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Compact Tori Associated to Hyperkähler Manifolds of Kummer Type

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For X a hyperkähler manifold of Kummer type, let $J^3(X)$ be the intermediate Jacobian associated to $H^3(X)$. We prove that $H^2(X)$ can be embedded into $H^2(J^3(X))$. We show that there exists a natural smooth quadric $Q(X)$ in the projectivization of $H^3(X)$, such that Gauss–Manin parallel transport identifies the set of projectivizations of $H^{2,1}(Y)$, for Y a deformation of X, with an open subset of a linear section of $Q^+(X)$, one component of the variety of maximal linear subspaces of $Q(X)$. We give a new proof of a result of Mongardi restricting the action of monodromy on $H^2(X)$. Lastly, we show that if X is projective, then $J^3(X)$ is an abelian fourfold of Weil type.

1 Introduction

1.1 Background and motivation

Let X be a hyperkähler manifold, that is (for us) a simply connected compact Kähler manifold carrying a holomorphic symplectic form whose cohomology class spans $H^{2,0}(X)$. The Kuga–Satake construction [5, 12] associates to X a compact complex torus $\mathrm{KS}(X)$ and an inclusion of Hodge structures $H^2(X) \subset H^1(\mathrm{KS}(X)) \otimes H^1(\mathrm{KS}(X)^\vee)$. The definition of $\mathrm{KS}(X)$ is transcendental: one constructs a weight 1 H.S. out of the weight 2 H.S. on $H^2(X)$. If X is projective with ample line bundle L, the Kuga–Satake construction

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applied to the primitive cohomology $H^2(X)_{pr}$ produces an abelian variety $\mathrm{KS}(X,L)$ and an injective homomorphism of H.S.'s

$$H^2(X)_{pr} \subset H^1(\mathrm{KS}(X,L)) \otimes H^1(\mathrm{KS}(X,L)). \tag{1.1.1}$$

One might wonder whether it is possible to relate the geometry of X and that of $\mathrm{KS}(X)$ or of $\mathrm{KS}(X,L)$. A famous instance of such a relation is provided by Deligne's proof of the Weil conjectures for (projective) $K3$ surfaces starting from the validity of the Weil conjectures for abelian varieties [5]. In this respect we notice that if X is projective, the Hodge conjecture predicts the existence of a Kuga–Satake algebraic cycle on $X \times \mathrm{KS}(X,L) \times \mathrm{KS}(X,L)$ realizing the homomorphism of H.S.'s in (1.1.1).

There are very few families of hyperkähler manifolds for which one has a geometric description of the corresponding Kuga–Satake varieties and a proof of existence of a Kuga–Satake algebraic cycle: Kummer surfaces [18] and $K3$ surfaces obtained as minimal desingularization of the double cover of a plane ramified over 6 lines [21].

The present paper grew out of the desire to understand the Kuga–Satake torus associated to hyperkähler manifolds of Kummer type, that is deformations of the $2n$-dimensional generalized Kummer manifold associated to an abelian surface (for $n \geqslant 2$).

Among known examples of hyperkähler manifolds, those of Kummer type are distinguished by the fact that they have non-zero odd cohomology. Let X be such a manifold. Then $b_3(X) = 8$, and hence there is an associated four dimensional intermediate Jacobian $J^3(X)$. Most of our paper is actually concerned with $J^3(X)$. Our starting point is the proof that there is an analogue of the key cohomological property of the Kuga–Satake torus (see (1.1.1)) valid with $J^3(X)$ replacing the Kuga–Satake torus. From this, it follows that if X is projective with polarization L, then $\mathrm{KS}(X,L)$ is isogenous to $J^3(X)^4$. Thus, $J^3(X)$ is a smaller dimensional version of the Kuga–Satake torus. Moreover, it is easier to relate geometrically X to $J^3(X)$ than it is to relate it to $\mathrm{KS}(X)$ (or $\mathrm{KS}(X,L)$), for example via the Abel–Jacobi map.

We will give an explicit recipe that produces the weight 1 H.S. on $J^3(X)$ in terms of the weight 2 H.S. on $H^2(X)$.

One fact that we discovered is that if X is projective, then $J^3(X)$ is an abelian fourfold of Weil type. More precisely, as (X,L) varies in a complete family of polarized hyperkählers of Kummer type with fixed discrete invariants, the corresponding polarized intermediate Jacobians $J^3(X)$ sweep out a complete family of polarized abelian fourfolds of Weil type with fixed discrete invariants. Notice that the number of moduli for both families is equal to 4. This result suggests that we will be able to describe explicitly locally complete families of projective hyperkählers of Kummer type starting from the locally complete families of abelian fourfolds of Weil type, which are known [22]. In this respect, we notice that several locally complete families of projective hyperkählers have been explicitly described, but the varieties in those families are all of $K3^{[n]}$ type (deformations of the Hilbert scheme of length n subschemes of a $K3$ surface).

There is a series of papers related to the present work. The 1st one is [27]. Following the proof of Theorem 9.2 of that paper, one shows that the Kuga–Satake $\mathrm{KS}(X,L)$ of a polarized HK of Kummer type (X,L) is the 4th power of an abelian fourfold of Weil type. Since $\mathrm{KS}(X,L)$ is isogenous to $J^3(X)^4$, it follows that $J^3(X)$ is of Weil type. However, we would like to stress that we have precise results on the integral Hodge structure on $J^3(X)$, not only up to isogeny. Another paper related to this work is [13]. Lastly, the recent preprint [15] is strictly related to our work.

1.2 Main results

Let X be a hyperkähler manifold of dimension at least 4, deformation equivalent to a generalized Kummer variety (following established terminology, we say that X is of Kummer type). Then $b^3(X) = 8$, see [9], and of course $H^{3,0}(X) = 0$. Thus,

$$J^3(X) = H^3(X)/\big(H^{2,1}(X) + H^3(X;\mathbb{Z})\big) \tag{1.2.1}$$

is a four dimensional compact complex torus. If X is projective and L is an ample line bundle on X, then $J^3(X)$ is an abelian fourfold (all of $H^3(X)$ is primitive because $H^1(X) = 0$), and we let Θ_L be the polarization defined by L.

Recall that, given an HK manifold X, there is a class $q_X^\vee \in H^{2,2}_{\mathbb{Q}}(X)$ that corresponds to the Beauville–Bogomolov–Fujiki (BBF) quadratic form of X (see Subsection 2.2 for details). Now assume that X is of Kummer type, of dimension $2n$. Then $\overline{q}_X := 2(n+1)q_X^\vee$ is an integral class (see Definition 2.4). Let

$$\phi\colon \bigwedge^2 H^3(X) \longrightarrow H^2(X)^\vee \tag{1.2.2}$$

be the composition of the map

$$\begin{array}{ccc} \bigwedge^2 H^3(X) & \longrightarrow & H^{4n-2}(X) \\ \gamma\wedge\gamma' & \mapsto & \gamma\smile\gamma'\smile\overline{q}_X^{\,n-2} \end{array}$$

and the map $H^{4n-2}(X) \to H^2(X)^\vee$ defined by cup product followed by integration.

Theorem 1.1. Let X be an HK manifold of Kummer type of dimension $2n$.

(1) The map ϕ is surjective, and hence its transpose defines an inclusion of integral Hodge structures

$$H^2(X) \subset \bigwedge^2 H^1(J^3(X)). \tag{1.2.3}$$

(2) The set

$$\mathbf{\Omega}(X) := \left\{[\gamma] \in \mathbb{P}(H^3(X)) \mid \phi(\gamma \wedge H^3(X)) \neq H^2(X)^\vee\right\} \tag{1.2.4}$$

is a smooth quadric hypersurface in $\mathbb{P}(H^3(X))$.

(3) The projectivization of $H^{2,1}(X)$ is a maximal linear subspace of $\mathbf{\Omega}(X)$.

If X is an HK manifold of Kummer type, let $\mathbf{\Omega}^+(X)$ be the irreducible component of the variety parametrizing maximal dimensional linear subspaces of $\mathbf{\Omega}(X)$ containing $\mathbb{P}(H^{2,1}(X))$ (this definition makes sense by Theorem 1.1). We recall that

$$\epsilon\colon \mathbf{\Omega}^+(X) \hookrightarrow \mathbb{P}(S^+(X)), \tag{1.2.5}$$

where $S^+(X)$ is one of the two spinor representations of $O(\mathbf{\Omega}(X))$. Recall also that $S^+(X)$ is 8 dimensional. Since $H^3(X)$ has an integral structure, so does $S^+(X)$. There is a unimodular integral quadratic form $\mathbf{q}_X^+$ on $S^+(X)$ (unique up to multiplication by ± 1) such that $\mathbf{\Omega}^+(X)$ is the set of zeroes of $\mathbf{q}_X^+$. Moreover, if $\pi\colon \mathscr{X} \to B$ is a family of HK manifolds of Kummer type, the flat connection on $R^3\pi_*\mathbb{Z}$ induces a flat connection on the fibration $S^+(\pi) \to B$ with fiber $S^+(\pi^{-1}(b))$ over b. Next, we make following.

Key observation 1.2. Let ϕ be the map in (1.2.2). Then $\phi\left(\bigwedge^2 H^{2,1}(X)\right)$ is equal the one-dimensional subspace $\mathrm{Ann} F^1 H^2(X)$.

In fact, $\phi\left(\bigwedge^2 H^{2,1}(X)\right)$ is contained in $\mathrm{Ann} F^1 H^2(X)$ because ϕ is a morphism of Hodge structures, and equality follows from surjectivity of ϕ. Notice that Item (3) of Theorem 1.1 follows from the Key observation 1.2.

The result below is motivated by the Key observation 1.2.

Theorem 1.3. Let X be an HK manifold of Kummer type of dimension $2n$, and let $S^+(X)$ be the spinor representation of $O(\mathbf{\Omega}(X))$ such that we have the embedding in (1.2.5). There exists a monodromy invariant codimension 1 subspace $T^+(X) \subset S^+(X)$ defined over $\mathbb{Z}$ such that the following hold:

(1) Given a 4 dimensional vector subspace $\Gamma \subset H^3(X)$, the subspace $\phi\big(\bigwedge^2 \Gamma\big)$ has dimension 1 if and only if $\mathbb{P}(\Gamma) = \epsilon([\sigma])$ for a point $[\sigma] \in \mathbb{P}(T^+(X)) \cap \mathbf{\Omega}^+(X)$, where ϵ is the embedding in (1.2.5). If this is the case, then $\phi\big(\bigwedge^2 \Gamma\big) = [\sigma]$ (this makes sense because of the description of $S^+(X)$ in Subsection 3.6).

(2) There exist an isomorphism $i\colon H^2(X)^\vee \xrightarrow{\sim} T^+(X)$ defined over $\mathbb{Q}$, invariant up to sign under monodromy, and a choice of "sign" for $\mathbf{q}_X^+$, such that the pull-back via i of $\mathbf{q}_X^+$ is equal to the dual of the BBF quadratic form.

Item (1) of Theorem 1.3 amounts to an explicit description of the weight 1 Hodge structure on $H^1(J^3(X))$ in terms of the weight 2 Hodge structure on $H^2(X)$.

The result below was first proved by Mongardi by other methods. We will show that it is a simple consequence of Theorem 1.3.

Corollary 1.4 (Mongardi [17]). Let X be an HK of Kummer type. Let $\rho \in O(H^2X;\mathbb{Z}), q_X)$ be a monodormy operator. Then either ρ acts trivially on the discriminant group $H^2(X;\mathbb{Z})^\vee/H^2(X;\mathbb{Z})$ (here $H^2(X;\mathbb{Z})$ is embedded into $H^2(X;\mathbb{Z})^\vee$ by the BBF quadratic form) and it has determinant 1 or it acts as multiplication by -1 on the discriminant group and it has determinant -1.

Below is our last main result.

Theorem 1.5. Let X be a hyperkähler variety of Kummer type, of dimension $2n$, and let L be an ample line bundle on X. Then $(J^3(X), \Theta_L)$ is of Weil type, with an inclusion

$$\mathbb{Q}\sqrt{-2(n+1)q_X(L)} \subset \mathrm{End}\big(J^3(X), \Theta_L\big)_{\mathbb{Q}},$$

where $q_X(L)$ is the value of the BBF quadratic form on $c_1(L)$. By varying (X, L), one gets a complete (up to isogeny) family of four dimensional abelian varieties of Weil type with associated field $\mathbb{Q}[\sqrt{-2(n+1)q_X(L)}]$ and trivial determinant (i.e., the discriminant of the associated hermitian form is the norm of a non-zero element of the field $\mathbb{Q}[\sqrt{-2(n+1)q_X(L)}]$, see Subsection 5.1). Moreover, the Kuga–Satake variety $\mathrm{KS}(X, L)$ is isogenous to $J^3(X)^4$.

Remark 1.6. Underlying Theorem 1.5 is a (classical) isomorphism between the period spaces for polarized HK's of Kummer type and polarized abelian fourfolds of Weil type.

In particular, we have an infinitesimal Torelli Theorem for polarized HK's of Kummer type in terms of the Hodge structure on H^3. A careful study of the monodromy on the integral H^3 of HK's of Kummer type should produce a Global Torelli Theorem in terms of the Hodge structure on H^3.

1.3 Organization of the paper

Most of Section 2 is devoted to the proof of results on the cohomology of HK's of Kummer type. After recalling the definition of generalized Kummers and establishing basic notation, we compute the constants that enter into the formula for certain integrals on an HK of Kummer type (see Proposition 2.3). In Subsection 2.3 we describe explicitly the integral 3rd cohomolgy group of a generalized Kummer. In dimension 4 this was done by Kapfer and Menet [11]. We extend their result to arbitrary dimension by adapting arguments of Totaro [24]. In Subsection 2.4 we show that, by invariance under the monodromy group of compact complex tori, the map ϕ in (1.2.2) for $2n$-dimensional HK's of Kummer type has a "shape" that depends on an apriori unknown $\vartheta(\overline{q}^{n-2}) \in \mathbb{Z}^3$. In Subsections 2.5, 2.6 and 2.7 we compute the 1st two entries of $\vartheta(\overline{q}^{n-2})$ (the 3rd entry will be determined up to sign in Subsection 3.4). Most of the effort goes in a painful computation of the cup product of certain cohomology classes on a generalized Kummer. In order to do this we rely on the explicit description of the cohomology ring of Hilbert schemes of smooth projective surfaces with trivial canonical bundle given by Lehn and Sorger [14]. The last subsection of Section 2 contains the proof of Theorem 1.1.

In Section 3 we prove Theorem 1.3 and Corollary 1.4. Actually we discuss an "abstract" map, which has the same shape as ϕ, depending on a choice of $\vartheta \in \mathbb{Z}^3$ with no vanishing entry. In such a set-up, we have a way of explicitly associating to a weight-2 H.S. of $K3$ type a weight-1 H.S. If the weight-2 H.S. is polarized, then the weight-1 H.S. is also polarized.

In the short Section 4 we compute the elementary divisors of the natural polarization of $J^3(X)$ for a polarized HK fourfold X.

Section 5 is devoted to the proof of Theorem 1.5. Actually we prove, more generally, that the polarized weight-1 H.S.'s constructed in Section 3 (depending on a $\vartheta \in \mathbb{Z}^3$ with no vanishing entry) are of Weil type.

1.4 Conventions

We work over $\mathbb{C}$: projective varieties will be complex projective varieties.

Throughout the present paper, A is an abelian surface.

Notation: in dealing with cohomology, we omit to mention the ring of coefficients when we consider complex coefficients.

Let Λ be a lattice. The divisibility of a non-zero $v \in \Lambda$ is the positive generator of (v, Λ); we denote it by $\mathrm{div}(v)$.

Let $n \in \mathbb{N}_+$. The double factorial of n is equal to

$$n!! := n \cdot (n-2) \cdot \ldots \cdot \left(n - 2\left\lfloor \frac{n-1}{2} \right\rfloor\right). \tag{1.4.1}$$

It is convenient to set $0!! := 1$ and $(-1)!! := 1$.

2 Generalized Kummers and their cohomology

2.1 Hilbert schemes parametrizing subschemes of finite length

Let S be a smooth projective surface. Let $S^{[n]}$ be the Hilbert scheme parametrizing length-n subschemes of S, and let $S^{(n)}$ be the symmetric n-th power of S. Let $[Z] \in S^{[n]}$. We let $|Z|$ be the cycle be associated to $[Z]$, that is the image of the Hilbert–Chow map $\widetilde{\mathfrak{h}}_n \colon S^{[n]} \to S^{(n)}$. Let $\widetilde{\Delta}_n(S) \subset S^{[n]}$ be the prime divisor parametrizing non-reduced schemes. The divisor class of $\widetilde{\Delta}_n(S)$ is divisible by 2. We let $\widetilde{\xi}_n(S) \in H^2(S^{[n]}; \mathbb{Z})/\mathrm{Tors}$ be characterized by

$$2\widetilde{\xi}_n(S) = c_1\left(\mathscr{O}_{S^{[n]}}(\widetilde{\Delta}_n(S))\right). \tag{2.1.1}$$

Let $\pi \colon S^n \to S^{(n)}$ be the quotient map and $p_i \colon S^n \to S$ be the i-th projection. Given $\alpha \in H^m(S; R)$, let $\alpha^{(n)} \in H^m(S^{(n)}; R)$ be characterized by the formula

$$\pi^* \alpha^{(n)} = \sum_{i=1}^{n} p_i^* \alpha. \tag{2.1.2}$$

(Here R is a commutative ring.) Let

$$\begin{array}{ccc} H^m(S; R) & \xrightarrow{\widetilde{\mu}_m} & H^m\left(S^{[n]}; R\right) \\ \alpha & \mapsto & \widetilde{\mathfrak{h}}_n^* \alpha^{(n)}. \end{array} \tag{2.1.3}$$

For $n \geqslant 2$, let

$$\Gamma_n(S) := \left\{(W, Z) \in S^{[2]} \times S^{[n]} \mid W \in \widetilde{\Delta}_2 \quad W \subset Z\right\}. \tag{2.1.4}$$

Then $\Gamma_n(S)$ is irreducible of dimension $2n-1$. Let $p\colon \Gamma_n(S)\to S$ be the map sending (W,Z) to the support of W, and let $q\colon \Gamma_n(S)\to S^{[n]}$ be the projection. We let

$$\begin{array}{ccc} H^{m-2}(S;R) & \xrightarrow{\widetilde{\nu}_m} & H^m(S^{[n]};R) \\ \beta & \mapsto & \mathrm{PD}\big(q_*([\Gamma_n(S)]\cap p^*\beta\big), \end{array} \tag{2.1.5}$$

where PD means Poincaré dual.

2.2 Generalized Kummers

Let A be an abelian surface. Let $\sigma_r\colon A^{(r)}\to A$ be the summation map (in the group A). The n-th generalized Kummer variety is

$$K_n(A) := \big\{[Z]\in A^{[n+1]} \mid \sigma_{n+1}(|Z|)=0\big\}.$$

Beauville [1] proved that $K_n(A)$ is a hyperkähler variety of dimension $2n$. Let

$$\Delta_n(A) := \widetilde{\Delta}_{n+1}(A)\cap K_n(A), \qquad \xi_n(A) := \widetilde{\xi}_{n+1}(A)|_{K_n(A)} \tag{2.2.1}$$

and

$$\begin{array}{ccc} H^m(A;R) & \xrightarrow{\mu_m} & H^m\big(K_n(A);R\big) \\ \alpha & \mapsto & \widetilde{\mu}_m(\alpha)|_{K_n(A)}. \end{array} \qquad \begin{array}{ccc} H^{m-2}(A;R) & \xrightarrow{\nu_m} & H^m\big(K_n(A);R\big) \\ \beta & \mapsto & \widetilde{\nu}_m(\beta)|_{K_n(A)}. \end{array} \tag{2.2.2}$$

Now suppose that $n\geqslant 2$. We recall some well-known results on the cohomology of $K_n(A)$ (a reference is [3, pp. 6-12]). We have a direct sum decomposition

$$H^2(K_n(A);\mathbb{Z}) = \mu_2\big(H^2(A;\mathbb{Z})\big)\oplus \mathbb{Z}\xi_n(A). \tag{2.2.3}$$

Moreover, the map μ_2 for $R=\mathbb{C}$ is a homomorphisms of integral Hodge structures. The BBF bilinear form $(,)$ is given by

$$\big(\mu_2(\alpha)+x\xi_n,\mu_2(\beta)+y\xi_n\big) = \left(\int_A \alpha\wedge\beta\right) - 2(n+1)xy, \quad \alpha,\beta\in H^2(A),\ x,y\in\mathbb{C}, \tag{2.2.4}$$

and the normalized Fujiki constant of $K_n(A)$ equals $n+1$, that is

$$\int_{K_n(A)} \alpha^{2n} = (n+1)(2n-1)!!\,(\alpha,\alpha)^n \quad \forall\alpha\in H^2(K_n(A);\mathbb{C}). \tag{2.2.5}$$

Remark 2.1. Let W be a complex vector space, equipped with a bilinear symmetric form $(,)$. Let us say that two permutations $\sigma,\tau\in\mathscr{S}_{2r}$ are $\sim$-equivalent if we have equality of multilinear symmetric functions

$$(\alpha_{\sigma(1)},\alpha_{\sigma(2)})\cdot\ldots\cdot(\alpha_{\sigma(2r-1)},\alpha_{\sigma(2r)})=(\alpha_{\tau(1)},\alpha_{\tau(2)})\cdot\ldots\cdot(\alpha_{\tau(2r-1)},\alpha_{\tau(2r)}). \tag{2.2.6}$$

Let $\widetilde{\mathscr{S}}_{2r}$ be a set of representatives for $\sim$-equivalence classes, and let $P\colon W^{2r}\to\mathbb{C}$ be the multilinear symmetric function defined by

$$P(\alpha_1,\ldots,\alpha_{2r}):=\sum_{\sigma\in\widetilde{\mathscr{S}}_{2r}}(\alpha_{\sigma(1)},\alpha_{\sigma(2)})\cdot\ldots\cdot(\alpha_{\sigma(2r-1)},\alpha_{\sigma(2r)}). \tag{2.2.7}$$

Then P is the polarization of the polynomial $\alpha\mapsto(2r-1)!!\,(\alpha,\alpha)^r$, that is

$$P(\alpha,\ldots,\alpha)=(2r-1)!!\,(\alpha,\alpha)^r. \tag{2.2.8}$$

In particular, we have

$$\int_{K_n(A)}\alpha_1\smile\ldots\smile\alpha_{2n}=(n+1)\sum_{\sigma\in\widetilde{\mathscr{S}}_{2n}}(\alpha_{i_1},\alpha_{i_2})\cdot\ldots\cdot(\alpha_{i_{2n-1}},\alpha_{i_{2n}}). \tag{2.2.9}$$

In fact, both the left- and the right-hand side of (2.2.9) are multilinear symmetric functions $H^2(K_n(A))^{2n}\to\mathbb{C}$, and by (2.2.5) and (2.2.8) they are equal when computed on diagonal elements $(\alpha,\ldots,\alpha)$.

Now let X be a $2n$-dimensional hyperkähler manifold of Kummer type. The bilinear form $(,)$ defines an isomorphism $H^2(X)\xrightarrow{\sim}H^2(X)^\vee$. The inverse $H^2(X)^\vee\xrightarrow{\sim}H^2(X)$ defines an element in $\mathrm{Sym}^2H^2(X)$, whose image by the cup-product map $\mathrm{Sym}^2H^2(X)\to H^4(X)$ is a class in $H^{2,2}_{\mathbb{Q}}(X)$ that we denote by $q_X^\vee$, or $q^\vee$ if there is no danger of misunderstanding. An explicit expression for $q^\vee_{K_n(A)}$ is obtained as follows. Let e_1,f_1,e_2,f_2,e_3,f_3 be a standard basis of $H^2(A;\mathbb{Z})$, that is

$$\int_A e_i^2=0,\quad\int_A e_i\smile f_i=1,\quad\langle e_i,f_i\rangle\text{ is orthogonal to }\langle e_j,f_j\rangle\ \text{ if }\ i\neq j. \tag{2.2.10}$$

(Notice that each $\langle e_i, f_i\rangle$ is a hyperbolic plane.) Then

$$q^{\vee}_{K_n(A)} = 2\sum_{i=1}^{3} \mu_2(e_i) \smile \mu_2(f_i) - \frac{1}{2(n+1)}\xi_n^2. \tag{2.2.11}$$

Before proving a result on powers of $q^{\vee}_{K_n(A)}$, we need an identity whose proof was kindly provided by Ruggero Bandiera.

