An introduction to Algebraic Geometry - Varieties

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Introduction

Background

Algebraic geometry studies solutions of systems polynomial equations in a finite set of variables. In a sense, algebraic geometry originated from Descartes' introduction of coordinates, because we view the set of solutions as a geometric object. It is much more convenient to study solutions of polynomial equations in the homogeneous coordinates of points of a projective space, even if one is initially interested in the solutions which belong to an affine space, and hence also projective geometry playes a key rôle in algebraic geometry.

However, the problems that really started algebraic geometry as we know it have to do with the computation of certain integrals. To explain this, consider the following indefinite integrals:

$$\int \frac{dx}{\sqrt{1+x^2}}, \quad \int \frac{dx}{\sqrt{1+x^3}}.$$

We may integrate the first one by the substitution $x = \frac{2t}{1-t^2}$, and we get

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{2dt}{1-t^2} = \log\left(\frac{1+t}{1-t}\right) + c = \log(x + \sqrt{1+x^2}) + c.$$

The intriguing fact discovered by Fagnano and Euler is that, although no susbtitution by a rational function will reduce the second integral to an elementary integral, there exists an addition formula

$$\int_0^a \frac{dx}{\sqrt{1+x^3}} + \int_0^b \frac{dx}{\sqrt{1+x^3}} = \mathrm{cost} + \int_0^c \frac{dx}{\sqrt{1+x^3}},$$

where $c = R(a, b, \sqrt{a^3+1}, \sqrt{b^3+1})$ is a rational funzction of $a, b, \sqrt{a^3+1}, \sqrt{b^3+1}$. The formula is analogous to the addition formula for logarithms, i.e. $\int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \int_1^{ab} \frac{dx}{x}$, and it holds for an analogous reason, i.e. the existence of maps $(x, y) \mapsto (\varphi(x, y), \psi(x, y))$ with φ, ψ rational functions, mapping to itself the curve $\{(a, b) \mid b^2 = a^3 + 1\}$, acting transitively on the points of such curve, and leaving invariant the differential that we are integrating (the maps are analogous to the maps $x \mapsto \lambda x$, which leave invariant the differential $\frac{dx}{x}$).

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Conventions

$\mathbb{K}$ is an algebraically closed field, and $\mathbb{K}[Z_0, \ldots, Z_n]$ is the $\mathbb{K}$-algebra of polynomials in $Z_0, \ldots, Z_n$ with coefficients in $\mathbb{K}$. Let $\mathbb{K}[Z_0, \ldots, Z_n]_d \subset \mathbb{K}[Z_0, \ldots, Z_n]$ be the degree-d subspace of the algebra of polynomials, i.e. the set of polynomials F such that $F(\lambda Z) = \lambda^d F(Z)$ for all $\lambda \in \mathbb{K}$ and $Z \in \mathbb{K}^{n+1}$. Thus we have the direct sum decomposition

$$\mathbb{K}[Z_0, \ldots, Z_n] = \bigoplus_{d=0}^{\infty} \mathbb{K}[Z_0, \ldots, Z_n]_d.$$

A polynomial in $\mathbb{K}[Z_0, \dots, Z_n]$ is *homogeneous* if it belongs to one of the above direct summands. An ideal $I \subset \mathbb{K}[Z_0, \dots, Z_n]$ is *homogeneous* if

$$I = \bigoplus_{d=0}^{\infty} I \cap \mathbb{K}[Z_0, \dots, Z_n]_d, \tag{0.0.1}$$

i.e. if it is generated by homogeneous elements.

Chapter 1

Algebraic varieties and regular maps

We are interested in understanding solutions $z_1, \ldots, z_n$ of a family of polynomial equations

$$f_1(z_1, \ldots, z_n) = 0, \ldots, f_r(z_1, \ldots, z_n) = 0.$$

The entries z_i are unknowns in a field $\mathbb{K}$, which we assume to be algebraically closed, e.g. $\mathbb{K} = \mathbb{C}$, and each f_i is an element of $\mathbb{K}[z_1, \ldots, z_n]$. Of course, one may consider an arbitrary field, and consider solutions with entries in that field, but the proper setting for this kind of questions is that of schemes.

In order to understand the geometry of a set of solutions of polynomial equations, it is convenient to replace affine space $\mathbb{A}^n_{\mathbb{K}}$ by projective space $\mathbb{P}^n_{\mathbb{K}}$, and consider the set of points in $\mathbb{P}^n_{\mathbb{K}}$ which are solutions of polynomial equations in the homohgeneous coordinates. The reason is that $\mathbb{P}^n_{\mathbb{C}}$, with the classical topology, is compact, and in general $\mathbb{P}^n_{\mathbb{K}}$ has an algebraic property which replaces compactness over $\mathbb{C}$.

We explain this with Bézout's Theorem, a result which holds for solutions of polynomial equations in $\mathbb{P}^n_{\mathbb{K}}$ but not in $\mathbb{A}^n_{\mathbb{K}}$. If $F_1, \ldots, F_n$ are homogeneous non costant polynomials in $Z_0, \ldots, Z_n$, then there exists a common solution of the polynomials

$$F_1(Z_0, \ldots, Z_n) = 0, \ldots, F_n(Z_0, \ldots, Z_n) = 0,$$

and moreover, either the set of common solutions is infinite, or it has cardinality $\deg F_1 \cdot \ldots \cdot F_n$, provided one assignes a suitable multiplicity to each common solution. No analogous result holds for solutions of polynomial equations in $\mathbb{A}^n$ (take f_1, f_2, where $f_2 = f_1 + 1$), and the reason is that, some (ora all) of the common solutions might be "at infinity".

Thus we will start by considering solutions of polynomial equations in a projective space.

We will omit $\mathbb{K}$ from the notation for affine and projective space.

1.1 The Zariski topology

Let $F \in \mathbb{K}[Z_0, \ldots, Z_n]_d$. Let $x \in \mathbb{P}^n$ be represented by a non zero $Z \in \mathbb{K}^{n+1}$. Then $F(Z) = 0$ if and only if $F(\lambda Z) = 0$ for every $\lambda \in \mathbb{K}^*$, because $F(\lambda Z) = \lambda^d F(Z)$. Hence, although $F(x)$ is not defined, it makes to state that $F(x) = 0$ or $F(x) \neq 0$. Let $I \subset \mathbb{K}[Z_0, \ldots, Z_n]$ be a homogeneous ideal; we let

$$V(I) := \{x \in \mathbb{P}^n \mid F(x) = 0 \ \ \forall \text{ homogeneous } F \in I\}.$$

By Hilbert's basis Theorem A.3.6, a homogeneous ideal I is generated by a finite set of homogeneous polynomials $F_1, \ldots, F_r$, i.e. $I = (F_1, \ldots, F_r)$. It follows that

$$V(I) = V(F_1, \ldots, F_r) := \{x \in \mathbb{P}^n \mid F_1(x) = \ldots = F_r(x) = 0\}$$

is the set of solutions of a *finite* system of algebraic equations.

Proposition 1.1.1. *The collection of subsets $V(I) \subset \mathbb{P}^n$, where I runs through the collection of homogeneous ideals of $\mathbb{K}[Z_0, \ldots, Z_n]$, satisfies the axioms for the closed subsets of a topological space.*

Proof. We have $\varnothing = V((1))$, $\mathbb{P}^n = V((0))$.

Let I, J be homogeneous ideals; we claim that $V(I) \cup V(J) = V(I \cap J)$. We have $V(I), V(J) \subset V(I \cap J)$, because $I, J \supset I \cap J$; thus $V(I) \cup V(J) \subset V(I \cap J)$. Hence it suffices to show that if $x \in V(I \cap J)$ and $x \notin V(I)$, then $x \in V(J)$. Since $x \notin V(I)$, there exists $F \in I$ such that $F(x) \neq 0$. If $G \in J$, then $F \cdot G \in I \cap J$, and thus $(F \cdot G)(x) = 0$ because $x \in V(I \cap J)$; since $F(x) \neq 0$, it follows that $G(x) = 0$. This proves that $x \in V(J)$.

Lastly, let $\{I_t\}_{t \in T}$ be a family of homogeneous ideals of $\mathbb{K}[Z_0, \ldots, Z_n]$. Then

$$\bigcap_{t \in T} V(I_t) = V(\langle \{I_t\}_{t \in T} \rangle),$$

where $\langle \{I_t\}_{t \in T} \rangle$ is the (homogeneous) ideal generated by the collection of the I_t's. □

Definition 1.1.2. The *Zariski topology* of $\mathbb{P}^n$ is the topology whose closed sets are the sets $V(I) \subset \mathbb{P}^n$, where I runs through the collection of homogeneous ideals of $\mathbb{K}[Z_0, \ldots, Z_n]$. The Zariski topology of a subset $A \subset \mathbb{P}^n$ is the topology induced by the Zariski topology of $\mathbb{P}^n$.

