres of $p: \mathscr{Z} \to B$.

Lemma 3.4 *Let S be a projective surface, B a quasi-projective variety. Let $\mathscr{E}$ be a rank-two vector bundle on $S \times B$. Then there exist: a dense subset $U \subset B$, a projective variety $\bar{U} \supset U$, with U dense in $\bar{U}$, and a $\bar{U}$-flat sheaf, $\bar{\mathscr{E}}$, on $S \times \bar{U}$ such that $\bar{\mathscr{E}}|_{S \times U} \cong \mathscr{E}|_{S \times U}$.*

Proof. Let p_S, p_B be the projections of $S \times B$ on S and B, respectively. Let H be an ample divisor on S. There exists an N_0 such that if $n \geqq N_0$, then $R^i(p_B)_*(\mathscr{E} \otimes p_S^*[nH]) = 0$ for $i > 0$. Let $b \in B$, then, as is easily verified, there exists N_1 such that if $n \geqq N_1$ then $\mathscr{E} \otimes p_S^*[nH]|_{S \times \{b\}}$ has sections with zero-dimensional zero-locus. Choose $n \geqq \max\{N_0, N_1\}$, then there exists an open dense $B_0 \subset B$ such that $\mathscr{E}_0 = \mathscr{E}|_{S \times B_0}$ has a section with zero-locus $\mathscr{Z}$ such that for all $b \in B_0$ the intersection $Z_b = \mathscr{Z} \cdot (S \times \{b\})$ is a zero-dimensional subscheme of S. Hence for all $b \in B_0$ the bundle $E_b = \mathscr{E}|_{S \times \{b\}}$ fits into the exact sequence

$$0 \to \mathcal{O}_S \to E_b \to I_{Z_b} \otimes [nH] \to 0\ .$$

We will prove the lemma by constructing a projective variety $\tilde{V}$ parametrizing all non-trivial extensions of $I_Z \otimes [nH]$ by $\mathcal{O}_S$, where Z is any zero-dimensional subscheme of S of length equal to the length of Z_b. The only properties of $[nH]$ that we will use will be that $h^0([-nH]) = h^1([-nH]) = 0$. Hence we might as well parametrize all non-trivial extensions

$$0 \to \mathcal{O}_S \to F \to I_Z \otimes [D] \to 0$$

where D is a fixed divisor such that $h^0(-D) = h^1(-D) = 0$.

Fix a positive integer c. Let $Z \subset S$ be a zero-dimensional subscheme of length c. By Serre duality the set of extensions $\mathrm{Ext}^1(I_Z \otimes [D], \mathcal{O}_S)$ is canonically identified with $H^1(I_Z \otimes [D + K_S])^*$. From the hypothesis that $h^1(-D) = 0$ we get the exact sequence

$$0 \to H^1(I_Z \otimes [D + K_S])^* \to H^0((D + K_S)|_Z)^* \xrightarrow{f_Z} H^0(D + K_S)^*$$

hence $\mathrm{Ext}^1(I_Z \otimes [D], \mathcal{O}_S)$ is identified with the kernel of f_Z. Let $\mathscr{Z} \subsetneq S \times \mathrm{Hilb}^c(S)$ be the universal subscheme. Let p_S, p_H be the projections of $S \times \mathrm{Hilb}^c(S)$ on S and $\mathrm{Hilb}^c(S)$, respectively. On $S \times \mathrm{Hilb}^c(S)$ we have the map of sheaves $p_S^*[D + K_S] \to I_{\mathscr{Z}} \otimes p_S^*[D + K_S]$, hence we get the following map of vector bundles over $\mathrm{Hilb}^c(S)$

$$F: (p_H)_* p_S^*[D + K_S] \to (p_H)_* I_{\mathscr{Z}} \otimes p_S^*[D + K_S] .$$

We let $V \subset \mathrm{Hilb}^c(S)$ be the degeneracy locus of F (i.e. the locus of points where F has a non-trivial kernel). Let $[Z] \in \mathrm{Hilb}^c(S)$, then the fiber of F over $[Z]$ is naturally identified with f_Z. Then, by the discussion above, $[Z] \in V$ if and only if $\mathrm{Ext}^1(I_Z \otimes [D], \mathcal{O}_S)$ is non-trivial. Next we let $\tilde{V} \subset \mathbf{P}((p_H)_* p_S^*[D + K_S]) \times V$ be the locus of couples $([\sigma], [Z])$ such that $[Z] \in V$ and $\sigma \in \ker f_Z$. Let $\pi: \tilde{V} \to V$ be the natural projection. Let $\tilde{Z} \subset S \times \tilde{V}$ be the pull-back of $\mathscr{Z}|_{S \times V}$, i.e. $\tilde{\mathscr{Z}} = (\mathrm{id}_S \times \pi)^* \mathscr{Z}$. Let $\pi_S, \pi_{\tilde{V}}$ be the projections of $S \times \tilde{V}$ to S and $\tilde{V}$ respectively. By definition of $\tilde{V}$ there exists a line bundle $[M]$ on $\tilde{V}$ such that there exists a global nowhere zero tautological section of the relative Ext-sheaf on $\tilde{V}$ given by $\mathrm{Ext}^1_{\mathcal{O}_{\tilde{V}}}(I_{\tilde{Z}} \otimes \pi_S^*[D] \otimes \pi_{\tilde{V}}^*[M], \mathcal{O}_{S \times \tilde{V}})$. Now reasoning as in [13], since $h^0(-D) = 0$, we conclude that there is a global extension

$$0 \to \mathcal{O}_{S \times \tilde{V}} \to \mathscr{F} \to I_{\tilde{Z}} \otimes \pi_S^*[D] \otimes \pi_{\tilde{V}}^*[M] \to 0 \tag{5}$$

such that the restriction of (5) to $S \times ([\sigma], [Z])$ is isomorphic to the extension corresponding to σ. Clearly $\mathscr{F}$ is $\tilde{V}$-flat (the Chern classes of the restriction of $\mathscr{F}$ to any slice $S \times \{x\}$ are given by $c_1 = c_1(D)$ and $c_2 = c$). The extension $\mathscr{F}$ is universal in an obvious sense.

Now we go back to our vector bundle $\mathscr{E}_0$ on $S \times B_0$. The global section of $\mathscr{E}_0$ induces a morphism $\varphi: B_0 \to \tilde{V}$, where $\tilde{V}$ parametrizes all non-trivial extensions of the form

$$0 \to \mathcal{O}_S \to F \to I_Z \otimes [nH] \to 0 .$$

The morphism φ is such that

$$\mathscr{E}_0 \cong (\mathrm{id}_S \times \varphi)^* \mathscr{F} \otimes p_{B_0}^*[N] ,$$

where $[N]$ is a line bundle on B_0. There exists a projective $\bar{U} \supset B_0$ such that φ extends to a morphism $\bar{\varphi}: \bar{U} \to \tilde{V}$. Then $\bar{\mathscr{E}} = (\mathrm{id}_S \times \bar{\varphi})^* \mathscr{F}$ is a $\bar{U}$-flat sheaf on $S \times \bar{U}$ such that $\bar{\mathscr{E}}|_{S \times B_0} \cong \mathscr{E}_0 \otimes p_{B_0}^*[N]$. Now let $U \subset B_0$ be an open dense subset such that $[N]$ is trivial on U. Then $\bar{\mathscr{E}}|_{S \times U} \cong \mathscr{E}|_{S \times U}$, q.e.d.

Remark 3.4 Assume B is smooth, then clearly we can choose $\bar{U}$ to be smooth.

Theorem 3.3 *Let S be a regular projective surface. Let ω be a holomorphic two-form on S. Let B be a smooth quasi-projective variety. Let $\mathscr{E}$ be a rank-two vector bundle on $S \times B$. Let $\mathscr{Z} \subset S \times B$ be the zero-locus of a section $\sigma \in H^0(\mathscr{E})$. Assume that for the generic $b \in B$ the intersection $\mathscr{Z} \cdot (S \times \{b\})$ is zero-dimensional, then*

$$\omega_{\mathscr{Z}} = \frac{1}{4\pi^2} \tau_{\mathscr{E}}(\omega) .$$

Proof. We apply the above lemma to the vector bundle $\mathscr{E}$. By the remark we can assume $\bar{U} \supset U$ to be smooth. Let $\mathscr{Z}_i$ be the irreducible components of $\mathscr{Z}$, so that

$[\mathscr{Z}] = \sum_i n_i[\mathscr{Z}_i]$ for some positive coefficients n_i. By shrinking U, if necessary, we can assume that $\mathscr{Z}_i$ intersects $S \times \{u\}$ in a zero-dimensional reduced subscheme for all $u \in U$. We will show that $\tau_{\mathscr{E}}(\omega)|_U = 4\pi^2 \omega_{\mathscr{Z}}|_U$. Since U is dense in B, this will imply the theorem.

Clearly $\tau_{\mathscr{E}}(\omega)|_U = \tau_{\bar{\mathscr{E}}}(\omega)|_U$. The subscheme $\mathscr{Z} \subset S \times U$ defines the holomorphic two-form $\omega_{\mathscr{Z}}$ on B, but, since U is dense in $\bar{U}$ it also defines a holomorphic two-form $\omega_{\mathscr{Z}}(\bar{U})$ on $\bar{U}$ such that $\omega_{\mathscr{Z}}|_U = \omega_{\mathscr{Z}}(\bar{U})|_U$. Our goal then is to prove that $\omega_{\mathscr{Z}}(\bar{U}) = \frac{1}{4\pi^2}\tau_{\bar{\mathscr{E}}}(\omega)$.

Since $\mathscr{Z}$ is the zero-locus of a section of $\mathscr{E}$ we have that $c_2(\mathscr{E})$ is Poincaré dual to the cycle $[\mathscr{Z}]$. Hence the cohomology class $v_{\bar{\mathscr{E}}}(\Gamma)|_U$ is represented by $\omega_{\mathscr{Z}}(\bar{U})|_U$. On the other hand the sheaf $\bar{\mathscr{E}}|_{S\times U}$ is locally free, hence by Theorem 3.1 we have that $v_{\bar{\mathscr{E}}}(\Gamma)|_U$ is also represented by $\frac{1}{4\pi^2}[\tau_{\bar{\mathscr{E}}}(\omega)]|_U$. Thus $[\omega_{\mathscr{Z}}(\bar{U})]|_U = \frac{1}{4\pi^2}[\tau_{\bar{\mathscr{E}}}(\omega)]|_U$. By Lemma 3.2 we conclude that $[\omega_{\mathscr{Z}}(\bar{U})] = \frac{1}{4\pi^2}[\tau_{\bar{\mathscr{E}}}(\omega)]$, q.e.d.

Theorem 3.4 *Let S be a regular projective surface. Let B be a smooth quasi-projective variety. Let $\omega \in H^0(\Omega_S^2)$. Let $\mathscr{E}$, $\mathscr{F}$ be rank two vector bundles on $S \times B$. Let $C \subset S$ be an effective divisor and let L be a line bundle on $C \times B$. Assume that $\mathscr{E}$ and $\mathscr{F}$ fit into the exact sequence*

$$0 \to \mathscr{E} \xrightarrow{f} \mathscr{F} \to \mathscr{O}_{C\times B}(L) \to 0$$

i.e. $\mathscr{E}$ is obtained from $\mathscr{F}$ by an elementary modification. Then

$$\tau_{\mathscr{E}}(\omega) = \tau_{\mathscr{F}}(\omega)\ .$$

Proof. As is easily checked there exists a line bundle N on $S \times B$ with the following properties:

(i) $N \cong p_S^* N_1 \otimes p_B^* N_2$, where p_S, p_B are the projections of $S \times B$ on S and B, and N_1, N_2 are line bundles on S, B respectively.

(ii) there exists a section $\sigma \in H^0(\mathscr{E} \otimes N)$ such that its zero-locus $\mathscr{Z} \subset S \times B$ intersects the generic slice $S \times \{b\}$ in a zero-dimensional subscheme.

(iii) let $\mathscr{W} \subset S \times B$ be the zero-locus of the section $f(\sigma) \in H^0(\mathscr{F} \otimes N)$, then for the generic $b \in B$ the intersection $\mathscr{W} \cap (S \times \{b\})$ is zero-dimensional.

By Remark 3.3 we have that $\tau_{\mathscr{E}\otimes N}(\omega) = \tau_{\mathscr{E}}(\omega)$ and $\tau_{\mathscr{F}\otimes N}(\omega) = \tau_{\mathscr{F}}(\omega)$. Hence the thesis of the theorem is equivalent to $\tau_{\mathscr{E}\otimes N}(\omega) = \tau_{\mathscr{F}\otimes N}(\omega)$. By Theorem 3.3 we have that $\tau_{\mathscr{E}\otimes N}(\omega) = 4\pi^2\omega_{\mathscr{Z}}$ and $\tau_{\mathscr{F}\otimes N}(\omega) = 4\pi^2\omega_{\mathscr{W}}$, hence we must show that $\omega_{\mathscr{Z}} = \omega_{\mathscr{W}}$. Since the restrictions of $\mathscr{E}$ and $\mathscr{F}$ to $(S \times B)\backslash(C \times B)$ are isomorphic we can write $[\mathscr{W}] = [\mathscr{Z}] + [\mathscr{Z}']$, where $\mathscr{Z}'$ is a subscheme supported on $C \times B$. Clearly $\omega_{\mathscr{Z}'} = 0$, so we conclude that $\omega_{\mathscr{Z}} = \omega_{\mathscr{W}}$.

We conclude this section by examining the case of a B-flat rank-two torsion-free sheaf $\mathscr{E}$ on $S \times B$, with the assumption that for all $b \in B$ the restriction $\mathscr{E}|_{S\times\{b\}}$ is not locally free. Let $\Gamma \in H_2(S)$ be Poincaré dual to the holomorphic two-form ω. We want to give a representative of $v_{\mathscr{E}}(\Gamma)$ similar to the representative $\frac{1}{4\pi^2}\tau_{\mathscr{E}}(\omega)$ in the case that $\mathscr{E}$ is locally free. This representation will be useful in Sects. 5 and 6.

Lemma 3.5 *Let S be a projective surface. Let B be a quasi-projective variety. Let $\mathscr{E}$ be a rank-two torsion-free B-flat sheaf on $S \times B$. Then there exists an open dense subset $U \subset B$ such that the following hold. The sheaf $\mathscr{E}^{**}|_{S \times U}$ is locally free. For each $u \in U$ the sequence*

$$0 \to \mathscr{E}|_{S \times \{u\}} \xrightarrow{f} \mathscr{E}^{**}|_{S \times \{u\}} \to \mathscr{Q}|_{S \times \{u\}} \to 0$$

is exact and is identified with the canonical sequence of $\mathscr{E}|_{S \times \{u\}}$. Furthermore, let $\mathscr{Z} \subset S \times U$ be the subscheme whose ideal sheaf is $\mathrm{Ann}(\mathscr{Q})$, then the projection $p: \mathscr{Z} \to U$ is flat.

Proof. The double dual $\mathscr{E}^{**}$ is reflexive, hence its singularity set has codimension at least three. Thus there exists an open dense $U_1 \subset B$ such that $\mathscr{E}^{**}|_{S \times U_1}$ is locally free. Since $\mathscr{E}$ is torsion-free the canonical map into $\mathscr{E}^{**}$ is an injection, let $\mathscr{Q}_1$ be the quotient. The support of $\mathscr{Q}_1$, $\mathrm{Supp}\,\mathscr{Q}_1$, is the singularity set of $\mathscr{E}$, since $\mathscr{E}$ is torsion-free we conclude that $\mathscr{Q}_1$ is supported on a set of codimension at least two. Hence there exists a dense open $U_2 \subset U_1$ such that for each $u \in U_2$ we have that $\mathrm{Supp}\,\mathscr{Q}_1 \cap (S \times \{u\})$ is zero-dimensional (or empty, but in that case we are done, just let $U = U_2$). Let $\mathscr{Q}_2 = \mathscr{Q}_1|_{S \times U_2}$, let $\mathscr{Z}_2 \subset S \times U_2$ be the subscheme whose ideal sheaf is $\mathrm{Ann}\,\mathscr{Q}_2$. Then there exists an open dense subset $U_3 \subset U_2$ such that, if $\mathscr{Z}_3 = \mathscr{Z}_2|_{S \times U_3}$, then the projection $p_3: \mathscr{Z}_3 \to U_3$ is flat. Finally there exists an open dense subset $U \subset U_3$ such that the sheaf $\mathscr{Q} = \mathscr{Q}_2|_{S \times U}$ is U-flat. We claim that U satisfies the conditions in the lemma. In fact let $u \in U$, then since $\mathscr{Q}$ is U-flat the following sequence is exact:

$$0 \to \mathscr{E}|_{S \times \{u\}} \to \mathscr{E}^{**}|_{S \times \{u\}} \to \mathscr{Q}|_{S \times \{u\}} \to 0 .$$

The sheaf $\mathscr{E}^{**}|_{S \times \{u\}}$ is identified with $\mathscr{E}|_{S \times \{u\}}$ outside $\mathscr{Z} \cap (S \times \{u\})$ a set of codimension two. Hence there is an isomorphism of $\mathscr{E}^{**}|_{S \times \{u\}}$ with $(\mathscr{E}|_{S \times \{u\}})^{**}$ outside this same codimension two set. Since they are both locally free Hartog's Theorem implies that the isomorphism extends to all of $S \times \{u\}$.

The other conditions are satisfied by construction.