Lemma 2.2 (Ruggero Bandiera). Let k and $\ell \leq n$ be natural numbers. Then

$$\sum_{i=0}^{\ell} \binom{\ell}{i} \frac{(2i+2k)!!}{(2k)!!} \frac{(2n-2i-1)!!}{(2n-2\ell-1)!!} = \frac{(2n+2k+1))!!}{(2n-2\ell+2k+1)!!}. \tag{2.2.12}$$

Proof. For fixed natural numbers k, ℓ the left- and right-hand sides of (2.2.12) are polynomials in n (of degree ℓ), that we denote p^k_ℓ and q^k_ℓ, respectively. In particular, $p^k_\ell(x)$ and $q^k_\ell(x)$ makes sense for any x, not only for x an integer greater than ℓ. One proves that $p^k_\ell = q^k_\ell$ by induction on ℓ arguing as follows. First, $p^k_0 = q^k_0$ because they are both equal to the constant polynomial 1. A straightforward computation shows that

$$p^k_\ell(n+1) - p^k_\ell(n) = 2\ell p^k_{\ell-1}(n), \quad q^k_\ell(n+1) - q^k_\ell(n) = 2\ell q^k_{\ell-1}(n), \quad \ell \geqslant 1,$$

and hence by the inductive hypothesis the difference operators of p^k_ℓ and of q^k_ℓ are equal. Since

$$p^k_\ell(n)\left(\frac{2\ell-1}{2}\right) = q^k_\ell(n)\left(\frac{2\ell-1}{2}\right),$$

it follows that $p^k_\ell = q^k_\ell$. ■

Proposition 2.3. Let X be a $2n$-dimensional hyperkähler manifold of Kummer type. Then, for all $\gamma \in H^2(X)$

$$\int_{[X]} (q^{\vee})^{\ell} \smile \gamma^{2n-2\ell} = (n+1)\frac{(2n+5)!!}{(2n+5-2\ell)!!}(2n-2\ell-1)!!\, q(\gamma)^{n-\ell}.$$

Proof. By a theorem of Fujiki (see Remark 4.12 in [8], or 1.11 in [10]), there exists a rational number C^{ℓ}_n (independent of X) such that

$$\int_{[X]} (q^{\vee})^{\ell} \smile \gamma^{2n-2\ell} = C^{\ell}_n \cdot q(\gamma)^{n-\ell} \qquad \forall \gamma \in H^2(X). \tag{2.2.13}$$

In order to determine C_n^ℓ, it suffices to compute the left-hand side of (2.2.13) for one X and one $\gamma \in H^2(X)$ such that $q(\gamma) \neq 0$. We will do the computation for $X = K_n(A)$ and $\gamma = \xi_n$. Let

$$\sigma_n := \sum_{i=1}^{3} \mu_2(e_i) \smile \mu_2(f_i) \in H_{\mathbb{Z}}^{2,2}(K_n(A)).$$

Thus,

$$(q_{K_n(A)}^\vee)^\ell = \left(2\sigma_n - \frac{1}{2(n+1)}\xi_n^2\right)^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} 2^i \left(-\frac{1}{2(n+1)}\right)^{\ell-i} \sigma_n^i \smile \xi_n^{2(\ell-i)}.$$

A straightforward computation shows that

$$\int_{K_n(A)} \sigma_n^i \smile \xi_n^{2n-2i} = (n+1)\frac{1}{2} i!\,(i+2)(i+1)(-2(n+1))^{n-i}(2n-2i-1)!\,!\,.$$

With some manipulations, it follows that

$$\int_{K_n(A)} (q_{K_n(A)}^\vee)^\ell \smile \xi_n^{2n-2\ell} = (n+1)q(\xi_n)^{n-\ell} \cdot \sum_{i=0}^{\ell} \frac{\ell!}{(\ell-i)!}(i+2)(i+1)2^{i-1}(2n-2i-1)!\,!\,.$$

Thus, it remains to show that

$$\sum_{i=0}^{\ell} \frac{\ell!}{(\ell-i)!}(i+2)(i+1)2^{i-1}(2n-2i-1)!\,! = \frac{(2n+5)!\,!}{(2n+5-2\ell)!\,!}(2n-2\ell-1)!\,!\,.$$

The above equality follows at once from the case $k = 2$ of Lemma 2.2.

Definition 2.4. If X is a $2n$-dimensional hyperkähler manifold of Kummer type, let $\overline{q}_X := 2(n+1)q_X^\vee$.

The point of the above definition is that $\overline{q} \in H_{\mathbb{Z}}^{2,2}(K_n(A))$ (by (2.2.11)).

2.3 On the integral cohomology of generalized Kummers

We will prove the following two results.

Proposition 2.5 (Contained in [11] for $n = 2$.). Let $\beta \in H^1(A; \mathbb{Z})$. Then $\nu_3(\beta)$ is divisible by 2 in $H^3(K_n(A); \mathbb{Z})$.

Remark 2.6. Let $\beta \in H^1(A;\mathbb{Z})$. By Proposition 2.5, there is a well-defined $\nu_3(\beta)/2 \in H^3(K_n(A);\mathbb{Z})/\mathrm{Tors}$

Theorem 2.7 (Proposition 6.2 in [11] for $n = 2$.). The map

$$\begin{array}{ccc} H^3(A) \oplus H^1(A)(-1) & \xrightarrow{\mathsf{F}} & H^3(K_n(A))/\mathrm{Tors} \\ (\alpha,\beta) & \mapsto & \mu_3(\alpha)+\nu_3(\beta)/2 \end{array} \tag{2.3.1}$$

is an isomorphism of integral Hodge structures.

We recall that $A^{(r)}$ is naturally stratified, with strata indexed by partitions of r. The stratification of $A^{(r)}$ defines a stratification of $A^{[r]}$ via pull back by the Hilbert–Chow map. Let $\lambda = (\lambda_1,\ldots,\lambda_s)$ be a partition of r, where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_s$. The stratum $A^{[r]}_\lambda$ is equal to the set of Z such that $|Z| = \lambda_1 a_1 + \ldots + \lambda_s a_s$, where the points $a_1,\ldots,a_s \in A$ are pairwise distinct. Each stratum is irreducible, and

$$\dim A^{[r]}_\lambda = r + s. \tag{2.3.2}$$

Since $\{A^{[r]}_\lambda\}_{\lambda\in\mathscr{P}_r}$ is a stratification, the dimension formula (2.3.2) shows that

$$U_r := A^{[r]}_{(1,\ldots,1)} \sqcup A^{[r]}_{(2,1,\ldots,1)}, \quad V_r := A^{[r]}_{(1,\ldots,1)} \sqcup A^{[r]}_{(2,1,\ldots,1)} \sqcup A^{[r]}_{(3,1,\ldots,1)} \sqcup A^{[r]}_{(2,2,1,\ldots,1)} \tag{2.3.3}$$

are open dense subsets of $A^{[r]}$. Notice that $V_r \supset U_r$.

Lemma 2.8. The restriction map $H^3(A^{[r]};\mathbb{Z}) \to H^3(U_r;\mathbb{Z})$ is an isomorphism.

Proof. The complement of $V_r \subset A^{[r]}$ has codimension 3; it follows by a standard argument that the map $H^3(A^{[r]};\mathbb{Z}) \to H^3(V_r;\mathbb{Z})$ is an isomorphism. Thus, it suffices to prove that the map $H^3(V_r,\mathbb{Z}) \to H^3(U_r,\mathbb{Z})$ is an isomorphism. If $r \le 2$, then $V_r = U_r$, and there is nothing to prove. Assume that $r \geqslant 3$. We have the exact sequence

$$H^3(V_r,U_r;\mathbb{Z}) \longrightarrow H^3(V_r;\mathbb{Z}) \longrightarrow H^3(U_r;\mathbb{Z}) \longrightarrow H^4(V_r,U_r;\mathbb{Z}) \xrightarrow{\rho_r} H^4(V_r;\mathbb{Z})$$

By excision and Thom's isomorphism, $H^3(V_r,U_r;\mathbb{Z}) = 0$, and

$$H^4(V_r,U_r;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } r=3, \\ \mathbb{Z}^2 & \text{if } r \geqslant 4, \end{cases} \quad \mathrm{Im}\rho_r = \begin{cases} \langle [A^{[r]}_{(3,1,\ldots,1)}]\rangle, & \text{if } r=3, \\ \langle [A^{[r]}_{(3,1,\ldots,1)}],[A^{[r]}_{(2,2,1,\ldots,1)}]\rangle, & \text{if } r \geqslant 4, \end{cases}$$

where $[A^{[r]}_{(3,1,\ldots,1)}]$ and $[A^{[r]}_{(2,2,1,\ldots,1)}]$ are the fundamental classes of $A^{[r]}_{(3,1,\ldots,1)}$ and $A^{[r]}_{(2,2,1,\ldots,1)}$ respectively. In order to finish the proof it suffices to show that ρ_r is injective, that is that $[A^{[r]}_{(3,1,\ldots,1)}], [A^{[r]}_{(2,2,1,\ldots,1)}]$ are independent over $\mathbb{Z}$ (if $r = 3$ this is to be interpreted as stating that $[A^{[r]}_{(3,1,\ldots,1)}]$ is not a torsion class). We may assume that $A = E \times F$, where E, F are elliptic curves. Given $q \in F$, we let $i_q \colon E \hookrightarrow E \times F$ be defined by $i_q(e) := (e, q)$. Let $D \subset E^{(3)}$ be a generic very ample divisor. Then D meets the curve $\{3p \mid p \in E\}$ in a finite non-empty set. Ket $x_1, \ldots, x_{r-3}$ be pairwise distinct points of $A \setminus i_q(E)$. Let $\Sigma \subset A^{[r]}$ be defined by

$$\Sigma := \left\{i_q(Z) \sqcup \{x_1, \ldots, x_{r-3}\} \mid Z \in D\right\}.$$

If $r \geqslant 4$, let $q_1, q_2 \in F$ be distinct, and let $y_1, \ldots, y_{r-4}$ be pairwise distinct points of $A \setminus i_{q_1}(E) \setminus i_{q_2}(E)$. Let $\phi \colon E \to \mathbb{P}^1$ be a degree 2 map. We let $D_E \subset E^{(2)}$ be the g^1_2 defined by ϕ, that is $D_E := \{\phi^*(p) \mid p \in \mathbb{P}^1\}$. Let $\Omega \subset A^{[r]}$ be defined by

$$\Omega := \left\{i_{q_1}(W) \sqcup i_{q_2}(Z) \sqcup \{y_1, \ldots, y_{r-4}\} \mid W, Z \in D_E\right\}.$$

Both Σ and Ω are projective and have pure dimension 2. Thus, we may evaluate the classes $[A^{[r]}_{(2,2,1,\ldots,1)}]$ and $[A^{[r]}_{(3,1,\ldots,1)}]$ on Σ and Ω. Notice that Σ meets $A^{[r]}_{(3,1,\ldots,1)}$ in a finite non-empy set, and that Ω meets $A^{[r]}_{(2,2,\ldots,1)}$ in a finite non-empty set, and it does not meet $A^{[r]}_{(3,1,\ldots,1)}$. It follows that the 2×2 matrix describing the evaluation of the classes $[A^{[r]}_{(3,1,\ldots,1)}]$, $[A^{[r]}_{(2,2,1,\ldots,1)}]$ on Σ and Ω is non-degenerate. (If $r = 3$, this is to be interpreted as stating that the evaluation of the class $[A^{[r]}_{(3,1,\ldots,1)}]$ on Σ is non-zero). This proves that ρ_r is injective. ■

Proof of Proposition 2.5. It suffices to prove that $\widetilde{\nu}_3(\beta) \in H^3(A^{[n+1]}; \mathbb{Z})$ is divisible by 2. Let $U_{n+1} \subset A^{[n+1]}$ be the open dense subset defined in (2.3.3); by Lemma 2.8 it suffices to prove that $\widetilde{\nu}_3(\beta)|_{U_{n+1}}$ is divisible by 2 in $H^3(U_{n+1}; \mathbb{Z})$. Let $\overline{\beta} \in H^1(A; \mathbb{F}_2)$ be the reduction modulo 2 of β; thus, $\widetilde{\nu}_3(\overline{\beta})$ is the reduction modulo 2 of $\widetilde{\nu}_3(\beta)$. We must show that

$$\widetilde{\nu}_3(\overline{\beta})|_{U_{n+1}} = 0. \tag{2.3.4}$$

A piece of the long exact sequence of cohomology with $\mathbb{F}_2$ coefficients for the couple $(U_{n+1}, A^{[n+1]}_{(1,1,\ldots,1)})$ reads

$$H^2\big(A^{[n+1]}_{(1,1,\ldots,1)}; \mathbb{F}_2\big) \xrightarrow{\partial} H^3\big(U_{n+1}, A^{[n+1]}_{(1,1,\ldots,1)}; \mathbb{F}_2\big) \xrightarrow{\pi} H^3\big(U_{n+1}; \mathbb{F}_2\big) \longrightarrow H^3\big(A^{[n+1]}_{(1,1,\ldots,1)}; \mathbb{F}_2\big) \tag{2.3.5}$$

Thom's isomorphism gives an identification

$$H^3\big(U_{n+1}, A^{[n+1]}_{(1,1,\ldots,1)}; \mathbb{F}_2\big) \cong H^1\big(A^{[n+1]}_{(2,1,\ldots,1)}; \mathbb{F}_2\big). \tag{2.3.6}$$

Let $\tau\colon A^{[n+1]}_{(2,1,\ldots,1)} \to A$ be the composition of $A^{[n+1]}_{(2,1,\ldots,1)} \to A \times A^{(n-1)}$ (the restriction of Hilbert–Chow) and the projection $A \times A^{(n-1)} \to A$. Then

$$\pi\big(\tau^*(\overline{\beta})\big) = \widetilde{\nu}_3(\overline{\beta})|_{U_{n+1}}. \tag{2.3.7}$$

(The above equation makes sense by (2.3.6)). By Lemma 3.1 in [24] (Totaro's Lemma is stated for $n = 1$, but the same proof gives the statement in general), $\tau^*(\overline{\beta}) \in \operatorname{Im}(\partial)$, and hence (2.3.4) holds. ■

Proof of Theorem 2.7. The map in (2.3.1) is a morphism of Hodge structures, integral by Proposition 2.5, hence we are left with the task of proving that it defines an isomorphism between $H^3(A;\mathbb{Z}) \oplus H^1(A;\mathbb{Z})$ and $H^3(K_n(A);\mathbb{Z})/\mathrm{Tors}$. We proceed as in the proof of Proposition 6.2 in [11].

Let $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ be an oriented basis of $H^1(A;\mathbb{Z})$, that is such that $\eta_1 \smile \ldots \smile \eta_4$ is the orientation class. Let $\{\eta_1^\vee, \ldots, \eta_4^\vee\}$ be the dual basis. The isomorphism

$$\begin{array}{ccc} H^3(A) & \xrightarrow{\sim} & H^1(A)^\vee \\ \alpha & \mapsto & \left(\beta \mapsto \int_A \alpha \smile \beta\right) \end{array} \tag{2.3.8}$$

allows us to view each $\eta_i^\vee$ as an element of $H^3(A;\mathbb{Z})$. Let $\Sigma_1, \ldots, \Sigma_4 \subset A$ be generic smooth oriented 1-manifolds representing the Poincaré duals of $\eta_1^\vee, \ldots, \eta_4^\vee$, and let $\Omega_1, \ldots, \Omega_4 \subset A$ be generic smooth oriented 2-manifolds representing the Poincaré duals of $\eta_4 \smile \eta_1, \eta_1 \smile \eta_2, \eta_2 \smile \eta_3, \eta_3 \smile \eta_4$. Choose generic distinct points $x_1, \ldots, x_{n-2}, y_1, \ldots, y_{n-2} \in A$. Let $\Gamma_1, \ldots, \Gamma_4, \Theta_1, \ldots, \Theta_4 \subset K_n(A)$ be the smooth oriented 3 manifolds

$$\begin{aligned} \Gamma_i &:= \big\{(Z_0 \sqcup \{x_1, \ldots, x_{n-2}\}) \in K_n(A) \mid Z_0 \cap \Sigma_i \neq \emptyset,\ Z_0 \cap \Omega_i \neq \emptyset\big\}, \\ \Theta_j &:= \big\{(Z_0 \sqcup \{y_1, \ldots, y_{n-2}\}) \in K_n(A) \mid |Z_0| = 2p + q,\ \ p \in \Sigma_j\big\}. \end{aligned}$$

A straightforward computation shows that the 8×8 matrix whose entries are the evaluations of the classes $\mu_3(\eta_1^\vee), \ldots, \mu_3(\eta_4^\vee), \nu_3(\eta_1/2), \ldots, \nu_3(\eta_4/2)$ on the 3-homology classes represented by $\Sigma_1, \ldots, \Sigma_4, \Theta_1, \ldots, \Theta_4$ is a matrix $\left(\begin{smallmatrix} C & * \\ 0_{4,4} & D \end{smallmatrix}\right)$, where C, D are diagonal matrices with entries ± 1 on the diagonals. This proves that the image of $H^3(A;\mathbb{Z}) \oplus H^1(A;\mathbb{Z})$ under the map in (2.3.1) is a rank 8 saturated subgroup of $H^3(K_n(A);\mathbb{Z})/\mathrm{Tors}$. By Göttsche [9] the rank of the latter is 8, and hence Theorem 2.7 follows. ■

2.4 Structure of ϕ for X a generalized Kummer

Let X be an HK of Kummer type, of dimension $2n$. Let $\mathfrak{U}_X \in H^{2n-4,2n-4}_{\mathbb{Z}}(X)$ be an integral Hodge class which remains of Hodge type for all deformations of X. Thus, $\mathfrak{U}_X$ might be $\overline{q}^{n-2}$, where $\overline{q}$ is as in Definition 2.4, or a weight $4n-8$ poynomial in the Chern classes of X. We let

$$\phi(\mathfrak{U}_X)\colon \bigwedge^2 H^3(X) \longrightarrow H^2(X)^\vee \tag{2.4.1}$$

be the composition of the map

$$\begin{array}{ccc} \bigwedge^2 H^3(X) & \longrightarrow & H^{4n-2}(X) \\ \gamma \wedge \gamma' & \mapsto & \gamma \smile \gamma' \smile \mathfrak{U}_X \end{array}$$

and the map $H^{4n-2}(X) \to H^2(X)^\vee$ defined by cup product. Then $\phi(\mathfrak{U}_X)$ is a morphism of Hodge structures, flat for the Gauss–Manin connnection.

Now let A be an abelian surface, and let $\mathfrak{U} = \mathfrak{U}_{K_n(A)}$. By Proposition 2.5 have the isomorphism $\mathsf{F}\colon H^3(A) \oplus H^1(A) \xrightarrow{\sim} H^3(K_n(A))$. We let

$$\begin{array}{ccc} \bigwedge^2 (H^3(A) \oplus H^1(A)) & \xrightarrow{\Phi(\mathfrak{U})} & H^2(K_n(A))^\vee \\ (\alpha, \beta) \wedge (\alpha', \beta') & \mapsto & \phi(\mathfrak{U})(\mathsf{F}(\alpha,\beta) \wedge \mathsf{F}(\alpha',\beta')) \end{array}$$

We will describe the general structure of $\Phi(\mathfrak{U})$.

Notation 2.9. Let $H^2(K_n(A))^\vee = H^2(A)^\vee \oplus \mathbb{C}\xi_n^\vee$ be the direct sum decomposition dual to (2.2.3) (tensored with $\mathbb{C}$). (Note: $\xi_n^\vee$ takes the value 1 on ξ_n.)

The codomain of $\Phi(\mathfrak{U})$ is identified with $H^2(A)^\vee \oplus \mathbb{C}\xi_n^\vee$. On the other hand $H^2(A)$ is naturally identified with $\bigwedge^2 H^1(A)$, hence $H^2(A)^\vee$ is naturally identified with $\bigwedge^2 H^1(A)^\vee$. Let λ be the inverse of the isomorphism in (2.3.8). We have the isomorphism $\bigwedge^2 \lambda\colon \bigwedge^2 H^1(A)^\vee \xrightarrow{\sim} \bigwedge^2 H^3(A)$, hence we may write

$$\Phi(\mathfrak{U})\colon \bigwedge^2 \left(H^3(A) \oplus H^1(A)\right) \longrightarrow \bigwedge^2 H^3(A) \oplus \mathbb{C}\xi_n^\vee. \tag{2.4.2}$$

Definition 2.10. Let $\iota\colon \bigwedge^2 H^1(A) \xrightarrow{\sim} \bigwedge^2 H^3(A)$ be the composition

$$\bigwedge^2 H^1(A) \xrightarrow{\sim} \bigwedge^2 H^1(A)^\vee \xrightarrow{\bigwedge^2 \lambda} \bigwedge^2 H^3(A), \tag{2.4.3}$$

where the 1st map is defined by wedge-product $\bigwedge^2 H^1(A) \times \bigwedge^2 H^1(A) \longrightarrow \mathbb{C}$.

Proposition 2.11. There exists $\vartheta(\mathfrak{U}) = (\vartheta_1(\mathfrak{U}), \vartheta_2(\mathfrak{U}), \vartheta_3(\mathfrak{U})) \in \mathbb{Z}^3$ such that

$$\Phi(\mathfrak{U})((\alpha,\beta)\wedge(\alpha',\beta')) = \vartheta_1(\mathfrak{U})\alpha\wedge\alpha' + \vartheta_2(\mathfrak{U})\iota(\beta\wedge\beta') + \vartheta_3(\mathfrak{U})(\langle\alpha,\beta'\rangle - \langle\alpha',\beta\rangle))\xi_n^\vee \tag{2.4.4}$$

for all $(\alpha,\beta), (\alpha',\beta') \in H^3(A)\oplus H^1(A)$, where $\langle\alpha',\beta\rangle$, $\langle\beta',\alpha\rangle$ make sense by (2.3.8).

Proof. Let $\Phi_i(\mathfrak{U})$ be $\Phi(\mathfrak{U})$ restricted to the i-th summand of the decomposition

$$\bigwedge^2 (H^3(A)\oplus H^1(A)) = \bigwedge^2 H^3(A) \oplus \bigwedge^2 H^1(A) \oplus H^3(A)\otimes H^1(A).$$

Each $\Phi_i(\mathfrak{U})$ is equivariant for the natural action of the monodromy group of 2D compact complex tori on domain and codomain. This makes sense because the generalized Kummer $K_n(T)$ is well defined for an arbitrary 2D compact complex torus. Since the images of the monodromy group in $H^1(A;\mathbb{Z})$ and $H^3(A;\mathbb{Z})$ are the full integral special linear groups, each of the maps $\Phi_i(\mathfrak{U})$ is equivariant for the natural actions of the groups $\mathrm{SL}(H^1(A))$ and $\mathrm{SL}(H^3(A))$. It follows that there exist $\vartheta_1(\mathfrak{U}), \vartheta_2(\mathfrak{U}), \vartheta_3(\mathfrak{U}) \in \mathbb{C}$ such that $\Phi_1(\mathfrak{U}) = (\vartheta_1(\mathfrak{U})\mathrm{Id}, 0)$, $\Phi_2(\mathfrak{U}) = (\vartheta_2(\mathfrak{U})\iota, 0)$, and $\Phi_3(\mathfrak{U})((\alpha,0)\wedge(0,\beta')) = \vartheta_3(\mathfrak{U})\langle\alpha,\beta'\rangle\xi_n^\vee$. Since $\Phi(\mathfrak{U})$ is integral, one gets that each $\vartheta_i(\mathfrak{U})$ is an integer. ■

Definition 2.12. Let X be an HK manifold of Kummer type of dimension $2n$. Deforming X to $K_n(A)$, we may set unambiguously $\vartheta(\overline{q}_X^{n-2}) := \vartheta(\overline{q}_{K_n(A)}^{n-2})$, where $\vartheta(\overline{q}_{K_n(A)}^{n-2})$ is the triple of integers defined in Proposition 2.11.