Remark 1.1.3. If $\mathbb{K} = \mathbb{C}$, the Zariski topology is weaker than the classical topology of $\mathbb{P}^n$. In fact, unless $n = 0$, the Zariski is much weaker than the classical topology, in particular it is *not* Hausdorff.

Remark 1.1.4. We will always identify $\mathbb{A}^n$ with the open subset $(\mathbb{P}^n \backslash V(Z_0)) \subset \mathbb{P}^n$. Thus $\mathbb{A}^n$ has a Zariski topology, that we describe below. Let $J \subset \mathbb{K}[z_1, \ldots, z_n]$ be an an ideal, in general *not* homogeneous. We let

$$V(J) := \{z \in \mathbb{A}^n \mid f(z) = 0 \quad \forall f \in J\}. \tag{1.1.1}$$

By Hilbert's basis Theorem, every ideal $J \subset \mathbb{K}[z_1, \ldots, z_n]$ is finitely generated, and if $J = (f_1, \ldots, f_r)$, then

$$V(J) = V(f_1, \ldots, f_r) := \{z \in \mathbb{A}^n \mid f(z) = 0 \quad \forall f \in J\}.$$

(The notation conflicts with the notation employed for closed subsets of $\mathbb{P}^n$, but it will always be clear form the context whether $V(J)$ is a subset of a projective space or of an affine space.)

A subet $X \subset \mathbb{A}^n$ is closed if and only if there exist an ideal $J \subset \mathbb{K}[z_1, \ldots, z_n]$ such that $X = V(J)$. In fact, if X is closed, say $X = (\mathbb{P}^n \backslash V(Z_0)) \cap V(F_1, \ldots, F_r)$, where $F_j \subset \mathbb{K}[Z_0, Z_1, \ldots, Z_n]$ are homogeneous, then $X = V(f_1, \ldots, f_r)$, where

$$f_j(z_1, \ldots, z_n) := F(1, z_1, \ldots, z_n).$$

Conversely, consider $V(f_1, \ldots, f_r)$. For $j \in \{1, \ldots, r\}$, let d_j be the degree of f_j. Then

$$F_j(Z_0, \ldots, Z_n) := Z_0^{d_j} f\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right)$$

is a homogegenous polynomial of degree d_j. Since $V(F_1, \ldots, F_r) \subset \mathbb{P}^n$ is closed, and

$$V(f_1, \ldots, f_r) = (\mathbb{P}^n \backslash V(Z_0)) \cap V(F_1, \ldots, F_r),$$

we get that $V(f_1, \ldots, f_r)$ is closed in $\mathbb{A}^n$.

Example 1.1.5. A subset $X \subset \mathbb{P}^n$ is a *hypersurface* if it is equal to $V(F)$, where F is a non constant homogeneous polynomial. Similarly, a subset $X \subset \mathbb{A}^n$ is a *hypersurface* if it is equal to $V(f)$, where f is a non constant polynomial (in general not homogeneous).

A picture of a hypersurface in $\mathbb{A}^2$ is in Figure 1.1. Notice that (x, y) are the affine coordinates - in general, whenever we consider affine or projective space of small dimension, we will denore affine or homogeneous coordinates by letters $x, y, z, \ldots$ and $X, Y, Z, \ldots$ respectively.

What is the field $\mathbb{K}$? The picture shows points with real coordinates. We can view the picture as a "slice" of the corresponding hypersurface over $\mathbb{C}$, or as the closure (either in the Zariski or the classical topology) of the corresponding hypersurface over the algebriac closure of the rationals $\overline{\mathbb{Q}}$.

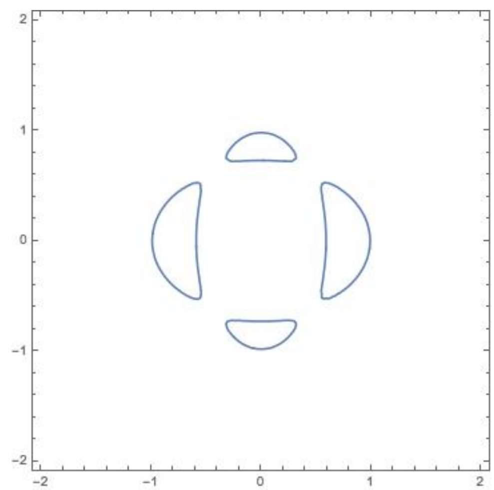

Figure 1.1: $(x^2 + 2y^2 - 1)(3x^2 + y^2 - 1) + \frac{3}{100} = 0$

Given a subset $A \subset \mathbb{P}^n$, let

$$I(A) := \langle F \in \mathbb{K}[Z_0, \ldots, Z_n] \mid F \text{ is homogeneous and } F(p) = 0 \text{ for all } p \in A \rangle, \tag{1.1.2}$$

where $\langle , \rangle$ means "the ideal generated by". Clearly $I(A)$ is a homogeneous ideal of $\mathbb{K}[Z_0, \ldots, Z_n]$, and $V(I(A))$ is the closure of A in the Zariski topology.

Definition 1.1.6. A *quasi-projective* variety is a Zariski locally closed subset of a projective space, i.e. $X \subset \mathbb{P}^n$ such that $X = U \cap Y$, where $U, Y \subset \mathbb{P}^n$ are Zariski open and Zariski closed respectively.

Example 1.1.7. By Remark 1.1.4, every subset $V(J) \subset \mathbb{A}^n$, where $J \subset \mathbb{K}[z_1, \ldots, z_n]$ is an ideal, is a quasi projective variety.

Definition 1.1.8. Let $X \subset \mathbb{P}^n$ be a closed subset. A *principal open subset* of X is an open $U \subset X$ which is equal to

$$X_F := X \backslash V(F),$$

where $F \in \mathbb{K}[Z_0, \ldots, Z_n]$ is a homogeneous polynomial of *strictly positive degree.* In general, if $X \subset \mathbb{P}^n$ is locally closed, a principal open subset of X is an open $U \subset X$ which is equal to $\overline{X}_F$, for a homogeneous polynomial $F \in \mathbb{K}[Z_0, \ldots, Z_n]$ of strictly positive degree.

Claim 1.1.9. *Let $X \subset \mathbb{P}^n$ be locally closed. The collection of principal open subsets of X is a basis of the Zariski topology of X.*

Proof. Let $U \subset X$ be open. Then U is open in $\overline{X}$. Hence it suffices to prove the claim for X closed. We have $U = X \backslash W$, where W is closed. Let $W = V(I)$, where $I \subset \mathbb{K}[Z_0, \ldots, Z_n]$ is a homogeneous ideal. Let $J \subset \mathbb{K}[Z_0, \ldots, Z_n]$ be the homogeneous ideal generated by all products $F \cdot Z_i$, where $F \in I$, and $i \in \{0, \ldots, n\}$. Then $V(J) = V(I) = W$, and J is generated by a *non empty* finite set of homogeneous polynomials $F_1, \ldots, F_r$. Then

$$U = X \backslash V(F_1, \ldots, F_r) = X_{F_1} \cup X_{F_2} \cup \ldots \cup X_{F_r}.$$

□

Remark 1.1.10. If V is a finite dimensional complex vector space, the Zariski topology on $\mathbb{P}(V)$ is defined by imitating what was done for $\mathbb{P}^n$: one associates to a homogeneous ideal $I \subset \operatorname{Sym} V^\vee$ the set of zeroes $V(I)$, etc. Similarly one defines the Zariski topology on a finite dimensional complex affine space. Everything that we do in the present chapter applies to this situation, but for the sake of concreteness we formulate it for $\mathbb{P}^n$ and $\mathbb{A}^n$.

1.2 Decomposition into irreducibles

A proper closed subset $X \subset \mathbb{P}^1$ (or $X \subset \mathbb{A}^1$) is a finite set of points. In general, a quasi projective variety is a finite union of closed subsets which are irreducible, i.e. are not the union of proper closed subsets. In order to formulate the relevant result, we give a few definitions.

Definition 1.2.1. Let X be a topological space. We say that X is *reducible* if either $X = \varnothing$ or there exist proper closed subsets $Y, W \subset X$ such that $X = Y \cup W$. We say that X is *irreducible* if it is not reducible.

Example 1.2.2. Projective space $\mathbb{P}^n$ with the euclidean (classical) topology is reducible except if $n = 0$. On the other hand, $\mathbb{P}^n$ with the Zariski topology is irreducible for any n. In fact suppose that $\mathbb{P}^n = Y \cup W$ with Y and W proper closed subsets. Then there exist $F \in I(Y)$ such that $F(p) \neq 0$ for one (at least) $p \in W$ and $g \in I(W)$ such that $g(q) \neq 0$ for one (at least) $q \in Y$. Then $fg = 0$ because $\mathbb{P}^n = Y \cup W$; that is a contradiction because $\mathbb{K}[Z_0, \ldots, Z_n]$ is an integral domain.