Let S, B, $\mathscr{E}$, U, $\mathscr{Z}$ be as in the above lemma. Let $\mathscr{Z}_i$ be the irreducible components of $\mathscr{Z}$. Let $p_i: \mathscr{Z}_i \to U$ be the restriction to $\mathscr{Z}_i$ of the projection, it is flat for all i. Let $d_i = \ell(\mathscr{Z}_i \cdot (S \times \{u\}))$, where $u \in U$. Then each $\mathscr{Z}_i$ defines a morphism $\alpha_i: U \to \mathrm{Hilb}^{d_i}(S)$. Let $\alpha: U \to \times_i \mathrm{Hilb}^{d_i}(S)$ be the product morphism.

Now let $\omega \in H^0(\Omega_S^2)$. Recall that on each Hilbert scheme $\mathrm{Hilb}^{d_i}(S)$ we have the holomorphic two-form denoted by $\omega^{(d_i)}$.

Proposition 3.2 *Let S be a regular projective surface. Let $\omega \in H^0(\Omega_S^2)$ and let $\Gamma \in H_2(S)$ be its Poincaré dual. Let B be a smooth projective variety. Let $\mathscr{E}$ be a B-flat rank-two torsion-free sheaf on $S \times B$. Let $U \subset B$ be the open dense subset of Lemma* 3.5. *Define $\alpha: U \to \times_i \mathrm{Hilb}^{d_i}(S)$ as in the above discussion. Then there exist positive integers n_i such that the holomorphic two-form on U given by*

$$\frac{1}{4\pi^2}\tau_{\mathscr{E}^{**}}(\omega) + \alpha^* \sum_i n_i \omega^{(d_i)}$$

extends to all of B, and the extended form represents $\nu_{\mathscr{E}}(\Gamma)$ in de Rham cohomology.

Proof. Let p_S, p_B be the projections of $S \times B$ on S and B, respectively. There exist line bundles L, M on S and B such that $\mathscr{E} \otimes p_S^*(L) \otimes p_B^*(M)$ has a section Σ whose zero-locus $\mathscr{W}$ intersects the generic slice $S \times \{b\}$ in a zero-dimensional subscheme. Let $U \subset B$ be the open dense subset given by Lemma 3.5. By shrinking U, if necessary, we can assume that for all $b \in U$ the intersection $\mathscr{W} \cdot (S \times \{b\})$ is a zero-dimensional subscheme of S. Let $f: \mathscr{E} \otimes p_S^*(L) \otimes p_B^*(M) \to \mathscr{E}^{**} \otimes p_S^*(L) \otimes p_B^*(M)$ be the canonical injection, then $f(\Sigma)$ is a section of $\mathscr{E}^{**} \otimes p_S^*(L) \otimes p_B^*(M)$. Let $\mathscr{W}_0 \subset S \times B$ be the zero-locus of $f(\Sigma)$, clearly if $b \in U$ then $\mathscr{W}_0 \cap (S \times \{b\})$ is zero-dimensional. There exist positive integers n_i such that

$$[\mathscr{W}] = [\mathscr{W}_0] + \sum_i n_i [\mathscr{Z}_i] .$$

Consider the holomorphic two-form (on B) given by $\omega_{\mathscr{W}} = \omega_{\mathscr{W}_0} + \sum_i n_i \omega_{\mathscr{Z}_i}$.

Claim. The two-form $\omega_{\mathscr{W}}$ represents, in de Rham cohomology the cohomology class $v_{\mathscr{E}}(\Gamma)$.

Since $\mathscr{W}$ is the zero-locus of a section of $\mathscr{E} \otimes p_S^*(L) \otimes p_B^*(M)$ we have that $c_2(\mathscr{E} \otimes p_S^*(L) \otimes p_B^*(M)) = [\mathscr{W}]$. Hence

$$c_2(\mathscr{E}) = [\mathscr{W}] - c_1(\mathscr{E}) \cdot c_1(p_S^*(L) \otimes p_B^*(M)) - c_1(p_S^*(L) \otimes p_B^*(M))^2 .$$

Since Γ is perpendicular to the first Chern class of any line bundle on S, we conclude that

$$c_2(\mathscr{E})/\Gamma = [\mathscr{W}]/\Gamma .$$

Hence $[\omega_{\mathscr{W}}]|_U = v_{\mathscr{E}}(\Gamma)|_U$. Since $v_{\mathscr{E}}(\Gamma)$ is of type (2, 0) we conclude that $[\omega_{\mathscr{W}}] = v_{\mathscr{E}}(\Gamma)$ on all of B.

By the claim $v_{\mathscr{E}}(\Gamma)$ is represented by $\omega_{\mathscr{W}_0} + \alpha^*(\sum_i n_i \omega^{(d_i)})$ (where α is defined as above). Consider the bundle over U given by $\mathscr{G} = \mathscr{E}^{**} \otimes p_S^*(L) \otimes p_B^*(M)$. By Theorem 3.3 the two-form $\omega_{\mathscr{W}_0}$ is an extension of $\frac{1}{4\pi^2} \tau_{\mathscr{G}}(\omega)$ to all of B. By Remark 3.3 we have that $\tau_{\mathscr{E}^{**}}(\omega) = \tau_{\mathscr{G}}(\omega)$, this finishes the proof.

Remark 3.5 Let H be a polarization on S. Assume that for every $b \in B$ we have that $\mathscr{E}|_{S \times \{b\}}$ is H-semistable. Then $(\mathscr{E}|_{S \times \{b\}})^{**}$ is H-slope-semistable. Thus, by applying Lemma 3.5 and then shrinking the open subset $U \subset B$ if necessary, we can assume the following. The thesis of Lemma 3.5 is verified by $\mathscr{E}|_{S \times U}$ and furthermore the vector bundle $\mathscr{E}^{**}|_{S \times U}$ induces a morphism $\beta: U \to Y$, where either $Y = M(S, H; c_1, c')$ or $Y = P(c_1, c')$ where $c_1 = c_1(\mathscr{E}|_{S \times \{b\}})$ and $c' = c_2(\mathscr{E}^{**}|_{S \times \{b\}})$ for $b \in U$.

We then set $f = \alpha \times \beta$, so $f: U \to \times_i \mathrm{Hilb}^{d_i}(S) \times Y$. An immediate consequence of the above lemma and the functoriality of the τ-map is the following

Corollary 3.2 *Let (S, H) be a polarized regular projective surface. Let B be a smooth projective variety. Let $\mathscr{E}$ be a B-flat rank-two torsion-free sheaf on $S \times B$. Let $\Gamma \in H_2(S)$ be the Poincaré dual of a holomorphic two-form on S. Let $U \subset B$ and $f: U \to \times_i \mathrm{Hilb}^{d_i}(S) \times Y$ be as in the above remark. Then there exists a holomorphic two-form, λ on $\times_i \mathrm{Hilb}^{d_i}(S) \times Y$ such that $f^* \lambda$ extends to a holomorphic form on all of B and such that $f^* \lambda$ represents $v_{\mathscr{E}}(\Gamma)$.*

Remark 3.6 The facts proved in this section have well-known analogues in the theory of line bundles over a curve.

Let C be a smooth projective curve, then the analogous of the τ-map is the following. Let B be an algebraic variety. Let $\mathscr{L}$ be a line bundle on $C \times B$. Let $\omega \in H^0(\Omega_C^1)$. We can mimic the definition above to define a section $\tau_{\mathscr{L}}(\omega) \in H^0(\Omega_B^1)$. Let $f: \Theta_B \to H^1(\mathcal{O}_C) \cong H^{0,1}(C)$ be the Kodaira-Spencer map. Let $v \in \Theta_b$, then we set

$$\tau_{\mathscr{L}}(\omega)(v) = \operatorname{tr}(f(v) \cup \omega) = \int_C f(v) \wedge \omega \, .$$

Then the τ-map for a curve reduces $\left(\text{up to a factor of } \frac{i}{2\pi}\right)$ to the well-known isomorphism between $H^0(\Omega_C^1)$ and $H^0(\Omega^1_{\operatorname{Pic}^d(C)})$. The analogous of Theorem 3.3 is given by the infinitesimal version of Abel's theorem, while Theorem 3.4 corresponds to the translation invariance of the global holomorphic one-forms on $\operatorname{Pic}^d(C)$.

4 Proof of Theorems 2.4, 2.7

Let $M = M(S, H; c_1, c_2)$. Let $D \in |K_S|$ be a smooth curve. Assume that M is K_S-good, then in each component of M there is a point representing a vector bundle E_0 such that $h^2(\operatorname{ad} E_0 \otimes [-K_S]) = 0$. Thus $\rho_D: (M, [E_0]) \to \mathbf{Def}^0(E_0|_D)$ has surjective differential near $[E_0]$. Notice that since $|K_S|$ is effective, $h^2(\operatorname{ad} E_0) = 0$, hence M is smooth at $[E_0]$. Hence, by Corollary 2.5 there exists $[E] \in M$ such that M is smooth at $[E]$ and $E|_D$ is the generic stable bundle with $\det E|_D \cong \det E_0|_D$. Let's consider the exact sequence

$$0 \to H^0(\operatorname{ad} E \otimes [K_S]|_D) \to H^1(\operatorname{ad} E) \xrightarrow{f_\sigma} H^1(\operatorname{ad} E(K_S)) \to H^1(\operatorname{ad} E \otimes [K_S]|_D) \to 0 \tag{1}$$

where f_σ is multiplication by a section $\sigma \in H^0(K_S)$ such that $(\sigma) = D$. The tangent space, $T_{[E]}M$, to M at $[E]$ is isomorphic to $H^1(\operatorname{ad} E)$. We can view the skew-symmetric form $\tau_M(\omega)$ at $[E]$ as the map $\tau_M(\omega)_{[E]}: T_{[E]} \to (T_{[E]})^*$ obtained by composing f_σ with the pairing $H^1(\operatorname{ad} E \otimes [K_S]) \otimes H^1(\operatorname{ad} E) \to \mathbf{C}$. By Serre duality and the exact sequence (1) then $\dim \ker \tau_M(\omega)_{[E]} = h^0(\operatorname{ad} E \otimes [K_S]|_D)$.

We will prove Theorem 2.7 by showing that if M is K_S-good, $D \in |K_S|$ is smooth and irreducible of genus at least two, then $h^0(\operatorname{ad} E \otimes [K_S]|_D) = 0, 1$, according to whether $\dim M$ is even or odd.

For this purpose, notice that, since by hypothesis the moduli space M is good, $\dim M = 4c_2 - c_1^2 - 3(p_g + 1)$, where p_g is the geometric genus of S. By adjunction the line bundle $\theta = K_S|_D$ is a theta characteristic on D. The exact sequence

$$0 \to H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_S(K_S)) \to H^0(\mathcal{O}_D(K_S)) \to 0$$

gives $h^0(\theta) = p_g - 1$. Let $P = \det E|_D$, where $[E] \in M$ is any point. Then $\deg P = c_1 \cdot K_S \equiv c_1^2 \operatorname{mod}(2)$. Hence $h^0(\theta) + \deg P \equiv \dim M \operatorname{mod}(2)$. Then Theorem 2.7 will follow from the following theorem, of which we postpone the proof.

Theorem 4.1 *Let D be a smooth connected curve of genus $g \geqq 2$, P a line bundle on D, θ a theta characteristic on D. Let $M(D; P)$ be the moduli space of stable rank two vector bundles, E, on D with* $\det E \cong P$. *Let* $[E] \in M(D; P)$ *be generic, then* $h^0(\operatorname{ad} E \otimes \theta) = 0$ *or* 1 *according to whether* $h^0(\theta) + \deg P$ *is even or odd.*

Remark 4.1 In [20] there is a proof of Theorem 4.1 in the case in which $h^0(\theta) + \deg P$ is odd. The general methods developed in that paper can be adapted to give a proof also in the case of $h^0(\theta) + \deg P$ even. For completeness, and also because the whole machinery of [20] is not needed in our case, we will give a proof of Theorem 4.1.

Now let's show how Theorem 2.4 follows from Theorem 2.7. Assume M is K_S-good and even-dimensional, let d be its dimension. Let $\Gamma \in H_2(S)$ be the Poincaré dual of ω. Let $\pi\colon \tilde{M} \to \bar{M}(S, H; c_1, c_2)$ be a desingularization of the compactification of M. By Theorem 3.2

$$\delta_{\tilde{M}}(\Gamma + \bar{\Gamma}) = \left(\frac{1}{4\pi^2}\right)^d \int_{\tilde{M}} \underbrace{(\tau_{\tilde{M}}(\omega) + \overline{\tau_{\tilde{M}}(\omega)}) \wedge \ldots \wedge (\tau_{\tilde{M}}(\omega) + \overline{\tau_{\tilde{M}}(\omega)})}_{d} . \tag{2}$$

By Theorem 2.7 there is an open dense subset of $\tilde{M}$ over which the two-form $\tau_{\tilde{M}}(\omega) + \overline{\tau_{\tilde{M}}(\omega)}$ is symplectic, hence the integral in (2) is positive. This finishes the proof.

Notice that we also get the following

Corollary 4.1 *Let (S, H) be a polarized projective regular surface. Suppose that there exists a smooth connected curve $D \in |K_S|$ of genus at least two. Assume that $\bar{M}(S, H; c_1, c_2)$ is non-empty, K_S-good and even-dimensional, then it has Kodaira dimension at least zero.*

Proof of Theorem 4.1

We will need the following

Lemma 4.1 *Let $\mathscr{E}$ be a rank two vector bundle on $D \times T$, where T is connected, let θ be a theta characteristic on D. Then $h^0(\operatorname{ad} E_t \otimes \theta)$ is constant modulo 2, where* $\operatorname{ad} E_t = \operatorname{ad} \mathscr{E}|_{D \times \{t\}}$.

Proof. The proof is an extension of the proof of the constancy, modulo two, of $h^0(\theta_t)$, where θ_t is a theta characteristic on a variable curve (see [1]). Choose an effective divisor $X = x_1 + \ldots + x_n$ on D, where n is very large. Let $F_t = \operatorname{ad} E_t \otimes \theta$, $V_t = H^0(F_t(X)/F_t(-X))$, clearly $\dim V_t = 6n$. On V_t we have a natural non-degenerate bilinear symmetric form Q, namely if $\sigma, \tau \in V_t$ then

$$Q(\sigma, \tau) = \sum_i \operatorname{Res}_{x_i} \operatorname{tr}(\sigma\tau) .$$

Let $A_t \subset V_t$ be $H^0(F_t/F_t(-X))$, then $\dim A_t = 3n$ and A_t is isotropic for Q, i.e. it is a maximal isotropic subspace. Let $B_t \subset V_t$ be the image of $H^0(F_t(X))$, then, by the residue theorem, B_t is also isotropic for Q. Consider the long exact sequence of cohomology groups associated to the exact sequence

$$0 \to F_t(-X) \to F_t(X) \to F_t(X)|_{2X} \to 0 .$$

Obviously $F_t(X)/F_t(-X) \cong F_t(X)|_{2X}$. If $n \gg 0$ then $h^0(F_t(-X)) = 0$, hence $\dim B_t = h^0(F_t(X)) = 3n$, i.e. B_t is also a maximal isotropic subspace. Clearly $H^0(F_t) = A_t \cap B_t$, so the result follows from the well known fact that if Q is a non-degenerate bilinear symmetric form on an even dimensional vector space V, and A_t, $B_t \subset V$ are maximal isotropic subspaces varying continuously, then $\dim A_t \cap B_t$ is constant modulo two.

Corollary 4.2 *Let E be a rank two vector bundle on D, with $\det E \cong P$, then*

$$h^0(\operatorname{ad} E(\theta)) \equiv h^0(\theta) + \deg P \mod(2) .$$

Proof. By the lemma if F is any rank two vector bundle with $\deg F = \deg E$ then $h^0(\operatorname{ad} F(\theta)) \equiv h^0(\operatorname{ad} E(\theta)) \operatorname{mod}(2)$; let $F = L \oplus P \otimes L^{-1}$ then

$$\operatorname{ad} F \otimes \theta \cong \theta \oplus (P \otimes L^{-2} \otimes \theta) \oplus (P^{-1} \otimes L^2 \otimes \theta)$$

hence $h^0(\operatorname{ad} F \otimes \theta) = h^0(\theta) + \chi(P \otimes L^{-2} \otimes \theta) = h^0(\theta) + \deg P - 2\deg L \equiv h^0(\theta) + \deg P \operatorname{mod}(2)$.

Proposition 4.1 *Let*

$$V = \{(E, \varphi) | E \text{ is stable, } \det E \cong P,\ \varphi\colon E \to E \otimes \theta,\ \varphi \neq 0\}/\text{isomorphism}$$

then $\dim V \leqq 3g - 3$.

Proof. We distinguish two cases, according to the behaviour of $\det \varphi$.