2.5 The cohomology ring of $A^{[m]}$.

Let S be a smooth projective surface with torsion canonical class. Lehn and Sorger [14] have identified the cohomology ring of $S^{[m]}$ with a ring functorially associated to $H(S)$, the cohomology ring of S. In the present subsection we recall the construction for an abelian surface A—there is one simplification, because the Euler characteristic vanishes. Throughout this subsection we will adhere to the notation of [14] (with the exception that they consider rational cohomology). Accordingly, we shift the grading of $H(A)$ by 2:

$$\deg H^p(A) := p-2, \tag{2.5.1}$$

2.5.1 *The ring $H(A)^{[m]}$*

Let I be a finite set. One sets

$$H(A)^{\otimes I} := H(A^I). \tag{2.5.2}$$

Suppose that I has cardinality r. Let $[r] := \{1, 2, \ldots, r\}$. A choice of bijection $f\colon [r] \xrightarrow{\sim} I$ defines an isomorphism $H(A)^{\otimes r} \xrightarrow{\sim} H(A)^{\otimes I}$. We define a grading of $H(A)^{\otimes I}$ according to (2.5.1), that is

$$\deg H^{p_1}(A) \otimes \ldots \otimes H^{p_r}(A) = p_1 + \ldots + p_r - 2r. \tag{2.5.3}$$

The degree of a homogeneous element $\alpha \in H(A)^{\otimes I}$ is denoted $|\alpha|$. One defines

$$\begin{array}{ccc} \underbrace{H(A) \otimes \ldots \otimes H(A)}_{r} & \xrightarrow{T_r} & \mathbb{C} \\ \alpha_1 \otimes \ldots \otimes \alpha_r & \mapsto & \left(-\int_A \alpha_1\right) \cdot \ldots \cdot \left(-\int_A \alpha_r\right) \end{array}$$

(notice the minus signs). Given a finite set I of cardinality r, we may define $T_I\colon H(A)^{\otimes I} \to \mathbb{C}$ by choosing a bijection $[r] \xrightarrow{\sim} I$, and T_I is clearly independent of the bijection. Notice that T_I is a non-degenerate bilinear form.

Let I, J be finite sets, and let $f\colon I \to J$ be a surjection; by taking the cup-product map $H(A)^{f^{-1}(j)} \to H(A)$ for every $j \in J$ (see p. 307 of [14]), one defines a map

$$f^*\colon H(A)^{\otimes I} \to H(A)^{\otimes J}. \tag{2.5.4}$$

Let

$$f_*\colon H(A)^{\otimes J} \to H(A)^{\otimes I} \tag{2.5.5}$$

be the adjoint of f^* with respect to the non-degenerate bilinear forms T_J and T_I. In particular, let $\Delta_r\colon H(A)^{\otimes r} \to H(A)$ be the multiplication map. Then $\Delta_{r,*}\colon H(A) \to H(A)^{\otimes r}$ is the adjoint of the multiplication map:

$$T(\Delta_{r,*}(\alpha) \cdot \beta_1 \otimes \ldots \otimes \beta_r) = -\int_A \alpha \smile \beta_1 \smile \ldots \smile \beta_r. \tag{2.5.6}$$

Next, let

$$H(A)\{\mathscr{S}_m\} := \bigoplus_{\pi \in \mathscr{S}_m} H(A)^{\otimes \langle \pi \rangle \backslash [m]} \cdot \pi. \tag{2.5.7}$$

Here $\langle \pi \rangle \backslash [m]$ is the set of orbits in $[m]$ of the subgroup of $\mathscr{S}_m$ spanned by π. If $\zeta \in H(A)^{\otimes \langle \pi \rangle \backslash [m]}$ is homogeneous, the degree of $\zeta\pi$ is defined to be $|\zeta|$.

One defines a multiplication on $H(A)\{\mathscr{S}_m\}$ proceeding as follows (see Proposition 2.13 of [14]). Let $\pi,\rho\in\mathscr{S}_m$. The graph defect $g(\pi,\rho)\colon\langle\pi,\rho\rangle\backslash[m]\to\mathbb{N}$ is the function (see Lemma 2.7 in [14]) defined by

$$g(\pi,\rho)(B)=\frac{1}{2}\big(|B|+2-|\langle\pi\rangle\backslash B|-|\langle\rho\rangle\backslash B|-|\langle\pi\rho\rangle\backslash B|\big).$$

The surjections $\langle\pi\rangle\backslash[m]\longrightarrow\langle\pi,\rho\rangle\backslash[m]$ and $\langle\rho\rangle\backslash[m]\longrightarrow\langle\pi,\rho\rangle\backslash[m]$ define maps

$$f^{\pi,\langle\pi,\rho\rangle}\colon H(A)^{\otimes\langle\pi\rangle\backslash[m]}\to H(A)^{\otimes\langle\pi,\rho\rangle\backslash[m]},\quad f^{\rho,\langle\pi,\rho\rangle}\colon H(A)^{\otimes\langle\rho\rangle\backslash[m]}\to H(A)^{\otimes\langle\pi,\rho\rangle\backslash[m]}$$

(see (2.5.4)), and the surjection $\langle\pi\rho\rangle\backslash[m]\longrightarrow\langle\pi,\rho\rangle\backslash[m]$ defines

$$f_{\langle\pi,\rho\rangle,\langle\pi\rho\rangle}\colon H(A)^{\otimes\langle\pi,\rho\rangle\backslash[m]}\to H(A)^{\otimes\langle\pi\rho\rangle\backslash[m]},$$

as in (2.5.5). One defines $\mu_{\pi,\rho}\colon H(A)^{\otimes\langle\pi\rangle\backslash[m]}\otimes H(A)^{\otimes\langle\rho\rangle\backslash[m]}\to H(A)^{\otimes\langle\pi\rho\rangle\backslash[m]}$ by setting

$$\mu_{\pi,\rho}(a\otimes b):=\begin{cases}f_{\langle\pi,\rho\rangle,\langle\pi\rho\rangle}(f^{\pi,\langle\pi,\rho\rangle}(a)\cdot f^{\rho,\langle\pi,\rho\rangle}(b)) & \text{if }\ g(\pi,\rho)=0,\\ 0 & \text{otherwise.}\end{cases}$$

The multiplication on $H(A)\{\mathscr{S}_m\}$ is defined by setting

$$\zeta\pi\cdot\xi\rho:=\mu_{\pi,\rho}(\zeta,\xi)\pi\rho.$$

The group $\mathscr{S}_m$ acts on $H(A)\{\mathscr{S}_m\}$, see p. 310 of [14], and one sets

$$H(A)^{[m]}=\big(H(A)\{\mathscr{S}_m\}\big)^{\mathscr{S}_m}. \tag{2.5.8}$$

The restriction of multiplication to $H(A)^{[m]}$ is graded commutative and homogeneous of degree $2m$, see Proposition 2.13 and Proposition 2.15 of [14]. Let

$$\bigoplus_{m=0}^{\infty}H(A)^{[m]}\xrightarrow{\Gamma}\bigoplus_{m=0}^{\infty}H\big(A^{[m]}\big)=:\mathbb{H} \tag{2.5.9}$$

be the isomorphism of vector spaces defined on p. 318 of [14].

Theorem 2.13 (Lehn-Sorger [14]). The map in (2.5.9) is an isomorphism of graded commutative rings, provided we define $\deg H^p(A^{[m]}):=p-2m$.

2.5.2 *Product of certain elements of $H(A)\{\mathscr{S}_m\}$*

Let $\tau \in \mathscr{S}_m$. We define a total ordering $\preceq$ on the orbit set $\langle\tau\rangle\backslash[m]$ by setting $I \preceq J$ if $\min(I) \le \min(J)$. Thus, letting p be the cardinality of $\langle\tau\rangle\backslash[m]$, we have a preferred isomorphism

$$H(A)^{\otimes p} \xrightarrow{\sim} H(A)^{\otimes\langle\tau\rangle\backslash[m]}. \tag{2.5.10}$$

Definition 2.14. Keep notation as above, and let $\beta_1, \ldots, \beta_p \in H(A)$. We may view $\beta_1 \otimes \ldots \otimes \beta_p$ as an element of $H(A)^{\otimes\langle\tau\rangle\backslash[m]}$ because of (2.5.10). This understood, we let $\beta_1 \otimes \ldots \otimes \beta_p\tau$ be the corresponding element of $H(A)\{\mathscr{S}_m\}$.

Given $\alpha \in H(A)$ and $1 \le i \le m$, we let

$$p_i^*(\alpha) := 1 \otimes \ldots \otimes 1 \otimes \underset{i}{\alpha} \otimes 1 \otimes \ldots \otimes 1 \in H(A)^{\otimes m}. \tag{2.5.11}$$

Definition 2.15. Let $\xi \in H(A)$. For $1 \le i < j \le (m+1)$, let $\Delta_*^{ij}(\xi) \in H(A)^{\otimes m}$ be the image of $\Delta_{2,*}(\xi)$ under the homomorphism $H(A)^{\otimes 2} \to H(A)^{\otimes m}$ mapping $a \otimes b$ to $p_i^*(a) \cdot p_j^*(b)$. Similarly, for $1 \le h < k < l \le m$, let $\Delta_*^{hkl}(\xi) \in H(A)^{\otimes m}$ be the image of $\Delta_{3,*}(\xi)$ under the homomorphism $H(A)^{\otimes 3} \to H(A)^{\otimes m}$ mapping $a \otimes b \otimes c$ to $p_h^*(a) \cdot p_k^*(b) \cdot p_l^*(c)$.

Let $1 \le i < j \le m$ and $1 \le h < k \le m$. Then (cf. Example 2.17 in [14]):

$$\big(p_i^*(\beta)(ij)\big) \cdot \big(p_h^*(\beta')(hk)\big) = \begin{cases} \Delta_*^{ij}(\beta \smile \beta')\mathrm{Id} & \text{if } \{i,j\} = \{h,k\}, \\ p^*_{\min\{i,j,h,k\}}(\beta \smile \beta')(ij) \cdot (hk) & \text{if } |\{i,j\} \cap \{h,k\}| = 1, \\ z_{ijhk}(\beta,\beta')(ij) \cdot (hk) & \text{if } \{i,j\} \cap \{h,k\} = \emptyset, \end{cases} \tag{2.5.12}$$

where

$$z_{ijhk}(\beta,\beta') := \begin{cases} p_i^*\beta \cdot p_h^*\beta' & \text{if } i < h < j \text{ or } h < i < k, \\ p_i^*\beta \cdot p_{h-1}^*\beta' & \text{if } i < j < h, \\ p_{i-1}^*\beta \cdot p_h^*\beta' & \text{if } h < k < i. \end{cases} \tag{2.5.13}$$

Let $1 \le i < j < k \le m$. Then (cf. Example 2.17 in [14]):

$$\big(p_i^*(\beta)(ijk)\big) \cdot \big(p_i^*(\beta')(kji)\big) = \Delta_*^{ijk}(\beta \smile \beta')\mathrm{Id}. \tag{2.5.14}$$

Lastly, let $1 \leq i < j \leq m$ and $1 \leq h < k \leq m$. Let $\beta, \beta', \gamma, \gamma' \in H(A)$, with γ and β' homogeneous. Then

$$\left(p_i^*(\beta)(ij)\right) \cdot \left(p_h^*(\beta')(hk)\right) \cdot \left(p_i^*(\gamma)(ij)\right) \cdot \left(p_h^*(\gamma')(hk)\right) = (-1)^{|\beta'| \cdot |\gamma|} \Delta_*^{ij}(\beta \smile \gamma) \cdot \Delta_*^{hk}(\beta' \smile \gamma')\mathrm{Id}. \tag{2.5.15}$$

2.5.3 *Cohomology classes and Grojnowski-Nakajima operators*

We describe elements of $H(A)^{[m]}$ that correspond to classes in $H(A^{[m]})$ that are relevant for our computations. In order to avoid misunderstandings, we let $\widetilde{\mu}_r^{[m]} \colon H^r(A) \to H^r(A^{[m]})$ be the map that was previously denoted by $\widetilde{\mu}_r$ (we add the superscript $[m]$), and similarly we let $\widetilde{\nu}_r^{[m]} \colon H^{r-2}(A) \to H^r(A^{[m]})$ be the map that was previously denoted by $\widetilde{\nu}_r$.

Notice that $\sum_{i=1}^m p_i^*(\alpha)\mathrm{Id} \in H(A)^{[m]}$.

Proposition 2.16. Keep notation as above, and let $\alpha \in H^r(A)$. Then

$$\Gamma\left(\sum_{i=1}^m p_i^*(\alpha)\mathrm{Id}\right) = \widetilde{\mu}_r^{[m]}(\alpha). \tag{2.5.16}$$

Proof. Given $\ell \in \mathbb{N}$ and $\gamma \in H(A)$, let $\mathfrak{p}_{-\ell}(\gamma) \colon \mathbb{H} \to \mathbb{H}$ be the Grojnowski–Nakajima operator, see p. 315 in [14]. Let $\mathbf{1} \in H^0(A^{[0]})$ be the function $\{\emptyset\} \to \mathbb{C}$ with value 1 (the vacuum). By definition of Γ, the proposition follows from the easily verified equality

$$\underbrace{\mathfrak{p}_{-1}(1) \cdot \ldots \cdot \mathfrak{p}_{-1}(1)}_{m-1} \cdot \mathfrak{p}_{-1}(\alpha) \cdot \mathbf{1} = (m-1)!\, \widetilde{\mu}_r^{[m]}(\alpha).$$

For $\beta \in H(A)$, let

$$c_m(\beta) := \sum_{1 \leq i < j \leq m} p_i^*(\beta)(ij). \tag{2.5.17}$$

(This is the only place where our notation differs from that of [14], our $c_m(1)$ is denoted $-\epsilon_{m,2}$, see p. 319 op. cit.) Notice that $c_m(\beta) \in H(A)^{[m]}$.

Proposition 2.17. Let $\beta \in H^{r-2}(A)$, and keep notation as above. Then

$$\Gamma(c_m(\beta)) = \frac{1}{2}\widetilde{\nu}_r^{[m]}(\beta). \tag{2.5.18}$$

Proof. By definition of Γ, the proposition follows from the equality

$$\underbrace{\mathfrak{p}_{-1}(1)\cdot\ldots\cdot\mathfrak{p}_{-1}(1)}_{m-2}\cdot\mathfrak{p}_{-2}(\beta)\cdot\mathbf{1}=(m-2)!\,\widetilde{\nu}_r^{[m]}(\beta)$$

Let $\beta,\beta'\in H(A)$. The following formula (which holds by (2.5.12)) will be handy:

$$c_m(\beta)\cdot c_m(\beta')=\sum_{1\le i<j\le m}\Delta_*^{ij}(\beta\smile\beta')\mathrm{Id}++\sum_{\substack{\{h,k,l\}\subset\{1,\ldots,m\}\\ |\{h,k,l\}|=3}}p^*_{\min\{h,k,l\}}(\beta\cup\beta')(hkl)$$
$$+\sum_{\substack{1\le i<j\le m\\ 1\le h<k\le m\\ \{i,j\}\cap\{h,k\}=\emptyset}}z_{ijhk}(\beta,\beta')(ij)(hk). \tag{2.5.19}$$

(Note: in the 2nd summation every order 3 cyclic permutation appears 3 times.)

Let $\eta_{A^{[m]}}\in H^{4m}(S^{[m]})$ be the fundamental class; it follows directly from the definition of Γ (see the definition of Φ on p. 311 of [14]) that

$$\Gamma\left(\eta_A^{\otimes m}\mathrm{Id}\right)=\frac{1}{m!}\eta_{A^{[m]}}. \tag{2.5.20}$$

2.6 Computation of $\vartheta_1(\overline{q}_X^{n-2})$

Let X be a $2n$-dimensional hyperkähler manifold of Kummer type, and let $\vartheta(\overline{q}_X^{n-2})$ be as in Definition 2.12.

Proposition 2.18. Let X be a $2n$-dimensional hyperkähler manifold of Kummer type, where $n\geqslant 2$. Then

$$\vartheta_1\left(\overline{q}_X^{n-2}\right)=-2^{n-2}(n+1)^{n-2}\frac{(2n+3)!!}{7!!}. \tag{2.6.1}$$

We notice that, in order to prove Theorem 1.1, it suffices to know that $\vartheta_1(\overline{q}_X^{n-2})$ and $\vartheta_2(\overline{q}_X^{n-2})$ are non-zero (but we do not know how to establish non-vanishing of $\vartheta_i(\overline{q}_X^{n-2})$ without actually computing it), while in the proof of Theorem 1.3 we need to know that $\vartheta_2(\overline{q}_X^{n-2})/\vartheta_1(\overline{q}_X^{n-2})=n+1$.

The proof of Proposition 2.18 is given at the end of the subsection. We start by going through some preliminary results. Let $H(K_n(A))_{(2)}\subset H(K_n(A))$ be the graded $\mathbb{C}$-algebra generated by $H^2(K_n(A))$. By a Theorem of Verbitsky [2, 25], the restriction of the Poincaré pairing to $H(K_n(A))_{(2)}$ is perfect, and the kernel of the natural map $\mathrm{Sym}H^2(K_n(A))\to H(K_n(A))_{(2)}$ is generated by all elements α^{n+1}, where $q(\alpha)=0$. Now suppose that $p \le n$. Then the map $\mathrm{Sym}^p H^2(K_n(A)) \to H(K_n(A))^{2p}_{(2)}$ is an isomorphism, and hence we have a direct sum decomposition

$$H^{2p}(K_n(A)) = \mathrm{Sym}^p H^2(K_n(A)) \oplus \left(H(K_n(A))^{\perp}_{(2)}\right)^{2p},$$

where orthogonality is with respect to the Poincaré pairing. Let

$$\Pi_p \colon H^{2p}(K_n(A)) \longrightarrow \mathrm{Sym}^p H^2(K_n(A))$$

be the projection. Let ι be as in Definition 2.10.

Lemma 2.19. Let $n \geqslant 3$. There exist $C_i(n), D_i(n) \in \mathbb{Q}$ for $i \in \{1,2,3\}$ such that for all $\alpha, \alpha' \in H^3(A)$ and $\beta, \beta' \in H^1(A)$,

$$\begin{aligned}
\Pi_3\big(\mu_3(\alpha) \smile \mu_3(\alpha')\big) &= C_1(n) q^{\vee} \smile \mu_2\big(\iota^{-1}(\alpha \wedge \alpha')\big) + D_1(n)\mu_2\big(\iota^{-1}(\alpha \wedge \alpha')\big) \smile \xi_n^2,\\
\Pi_3\big(\nu_3(\beta) \smile \nu_3(\beta')\big) &= C_2(n) q^{\vee} \smile \mu_2(\beta \smile \beta') + D_2(n)\mu_2(\beta \smile \beta') \smile \xi_n^2,\\
\Pi_3\big(\mu_3(\alpha) \smile \nu_3(\beta)\big) &= C_3(n)\left(\int_A \alpha \smile \beta\right) q^{\vee} \smile \xi_n + D_3(n)\left(\int_A \alpha \smile \beta\right)\xi_n^3.
\end{aligned}$$

Proof. Let $\Psi_1 \colon \bigwedge^2 H^3(A) \to \mathrm{Sym}^3 H^2(K_n(A))$, $\Psi_2 \colon \bigwedge^2 H^1(A) \to \mathrm{Sym}^3 H^2(K_n(A))$, and $\Psi_3 \colon H^3(A) \otimes H^1(A) \to \mathrm{Sym}^3 H^2(K_n(A))$ be the linear maps which have values $\Pi_3(\mu_3(\alpha) \smile \mu_3(\alpha'))$, $\Pi_3(\nu_3(\beta) \smile \nu_3(\beta'))$ and $\Pi_3(\mu_3(\alpha) \smile \nu_3(\beta))$ on decomposable vectors $\alpha \wedge \alpha'$, $\beta \wedge \beta'$ and $\alpha \otimes \beta$, respectively. Because of (2.2.3), we write the codomain of Ψ_i as

$$\mathrm{Sym}^3 H^2(A) \oplus \left(\mathrm{Sym}^2 H^2(A) \otimes \mathbb{C}\xi_n\right) \oplus \left(H^2(A) \otimes \mathbb{C}\xi_n^2\right) \oplus \mathbb{C}\xi_n^3. \tag{2.6.2}$$

The map Ψ_i is equivariant for the action of the monodromy group of 2D compact complex tori on domain and codomain. Since the monodromy group is $\mathrm{SL}H^3(A;\mathbb{Z})$, Ψ_i is equivariant for the action of $\mathrm{SL}H^3(A)$. The domains of Ψ_1 and Ψ_2 are irreducible representations of $\mathrm{SL}H^3(A)$. Decomposing each summand of (2.6.2) into a direct sum of irreducible $\mathrm{SL}H^3(A)$ representations, one gets the 1st two equations. The decomposition into irreducible summands of the domain of Ψ_3 is $\mathrm{End}_0(H^3(A)) \oplus \mathbb{C}\mathrm{Id}_{H^3(A)}$. Of these two representations, only the trivial one appears in the decomposition of (2.6.2), and the third equation follows.

Throughout the present subsection we let $\{\eta_1, \ldots, \eta_4\}$ be an oriented basis of $H^1(A)$, that is such that $\eta := \eta_1 \smile \ldots \smile \eta_4$ is the fundamental class of A.

Proposition 2.20. Let $\alpha, \alpha' \in H^3(A)$, and $\gamma \in H^2(A)$. If $n \geqslant 2$, then

$$\int_{K_n(A)} \mu_3(\alpha) \smile \mu_3(\alpha') \smile \mu_2(\gamma)^{2n-3} = -(2n-3)!!\left(\int_A \iota^{-1}(\alpha \wedge \alpha') \smile \gamma\right) \cdot \left(\int_A \gamma^2\right)^{n-2}. \quad (2.6.3)$$

Proof. The required computation is done on A^{n+1} by the following argument. Let

$$\sigma_{n+1} : A^{(n+1)} \to A, \qquad \widehat{\sigma}_{n+1} : A^{n+1} \to A$$

be the summation maps, and let

$$W_{n+1}(A) := \sigma_{n+1}^{-1}(0), \qquad \widehat{W}_{n+1}(A) := \widehat{\sigma}_{n+1}^{-1}(0). \quad (2.6.4)$$

The restriction of the Hilbert–Chow map to $K_n(A)$ is a map $\mathfrak{h}_n : K_n(A) \to W_{n+1}(A)$ of degree 1 and, for $\lambda \in H^k(A)$, the class $\mu_k(\lambda)$ is equal to $\mathfrak{h}_n^*(\lambda^{(n+1)}|_{W_{n+1}(A)})$. Hence, the computation may be done on $W_{n+1}(A)$. The natural map $\widehat{W}_{n+1}(A) \to W_{n+1}(A)$ has degree $(n+1)!$, and therefore the computation may be done on $\widehat{W}_{n+1}(A)$. Lastly, we may compute on A^{n+1}, because the relevant classes on $\widehat{W}_{n+1}(A)$ are the restrictions of classes on A^{n+1}.

Let $p_i : A^{n+1} \to A$ be the i-th projection. Then

$$\omega := \sum_{a=1}^{n+1} p_a^*(\eta_1) \cup \sum_{b=1}^{n+1} p_b^*(\eta_2) \cup \sum_{c=1}^{n+1} p_c^*(\eta_3) \cup \sum_{d=1}^{n+1} p_d^*(\eta_4) \quad (2.6.5)$$

is the Poincaré dual of $\widehat{W}_{n+1}(A)$. Thus, (2.6.3) is equivalent to the following equality:

$$\int_{A^{n+1}} \left(\sum_{r=1}^{n+1} p_r^*\alpha\right) \smile \left(\sum_{s=1}^{n+1} p_s^*\alpha'\right) \smile \left(\sum_{t=1}^{n+1} p_t^*\gamma\right)^{2n-3} \smile \omega$$
$$= -(n+1)! \cdot (2n-3)!!\left(\int_A \iota^{-1}(\alpha \wedge \alpha') \smile \gamma\right) \cdot \left(\int_A \gamma \smile \gamma\right)^{n-2}. \quad (2.6.6)$$

By Lemma 2.19, it suffices to prove that (2.6.6) holds for

$$\alpha = \eta_1 \smile \eta_2 \smile \eta_3, \qquad \alpha' = \eta_1 \smile \eta_2 \smile \eta_4. \quad (2.6.7)$$

Notice that

$$\iota^{-1}(\alpha \wedge \alpha') = \eta_1 \smile \eta_2. \tag{2.6.8}$$

The integrand in the left-hand side of (2.6.6) is the sum of monomials given by products of the addends of the factors. Each non-vanishing monomial is equal to

$$p_r^*(\alpha \smile \eta_4) \smile p_s^*(\alpha' \smile \eta_3) \smile p_{t_1}^*(\gamma^2) \smile \ldots \smile p_{t_{n-2}}^*(\gamma^2) \smile p_{t_{n-1}}^*(\gamma \smile \eta_1 \smile \eta_2), \tag{2.6.9}$$

where $\{r, s, t_1, \ldots, t_{n-1}\} = \{1, \ldots, n+1\}$.