Definition 1.2.3. Let X be a topological space. An *irreducible decomposition of* X consists of a decomposition (possibly empty)

$$X = X_1 \cup \cdots \cup X_r \tag{1.2.1}$$

where each X_i is a closed irreducible subset of X (irreducible with respect to the induced topology) and moreover $X_i \not\subset X_j$ for all $i \neq j$.

We will prove the following result.

Theorem 1.2.4. *Let $A \subset \mathbb{P}^n$ with the (induced) Zariski topology. Then A admits an irreducible decomposition, and such a decomposition is unique up to reordering of components.*

The key step in the proof of Theorem 1.2.4 is the following remarkable consequence of Hilbert's basis Theorem A.3.6.

Proposition 1.2.5. *Let $A \subset \mathbb{P}^n$, and let $A \supset X_0 \supset X_1 \supset \ldots \supset X_m \supset \ldots$ be a descending chain of Zariski closed subsets of A, i.e $X_m \supset X_{m+1}$ for all $m \in \mathbb{N}$. Then the chain is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such that $X_m = X_{m_0}$ for $m \geqslant m_0$.*

Proof. Let $\overline{X}_i$ be the closure of X_i in $\mathbb{P}^n$. Then $X_i = A \cap \overline{X}_i$, because X_i is closed in A. Hence we may replace X_i by $\overline{X}_i$, or equivalently we may suppose that the X_i are closed in $\mathbb{P}^n$. Let $I_m = I(X_m)$. Then $I_0 \subset I_1 \subset \ldots \subset I_m \subset \ldots$ is an ascending chain of (homogeneous) ideals of $\mathbb{K}[Z_0, \ldots, Z_n]$. By Hilbert's basis Theorem and Lemma A.3.3 the ascending chain of ideals is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such that $I_{m_0} = I_m$ for $m \geqslant m_0$. Thus $X_{m_0} = V(I_{m_0}) = V(I_m) = X_m$ for $m \geqslant m_0$. □

Proof of Theorem 1.2.4. If A is empty, then it is the empty union (of irreducibles). . Next, suppose that A is not empty and that it does not admit an irreducible decomposition; we will arrive at a contradiction. First A in reducible, i.e. $A = X_0 \cup W_0$ with $X_0, W_0 \subset A$ proper closed subsets. If both X_0 and W_0 have an irreducible decomposition, then A is the union of the irreducible components of X_0 and W_0, contradicting the assumption that A does not admit an irreducible decomposition. Hence one of X_0, W_0, say X_0, does *not* have an irreducible decomposition. In particular X_0 is reducible. Thus $X_0 = X_1 \cup W_1$ with $X_1, W_1 \subset X_0$ proper closed subsets, and arguing as above, one of X_1, W_1, say X_1, does not admit a decomposition into irredicbles. Iterating, we get a strictly descending chain of closed subsets

$$A \supsetneq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_m \supsetneq X_{m+1} \supsetneq \cdots$$

This contradicts Proposition 1.2.5. This proves that X has a decomposition into irreducibles $X = X_1 \cup \ldots \cup X_r$.

By discarding X_i's which are contained in X_j with $i \neq j$, we may assume that if $i \neq j$, then X_i is not contained in X_j.

Lastly, let us prove that such a decomposition is unique up to reordering, by induction on r. The case $r = 1$ is trivially true. Let $r \geqslant 2$. Suppose that $X = Y_1 \cup \ldots \cup Y_s$, where each Y_j is Zariski closed irreducible, and $Y_j \not\subset Y_k$ if $j \neq k$. Since Y_s is irreducible, there exists i such that $Y_s \subset X_i$. We may assume that $i = r$. By the same argument, there exists j such that $X_r \subset Y_j$. Thus $Y_s \subset X_r \subset Y_j$. It follows that $j = s$, and hence $Y_s = X_r$. It follows that $X_1 \cup \ldots \cup X_{r-1} = Y_1 \cup \ldots \cup Y_{s-1}$, and hence the decomposition is unique up to reordering by the inductive hypothesis. □

Definition 1.2.6. Let X be a quasi projective variety, and let

$$X = X_1 \cup \ldots \cup X_r$$

be an irreducible decomposition of X. The X_i's are the *irreducible components of* X (this makes sense because, by Theorem 1.2.4, the collection of the X_i's is uniquely determined by X).

We notice the following consequence of Proposition 1.2.5.

Corollary 1.2.7. *A quasi projective variety X (with the Zariski topology) is quasi compact, i.e. every open covering of X has a finite subcover.*

The following result makes a connection between irreducibility and algebra.

Proposition 1.2.8. *A subset $X \subset \mathbb{P}^n$ is irreducible if and only if $I(X)$ is a prime ideal.*

Proof. The proof has essentially been given in Example 1.2.2. Suppose that X is irreducible. In particular $X \neq \varnothing$ (by definition), and hence $I(X)$ is a proper ideal of $\mathbb{K}[Z_0, \ldots, Z_n]$. We must prove that $\mathbb{K}[Z_0, \ldots, Z_n]/I(X)$ is an integral domain. Suppose the contrary. Then there exist

$$F, G \in (\mathbb{K}[Z_0, \ldots, Z_n] \backslash I(X)) \tag{1.2.2}$$

such that

$$F \cdot G \in I(X). \tag{1.2.3}$$

By (1.2.3), we have $X = (X \cap V(F)) \cup (X \cap V(G))$, and both $X \cap V(F)$, $X \cap V(G)$ are proper closed subsets of X by (1.2.2). This proves that if X is irreducible, then $I(X)$ is a prime ideal.

Next, assume that X is reducible; we must prove that $I(X)$ is not prime. If $X = \varnothing$, then $I(X) = \mathbb{K}[Z_0, \ldots, Z_n]$ and hence $I(X)$ is not prime. Thus we may assume that $X \neq \varnothing$, and hence there exist proper closed subset $Y, W \subset X$ such that $X = Y \cup W$. Since $Y \not\subset W$ and $W \not\subset Y$, there exist $F \in (I(Y) \backslash I(W))$ and $G \in (I(W) \backslash I(Y))$. It follows that both (1.2.2) and (1.2.3) hold, and hence $I(X)$ is not prime. □

Remark 1.2.9. Let $I := (Z_0^2) \subset \mathbb{K}[Z_0, Z_1]$. Then $V(I) = \{[0,1]\}$ is irreducible although I is *not* prime. Of course $I(V(I))$ is prime, it equals (Z_0).

Remark 1.2.10. Let $X \subset \mathbb{A}^n$. Let $I(X) \subset \mathbb{K}[z_1, \ldots, z_n]$ be the ideal of polynomials vanishing on X. Then X is irreducible if and only if $I(X)$ is a prime ideal. The proof is analogous to the proof of Proposition 1.2.8. One may also directly relate $I(X)$ with the ideal $J \subset \mathbb{K}[Z_0, \ldots, Z_n]$ generated by homogeneous polynomials vanishing on X (as subset of $\mathbb{P}^n$), and argue that $I(X)$ is prime if and only if J is.

Example 1.2.11. Let $V(F) \subset \mathbb{P}^n$ be a hypersurface, and let $F_1, \ldots, F_r$ be the distinct prime factors of the decomposition of F into a products of primes (recall that $\mathbb{K}[Z_0, \ldots, Z_n]$ is a UFD, by Corollary A.2.2). The irreducible decomposition of $V(F)$ is

$$V(F) = V(F_1) \cup \ldots \cup V(F_r).$$

In fact, each $V(F_i)$ is irreducible by Proposition 1.2.8. What is not obvious is that $V(F_i)n^o \subset V(F_j)$ if F_i, F_j are non associated primes. This follows from Hilbert's Nullstellensatz, i.e. Theorem A.4.1 (or by a simpler argument involving only unique factorization in the ring of polynomials).

1.3 Regular maps

Definition 1.3.1. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be quasi projective varieties. A map $f\colon X \to Y$ is *regular at* $a \in X$ if there exist an open $U \subset X$ containing a and $F_0, \ldots, F_m \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ such that for all $[Z] \in U$ $(F_0(Z), \ldots, F_m(Z)) \neq (0, \ldots, 0)$, and

$$f([Z]) = [F_0(Z), \ldots, F_m(Z)]. \tag{1.3.1}$$

The map f is *regular* if it is regular at each point of X.

The identity map of a quasi projective variety is regular (choose $F_j(Z) = Z_j$). If $f\colon X \to Y$ and $g\colon Y \to W$ are regular maps of quasi projective varieties, the composition $g \circ f\colon X \to W$ is regular, because the composition of polynomial functions is a polynomial function. Thus we have the category of quasi projective varieties. In particular we have the notion of isomorphism between quasi projective varieties.