First case. The number of moduli of (E, φ) *with* $\det \varphi = 0$ *is at most* $\frac{5}{2}g - \frac{5}{2}$. Let $L^{-1} = \ker \varphi$, since E is stable $\deg L > \frac{-1}{2} \deg P$. We have the following diagram

$$\begin{array}{ccccccccc} 0 & \to & L^{-1} & \to & E & \xrightarrow{f} & P \otimes L & \to & 0 \\ & & & & \varphi \downarrow & & & & \\ 0 & \to & L^{-1} \otimes \theta & \to & E \otimes \theta & \xrightarrow{f'} & P \otimes L \otimes \theta & \to & 0 . \end{array}$$

Let $e \in H^1(P^{-1} \otimes L^{-2})$ be the extension class defining the top exact sequence. Let $\varphi'\colon P \otimes L \to E \otimes \theta$ be the quotient morphism and let $\psi = f'\varphi' \in H^0(\theta)$. Consider the long exact sequence

$$0 \to H^0(P^{-1} \otimes L^{-2} \otimes \theta) \to \operatorname{Hom}(P \otimes L, E \otimes \theta)$$

$$\to H^0(\theta) \xrightarrow{\partial} H^1(P^{-1} \otimes L^{-2} \otimes \theta) .$$

The coboundary map ∂ is cup product with the extension class e. Now let's count the moduli of (E, φ) with L and ψ fixed ($\psi \neq 0$). Let $\alpha \in H^0(P \otimes L^2 \otimes \theta)$, since $\partial(\psi) = 0$, $(\alpha \otimes \psi) \cup e = 0$, hence, by Serre duality, $[e] \in K_\psi = \mathbb{P}(\operatorname{Ann}(H^0(P \otimes L^2 \otimes \theta) \otimes \psi))$, obviously

$$\dim K_\psi = h^0(P \otimes L^2 \otimes K_D) - h^0(P \otimes L^2 \otimes \theta) - 1 .$$

By stability $\deg(P \otimes L^2 \otimes K_D) > 2g - 2$, hence $h^0(P \otimes L^2 \otimes K_D) = \deg P + 2\deg L + g - 1$. Furthermore $h^0(P \otimes L^2 \otimes \theta) = \chi(P \otimes L^2 \otimes \theta) + h^1(P \otimes L^2 \otimes \theta)$, hence

$$\dim K_\psi = g - 2 - h^1(P \otimes L^2 \otimes \theta) \leqq g - 2 .$$

Now let L vary in $\mathrm{Pic}(D)$ and ψ in $H^0(\theta)\setminus\{0\}$, then (taking into account the homotheties of E, which give an isomorphism between (E, φ) and $(E, \lambda\varphi)$, where $\lambda \in \mathbf{C}^*$) we get

$$\dim\{(E, \varphi) | \det\varphi = 0, \psi \neq 0\}/\text{isom.} \leqq 2g - 3 + h^0(\theta) .$$

Applying Clifford's Theorem to θ we get that

$$\dim\{(E, \varphi) | \det\varphi = 0, \psi \neq 0\}/\text{isom.} \leqq \tfrac{5}{2}g - \tfrac{5}{2} .$$

On the other hand let $T = \{(E, \varphi) | \psi = 0\}/\text{isom.}$, then $\dim T \leqq h^0(P^{-1} \otimes L^{-2} \otimes \theta) + g - 1 \leqq \frac{3}{2}g - \frac{1}{2}$. We conclude that the number of moduli is at most $\frac{5}{2}g - \frac{5}{2}$, which is less than $3g - 3$ if $g \geqq 2$.

Second case. The number of moduli of $\mathscr{A} = \{(E, \varphi) | \det\varphi \not\equiv 0\}$ *is at most* $3g - 3$. We stratify $\mathscr{A} = \mathscr{A}_1 \cup \ldots \cup \mathscr{A}_n$ according to the number of zeroes $Z_1, \ldots, Z_n$ (the Z_i's might not be distinct) of φ. Clearly $0 \leqq n \leqq g - 1$, let $L = \theta(-Z_1 - \ldots - Z_n)$, so that $n = g - 1 - \deg L$, then φ defines a map $\varphi' : E \to E \otimes L$ which is nowhere zero. We must prove that

$$\dim \mathscr{A}_n + n \leqq 3g - 3 .$$

Assume $\varphi : E \to E \otimes L$ is nowhere zero and $0 \leqq \deg L \leqq g - 1$. The characteristic polynomial of φ is $\lambda^2 + \det\varphi = 0$, where λ is a fiber coordinate on L. Let's first examine the map φ locally. Let $P \in D$ be a point on the curve and $\mathcal{O} = \mathcal{O}_{D,P}$ the local ring of D at P, let E be a rank two locally free sheaf on $\mathrm{Spec}\,\mathcal{O}$ and $\varphi : E \to E$ a traceless endomorphism, non-zero on the closed point, with $\det\varphi = -t^{2n}$, $n \geqq 1$, where t is a uniformizing parameter. Let the two eigenvector (rank one) sub-linebundles be E_1 and E_2, there is a natural injective map $\alpha : E_1 \oplus E_2 \to E$, we claim that the quotient is $\mathcal{O}/m^n$, where m is the maximal ideal. To see this let

$$\varphi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

then, since $a^2 + bc = t^{2n}$ and g.c.d. $(a, b, c) = 1$, one at least among b, c is a unit, say b (if c is a unit the argument is analogous). The eigenbundle corresponding to the eigenvalue t^n is generated by $v_1 = (b, t^n - a)$; if the eigenvalue is $-t^n$ the corresponding eigenbundle is generated by $v_2 = (-b, t^n + a)$, so $v_1 \wedge v_2 = 2bt^n$, since b is a unit the claim follows.

Now let

$$(\det\varphi) = \sum_{i=1}^{p} 2m_i P_i + \sum_{j=1}^{q} (2n_j + 1) Q_j, \quad B = \sum_j Q_j .$$

Let $\pi : \tilde{D} \to D$ be the normalization of the double cover of D determined by the characteristic polynomial of φ, it is smooth and connected, ramified over B. The eigenbundles E_1, E_2 of the pull-back of φ to $\tilde{D}$ are sublinebundles of π^*E and one has the exact sequence

$$0 \to E_1 \oplus E_2 \xrightarrow{\alpha} \pi^*E \to \mathcal{O}_Z \to 0$$

where $Z = \sum_i m_i(P_i' + P_i'') + \sum_j (2n_j + 1)Q_j'$, $\pi^{-1}(P_i) = \{P_i', P_i''\}$, $\pi^{-1}(Q_j) = \{Q_j'\}$. Dually we have

$$0 \to \pi^*E^* \xrightarrow{\alpha^t} E_1^* \oplus E_2^* \xrightarrow{f} \mathcal{O}_Z \to 0\ .$$

On $\tilde{D}$ we have that $\det \pi^*(\varphi) = \sigma^2$ and $\pi^*(\varphi)$ is induced by $(\sigma, -\sigma)$: $E_1^* \oplus E_2^* \to (E_1^* \oplus E_2^*) \otimes L$. The map f is determined by data at the points P_i', P_i'', Q_j'. As in the previous case each P_i', P_i'' contributes m_i moduli but, since $\pi^*(E)$ is invariant under the action of the covering involution ι, the map f at P_i' determines f at P_i'', hence there are only m_i moduli for each i. At the point Q_j' the map f is determined by f_j: $E_1^*/m_j^{2n_j+1} \oplus E_2^*/m_j^{2n_j+1} \to \mathcal{O}_{(2n_j+1)Q_j'}$ (m_j is the maximal ideal at Q_j'). Let $f_j(x, y) = ax + by$, where x, y are fibre coordinates on $E_1^*/m_j^{2n_j+1}$ and $E_2^*/m_j^{2n_j+1}$ respectively. The action of ι on (x, y) is given by $\iota^*(x, y) = (y, x)$, since ι leaves π^*E^* invariant we see that $a^2 - b^2 = 0$, this implies that $a - b \equiv 0 \bmod (t^{n+1})$ or $a + b \equiv 0 \bmod (t^{n+1})$, where t is a uniformizing parameter at Q_j'. Hence there are $3n_j + 1$ moduli for f_j, but multiplying by a unit of $\mathcal{O}_{(2n_j+1)Q_j'}$ doesn't affect the isomorphism class of π^*E, hence each j contributes n_j moduli. Now we count the number of moduli: we have $\sum_i m_i + \sum_j n_j$ moduli determining f. In order to count the moduli of E_1 notice that $E_1 \otimes E_2 \cong \pi^*(P)(-Z)$, hence $E_1 \otimes \iota^*(E_1) \cong \pi^*(P)(-Z)$, so E_1 has $g(\tilde{D}) - g(D)$ moduli. The genus of $\tilde{D}$ is $\tilde{g} = 2g + \frac{1}{2}q - 1$, so we get that

$$\begin{aligned} \#\ of\ moduli &\leqq g + \tfrac{1}{2}q - 1 + \sum_i m_i + \sum_j n_j + h^0(L^2) + g - 2 - \deg L \\ &= 2g - 3 + h^0(L^2) \\ &\leqq 3g - 3 \end{aligned}$$

where in the first equality we have used the fact that $\sum_i 2m_i + \sum_j 2n_j + q = 2\deg L$, and for the last inequality one just notices that $\deg L^2 \leqq 2g - 2$ so $h^0(L^2) \leqq g$.

Remark 4.2 Notice that in bounding the number of moduli in the second case we didn't make use of the assumption that E is stable.

In order to conclude the proof of Theorem 4.1, let d be $h^0(\mathrm{ad}\,E \otimes \theta)$ of the generic stable E. Then, since $\dim M(D; P) = 3g - 3$ and since (E, φ) is identified with $(E, \lambda\varphi)$ for any $\lambda \in \mathbf{C}^*$, we have that

$$d - 1 + 3g - 3 \leqq 3g - 3$$

i.e. either $d = 0$ or $d = 1$. By Corollary 4.2 if $h^0(\theta) + \deg P$ is even then $d = 0$, if $h^0(\theta) + \deg P$ is odd then $d = 1$.

Remark 4.3 Let $M = M(S, H; c_1, c_2)$. A natural question to ask is: how degenerate can $\tau_M(\omega)$ be if the hypotheses in Theorem 2.7 are not satisfied? If S is a K3 surface, then M is always K_S-good, and $\tau_M(\omega)$ is non-degenerate on all of M (this was proved by Mukai [17]).

Now let's briefly examine the case of a minimal regular elliptic surface S and the moduli space $M_c = M(S, H; 0, c)$. Let $F_1, \ldots, F_r$, be the multiple fibers of the elliptic fibration, with multiplicities $m_1, \ldots, m_r$. Then the generic $D \in |K_S|$ is of the form

$$D = D_1 + \ldots + D_s + (m_1 - 1)F_1 + \ldots + (m_r - 1)F_r$$

where the D_i's are smooth fibers of the elliptic fibration. The number s of reduced fibers is related to the geometric genus of S by $s = p_g(S) - 1$.

Assume that M_c is K_S-good, then for generic $E \in M_c$ we have that $h^0(\operatorname{ad} E \otimes K_S|D) = s$. Hence if ω is a holomorphic two-form with $(\omega) = D$ the kernel of $\tau_{M_c}(\omega)$ at $[E]$ has dimension s. This fact can be related to the work of Friedman and Morgan [10] on Donaldson's polynomials of simply connected minimal elliptic surfaces in the following way.

Let $d(c) = 4c - 3\chi(\mathcal{O}_S) = \dim M(S, H; 0, c)$. Let $\Gamma \in H_2(S)$ be the Poincaré dual of ω. Let $\alpha_1, \ldots, \alpha_p \in H_2(S)$ be p classes, where $p < s$. Assume that c is odd so that $\delta_c(S, H)$ is defined. Then, since the kernel of $\tau_M(\omega)$ at any point has dimension at least equal to s we have that

$$\delta_c(S, H)(\alpha_1, \ldots, \alpha_p, \underbrace{\Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}}_{d(c) - p}) = 0 .$$

On the other hand Friedman and Morgan have proved the following. Let $q \in \operatorname{Sym}^2(H^2(S))$ be the intersection form and let $k = c_1(K_S)$, both thought of as polynomials on $H_2(S)$. Let $n = \frac{d(c)}{2} - \frac{s}{2}$. Then there exist (rational) coefficients a_i such that

$$\gamma_c(S) = \sum_{i=0}^{n} a_i q^i k^{d-2i} .$$

Thus, if $p < s$ we get that

$$\gamma_c(S)(\alpha_1, \ldots, \alpha_p, \underbrace{\Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}}_{d(c) - p}) = 0 .$$

This is evidence in favor of the hypothesis that there is a strict relationship between the algebro-geometric polynomials $\delta_c(S, H)$ and Donaldson's polynomials $\gamma_c(S)$.

Finally one can ask whether $\tau_M: H^0(\Omega_S^2) \to H^0(\Omega_M^2)$ is injective. If we fix c_1, then for $c_2 \gg 0$ the answer is yes by the results of Zuo [14]. The methods of our paper should give stronger results than Zuo's, but we will not pursue this here.

5 Proof of Theorem 2.6

We start by sketching the proof of the theorem. Let S be a surface satisfying the hypotheses of Theorem 2.6. Let H be a polarization on S. Let $M_c = M(S, H; 0, c)$ and let $\bar{M}_c = \bar{M}(S, H; 0, c)$. Let $\varphi: \tilde{M}_c \to \bar{M}_c$ be a desingularization.

The first step in the proof is to construct suitable representatives of the Poincaré dual of $\nu_{\tilde{M}_c}(k[C_0])^p$, where k is a large integer. We will show that if $p \leqq g+1$ we can represent $\nu_{\tilde{M}_c}(k[C_0])^p$ by an algebraic codimension-p cycle $[A]+[B]+[Y]$, where $[A]$ is effective and not zero, $[B]$ is effective. By Theorem 3.2 we can compute

$$\delta_c(S,H)(\underbrace{k[C_0],\dots,k[C_0]}_{p},\underbrace{\Gamma+\bar{\Gamma},\dots,\gamma+\bar{\Gamma}}_{d(c)-p})$$

by integrating $\wedge^{d(c)-p}\frac{1}{4\pi^2}(\tau_{\tilde{M}_c}(\omega)+\overline{\tau_{\tilde{M}_c}(\omega)})$ over $[A]$, $[B]$, $[Y]$. Proposition 5.3 shows that if c is big enough then the integral over $[Y]$ is zero. Hence in order to prove the theorem it will suffice to show that the integral over $[A]$ is positive. Let's describe briefly how $[A]$ is obtained. Choose a smooth curve $C\in|C_0|$, then restriction of sheaves defines a rational map ρ_C from $\bar{M}_c$ to $\bar{M}(C;0)$, the moduli space of semistable bundles on C with trivial determinant. Let Θ be the theta divisor on $\bar{M}(C;0)$. The proper transform of a divisor in $|k\Theta|$ is a divisor on $\tilde{M}_c$. The cycle $[A]$ is obtained by intersecting p such divisors (more precisely it is a nonempty union of components of the intersection). Now let $\omega\in H^0(\Omega_S^2)$ be such that (ω) is a smooth irreducible curve of genus at least two. By Theorem 2.7 we know that $\tau_{\tilde{M}_c}(\omega)$ is non-degenerate and by our construction it will be non-degenerate at the generic point of A. One also knows that Θ is ample on $\bar{M}(C;0)$. This will imply that in order to prove the theorem we must bound the dimension of the kernel of the restriction of $\tau_{\tilde{M}_c}(\omega)$ to fibers of the restriction of ρ_C to A. There are two ingredients in the proof that the restriction of $\tau_{\tilde{M}_c}(\omega)$ to such a fiber has kernel of small dimension: the invariance of the forms $\tau_{\mathscr{E}}(\omega)$ under elementary modifications (Theorem 3.4) and the non-degeneracy of $\tau_M(\omega)$ for moduli spaces M satisfying the hypotheses of Theorem 2.7.

Our first task is to discuss how to represent the Poincaré dual of $\nu_{\mathscr{E}}(C)$ for a curve $C\subset S$ and a sheaf $\mathscr{E}$ on $S\times B$. Our discussion will be slightly more general than what is strictly needed for this section, but it will be useful in the next section.

Let B be an algebraic variety. Let $\mathscr{E}$ be a B-flat rank-two sheaf on $S\times B$ such that for all $b\in B$ we have that $c_1(\mathscr{E}|_{S\times\{b\}})=0$. Let $C\subset S$ be a smooth curve of genus g and let $L\in \mathrm{Pic}^{g-1}(C)$. We define the algebraic subset $\Delta_{\mathscr{E}}^{\mathrm{red}}(C,L)\subset B$ by

$$\Delta_{\mathscr{E}}^{\mathrm{red}}(C,L)=\{b\in B\,|\,h^0((\mathscr{E}|_{C\times\{b\}})\otimes L)>0\}\ .$$

If $\Delta_{\mathscr{E}}^{\mathrm{red}}(C,L)$ is not all of B then it is either of codimension one or empty (notice that $\chi((\mathscr{E}|_{C\times\{b\}})\otimes L)=0$). In this case there exists a Cartier divisor $\Delta_{\mathscr{E}}(C,L)$ on B, supported on $\Delta_{\mathscr{E}}^{\mathrm{red}}(C,L)$, with a positive coefficient for each irreducible component of $\Delta_{\mathscr{E}}^{\mathrm{red}}(C,L)$, such that

$$\nu_{\mathscr{E}}(C)=[\Delta_{\mathscr{E}}(C,L)]\ .$$

This is proved by applying the Grothendieck-Riemann-Roch Theorem to $\mathscr{E}|_{C\times B}$. Let $\iota\colon C\times B\hookrightarrow S\times B$ be the inclusion. Let p_C, p_B be the projections of $C\times B$ to C and B, respectively. An application of the Grothendieck-Riemann-Roch Theorem shows that

$$c_2(\mathscr{E})/[C]=-c_1(\det(p_B)_!(\iota^*\mathscr{E}\otimes p_C^*(L)))\ . \tag{1}$$

Then one shows [10] that $(\det(p_B)_!(\iota^*\mathscr{E}\otimes p_C^*(L)))^*$ has a section whose zero-locus is a divisor $\Delta_{\mathscr{E}}(C,L)$ supported on $\Delta_{\mathscr{E}}^{\mathrm{red}}(C,L)$. The multiplicity of an irreducible component of $\Delta_{\mathscr{E}}^{\mathrm{red}}(C,L)$ is given by the length of $R^1(p_B)_*(\iota^*\mathscr{E}\otimes p_C^*(L))$ at the generic point of the component, hence it is positive.