By (2.6.8), the integral over A^{n+1} of the class in (2.6.9) equals

$$-\left(\int_A \iota^{-1}(\alpha \wedge \alpha') \smile \gamma\right) \cdot \left(\int_A \gamma^2\right)^{n-2}.$$

Since $(n+1)!\,(2n-3)!!$ such integrals appear, the proposition follows. ■

Proposition 2.21. Let $\alpha, \alpha' \in H^3(A)$, and $\gamma \in H^2(A)$. If $n \geqslant 3$, then

$$\int_{K_n(A)} \mu_3(\alpha) \smile \mu_3(\alpha') \smile \mu_2(\gamma)^{2n-5} \smile \xi_n^2$$

$$= 2(n+1) \cdot (2n-5)!! \left(\int_A \iota^{-1}(\alpha \wedge \alpha') \smile \gamma\right) \cdot \left(\int_A \gamma^2\right)^{n-3}. \tag{2.6.10}$$

Proof. Let Ω be the number such that

$$c_{n+1}(1)^2 \cdot \left(\sum_{i=1}^{n+1} p_i^*(\alpha)\mathrm{Id}\right) \cdot \left(\sum_{i=1}^{n+1} p_i^*(\alpha')\mathrm{Id}\right) \cdot \left(\sum_{i=1}^{n+1} p_i^*(\gamma)\mathrm{Id}\right)^{2n-5}$$
$$\cdot \prod_{s=1}^{4} \left(p_1^*(\eta_s)\mathrm{Id} + \ldots + p_{n+1}^*(\eta_s)\mathrm{Id}\right) = \Omega \eta^{\otimes(n+1)}. \tag{2.6.11}$$

By the results recalled in Subsection 2.5, the integral in the left-hand side of (2.6.10) is equal to $\Omega/(n+1)!$. Now consider Equation (19) for $\beta = \beta' = 1$, and plug it into the left-hand side of (2.6.11): the terms in the right-hand side of (19) that involve non-trivial permutations will give zero when multiplied by the other factors, hence we get that the left-hand side of (2.6.11) is equal to the sum, for $1 \leq i < j \leq (n+1)$, of the products obtained by substituting $c_{n+1}(1)^2$ with $\Delta_*^{ij}(1)\mathrm{Id}$ in the left-hand side of (2.6.11). Since there are $n(n+1)/2$ such terms, and each contributes (by symmetry) the same amount to Q, it follows that

$$\int_{K_n(A)} \mu_3(\alpha) \smile \mu_3(\alpha') \smile \mu_2(\gamma)^{2n-5} \smile \xi_n^2 = \frac{1}{(n-1)! \cdot 2} \int_{A^{n+1}} \Delta_*^{n,(n+1)}(1) \smile \left(\sum_{i=1}^{n+1} p_i^*(\alpha)\right)$$
$$\smile \left(\sum_{i=1}^{n+1} p_i^*(\alpha')\right) \smile \left(\sum_{i=1}^{n+1} p_i^*(\gamma)\right)^{2n-5} \smile \prod_{s=1}^{4} \left(p_1^*(\eta_s) + \ldots + p_{n+1}^*(\eta_s)\right). \quad (2.6.12)$$

Notice that $-\Delta_*^{n,(n+1)}(1)$ is the Poincaré dual of

$$\begin{array}{ccc} \{a \in A^{n+1} \mid a_n = a_{n+1}\} & \xrightarrow{\sim} & A^n \\ a & \mapsto & (a_1, \ldots, a_n). \end{array} \quad (2.6.13)$$

Letting ν be the cohomology class on A^n given by

$$\nu := \left(p_1^*(\gamma) + \ldots + p_{n-1}^*(\gamma) + 2p_n^*(\gamma)\right)^{2n-5} \smile \prod_{s=1}^{4} \left(p_1^*(\eta_s) + \ldots + p_{n-1}^*(\eta_s) + 2p_n^*(\eta_s)\right), \quad (2.6.14)$$

it follows that the integral in the right-hand side of (12) is equal to

$$-\int_{A^n} \left(p_1^*(\alpha) + \ldots + p_{n-1}^*(\alpha) + 2p_n^*(\alpha)\right) \smile \left(p_1^*(\alpha') + \ldots + p_{n-1}^*(\alpha') + 2p_n^*(\alpha')\right) \smile \nu. \quad (2.6.15)$$

The integrand in (2.6.15) equals

$$\sum_{\substack{i \neq j \\ 1 \leq i,j \leq (n-1)}} p_i^*(\alpha) \smile p_j^*(\alpha') \smile \nu + 2\sum_{i=1}^{n-1} p_i^*(\alpha) \smile p_n^*(\alpha') \smile \nu + 2\sum_{j=1}^{n-1} p_n^*(\alpha) \smile p_j^*(\alpha') \smile \nu. \quad (2.6.16)$$

Since ν is $\mathscr{S}_{n-1}$-invariant, it follows that the integral in (2.6.15) equals

$$(n-1)(n-2)\int_{A^n} p_1^*(\alpha) \smile p_2^*(\alpha') \smile \nu + 2(n-1)\int_{A^n} p_1^*(\alpha) \smile p_n^*(\alpha') \smile \nu$$
$$+ 2(n-1)\int_{A^n} p_n^*(\alpha) \smile p_1^*(\alpha') \smile \nu. \quad (2.6.17)$$

By Lemma 2.19, it suffices to prove that (2.6.10) holds for α, α' as in (2.6.7). Expanding ν as a sum of monomials, one gets that

$$\begin{aligned} p_1^*(\alpha) \smile p_2^*(\alpha') \smile \nu = \sum_{i=3}^{n-1} & 4(n-3)!\,(2n-5)!!\,p_1^*(\alpha \smile \eta_4) \smile p_2^*(\alpha' \smile \eta_3) \smile p_3^*(\gamma^2) \smile \ldots \\ & \smile p_{i-1}^*(\gamma^2) \smile p_i^*(\gamma \smile \eta_1 \smile \eta_2) \smile p_{i+1}^*(\gamma^2) \smile \ldots \smile p_n^*(\gamma^2) \\ & + 8(n-3)!\,(2n-5)!!\,p_1^*(\alpha \smile \eta_4) \smile p_2^*(\alpha' \smile \eta_3) \smile p_3^*(\gamma^2) \smile \ldots \\ & \smile p_{n-1}^*(\gamma^2) \smile p_n^*(\gamma \smile \eta_1 \smile \eta_2). \end{aligned} \tag{2.6.18}$$

Thus, recalling (2.6.8), Equation (18) gives

$$\int_{A^n} p_1^*(\alpha) \smile p_2^*(\alpha') \smile \nu = -4(n-1)(n-3)!\,(2n-5)!!\left(\int_A \gamma \smile \iota^{-1}(\alpha \wedge \alpha')\right)\left(\int_A \gamma^2\right)^{n-3}. \tag{2.6.19}$$

Similarly (see [20]), one gets that

$$\begin{aligned} \int_{A^n} p_1^*(\alpha) \smile p_n^*(\alpha') \smile \nu &= \int_{A^n} p_n^*(\alpha) \smile p_1^*(\alpha') \smile \nu \\ &= -2(n-2)!\,(2n-5)!!\left(\int_A \gamma \smile \iota^{-1}(\alpha \wedge \alpha')\right)\left(\int_A \gamma^2\right)^{n-3}. \end{aligned} \tag{2.6.20}$$

By (2.6.17), the integral in the right-hand side of (12) is equal to

$$4(n+1)(n-1)!\,(2n-5)!!\left(\int_A \gamma \smile \iota^{-1}(\alpha \wedge \alpha')\right)\left(\int_A \gamma^2\right)^{n-3}.$$

■

Corollary 2.22. Let $n \geqslant 3$. Then (notation as in Lemma 2.19)

$$C_1(n) = -\frac{1}{(n+1)(2n+5)}, \qquad D_1(n) = 0.$$

Proof. By Lemma 2.19, we have

$$\int_{K_n(A)} \mu_3(\alpha) \smile \mu_3(\alpha') \smile \mu_2(\gamma)^{2n-3} = C_1(n) \int_{K_n(A)} q^\vee \smile \mu_2(\iota^{-1}(\alpha \wedge \alpha')) \smile \mu_2(\gamma)^{2n-3} + \\ + D_1(n) \int_{K_n(A)} \mu_2(\iota^{-1}(\alpha \wedge \alpha')) \smile \xi_n^2 \smile \mu_2(\gamma)^{2n-3}, \quad (2.6.21)$$

and

$$\int_{K_n(A)} \mu_3(\alpha) \smile \mu_3(\alpha') \smile \mu_2(\gamma)^{2n-5} \smile \xi_n^2 = C_1(n) \int_{K_n(A)} q^\vee \smile \mu_2(\iota^{-1}(\alpha \wedge \alpha')) \\ \smile \mu_2(\gamma)^{2n-5} \smile \xi_n^2 + +D_1(n) \int_{K_n(A)} \mu_2(\iota^{-1}(\alpha \wedge \alpha')) \smile \mu_2(\gamma)^{2n-5} \smile \xi_n^4. \quad (2.6.22)$$

Each of the integrals appearing in the right-hand side of the above equations may be computed by invoking the case $\ell = 1$ of Proposition 2.3 or Equation (2.2.5) (see also Remark 2.1). By Proposition 2.20 and Proposition 2.21, it follows that $C_1(n)$ and $D_1(n)$ are the solutions of the system of linear equations

$$\begin{aligned} -(2n-3)!! &= (n+1)(2n+5)\cdot(2n-3)!!\,C_1(n) - 2(n+1)^2\cdot(2n-3)!!\,D_1(n), \\ 2(n+1)\cdot(2n-5)!! &= -2(n+1)^2(2n+5)\cdot(2n-5)!!\,C_1(n) + 12(n+1)^3\cdot(2n-5)!!\,D_1(n). \end{aligned} \quad (2.6.23)$$

Solving for $C_1(n)$ and $D_1(n)$ one gets the formulae of the proposition. ■

Proof of Proposition 2.18. We must prove that if $\alpha, \alpha' \in H^3(A)$ and $\gamma \in H^2(A)$, then

$$\int_{K_n(A)} \mu_3(\alpha) \smile \mu_3(\alpha') \smile \overline{q}^{n-2} \smile \mu_2(\gamma) = -2^{n-2}(n+1)^{n-2}\frac{(2n+3)!!}{7!!}\int_A \iota^{-1}(\alpha \wedge \alpha') \smile \gamma. \quad (2.6.24)$$

If $n = 2$, Equation (2.6.24) follows directly from (2.6.3). If $n \geqslant 3$, first recall that $\overline{q}_X = 2(n+1)q_X^\vee$ by Definition 2.4. Next, one applies Lemma 2.19. In fact, one assigns to $C_1(n)$ and $D_1(n)$ the values given by Corollary 2.22, and then one applies (2.2.5) (see also Remark 2.1) and Proposition 2.3 in order to carry out the required computations. ■

2.7 Computation of $\vartheta_2(\overline{q}^{n-2})$

We will prove the following result.

Proposition 2.23. Let X be a $2n$-dimensional hyperkähler manifold of Kummer type, where $n \geqslant 2$. Then

$$\vartheta_2(\overline{q}_X^{n-2}) = -2^{n-2}(n+1)^{n-1}\frac{(2n+3)!\,!}{7!\,!}. \tag{2.7.1}$$

The proof of Proposition 2.23 will be given at the end of this subsection. Throughout the subsection, $\{\eta_1, \ldots, \eta_4\}$ is an oriented basis of $H^1(A)$, that is $\eta := \eta_1 \smile \ldots \smile \eta_4$ is the fundamental class of A.

Proposition 2.24. Let $\beta, \beta' \in H^1(A)$ and $\gamma \in H^2(A)$. If $n \geqslant 2$, then

$$\int\limits_{K_n(A)} \nu_3(\beta) \smile \nu_3(\beta') \smile \mu_2(\gamma)^{2n-3} = -4(n+1)(2n-3)!\,!\left(\int_A \beta \smile \beta' \smile \gamma\right) \cdot \left(\int_A \gamma^2\right)^{n-2}. \tag{2.7.2}$$

Proof. Let M be the integer such that

$$c_{n+1}(\beta) \cdot c_{n+1}(\beta') \cdot \left(\sum_{i=1}^{n+1} p_i^*(\gamma)\mathrm{Id}\right)^{2n-3} \cdot \prod_{s=1}^{4}\left(\sum_{j=1}^{n+1} p_j^*(\eta_s)\mathrm{Id}\right) = M\eta^{\otimes(n+1)}\mathrm{Id}. \tag{2.7.3}$$

By Proposition 2.16, Proposition 2.17 and (2.5.20), we have

$$\int\limits_{K_n(A)} \nu_3(\beta) \smile \nu_3(\beta') \smile \mu_2(\gamma)^{2n-3} = \frac{4M}{(n+1)!}. \tag{2.7.4}$$

Let us compute M. By (19)

$$\begin{aligned} M &= \sum_{1\le h<k\le(n+1)} \int_{A^{n+1}} \Delta_*^{hk}(\beta \smile \beta') \smile \left(\sum_{i=1}^{n+1} p_i^*(\gamma)\right)^{2n-3} \smile \prod_{s=1}^{4}\left(\sum_{j=1}^{n+1} p_j^*(\eta_s)\right) \\ &= \frac{n(n+1)}{2}\int_{A^{n+1}} \Delta_*^{12}(\beta \smile \beta') \smile \left(\sum_{i=1}^{n+1} p_i^*(\gamma)\right)^{2n-3} \smile \prod_{s=1}^{4}\left(\sum_{j=1}^{n+1} p_j^*(\eta_s)\right). \end{aligned} \tag{2.7.5}$$

Since

$$\Delta_*^{12}(\beta \smile \beta') = -p_1^*(\beta \smile \beta') \smile \text{ P.D. } \{a \in A^{n+1} \mid a_1 = a_2\},$$

(here P.D. stands for "Poincaré dual"), it follows that

$$\int_{K_n(A)} \nu_3(\beta) \smile \nu_3(\beta') \smile \mu_2(\gamma)^{2n-3} = -\frac{2}{(n-1)!} \int_{A^n} p_1^*(\beta \smile \beta') \smile (2p_1^*(\gamma) + p_2^*(\gamma) + \dots$$
$$+ p_n^*(\gamma))^{2n-3} \smile \prod_{s=1}^{4} (2p_1^*(\eta_s) + p_2^*(\eta_s) + \dots + p_n^*(\eta_s)). \tag{2.7.6}$$

By Lemma 2.19 it suffices to prove that (2.7.2) holds for

$$\beta = \eta_1, \qquad \beta' = \eta_2. \tag{2.7.7}$$

The integrand in the right-hand side of (6) is equal to

$$2(n-2)!\,(2n-3)!!\Bigg[\sum_{i=2}^{n} p_1^*(\beta \smile \beta' \smile \gamma) \smile p_2^*(\gamma^2) \smile \dots \smile p_{i-1}^*(\gamma^2) \smile p_{i+1}^*(\gamma^2) \smile \dots$$
$$\smile p_n^*(\gamma^2) \smile p_i^*(\eta) + \sum_{i=2}^{n} 2p_1^*(\beta \smile \beta') \smile p_2^*(\gamma^2) \smile \dots \smile p_{i-1}^*(\gamma^2)$$
$$\smile p_i^*(\gamma \smile \eta_1 \smile \eta_2) \smile p_{i+1}^*(\gamma^2) \smile \dots \smile p_n^*(\gamma^2) \smile p_1^*(\eta_3 \smile \eta_4)$$
$$+ \sum_{2 \le i < j \le n} 2p_1^*(\beta \smile \beta' \smile \gamma) \smile p_2^*(\gamma^2) \smile \dots \smile p_{i-1}^*(\gamma^2) \smile p_i^*(\gamma)$$
$$\smile p_{i+1}^*(\gamma^2) \smile \dots \smile p_{j-1}^*(\gamma^2) \smile p_j^*(\gamma) \smile p_{j+1}^*(\gamma^2) \smile \dots$$
$$\smile p_n^*(\gamma^2) \smile \nu_{ij}\Bigg], \tag{2.7.8}$$

(do not overlook the square brackets!) where

$$\nu_{ij} := \sum_{\substack{1 \le a < b \le 4 \\ 1 \le c < d \le 4 \\ \{a,b,c,d\} = \{1,\dots,4\}}} (-1)^{a+b-1} p_i^*(\eta_a \smile \eta_b) \smile p_j^*(\eta_c \smile \eta_d). \tag{2.7.9}$$

Thus,

$$\int_{A^n} p_1^*(\beta \smile \beta') \smile (2p_1^*(\gamma) + p_2^*(\gamma) + \dots + p_n^*(\gamma))^{2n-3} \smile \prod_{s=1}^{4} (2p_1^*(\eta_s) + p_2^*(\eta_s) + \dots + p_n^*(\eta_s))$$
$$= 2(n+1)(n-1)!\,(2n-3)!!\left(\int_A \beta \smile \beta' \smile \gamma\right) \cdot \left(\int_A \gamma \smile \gamma\right)^{n-2},$$

and the proposition follows from (6). ∎

Proposition 2.25. Let $\beta, \beta' \in H^1(A)$, and $\gamma \in H^2(A)$. If $n \geqslant 3$, then

$$\int_{K_n(A)} \nu_3(\beta) \smile \nu_3(\beta') \smile \mu_2(\gamma)^{2n-5} \smile \xi_n^2 = 8(n+1)^2(2n-5)!!\left(\int_A \beta \smile \beta' \smile \gamma\right) \cdot \left(\int_A \gamma^2\right)^{n-3}. \tag{2.7.10}$$

Proof. Let P be the integer such that

$$c_{n+1}(\beta) \cdot c_{n+1}(\beta') \cdot c_{n+1}(1) \cdot c_{n+1}(1) \cdot \left(\sum_{i=1}^{n+1} p_i^*(\gamma)\mathrm{Id}\right)^{2n-5} \cdot \prod_{s=1}^{4}\left(\sum_{j=1}^{n+1} p_j^*(\eta_s)\mathrm{Id}\right) = P\eta^{\otimes(n+1)}\mathrm{Id}. \tag{2.7.11}$$

By Proposition 2.16, Proposition 2.17 and (2.5.20), we have

$$\int_{K_n(A)} \nu_3(\beta) \smile \nu_3(\beta') \smile \mu_2(\gamma)^{2n-5} \smile \xi_n^2 = \frac{4P}{(n+1)!}. \tag{2.7.12}$$

Let us compute P. By (19) and the formulae in Subsubsection 2.5.2, we have

$$\begin{aligned} c_{n+1}(\beta) \cdot c_{n+1}(\beta') \cdot c_{n+1}(1) \cdot c_{n+1}(1) = & \sum_{\substack{1 \leq a < b \leq (n+1) \\ 1 \leq c < d \leq (n+1)}} \Delta_*^{ab}(\beta \smile \beta') \cdot \Delta_*^{cd}(1)\mathrm{Id} \\ & + 18\left(\sum_{1 \leq h < k < l \leq (n+1)} \Delta_*^{hkl}(\beta \smile \beta')\mathrm{Id}\right) + 2\left(\sum_{\substack{1 \leq r < s \leq (n+1) \\ 1 \leq t < u \leq (n+1) \\ \{r,s\} \cap \{t,u\} = \emptyset}} \Delta_*^{rs}(\beta) \cdot \Delta_*^{tu}(\beta')\mathrm{Id}\right) + \mathscr{R}, \end{aligned}$$

where the remainder $\mathscr{R}$ is a sum of terms involving non-trivial permutations.

Let $\tau := \left(\sum_{i=1}^{n+1} p_i^*(\gamma)\mathrm{Id}\right)^{2n-5} \cdot \prod_{s=1}^4 \left(\sum_{j=1}^{n+1} p_j^*(\eta_s)\mathrm{Id}\right)$, where p_i is projection to the i-th factor. Since τ is $\mathscr{S}_{n+1}$-invariant,

$$\begin{aligned} P = & (n+1)n(n-1)\left(\int_{A^{n+1}} \Delta_*^{12}(\beta \smile \beta') \smile \Delta_*^{13}(1) \smile \tau\right) \\ & + \frac{1}{4}(n+1)n(n-1)(n-2)\left(\int_{A^{n+1}} \Delta_*^{12}(\beta \smile \beta') \smile \Delta_*^{34}(1) \smile \tau\right) \\ & + 3(n+1)n(n-1)\left(\int_{A^{n+1}} \Delta_*^{123}(\beta \smile \beta') \smile \tau\right) \\ & + \frac{1}{2}(n+1)n(n-1)(n-2)\left(\int_{A^{n+1}} \Delta_*^{12}(\beta) \smile \Delta_*^{34}(\beta') \smile \tau\right). \end{aligned} \tag{2.7.13}$$

(Notice that $\Delta_*^{ab}(\beta \smile \beta') \cdot \Delta_*^{ab}(1) = 0$ for dimension reasons.)

Next, notice that $\Delta_{r,*}(1)$ is the Poincaré dual of the small diagonal in A^r multiplied by $(-1)^{r+1}$. Moreover, if $\gamma \in H(A)$ then $\Delta_{r,*}(\gamma) = p_1^*(\gamma) \smile \Delta_{r,*}(1)$. It follows that the integrals in (13) are equal to integrals over the subset $\{(x, x, x, y_1, \dots, y_{n-2}\} \subset A^{n+1}$, or the subset $\{(x, x, y, y, z_1, \dots, z_{n-3}\} \subset A^{n+1}$. More precisely, let τ_3 and $\tau_{2,2}$ be the cohomology classes on A^{n-1} given by

$$\begin{aligned}
\tau_3 \quad &:= \quad \left(3p_1^*(\gamma) + \sum_{i=2}^{n-1} p_i^*(\gamma)\right)^{2n-5} \smile \prod_{s=1}^{4} \left(3p_1^*(\eta_s) + \sum_{j=2}^{n-1} p_j^*(\eta_s)\right), \\
\tau_{2,2} \quad &:= \quad \left(2p_1^*(\gamma) + 2p_2^*(\gamma) + \sum_{i=3}^{n-1} p_i^*(\gamma)\right)^{2n-5} \smile \prod_{s=1}^{4} \left(2p_1^*(\eta_s) + 2p_2^*(\eta_s) + \sum_{j=3}^{n-1} p_j^*(\eta_s)\right).
\end{aligned}$$

Then (13) reads

$$\begin{aligned}
P &= (n+1)n(n-1)\left(\int\limits_{A^{n-1}} p_1^*(\beta \smile \beta') \smile \tau_3\right) \\
&+ \frac{1}{4}(n+1)n(n-1)(n-2)\left(\int\limits_{A^{n-1}} p_1^*(\beta \smile \beta') \smile \tau_{2,2}\right) \\
&+ 3(n+1)n(n-1)\left(\int\limits_{A^{n-1}} p_1^*(\beta \smile \beta') \smile \tau_3\right) \\
&+ \frac{1}{2}(n+1)n(n-1)(n-2)\left(\int\limits_{A^{n-1}} p_1^*(\beta) \smile p_2^*(\beta') \smile \tau_{2,2}\right).
\end{aligned} \tag{2.7.14}$$

By Lemma 2.19, it suffices to prove that (2.7.10) holds for

$$\beta = \eta_1, \qquad \beta' = \eta_2. \tag{2.7.15}$$

Straightforward computations (see [20]) give that

$$\int\limits_{A^{n-1}} p_1^*(\beta \smile \beta') \smile \tau_3 = 3(n+1)(n-2)!\,(2n-5)!\,!\left(\int_A \beta \smile \beta' \smile \gamma\right) \cdot \left(\int_A \gamma \smile \gamma\right)^{n-3}. \tag{2.7.16}$$

$$\int_{A^{n-1}} p_1^*(\beta\smile\beta')\smile\tau_{2,2}=8(n+1)(n-1)(n-3)!\,(2n-5)!!\left(\int_A\beta\smile\beta'\smile\gamma\right)\cdot\left(\int_A\gamma\smile\gamma\right)^{n-3}. \tag{2.7.17}$$

$$\int_{A^{n-1}} p_1^*(\beta)\smile p_2^*(\beta')\smile\tau_{2,2}=-16(n+1)(n-3)!\,(2n-5)!!\left(\int_A\beta\smile\beta'\smile\gamma\right)\cdot\left(\int_A\gamma\smile\gamma\right)^{n-3}. \tag{2.7.18}$$

Thus, $P=2(n+1)!\,(n+1)^2(2n-5)!!$ by (14), (2.7.16), (2.7.17), and (2.7.18), and the proposition follows from (2.7.12). ■

Corollary 2.26. Let $n\geqslant 3$. Then (notation as in Lemma 2.19)

$$C_2(n)=-\frac{4}{(2n+5)},\qquad D_2(n)=0.$$

Proof. By Propositions 2.20, 2.21, 2.24, and 2.25 we have

$$\begin{aligned}\int_{K_n(A)}\nu_3(\beta)\smile\nu_3(\beta')\smile\mu_2(\gamma)^{2n-3} &= 4(n+1)\int_{K_n(A)}\mu_3(\alpha)\smile\mu_3(\alpha')\smile\mu_2(\gamma)^{2n-3},\\ \int_{K_n(A)}\nu_3(\beta)\smile\nu_3(\beta')\smile\mu_2(\gamma)^{2n-5}\smile\xi_n^2 &= 4(n+1)\int_{K_n(A)}\mu_3(\alpha)\smile\mu_3(\alpha')\smile\mu_2(\gamma)^{2n-5}\smile\xi_n^2.\end{aligned}$$

Thus, going through the proof of Corollary 2.22 one gets that $C_2(n)$ and $D_2(n)$ satisfy the system of linear equations obtained from (2.6.23) by multiplying the left-hand terms by $4(n+1)$ and replacing $C_1(n)$ $D_1(n)$ by $C_2(n)$, $D_2(n)$, respectively. Hence, $C_2(n)=4(n+1)C_1(n)$ and $D_2(n)=4(n+1)D_1(n)$. Hence, the result follows from corollary 2.22. ■

Proof of Proposition 2.23. We must prove that $\vartheta_2(\overline{q}^{n-2})=(n+1)\vartheta_1(\overline{q}^{n-2})$. This holds because $C_2(n)=4(n+1)C_1(n)$ and $D_2(n)=4(n+1)D_1(n)$. (Recall that $\mathsf{F}(0,\beta)=\nu(\beta)/2$, where F is the isomorphism in (2.3.1).) ■

2.8 Proof of the 1st main result

We prove Theorem 1.1.