Example 1.3.2. Let $X \subset \mathbb{A}^n$ be a locally closed subset (recall that $\mathbb{A}^n = \mathbb{P}^n_{Z_0}$). Then $f\colon X \to \mathbb{P}^m$ is a regular map if and only if, given any $a \in X$, there exist $f_0, \ldots, f_m \in \mathbb{K}[z_1, \ldots, z_n]$ (in general *not* homogeneous) such that on an open subset $U \subset X$ containing a we have

$$f(z) = [f_0(z), \ldots, f_m(z)]. \tag{1.3.2}$$

(This includes the statement that $V(f_1, \ldots, f_m) \cap U = \emptyset$.) In fact, if f is regular there exist homogeneous $F_0, \ldots, F_m \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ such that $f([1, z]) = [F_0(1, z), \ldots, F_m(1, z)]$, and it suffices to let $f_j(z) := F_j(1, z)$. Conversley, if (1.3.2) holds, then

$$f([Z_0, Z_1, \ldots, Z_n]) = [Z_0^d, Z_0^d f_1\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right), \ldots, Z_0^d f_m\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right)], \tag{1.3.3}$$

and for d is large enough, each of the rational functions appearing in (1.3.3) is actually a homogeneous polynomial of degree d.

Example 1.3.3. Let $X \subset \mathbb{A}^n$ be a locally closed subset and let $f\colon X \to \mathbb{P}^m$ be a map such that $f(X) \subset \mathbb{P}^m_{T_0}$ (we let $[T_0, \ldots, T_m]$ be homogeneous coordinates on $\mathbb{P}^m$). Then f is regular if and only if locally there exist $f_0, \ldots, f_m \in \mathbb{K}[z_1, \ldots, z_n]$ (in general *not* homogeneous) such that, in affine coordinates $(\frac{T_1}{T_0}, \ldots, \frac{T_m}{T_0})$, we have

$$f(z) = \left(\frac{f_1(z)}{f_0(z)}, \ldots, \frac{f_m(z)}{f_0(z)}\right). \tag{1.3.4}$$

Example 1.3.4. Let $f \in \mathbb{K}[z_1, \ldots, z_n]$. Let $Y := V(f(z_1, \ldots, z_n) \cdot z_{n+1} - 1) \subset \mathbb{A}^{n+1}$. The map

$$\begin{array}{ccc} \mathbb{A}^n \backslash V(f) & \longrightarrow & Y \\ (z_1, \ldots, z_n) & \mapsto & (z_1, \ldots, z_n, \frac{1}{f(z_1, \ldots, z_n)}) \end{array}$$

is an isomorphism.

Example 1.3.5. Let

$$\mathcal{C}_n = \left\{[\xi_0, \ldots, \xi_n] \in \mathbb{P}^n \mid \operatorname{rk}\begin{pmatrix} \xi_0 & \xi_1 & \cdots & \xi_{n-1} \\ \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix} \leqslant 1\right\}. \tag{1.3.5}$$

Since a matrix has rank at most 1 if and only if all the determinants of its 2×2 minors vanish it follows that $\mathbb{K}_n$ is closed. We have a regular map

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\varphi_n} & \mathbb{K}_n \\ [s,t] & \mapsto & [s^n, s^{n-1}t, \ldots, t^n] \end{array} \tag{1.3.6}$$

Let us prove that φ_n is an isomorphism. Let $\psi_n \colon \mathcal{C}_n \to \mathbb{P}^1$ be defined as follows:

$$\psi_n([\xi_0, \ldots, \xi_n]) = \begin{cases} [\xi_0, \xi_1] & \text{if } [\xi_0, \ldots, \xi_n] \in \mathcal{C}_n \cap \mathbb{P}^n_{\xi_0} \\ [\xi_{n-1}, \xi_n] & \text{if } [\xi_0, \ldots, \xi_n] \in \mathcal{C}_n \cap \mathbb{P}^n_{\xi_n} \end{cases}$$

Of course one has to check that the two expressions coincide for points in $\mathbb{K}_n \cap \mathbb{P}^n_{\xi_0} \cap \mathbb{P}^n_{\xi_n}$: from (1.3.5) we get that $\xi_0 \cdot \xi_n - \xi_1 \xi_{n-1}$ vanishes on $\mathbb{K}_n$ and this shows the required compatibility. One checks easily that $\psi_d \circ \varphi_n = \mathrm{Id}_{\mathbb{P}^1}$ and $\varphi_n \circ \psi_n = \mathrm{Id}_{\mathbb{K}_n}$; thus φ_n defines an isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \mathbb{K}_n$.

Unless we are in the trivial case $n = 1$, it is *not* possible to define ψ_n *globally* as

$$\psi_n([\xi_0, \ldots, \xi_n]) = [P(\xi_0, \ldots, \xi_n), Q(\xi_0, \ldots, \xi_n)], \tag{1.3.7}$$

with $P, Q \in \mathbb{K}[\xi_0, \ldots, \xi_n]_e$ not vanishing simultaneously on $\mathbb{K}_n$. In fact suppose that (1.3.7) holds, and let

$$p(s,t) := P(s^n, \ldots, t^n), \quad q(s,t) := Q(s^n, \ldots, t^n).$$

The polynomials $p(s,t), q(s,t)$ are homogeneous of degree de, they do not vanish simultaneously on a non zero (s_0, t_0), and forall $[s,t] \in \mathbb{P}^1$ we have $[p(s,t), q(s,t)] = [s,t]$. It follows that $p(s,t) = s \cdot r(s,t)$ and $q(s,t) = t \cdot r(s,t)$, where $r(s,t)$ has no non trivial zeroes, i.e. $r(s,t)$ is constant. In particular $de = \deg p = \deg q = 1$, and hence $d = 1$.

Example 1.3.6. We recall the formula

$$\dim \mathbb{K}[Z_0, \ldots, Z_n]_d = \binom{d+n}{n}. \tag{1.3.8}$$

(See Exercise 1.8.1 for a proof.) Let $N(n;d) := \binom{d+n}{n} - 1$. Let

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu_d^n} & \mathbb{P}^{N(n;d)} \\ [Z] & \mapsto & [Z_0^d, Z_0^{d-1}Z_1, \ldots, Z_n^d] \end{array} \tag{1.3.9}$$

be defined by all homogeneous monomials of degree d - this is a *Veronese map*. Clearly ν_d^n is regular.

The homogeneous coordinates on $\mathbb{P}^{N(n;d)}$ appearing in (1.3.9) are indiced by length $n+1$ multiindices $I = (i_0, \ldots, i_n)$ such that $\deg I := i_0 + \ldots + i_n = d$; we denote them by $[\ldots, \xi_I, \ldots]$. Let $\mathscr{V}_d^n \subset \mathbb{P}^{N(n;d)}$ be the closed subset defined by

$$\mathscr{V}_d^n := V(\ldots, \xi_I \cdot \xi_J - \xi_K \cdot \xi_L, \ldots),$$

where I, J, L, K run through all multiindices such that $I + J = K + L$. Clearly $\nu_d^n(\mathbb{P}^n) \subset \mathscr{V}_d^n$. Let us show that ν_d^n is an isomorphism onto $\mathscr{V}_d^n$.

Given a length $n+1$ multiindex H of degree $d-1$, we let $H_s := H + e_s$, where, for $e_0, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$, i.e. e_s has alla entries equal to 0, except for the entry at place $s+1$, which is equal to 1. For $s \in \{0, \ldots, n\}$, let

$$\begin{array}{ccc} \mathscr{V}_d^n \backslash V(\xi_{H_0}, \ldots, \xi_{H_n}) & \xrightarrow{\varphi_d^n(H)} & \mathbb{P}^n \\ [\ldots, \xi_I, \ldots] & \mapsto & [\xi_{H_0}, \ldots, \xi_{H_n}] \end{array}$$

Let H, H' be length $n+1$ multiindices of degree $d-1$. It follows from the equations defining $\mathscr{V}_d^n$ that $\varphi_d^n(H)([z]) = \varphi_d^n(H')([z])$ for all $[Z]$ which is in the domain of $\varphi_d^n(H)$ and $\varphi_d^n(H')$. Thus the $\varphi_d^n(H)$'s define a regular map $\varphi_d^n \colon \mathscr{V}_d^n \to \mathbb{P}^n$. We claim that

$$\begin{aligned} \varphi_d^n \circ \nu_d^n &= \mathrm{Id}_{\mathbb{P}^n} && (1.3.10)\\ \nu_d^n \circ \varphi_d^n &= \mathrm{Id}_{\mathbb{P}^{N(n;d)}}\,. && (1.3.11)\end{aligned}$$

The first equality is easily checked. In order to check the second equality, one may proceed as follows. Let $[\xi] = [\ldots, \xi_I, \ldots] \in \mathscr{V}_d^n$ be a point such that $\xi_{de_s} \neq 0$ for some $s \in \{0, \ldots, n\}$. Then it is not difficult to show that there exists $[Z] \in \mathbb{P}^n$ such that $[\xi] = \nu_d^n([z])$. By (1.3.10), it follows that $\nu_d^n \circ \varphi_d^n([\xi]) = [\xi]$. Hence it suffices to prove that if $[\xi] \in \mathscr{V}_d^n$, then there exists $s \in \{0, \ldots, n\}$ such that $\xi_{de_s} \neq 0$. Thus, we must show that if $\ldots, \xi_I, \ldots$ are such that $\xi_I \cdot \xi_J = \xi_K \cdot \xi_L$ whenever $I + J = K + L$, and $\xi_{de_s} = 0$ for all $s \in \{0, \ldots, n\}$, then $\xi_I = 0$ for all multiindices I. This is easily proved by "descending induction" on the maximum of $i_0, \ldots, i_n$, by using a suitable relation $\xi_I^2 = \xi_K \cdot \xi_L$ (if the maximum is d, then $\xi_I = 0$ by hypothesis).