Let $\pi_B\colon S\times B\to B$ be the projection. Let M be a line bundle on B and let $\mathscr{E}'=\mathscr{E}\otimes\pi_B^*(M)$. Then it is easily checked that, since $c_1(\mathscr{E}|_{S\times\{b\}})=0$, $\Delta_{\mathscr{E}'}(C,L)=\Delta_{\mathscr{E}}(C,L)$. This shows that we can define unambiguously a divisor $\Delta_{\tilde{M}_c}(C,L)$ on $\tilde{M}_c$ by setting $\Delta_{\tilde{M}_c}(C,L)=\Delta_{\tilde{\mathscr{E}}}(C,L)$, where $\tilde{\mathscr{E}}$ is the pull-back to $S\times\tilde{M}_c$ of a universal sheaf on $S\times\bar{M}$. Obviously $\Delta_{\tilde{M}_c}(C,L)$ is Poincaré dual to $\nu_{\tilde{M}_c}(C)$.

We now relate these representatives to the theta divisor on $\bar{M}(C;0)$.

By the results in [6] $\mathrm{Pic}(\bar{M}(C;0))$ is generated by the ample theta divisor Θ which is characterized by the following property. Let $\mathscr{F}$ be a rank-two vector bundle on $C\times B$ such that for all $b\in B$ the bundle $\mathscr{F}|_{C\times\{b\}}$ is stable. Let $f\colon B\to\bar{M}(C;0)$ be the induced morphism. Choose $L\in\mathrm{Pic}^{g-1}(C)$, then

$$f^*(\Theta)\cong\det((p_B)_!(\mathscr{F}\otimes p_C^*(L)))^*\ .$$

For notational simplicity let's introduce the following

Assumption 5.1 The surface S is regular, projective and it has a base-point free pencil $|C_0|$, where C_0 is a smooth connected curve of genus $g\geqq 2$.

Definition 5.1 *Let S be a surface satisfying Assumption* 5.1. *Let c be odd. Let $\mathscr{E}$ be a universal sheaf on $S\times\bar{M}_c$ and let $\tilde{\mathscr{E}}$ be the pull-back of $\mathscr{E}$ to $S\times\tilde{M}_c$. If $x\in\tilde{M}_c$ we let $E_x=\tilde{\mathscr{E}}|_{S\times\{x\}}$.* Let $C\in|C_0|$ *be a smooth curve, then we set*

$$S_C=\{x\in\tilde{M}_c\,|\,E_x|_C\text{ is not locally free}\}$$

$$F_C=\tilde{M}_c\backslash S_C$$

$$F_C^s=\{x\in F_C\,|\,E_x|_C\text{ is stable}\}$$

$$F_C^{ss}=\{x\in F_C\,|\,E_x|_C\text{ is semistable}\}$$

$$F_C^u=\{x\in F_C\,|\,E_x|_C\text{ is unstable}\}$$

$$\Sigma=\{x\in\tilde{M}_c\,|\,h^0(\mathrm{ad}\,E_x^{**}\otimes[K_S+C_0])>0\}\ .$$

Remark 5.1 By Corollary 2.1 the exceptional locus of φ is contained in Σ, hence the restriction $\varphi|_{(\tilde{M}_c\backslash\Sigma)}\colon\tilde{M}_c\backslash\Sigma\to\bar{M}_c$ is an isomorphism between $\tilde{M}_c\backslash\Sigma$ and an open subset of $\bar{M}_c$.

Lemma 5.1 *Assume M_c is C_0-good, then the codimension of $F_C^u\backslash\Sigma$ in $\tilde{M}_c$ is at least $g+1$.*

Proof. Let $x\in F_C$, then $\tilde{\mathscr{E}}$ defines, by restriction, a morphism $\rho_C\colon(F_C,x)\to\mathbf{Def}^0(E_x|_C)$. Now assume $x\in F_C\backslash\Sigma$, then ρ_C has surjective differential at x. To see this first notice that, since $x\notin\Sigma$ it is enough to show that $\rho_C\colon(\bar{M}_c,\varphi(x))\to\mathbf{Def}^0(E_x|_C)$ has surjective differential at $\varphi(x)$. Every first-order deformation of E_x^{**} lifts to a first-order deformation of E_x, since $h^0(\mathrm{ad}\,E_x^{**}\otimes[K_S+C])=0$ we conclude that the differential is surjective. Since $\bar{M}_c$ is smooth at $[E_x]$ we conclude that the image of $\rho_C: (\bar{M}_c, \varphi(x)) \to \mathbf{Def}^0(E_x|_C)$ contains a neighborhood of the origin in $\mathbf{Def}^0(E_x|_C)$. The same statement holds for $\rho_C: (\tilde{M}_c, x) \to \mathbf{Def}^0(E_x|_C)$. Hence by Proposition 2.4 the codimension of $F_C^u \backslash \Sigma$ in $\tilde{M}_c$ is at least $g + 1$.

Notation. Let S be a surface satisfying Assumption 5.1. Let $C \in |C_0|$ be a smooth curve. For the rest of this section we will denote by $\rho_C: (F_C^{ss} \backslash \Sigma) \to \bar{M}(C; 0)$ the restriction of ρ_C to $F_C^{ss} \backslash \Sigma$.

We decompose $\tilde{M}_c$ as follows:

$$\tilde{M}_c = (F_C^{ss} \backslash \Sigma) \cup (F_C^u \backslash \Sigma) \cup S_C \cup \Sigma \,. \tag{2}$$

Proposition 5.1 *Let S satisfy Assumption* 5.1. *Let c be odd and such that M_c is C_0-good. Let $C \in |C_0|$ be a smooth curve such that S_C has codimension at least two in $\tilde{M}_c$. Let Θ be the theta divisor on $\bar{M}(C; 0)$ and let k be such that $k\Theta$ is ample. Then there exists a divisor $D \in |k\Theta|$ such that $\rho_C(F_C^{ss} \backslash \Sigma)$ is not contained in D. Given such a D we can represent the Poincaré dual of $\varphi^*(v_{\bar{M}_c}(k[C_0]))$ as*

$$\text{P.D.}\varphi^*(v_{\bar{M}_c}(k[C_0])) = [\overline{\rho_C^*(D)}] + [\text{divisor supported on } \Sigma] \,.$$

Furthermore, if $x \in F_C^{ss}$ is given, we can choose D so that $x \in \overline{\rho_C^*(D)}$.

Proof. That there exists a divisor D satisfying our conditions follows from the hypothesis that $k\Theta$ is ample and Lemma 5.1. Let $\iota: C \times \tilde{M}_c \hookrightarrow S \times \tilde{M}_c$ be the inclusion. Let $p_C, p_{\tilde{M}_c}$ be the projections of $C \times \tilde{M}_c$ onto C and $\tilde{M}_c$, respectively. Let $L \in \text{Pic}^{g-1}(C)$. Consider the line bundle on $\tilde{M}_c$ given by

$$\mathscr{L} = \mathcal{O}_{\tilde{M}_c}(\overline{\rho_C^*(D)}) \otimes (\det((p_{\tilde{M}_c})_!(\iota^* \tilde{\mathscr{E}} \otimes p_C^*(L)))^{\otimes k} \,.$$

By our previous discussion its restriction to $F_C^{ss} \backslash \Sigma$ is trivial. By our choice of C we have that $\text{cod}(S_C, \tilde{M}_c) \geqq 2$. By Lemma 5.1 we have that $\text{cod}((F_C^u \backslash \Sigma), \tilde{M}_c) \geqq g + 1$. Thus, by the decomposition (2), the restriction of $\mathscr{L}$ to $\tilde{M}_c \backslash \Sigma$ is trivial. The last statement is clear, since we are assuming that $k\Theta$ is ample.

Proposition 5.2 *Let S be a surface satisfying Assumption* 5.1. *Let c be an odd integer such that M_c is C_0-good. Let $C \in |C_0|$ be a smooth curve such that S_C has codimension two in $\tilde{M}_c$. Let k be a positive integer such that $k\Theta$ is ample on $\bar{M}(C; 0)$. Then, given any integer p, there exist divisors $D_1, \ldots, D_p \in |k\Theta|$ such that $D_1 \cap \ldots \cap D_p \cap \rho_C(F_C^{ss} \backslash \Sigma)$ is of codimension p in $\rho_C(F_C^{ss} \backslash \Sigma)$. If $p \leqq g + 1$ and the D_i's are as above then we can represent the Poincaré dual of $\varphi^* v_{\bar{M}_c}(k[C_0])^p$ as*

$$\text{P.D.}\, \varphi^* v_{\bar{M}_c}(k[C_0])^p = [A] + [B] + [Y] \tag{3}$$

where $A = \overline{\rho_C^*(D_1 \cdot D_2 \ldots D_p)}$, $[B]$ is an effective codimension-p cycle and $[Y]$ is a codimension-p cycle (not necessarily effective) supported on Σ. Furthermore, if $x \in F_C^{ss} \backslash \Sigma$ is a given point, we can choose the divisors D_i so that $x \in \overline{\rho_C^*(D_1 \cdot D_2 \ldots D_p)}$.

Proof. By "induction" on p. If p is one then (3) reduces to Proposition 5.1. Assume that (3) holds for $p \leqq g$. Then

$$\begin{aligned}\text{P.D.}\, \varphi^* v_{\bar{M}_c}(k[C_0])^{p+1} &= [A] \cdot (\text{P.D.}\, \varphi^* v_{\bar{M}_c}(k[C_0])) \\ &\quad + [B] \cdot (\text{P.D.}\, \varphi^* v_{\bar{M}_c}(k[C_0])) + [Y] \cdot (\text{P.D.}\, \varphi^* v_{\bar{M}_c}(k[C_0]))\end{aligned}$$

with $A = \overline{\rho_C^*(D_1 \cdot D_2 \ldots D_p)}$. Applying Proposition 5.1 and Lemma 5.1 we get that

$$[A] \cdot \text{P.D.}(\varphi^* v_{\bar{M}_c}(k[C_0])) = [\overline{\rho_C^*(D_1 \cdot D_2 \ldots D_{p+1})}] + [B'] + [Y']$$

where B' is an effective codimension-$(p+1)$ cycle and C' is a (not necessarily effective) codimension-$(p+1)$ cycle supported on Σ.

Now we must deal with the term $[B]\cdot(\text{P.D.}\,\varphi^* v_{\bar{M}_c}(k[C_0]))$. Clearly we can assume that no irreducible component of B is contained in Σ.

Claim. Let $p \leqq g$, let $\Omega \subset \tilde{M}_c$ be a codimension p irreducible subvariety not contained in Σ. Then there exists a smooth curve $C' \in |C_0|$ and a divisor $D' \in |k\Theta'|$ such that $\rho_{C'}^*(D')$ does not contain Ω, where Θ' is the theta divisor on $\bar{M}(C'; 0)$.

First of all choose C' so that Ω is not contained in $S_{C'}$. Let $U = \Omega \backslash (S_{C'} \cup \Sigma)$, it is dense in Ω. By Lemma 5.1 U is not contained in $F_{C'}^u$, hence the result follows from a theorem of Raynaud (Proposition 1.6.2 [21]), which shows that given any point $[E] \in \bar{M}(C'; 0)$ there exists a line bundle $L \in \mathrm{Pic}^{g-1}(C')$ such that $h^0(E \otimes L) = 0$, i.e. there exists a divisor $D_0' \in |\Theta'|$ not containing $[E]$.

Going back to the proof of the proposition, let B_i be an irreducible component of B. Apply the claim to B_i, let C_i', D_i' be the corresponding curve and divisor. By Proposition 5.1 we can write

$$[B_i]\cdot \text{P.D.}(\varphi^*(k[C_0])) = [B_i]\cdot[\overline{\rho_{C_i'}^*(D_i')}] + [Y_i']$$

where $\overline{B_i \cdot \rho_{C_i'}^*(D_i')}$ is an effective codimension-$(p+1)$ cycle, and Y_i' is a (not necessarily effective) codimension-$(p+1)$ cycle supported on Σ. The last statement is clear, since we are assuming that $k\Theta$ is ample.

Proposition 5.3 *Let S be a surface satisfying Assumption* 5.1. *Let p be any given integer. There exists n_p such that if c is odd, $c > n_p$ and $N \subset \Sigma$ is any codimension-p subvariety, then*

$$\int_{[N]} \underbrace{\varphi^*(v(\Gamma + \bar{\Gamma})) \cup \ldots \cup \varphi^*(v(\Gamma + \bar{\Gamma}))}_{d(c)-p} = 0$$

where $d(c) = \dim \tilde{M}_c$.

Proof. Let $\psi\colon \tilde{N} \to N$ be a desingularization of N. Consider the sheaf on $S \times \tilde{N}$ given by $\mathscr{F} = (\mathrm{id}_S \times \psi)^* \tilde{\mathscr{E}}$. By Lemma 3.3 we have that

$$\int_{[N]} \underbrace{\varphi^*(v(\Gamma + \bar{\Gamma})) \cup \ldots \cup \varphi^*(v(\Gamma + \bar{\Gamma}))}_{d(c)-p} = \int_{[\tilde{N}]} \underbrace{v_{\mathscr{F}}(\Gamma + \bar{\Gamma}) \wedge \ldots \wedge v_{\mathscr{F}}(\Gamma + \bar{\Gamma})}_{d(c)-p}.$$

First of all, by Corollary 2.2 there exists m_0 such that if $c > m_0$ then M_c is good, hence $d(c) = 4c - 3\chi(\mathcal{O}_S)$.

We distinguish two cases, depending on whether $F_x = \mathscr{F}|_{S \times \{x\}}$ is free for the generic $x \in \tilde{N}$ or not.

Assume that there is a non-empty Zariski open subset $U \subset \tilde{N}$ such that if $x \in U$ then F_x is locally free. By Lemmas 3.2, 3.3 and Theorem 3.2 we know that $v_{\mathscr{F}}(\Gamma)$ is represented by $\frac{1}{4\pi^2}\tau_{\mathscr{F}}(\omega)$. Thus we are reduced to showing that $\tau_{\mathscr{F}}(\omega)$ is degenerate. Since for every x the sheaf F_x is H-stable it is also H-slope-semistable. Then by shrinking U we can assume that for every $x \in U$ either F_x is H-slope-stable or properly H-slope-semistable. In the first case the vector bundle $\mathscr{F}|_{S \times U}$ defines a morphism $\beta\colon U \to M_c$ (which in fact equals the composition $\varphi\psi$) such that $\mathscr{F}$ is the pull-back of the universal sheaf $\mathscr{E}$ via $\mathrm{id}_S \times \beta$. By definition of Σ we have that $h^0(\mathrm{ad}\, F_x \otimes K_S \otimes [C_0]) > 0$ for every $x \in U$. Hence by Theorem 2.3 there exist constants A, B such that $\dim \alpha(U) \leqq 3c + A\sqrt{c} + B$. Thus there exists $m_1 \geqq m_0$ such that if $c > m_1$ then $d(c) - p = 4c - 3\chi(\mathcal{O}_S) - p > 3c + A\sqrt{c} + B$. By the functoriality of the τ-map we conclude that if $c > m_1$ then $\tau_{\mathscr{F}}(\omega)$ is degenerate on U and hence on N. The case in which F_x is properly H-slope-semistable is treated similarly, by appealing to Remark 2.3. Thus there exists $m_2 \geqq m_0$ such that if $c > m_2$ then $\tau_{\mathscr{F}}(\omega)$ is degenerate.

Now we deal with the case in which for all $x \in \tilde{N}$ the sheaf F_x is not locally free. We apply Corollary 3.2 in order to give a holomorphic representative of $\nu_{\mathscr{F}}(\Gamma)$. Thus we have an open dense subset $U \subset N$ and an exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}^{**} \to \mathscr{Q} \to 0$$

such that for all $x \in U$ it restricts to the canonical exact sequence of $\mathscr{F}|_{S \times \{x\}}$. Following the notation of Corollary 3.2 we also have a morphism $f\colon U \to \times_i \mathrm{Hilb}^{d_i}(S) \times Y$ and a holomorphic two-form, λ, on $\times_i \mathrm{Hilb}^{d_i}(S) \times Y$ such that $f^*\lambda$ extends to all of $\tilde{N}$ and it represents $\nu_{\mathscr{F}}(\Gamma)$. The map f is defined by $f = \alpha \times \beta$, where α is defined by associating to E_x the cycle $Z(E_x)(Z(E_x)$ is the subscheme of S of points where E_x is not locally free) and β is defined by associating to E_x the isomorphism class of E_x^{**}.

We will show that there exists m_4 such that if $c > m_4$ then $\dim f(U) < \dim U$.

Let $c' = c_2(F_x^{**})$, where $x \in U$. Then $\sum_i d_i \leqq c - c'$. In fact $c - c' = \ell(\mathscr{Q}|_{S \times \{x\}})$ and a moment's thought shows that $\ell(\mathscr{Q}|_{S \times \{x\}}) \geqq \sum_i d_i$.

Assume first that F_x^{**} is H-slope-stable for the generic $x \in U$. By shrinking U we can assume that F_x^{**} is H-slope-stable for every $x \in U$. Since $N \subset \Sigma$ we have that $h^0(\mathrm{ad}\, F_x^{**} \otimes [K_S + C_0]) > 0$ for all $x \in U$. Hence by Theorem 2.3 we conclude that there exist constants A, B (we need to observe that $c' < c$) such that $\dim \beta(U) < 3c' + A\sqrt{c'} + B$. Thus

$$\dim \times_i \mathrm{Hilb}^{d_i}(S) \times \beta(U) < 3c + A\sqrt{c} + B$$

where we have used the fact that $\sum_i d_i \leqq c - c'$ and $c' < c$. Thus there exists m_3 such that if $c > m_3$ then

$$\dim \tilde{N} = d(c) - p = 4c - 3\chi(\mathcal{O}_S) - p > \dim(\times_i \mathrm{Hilb}^{d_i}(S) \times \alpha(U))$$

so that $\dim f(U) < \dim U$.