Let us prove Item (1), that is surjectivity of ϕ. Since $\vartheta_1(\overline{q}_X^{n-2})$ and $\vartheta_2(\overline{q}_X^{n-2})$ are non-zero, the map ϕ is non-zero. Let X_0 be very general. Then there is no non-trivial sub Hodge structure of $H^2(X_0)^\vee$, and hence ϕ is surjective. Since ϕ is flat for the Gauss-Manin connection, it follows that ϕ is surjective for every X.

Before going to Item (2), we note the following.

Lemma 2.27. Let X be an HK of Kummer type, of dimension $2n$. Then $\vartheta_3(\overline{q}_X^{n-2})$ is non-zero.

Proof. We may assume that $X = K_n(A)$. Since ϕ is surjective, it follows that $\vartheta_3(\overline{q}_X^{n-2})$ is non-zero. ■

Remark 2.28. We will compute $\vartheta_3(\overline{q}_X^{n-2})$ up to sign, see corollary 3.7.

In proving Item (2), we may assume that X is a generalized Kummer $K_n(A)$, Identify $H^3(A;\mathbb{Z}) \oplus H^1(A;\mathbb{Z})$ with $H^3(K_n(A);\mathbb{Z})$ via Theorem 2.7, see (2.3.1). Identify $H^1(A;\mathbb{Z})$ with $H^3(A;\mathbb{Z})^\vee$ via (2.3.8). Given these identifications, we define the following integral unimodular quadratic form:

$$\begin{array}{ccc} H^3(K_n(A))/\mathrm{Tors} & \xrightarrow{\mathbf{q}_{K_n(A)}} & \mathbb{C} \\ (\alpha,\beta) & \mapsto & 2\beta(\alpha) \end{array} \tag{2.8.19}$$

Let $\gamma \in H^3(K_n(A))$. Then, since the components of $\vartheta(\overline{q}^{n-2})$ are non-zero,

$$\dim\phi(\gamma \wedge H^3(K_n(A)) = \begin{cases} 0 & \text{if } \gamma = 0, \\ 4 & \text{if } \gamma \neq 0 \text{ and } \mathbf{q}_{K_n(A)}(\gamma) = 0, \\ 7 & \text{if } \mathbf{q}_{K_n(A)}(\gamma) \neq 0. \end{cases} \tag{2.8.20}$$

Item (2) of Theorem 1.1 follows.

Lastly, we prove Item (3). Let $0 \neq \gamma \in H^{2,1}(X)$. Then $\phi(\gamma \wedge H^{2,1}(X)) \subset \mathrm{Ann}(F^1H^2(X))$, and hence $\dim\phi(\gamma \wedge H^3(X)) \leq 5$. Thus, $[\gamma] \in \mathbf{Q}(X)$.

3 Reconstructing $H^3(X)$ from $H^2(X)$

3.1 Summary

In the present section we will prove Theorem 1.3. We assume that we are given an abstract version of the map in (2.4.4), that is a linear map

$$\bigwedge^2(V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee) \xrightarrow{\Phi_\vartheta} \bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C}, \tag{3.1.1}$$

depending on a choice of $\vartheta := (\vartheta_1, \vartheta_2, \vartheta_3) \in \mathbb{Z}^3$, where V is a free $\mathbb{Z}$-module of rank 4. Assuming that all the components of ϑ are non-zero, we determine for which 4D subspaces Γ of the domain the image $\Phi_\vartheta(\bigwedge^2\Gamma)$ is one-dimensional. The motivation is the Key observation 1.2. Next, we equip the codomain of Φ_ϑ with a non-degenerate quadratic form modelled on the BBF of generalized Kummers, and we get a corresponding open subset of a quadric, call it $\mathscr{D}$, parametrizing weight 2 Hodge structure of $K3$ type. We show that for Γ as above, $\Phi_\vartheta(\bigwedge^2\Gamma)$ is an element of $\mathscr{D}$, and that conversely every element of $\mathscr{D}$ comes from a unique Γ. Thus, associated to each point of $\mathscr{D}$ there is an integral effective weight 1 Hodge structure, and hence a compact complex torus. In the last subsection we prove Theorem 1.3.

3.2 Set up

Keeping notation as above, let $\mathrm{vol}\colon \bigwedge^4 V \xrightarrow{\sim} \mathbb{Z}$ be a volume form. Let $(,)$ be the bilinear symmetric non-degenerate form on $\bigwedge^2 V$ defined by $(\alpha,\beta) := \mathrm{vol}(\alpha\wedge\beta)$. We extend bilinearly $(,)$ to $\bigwedge^2 V_{\mathbb{C}}$, where $V_{\mathbb{C}} := V\otimes_{\mathbb{Z}}\mathbb{C}$, and we denote it by the same symbol. Let

$$\iota\colon \bigwedge^2 V_{\mathbb{C}}^{\vee} \xrightarrow{\sim} \bigwedge^2 V_{\mathbb{C}} \tag{3.2.1}$$

be the isomorphism defined by $(,)$.

We define the map Φ_ϑ in (3.1.1) to be the one induced by the bilinear antisymmetric map

$$\begin{array}{ccc} (V_{\mathbb{C}}\oplus V_{\mathbb{C}}^{\vee})\times(V_{\mathbb{C}}\oplus V_{\mathbb{C}}^{\vee}) & \longrightarrow & \bigwedge^2 V_{\mathbb{C}}\oplus\mathbb{C} \\ ((v,g),(w,h)) & \mapsto & (\vartheta_1 v\wedge w+\vartheta_2\iota(g\wedge h),\vartheta_3(g(w)-h(v))) \end{array} \tag{3.2.2}$$

Remark 3.1. Let X be an HK manifold of Kummer type of dimension $2n$. By Proposition 2.11, there exist isomorphisms $H^3(X;\mathbb{Z})\cong V\oplus V^{\vee}$ and $H^2(X;\mathbb{Z})^{\vee}\cong(\bigwedge^2 V\oplus\mathbb{Z})$, such that ϕ (see (1.2.2)) gets identified with $\Phi_{\vartheta(\overline{q}_X^{n-2})}$.

Notation 3.2. Keeping notation as above, we let $\zeta := (0,1)\in(\bigwedge^2 V_{\mathbb{C}}\oplus\mathbb{C})$.

3.3 A result in linear algebra

Let $f\colon V_{\mathbb{C}}\to V_{\mathbb{C}}^{\vee}$ be a linear map. We let $\omega_f\in\bigwedge^2 V_{\mathbb{C}}^{\vee}$ be the antisymmetric form defined by f, that is

$$\omega_f(v,w)=\frac{1}{2}\left(\langle f(w),v\rangle-\langle f(v),w\rangle\right), \tag{3.3.1}$$

where $\langle,\rangle$ denotes the natural pairing between $V_{\mathbb{C}}^{\vee}$ and $V_{\mathbb{C}}$.

We let $f = f_+ + f_-$ be the decomposition into the sum of a symmetric and an antisymmetric linear map. Notice that $\omega_{f_-} = \omega_f$.

Now assume that f is antisymmetric. The Pfaffian of f is the Pfaffian of (any) matrix associated to f by the choice of a basis $\{v_1, \ldots, v_4\}$ of $V_{\mathbb{C}}$ of volume 1, and the dual basis $\{v_1^\vee, \ldots, v_4^\vee\}$ of $V_{\mathbb{C}}^\vee$. We denote the Pfaffian of f by $\mathrm{Pf}(f)$. If $\omega_f = \sum_{1\le i<j\le 4} a_{ij} v_i^\vee \wedge v_j^\vee$, then

$$\mathrm{Pf}(f) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}. \tag{3.3.2}$$

Proposition 3.3. Keep notation as above, and assume that ϑ_1, ϑ_2 and ϑ_3 are non-zero. Let $\Gamma \subset (V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee)$ be a vector subspace of dimension 4. Then $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is a one-dimensional subspace of $\bigwedge^2 V \oplus \mathbb{C}$ if and only if one of the following holds:

(1) There exists an antisymmetric non-degenerate linear map $f\colon V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee$ such that

$$\vartheta_1 = \vartheta_2 \cdot \mathrm{Pf}(f), \tag{3.3.3}$$

and

$$\Gamma := \{(v, f(v)) \mid v \in V_{\mathbb{C}}\}. \tag{3.3.4}$$

If this is the case, then

$$\Phi_\vartheta\left(\bigwedge^2 \Gamma\right) = \mathrm{span}\{\vartheta_2 \iota(\omega_f) - 2\vartheta_3\zeta\}. \tag{3.3.5}$$

(2) $\Gamma = U \oplus U^\perp$, where $U \subset V_{\mathbb{C}}$ is a 2D subspace, and $U^\perp \subset V_{\mathbb{C}}^\vee$ is the annihilator of U. If this is the case, then

$$\Phi_\vartheta\left(\bigwedge^2 \Gamma\right) = \bigwedge^2 U = \iota\left(\bigwedge^2 U^\perp\right). \tag{3.3.6}$$

Proof. Suppose that (3.3.4) holds. Then decomposable elements of $\bigwedge^2 \Gamma$ are given by $(v, f(v)) \wedge (w, f(w)$ for $v, w \in V_{\mathbb{C}}$, and

$$\Phi_\vartheta\left((v, f(v)) \wedge (w, f(w))\right) = \vartheta_1 v \wedge w + \vartheta_2 \iota(f(v) \wedge f(w)) - 2\vartheta_3 \omega_f(v, w)\zeta. \tag{3.3.7}$$

It follows that

$$\text{if } f_- \neq 0, \text{then } \Phi_\vartheta(\bigwedge^2 \Gamma) \text{ contains a vector } \alpha + s\zeta \text{ where } \alpha \in \bigwedge^2 V_{\mathbb{C}} \text{ and } s \neq 0. \tag{3.3.8}$$

Now, we do a case-by-case analysis - always assuming that (3.3.4) holds.

(1) f_- is non-degenerate. Let $v, w, a, b \in V_{\mathbb{C}}$; then

$$\mathrm{Pf}(f_-)\mathrm{vol}(v \wedge w \wedge a \wedge b) = \omega_f(v, w) \cdot \omega_f(a, b) - \omega_f(v, a) \cdot \omega_f(w, b) + \omega_f(v, b) \cdot \omega_f(w, a),$$

that is

$$\mathrm{Pf}(f_-)v \wedge w = \iota\left(\omega_f(v, w) \cdot \omega_f - f_-(v) \wedge f_-(w)\right).$$

By hypothesis $\mathrm{Pf}(f_-) \neq 0$, hence

$$v \wedge w = \mathrm{Pf}(f_-)^{-1}\iota\left(\omega_f(v, w) \cdot \omega_f - f_-(v) \wedge f_-(w)\right). \tag{3.3.9}$$

It follows that

$$\begin{aligned}\Phi_\vartheta\left((v, f(v)) \wedge (w, f(w))\right) = \vartheta_1 \cdot \mathrm{Pf}(f_-)^{-1}\omega_f(v, w)\iota(\omega_f) + \vartheta_2\iota(&f_+(v) \wedge f_+(w) \\ &+ f_+(v) \wedge f_-(w) + f_-(v) \wedge f_+(w)) \\ &+ (\vartheta_2 - \vartheta_1 \cdot \mathrm{Pf}(f_-)^{-1})\iota(f_-(v) \wedge f_-(w)) \\ &- 2\vartheta_3\omega_f(v, w)\zeta.\end{aligned} \tag{3.3.10}$$

Next, we distinguish between the two subcases: f antisymmetric or f not antisymmetric.

(a) $f_+ = 0$. Equation (10) shows that if (3.3.4) and (3.3.3) hold, and f is antisymmetric non-degenerate, then $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is one-dimensional, given by (3.3.5).

(a) Let us prove that if $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is one-dimensional, (3.3.4) holds, and f is antisymmetric non-degenerate, then (3.3.3) holds. Let $v, w \in V_{\mathbb{C}}$ be linearly independent, and such that $\omega_f(v, w) = 0$. Then the right-hand side of (10) is equal to

$$(\vartheta_2 - \vartheta_1 \cdot \mathrm{Pf}(f_-)^{-1})\iota(f_-(v) \wedge f_-(w)).$$

Since $f_-(v) \wedge f_-(w)$ is not zero (recall that f_- is non-degenerate by hypothesis), the above vector and the vector in (3.3.8) are linearly dependent only if $\vartheta_2 - \vartheta_1 \cdot \mathrm{Pf}(f_-)^{-1} = 0$, that is (3.3.3) holds.

(b) $f_+ \neq 0$. Assume that $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is one-dimensional; we will reach a contradiction. Let $v, w \in V_{\mathbb{C}}$ be such that $\omega_f(v,w) = 0$. By (10), $\Phi_\vartheta(\bigwedge^2 \Gamma)$ contains the vector

$$\vartheta_2 \iota((f_+(v) \wedge f_+(w) + f_+(v) \wedge f_-(w) + f_-(v) \wedge f_+(w)) \\ + (\vartheta_2 - \vartheta_1 \cdot \mathrm{Pf}(f_-)^{-1}) f_-(v) \wedge f_-(w). \quad (3.3.11)$$

By (3.3.8), we get that the vector in (3.3.11) is zero. Multiplying the vector in (3.3.11) by $f_-(v)$, we get (recall that by hypothesis $\vartheta_2 \neq 0$)

$$f_+(v) \wedge f_-(v) \wedge f(w) = 0. \quad (3.3.12)$$

We claim that this is a contradiction, that is that there exist $v, w \in V_{\mathbb{C}}$ such that $\omega_f(v,w) = 0$, and $f_+(v), f_-(v), f(w)$ are linearly independent. In fact, since f_- is non-degenerate and f_+ is non-zero, $f_+(v), f_-(v)$ are linearly independent for generic v. Now suppose first that f is non-degenerate. Let v be such that $f_+(v), f_-(v)$ are linearly independent. Let $v^\perp \subset V_{\mathbb{C}}$ be the orthogonal to v with respect to ω_f. Then $v^\perp$ is 3D, and hence so is $f(v^\perp)$. Thus, there exists $u \in f(v^\perp) \backslash \mathrm{span}\{f_+(v), f_-(v)\}$. Letting $w := f^{-1}(u)$, we get that $\omega_f(v,w) = 0$, and $f_+(v), f_-(v), f(w)$ are linearly independent.

(b) One argues similarly if $\mathrm{rk} f \in \{2,3\}$ (since $f_- = (f - f^t)/2$ is non-degenerate, $\mathrm{rk} f \geqslant 2$). More precisely, if $\mathrm{rk} f = 3$, and $\ker f$ is generated by v_0, we repeat the argument above, with v a generic vector not in $v_0^\perp$. If $\mathrm{rk} f = 2$ we let v be a generic vector in $V_{\mathbb{C}}$. Then $f_-(v) \notin \mathrm{Im} f$, the span of $f_+(v)$ and $f_-(v)$ intersects $\mathrm{Im} f$ in a one-dimensional space, and $v^\perp$ does not contain $\ker f$. In particular $f(v^\perp) = \mathrm{Im} f$, and hence if $w \in v^\perp$ is generic, then $f(w)$ is not contained in the span of $f_+(v), f_-(v)$.

(2) $\mathrm{rk}(f_-) = 2$. Suppose that $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is one-dimensional; we will reach a contradiction. Let $v, w \in V_{\mathbb{C}}$ be such that $\omega_f(v,w) = 0$; then $\Phi_\vartheta((v, f(v)) \wedge (w, f(w))) = 0$ by (3.3.8). Let $a, b \in V_{\mathbb{C}}$; multiplying the 1st component of $\Phi_\vartheta((v, f(v)) \wedge (w, f(w)))$ (as given in (3.2.2)) by $a \wedge b$, we get that

$$\vartheta_1 \mathrm{vol}(v \wedge w \wedge a \wedge b) + \vartheta_2(\langle f(v), a\rangle \cdot \langle f(w), b\rangle - \langle f(v), b\rangle \cdot \langle f(w), a\rangle) = 0. \quad (3.3.13)$$

Now let $v \in V_{\mathbb{C}}$ be a non-zero element of the (2D-) kernel of f_{-}, that is such that $f(v) = f^t(v)$. Then $f^{-1}\mathrm{Ann}(v)$ has dimension at least 3, hence there exist $a, b \in f^{-1}\mathrm{Ann}(v)$ such that v, a, b are linearly independent. Complete $\{v, a, b\}$ to a basis $\{v, a, b, w\}$ of $V_{\mathbb{C}}$. Then the left-hand side of (3.3.13) is non-zero. In fact $\langle f(v), a\rangle = \langle f(a), v\rangle = 0$ because $f(v) = f^t(v)$ and $a \in f^{-1}\mathrm{Ann}(v)$. Similarly $\langle f(v), b\rangle = 0$. Hence, the left-hand side of (3.3.13) is equal to $\vartheta_1 \mathrm{vol}(v \wedge w \wedge a \wedge b)$, which is non-zero because $\{v, a, b, w\}$ is a basis of $V_{\mathbb{C}}$ (and $\vartheta_1 \neq 0$). On the other hand $\omega_f(v, w) = 0$ because v is in the kernel of f_{-}. That is a contradiction, and hence $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is not one-dimensional.

(3) $f_{-} = 0$. Then f is symmetric, and hence we may diagonalize f. An explicit computation shows that $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is not one-dimensional.

Suppose that there does not exist a linear map $f\colon V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}$ such that (3.3.4) holds. Thus, $\Gamma \cap (V_{\mathbb{C}}, 0)$ is non-trivial. An easy case-by-case analysis shows that Item (2) holds. Viceversa, it is clear that if Item (2) holds, then $\Phi_\vartheta(\bigwedge^2 \Gamma)$ is one-dimensional, given by (3.3.6). ■

3.4 From weight 2 to weight 1

Let $m \in \mathbb{Q}_{+}$. We let $(,)$ be the bilinear symmetric non-degenerate form on $(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C}) \times (\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C})$ defined by

$$(\alpha + x\zeta, \beta + y\zeta) := \mathrm{vol}(\alpha \wedge \beta) - mxy. \tag{3.4.1}$$

Example 3.4. Let A be an abelian surface. Let $\mathrm{vol}\colon \bigwedge^4 H^1(A;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ be defined by the orientation of A, and let $V = H^1(A;\mathbb{Z})$. Then $\bigwedge^2 V$ is identified with $H^2(A;\mathbb{Z})$. If we let $\zeta = \xi_n$, and $m = 2(n+1)$, the BBF quadratic form on $H^2(K_n(A))$ is identified with (3.4.1), see (2.2.4).

Example 3.5. Let A be an abelian surface. Let $V = H^3(A;\mathbb{Z}) \cong H^1(A;\mathbb{Z})^{\vee}$, where the isomorphism is defined by (2.3.8). Let $\zeta = \xi_n^{\vee}$, see Notation 2.9. Then we have an isomorphism

$$\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C}\zeta \xrightarrow{\sim} H^2(K_n(A))^{\vee}. \tag{3.4.2}$$

Since the BBF bilinear symmetric form is non-degenerate, it defines a dual BBF rational quadratic form on $H^2(K_n(A))^{\vee}$. If $m = \frac{1}{2(n+1)}$, the dual BBF quadratic form is identified with (3.4.1), see (2.2.4).

Complex conjugation defines a conjugation operator on $\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C}$. Let

$$\mathscr{D} := \left\{ [\sigma] \in \mathbb{P}\left(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C} \right) \middle| (\sigma, \sigma) = 0, \quad (\sigma, \overline{\sigma}) > 0 \right\}. \tag{3.4.3}$$

Then $\mathscr{D}$ is a connected complex manifold of dimension 5. We recall that $\mathscr{D}$ parametrizes integral weight 2 Hodge structures $(\bigwedge^2 V \oplus \mathbb{Z}, H^{p,q})$ of $K3$ type as follows. Given $[\sigma] \in \mathscr{D}$, we let

$$H^{2,0}_{[\sigma]} := [\sigma], \quad H^{1,1}_{[\sigma]} := \{\sigma, \overline{\sigma}\}^{\perp}, \quad H^{0,2}_{[\sigma]} := [\overline{\sigma}]. \tag{3.4.4}$$

Proposition 3.6. Suppose that ϑ_1, ϑ_2, and ϑ_3 are non-zero. Let $[\sigma] \in \mathscr{D} \setminus \zeta^{\perp}$, and assume that there exists an integral (effective) Hodge structure $(V \oplus V^{\vee}, H^{p,q})$ of weight 1 such that Φ_{ϑ} is a morphism of Hodge structures. Then

$$\vartheta_1 \cdot \vartheta_2 = 2m\vartheta_3^2. \tag{3.4.5}$$

Proof. Since Φ_{ϑ} is a morphism of Hodge structures, $\Phi_{\vartheta}(\bigwedge^2 H^{1,0}_{[\sigma]}(\vartheta)) \subset H^{2,0}_{[\sigma]}$, and we have equality because Φ_{ϑ} is surjective. Since $H^{1,0}_{[\sigma]}(\vartheta)$ is a 4D subspace of $V_{\mathbb{C}} \oplus V^{\vee}_{\mathbb{C}}$, and $H^{2,0}_{[\sigma]}$ is one-dimensional, we may apply Proposition 3.3; we get that there exists an antisymmetric non-degenerate map $f \colon V_{\mathbb{C}} \to V^{\vee}_{\mathbb{C}}$ such that

$$H^{2,0}_{[\sigma]} = \Phi_{\vartheta}\left(\bigwedge^2 H^{1,0} \right) = \operatorname{span}\left\{ \vartheta_2 \iota(\omega_f) - 2\vartheta_3 \zeta \right\}.$$

Thus,

$$0 = (\vartheta_2 \iota(\omega_f) - 2\vartheta_3\zeta, \vartheta_2\iota(\omega_f) - 2\vartheta_3\zeta) = \vartheta_2^2 \operatorname{vol}(\iota(\omega_f) \wedge \iota(\omega_f)) - 4m\vartheta_3^2. \tag{3.4.6}$$

Let $\operatorname{vol}^{\vee} \colon \bigwedge^4 V^{\vee}_{\mathbb{C}} \xrightarrow{\sim} \mathbb{C}$ be the volume form dual to vol. Since $\operatorname{vol}(\iota(\omega_f) \wedge \iota(\omega_f)) = \operatorname{vol}^{\vee}(\omega_f \wedge \omega_f) = 2\operatorname{Pf}(f)$, Equation (3.4.5) follows from (3.4.6) and (3.3.3). ■

Corollary 3.7. Let X be a $2n$-dimensional hyperkähler manifold of Kummer type, where $n \geqslant 2$. Then

$$\vartheta\left(\overline{q}_X^{n-2}\right) = -2^{n-2}(n+1)^{n-2}\frac{(2n+3)!!}{7!!}(1, (n+1), (-1)^{\epsilon_n}(n+1)), \tag{3.4.7}$$

for some $\epsilon_n \in \{0, 1\}$.