Example 1.3.7. Assume that $\operatorname{char} \mathbb{K} = p > 0$. Let $X = V(G_1, \ldots, G_r) \subset \mathbb{P}^n$ be a closed subset defined by homogeneous $G_1, \ldots, G_r \in \mathbb{F}_p[Z_0, \ldots, Z_n]$ (we require that the coefficients of the G_i's belong to the prime field $\mathbb{F}_p$). Then we may define the *Frobenius map* : $X \to X$ by setting

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ [Z] & \mapsto & [Z_0^p, \ldots, Z_i^p, \ldots, Z_n^p]. \end{array}$$

In fact, if $G_i = \sum_I a_J Z^J$, then

$$G_i(Z_0^p, \ldots, Z_i^p, \ldots, Z_n^p) = \sum_I a_J (Z^J)^p = \sum_I a_J^p (Z^J)^p = G_i(Z_0, \ldots, Z_i, \ldots, Z_n)^p = 0.$$

More generally, if all the coefficients of the G_i's are contained in $\mathbb{F}_{p^r}$ (e.g. if $\mathbb{K}$ is the algebraic closure of $\mathbb{F}_p$), then we may define $F \colon X \to X$ replacing the exponent p by p^r. Notice that F is bijective, but it is not an isomorphism.

Proposition 1.3.8. *A regular map of quasi projective varieties is Zariski continuous.*

Proof. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be Zariski locally closed, and let $f \colon X \to Y$ be a regular map. We must prove that if $C \subset Y$ is Zariski closed, then $f^{-1}C$ is Zariski closed in X. Let $U \subset W$ be an open subset such that (1.3.1) holds. Let us show that $\phi^{-1}C \cap U$ is closed in U. Since C is closed $C = V(I) \cap Y$ where $I \subset \mathbb{K}[T_0, \ldots, T_m]$ is a homogeneous ideal. Thus

$$\phi^{-1}C \cap U = \{[Z] \in U \mid P(F_0(Z), \ldots, F_m(Z)) = 0 \; \forall P \in I\}.$$

Since each $P(F_0(Z), \ldots, F_m(Z))$ is a homogeneous polynomial, we get that $\phi^{-1}C \cap U$ is closed in U.

By definition of regular map X can be covered by Zariski open sets U_α such that (1.3.1) holds with U replaced by U_α. We have proved that $C_\alpha := \phi^{-1}C \cap U_\alpha$ is closed in U_α for all α. It follows that $\phi^{-1}C$ is closed. In fact let $\overline{C}_\alpha \subset X$ be the closure of C_α and $D_\alpha := X \backslash U_\alpha$. Since C_α is closed in U_α we have

$$\overline{C}_\alpha \cap U_\alpha = C_\alpha = \phi^{-1}C \cap U_\alpha. \tag{1.3.12}$$

Moreover D_α is closed in X because U_α is open. By (1.3.12) we have

$$\phi^{-1}C = \bigcap_\alpha \left(\overline{C}_\alpha \cup D_\alpha\right).$$

Thus $\phi^{-1}C$ is an intersection of closed sets and hence is closed. $\square$

The following lemma will be useful later on. The easy proof is left to the reader.

Lemma 1.3.9. *Let $f\colon X \to Y$ be a map between quasi projective varieties. Suppose that $Y = \bigcup_{i\in I} U_i$ is an open cover, that $f^{-1}U_i$ is open in X for each $i \in I$ and that the restriction*

$$\begin{array}{ccc} f^{-1}U_i & \longrightarrow & U_i \\ x & \mapsto & f(x) \end{array}$$

is regular for each $i \in I$. Then f is regular.

Definition 1.3.10. A quasi projective variety is

- an *affine variety* if it is isomorphic to a closed subset of an affine space (as usual we view $\mathbb{A}^n$ as the open subset $\mathbb{P}^n_{Z_0} \subset \mathbb{P}^n$),
- a *projective variety* if it is isomorphic to a closed subset of a projective space.

Example 1.3.11. Let $F \in \mathbb{K}[Z_0, \ldots, Z_n]$ be a homogeneous polynomial of strictly positive degree. The principal open subset $\mathbb{P}^n_F$ (see Definition 1.1.8) is an affine variety. In fact, let $\nu^n_d\colon \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{d+n}{n}-1}$ be the Veronese map, see (1.3.9), and let $\mathscr{V}^n_d := \operatorname{Im}\nu^n_d$ be the corresponding Veronese variety. As shown in Example 1.3.6 the map $\mathbb{P}^n \to \mathscr{V}^n_d$ defined by ν^n_d is an isomorphism. It follows that the restriction of ν^n_d to $\mathbb{P}^n_F$ defines an isomorphism between $\mathbb{P}^n_F$ and $\mathscr{V}^n_d\backslash H$, where $H \subset \mathbb{P}^{\binom{d+n}{n}-1}$ is a suitable hyperplane section. Equivalently, $\mathbb{P}^n_F$ is isomorphic to the intersection of the affine space $\mathbb{P}^{\binom{d+n}{n}-1}\backslash H$ and the closed set $\mathscr{V}^n_d$, and hence is an affine variety.

If $Y \subset \mathbb{P}^n$ is closed, and $F \in \mathbb{K}[Z_0, \ldots, Z_n]$ is homogeneous of strictly positive degree d, it follows that the principal open set $Y_F = Y\backslash V(F)$ is an affine variety. In fact, since ν^n_d is an isomorphism $\nu^n_d(Y_F)$ is closed in the affine variety $\mathscr{V}^n_d\backslash H$, and hence is itself affine. Moreover, the restriction of ν^n_d to Y_F defines an isomorphism Y_F and the affine variety $\nu^n_d(Y_F)$.

Claim 1.1.9 and Example 1.3.11 give the following result.

Proposition 1.3.12. *The open affine subsets of a quasi projective variety form a basis of the Zariski topology.*

In a certain sense, open affine subsets of a quasi projective variety are similar to the open subsets of a complex manifold given by charts of a holomorphic atlas.

1.4 Regular functions on affine varieties

Definition 1.4.1. A *regular function* on a quasi projective variety X is a regular map $X \to \mathbb{K}$.

Let X be a non empty quasi projective variety. The set of regular functions on X with pointwise addition and multiplication is a $\mathbb{K}$-algebra, named the *ring of regular functions* of X. We denote it by $\mathbb{K}[X]$.

If X is a projective variety, then it has few regular functions. In fact we will prove (see Corollary 1.6.6) that every regular function on X is locally constant. On the other hand, affine varieties have plenty of functions. In fact if $X \subset \mathbb{A}^n$ is closed we have an inclusion

$$\mathbb{K}[z_1, \ldots, z_n]/I(X) \hookrightarrow \mathbb{K}[X]. \tag{1.4.1}$$

Theorem 1.4.2. *Let $X \subset \mathbb{A}^n$ be closed. Then* (1.4.1) *is an equality, i.e. every regular function on X is the restriction of a polynomial function on $\mathbb{A}^n$.*

Before proving Theorem 1.4.2, we notice that, if $X \subset \mathbb{A}^n$ is closed, the Nullstellensatz for $\mathbb{K}[z_1, \ldots, z_n]$ implies a Nullstellensatz for $\mathbb{K}[z_1, \ldots, z_n]/I(X)$. First a definition: given an ideal $J \subset (\mathbb{K}[z_1, \ldots, z_n]/I(X))$ we let

$$V(J) := \{a \in X \mid f(a) = 0 \quad \forall f \in J\}\,.$$

The following result follows at once from the Nullstellensatz.

Proposition 1.4.3 (Nullstellensatz for a closed subset of $\mathbb{A}^n$)**.** *Let $X \subset \mathbb{A}^n$ be closed, and let $J \subset (\mathbb{K}[z_1, \ldots, z_n]/I(X))$ be an ideal. Then*

$$\left\{f \in (\mathbb{K}[z_1, \ldots, z_n]/I(X)) \mid f_{|V(J)} = 0\right\} = \sqrt{J}.$$

(The radical $\sqrt{J}$ is taken inside $\mathbb{K}[z_1, \ldots, z_n]/I(X)$.) In particular $V(J) = \varnothing$ if and only if $J = (1)$.