Similarly, there exists $m_4 \geqq m_3$ such that if $c > m_4$ and F_x^{**} is properly H-slope-semistable for all $x \in U$ then $\dim f(U) < \dim U$.

Thus the proposition holds if we set $n_p = \max\{m_0, m_1, m_2, m_3\}$.

The following is an immediate consequence of Propositions 5.2, 5.3.

Corollary 5.1 *Let (S, H) be a polarized surface satisfying Assumption 5.1. Let $p \leqq g + 1$, where g is the genus of C_0. Let $\omega \in H^0(\Omega_S^2)$ and let $\Gamma \in H_2(S)$ be its Poincaré dual. There exists n_p such that if $c > n_p$ then M_c is C_0-good and furthermore*

$$\delta_c(S, H)(\underbrace{[C_0], \ldots, [C_0]}_{p}, \underbrace{\Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}}_{d(c) - p}) \geqq 0\,.$$

At this point we outline in greater detail the proof of Theorem 2.6. Let S be a surface satisfying Assumption 5.1 and let H be a polarization on S. Assume that $p_g(S)$ is odd and that there exists $\omega \in H^0(\Omega_S^2)$ such that $D = (\omega)$ is a smooth irreducible curve of genus at least two. Let i be such that $2i \leqq \min\{g+1, g-3\}$. We will prove that there exists a number n such that if $c > n$ (and c is odd) we can represent the Poincaré dual of $v_{\tilde{M}_c}(\omega)^{2i}$ as in Proposition 5.2 in such a way that

$$\int_{[A]} \underbrace{(\tau_{\tilde{M}_c}(\omega) + \overline{\tau_{\tilde{M}_c}(\omega)}) \wedge \ldots \wedge (\tau_{\tilde{M}_c}(\omega) + \overline{\tau_{\tilde{M}_c}(\omega)})}_{d(c)-p} > 0 .$$

This will be achieved as follows. Assume that c is odd and big enough so that $\bar{M}_c$ is $(C_0 + K_S)$-good. Let $C \in |C_0|$ be a smooth curve. Then by our hypotheses and Theorem 2.7 there exists a Zariski open dense subset $U \subset F_C^s \backslash \Sigma$ such that $\tau_{\tilde{M}_c}(\omega) + \overline{\tau_{\tilde{M}_c}(\omega)}$ is symplectic on U. We can represent $\varphi^* v_{\bar{M}_c}(k[C_0])^{2i}$ as in Proposition 5.2, with $A = \rho_C^*(D_1 \cdot D_2 \ldots D_{2i})$ non-empty. We choose the D_i so that there exists $x \in A \backslash \Sigma$ with $E_x|_C$ stable and $\rho_C(x) \in D_i$ for all i. The map $\varphi \colon \tilde{M}_c \to \bar{M}_c$ is an isomorphism near x, hence the tangent space to $\tilde{M}_c$ at x is identified with $H^1(\text{ad}\, E_x)$. Consider the exact sequence

$$0 \to H^1(\text{ad}\, E_x \otimes [-C_0]) \to H^1(\text{ad}\, E_x) \xrightarrow{f} H^1(\text{ad}\, E_x|_C) \to 0 .$$

Since $E_x|_C$ is stable the map f is the differential of ρ_C at x.

We are assuming that $|k\Theta|$ is very ample on $\bar{M}(C; 0)$, hence by appropriately choosing $D_1, \ldots, D_{2i}$, the tangent space to $D_1 \cap \ldots \cap D_{2i}$ at $[E_x|_C]$ can be any assigned codimension $2i$ subspace of the tangent space to $\bar{M}(C; 0)$ at $[E_x|_C]$. Thus, by the above exact sequence, one can choose the D_n's so that the tangent space to $\rho^*(D_1 \cdot D_2 \ldots D_{2i})$ at x is any preassigned codimension-$2i$ subspace containing $H^1(\text{ad}\, E_x \otimes [-C_0])$. At this point notice that one has the following

Lemma 5.2 *Let V be a vector space and α a non-degenerate symplectic form on V. Let $W \subset W' \subset V$ be subspaces. Assume that the restriction of α to W is non-degenerate. Let n be an integer such that* $2n \leqq \text{cod}(W', V) - \text{cod}(W, W')$. *Then there exists a subspace $W'' \subset W$ containing W' such that* $\text{cod}(W'', V) = 2n$ *and the restriction of α to W'' is non-degenerate.*

We will first show that there is a component of $\bar{M}_c$ and a point $[E]$ in the intersection of this component with U with the following properties. There is a subspace $W \subset H^1(\text{ad}\, E \otimes [-C_0])$ with $\text{cod}(W, H^1(\text{ad}\, E \otimes [-C_0])) = g + 3$, the restriction of $\tau_{\tilde{M}_c}(\omega)$ to W is non-degenerate and $h^0(\text{ad}\, E \otimes [C_0 + K_S]) = 0$.

The basic idea is to construct a (locally closed) algebraic subset $\Sigma \subset M_c$ such that for all $[E] \in X$ we have

$$E|_C \cong \mathcal{O}_C(D) \oplus \mathcal{O}_C(-D)$$

where D is a fixed divisor of degree g on C. Each of these bundles is obtained from a bundle F, where $[F] \in M(S, H; [C], c - g)$, by an elementary modification along C. The invariance of the two-forms $\tau_{\mathscr{E}}(\omega)$ under elementary modifications

(Theorem 3.4) shows that the restriction of $\tau_{M_c}(\omega)$ to the tangent space to X at $[E]$ is non-degenerate. On the other hand such a tangent space $T_{[E]}(X)$ is contained in the kernel of the differential of $\rho_C\colon (M_c, [E]) \to \mathbf{Def}^0(E|_C)$. Then we will conclude by the density of F_C^s and an upper semicontinuity argument that there exists $[E'] \in F_C^s \backslash \Sigma$ such that $\tau_{M_c}(\omega)$ is non-degenerate at $[E']$ and such that $H^1(\operatorname{ad} E' \otimes [-C_0])$ contains a subspace of dimension equal to the dimension of $T_{[E]}(X)$ on which the restriction of $\tau_{M_c}(\omega)$ is non-degenerate.

We first need the following lemmas.

Lemma 5.3 *Let C be a curve of genus $g \geqq 2$. Let D be a divisor on C of degree $d \geqq g$. Let $N_D \subset M(C; 0)$ be the subset of isomorphism classes of vector bundles V such that* $\dim \operatorname{Hom}(V, \mathcal{O}_C(D)) = \chi(V^* \otimes [D]) = 2(d-g) + 2$ *and such that there exists* $\varphi \in \operatorname{Hom}(V, \mathcal{O}_C(D))$ *with* $\varphi\colon V \to \mathcal{O}_C(D)$ *surjective. Then N_D is open and non-empty.*

Proof. The condition defining N_D is clearly open, we must show that N_D is not empty. Consider

$$V_A = \mathcal{O}_C(A) \oplus \mathcal{O}_C(-A)$$

where A is a divisor of degree zero. We can certainly choose A so that both $\mathcal{O}_C(D + A)$ and $\mathcal{O}_C(D - A)$ are non-special and the linear systems $|D + A|, |D - A|$ have no common base-point. With this choice of A, $\dim \operatorname{Hom}(V_A, \mathcal{O}_C(D)) = \chi(V_A^* \otimes \mathcal{O}_C(D))$ and there exist a surjective $\varphi\colon V_A \to \mathcal{O}_C(D)$. Now consider $\mathbf{Def}^0(V_A)$; by semicontinuity there is a neighborhood $U \subset \mathbf{Def}^0(V_A)$ of the origin such that for all $u \in U$ we have that $\dim \operatorname{Hom}(V_u, \mathcal{O}_C(D)) = \chi(V_u^* \otimes \mathcal{O}_C(D))$, where V_u is the bundle parametrized by u. By Proposition 2.4 there is an open non-empty subset $U' \subset U$ parametrizing stable bundles. By openness of versality the morphism from U' to $M(C; 0)$ maps U' to an open subset of $M(C; 0)$. This proves the lemma.

Definition 5.2 Let (S, H) be a polarized regular projective surface, set $M = M(S, H; c_1, c_2)$. Let $C \subset S$ be a smooth irreducible curve of genus $g \geqq 2$. Let L be a line bundle on S such that $c_1(L) = c_1$ (since S is regular the isomorphism class of L is uniquely determined). Assume that $L|_C \cong \mathcal{O}_C$. Let D be a divisor on C of degree $d \geqq g$. We set

$$\mathrm{U}(M; C, D) = \{[F] \in M \,|\, F|_C \text{ is stable, } [F|_C] \in N_D\} .$$

Let $\mathbf{P}(M; C, D)$ be the $\mathbf{P}^{2(d-g)+1}$-bundle over $U(M; C, D)$ whose fiber over $[F]$ is $\mathbf{P}(H^0((F|_C)^* \otimes \mathcal{O}_C(D)))$. Furthermore let $\mathcal{U}(M; C, D) \subset \mathbf{P}(M; C, D)$ the open subset of couples $(F, [\varphi])$ with φ surjective.

Proposition 5.4 *Let (S, H), M, C be as above. Assume that M is C-good, then $U(M; C, D)$ is dense in M and* $\dim \mathcal{U}(M; C, D) = \dim M + 2(d-g) + 1$.

Proof. Applying Proposition 2.4 and using the hypothesis that M is C-good we conclude that there is an open dense subset $M^0 \subset M$ such that if $[E] \in M^0$ then $h^2(\operatorname{ad} E \otimes [-C]) = 0$ and $E|_C$ is stable. Thus the image of the morphism $\rho_C\colon M^0 \to M(C; 0)$ induced by restriction is a dense subset of $M(C; 0)$. By Lemma 5.3 we conclude that $U(M; C, D)$ is dense in M. The formula for the dimension of $\mathcal{U}(M; C, D)$ follows immediately.

Corollary 5.2 *Let (S, H), M and C be as above. Fix $c_1 \in H^{1,1}(S, \mathbf{Z})$. There exists n such that if $c_2 > n$ the space $U(M; C, D)$ is dense in M and* $\dim \mathscr{U}(M; C, D) = \dim M + 2(d - g) + 1$.

Proof. By Corollary 2.2 there exists n such that if $c_2 > n$ then the moduli space $M(S, H; c_1, c_2)$ is C-good. The corollary follows from the previous proposition.

Let $(F, [\varphi]) \in \mathscr{U}(M; C, D)$, it presents $F|_C$ as an extension

$$0 \to \mathcal{O}_C(-D) \to F|_C \xrightarrow{\varphi} \mathcal{O}_C(D) \to 0 . \tag{4}$$

Let $E = T(F, \varphi)$ be the elementary modification of F defined by the exact sequence

$$0 \to E \to F \to \mathcal{O}_C(D) \to 0 . \tag{5}$$

An easy computation shows that $c_1(E) = c_1(F) - [C]$ and $c_2(E) = c_2(F) + d$.
Let $[C]|_C \cong \mathcal{O}_C(N)$. Restricting E to C, we get

$$0 \to \mathcal{O}_C(D - N) \to E|_C \xrightarrow{\psi} \mathcal{O}_C(-D) \to 0 . \tag{6}$$

The elementary modification inverse to (5) is given by

$$0 \to F \otimes [-C] \to E \xrightarrow{\psi} \mathcal{O}_C(-D) \to 0 . \tag{7}$$

Definition 5.3 We define the map

$$[T]: \mathscr{U}(M; C, D) \to \{\text{rank two vector bundles}\}/\text{isomorphism}$$

by letting $[T]([F], [\varphi])$ be the isomorphism class of $T(F, \varphi)$.

Lemma 5.4 *Assume that $h^1(2D - N) = 0$, then the map $[T]$ is injective.*

Proof. We must show that the isomorphism class of E determines F up to isomorphism and $\varphi: F \to \mathcal{O}_C(D)$ up to scalar multiplication.

The extension class of (6) is in $H^1(2D - N)$. Since $h^1(2D - N) = 0$ the extension is trivial, i.e. $E|_C \cong \mathcal{O}_C(D - N) \otimes \mathcal{O}_C(-D)$. By Serre duality $h^0(K_C + N - 2D) = 0$, since we are assuming that the genus of C is at least two it follows that $h^0(N - 2D) = 0$. Thus $\psi: E|_C \to \mathcal{O}_C(-D)$ is unique up to scalar multiplication. By the exact sequence (7) we conclude that the isomorphism class of E uniquely determines the isomorphism class of F. In order to reconstruct φ notice that the sequence (7) determines the exact sequence

$$0 \to \mathcal{O}_C(-D - N) \to F \otimes [-C]|_C \xrightarrow{\alpha} \mathcal{O}_C(D - N) \to 0$$

where α is a multiple of φ. Thus the isomorphism class of E determines φ up to scalar multiplication.

Definition 5.4 We let $\mathscr{V}(M; C, D) \subset \mathscr{U}(M; C, D)$ be given by

$$\mathscr{V}(M; C, D) = \{([F], [\varphi]) \in \mathscr{U}(M; C, D) \mid T([F], [\varphi]) \text{ is } H-\text{slope}-\text{stable}\} .$$

Clearly $\mathscr{V}(M; C, D)$ is an open subset of $\mathscr{U}(M; C, D)$.

Lemma 5.5 *Let* (S, H), $M = M(S, H; c_1, c_2)$, C *be as above. Fix* $c_1 \in H^{1,1}(S, \mathbf{Z})$. *Let* D *be a divisor on* C *of degree* $d \geqq g$. *There exists an* n *such that if* $c_2 > n$ *then* $\mathscr{V}(M; C, D)$ *has dimension equal to* $\dim M + 2(d - g) + 1$.

Proof. By Corollary 2.2 and Proposition 5.4 there exists n' such that if $c_2 > n'$ then $\dim U(M; C, D) = \dim M = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$. Let $\pi: \mathscr{U}(M; C, D) \to U(M; C, D)$ be the natural projection (i.e. $\pi([F], [\varphi]) = [F]$). We will show that there exists $n \geqq n'$ such that if $c_2 > n$ then $\pi(\mathscr{U}(M; C, D) \backslash \mathscr{V}(M; C, D))$ is a proper subset of $U(M; C, D)$. This clearly implies the lemma.

Let $[F] \in \pi(\mathscr{U}(M; C, D) \backslash \mathscr{V}(M; C, D))$. Then $E = T(F, \varphi)$ is not H-slope-stable, i.e. there exists a line bundle L on S which injects into E and such that $L \cdot H \geqq \frac{1}{2}(c_1 - [C]) \cdot H$. Thus, by the exact sequence defining E, there is a non-zero map $L \to F$. By Proposition 2.3 we conclude that the number of moduli, $n(c_2)$, of such bundles F satisfies the inequality

$$n(c_2) < 3c_2 + A\sqrt{c_2} + B$$

for some numbers A and B. Clearly there exists an $n \geqq n'$ such that if $c_2 > n$ then $4c_2 - c_1^2 - 3\chi(\mathcal{O}_S) > 3c_2 + A\sqrt{c_2} + B$. Then if $c_2 > n$ we have that $\dim U(M; C, D) > 3c_2 + A\sqrt{c_2} + B$, thus $\pi(\mathscr{U}(M; C, D) \backslash \mathscr{V}(M; C, D))$ is a proper subvariety of $U(M; C, D)$ and we are done.

Corollary 5.3 *Let* (S, H) *be a polarized projective surface. Fix* $c_1 \in H^{1,1}_{\mathbf{Z}}(S)$. *There is a number* $n(H, c_1)$ *such that if* $c_2 > n(H, c_1)$ *then* $M(S, H; c_1, c_2)$ *is not empty.*

Proof. Let L be a line bundle on S. By tensoring with L we get an isomorphism between $M(S, H; c_1, c_2)$ and $M(S, H; c_1 + 2c_1(L), c_2 + c_1 \cdot c_1(L) + c_1(L) \cdot c_1(L))$. Thus, choosing L to be sufficiently negative, we can assume that $c_1 = c_1([-C])$, where $C \subset S$ is a curve of genus $g \geqq 2$. Fix a divisor D on C of degree $d \geqq g$. By Gieseker's Theorem 2.2 there exists n such that if $c > n$ then $M_c = M(S, H; 0, c)$ is not empty. Thus by Lemma 5.6 there exists $n' \geqq n$ such that if $c > n'$ then $\mathscr{V}(M_c; C, D)$ is not empty. If $([F], [\varphi]) \in \mathscr{V}(M_c; C, D)$ then $[T]([F], [\varphi]) \in M(S, H; -[C], c + d)$, so we are done.

Definition 5.5 Let (S, H), M, C be as above. Let L be a divisor on S. Let $C \subset S$ be a smooth irreducible curve of genus $g \geqq 2$. Then we let $\mathscr{W}(M, L; C, D)$ be the subset of $\mathscr{V}(M; C, D)$ defined by

$$\mathscr{W}(M, L; C, D) =$$

$$\{([F], [\varphi]) \in \mathscr{V}(M; C, D) | h^2(\operatorname{ad} E \otimes [-L]) = 0, \text{ where } E = T(F, \varphi)\}\ .$$

Lemma 5.6 *Let* (S, H), M, L *and* C *be as above. Let* D *be a divisor on* C *of degree* $d \geqq g$ *such that* $h^1(2D - N) = 0$, *where* $N \cong \mathcal{O}_C(C)$. *Fix* $c_1 \in H^{1,1}(S, \mathbf{Z})$. *There exists* n *such that if* $c_2 > n$ *then* $\mathscr{W}(M, L; C, D)$ *is an open dense subset of* $\mathscr{V}(M; C, D)$ *of dimension equal to* $\dim M + 2(d - g) + 1$.