Proof. The values of $\vartheta_1(\overline{q}_X^{n-2})$ and $\vartheta_2(\overline{q}_X^{n-2})$ are given by Proposition 2.18 and Proposition 2.23, respectively. It remains to compute $\vartheta_3(\overline{q}_X^{n-2})$ up to sign. By Lemma 2.27, $\vartheta_3(\overline{q}_X^{n-2})$ is non-zero. By Example 3.5 and Proposition 3.6, we get that

$$(n+1)\vartheta_1\left(\overline{q}_X^{n-2}\right)\cdot\vartheta_2\left(\overline{q}_X^{n-2}\right)=\vartheta_3\left(\overline{q}_X^{n-2}\right)^2,$$

and hence $\vartheta_3(\overline{q}_X^{n-2})=\pm 2^{n-2}(n+1)^{n-1}\frac{(2n+3)!!}{7!!}$. ■

Remark 3.8. Let $\alpha\in H^3(A)$ and $\beta\in H^1(A)$. A straightforward computation (similar to the computations carried out to prove Propositions 2.21, 2.23, and 2.25, but much simpler) gives that

$$\int_{K_2(A)}\mu_3(\alpha)\smile\nu_3(\beta)\smile\xi_2=-6\int_A\alpha\smile\beta. \tag{3.4.8}$$

Thus, $\vartheta_3(1)=-3$. Equivalently $\epsilon_2=1$.

Theorem 3.9. Suppose that ϑ_1, ϑ_2, and ϑ_3 are non-zero, and that (3.4.5) holds. Let $[\sigma]\in\mathscr{D}$ (see (3.4.3)). There exists a unique integral effective Hodge structure

$$\left(V\oplus V^\vee, H^{p,q}_{[\sigma]}(\vartheta)\right) \tag{3.4.9}$$

of weight 1 (effective means that $h^{p,q}=0$ if p or q is negative) with the property that Φ_ϑ is a morphism of integral Hodge structures, and it is described as follows:

(1) If $\sigma\notin\zeta^\perp$ (see Notation 3.2), rescale σ so that $\sigma=\vartheta_2\alpha-2\vartheta_3\zeta$, where $\alpha\in\bigwedge^2 V_{\mathbb{C}}$. Thus, $\alpha\wedge\alpha\neq 0$ because σ is isotropic. Let $f\colon V_{\mathbb{C}}\to V_{\mathbb{C}}^\vee$ be the antisymmetric non-degenerate map such that $\iota(\omega_f)=\alpha$. Then

$$H^{1,0}_{[\sigma]}(\vartheta)=\left\{(v,f(v))\mid v\in V_{\mathbb{C}}\right\},\qquad H^{0,1}_{[\sigma]}(\vartheta):=\overline{H^{1,0}_{[\sigma]}(\vartheta)}. \tag{3.4.10}$$

(2) If $\sigma\in\zeta^\perp$, that is σ is a decomposable element of $\bigwedge^2 V_{\mathbb{C}}$, then

$$H^{1,0}_{[\sigma]}(\vartheta)=U\oplus U^\perp,\qquad H^{0,1}_{[\sigma]}(\vartheta):=\overline{H^{1,0}_{[\sigma]}(\vartheta)}, \tag{3.4.11}$$

where $U\in\mathrm{Gr}(2,V_{\mathbb{C}})$ is the unique element such that $\bigwedge^2 U=[\sigma]$.

Proof. Suppose that there exists an effective integral weight 1 Hodge structure (3.4.9) such that Φ_ϑ is morphism of Hodge structures. Then $\Phi_\vartheta(\bigwedge H^{1,0}_{[\sigma]}(\vartheta))\subset H^{2,0}_{[\sigma]}$, and we have equality because Φ_ϑ is surjective. Since $H^{1,0}_{[\sigma]}(\vartheta)$ is a 4D subspace of $V_{\mathbb{C}} \oplus V^\vee_{\mathbb{C}}$, and $H^{2,0}_{[\sigma]}$ is one-dimensional, we may apply Proposition 3.3; we get that either (1) or (2) holds.

Next, we show that (3.4.9) does indeed define an effective integral weight 1 Hodge structure, and that Φ_ϑ is morphism of Hodge structures. If $\sigma \in \zeta^\perp$, the verification is straightforward. Thus, we assume that $\sigma \notin \zeta^\perp$. The proof of Proposition 3.6 gives that (3.3.3) holds. Thus, by Proposition 3.3

$$\Phi_\vartheta\left(\bigwedge H^{1,0}_{[\sigma]}(\vartheta)\right) = H^{2,0}_{[\sigma]}. \tag{3.4.12}$$

Since Φ_ϑ is real, it follows that $\Phi_\vartheta(\bigwedge H^{0,1}_{[\sigma]}(\vartheta)) = H^{0,2}_{[\sigma]}$. Let us prove that

$$\Phi_\vartheta\left(H^{1,0}_{[\sigma]}(\vartheta) \wedge H^{0,1}_{[\sigma]}(\vartheta)\right) \subset H^{1,1}_{[\sigma]}. \tag{3.4.13}$$

It suffices to prove that

$$(\vartheta_2\iota(\omega_f) - 2\vartheta_3\zeta)\perp\Phi_\vartheta\left(H^{1,0}_{[\sigma]}(\vartheta) \wedge H^{0,1}_{[\sigma]}(\vartheta)\right). \tag{3.4.14}$$

Let $(v,f(v)),(w,f(w)) \in H^{1,0}_{[\sigma]}(\vartheta)$. Then

$$\begin{aligned}&\Big(\vartheta_2\iota(\omega_f) - 2\vartheta_3\zeta, \Phi_\vartheta((v,f(v)) \wedge (\overline{w},\overline{f}(w)))\Big)\\ &\qquad = \vartheta_2\Big(\vartheta_2\mathrm{vol}^\vee(\omega_f \wedge f(v) \wedge \overline{f}(w)) - \vartheta_1\omega_{\overline{f}}(v,w)\Big).\end{aligned} \tag{3.4.15}$$

The right-hand side vanishes because of the formula (proved by a straightforward computation)

$$\mathrm{vol}^\vee(\omega_f \wedge f(a) \wedge \overline{f}(b)) = \mathrm{Pf}(f)\omega_{\overline{f}}(a,b). \tag{3.4.16}$$

It remains to prove that

$$H^{1,0}_{[\sigma]}(\vartheta) \cap (V_{\mathbb{R}} \oplus V^\vee_{\mathbb{R}}) = \{0\}. \tag{3.4.17}$$

Suppose the contrary. It follows that there exists a basis $\{v_1,v_2,v_3,v_4\}$ of $V_{\mathbb{R}}$ such that $\omega_f = v_1^\vee \wedge v_2^\vee + cv_3^\vee \wedge v_4^\vee$. By (3.3.3), $\mathrm{Pf}(f)$ is real, and hence $c \in \mathbb{R}$. It follows that $(\sigma,\overline{\sigma}) = (\sigma,\sigma) = 0$, and that contradicts the hypothesis that $[\sigma] \in \mathscr{D}$. ■

Example 3.10. Let $K_n(A)$ be a generalized Kummer of dimension $2n$. We adopt the identifications of Example 3.5, and we set $\vartheta = \vartheta(\overline{q}^{n-2})$. Let $\Gamma \subset (V_{\mathbb{C}} \oplus V^\vee_{\mathbb{C}})$ be the graph of a non-degenerate linear map such that there exists an HK of Kummer type of dimension

$2n$ and a Gauss–Manin parallel transport operator $H^3(X) \xrightarrow{\sim} (V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee)$ sending $H^{2,1}(X)$ to Γ. By Key observation 1.2 and Proposition 3.3, f is skew-symmetric and

$$\mathrm{Pf}(f) = \frac{\vartheta_1(\overline{q}^{n-2})}{\vartheta_2(\overline{q}^{n-2})} = \frac{1}{(n+1)}. \tag{3.4.18}$$

Conversely, let f be a generic skew-symmetric map as above, such that (3.4.18) holds, and let Γ be the graph of f. Then by Theorem 3.9 there exists an HK of Kummer type of dimension $2n$ and a Gauss–Manin parallel transport operator $H^3(X) \xrightarrow{\sim} (V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee)$ sending $H^{2,1}(X)$ to Γ.

3.5 The compact complex torus associated to a point of $\mathscr{D}$

Definition 3.11. Keep notation and hypotheses as in Theorem 3.9. We let

$$J_{[\sigma]}(\vartheta) := (V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee)/(H^{1,0}_{[\sigma]}(\vartheta) + (V \oplus V^\vee)). \tag{3.5.1}$$

Thus, $J_{[\sigma]}(\vartheta)$ is a compact complex torus of dimension 4.

Example 3.12. Set $m = 1/2(n+1)$, and $\vartheta = \vartheta(\overline{q}_X^{n-2})$, where X is an HK of Kummer type, of dimension $2n$. Going through the identifications in Example 3.5, we may identify $\mathrm{Ann}F^1(X)$ with a point $[\sigma] \in \mathscr{D}$. We recall that the integral cohomology $H^3(X;\mathbb{Z})$ is identified with $V \oplus V^\vee$, see Theorem 2.7. Thus, we have an isomorphism

$$J_{[\sigma]}(\vartheta) \xrightarrow{\sim} J^3(X). \tag{3.5.2}$$

For a very general $[\sigma] \in \mathscr{D}$ the torus $J_{[\sigma]}(\vartheta)$ is not projective. We will prove that if there exists a rational class of positive square in the orthogonal $\sigma^\perp$, then $J_{[\sigma]}(\vartheta)$ is an abelian variety.

Definition 3.13. Let $h \in (\bigwedge^2 V \oplus \mathbb{Z})^\vee$ be non-zero. We let

$$\mathscr{D}_h := \left\{[\sigma] \in \mathscr{D} \mid \langle h, \sigma \rangle = 0\right\},$$

where $\langle , \rangle$ is the duality pairing.

Definition 3.14. Since the bilinear symmetric form defined in (3.4.1) is non-degenerate, it defines a rational isomorphism

$$\left(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C}\right)^{\vee} \xrightarrow{\sim} \left(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C}\right), \tag{3.5.3}$$

and hence also a rational bilinear symmetric form on $(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C})^{\vee}$, that will be denoted $(,)^{\vee}$.

Remark 3.15. Let $h \in (\bigwedge^2 V \oplus \mathbb{Z})^{\vee}$ be a class of (strictly) positive square. Then $\mathscr{D}_h$ is not connected, in fact it has two connected components, interchanged by conjugation. Each connected component of $\mathscr{D}_h$ is a Type IV bounded symmetric domain.

Proposition 3.16. Let $h \in (\bigwedge^2 V \oplus \mathbb{Z})^{\vee}$ be a class of strictly positive square. Let $\sigma \in \mathscr{D}_h$. Then

$$\langle i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha}), h\rangle \neq 0 \quad \forall \alpha \in H^{1,0}_{[\sigma]}(\vartheta) \setminus \{0\}. \tag{3.5.4}$$

Proof. First of all, notice that $i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha}) \in (\bigwedge^2 V_{\mathbb{R}} \oplus \mathbb{R})$. Suppose that (3.5.4) does not hold, and that $\alpha \in H^{1,0}_{[\sigma]}(\vartheta)$ provides a counterexample, we will arrive at a contradiction. Let $\ell \in (\bigwedge^2 V \oplus \mathbb{Q})$ be the class corresponding to h via the isomorphism in (3.5.3). The restriction to $(\bigwedge^2 V_{\mathbb{R}} \oplus \mathbb{R})$ of the bilinear symmetric form $(,)$ has signature $(3,4)$. The real subspace $W \subset (\bigwedge^2 V_{\mathbb{R}} \oplus \mathbb{R})$ spanned by ℓ and $\{c\sigma + \overline{c\sigma} \mid c \in \mathbb{C}\}$ is 3D and the restriction of $(,)$ to W is positive definite because $(\ell, \ell) > 0$. Since Φ_{ϑ} is a morphism of Hodge structures, and since (3.5.4) does not hold, $i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha})$ is orthogonal to W. It follows that $(i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha}), i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha})) \leq 0$, with equality only if $i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha}) = 0$. Let $\alpha = (v, g)$, where $v \in V_{\mathbb{C}}$ and $g \in V_{\mathbb{C}}^{\vee}$. By Theorem 3.9

$$g(v) = 0. \tag{3.5.5}$$

We have

$$\Phi_{\vartheta}(\alpha \wedge \overline{\alpha}) = \vartheta_1 v \wedge \overline{v} + \vartheta_2 \iota(g \wedge \overline{g}) + 2\vartheta_3 i \operatorname{Im} g(\overline{v})\zeta. \tag{3.5.6}$$

(In the above equation $\operatorname{Im} g(\overline{v})$ is the imaginary part of $g(\overline{v})$.) Thus,

$$\begin{aligned}(i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha}), i\Phi_{\vartheta}(\alpha \wedge \overline{\alpha})) = &-\mathrm{vol}((\vartheta_1 v \wedge \overline{v} + \vartheta_2 \iota(g \wedge \overline{g}) \wedge (\vartheta_1 v \wedge \overline{v} + \vartheta_2 \iota(g \wedge \overline{g})) \\ &- 4m\vartheta_3^2(\operatorname{Im} g(\overline{v}))^2 \\ &= 2\vartheta_1\vartheta_2|g(\overline{v})|^2 - 4m\vartheta_3^2(\operatorname{Im} g(\overline{v}))^2 = 2\vartheta_1\vartheta_2(|g(\overline{v})|^2 - (\operatorname{Im} g(\overline{v}))^2).\end{aligned} \tag{3.5.7}$$

(The 2nd-to-last equality follows from (3.5.5), the last equality follows from (3.4.5).) Since $(i\Phi_{\vartheta}(\alpha\wedge\overline{\alpha}), i\Phi_{\vartheta}(\alpha\wedge\overline{\alpha}))\le 0$, with equality only if $\Phi_{\vartheta}(\alpha\wedge\overline{\alpha})=0$, it follows that

$$g(\overline{v}) = 0, \tag{3.5.8}$$

$$\vartheta_1 v\wedge\overline{v}+\vartheta_2\iota(g\wedge\overline{g}) = 0. \tag{3.5.9}$$

By (3.4.17) one (at least) among $v\wedge\overline{v}$ and $g\wedge\overline{g}$ is non-zero. Since ϑ_1 and ϑ_2 are both non-zero, it follows that

$$v\wedge\overline{v}\neq 0,\qquad g\wedge\overline{g}\neq 0. \tag{3.5.10}$$

Either Item (1) or Item (2) of Theorem 3.9 holds. We deal separately with the two cases.

Suppose that Item (1) holds. There exists a basis $\{v_1,\ldots,v_4\}$ of $V_{\mathbb{R}}$ such that $v=v_1+iv_2$ and $\mathrm{vol}(v_1\wedge\ldots\wedge v_4)=1$. Let $A=(a_{ij})$ be the matrix of $f\colon V_{\mathbb{C}}\to V_{\mathbb{C}}^{\vee}$ with respect to the bases $\{v_1,\ldots,v_4\}$ and $\{v_1^{\vee},\ldots,v_4^{\vee}\}$. Thus, $A^t=-A$. The linear function g is equal to $f(v)$. Since $\langle f(v),\overline{v}\rangle=0$, we have $a_{12}=0$. Thus,

$$\mathrm{Pf}(f)=a_{14}a_{23}-a_{13}a_{24}=\frac{\vartheta_1}{\vartheta_2}. \tag{3.5.11}$$

(The 2nd equality follows from (3.3.3).) Next we notice that by (3.4.5), the inequality $(\vartheta_2\iota(\omega_f)-2\vartheta_3\zeta,\vartheta_2\overline{\iota(\omega_f)}-2\vartheta_3\zeta)>0$ is equivalent to

$$\left(\iota(\omega_f),\overline{\iota(\omega_f)}\right)>\frac{2\vartheta_1}{\vartheta_2}. \tag{3.5.12}$$

We will write the above inequality in an equivalent form. Straightforward computations give

$$\begin{aligned}\begin{vmatrix} a_{13}-\overline{a}_{13} & a_{14}-\overline{a}_{14}\\ a_{23}-\overline{a}_{23} & a_{24}-\overline{a}_{24}\end{vmatrix} &= -\mathrm{Pf}(f)-\overline{\mathrm{Pf}(f)}+2\mathrm{Re}(a_{14}\overline{a}_{23}-a_{13}\overline{a}_{24})\\ &= -\frac{2\vartheta_1}{\vartheta_2}+2\mathrm{Re}(a_{14}\overline{a}_{23}-a_{13}\overline{a}_{24})=-\frac{2\vartheta_1}{\vartheta_2}+\left(\iota(\omega_f),\overline{\iota(\omega_f)}\right).\end{aligned} \tag{3.5.13}$$

Let D be the real number such that

$$4D = \begin{vmatrix} a_{13} - \overline{a}_{13} & a_{14} - \overline{a}_{14} \\ a_{23} - \overline{a}_{23} & a_{24} - \overline{a}_{24} \end{vmatrix}$$

By (3.5.12) and (13), we have

$$D > 0. \tag{3.5.14}$$

Straightforward computations give

$$\begin{aligned} v \wedge \overline{v} &= -2iv_1 \wedge v_2, \\ \iota(f(v) \wedge \overline{f(v)}) &= 2i\left(\mathrm{Re}(a_{14}\overline{a}_{23} - a_{13}\overline{a}_{24}) + \mathrm{Im}(a_{13}\overline{a}_{14} + a_{23}\overline{a}_{24})\right) v_1 \wedge v_2. \end{aligned}$$

Using (13), we get that (3.5.9) holds if and only if

$$2D + \mathrm{Im}(a_{13}\overline{a}_{14} + a_{23}\overline{a}_{24}) = 0. \tag{3.5.15}$$

Now let

$$a_{13} = x_1 + iy_1, \quad a_{14} = x_2 + iy_2, \quad a_{23} = x_3 + iy_3, \quad a_{24} = x_4 + iy_4, \quad x_k, y_k \in \mathbb{R}.$$

Writing (3.5.15) and the equation $\mathrm{ImPf}(f) = 0$ in terms of $x_1, \ldots, x_4, y_1, \ldots, y_4$, we get that

$$x_1y_2 - x_2y_1 = -x_3y_4 + x_4y_3 + 2D, \tag{3.5.16}$$

$$x_1y_4 - x_2y_3 = x_3y_2 - x_4y_1. \tag{3.5.17}$$

By Cramer's formula

$$Dx_1 = -x_3y_3y_4 + x_4y_3^2 - x_3y_1y_2 + x_4y_1^2 + 2Dy_3, \tag{3.5.18}$$

$$Dx_2 = -x_3y_4^2 + x_4y_3y_4 - x_3y_2^2 + x_4y_1y_2 + 2Dy_4. \tag{3.5.19}$$

Writing out the equation $\mathrm{RePf}(f) = \vartheta_1/\vartheta_2$ in terms of $x_1, \ldots, x_4, y_1, \ldots, y_4$, multiplying it by D, replacing Dx_1 by the expression in the right-hand side of (3.5.18), and Dx_2 by the expression in the right-hand side of (3.5.19), we get that

$$-(D - x_3y_4 + x_4y_3)^2 - (x_3y_2 - x_4y_1)^2 = D\frac{\vartheta_1}{\vartheta_2}. \tag{3.5.20}$$

Since D and ϑ_1/ϑ_2 are strictly positive by (3.5.14) and (3.4.5), respectively, the above equation is absurd. We have reached a contradiction if Item (1) of Theorem 3.9 holds.

Now suppose that Item (2) holds. By (3.4.17) both $v \wedge \overline{v}$ and $g \wedge \overline{g}$ are non-zero. Complete v to a basis $\{v, w\}$ of U. The inequality $(\sigma, \overline{\sigma}) > 0$ translates into

$$\mathrm{vol}(v \wedge w \wedge \overline{v} \wedge \overline{w}) > 0. \tag{3.5.21}$$

Since $g(v) = g(\overline{v}) = 0$, we have

$$\iota(g \wedge \overline{g}) = \lambda v \wedge \overline{v}, \quad \lambda \in \mathbb{R}^*. \tag{3.5.22}$$

Moreover,

$$\mathrm{vol}(\lambda v \wedge \overline{v} \wedge w \wedge \overline{w}) = \langle g \wedge \overline{g}, w \wedge \overline{w}\rangle = g(w) \cdot \overline{g}(\overline{w}) - \overline{g}(w) \cdot g(\overline{w}) = -|g(\overline{w})|^2 < 0.$$

(Recall that $g \in U^{\perp}$.) By (3.5.21) it follows that $\lambda > 0$. This contradicts (3.5.9) because ϑ_1 and ϑ_2 have the same sign by (3.4.5). ■

Let $h \in (\bigwedge^2 V \oplus \mathbb{Z})^{\vee}$. We define the following skew-symmetric bilinear form on $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^{\vee}$:

$$\langle \alpha, \beta \rangle_{\vartheta, h} := \langle h, \Phi_{\vartheta}(\alpha \wedge \beta)\rangle. \tag{3.5.23}$$

Note that $\langle , \rangle$ in the right-hand side of the above equation denotes the duality pairing.

Definition 3.17. Let $h \in (\bigwedge^2 V \oplus \mathbb{Z})^{\vee}$. Let $\sigma \in \mathscr{D}_h$. Restricting the skew-symmetric bilinear form in (3.5.23) to $H^{0,1}_{[\sigma]}(\vartheta) \times H^{1,0}_{[\sigma]}(\vartheta)$, that is the product of the tangent space at the origin of $J_{[\sigma]}(\vartheta)$ and its complex conjugate, we get a translation invariant rational $(1,1)$-form on $J_{[\sigma]}(\vartheta)$. We let

$$\Theta_{[\sigma]}(\vartheta) \in H^{1,1}_{\mathbb{Q}}(J_{[\sigma]}(\vartheta)) \tag{3.5.24}$$

be the corresponding cohomology class.

Proposition 3.18. Let $h \in (\bigwedge^2 V \oplus \mathbb{Z})^\vee$ be a class of positive square. For one of the two connected components of $\mathscr{D}_h$, call it $\mathscr{D}_h^+$, the following holds. Let $[\sigma] \in \mathscr{D}_h^+$; then the cohomology class $\Theta_{[\sigma]}(\vartheta)$ is ample on $J_{[\sigma]}(\vartheta)$.

Proof. Let $\mathscr{D}_h^1, \mathscr{D}_h^2$ be the two connected components of $\mathscr{D}_h$. The set $\mathscr{V}^j$ of couples $([\sigma], \alpha)$ where $[\sigma] \in \mathscr{D}_h^j$ and $0 \neq \alpha \in H^{0,1}_{[\sigma]}(\vartheta)$ is the complement of the zero section in a complex vector bundle over the connected space $\mathscr{D}_h^j$. Thus, $\mathscr{V}^j$ is connected. By Proposition 3.16, the real number

$$\langle i\Phi_\vartheta(\alpha \wedge \overline{\alpha}), h\rangle \tag{3.5.25}$$

is either strictly positive for all $([\sigma], \alpha) \in \mathscr{V}_h^j$, or always strictly negative. Conjugation $([\sigma], \alpha) \mapsto ([\overline{\sigma}], \overline{\alpha})$ maps bijectively $\mathscr{V}^1$ to $\mathscr{V}^2$. Since conjugation changes sign to the number in (3.5.25), the proposition follows. ∎

Example 3.19. Let us go back to Example 3.12, and assume that X is projective. Let L be an ample line bundle on X. Referring to Example 3.5, $c_1(L)$ gets identified with an element of $h \in (\bigwedge^2 V_{\mathbb{Z}} \oplus \mathbb{Z}\zeta)^\vee$ (see (3.4.2)) of positive square. By Remark 3.1, the bilinear form (3.5.23) is identified, via the isomorphism in (3.5.2), with the bilinear form

$$\begin{array}{ccc} H^3(X) \times H^3(X) & \longrightarrow & \mathbb{C} \\ (\alpha, \beta) & \mapsto & \int_{[X]} \alpha \smile \beta \smile \overline{q}_X^{n-2} \smile c_1(L). \end{array} \tag{3.5.26}$$

It follows that if $n = 2$, the isomorphism in (3.5.2) matches $\Theta_{[\sigma]}(\vartheta)$ and the polarization Θ_L of $J^3(X)$. Later on we will show that an analogous statement holds also for $n > 2$.

3.6 A rank 7 sub local system of the local system with fiber $S^+(X)$

Let $\mathbf{q}$ be the integral unimodular bilinear symmetric form on $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee$ defined by

$$\begin{array}{ccc} V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee & \xrightarrow{\mathbf{q}} & \mathbb{C} \\ (v, \ell) & \mapsto & 2\ell(v) \end{array} \tag{3.6.1}$$

(See (2.8.1).) Let $\mathbf{Q} := V(q) \subset \mathbb{P}(V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee)$ be the zero locus of the quadratic form q. One of the two spinor representations of $O(\mathbf{q})$ may be identified with $S^+ := \bigwedge^{ev} V_{\mathbb{C}} = \mathbb{C} \oplus \bigwedge^2 V_{\mathbb{C}} \oplus \bigwedge^4 V_{\mathbb{C}}$. We recall the identification of a specific quadric hypersurface in $\mathbb{P}(S^+)$ with one of the two irreducible components of the variety parametrizing 3D linear subspaces of $\mathbf{Q}$, see §20.3 in [7]. Denote elements of $\bigwedge^{ev} V_{\mathbb{C}}$ as follows:

$$\alpha+\eta+\beta, \qquad \alpha\in\mathbb{C},\ \eta\in\bigwedge^2 V_{\mathbb{C}},\ \beta\in\bigwedge^4 V_{\mathbb{C}}. \tag{3.6.2}$$

Let $\mathbf{q}^+$ be the integral unimodular bilinear symmetric form on $\bigwedge^{ev} V_{\mathbb{C}}$ defined by

$$\begin{array}{ccc} \bigwedge^{ev} V_{\mathbb{C}} & \xrightarrow{\mathbf{q}^+} & \mathbb{C} \\ \alpha+\eta+\beta & \mapsto & \mathrm{vol}(\eta\wedge\eta-2\alpha\beta) \end{array} \tag{3.6.3}$$

Let $\mathbf{Q}^+\subset\mathbb{P}(\bigwedge^{ev} V_{\mathbb{C}})$ be the set of zeroes of $\mathbf{q}^+$.