The following example makes it clear that Proposition 1.4.3 must play a rôle in the proof of Theorem 1.4.2. Let $X \subset \mathbb{A}^n$ be closed. Suppose that $g \in \mathbb{K}[z_1, \ldots, z_n]$ and that $g(a) \neq 0$ for all $a \in Z$. Then $1/g \in \mathbb{K}[X]$ and hence Theorem 1.4.2 predicts the existence of $f \in \mathbb{K}[z_1, \ldots, z_n]$ such that $g^{-1} = f_{|X}$. By Proposition 1.4.3, $(g) = (1)$ in $\mathbb{K}[z_1, \ldots, z_n]/I(X)$, because $V(\overline{g}) = \varnothing$, where $\overline{g} := g_{|X}$. hence there exists $f \in \mathbb{K}[z_1, \ldots, z_n]$ such that $\overline{f} \cdot \overline{g} = 1$, where $\overline{f} := f_{|X}$, i.e. $g^{-1} = f_{|X}$

Proof of Theorem 1.4.2. Let $\varphi \in \mathbb{K}[X]$. We claim that there exist $f_i, g_i \in \mathbb{K}[z_1, \ldots, z_n]$ for $1 \leqslant i \leqslant d$ such that

1. $X = \bigcup_{1 \leqslant i \leqslant d} X_{g_i}$, i.e. $V(g_1, \ldots, g_d) \cap X = \varnothing$,
2. for all $a \in X_{g_i}$ we have $\varphi(a) = \frac{f_i(a)}{g_i(a)}$,
3. for $1 \leqslant i \leqslant j$ we have $(g_j f_i - g_i f_j)|_X = 0$.

(Notice: the last item implies that on $X_{g_i} \cap X_{g_j}$ we have $f_i/g_i = f_j/g_j$.) For $i = 1, \ldots, d$ let $\overline{g}_i := g_{i|X}$ and $\overline{f}_i := f_{i|X}$. Then

$$\overline{g}_i \varphi = \overline{f}_i. \tag{1.4.2}$$

In fact by Item (1) it suffices to check that (1.4.2) holds on X_{f_j} for $j = 1, \ldots, d$. For $j = i$ it holds by Item (2), for $j \neq i$ it holds by Item (3). (Notice: if we do not assume that Item (3) holds we only know that (1.4.2) holds on $U_j \cap U_i$.) By Proposition 1.4.3 we have that $(\overline{g}_1, \ldots, \overline{g}_d) = (1)$, i.e. there exist $h_1, \ldots, h_d \in \mathbb{K}[z_1, \ldots, z_n]$ such that

$$1 = \overline{h}_1 \overline{g}_1 + \cdots + \overline{h}_d \overline{g}_d.$$

where $\overline{h}_i := h_{i|X}$. Multiplying by φ both sides of the above equality and remembering (1.4.2) we get that

$$\varphi = \overline{h}_1 \overline{g}_1 \varphi + \cdots + \overline{h}_d \overline{g}_d \varphi = \overline{h}_1 \overline{f}_1 + \ldots + \overline{h}_1 \overline{f}_d = (h_1 f_1 + \cdots + h_d f_d)_{|X}. \tag{1.4.3}$$

It remains to prove that there exist $f_i, g_i \in \mathbb{K}[z_1, \ldots, z_n]$ with the properties stated above. By definition of regular function there exist an open covering of X, and for each set U of the open cover a couple $\alpha, \beta \in \mathbb{K}[z_1, \ldots, z_n]$ such that $\varphi(x) = \alpha(x)/\beta(x)$ for all $x \in U$ (it is understood that $\beta(x) \neq 0$ for all $x \in U$). By Remark 1.4.4 we may cover U by open affine sets $X_{\gamma_1}, \ldots, X_{\gamma_r}$. Since $V(\beta) \subset \bigcap_{i=1}^{r} V(\gamma_i)$ the Nullstellensatz gives that, for each i, there exist $N_i > 0$ and $\mu_i \in \mathbb{K}[z_1, \ldots, z_n]$ such that $\gamma_i^{N_i} = \mu_i \beta$ and hence $\varphi(x) = \mu_i(x)\alpha(x)/\gamma_i(x)^N$ for all $x \in X_{\gamma_i}$. Since $X_{\gamma_i} = X_{\gamma_i^N}$ we get that we have covered X by principal open sets $X_{g'}$ such that $\varphi = f'/g'$ for all $x \in X_{g'}$, where $f' \in \mathbb{K}[z_1, \ldots, z_n]$ (of course f' depends on g'). By Corollary 1.2.7, the open covering has a finite subcovering, corresponding to $f'_1, g'_1, \ldots, f'_d, g'_d$. Now let

$$f_i := f'_i g'_i, \qquad g_i := (g'_i)^2.$$

Clearly Items (1) and (2) hold. In order to check Item (3) we write

$$(g_j f_i - g_i f_j)|_X = ((g'_j)^2 f'_i g'_i - (g'_i)^2 f'_j g'_j)|_X = ((g'_i g'_j)(f'_i g'_j - f'_j g'_i))|_X.$$

Since $\varphi(z) = f'_i(z)/g'_i(z) = f'_j(z)/g'_j(z)$ for all $z \in X_{g'_i} \cap X_{g'_j}$ the last term vanishes on $X_{g'_i} \cap X_{g'_j}$, on the other hand it vanishes also on $(X \backslash X_{g'_i} \cap X_{g'_j}) = X \cap V(g'_i g'_j)$ because of the factor $(g'_i g'_j)$. □

We end the present section with a couple of consequences of Theorem 1.4.2.

First we give a more explicit version of Proposition 1.3.12 in the case that the quasi projective variety itself is affine. Given a quasi projective variety X, and $f \in \mathbb{K}[X]$, let

$$X_f := X \backslash V(f), \tag{1.4.4}$$

where $V(f) := \{x \in X \mid f(x) = 0\}$. The following remark is easily verified.

Remark 1.4.4. Let $X \subset \mathbb{A}^n$ be closed (and hence an affine variety). Let $f \in \mathbb{K}[X]$, and hence by Theorem 1.4.2 there exists $\widetilde{f} \in \mathbb{K}[z_1, \ldots, z_n]$ such that $\widetilde{f}_{|X} = f$. Let $Y \subset \mathbb{A}^{n+1}$ be the subset of solutions of $g(z_1, \ldots, z_n) = 0$ for all $g \in I(X)$, and the extra equation $f(z_1, \ldots, z_n) \cdot z_{n+1} - 1 = 0$. Then the map

$$\begin{array}{ccc} X_f & \longrightarrow & Y \\ (z_1, \ldots, z_n) & \mapsto & (z_1, \ldots, z_n, \frac{1}{f(z_1, \ldots, z_n)}) \end{array}$$

is an isomorphism. In particular X_f is an open affine subset of X. Moreover, the open affine subset X_f, for $f \in \mathbb{K}[X]$ form a basis for the Zariski topology of X.

Notice that, by Theorem 1.4.2 and the above isomorphism, every regular function on X_f is given by the restriction to X_f of $\frac{g}{f^m}$, where $g \in \mathbb{K}[X]$ and $m \in \mathbb{N}$.

Next, we give a few remarkable consequences of Theorem 1.4.2.

Proposition 1.4.5. *Let R be a finitely generated $\mathbb{K}$ algebra without nilpotents. There exists an affine variety X such that $\mathbb{K}[X] \cong R$ (as $\mathbb{K}$ algebras).*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be generators (over $\mathbb{K}$) of R, and let $\varphi\colon \mathbb{K}[z_1, \ldots, z_n] \to R$ be the surjection of algebras mapping z_i to α_i. The kernel of φ is an ideal $I \subset \mathbb{K}[z_1, \ldots, z_n]$, which is radical because R has no nilpotents. Let $X := V(I) \subset \mathbb{A}^n$. Then $\mathbb{K}[X] \cong R$ by Theorem 1.4.2. □

In order to introduce the next result, consider a regular map $f\colon X \to Y$ of (non empty) quasi projective varieties. The pull-back $f^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is the homomorphism of $\mathbb{K}$-algebras defined by $f^*(\varphi) := \varphi \circ f$.

Proposition 1.4.6. *Let Y be an affine variety, and let X be a quasi projective variety. The map*

$$\begin{array}{ccc} \{f\colon X \to Y \mid f \text{ regular}\} & \longrightarrow & \{\varphi\colon \mathbb{K}[Y] \to \mathbb{K}[X] \mid \varphi \text{ homomorphism of } \mathbb{K}\text{-algebras}\} \\ f & \mapsto & f^* \end{array} \tag{1.4.5}$$

is a bijection.