Proof. By Lemma 5.5 and Corollary 5.3 there exists n' such that if $c_2 > n'$ then $\mathscr{V}(M; C, D)$ is not empty of dimension equal to $\dim M + 2(d - g) + 1$. Restricting

$[T]$ to $\mathscr{V}(M; C, D)$ we get a morphism $[T]: \mathscr{V}(M; C, D) \to M(S, H; c_1 - [C], c_2 + d)$. By Lemma 5.4 the map $[T]$ is injective. Hence

$$\dim[T](\mathscr{V}(M; C, D)) \geqq 4c_2 - c_1^2 - 3\chi(\mathscr{O}_S) + 2(d - g) + 1 .$$

A dimension count similar to the one given in the proof of the previous lemma concludes the proof.

For the rest of this section (S, H) will be a polarized surface satisfying Assumption 5.1 and with $p_g(S)$ is odd. We also assume that there exists a smooth irreducible curve $B \in |K_S|$ of genus at least two. We let $\omega \in H^0(\Omega_S^2)$ be a two-form such that $(\omega) = B$. We fix once and for all a smooth curve $C \in |C_0|$ such that S_C has codimension at least two in $\tilde{M}_c$. Recall that in order to prove Theorem 2.6 we must show the following: there exists an n such that if c is odd and $c > n$ then there exists $[E] \in M_c$ with the property that $E|_C$ is stable, $h^2(\operatorname{ad} E \otimes [-C]) = 0$, $\tau_{M_c}(\omega)$ is non-degenerate at $[E]$ and there is a subspace $W \subset H^1(\operatorname{ad} E \otimes [-C])$ of codimension equal to $g + 3$ on which $\tau_{M_c}(\omega)$ is non-degenerate.

We set $X_c = M(S, H; [C], c - g)$. Choose once and for all a divisor D on C of degree g. Let $\mathscr{W}_c = \mathscr{W}(X_c, K_S + C; C, D)$. By Corollary 2.2, Corollary 5.2 and Lemma 5.6 there exists n such that if $c > n$ then

(i) both M_c and X_c are non-empty, $K_S + C_0$-good and of dimension equal to $4c - 3\chi(\mathscr{O}_S)$, $4(c - g) - 3\chi(\mathscr{O}_S)$, respectively,

(ii) $\mathscr{W}_c$ is not empty and the natural projection $\pi: \mathscr{W}_c \to X_c$ maps $\mathscr{W}_c$ to an open dense subset of X_c.

Now notice that the morphism $[T]$ maps $\mathscr{W}_c$ to M_c, i.e. we have

$$[T]: \mathscr{W}_c \to M_c .$$

Furthermore we have that if $[E] \in [T](\mathscr{W}_c)$ then: $E|_C \cong \mathscr{O}_C(D) \otimes \mathscr{O}_C(-D)$ and $h^2(\operatorname{ad} E \otimes [-C_0 - K_S]) = 0$. In particular $[E]$ is a smooth point of M_c and the map $\rho_C: (M_c, [E]) \to \mathbf{Def}^0(E|_C)$ has surjective differential in a neighborhood of $[E]$. Finally, by our hypothesis on $B = (\omega)$ it follows (Theorem 2.7) that $\tau_{X_c}(\omega)$ and $\tau_{M_c}(\omega)$ are non-degenerate on open dense subsets of X_c and M_c respectively.

Lemma 5.7 *The following equality holds*:

$$[T]^* \tau_{M_c}(\omega) = \pi^* \tau_{X_c}(\omega) .$$

Proof. Let $([F_0], [\varphi_0])$ be a point of $\mathscr{W}_c$. There is an open (either in the étale or in the analytic topology) neighborhood, U_0, of $[F_0]$ in X_c, such that there is a universal sheaf $\mathscr{F}$ on $S \times U_0$. Let $\tilde{U}_0 = \pi^{-1}(U_0)$. Let $\mathscr{E}$ be a universal sheaf on $S \times M_c$. Let $p: S \times \tilde{U}_0 \to \tilde{U}_0$ be the projection. There exist a line bundle L on $\tilde{U}_0$ and a line bundle $\mathscr{L}$ on $Y = C \times \tilde{U}_0$ so that we have an exact sequence

$$0 \to [T]^* \mathscr{E} \to \pi^* \mathscr{F} \otimes p^*(L) \to \mathscr{O}_Y(\mathscr{L}) \to 0 \tag{8}$$

such that if $([F], [\varphi])$ is a point of $\tilde{U}_0$ the restriction of the exact sequence to $S \times ([F], [\varphi])$ is the elementary modification

$$0 \to E \to F \xrightarrow{\varphi} \mathscr{O}_C(D) \to 0 .$$

Applying Theorem 3.4 and Remark 3.3 to the exact sequence (8) we get that

$[T]^*\tau_{\mathscr{E}}(\omega) = \pi^*\tau_{\mathscr{F}}(\omega)$. Since $[T]^*\tau_{M_c}(\omega) = [T]^*\tau_{\mathscr{E}}(\omega)$ and $\pi^*\tau_{X_c}(\omega) = \pi^*\tau_{\mathscr{F}}(\omega)$ we are done.

By our assumptions there is an open dense subset $U' \subset X_c$ contained in the smooth locus of X_c such that $\tau_{X_c}(\omega)|_{U'}$ is non-degenerate. Since the image, $\pi(\mathscr{W}_c)$, of $\pi\colon \mathscr{W}_c \to X_c$ is an open dense subset and since $\mathscr{W}_c$ is an open subset of the projectivization of a vector bundle over $\pi(\mathscr{W}_c)$ there exists an open dense subset $U \subset U'$ such that $U \subset \pi(\mathscr{W}_c)$ and such that there exists a section $\tilde{\Sigma} \subset \mathscr{W}_c$ of π over U. Notice that, since $\tilde{\Sigma}$ is a section of π we have that $\pi^*\tau_{X_c}(\omega)|_{\tilde{\Sigma}}$ is non-degenerate.

Let $\Sigma = [T](\tilde{\Sigma})$. Let $[E] \in \Sigma$. Consider the map $\rho_C\colon (M_c, [E]) \to \mathbf{Def}^0(E|_C)$. Since $[E] \in [T](\mathscr{W}_c)$ there is an open neighborhood V of $[E]$ such that V is contained in the smooth locus of M_c and such that $\rho_C|_V$ has surjective differential everywhere. Let $\Theta(\rho)$ be the subbundle of the tangent bundle to V consisting of vectors tangent to the fibers of ρ_C, i.e.

$$\Theta(\rho) = \mathrm{Ker}((\rho_C)_*\colon \Theta_V \to \rho_C^*\,\Theta_{\mathbf{Def}^0(E|_C)})$$

where Θ_V, $\Theta_{\mathbf{Def}^0(E|_C)}$ are the tangent bundles of V and $\mathbf{Def}^0(E|_C)$ respectively. Since for every $[E'] \in \Sigma$ we have that $E'|_C \cong \mathcal{O}_C(D) \otimes \mathcal{O}_C(-D)$ it follows that $\Theta_{[E]}(\Sigma) \subset \Theta_{[E]}(\rho)$. As we have noticed the restriction of $\pi^*\tau_{X_c}(\omega)$ to $\Theta_{[E]}(\Sigma)$ is non-degenerate. By Lemma 5.7 we get that the restriction of $\tau_{M_c}(\omega)$ to $\Theta_{[E]}(\Sigma)$ is non-degenerate. Thus there exists an open neighborhood $U' \subset M_c$ of $[E]$ such that if $[E'] \in U'$ then there exists a subspace $W \subset \Theta_{[E']}(\rho)$ of the same dimension as Σ on which $\tau_{M_c}(\omega)$ is non-degenerate. Now notice that the restriction of the map $H^1(\mathrm{ad}\,E') \to H^1(\mathrm{ad}\,E'|_C)$ to $\Theta_{[E']}(\rho)$ is zero, hence $\Theta_{[E']}(\rho) \subset H^1(\mathrm{ad}\,E' \otimes [-C_0])$. Thus for every $[E'] \in U'$ there exists a subspace $W \subset H^1(\mathrm{ad}\,E' \otimes [-C_0])$ of dimension equal to $\dim X_c = \dim M_c - 4g$ on which the restriction of $\tau_{M_c}(\omega)$ is non-degenerate. Now, since $\tau_{M_c}(\omega)$ is non-degenerate on an open dense subset of M_c, and since M_c is assumed to be C_0-good, there exists an open dense subset $U \subset U'$ such that $\tau_{M_c}(\omega)$ is non-degenerate on U and for every $[E'] \in U$ we have that $E'|_C$ is stable. If $E'|_C$ is stable we have that $\mathrm{cod}(H^1(\mathrm{ad}\,E' \otimes [-C_0]), H^1(\mathrm{ad}\,E')) = 3g - 3$. Applying Lemma 5.2 to

$$W \subset H^1(\mathrm{ad}\,E' \otimes [-C_0]) \subset H^1(\mathrm{ad}\,E')$$

we conclude that for any n such that $n \leqq g - 3$ there exists a codimension $2n$ subspace $W'' \subset H^1(\mathrm{ad}\,E')$ containing $H^1(\mathrm{ad}\,E' \otimes [-C_0])$ and such that the restriction of $\tau_{M_c}(\omega)$ to W'' is non-degenerate. Thus we conclude that for any n such that $n \leqq \min\{\frac{1}{2}(g+1), g-3\}$ we can represent the Poincaré dual of $\varphi^*(\nu_{\tilde{M}_c}(k[C_0])^{2n}$ as in Proposition 5.2 in such a way that

$$\int_{[A]} \underbrace{(\tau_{\tilde{M}_c}(\omega) + \overline{\tau_{\tilde{M}_c}(\omega)}) \wedge \ldots \wedge (\tau_{\tilde{M}_c}(\omega) + \overline{\tau_{\tilde{M}_c}(\omega)})}_{d(c)-2n} > 0\,.$$

This concludes the proof of Theorem 2.6.

6 The polynomials δ_c for complete intersections

In this section we will examine more closely the polynomial $\delta_c(S, H)$ of a complete intersection S (of general type) in a projective space $\mathbb{P}^{r+2}$, where the polarization H is the pull-back to S of the hyperplane class on $\mathbb{P}^{r+2}$. We will prove Theorems 2.8

and 2.9. We do not know whether $\delta_c(S, H)$ is equal to Donaldson's polynomial, but the result of Theorem 2.8 is analogous to a theorem of Friedman and Morgan concerning the Donaldson polynomial of complete intersections. As we explained in the first section, if we knew that $\gamma_c(S) = \delta_c(S, H)$, then we would conclude from Theorem 2.9 that if S is a complete intersection with $p_g(S)$ odd, then $\pm c_1(K_S)$ is a diffeomorphism invariant (if $p_g(S)$ is even this has been proved by Friedman and Morgan [9]).

Throughout this section B will denote an irreducible component of the parameter space for smooth complete intersections of general type in the projective space $\mathbf{P}^{r+2}$. If $x \in B$ we let S_x be the surface parametrized by x. Then there are positive integers $(d_1, \ldots, d_r)$ such that each surface S_x is given by $S_x = V_1 \cap \ldots \cap V_r$, where $V_i \subset \mathbf{P}^{r+2}$ is a hypersurface of degree d_i. By adjunction the condition that S_x be of general type is equivalent to $\sum_i d_i > r + 3$. Let $f: \mathscr{S} \to B$ be the universal surface over B. Clearly on $\mathscr{S}$ we have a line bundle $\mathscr{H}$ such that for every $x \in B$ the restriction of $\mathscr{H}$ to S_x is the pull-back to S_x of the hyperplane class, which we denote by H_x. Then, by a theorem of Maruyama [15], given any integer c there exists a relative moduli space $\pi_{GM}: \mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c) \to B$ with the following properties:

(i) $\mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c)$ is proper over B

(ii) if $x \in B$ then the scheme-theoretic fiber $\pi^{-1}(x)$ is isomorphic to the moduli space of rank-two torsion free sheaves on S_x with $c_1 = 0$ and $c_2 = c$, which we denote by $M_{GM}(S_x, H_x; 0, c)$.

Let $\mathscr{M}(\mathscr{S}, \mathscr{H}; 0, c) \subset \mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c)$ be the open subset of points parametrizing slope-stable locally free sheaves. We let $\bar{\mathscr{M}}(\mathscr{S}, \mathscr{H}; 0, c)$ be the closure of $\mathscr{M}(\mathscr{S}, \mathscr{H}; 0, c)$ in $\mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c)$. We will denote by π the restriction of π_{GM} to $\bar{\mathscr{M}}(\mathscr{S}, \mathscr{H}; 0, c)$. Clearly π is proper. If $x \in B$ then $\bar{M}(S_x, H_x; 0, c) \subset \pi^{-1}(x)$. If x is a generic point of B then $\bar{M}(S_x, H_x; 0, c) = \pi^{-1}(x)$, but a priori it might happen that there exist points x such that $\bar{M}(S_x, H_x; 0, c)$ is a proper subscheme of $\pi^{-1}(x)$.

The following is an immediate consequence of Gieseker's Theorem 2.2.

Lemma 6.1 *There exists a number k_0 such that if $c > k_0$ then $\pi(\bar{\mathscr{M}}(\mathscr{S}, \mathscr{H}; 0, c)) = B$.*

Definition 6.1 We define $U(c) \subset B$ by setting

$$U(c) = \{x \in B \mid \pi^{-1}(x) = \bar{M}(S_x, H_x; 0, c)\} .$$

Definition 6.2 Let n be an integer. We define $G_n(c) \subset B$ by setting

$$G_n(c) = \{x \in B \mid M(S_x, H_x; 0, c) \text{ is } nH_x - \text{good}\} .$$

Lemma 6.2 *Let B be as above. Fix an integer n. Then there exists a number k_1 such that if $c > k_1$ then $G_n(c)$ contains a Zariski open non-empty subset of B.*

Proof. Let k_0 be as in Lemma 6.1. Then if $c > k_0$ we have that $U(c)$ is an open dense subset of B. Thus there exists $x \in B$ such that $x \in U(c)$ for all $c > k_0$. By Corollary 2.2 there exists a number k_0' such that if $c > k_0'$ then $M(S_x, H_x; 0, c)$ is nH_x-good. Let $k_1 = \max\{k_0, k_0'\}$.

Let $\mathscr{G}_c(c) \subset \mathscr{M}(\mathscr{S}, \mathscr{H}; 0, c)$ be the subset of points $\tilde{y}$ parametrizing vector bundles $E_{\tilde{y}}$ which are nH_y-good, where $y = \pi(\tilde{y})$. By the semicontinuity theorem $\mathscr{G}_n(c)$ is a Zariski open subset of $\mathscr{M}(\mathscr{S}, \mathscr{H}; 0, c)$. Since $\pi^{-1}(x) = \bar{M}(S_x, H_x; 0, c)$ and $M(S_x, H_x; 0, c)$ is nH_x-good we conclude that $\mathscr{G}_n(c)$ intersects in a dense open subset every irreducible component of $\mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c)$ containing a point of $\pi^{-1}(x)$. This implies that there is a Zariski open subset $\mathscr{U} \subset B$ containing x and such that if $y \in \mathscr{U}$ the moduli space $M(S_y, H_y; 0, c)$ is nH_y-good, q.e.d.

Let (S, H) be a polarized projective surface. Let c be odd, then by Remark 2.2 there exists a universal sheaf $\mathscr{E}$ on $S \times M_{GM}(S, H; 0, c)$. Then for any $\alpha \in H_2(S)$ we let $v_{M_{GM}(S,H;0,c)}(\alpha) = v_{\mathscr{E}}(\alpha)$. As is easily checked $v_{M_{GM}(S,H;0,c)}(\alpha)$ does not depend on the choice of a universal sheaf.

The following two propositions are proved in [16].

Proposition 6.1 *Let c be odd. Then one can define a relative v map. More precisely: let $\mathscr{U}$ be a contractible open (in the analytic topology) subset of B, then one can define a map $v: H_2(f^{-1}(\mathscr{U})) \to H^2(\pi^{-1}(\mathscr{U}))$ which, if $x \in B$, is compatible with the map $v_{M_{GM}(S_x, H_x; 0, c)}$, where $x \in \mathscr{U}$, via the isomorphism $H_2(S_x) \cong H_2(f^{-1}(\mathscr{U}))$.*

Proposition 6.2 *Let c be an integer. Assume that $G_0(c)$ contains an open dense subset of B. Then there exists an open dense subset $V(c) \subset B$ such that:*
(i) *If $x \in V(c)$ then $\pi^{-1}(x) = \bar{M}(S_x, H_x; 0, c)$ and it is of pure dimension equal to $d(c) = 4c - 3\chi(\mathcal{O}_S)$.*
(ii) *If $x, y \in V(c)$ the cycles on $\bar{\mathscr{M}}(\mathscr{S}, \mathscr{H}; 0, c)$ given by $[\pi^{-1}(x)]$, $[\pi^{-1}(y)]$ are homologous.*
(iii) *Assume that c is odd. Let $x, y \in V(c)$ and let γ be a path in $V(c)$ going from x to y. Let $\gamma_*: H_2(S_x) \to H_2(S_y)$ be the map induced by parallel transport, then as a consequence of* (ii) *we have that*

$$\delta_c(S_x, H_x)(\alpha_1, \ldots, \alpha_{d(c)}) = \delta_c(S_y, H_y)(\gamma_*(\alpha_1), \ldots, \gamma_*(\alpha_{d(c)})) .$$

The last property allows us to prove Theorem 2.8.