Given $\ell\in V_{\mathbb{C}}^{\vee}$ and $\eta\in\bigwedge^{\bullet} V_{\mathbb{C}}$, we let $\ell(\eta)$ be the contraction of ℓ and η. Given $[\alpha+\eta+\beta]\in\mathbf{Q}^+$, we let

$$Z_{[\alpha+\eta+\beta]} := \Big\{[(v,\ell)]\in\mathbb{P}(V_{\mathbb{C}}\oplus V_{\mathbb{C}}^{\vee}) \mid \alpha v+\ell(\eta)+\eta\wedge v+\ell(\beta)=0\Big\}. \tag{3.6.4}$$

In other words, $Z_{[\alpha+\eta+\beta]}$ is the projectivization of the subspace of $V\oplus V^{\vee}$ (embedded in the Clifford algebra $Cl(V_{\mathbb{C}}\oplus V_{\mathbb{C}}^{\vee},\mathbf{q})$) of vectors, which kill the element $(\alpha+\eta+\beta)\in S^+$.

Lemma 3.20. Let $[\alpha+\eta+\beta]\in\mathbf{Q}^+$, and suppose that $\eta\wedge\eta\neq 0$. Then $Z_{[\alpha+\eta+\beta]}$ is the graph of a non-degenerate skew-symmetric map $f\colon V_{\mathbb{C}}\to V_{\mathbb{C}}^{\vee}$ such that

$$\mathrm{vol}(\beta)\iota(\omega_f)=-\eta. \tag{3.6.5}$$

Proof. A straightforward argument shows that

$$Z_{[\alpha+\eta+\beta]} = \Big\{[(v,\ell)]\in\mathbb{P}(V_{\mathbb{C}}\oplus V_{\mathbb{C}}^{\vee}) \mid \alpha v+\ell(\eta)=0\Big\}. \tag{3.6.6}$$

From (3.6.6) we get that $Z_{[\alpha+\eta+\beta]}$ is the graph of a non-degenerate map $f\colon V_{\mathbb{C}}\to V_{\mathbb{C}}^{\vee}$. Since $0=\ell(\alpha v+\ell(\eta))=\alpha\ell(v)$, the map f is skew-symmetric.

Lastly, we prove (3.6.5). We may choose a basis $\{v_1,\ldots,v_4\}$ of $V_{\mathbb{C}}$ of volume 1 such that $\eta=v_1\wedge v_2+tv_3\wedge v_4$, for some $t\in\mathbb{C}^*$. A computation gives that

$$\omega_f=-\alpha v_1^{\vee}\wedge v_2^{\vee}-\frac{\alpha}{t}v_3^{\vee}\wedge v_4^{\vee}.$$

Equation (3.6.5) follows from the above equality. ■

The (well-known) result below follows easily from Lemma 3.20.

Proposition 3.21. Let $[\alpha + \eta + \beta] \in \mathbf{Q}^+$. Then $Z_{[\alpha+\eta+\beta]}$ is a 3D linear subspace of $\mathbf{Q}$, and it is the graph of a non-degenerate skew-symmetric map $f\colon V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}$ if and only if $\eta \wedge \eta \neq 0$. The map assigning $Z_{[\alpha+\eta+\beta]}$ to $[\alpha + \eta + \beta] \in \mathbf{Q}^+$ is an isomorphism between $\mathbf{Q}^+$ and one of the two irreducible components of the variety parametrizing maximal dimensional linear subspaces of $\mathbf{Q}$.

Remark 3.22. Let $U \subset V_{\mathbb{C}}$ be a 2D subspace. Let η be a generator of $\bigwedge^2 U \subset \bigwedge^2 V_{\mathbb{C}}$. Then $[0, \eta, 0] \in \mathbf{Q}^+$, and $Z_{[\eta]} = \mathbb{P}(U \oplus U^{\perp})$.

The following definition makes sense by the Key observation 1.2, Proposition 3.3, Lemma 3.20, and Remark 3.22.

Definition 3.23. Let X be an HK of Kummer type, of dimension $2n$. Let $T^+(X) \subset S^+(X)$ be the minimal vector subspace such that $\mathbb{P}(T^+(X))$ contains all $x \in \mathbf{Q}^+(X)$ parametrizing a 3D linear space $\mathbb{P}(\Gamma) \subset \mathbf{Q}(X)$ for which there exist an HK Y of Kummer type of dimension $2n$, and a parallel transport operator $g\colon H^3(Y) \xrightarrow{\sim} H^3(X)$ such that $g(H^{2,1}(X)) = \Gamma$.

Remark 3.24. Let $\pi : \mathscr{X} \to B$ be a family of HK's of Kummer type. Let $b_0, b_1 \in B$ and let $X_0 := \pi^{-1}(b_0), X_1 := \pi^{-1}(b_1)$. Let λ be an arc starting in b_0 and ending in b_1. By Definition 3.23 the induced isomorphism $S^+(\lambda)\colon S^+(X_0) \to S^+(X_1)$ maps $T^+(X_0)$ to $T^+(X_1)$.

Proposition 3.25. Keeping notation as above, the following hold:

(1) Suppose that $X = K_n(A)$. As usual, identify $H^3(A;\mathbb{Z}) \oplus H^1(A;\mathbb{Z})$ with $H^3(K_n(A);\mathbb{Z})$ via Theorem 2.7, and identify $H^1(A;\mathbb{Z})$ with $H^3(A;\mathbb{Z})^{\vee}$ via (2.3.8). Then

$$T^+(K_n(A)) = \left\{ (\alpha + \eta + \beta) \in \bigwedge^{ev} H^3(A) \mid (n+1)\alpha - \mathrm{vol}(\beta) = 0 \right\}. \tag{3.6.7}$$

(2) If $\pi : \mathscr{X} \to B$ is a family of HK's of Kummer type, then there is a sub local system of rank 7 of $S^+(\pi)$ with fiber $T^+(\pi^{-1}(b))$ over $b \in B$.

Proof. Let us prove Item (1). Suppose that $Z_{[\alpha+\eta+\beta]} = \mathbb{P}(\Gamma)$ where $\Gamma \subset (V_{\mathbb{C}} \oplus V_{\mathbb{C}}^{\vee})$ is as in Definition 3.23. By Proposition 3.3, either Γ is the graph of a non-degenerate antisymmetric map $f\colon V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}$ such that $\mathrm{Pf}(f) = 1/(n+1)$ (see Example 3.10 for the last equality), or $\Gamma = U \oplus U^{\perp}$, where $U \subset V_{\mathbb{C}}$ is a 2D subspace. In the 1st case, by (3.6.5) we have

$$\frac{1}{(n+1)}=\mathrm{Pf}(f)=\frac{1}{2}\mathrm{vol}(\omega_f\wedge\omega_f)=\frac{1}{2\mathrm{vol}(\beta)^2}\mathrm{vol}(\eta\wedge\eta)=\frac{\alpha}{\mathrm{vol}(\beta)}.$$

Thus, $[\alpha+\eta+\beta]$ belongs to the right-hand side of (3.6.7). In the 2nd case, the same holds because $\alpha=\beta=0$. This proves that $T^+(K_n(A))$ is contained in the right-hand side of (3.6.7). On the other hand (see Example 3.10) there is a dense (in the Zariski topology) open (in the classical topology) subset of $\mathbb{P}(T^+(K_n(A)))\cap\mathbf{\Omega}^+(K_n(A))$ parametrizing Γ's as in Definition 3.23. Item (1) follows.

As noticed in Remark 3.24, there is a sub local system of $S^+(\pi)$ with fiber $T^+(\pi^{-1}(b))$ over $b\in B$. It has rank 7 by Item (1). ■

3.7 Proof of the 2nd main result

Item (1) of Theorem 1.3 follows from Definition 3.23 and Item (2) of Proposition 3.25.

Next we proceed to prove Item (2) of Theorem 1.3. Assume first that $X=K_n(A)$. As usual identify $H^2(K_n(A))^\vee=\bigwedge^2 H^3(A)\oplus\mathbb{C}\xi_n^\vee$, see Example 3.5. With this identification, $S^+(K_n(A))=\mathbb{C}\oplus\bigwedge^2 H^3(A)\oplus\bigwedge^4 H^3(A)$. Let $\tau\in\bigwedge^4 H^3(A)$ be the element of volume 1. We let

$$\begin{array}{ccc}\bigwedge^2 H^3(A)\oplus\mathbb{C}\xi_n^\vee & \overset{i}{\hookrightarrow} & S^+(K_n(A))\\ \eta+x\xi_n^\vee & \mapsto & \frac{(-1)^{\epsilon_n}x}{2(n+1)}+\eta+\frac{(-1)^{\epsilon_n}x}{2}\tau,\end{array}\tag{3.7.1}$$

where $\epsilon_n\in\{0,1\}$ is as in corollary 3.7. By Item (1) of Proposition 3.25, the above map defines an isomorphism between $H^2(K_n(A))^\vee$ and $T^+(K_n(A))$. Let $\mathbf{q}^+_{K_n(A))}$ be the quadratic form on $S^+(K_n(A))$ defined by (3.6.3) with $V=H^3(A;\mathbb{Z})$. One checks easily that the restriction of $\mathbf{q}^+_{K_n(A))}$ to $H^2(K_n(A))^\vee$ is the dual of the BBF. This proves Item (2) of Theorem 1.3 for $X=K_n(A)$.

In order to prove Item (2) of Theorem 1.3 for an arbitrary HK of Kummer type, we first give the auxiliary result below.

Claim 3.26. The map i in (3.7.1) is equivariant up to sign for the action of monodromy.

Proof. Let $\pi\colon\mathscr{X}\to B$ be a family of HK's of Kummer type, with the fiber $\pi^{-1}(b_0)$ isomorphic to $K_n(A)$. Let λ be a loop in B based at b_0. Then λ defines a diffeomorphism $\lambda_*\colon K_n(A)\to K_n(A)$. Let $H^2(\lambda_*)^t$ be the action of λ_* on $H^2(K_n(A))^\vee$, and let $S^+(\lambda_*)$ be the action of λ_* on $S^+(K_n(A))$. We must prove that

$$S^+(\lambda_*)\circ i=\pm i\circ H^2(\lambda_*)^t.\tag{3.7.2}$$

First, $S^+(\lambda_*)$ maps $T^+(K_n(A))$ to itself, see Remark 3.24. Next, let $\Gamma \subset H^3(K_n(A))$ be a 4D vector subspace such that $\phi(\bigwedge^2\Gamma)$ has dimension 1. Then, $S^+(\lambda_*)\circ i(\phi(\bigwedge^2\Gamma)) = i\circ H^2(\lambda_*)^t\phi(\bigwedge^2\Gamma)$ by Item (2) (which has been proved for $X = K_n(A)$). The set of elements of $i(H^2(K_n(A))^\vee \cap \mathbf{\Omega}^+(K_n(A))$ of the form $\phi(\bigwedge^2\Gamma)$ for Γ as above is an open dense subset of $i(H^2(K_n(A))^\vee \cap \mathbf{\Omega}^+(K_n(A))$. Equation (3.7.2) follows. ■

Now we may define $i\colon H^2(X)^\vee \overset{i}{\hookrightarrow} S^+(X)$, for X an arbitrary HK of Kummer type, acting by parallel transport on the map i in (3.7.1). The map is well-defined up to sign, that is independent (up to sign) of the chosen path connecting $K_n(A)$ to X (in a connected family of HK's in which both $K_n(A)$ and X are fibers), because of Claim 3.26. Item (2) of Theorem 1.3 for X follows from the corresponding statements for $K_n(A)$.

Proof of Corollary 1.4. We may assume that $X = K_n(A)$. Thus, we have the identifications of Proposition 3.25. Let $\pi : \mathscr{X} \to B$ be a family of HK's of Kummer type, with fiber $\pi^{-1}(b_0)$ isomorphic to $K_n(A)$. Let λ be a loop in B based at b_0, and let $\lambda_*\colon K_n(A) \to K_n(A)$ be the associated diffeomorphism. We have associated maps

$$H^2(\lambda_*) \in O(H^2(K_n(A);\mathbb{Z}), q_{K_n(A)}), \quad S^+(\lambda_*) \in O(S^+(K_n(A)), \mathbf{q}^+_{K_n(A)}).$$

By (3.7.2) we have

$$H^2(\lambda_*)^t = \pm i^{-1}\circ S^+(\lambda_*)\circ i. \tag{3.7.3}$$

Next, we notice that the map i in (3.7.1) embeds $H^2(K_n(A);\mathbb{Z})$ as a saturated sublattice of $S^+(K_n(A))_{\mathbb{Z}} := \mathbb{Z}\oplus\bigwedge^2 V\oplus\bigwedge^4 V$. In fact, we have a chain of maps

$$H^2(K_n(A);\mathbb{Z}) \hookrightarrow H^2(K_n(A);\mathbb{Q}) \xrightarrow{\sim} H^2(K_n(A);\mathbb{Q})^\vee \hookrightarrow S^+(K_n(A)). \tag{3.7.4}$$

(The 1st map is the natural embedding, the 2nd map is defined by the non-degenerate BBF quadratic form, and the last map is defined by i.) The composition of the maps in (3.7.4) is an isometric embedding

$$j\colon H^2(K_n(A);\mathbb{Z}) \hookrightarrow S^+(K_n(A))_{\mathbb{Z}}.$$

The image of j is described as follows. Let $\tau \in \bigwedge^4 H^3(A;\mathbb{Z})$ be the element of volume 1. Let

$$u := 1-(n+1)\tau, \quad w := 1+(n+1)\tau. \tag{3.7.5}$$

Then $u, w \in S^+(K_n(A))_{\mathbb{Z}}$, $\mathrm{Im}(j) = u^\perp \cap S^+(K_n(A))_{\mathbb{Z}}$, and $w \in \mathrm{Im}(j)$.

Thus, we examine the restriction of $S^+(\lambda_*)$ to $u^\perp$. In order to simplify notation, we let $\rho := S^+(\lambda_*)$. Since ρ maps $\mathrm{Im}(j)$ to iself, we have $\rho(u) = \pm u$. Next, we notice that $\mathrm{Det}\rho = 1$, because by Item (3) of Theorem 1.1 monodromy does not exchange the two irreducible components of the variety parametrizing maximal linear subspaces of $\mathbf{Q}(K_n(A))$. It follows that the determinant of $H^2(\lambda_*)$ equals 1 if $\rho(u) = u$, and it equals (-1) if $\rho(u) = -u$.

Moreover the discriminant group of $H^2(K_n(A);\mathbb{Z})$ is generated by $\xi_n^\vee$, and is isomorphic to $\mathbb{Z}/(2n+2)$. Since $i(\xi_n^\vee) = w$, we must prove that

$$\rho(w) = \begin{cases} (1+2a(n+1))w + 2(n+1)y, \ \ a \in \mathbb{Z},\ y \in \mathrm{Im}(j) & \text{if } \rho(u) = u, \\ (-1+2a(n+1))w + 2(n+1)y, \ \ a \in \mathbb{Z},\ y \in \mathrm{Im}(j) & \text{if } \rho(u) = -u. \end{cases} \tag{3.7.6}$$

Suppose that $\rho(u) = u$. Then $\rho(w) - u = \rho(w-u) = \rho(2(n+1)\tau) = 2(n+1)\rho(\tau)$. Since $\rho(\tau) \perp u$, we get that there exist $a \in \mathbb{Z}$ and $y \in \mathrm{Im}(j)$ such that $\rho(\tau) = aw + \tau + y$. Thus,

$$\rho(w) = u + 2(n+1)\tau + 2a(n+1)w + 2(n+1)y = (1+2a(n+1))w + 2(n+1)y. \tag{3.7.7}$$

This proves (3.7.6) if $\rho(u) = u$. The proof in the case $\rho(u) = -u$ is similar. ■

4 Polarization Type of $J^3(X)$ for X of Dimension 4

If X is a polarized HK of Kummer type, then $J^3(X)$ is a 4D abelian variety, with a polarization associated to the polarization of X. In the present subsection we compute the discrete invariants (elementary divisors) of the polarization of $J^3(X)$, for X of dimension 4.

4.1 Set up

Let X be an HK fourfold of Kummer type, and let L be an ample line bundle on X. The skew-symmetric form

$$\begin{array}{ccc} H^3(X) \times H^3(X) & \xrightarrow{\langle,\rangle} & \mathbb{C} \\ (\alpha,\beta) & \mapsto & \int_X \alpha \smile \beta \smile c_1(L) \end{array} \tag{4.1.1}$$

defines a polarization Θ_L of $J^3(X)$ by the Hodge–Riemann bilinear relations. We may assume that X is a generalized Kummer $K_2(A)$. We recall that (2.2.3) and the map F in

(2.3.1) give identifications

$$H^2(K_2(A);\mathbb{Z}) = \bigwedge^2 H^1(A;\mathbb{Z}) \oplus \mathbb{Z}\xi_2, \tag{4.1.2}$$

$$H^3(K_2(A);\mathbb{Z}) = H^3(A;\mathbb{Z}) \oplus H^1(A;\mathbb{Z}). \tag{4.1.3}$$

We identify $H^1(A;\mathbb{Z})$ with $H^3(A;\mathbb{Z})^\vee$ via (2.3.8), and we set $V := H^3(A;\mathbb{Z})$. Let $\{v_1,\ldots,v_4\}$ be a basis of V such that $\mathrm{vol}(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = 1$. We may write

$$c_1(L) = c\big(ev_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee\big) + s\zeta^\vee, \tag{4.1.4}$$

where

$$c \in \mathbb{N}_+, \quad s \in \mathbb{Z}, \quad \gcd\{c,s\} = 1. \tag{4.1.5}$$

Let $(w,f),(w',f') \in (V \oplus V^\vee)$. By corollary 3.7 and Remark 3.8 we have

$$\langle (w,f),(w',f')\rangle = \langle -w\wedge w' - 3\iota(f\wedge f') - 3(\langle w,f'\rangle - \langle w',f\rangle)\zeta, c(ev_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee) + s\zeta^\vee\rangle.$$

In the above equation, the angle brackets in the left-hand side denote the polarization form in (4.1.1), those in the right-hand side denote the duality pairing. Let

$$\begin{aligned} &\alpha_1 := (0,v_1^\vee),\ \alpha_2 := (v_2,0),\ \alpha_3 := (v_4,0),\ \alpha_4 := (0,v_3^\vee), \\ &\beta_1 := (0,-v_2^\vee),\ \beta_2 := (v_1,0),\ \beta_3 := (v_3,0),\ \beta_4 := (0,-v_4^\vee). \end{aligned} \tag{4.1.6}$$

Then $\{\alpha_1,\ldots,\beta_4\}$ is a basis of $V \oplus V^\vee$, and both the span of the α_i's and the span of the β_j's are $\langle,\rangle_{\vartheta,h}$-isotropic subgroups of $V \oplus V^\vee$. The intersection matrix between the α_i's and the β_j's is equal to

$$(\langle \alpha_i,\beta_j\rangle_{\vartheta,h}) = \begin{pmatrix} 3c & 3s & 0 & 0 \\ 3s & c\cdot e & 0 & 0 \\ 0 & 0 & c & 3s \\ 0 & 0 & 3s & 3c\cdot e \end{pmatrix} \tag{4.1.7}$$

Let X be a 4 dimensional generalized Kummer. By Corollary 4.8 in [16] (the proof is in [15]), non-zero elements $\alpha \in H^2(X;\mathbb{Z})$ up to monodromy are classified by the value $q_X(\alpha)$ and by the divisibility $\mathrm{div}(\alpha)$ (see Subsection 1.4 for the definition of $\mathrm{div}(\alpha)$). The divisibility is an element of $\{1,2,3,6\}$. Thus, we distinguish four cases.

4.2 **Divisibility** 1

Suppose that

$$c_1(L) = ev_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee, \tag{4.2.1}$$

that is $c = 1$ and $s = 0$ in the notation of (4.1.4). Then $(c_1(L), c_1(L)) = 2e$, and $\mathrm{div}(c_1(L)) = 1$. Let $g := \gcd\{3, e\}$, and let x, y be integers such that $3x + ey = g$. A basis of $V \oplus V^\vee$ is given by

$$\{\alpha_3,\ x\alpha_1 + y\alpha_2,\ (e\alpha_1 - 3\alpha_2)/g,\ \alpha_4,\ \beta_3,\ \beta_1 + \beta_2,\ (ey\beta_1 - 3x\beta_2)/g,\ \beta_4\}.$$

The matrix of $\langle , \rangle$ in the above basis is equal to $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, where Δ is the 4×4 diagonal matrix with entries $1, g, 3e/g, 3e$.

4.3 **Divisibility** 2

Suppose that

$$c_1(L) = 2\left(ev_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee\right) + \zeta^\vee, \tag{4.3.1}$$

that is $c = 2$ and $s = 1$ in the notation of (4.1.4). Then $(c_1(L), c_1(L)) = 2(4e - 3)$, and $\mathrm{div}(c_1(L)) = 2$. Let $g := \gcd\{3, e\} = \gcd\{3, 2e\}$, and let $x, y \in \mathbb{Z}$ be such that $3x + 2ey = g$. A basis of $V \oplus V^\vee$ is given by

$$\{\alpha_3,\ x\alpha_1 + y\alpha_2,\ (2e\alpha_1 - 3\alpha_2)/g,\ (6e-3)\alpha_3 - \alpha_4,\ \beta_4 - \beta_3,\ \beta_2,\ (g\beta_1 - (6x+3y)\beta_2)/g,\ 3\beta_3 - 2\beta_4\}.$$

The matrix of $\langle , \rangle$ in the above basis is equal to $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, where Δ is the 4×4 diagonal matrix with entries $1, g, 3(4e-3)/g, 3(4e-3)$.

4.4 **Divisibility** 3

Suppose that

$$c_1(L) = 3(ev_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee) + \zeta^\vee, \tag{4.4.1}$$

that is $c = 3$ and $s = 1$ in the notation of (4.1.4). Then $(c_1(L), c_1(L)) = 6(3e - 1)$, and $\mathrm{div}(c_1(L)) = 3$. A basis of $V \oplus V^\vee$ is given by

$$\left\{\alpha_3,\ \alpha_1,\ \alpha_4 - \alpha_3,\ e\alpha_1 - \alpha_2,\ \beta_3,\ \beta_2,\ \beta_4 - \beta_3,\ \beta_1 - 3\beta_2\right\}.$$

The matrix of $\langle,\rangle$ in the above basis is equal to $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, where Δ is the 4×4 diagonal matrix with entries $3, 3, 3(3e-1), 3(3e-1)$.

4.5 Divisibility 6

Suppose that

$$c_1(L) = 6\left(ev_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee\right) + \zeta^\vee, \tag{4.5.1}$$

that is $c = 6$ and $s = 1$ in the notation of (4.1.4). Then $(c_1(L), c_1(L)) = 6(12e-1)$, and $\operatorname{div}(c_1(L)) = 6$. A basis of $V \oplus V^\vee$ is given by

$$\left\{\alpha_1,\, \alpha_3,\, 2\alpha_3 - \alpha_4,\, 2e\alpha_1 - \alpha_2,\, \beta_2,\, \beta_4,\, \beta_3 - 6e\beta_4,\, \beta_1 - 6\beta_2\right\}.$$

The matrix of $\langle,\rangle$ in the above basis is equal to $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, where Δ is the 4×4 diagonal matrix with entries $3, 3, 3(12e-1), 3(12e-1)$.

5 Weil Type

5.1 Abelian varieties of Weil type

We recall that a compact complex torus T of dimension $2g$ is of Weil type (see [30]) if there exists an endomorphism $\varphi\colon T \to T$ such that the following hold:

(1) $\varphi \circ \varphi = -D\mathrm{Id}_T$, where D is a strictly positive integer.
(2) The restriction of φ^* to $H^{1,0}(T)$ decomposes as the direct sum of $\pm\sqrt{-D}$ eigenspaces of the same dimension g.