Proof. We may assume that $Y \subset \mathbb{A}^n$ is closed; let $\iota\colon Y \hookrightarrow \mathbb{A}^n$ be the inclusion map. Suppose that $f, g\colon X \to Y$ are regular maps, and that $f^* = g^*$. Then $f^*(\iota^*(z_i)) = g^*(\iota^*(z_i))$ for $i \in \{1, \ldots, n\}$, and hence $f = g$. This proves injectivity of the map in (1.4.5). In order to prove surjectivity, let $\varphi\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ be a homomorphism of $\mathbb{K}$ algebras. Let $f_i := \varphi(\iota^*(z_i))$, and let $f\colon X \to \mathbb{A}^n$ be the regular map defined by $f(x) := (f_1(x), \ldots, f_n(x))$ for $x \in X$. Then $f(x) \in Y$ for all $x \in X$. In fact, since Y is closed, it suffices to show that $g(f(x)) = 0$ for all $g \in I(X)$. Now

$$g(f_1(x), \ldots, f_n(x)) = g(\varphi(\iota^*(z_1)), \ldots, \varphi(\iota^*(z_n)) = \varphi(g(\iota^*(z_1)), \ldots, \iota^*(z_n)) = \varphi(0) = 0.$$

(The second and last equality hold because φ is a homomorphism of $\mathbb{K}$-algebras.) Thus f is a regular map $f\colon X \to Y$ such that $f^*(\iota^*(z_i)) = \varphi(\iota^*(z_i))$ for $i \in \{1, \ldots, n\}$. By Theorem 1.4.2 the $\mathbb{K}$-algebra $\mathbb{K}[Y]$ is generated by $\iota^*(z_1), \ldots, \iota^*(z_n)$; it follows that $f^* = \varphi$. □

Corollary 1.4.7. *In Proposition 1.4.5, the affine variety X such that $\mathbb{K}[X] \cong R$ is unique up to isomorphism.*

1.5 Products

We will prove that the category of quasi projective varieties has (finite) products.

First let X, Y be affine varieties. Thus, we may assume that $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are closed subsets. Then $X \times Y \subset \mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$ is a closed subset, and the maps $X \times Y \to X$ and $X \times Y \to Y$ given by the two projections are regular. One checks easily that $X \times Y$ with the two projection maps is the product of X and Y in the category of quasi projective varieties (use Proposition 1.4.6). The ring of regular functions of $X \times Y$ is constructed from $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ as follows. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projections. The $\mathbb{K}$-bilinear map

$$\begin{array}{ccc} \mathbb{K}[X] \times \mathbb{K}[Y] & \longrightarrow & \mathbb{K}[X \times Y] \\ (f, g) & \mapsto & \pi_X^*(f) \cdot \pi_Y^*(g) \end{array} \tag{1.5.1}$$

induces a linear map

$$\mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y] \longrightarrow \mathbb{K}[X \times Y]. \tag{1.5.2}$$

Proposition 1.5.1. *The map in* (1.5.2) *is an isomorphism.*

Proof. We may assume that $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are closed subsets. Then $X \times Y \subset \mathbb{A}^{m+n}$ is closed subset, and hence the map in (1.5.2) is surjective by Theorem 1.4.2. It remains to prove injectivity, i.e. the following: if $A \subset \mathbb{K}[X]$ and $B \subset \mathbb{K}[Y]$ are finite-dimensional complex vector subspaces, then the map $A \otimes B \to \mathbb{K}[X \times Y]$ obtained by restriction of (1.5.2) is injective. Let $\{f_1, \ldots, f_a\}$, $\{g_1, \ldots, g_b\}$ be bases of A and B. By considering the maps

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{K}^a \\ z & \mapsto & (f_1(z), \ldots, f_a(z)) \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & \mathbb{K}^b \\ z & \mapsto & (g_1(z), \ldots, g_b(z)) \end{array} \tag{1.5.3}$$

we get that there exist $p_1, \ldots, p_a \in X$ and $q_1, \ldots, q_b \in Y$ such that the square matrices $(f_i(p_j))$ and $(g_i(q_j))$ are non-singular. By change of bases, we may assume that $f_i(p_j) = \delta_{ij}$ and $g_k(q_h) = \delta_{kh}$. Computing the values of $\pi_X^*(f_i) \cdot \pi_Y^*(g_j)$ on (p_s, q_t) for $1 \leqslant i, s \leqslant a$ and $1 \leqslant j, t \leqslant b$ we get that the functions $\ldots, \pi_X^*(f_i) \cdot \pi_Y^*(g_j), \ldots$ are linearly independent. Thus $A \otimes B \to \mathbb{K}[W \times Z]$ is injective. $\square$

Since every quasi projective variety has an open cover by affine varieties, one could try to define the product of quasi projective varieties X and Y by gluing together the products of the affine varieties in open coverings of X and Y. This is done in scheme theory, where schemes are algebriac varieties defined by atlases with charts given by affine schemes. However, one wants to show more, for example that the product of projective varieties is a projective variety. This is why we need the more elaborate construction presented below.

Let $\mathscr{M}_{m+1,n+1}$ be the vector space of complex $(m+1) \times (n+1)$ matrices. Let

$$\Sigma_{m,n} := \{[A] \in \mathbb{P}(\mathscr{M}_{m+1,n+1}) \mid \operatorname{rk} A = 1\}.$$

Then $\Sigma_{m,n}$ is a projective variety in $\mathbb{P}(\mathscr{M}_{m+1,n+1}) = \mathbb{P}^{mn+m+n}$. In fact the entries of a non zero matrix $A \in \mathscr{M}_{m+1,n+1}$ define homogegeous coordinates on $\mathbb{P}(\mathscr{M}_{m+1,n+1})$, and $\Sigma_{m,n}$ is the set of zeroes of determinants of all 2×2 minors of A. Let $[W] \in \mathbb{P}^m$ and $[Z] \in \mathbb{P}^n$; then $W^t \cdot Z$ is a complex $(m+1) \times (n+1)$ matrix of rank 1, determined up to recsaling. Thus we have the *Segre map*

$$\begin{array}{ccc} \mathbb{P}^m \times \mathbb{P}^n & \xrightarrow{\sigma_{m,n}} & \Sigma_{m,n} \\ ([W], [Z]) & \mapsto & [W^t \cdot Z] \end{array} \tag{1.5.4}$$

Proposition 1.5.2. *The map in* (1.5.4) *is a bijection.*

From now on, we identify $\mathbb{P}^m \times \mathbb{P}^n$ with the projective variety $\Sigma_{m,n}$. In particular $\mathbb{P}^m \times \mathbb{P}^n$ has a Zariski topology.

Claim 1.5.3. *A subset $X \subset \mathbb{P}^m \times \mathbb{P}^n$ is closed if and only if there exist bihomogeneous polynomials* [1]

$$F_1, \dots, F_r \in \mathbb{K}[W_0, \dots, W_m, Z_0, \dots, Z_n]$$

such that

$$X = V(F_1, \dots, F_r) := \{([W],[Z]) \in \mathbb{P}^n \times \mathbb{P}^m \mid 0 = F_1(W;Z) = \cdots = F_r(W;Z)\}. \tag{1.5.5}$$

Remark 1.5.4. If $m \neq 0$ and $n \neq 0$, then the Zariski topology on the product $\mathbb{P}^m \times \mathbb{P}^n$ is *not* the product topology. In fact it is finer than the product topology

Example 1.5.5. The diagonal $\Delta_{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^n$ is closed. In fact, Δ is the set of couples $([W],[Z])$ such that the matrix with rows W and Z has rank less than 2, and hence it is the zero locus of the bihomogeneous polynomials $W_iZ_j - W_jZ_i$ for $(i,j) \in \{0, \dots, n\}$. Notice that this is not in contrast with the fact that, if $n \neq 0$, the Zariski topology on $\mathbb{P}^n$ is *not* Hausdorff, because of Remark 1.5.4.

Claim 1.5.6. *The projections of $\mathbb{P}^m \times \mathbb{P}^n$ on its two factors are regular maps.*

Proof. Let a_{ij}, where $(i,j) \in \{0, \dots, m\} \times \{0, \dots, n\}$, be the homogeneous coordinates on $\mathbb{P}(\mathscr{M}_{m+1,n+1})$ given by the entries of a matrix $A \in \mathscr{M}_{m+1,n+1}$. Then

$$\mathbb{P}^m \times \mathbb{P}^n = \bigcup_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} (\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}. \tag{1.5.6}$$

On the open subset $(\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}$, the projections $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$, $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ are given by

$$\begin{array}{ccc} \mathbb{P}^m \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^m \\ [A] & \mapsto & [a_{0j}, \dots, a_{mj}] \end{array} \qquad \begin{array}{ccc} \mathbb{P}^m \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ [A] & \mapsto & [a_{i0}, \dots, a_{in}] \end{array}$$

respectively. □

Proposition 1.5.7. *Let X be a quasi projective variety, and let $f\colon X \to \mathbb{P}^m$ and $g\colon X \to \mathbb{P}^n$ be regular maps. Then*

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^m \times \mathbb{P}^n \\ x & \mapsto & (f(x), g(x)) \end{array} \tag{1.5.7}$$

is a regular map.