Proof of Theorem 2.8

Recall that we have to show that if $x \in V(c)$ then there exist rational coefficients a_i^x such that

$$\delta_c(S_x, H_x; 0, c) = \sum_{i=0}^{[\frac{d(c)}{2}]} a_i^x q_x^i k_x^{d(c)-2i} \tag{1}$$

and that a_i^x is independent of x.

By Lemma 6.2 there exists a number k such that if $c > k$ then $G_0(c)$ contains a Zariski open dense subset of B. So let c be odd and greater than k. Fix $x \in V(c)$, then by property (iii) in the above proposition we have that $\delta_c(S_x, H_x)$ is invariant under the monodromy representation of $\pi_1(V(c), x)$. As is proved in [9] the algebra of invariants for the monodromy representation of a complete intersection of general type is generated by q_x, k_x. Thus we get the equality (1). The coefficients are rational because $\delta_c(S_x, H_x)$ is defined over **Q**.

To prove that the coefficients are independent of x, let $y \in V(c)$ be another point, let γ be a path contained in $V(c)$ and joining x to y. By Proposition 6.2 we have that $\gamma_* \delta_c(S_x, H_x) = \delta_c(S_y, H_y)$. On the other hand $\gamma_* q_x = q_y$ and $\gamma_* k_x = k_y$. As is easily checked this implies that $a_i^x = a_i^y$.

Theorem 6.1 *Assume that the geometric genus of surfaces parametrized by B is odd. Then there exists a number k such that if c is odd and $c > k$ the following holds: there is an open dense subset $B(c) \subset B$ such that if $x \in B(c)$ we have*

$$\delta_c(S_x, H_x) = \sum_{i=0}^{[\frac{d(c)}{2}]} a_i q^i k^{d(c)-2i} . \tag{2}$$

with $a_{\frac{d(c)}{2}} > 0$.

Proof. Let $S_x = V_1 \cap \ldots \cap V_r \subset \mathbf{P}^{r+2}$ be the complete intersection corresponding to the point $x \in B$. Let d_i be the degree of the hypersurface V_i. Then, by adjunction, we have that $K_{S_x} \cong (\sum_i d_i - (r+3))H_x$. Let $n = \sum_i d_i - (r+3)$. By Lemma 6.2 there exists a number k such that if $c > k$ then $G_n(c)$ contains an open dense subset of B. Notice that, since $n > 0$ we have that $G_n(c) \subset G_0(c)$, hence if $c > k$ Proposition 6.2 applies. Thus there is an open dense subset of $V(c) \cap G_n(c)$, call it $B(c)$.

Now let c be odd with $c > k$. If $x \in B(c)$ then $\delta_c(S_x, H_x)$ is given by a polynomial as in (2).

On the other hand the generic divisor $D \in |K_{S_x}|$ is a smooth connected curve of genus at least six. Thus, since $p_g(S_x)$ is odd and $x \in G_n(c)$, Theorem 2.4 applies to the moduli space $\bar{M}(S_x, H_x; 0, c)$. Let $\omega \in H^0(\Omega^2_{S_x})$ be a holomorphic two-form such that (ω) is a smooth connected curve, and let Γ be the Poincaré dual of ω. Then $\delta_c(S_x, H_x)(\Gamma + \bar{\Gamma}) > 0$.

The expression (2) gives that $\delta_c(S_x, H_x)(\Gamma + \bar{\Gamma}) = a_{\frac{d(c)}{2}} q(\Gamma + \bar{\Gamma})^{\frac{d(c)}{2}}$. Since $q(\Gamma + \bar{\Gamma}) > 0$ we conclude that $a_{\frac{d(c)}{2}} > 0$.

Now we wish to apply Theorem 2.6 to complete intersections with odd geometric genus in order to draw some consequences on the coefficients a_i. We first notice that for any r-tuple $(d_1, \ldots, d_r)$ with $\sum_i d_i > r + 3$ there exists a complete intersection $S_x = V_1 \cap \ldots \cap V_r \subset \mathbf{P}^{r+2}$ (where V_i is a hypersurface of degree d_i) which has a base-point free pencil $|C_0|$ with C_0 a smooth connected curve of genus $g \geqq 4$ (we will discuss this later on). Let $D \in |K_{S_x}|$ be a smooth curve, let $\omega \in H^0(\Omega^2_{S_x})$ be such that $(\omega) = D$, and let Γ be its Poincaré dual. By Theorem 2.6, if c is odd and big enough, then if $j \leqq \min\{\frac{1}{2}(g+1), g-3\}$ we have that

$$\delta_c(S_x, H_x)(\underbrace{[C_0], \ldots, [C_0]}_{2j}, \underbrace{\Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}}_{d(c)-2j}) > 0 .$$

If we knew that $x \in V(c)$ then $\delta_c(S_x, H_x)$ would be given by the expression (2). But then, as is easily checked, we would conclude that a_i is positive for all $i \geqq \frac{1}{2}d(c) - \min\{\frac{1}{2}(g+1), g-3\}$.

Since we don't know whether we can assume that $x \in V(c)$ we now proceed to give a more roundabout argument. We will start by proving some lemmas.

Lemma 6.3 *There exists a number k such that if $c > k$ then the following holds. For all $x \in B$ we have that if $\tilde{x} \in \pi^{-1}(x) \backslash \bar{M}(S_x, H_x; 0, c)$ then the sheaf $E_{\tilde{x}}$ corresponding to $\tilde{x}$ is not locally free.*

Proof. Let $\mathscr{F} \subset \mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c)$ be the open subset parametrizing locally free sheaves. We must show that there exists a k such that if $c > k$ then the closure of

$\mathscr{F}$ in $\mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c)$ is equal to $\bar{\mathscr{M}}(\mathscr{S}, \mathscr{H}; 0, c)$. Let $\tilde{x} \in \mathscr{F} \setminus \mathscr{M}(\mathscr{S}, \mathscr{H}; 0, c)$ and let $E_{\tilde{x}}$ be the corresponding sheaf. Then, since $E_{\tilde{x}}$ is H_x-semistable (where $x = \pi(\tilde{x})$) but not H-slope-stable, it must be properly H-slope-semistable. As is easily checked, there is a parametrized version of Proposition 2.3, i.e. for a family of polarized surfaces parametrized by a scheme of finite type we can choose constants a, b so that the proposition holds for any surface in the family. In particular, letting $N(x, c)$ be the number of moduli of H_x-properly-semistable rank-two bundles on S_x with $c_1 = 0$, $c_2 = c$, there exist constants a, b such that for all $x \in B$ we have

$$N(x, c) < 3c + a\sqrt{c} + b .$$

Now let k be such that if $c > k$ then $4c - 3\chi(\mathcal{O}_{S_x}) > 3c + a\sqrt{c} + b$. Then by deformation theory we have that any point of $\mathscr{F}$ is contained in the closure of $\mathscr{M}(\mathscr{S}, \mathscr{H}; 0, c)$, q.e.d.

Lemma 6.4 *Assume $G_0(c)$ contains an open dense subset of B. Let $V(c)$ be the open subset of B given by Proposition 6.2. Let $x \in B$. Let $\Delta \subset B$ be an analytic disc (i.e. $\Delta \cong \{z \in \mathbf{C} \,|\, |z| < 1\}$) centered at x. Let $\tilde{x} \in M(S_x, H_x; 0, c) \subset \pi^{-1}(x)$ be a point such that, letting $[E_{\tilde{x}}]$ be the corresponding isomorphism class of vector bundles, we have $h^2(\mathrm{ad}\, E_x) = 0$. Then there exists a disc Δ_0 centered at x and contained in Δ such that there is a section $\tilde{\Delta}_0 \subset \pi^{-1}(\Delta_0)$ of π restricted to $\pi^{-1}(\Delta_0)$.*

Proof. By the universal property of the relative moduli space $\mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c)$ the scheme-theoretic fiber over x of π_{GM} is smooth at $\tilde{x}$. This implies in particular that $\pi_{GM}^{-1}(\Delta)$ is smooth at $\tilde{x}$. Hence it will suffice to show that there exists a smaller disc Δ' centered at x such that π has a multisection of over Δ' passing through $\tilde{x}$.

By openness of stability it is enough to show that there exists a rank two vector bundle $\mathscr{E}$ over $f^{-1}(\Delta)$ such that $\mathscr{E}|_{S_x} \cong E_{\tilde{x}}$. The proof of this is obtained by a straightforward modification of Proposition 2.1 and Lemma 5.1 in [12].

Lemma 6.5 *Assume that $G_0(c)$ contains an open dense subset of B, so that Proposition 6.2 applies. Let $y \in B \setminus V(c)$. Let $\Delta \subset B$ be an analytic disc centered at y such that $\Delta \setminus \{y\}$ is contained in $V(c)$. Then there is an algebraic cycle $Z = \sum_i m_i [Z_i]$ of dimension $d(c) = 4c - 3\chi(\mathcal{O}_S)$ supported on $\pi^{-1}(y) = M_{GM}(S_y, H_y; 0, c)$ such that if $x \in \Delta \setminus \{y\}$ then $[\pi^{-1}(x)]$ is homologous to Z. Furthermore each irreducible component of $\bar{M}(S_y, H_y; 0, c)$ appears in Z with coefficient one.*

Proof. Consider

$$Y = \overline{\pi_{GM}^{-1}(\Delta \setminus \{y\})} \subset \mathscr{M}_{GM}(\mathscr{S}, \mathscr{H}; 0, c) .$$

The restriction of π_{GM} to Y is proper. Let $x \in \Delta \setminus \{y\}$, then the fiber $\pi_{GM}^{-1}(x)$ has pure dimension equal to $d(c)$. Thus also the fiber of $\pi_{GM}^{-1}(y)$ has pure dimension equal to $d(c)$. Let Z_i be the irreducible components of this fiber. Then there exist positive integers m_i such that if $x \in \Delta \setminus \{y\}$ we have that $[\pi_{GM}^{-1}(x)]$ is homologous (in Y) to the algebraic cycle

$$Z = \sum_i m_i [Z_i] . \tag{3}$$

By the previous lemma we see that each irreducible component of $\bar{M}(S_y, H_y; 0, c) \subset \pi_{GM}^{-1}(y)$ is in the closure of $\pi_{GM}^{-1}(\Delta \setminus \{y\})$, hence it appears in the expression (3). Again by the previous lemma, there is a local section of π_{GM} over Δ passing through the generic point of each irreducible component of $\bar{M}(S_y, H_y; 0, c) \subset \pi^{-1}(y)$. This implies that each irreducible component of $\bar{M}(S_y, H_y; 0, c)$ appears with coefficient one.

Proposition 6.3 *Let $y \in B$. Assume that S_y has a base-point free pencil $|C_0|$, where C_0 is a smooth connected curve of genus g. Then there exists a number k such that the following holds for all c odd with $c > k$.*

The moduli space $\bar{M}(S_y, H_y; 0, c)$ is non-empty and good, hence $\delta_c(S_y, H_y; 0, c)$ is defined. The set $G_0(c)$ contains an open dense subset of B, so that Proposition 2.2 applies. Let $\Gamma \in H_2(S_y)$ be the Poincaré dual of a holomorphic two-form. Let $x \in V(c)$. Let γ be a path in B starting at y and ending at x, and let $\gamma_*: H_2(S_y) \to H_2(S_x)$ be the map given by parallel transport along γ. Then for all $j \leqq g + 1$ we have that

$$\delta_c(S_y, H_y)(\underbrace{[C_0], \ldots, [C_0]}_{j}, \underbrace{\Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}}_{d(c)-j})$$

$$= \delta_c(S_x, H_x)(\underbrace{\gamma_*[C_0], \ldots, \gamma_*[C_0]}_{j}, \underbrace{\gamma_*(\Gamma + \bar{\Gamma}), \ldots, \gamma_*(\Gamma + \bar{\Gamma})}_{d(c)-j}) \qquad (4)$$

where $d(c) = 4c - 3\chi(\mathcal{O}_{S_y})$.

Proof. By Corollary 2.2, Lemma 6.2 and Lemma 6.3 there exists k such that if $c > k$ then $\bar{M}(S_y, H_y; 0, c)$ is good, $G_0(c)$ contains an open dense subset of B and if $\tilde{y} \in \pi^{-1}(y) \backslash \bar{M}(S_y, H_y; 0, c)$ then the corresponding sheaf is not locally free. Now deform the path γ (leaving its end points fixed) so that the image is contained in $V(c)$, except for the initial point y. We can also assume that there is a number $0 < a \leqq 1$ such that $\gamma([0, a])$ is contained in an analytic disc Δ centered at y. Let $x_a = \gamma(a)$ and let γ_a be the path $\gamma|_{[0,a]}$. By Proposition 6.2 it is enough to prove that (4) holds with x, γ replaced by x_a, γ_a. Let $Z = \sum_i m_i[Z_i]$ be the cycle supported on $M_{GM}(S_y, H_y; 0, c)$ given by the above lemma. Let $\mathscr{E}_i$ be the restriction to $S_y \times Z_i$ of the universal sheaf on $S \times M_{GM}(S_y, H_y; 0, c)$. Let $v_i = v_{\mathscr{E}_i}$, then for each i we can define a polynomial δ_i on $H_2(S_y)$ by setting

$$\delta_i(\alpha) = \int_{[Z_i]} \underbrace{v_i(\alpha) \cup \ldots \cup v_i(\alpha)}_{d(c)} .$$

Let I be the set of indices corresponding to the irreducible components of $\bar{M}(S_y, H_y; 0, c)$ in the expression (3), and let J be the set of indices corresponding to the Z_i which are not contained in $\bar{M}(S_y, H_y; 0, c)$. Let $\beta_1, \ldots, \beta_{d(c)} \in H_2(S_{x_a})$, then, since if $i \in I$ the coefficient of $[Z_i]$ in Z is one we have that

$$\delta_c(S_{x_a}, H_{x_a})(\beta_1, \ldots, \beta_{d(c)}) = \delta_c((\gamma_a)_*^{-1}\beta_1, \ldots, (\gamma_a)_*^{-1}\beta_{d(c)})$$
$$+ \sum_{i \in J} m_i \delta_i((\gamma_a)_*^{-1}\beta_1, \ldots, (\gamma_a)_*^{-1}\beta_{d(c)}) .$$

At this point we change k, if necessary, so that the following proposition applies to (S_y, H_y).

Proposition 6.4 *Let (S, H) be a polarized regular projective surface with a base-point free pencil $|C_0|$, where $C_0 \subset S$ is a smooth connected curve of genus $g \geqq 2$. There exists a number k such that if $c > k$ then the following holds.*

Let X be a projective variety of dimension $d(c) = 4c - 3\chi(\mathcal{O}_S)$. Let $\mathscr{E}$ be an X-flat rank-two torsion-free sheaf on $S \times X$. If $x \in X$ let $E_x = \mathscr{E}|_{S \times \{x\}}$. Assume that for every $x \in X$ one has: $c_1(E_x) = 0$, $c_2(E_x) = c$, E_x is H-semistable and not locally free. Let ω be a holomorphic two-form on S and let Γ be its Poincaré dual. Then if $0 \leqq p \leqq g + 1$ we have that

$$\int_{[X]} \underbrace{\nu_{\mathscr{E}}([C_0]) \wedge \ldots \wedge \nu_{\mathscr{E}}([C_0])}_{p} \wedge \underbrace{\nu_{\mathscr{E}}(\Gamma + \bar{\Gamma}) \wedge \ldots \wedge \nu_{\mathscr{E}}(\Gamma + \bar{\Gamma})}_{d(c)-p} = 0 .$$

We postpone the proof of this lemma and proceed to deduce some consequences of Proposition 6.3.

Let S be a smooth complete intersections $S = V_1 \cap \ldots \cap V_r \subset \mathbf{P}^{r+2}$, where V_i is a hypersurface of degree d_i. Assume that the equation of V_1 is of the form $\lambda FN + \mu GL = 0$, where λ, μ are constants and F, G, L, N are homogeneous polynomials such that $\deg F = \deg G$, $\deg L = \deg N$, (and obviously $\deg F + \deg N = d_1$). Then $f = \dfrac{F}{G}$ defines an everywhere regular map from S to $\mathbf{P}^1$, i.e. a base-point free pencil $|C_0|$. The curve C_0 can be taken to be the complete intersection of $V_2, \ldots, V_r$ and the two hypersurfaces defined by $F = 0$ and $L = 0$. An application of Bertini's Theorem shows that if $\lambda FN + \mu GL$ and the V_i's $(i \geqq 2)$ are generic then S is smooth. Now, if S is of general type, i.e. $\sum_i d_i > (r + 3)$, one can always choose the degrees of F, G, L, N so that the genus of C_0 is at least four. Thus we are ready for the

Proof of Theorem 2.9

Let y be a point of B parametrizing a surface S_y which has a base-point free pencil $|C_0|$ with C_0 a smooth connected curve of genus $g > 3$. Let $\omega \in H^0(\Omega^2_{S_y})$ be such that (ω) is a smooth curve, and let Γ be its Poincaré dual. By Theorem 2.6 there exists a k_0 such that if c is odd and $c > k_0$ then the moduli space $\bar{M}(S_y, H_y; 0, c)$ is good and for $j \leqq \min\left\{\dfrac{g+1}{2}, g - 3\right\}$ we have that

$$\delta_c(S_y, H_y)(\underbrace{[C_0], \ldots, [C_0]}_{i}, \underbrace{\Gamma + \bar{\Gamma}, \ldots, \Gamma + \bar{\Gamma}}_{d(c)-i}) > 0 .$$

On the other hand there exists a k_1 such that if c is odd and $c > k_1$ then Proposition 6.4 applies to S_y. Let $k = \max\{k_0, k_1\}$, then the theorem is a consequence of Proposition 6.3.