Such a torus T has a 2D space of classes in $H^{g,g}_{\mathbb{Z}}(T)$, which are not in the ring generated by $H^{1,1}_{\mathbb{Z}}(T)$ unless $g = 1$. Voisin [29] proved that they provide counterexamples to the extension of the Hodge conjecture to compact Kähler manifolds. On the other hand, for certain families of abelian varieties of Weil type it is known that the Weil classes are algebraic [22]. As references for what follows, we recommend [26] and [23].

If A is an abelian variety of Weil type, with endomorphism φ, there exists a polarization Θ such that $\varphi^*\Theta \equiv d\Theta$. If this is the case, one says that (A, φ, Θ) is a polarized abelain variety of Weil type. Let us view the polarization Θ as a bilinear alternating function $E\colon H_1(A;\mathbb{Q}) \times H_1(A;\mathbb{Q}) \to \mathbb{Q}$. The endomorphism φ gives $H_1(A;\mathbb{Q})$

the structure of a vector space over the quadratic field $\mathbb{K} := \mathbb{Q}[\sqrt{-D}]$. One defines

$$\begin{array}{ccc} H_1(A;\mathbb{Q}) \times H_1(A;\mathbb{Q}) & \xrightarrow{H} & \mathbb{K} \\ (\alpha,\beta) & \mapsto & E(\alpha,\varphi_*\beta) + \sqrt{-D}E(\alpha,\beta) \end{array} \tag{5.1.1}$$

As is easily checked H is $\mathbb{K}$ linear in the 2nd entry, and $H(\beta,\alpha) = \overline{(\alpha,\beta)}$. Thus, H is a non-degenerate Hermitian form on the $\mathbb{K}$ vector space $H_1(A;\mathbb{Q})$. The determinant of the Hermitian matrix associated to H by a choice of $\mathbb{K}$-basis of $H_1(A;\mathbb{Q})$ is well-determined modulo multiplication by elements of $\mathrm{Nm}(\mathbb{K}^*)$. Thus, we may associate to H its determinant $\mathrm{Det}H \in \mathbb{Q}^*/\mathrm{Nm}(\mathbb{K}^*)$. We denote $\mathrm{Det}H$ by $\mathrm{Det}\Theta$.

Given an imaginary quadratic field $\mathbb{K}$, and an element of $\mathbb{Q}^*/\mathrm{Nm}(\mathbb{K}^*)$, one may construct a complete up to isogeny irreducible family of $2g$-dimensional polarized abelian varieties of Weil type (A,φ,Θ) with associated field $\mathbb{K}$, and assigned Det of the polarization, of dimension g^2. Complete up to isogeny means that every polarized abelian variety of Weil type (A,φ,Θ) with the given field and determinant is isogenous to one of the varieties in the family (of course the isogeny matches the endomorphisms and the polarizations).

5.2 The abelian variety associated to a point of $\mathscr{D}_h$ is of Weil type

We suppose that $\vartheta = (\vartheta_1,\vartheta_2,\vartheta_3) \in \mathbb{Z}^3$, with all entries non-zero. We suppose also that m is a (strictly) positive rational number, and that Equation (3.4.5) holds. We will adopt the notation of Section 3 without further notice. In particular $(,)^\vee$ is the bilinear symmetric form on $(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C})^\vee$ dual to $(,)$ (see Definition 3.14), and $\langle,\rangle$ denotes the natural perfect pairing between $(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C})^\vee$ and $(\bigwedge^2 V_{\mathbb{C}} \oplus \mathbb{C})$.

Theorem 5.1. Let $h \in (\bigwedge^2 V \oplus \mathbb{Z})^\vee$ be a vector of positive square, and assume that $\sigma \in \mathscr{D}_h^+$ (see Proposition Proposition 3.1 for the definition of $\mathscr{D}_h^+$). Let $J_{[\sigma]}(\vartheta)$ be the compact complex torus in Definition 3.11, and let $\Theta_{[\sigma]}(\vartheta) \in H^{1,1}_{\mathbb{Q}}(J_{[\sigma]}(\vartheta)$ be the ample class in Definition 3.17. Then $(J_{[\sigma]}(\vartheta),\Theta_{[\sigma]}(\vartheta))$ is of Weil type, with an embedding

$$\mathbb{Q}[\sqrt{-m(h,h)^\vee}] \subset \mathrm{End}(J_{[\sigma]}(\vartheta),\Theta_{[\sigma]}(\vartheta))_{\mathbb{Q}}. \tag{5.2.1}$$

The determinant of the polarization $\Theta_{[\sigma]}(\vartheta)$ is 1. By varying $\sigma \in \mathscr{D}_h^+$, one gets a complete (up to isogeny) family of 4 dimensional polarized abelian varieties of Weil type with associated field $\mathbb{Q}[\sqrt{-m(h,h)^\vee}]$, and $\mathrm{Det} \equiv 1$.

Before proving Theorem 5.1, we will go through a few elementary results. The proof of the next lemma is a straightforward exercise.

Lemma 5.2. Let $X = (x_{ij})$ be a 4×4 invertible antisymmetric matrix with coefficients in a field $\mathbb{K}$. Then

$$X^{-1} = \mathrm{Pf}(X)^{-1} \begin{pmatrix} 0 & -x_{34} & x_{24} & -x_{23} \\ x_{34} & 0 & -x_{14} & x_{13} \\ -x_{24} & x_{14} & 0 & -x_{12} \\ x_{23} & -x_{13} & x_{12} & 0 \end{pmatrix}$$

Lemma 5.3. Let X, Y be 4×4 antisymmetric matrices over a field $\mathbb{K}$ (if $\mathrm{char}\mathbb{K} = 2$, a matrix $X = (x_{ij})$ is antisymmetric if $X^t = -X$, and $x_{ii} = 0$ for all i). Then

$$(X \cdot Y)^2 - \frac{1}{2}\mathrm{Tr}(X \cdot Y) X \cdot Y + \mathrm{Pf}(X) \cdot \mathrm{Pf}(Y) 1_4 = 0. \tag{5.2.2}$$

(Note: $\frac{1}{2}\mathrm{Tr}(X \cdot Y) = -\sum_{1 \le i < j \le 4} x_{ij} y_{ij}$, hence it makes sense even if $\mathrm{char}\mathbb{K} = 2$.)

Proof. This is the content of the main result of [6], in the case of 4×4 matrices. In fact, let $p \in \mathbb{Z}[x_{ij}, y_{kh}][\lambda]$ be equal to $\mathrm{Pf}(X) \cdot \mathrm{Pf}(\lambda X^{-1} - Y)$. In [6] it is proved that $p(X \cdot Y) = 0$ (we replace λ by $X \cdot Y$). Expanding $\mathrm{Pf}(X) \cdot \mathrm{Pf}(\lambda X^{-1} - Y)$ (Lemma 5.2 will be handy), one gets the lemma. ■

Proof of Theorem 5.1. By the Theorem on elementary divisors, there exists a basis $\mathscr{B} = \{v_1, \ldots, v_4\}$ of V (of volume 1) such that

$$h = h_0 + s\zeta^\vee, \qquad h_0 = c(ev_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee), \quad c, e \in \mathbb{N}_+, \ s \in \mathbb{Z}. \tag{5.2.3}$$

Let $g\colon V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee$ be the antisymmetric map such that $\omega_g = h_0$. Notice that $g(V_{\mathbb{Q}}) = V_{\mathbb{Q}}^\vee$. For $N, b \in \mathbb{Q}$, let

$$\begin{array}{ccc} V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee & \xrightarrow{\Psi} & V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee \\ (v, \ell) & \mapsto & (g^{-1}(\ell) - bv, b\ell - Ng(v)) \end{array} \tag{5.2.4}$$

Then $\Psi(V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^\vee) = V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^\vee$, and

$$\Psi \circ \Psi = -(N - b^2)\mathrm{Id}_{V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee}. \tag{5.2.5}$$

The map Ψ induces an endomorphism of $J_{[\sigma]}(\vartheta)$ if

$$\Psi\left(H^{1,0}_{[\sigma]}\right) \subset H^{1,0}_{[\sigma]}. \tag{5.2.6}$$

If $[\sigma] \in \zeta^{\perp}$, then (5.2.6) holds for any choice of N, b. In fact, by Proposition 3.3, there exists a 2D subspace $U \subset V_{\mathbb{C}}$ such that $H^{1,0}_{[\sigma]}(\vartheta) = U \oplus U^{\perp}$, and $[\sigma] = \bigwedge^2 U$. Since $\langle h_0, \sigma\rangle = 0$ (because $[\sigma] \in \zeta^{\perp}$), and $\omega_g = h_0$, the subspace U is isotropic for the symplectic form ω_g; it follows that $g(U) = U^{\perp}$. This shows that $\Psi(u, 0) \subset H^{1,0}_{[\sigma]}(\vartheta)$ for all $u \in U$. Similarly, one checks tht $\Psi(0, \ell) \subset H^{1,0}_{[\sigma]}(\vartheta)$ for all $\ell \in U^{\perp}$. This proves that (5.2.6) holds if $[\sigma] \in \zeta^{\perp}$ and N, b are arbitrary.

Now let us assume that $\sigma \notin \zeta^{\perp}$. By Theorem 3.9, we may assume that

$$\sigma = \vartheta_2 \iota(\omega_f) - 2\vartheta_3\zeta, \quad H^{1,0}_{[\sigma]}(\vartheta) = \{(v, f(v)) \mid v \in V_{\mathbb{C}}\}, \tag{5.2.7}$$

where $f\colon V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}$ is an invertible antisymmetric map, and $\omega_f \in \bigwedge^2 V^{\vee}$ is the symplectic form associated to f. In particular (5.2.6) holds if and only if

$$g^{-1} \circ f \circ g^{-1} \circ f - 2b(g^{-1} \circ f) + N\mathrm{Id}_{V_{\mathbb{C}}} = 0. \tag{5.2.8}$$

There exist $N, b \in \mathbb{Q}$ such that (5.2.8) holds because of Lemma 5.3. In fact, let X, Y be the matrices of g and f respectively (with respect to the bases $\mathscr{B}, \mathscr{B}^{\vee}$). Because of the equality $\langle h, \vartheta_2\iota(\omega_f) - 2\vartheta_3\zeta\rangle = 0$, we have

$$\operatorname{Tr}(X^{-1} \cdot Y) = 2c^{-1}e^{-1}(y_{12} + ey_{34}) = 2c^{-2}e^{-1}\langle h_0, \iota(\omega_f)\rangle = 4c^{-2}e^{-1}s\frac{\vartheta_3}{\vartheta_2}. \tag{5.2.9}$$

Equation (5.2.9) is the key point: the trace on the left-hand side is rational because $[\sigma] \in \mathscr{D}_h$. Next, we have

$$\operatorname{Pf}(X^{-1}) = c^{-2}e^{-1}, \quad \operatorname{Pf}(Y) = \frac{\vartheta_2}{\vartheta_1}.$$

By Lemma 5.3 it follows that (5.2.8) holds if we set

$$N := c^{-2}e^{-1}\frac{\vartheta_1}{\vartheta_2}, \quad b := c^{-2}e^{-1}s\frac{\vartheta_3}{\vartheta_2}. \tag{5.2.10}$$

We have proved that if N, b are as above, then Ψ induces an endomorphism

$$\varphi\colon J_{[\sigma]}(\vartheta) \to J_{[\sigma]}(\vartheta) \tag{5.2.11}$$

such that $\varphi \circ \varphi = -(N - b^2)\mathrm{Id}$. Using (3.4.5), we get that

$$N - b^2 = c^{-4}e^{-2}\frac{\vartheta_3^2}{\vartheta_2^2}(2mc^2e - s^2) = c^{-4}e^{-2}\frac{\vartheta_3^2}{\vartheta_2^2}m(h,h)^\vee. \tag{5.2.12}$$

Thus, φ defines an embedding $\mathbb{Q}[\sqrt{-m(h,h)^\vee}] \subset \mathrm{End}(J_{[\sigma]}(\vartheta))_{\mathbb{Q}}$. In order to prove (5.2.1), it remains to show that

$$\langle \Psi(\alpha), \Psi(\beta)\rangle_{\vartheta,h} = (N - b^2)\langle \alpha, \beta\rangle_{\vartheta,h}. \tag{5.2.13}$$

Let N, b be as in (5.2.10), and let

$$\lambda_1 := b + i\sqrt{N - b^2}, \quad \lambda_2 := b - i\sqrt{N - b^2}.$$

Let $E_{\pm i\sqrt{N-b^2}} \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee$ be the Ψ-eigenspace with eigenvalue $\pm i\sqrt{N - b^2}$. An easy computation gives that

$$E_{i\sqrt{N-b^2}} = \left\{(v, \lambda_1 g(v)) \mid v \in V_{\mathbb{C}}\right\}, \quad E_{-i\sqrt{N-b^2}} = \left\{(v, \lambda_2 g(v)) \mid v \in V_{\mathbb{C}}\right\}. \tag{5.2.14}$$

We claim that

$$\langle\ ,\ \rangle_{\vartheta,h}|_{E_{\pm i\sqrt{N-b^2}}} = 0, \tag{5.2.15}$$

and that for $v, w \in V_{\mathbb{C}}$, we have

$$\langle (v, \lambda_1 g(v)), (w, \lambda_2 g(w)\rangle_{\vartheta,h} = \vartheta_1 c^{-2}e^{-1}(h,h)^\vee \omega_g(v,w). \tag{5.2.16}$$

In order to prove (5.2.15), let $j, k \in \{1, 2\}$, and let $v, w \in V_{\mathbb{C}}$. Then (keep in mind that $\omega_g = h_0$)

$$\begin{aligned}\langle (v, \lambda_j g(v)), (w, \lambda_k g(w))\rangle_{\vartheta,h} = {} & \vartheta_1\langle \omega_g, v \wedge w\rangle + \vartheta_2\lambda_j\lambda_k\langle \omega_g, \iota(g(v) \wedge g(w))\rangle \\ & - s\vartheta_3(\lambda_j + \lambda_k)\omega_g(v,w).\end{aligned} \tag{5.2.17}$$

We have $\langle \omega_g, v \wedge w\rangle = \omega_g(v,w)$, and a simple argument shows that

$$\langle \omega_g, \iota(g(v) \wedge g(w))\rangle = \mathrm{Pf}(g)\omega_g(v,w) = c^2 e\omega_g(v,w).$$

Thus, (17) reads

$$\langle (v, \lambda_j g(v)), (w, \lambda_k g(w)) \rangle_{\vartheta,h} = \omega_g(v,w)(\vartheta_2 c^2 e \lambda_j \lambda_k - s\vartheta_3(\lambda_j + \lambda_k) + \vartheta_1). \tag{5.2.18}$$

A minimal polynomial of λ_j is equal to $x^2 - 2bx + N$. By (5.2.10) we get that Equations (5.2.15) and (5.2.16) follow from (5.2.18). Equation (5.2.13) follows at once from (5.2.15) and (5.2.16).

We must also prove that the $\pm i\sqrt{N-b^2}$-eigenspaces of the action of Ψ on $H^{1,0}_{[\sigma]}(\vartheta)$ have dimension 2. Since $\mathscr{D}_h^+$ is irreducible, the dimensions of $\pm i\sqrt{N-b^2}$-eigenspaces are independent of $[\sigma] \in \mathscr{D}_h^+$. Hence, we may assume that $[\sigma] \in \zeta^\perp$. Thus, there exists a 2D subspace $U \subset V_{\mathbb{C}}$ such that $H^{1,0}_{[\sigma]}(\vartheta) = U \oplus U^\perp$. The statement about eigenspaces follows at once from (5.2.14). This finishes the proof that (5.2.1) holds.

Next, we prove that $\mathrm{Det}\Theta_{[\sigma]}(\vartheta) \equiv 1$. Let H be the Hermitian form defined by (5.1.1). We must compute the determinant of the Gram matrix of H relative to a basis of $V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^\vee$ as vector space over $\mathbb{Q}[i\sqrt{N-b^2}]$. Let $\mathscr{B} = \{v_1, \dots, v_4\}$ be the basis of V such that (5.2.3) holds. Then $\{(v_1, 0), \dots, (v_4, 0)\}$ is a basis of $V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^\vee$ as vector space over $\mathbb{Q}[i\sqrt{N-b^2}]$. A computation gives that the Gram matrix of H relative to the chosen $\mathbb{Q}[i\sqrt{N-b^2}]$-basis is equal to

$$\begin{pmatrix} 0 & i\vartheta_1 ce\sqrt{N-b^2} & 0 & 0 \\ -i\vartheta_1 ce\sqrt{N-b^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\vartheta_1 c\sqrt{N-b^2} \\ 0 & 0 & -i\vartheta_1 c\sqrt{N-b^2} & 0 \end{pmatrix}.$$

Thus, $\mathrm{Det}H \in \mathrm{Nm}(\mathbb{Q}[i\sqrt{N-b^2}])$.

It remains to show that, by varying $\sigma \in \mathscr{D}_h$, one gets a complete (up to isogeny) family of "polarized" tori of Weil type with fixed discrete invariants. Let $(V \oplus V^\vee, H^{p,q})$ be a weight 1 Hodge structure such that Ψ induces a homomorphism of $(V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee)/(H^{1,0} + V \oplus V^\vee)$, that is such that $\Psi(H^{1,0}) \subset H^{1,0}$, and such that the restriction of Ψ to $H^{1,0}$ has eigenspaces of equal dimensions (i.e., of dimension 2). Then $\dim(H^{1,0} \cap E_{\pm i\sqrt{N-b^2}}) = 2$. Moreover, since $H^{1,0}$ is isotropic for $\langle , \rangle_{\vartheta,h}$, we have

$$H^{1,0} \cap E_{-i\sqrt{N-b^2}} = (H^{1,0} \cap E_{i\sqrt{N-b^2}})^\perp.$$

(Recall that $\langle,\rangle_{\vartheta,h}$ gives a perfect pairing between $E_{i\sqrt{N-b^2}}$ and $E_{-i\sqrt{N-b^2}}$.) Let $\pi\colon (V_{\mathbb{C}} \oplus V_{\mathbb{C}}^{\vee}) \to V_{\mathbb{C}}$ be the projection. Let

$$W_{\pm} := \pi\left(H^{1,0} \cap E_{\pm i\sqrt{N-b^2}}\right).$$

Then $W_{\pm}$ are 2D subspaces of $V_{\mathbb{C}}$, and by (5.2.16) they are orthogonal for the symplectic form ω_g. Thus, either $W_+ \cap W_- = \{0\}$, or $W_+ = W_-$. In the former case $H^{1,0}$ is the graph of a non-degenerete skew-symmetric map $f\colon V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}$ such that $\mathrm{Pf}(f) = \vartheta_1/\vartheta_2$, in the latter case $H^{1,0} = U \oplus U^{\perp}$ for a 2D subspace (equal to $W_+ = W_-$) of $V_{\mathbb{C}}$. Hence, in both cases $H^{1,0} = H^{1,0}_{[\sigma]}(\vartheta)$ for some $[\sigma] \in \mathscr{D}$. Since $H^{1,0}$ is isotropic for $\langle,\rangle_{\vartheta,h}$, we have $[\sigma] \in \mathscr{D}_h$, and actually $[\sigma] \in \mathscr{D}_h^+$ by the ampleness of $\Theta_{[\sigma]}(\vartheta)$. ■

5.3 Proof of the 3rd main result

We prove Theorem 1.5. The isogeny between $\mathrm{KS}(X,L)$ and $J^3(X)^4$ follows from Item (1) of Theorem 1.1, and results of van Geemen, Voisin, and Charles. In fact, assume that the Hodge Tate group of the primitive Hodge structure $c_1(L)^{\perp} \subset H^2(X)$ is the special orthogonal group of $c_1(L)^{\perp}$ equipped with the BBF quadratic form. Then, by Theorem 9.2 in [27], $\mathrm{KS}(X,L)$ is isogenous to B^4, where B is a simple abelian fourfold of Weil Type. On the other hand, by Item (1) of Theorem 1.1, and by Proposition 6 of [28] (see also [4]), there exists a non-trivial homomorphism $J^3(X) \to B$, which is an isomorphism because B is simple. This proves that $\mathrm{KS}(X,L)$ is isogenous to $J^3(X)^4$ if the Hodge Tate group of $c_1(L)^{\perp} \subset H^2(X)$ is the special orthogonal group of $c_1(L)^{\perp}$. Since the very generic polarized (X,L) as above has Hodge Tate group equal to the special orthogonal group, it follows that $\mathrm{KS}(X,L)$ is isogenous to $J^3(X)^4$ for all (X,L).

Next, we prove the other statements of the theorem. There exists an isomorphism

$$\varphi\colon J_{[\sigma(X)]}(\vartheta) \xrightarrow{\sim} J^3(X),$$

where $[\sigma(X)] = H^{2,0}(X)$ is the period point of X, and $\vartheta = \vartheta(\overline{q}_X^{n-2})$. If $n = 2$, then $\varphi^*\Theta_L \equiv \Theta_{[\sigma(X)]}(\vartheta)$, and hence Theorem 1.5 follows from Theorem 5.1. Now let $n > 2$. Since the definitions of $\varphi^*\Theta_L$ and $\Theta_{[\sigma(X)]}(\vartheta)$ are different (see Example 3.19), we argue as follows. For a very generic X the Néron–Severi groups of $J_{[\sigma(X)]}(\vartheta)$ and of $J^3(X)$ have rank 1, and hence there exists $c \in \mathbb{Q}_+$ such that $\varphi^*\Theta_L \equiv c\Theta_{[\sigma(X)]}(\vartheta)$. Thus, Theorem 1.5 follows from Theorem 5.1.

Remark 5.4. The proof of Theorem 1.5 provides the following non-trivial statement. Let X be an HK of Kummer type, of dimension $2n$. Let $\gamma \in H^2(X;\mathbb{Q})$ be a class of positive square. Then there exists $c_\gamma \in \mathbb{Q}^*$ such that for all $\alpha, \beta \in H^3(X;\mathbb{Q})$

$$\int_X \alpha \smile \beta \smile \gamma^{2n-3} = c_\gamma \int_X \alpha \smile \beta \smile \gamma \smile (q_X^\vee)^{n-2}. \tag{5.3.1}$$

5.4 An example

We work out one example in order to emphasize that our procedure is very explicit. We assume that (X,L) is a polarized HK fourfold of Kummer type, and that $q_K(L) = 2$ and $c_1(L)$ has divisibility 1. By Theorem 1.5 there exists an injection $\mathbb{Q}[\sqrt{-3}] \subset \mathrm{End}(J^3(X),\Theta_L)_{\mathbb{Q}}$. Since $\mathrm{Det}\Theta_L \equiv 1$, it follows that $J^3(X)$ is isogenous to the Prym variety of an étale cyclic triple cover $\widetilde{C} \to C$, where C is a curve of genus 3 (possibly a stable curve), that is examples considered by Schoen [22], see also Section 7 in [26]. We recall that these abelian fourfolds are of Weil type, with an endomorphism which is a (non-trivial) cube root of Id, and the determinant of the Weil polarization is 1.

By Remark 3.8, we may identify $(J^3(X),L)$ with $(J_{[\sigma]}(\vartheta),\Theta_{[\sigma]}(\vartheta))$, where $[\sigma]$ is the period point of X, $\vartheta = (-1,-3,-3)$ and $h = v_1^\vee \wedge v_2^\vee + v_3^\vee \wedge v_4^\vee$ (we adopt the notation introduced in the proof of Theorem 5.1). Thus, (in the notation introduced in the proof of Theorem 5.1) $N = 1/3$ and $b = 0$. It follows that

$$\begin{array}{ccc} V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee & \xrightarrow{\Psi_0} & V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee \\ (v,\ell) & \mapsto & (3g^{-1}(\ell), -g(v)) \end{array}$$

defines an endomorphism of $(J^3(X),L)$ such that $\Psi_0 \circ \Psi_0 = -3\mathrm{Id}$. The endomorphism of $(V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee)$ defined by the cube root of Id given by $\Omega := -(\mathrm{Id} + \Psi_0)/2$ does not map $V \oplus V^\vee$ to itself, and hence does not descend to an endomorphism of $(J^3(X),L)$. On the other hand Ω does descend to an endomorphism of

$$Y := J_{[\sigma]}(\vartheta)/\{\overline{(v/2, g(v)/2)} \mid v \in V\}.$$

Now Y is one of the abelian fourfolds of Weil type considered by Schoen, and the quotient map $J_{[\sigma]}(\vartheta) \to Y$ has a kernel isomorphic to $\mathbb{F}_2^4$.

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