Remark 6.1 Let's examine the case of surfaces in $\mathbf{P}^3$ in greater detail. First notice that a surface $S_n \subset \mathbf{P}^3$ of degree n has odd geometric genus if and only if n is divisible by 4. Let F, G, L, N be generic homogeneous polynomials of degree $\frac{1}{2}n$.

Then $\lambda FN + \mu GL = 0$ defines a smooth surface S of degree n containing the base-point free pencil $|C_0|$ corresponding to the morphism $f: S \to \mathbf{P}^1$ given by $f = \frac{F}{G}$. As is easily checked $g(C_0) = \frac{1}{8}n^3 - \frac{1}{2}n^2 + 1$. Thus in this case we get that if c is odd and big enough then there exists an open dense $V(c) \subset B$ such that if $x \in V(c)$ we have

$$\delta_c(S_x, H_x) = \sum_{i=0}^{\frac{d(c)}{2}} a_i q^i k^{d(c)-2i}$$

with $a_i > 0$ for all i such that $i \geqq \frac{d(c)}{2} - (\frac{1}{16}n^3 - \frac{1}{4}n^2 + 1)$.

Now we proceed to prove Proposition 6.4.

Lemma 6.6 *Let (S, H) be a polarized regular projective surface with a base-point free pencil $|C_0|$, where $C_0 \subset S$ is a smooth connected curve of genus $g \geqq 2$. There exists a number k_0 such that if $c > k_0$ then the moduli space $M(S, H; 0, c)$ is good and the following holds. Let $C \in |C_0|$ be a smooth curve, let $U_C(c) \subset M(S, H; 0, c)$ be the subset of points $[E]$ such that $E|_C$ is unstable, then* $\operatorname{cod}(U_C(c), M(S, H; 0, c)) \geqq g + 1$.

Proof. By Theorem 2.3 there exists a k_0 such that if $c > k_0$ then the moduli space $M(S, H; 0, c)$ is C_0-good and the subset, Σ, of points $[E] \in M(S, H; 0, c)$ for which $h^2(\operatorname{ad} E \otimes [-C_0]) > 0$ is of codimension at least $g + 1$.

If $U_C(c) \subset \Sigma$ we are done, so assume $[E] \in U_C(c) \setminus \Sigma$. The differential of the map $\rho_C: (M(S, H; 0, c), [E]) \to \mathbf{Def}^0(E|_C)$ is surjective at $[E]$, because $[E] \notin \Sigma$. By Theorem 2.1 $M(S, H; 0, c)$ is smooth at $[E]$ hence we conclude that ρ_C is surjective near $[E]$. Hence by Proposition 2.4 we get that $\operatorname{cod}_{[E]}(U_C, M(S, H; 0, c)) \geqq g + 1$.

Lemma 6.7 *Let (S, H) be a polarized regular projective surface with a base-point free-pencil $|C_0|$, where $C_0 \subset S$ is a smooth connected curve of genus $g \geqq 2$. Then there exists a number k such that if $c > k$ the following holds.*

Let X be an irreducible projective variety such that $\dim X \geqq 4c - 3\chi(\mathcal{O}_S) - g - 1$. *Let $\mathscr{E}$ be a coherent X-flat rank-two torsion-free sheaf on $S \times X$. If $x \in X$ let $E_x = \mathscr{E}|_{S \times \{x\}}$. Assume that for every $x \in X$ one has: $c_1(E_x) = 0$, $c_2(E_x) = c$, E_x is H-semistable and is not locally free.*

Let $\omega \in H^0(\Omega_S^2)$. Let Γ be its Poincaré dual.

Assume that one of the following holds:

(i) *letting $c'' = c_2(E_x^{**})$ for the generic $x \in X$, one has that $c'' \leqq k_0$, where k_0 is as in the previous lemma*

(ii) *for the generic $x \in X$ the vector bundle E_x^{**} is properly H-slope-semistable*

(iii) *for every $C \in \{C \in |C_0| \mid C$ is smooth and $E_x|_C$ is locally free for generic $x \in X\}$ we have that $E_x|_C$ is unstable for the generic x.*

Then, if $A \subset X$ is any subvariety with $\dim A = d \geqq 4c - 3\chi(\mathcal{O}_S) - g - 1$, *we have that*

$$\int_{[A]} \underbrace{\nu_{\mathscr{E}}(\Gamma + \bar{\Gamma}) \cup \ldots \cup \nu_{\mathscr{E}}(\Gamma + \bar{\Gamma})}_{d} = 0 .$$

Proof. Let N be the maximum of the dimensions of moduli spaces $M(S, H; 0, c')$ where $c' \leqq k_0$. Let a, b be numbers such that we have

$$\dim P(0, c') < 3c' + a\sqrt{c'} + b$$

where $P(0, c')$ is the parameter space for properly H-slope-semistable rank-two vector bundles on S with $c_1 = 0$, $c_2 = c'$ (Remark 2.3). We choose k so that

$$k > \tfrac{3}{2}\chi(\mathcal{O}_S) + \tfrac{1}{2}(g + 1) + \tfrac{1}{2}N$$

and so that if $c > k$ then

$$c > a\sqrt{c} + b + 3\chi(\mathcal{O}_S) + g + 1\ .$$

Let $\varphi\colon \tilde{A} \to A$ be a desingularization of A. Let $\tilde{\mathscr{E}} = (\mathrm{id}_S \times \varphi)^*(\mathscr{E}|_{S\times A})$. By Lemma 3.3 we must show that $\int_{[\tilde{A}]} \underbrace{v_{\tilde{\mathscr{E}}}(\Gamma + \bar{\Gamma}) \cup \ldots \cup v_{\tilde{\mathscr{E}}}(\Gamma + \bar{\Gamma})}_{d} = 0$. Now let's apply the construction described in Remark 3.5 to $\tilde{\mathscr{E}}$. Hence we have an open dense subset $U \subset \tilde{A}$ such that $(\tilde{\mathscr{E}})^{**}|_{S\times U}$ is locally free, let $c' = c_2(\tilde{\mathscr{E}})^{**}|_{S\times\{x\}}$, where $x \in U$. We also have a morphism $f\colon U \to \times_i \mathrm{Hilb}^{d_i}(S) \times Y$ and a holomorphic two-form λ on $\times_i \mathrm{Hilb}^{d_i}(S) \times Y$ such that $f^*\lambda$ extends to all of $\tilde{A}$ and the resulting extension represents $v_{\tilde{\mathscr{E}}}(\Gamma)$.

We will show that $\dim f(U) < 4c - 3\chi(\mathcal{O}_S) - g - 1$, this clearly implies the lemma.

First of all we notice that $c' < c$ and that $\dim \times_i \mathrm{Hilb}^{d_i}(S) \leqq 2(c - c')$. Now let's first assume that either $c' \leqq n_0$ or $Y = P(c')$ (this covers cases (i) and (ii)). Then an easy computation shows that, by our choice of n, we have

$$\dim \times_i \mathrm{Hilb}^{d_i}(S) \times Y < 4c - 3\chi(\mathcal{O}_S) - g - 1\ .$$

Now let's examine case (iii). Since we have proved the lemma in the two previous cases we can assume that $Y = M_{c'}$. There exists a smooth $C \in |C_0|$ such that for generic $x \in \tilde{A}$ the sheaf $E_x|_C$ is locally free. By hypothesis then it is unstable for all such x. Let $\beta\colon U \to M_{c'}$ be the morphism defined in Remark 3.5 (i.e. $\beta(x) = [(\tilde{E}_x)^{**}]$). Then $\beta(U) \subset U_C(c')$. By Lemma 6.6 $\dim U_C(c') \leqq 4c' - 3\chi(\mathcal{O}_S) - g - 1$. An easy computation then shows that $\dim f(U) < 4c - 3\chi(\mathcal{O}_S) - g - 1$ as required.

Proof of Proposition 6.4

We claim that if $p \leqq g + 1$ then there exist distinct smooth curves $C_1, \ldots, C_r \in |C_0| (r \geqq p)$ and line bundles $L_i \in \mathrm{Pic}^{g-1}(C_i)$ with the following properties: we can represent the Poincaré dual of $v_{\mathscr{E}}([C_0])^p$ by an algebraic cycle $A = \sum_j m_j[A_j]$, where each A_j is an irreducible codimension-p subvariety of X such that either Lemma 6.7 applies to the restriction of $\mathscr{E}$ to $S \times A_j$ or else A_j is an irreducible component of $\Delta^{\mathrm{red}}_{\mathscr{E}}(C_{i_1}, L_{i_1}) \cap \ldots \cap \Delta^{\mathrm{red}}_{\mathscr{E}}(C_{i_p}, L_{i_p})$ for some choice of p distinct curves among the C_i's. Furthermore we can assume that for each $c' > k_0$ (where k_0 is as in Lemma 6.6) and for each s-tuple of distinct curves $C_{i_1}, \ldots, C_{i_s}$ the intersection $\Delta^{\mathrm{red}}_{M_{c'}}(C_{i_1}, L_{i_1}) \cap \ldots \cap \Delta^{\mathrm{red}}_{M_{c'}}(C_{i_s}, L_{i_s})$ has codimension s in $M_{c'}$.

We prove the claim by induction on p. If $p = 0$ there is nothing to prove. Assume the claim holds for $p < g + 1$. Hence the Poincaré dual of $\nu_{\mathscr{E}}([C_0])^p$ is represented by the algebraic cycle $\sum_j m_j[A_j]$. Consider one A_j. Assume Lemma 6.7 applies to the restriction of $\mathscr{E}$ to $S \times A_j$. Represent the intersection of the Poincaré dual of $\nu_{\mathscr{E}}([C_0])$ with $[A_j]$ by an algebraic cycle $\sum_k m_{jk}[A_{jk}]$ supported on A_j. Then Lemma 6.7 will still apply to each A_{jk}.

Now assume that Lemma 6.7 does not apply to A_j. Then there exists a smooth curve $C' \in |C_0|$ such that $E_x|_{C'}$ is locally free for the generic $x \in A_j$ and $E_x|_{C'}$ is semistable. Then, by openness of semistability, there exists such a C' not equal to any of the previously chosen curves C_i. Let $C_{r+1} = C'$. We can choose $L_{r+1} \in \mathrm{Pic}^{g-1}(C_{r+1})$ so that $A'_j = \Delta_{\mathscr{E}}^{\mathrm{red}}(C_{r+1}, L_{r+1}) \cap A_j$ is of codimension one in A_j. Furthermore we can choose (C_{r+1}, L_{r+1}) so that for all $k_0 < c' < c$ and for any $(s-1)$-tuple $C_{i_1}, \ldots, C_{i_{s-1}}$ of distinct curves the intersection $\Delta_{M_{c'}}^{\mathrm{red}}(C_{r+1}, L_{r+1}) \cap \Delta_{M_{c'}}^{\mathrm{red}}(C_{i_1}, L_{i_1}) \cap \ldots \cap \Delta_{M_{c'}}^{\mathrm{red}}(C_{i_{s-1}}, L_{i_{s-1}})$ has codimension s in $M_{c'}$. Thus we can represent the intersection of the Poincaré dual of $\nu_{\mathscr{E}}([C_0])$ with $[A_j]$ by an algebraic cycle $\sum_k m_{jk}[A_{jk}]$ where each A_{jk} is an irreducible component of $\Delta_{\mathscr{E}}^{\mathrm{red}}(C_{r+1}, L_{r+1}) \cap \Delta_{\mathscr{E}}^{\mathrm{red}}(C_{i_1}, L_{i_1}) \cap \ldots \cap \Delta_{\mathscr{E}}^{\mathrm{red}}(C_{i_p}, L_{i_p})$. This finishes the proof of the claim.

Now let the Poincaré dual of $\nu_{\mathscr{E}}([C_0])^p$ be represented by $A = \sum_j m_j[A_j]$ as above. Let $\varphi_j\colon \tilde{A}_j \to A_j$ be a desingularization of A_j. Let $\mathscr{E}_j$ be the pull-back of $\mathscr{E}$ to $S \times \tilde{A}_j$. We will show that $\int_{[\tilde{A}_i]} \underbrace{\nu_{\mathscr{E}_j}(\Gamma + \bar{\Gamma}) \wedge \ldots \wedge \nu_{\mathscr{E}_j}(\Gamma + \bar{\Gamma})}_{d(c)-p} = 0$. By Lemma 3.3 this will conclude the proof. If Lemma 6.7 applies to $\mathscr{E}_j$ there is nothing to prove. If Lemma 6.7 doesn't apply we consider the construction of Remark 3.5 for $\mathscr{E}_j$. Hence we have the open dense $U \subset \tilde{A}_j$, the morphism $f\colon U \to \times_i \mathrm{Hilb}^{d_i}(S) \times Y$ and a holomorphic two-form λ on $\times_i \mathrm{Hilb}^{d_i}(S) \times Y$ such that $f^*\lambda$ extends to $\tilde{A}_i$ and represents $\nu_{\mathscr{E}_j}(\Gamma)$. By hypothesis $Y = M_{c'}$ with $k_0 \leqq c' < c$. There are p distinct curves $C_{i_1}, \ldots, C_{i_p} \in |C_0|$ such that $A_j \subset \Delta_{\mathscr{E}}^{\mathrm{red}}(C_{i_s}, L_{i_s})$ for all $1 \leqq s \leqq p$. Thus for each s, if x is the generic point of $\tilde{A}_j$, then either $E_x|_{C_{i_s}}$ is not locally free or it is locally free and $[E_x^{**}] \in \Delta_{M_{c'}}(C_{i_s}, L_{i_s})$. Since the divisors $\Delta_{M_{c'}}^{\mathrm{red}}(C_{i_s}, L_{i_s})$ intersect properly and the curves $C_{i_1}, \ldots, C_{i_p}$ are distinct we have $\dim f(U) \leqq \dim(\times_i \mathrm{Hilb}^{d_i}(S) \times Y) - p < d(c) - p$. Hence $\int_{[A_j]} \underbrace{f^*(\lambda + \bar{\lambda}) \wedge \ldots \wedge f^*(\lambda + \bar{\lambda})}_{d(c)-p} = 0$, and we are done.

References

1. Arbarello, E., Cornalba, M., Griffiths, P.A., Harris, J.: Geometry of algebraic curves, vol. 1. (Grundlehren Math. Wiss., Bd. 267) Berlin Heidelberg New York: Springer 1985
2. Atiyah, M.F.: Complex analytic connections. Trans. Amer. Math. Soc. **85**, 181–207 (1957)
3. Beauville, A.: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differ. Geom. **18**, 755–782 (1983)
4. Donaldson, S.K.: Infinite determinants, stable bundles and curvature. Duke Math. J. **54** (No. 1), 231–247 (1987)
5. Donaldson, S.K.: Polynomial invariants for smooth four-manifolds. Topology **29** (No. 3), 257–315 (1990)
6. Drezet, J.M., Narasimhan, M.S.: Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. Invent. Math. **97**, 53–94 (1989)
7. Friedman, R.: Rank two vector bundles over regular elliptic surfaces. Invent. Math. **96**, 283–332 (1989)

8. Friedman, R.: Vector bundles over surfaces. (to be published)
9. Friedman, R., Moishezon, B., Morgan, J.: On the C^∞ invariance of the canonical class of certain algebraic surfaces. Bull. Am. Math. Soc. **17**, 283–286 (1987)
10. Friedman, R., Morgan, J.: Smooth four-manifolds and complex surfaces. (to be published)
11. Gieseker, D.: On the moduli of vector bundles on an algebraic surface. Ann. Math. **106**, 45–60 (1977)
12. Gieseker, D.: A construction of stable bundles on an algebraic surface. J. Differ. Geom. **27**, 137–154 (1988)
13. Hoppe, H.J., Spindler, H.: Modulräume stabiler 2-Bündel auf Regelflächen. Math. Ann. **249**, 127–140 (1980)
14. Kang Zuo: Regular two-forms on the moduli space of rank two stable bundles on an algebraic surface. Thesis, Max Planck Institut für Mathematik 1990
15. Maruyama, M.: Moduli of stable sheaves II. J. Math. Kyoto Univ. **18**, 557–614 (1978)
16. Morgan, J., O'Grady, K.: The smooth classification of elliptic surfaces with geometric genus equal to one. (to appear)
17. Mukai, S.: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. Math. **77**, 101–116 (1984)
18. Mukai, S.: On the moduli spaces of bundles on K3 surfaces, I, Vector bundles on algebraic varieties. Oxford: Oxford University Press 1987
19. O'Grady, K.: Donaldson's polynomials for K3 surfaces. (Preprint)
20. Oxbury, W.M.: Spectral curves of vector bundle endomorphisms. (Preprint)
21. Raynaud, M.: Sections des fibrés vectoriels sur une courbe. Bull. Soc. Math. Fr. **110**, 103–125 (1982)
22. Tyurin, A.N.: Symplectic structures on the varieties of moduli of vector bundles on algebraic surfaces with $p_g > 0$. Math. USSR Izv. **33**, 139–177 (1989)
23. Tyurin, A.N.: Algebraic geometric aspects of smooth structures I. The Donaldson polynomials. Russ. Math. Surv. **44**, 113–178 (1990)