Proof. We have the open cover of $\mathbb{P}^m \times \mathbb{P}^n$ given by (1.5.6), with open sets indicized by $\{0, \dots, m\} \times \{0, \dots, n\}$. By Lemma 1.3.9, it suffices to prove that, for each $(i,j) \in \{0, \dots, m\} \times \{0, \dots, n\}$, the following hold:

1. $(f \times g)^{-1}(\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}})$ is open in X.
2. The restriction
$$\begin{array}{ccc} (f \times g)^{-1}(\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}) & \longrightarrow & (\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}} \\ x & \mapsto & (f(x), g(x)) \end{array} \tag{1.5.8}$$
is regular.

We have

$$(f \times g)^{-1}((\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}}) = X \backslash (f^{-1}V(W_i) \cup g^{-1}V(Z_j)).$$

Both f and g are continuous, because they are regular, and hence $f^{-1}V(X_i)$ and $g^{-1}V(Y_j)$ are closed. It follows that Item (1) holds. The map

$$\begin{array}{ccc} \mathbb{A}^m \times \mathbb{A}^n & \longrightarrow & (\mathbb{P}^m \times \mathbb{P}^n)_{a_{ij}} \\ ((w_0, \dots, \widehat{w}_i, \dots, w_m), (z_0, \dots, \widehat{z}_j, \dots, z_n)) & \mapsto & ([w_0, \dots, w_{i-1}, 1, w_{i+1} \dots, w_m], [z_0, \dots, z_{j-1}, 1, z_{j+1}, \dots, z_n]) \end{array}$$

is an isomorphism commuting with the projections. Item (2) follows. □

[1] A polynomial $F \in \mathbb{K}[W;Z]$ is *bihomogeneous of degree* (d,e) if $F = \sum_{\substack{\deg I = d \\ \deg J = e}} a_{I,J} W^I Z^J$.

It follows that $\mathbb{P}^m \times \mathbb{P}^n$ with the two projections is the product of $\mathbb{P}^m$ and $\mathbb{P}^n$ in the category of quasi projective varieties.

Now suppose that $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are locally closed sets. It follows from Claim 1.5.5 that $Y \times Y \subset \mathbb{P}^m \times \mathbb{P}^n$ is locally closed, i.e. we have identified $W \times Z$ with a quasi-projective set. Moreover, the projections of $X \times Y$ to X and Y are regular, because they are the restrictions of the projections of $\mathbb{P}^m \times \mathbb{P}^n$ to $X \times Y$.

The proof of the following result is easy; we leave details to the reader.

Proposition 1.5.8. *Keep notation as above. The quasi projective variety $X \times Y$, with the projections to the two factors, is the product of X and Y in the category of quasi projective sets.*

Notice that if $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are closed then $X \times Y$ is closed in $\mathbb{P}^m \times \mathbb{P}^n$. Hence the product of projective varieties is a projective variety. On the othar hand, we have already observed that the product of affine varieties is an affine varietry.

Remark 1.5.9. Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ be locally closed sets. Let $\varphi\colon X \xrightarrow{\sim} X'$, $\psi\colon Y \xrightarrow{\sim} Y'$ be isomorphisms, where $X' \subset \mathbb{P}^a$ and $Y' \subset \mathbb{P}^b$ are locally closed sets. Then

$$\begin{array}{ccc} X \times Y & \longrightarrow & X' \times Y' \\ (p,q) & \mapsto & (\varphi(p), \psi(q)) \end{array} \tag{1.5.9}$$

is an isomorphism. This follows from the formal property of a categorical product. Thus the isomorphism class of $X \times Y$ is independent of the embeddings $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$. This is why we say that $X \times Y$ is *the* product of X and Y.

Since the product of two quasi projective varieties exists, also the product $X_1 \times \ldots \times X_r$ of a finite collection $X_1, \ldots, X_r$ of quasi-projective varieties exists; it is given by $(X_1 \times (X_2 \times (X_3 \ldots \times X_r) \ldots)$ (we may rearrange the parenthesis arbitrarily, and we will get an isomorphic variety).

Let X be a quasi projective variety, and let $\Delta_X \subset X \times X$ be the diagonal. It follows from Example 1.5.5 that Δ_X is closed in $X \times X$ (this is not in contradiction with the fact that, if X is not finite, then it is not Hausdorff, see Remark 1.5.4). This property of quasi projective varieties goes under the name of *properness*. The following is a consequence of properness.

Proposition 1.5.10. *Let X, Y be quasi projective varieties, and let f, g be regular maps $X \to Y$. If $f(x) = g(x)$ for x in a dense subset of X, then $f = g$.*

Proof. Let $\varphi\colon X \to Y \times Y$ be the map defined by $\varphi(x) := (f(x), g(x))$. Then φ is regular, because $Y \times Y$ is the categorical square of Y. Since Δ_Y is closed, $\varphi^{-1}(\Delta_Y)$ is closed. By hypothesis $\varphi^{-1}(\Delta_Y)$ contains a dense subset of X, hence it is equal to X, i.e. $f(x) = g(x)$ for all $x \in X$. $\square$

1.6 Elimination theory

Let M be a topological space. Then M is quasi compact, i.e. every open covering has a finite subcovering, if and only if M is universally closed, i.e. for any topological space T, the projection map $T \times M \to T$ is closed, i.e. it maps closed sets to closed sets. (See tag/005M in [**?**].)

A quasi projective variety X is quasi compact, but it is not generally true that, for a variety T, the projection $T \times X \to T$ is closed. In fact, let $X \subset \mathbb{P}^n$ be locally closed; then Δ_X, the diagonal of X, is closed in $X \times \mathbb{P}^n$, because it is the intersection of $X \times X \subset \mathbb{P}^n \times \mathbb{P}^n$ with the diagonal $\Delta_{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^n$, which is closed. The projection $X \times \mathbb{P}^n \to \mathbb{P}^n$ maps X to X, hence if X is not closed in $\mathbb{P}^n$, then X is not universally closed. This does not contradict the result in topology quoted above, because the Zariski topology of the product of quasi projective varieties is not the product topology.

The following key result states that projective varieties are the equivalent of compact topological spaces in the category of quasi projective varieties.

Theorem 1.6.1 (Main Theorem of elimination theory)**.** *Let T be a quasi-projective variety and let X be a closed subset of a projective space. Then the projection*

$$\pi \colon T \times X \to T$$

is closed.

Proof. By hypothesis we may assume that $X \subset \mathbb{P}^n$ is closed. It follows that $T \times X \subset T \times \mathbb{P}^n$ is closed. Thus it suffices to prove the result for $X = \mathbb{P}^n$. Since T is covered by open affine subsets, we may assume that T is affine, i.e. T is (isomorphic to) a closed subset of $\mathbb{A}^m$ for some m. It follows that it suffices to prove the proposition for $T = \mathbb{A}^m$. To sum up: it suffices to prove that if $X \subset \mathbb{A}^m \times \mathbb{P}^n$ is closed, then $\pi(X)$ is closed in $\mathbb{A}^m$, where $\pi \colon \mathbb{A}^m \times \mathbb{P}^n \to \mathbb{A}^m$ is the projection. We will show that $(\mathbb{A}^m \backslash \pi(X))$ is open. By Claim 1.5.5 there exist $F_i \in \mathbb{K}[t_1, \ldots, t_m, Z_0, \ldots, Z_n]$ for $i = 1, \ldots, r$, *homogeneous* as polynomial in $X_0, \ldots, X_n$ such that

$$X = \{(t, [Z]) \mid 0 = F_1(t, Z) = \ldots = F_r(t, Z)\}.$$

Suppose that $F_i \in \mathbb{K}[t_1, \ldots, t_m][Z_0, \ldots, Z_n]_{d_i}$ i.e. F_i is homogeneous of degree d_i in $Z_0, \ldots, Z_n$. Let $\bar{t} \in (T \backslash \pi(X))$. By Hilbert's Nullstellensatz, there exists $N \geqslant 0$ such that

$$(F_1(\bar{t}, Z), \ldots, F_r(\bar{t}, Z)) \supset \mathbb{K}[Z_0, \ldots, Z_n]_N. \tag{1.6.10}$$

We may assume that $N \geqslant d_i$ for $1 \leqslant i \leqslant r$. For $t \in \mathbb{A}^m$ let

$$\begin{array}{ccc} \mathbb{K}[Z_0, \ldots, Z_n]_{N-d_1} \times \ldots \times [Z_0, \ldots, Z_n]_{N-d_r} & \xrightarrow{\Phi(t)} & \mathbb{K}[Z_0, \ldots, Z_n]_N \\ (G_1, \ldots, G_r) & \mapsto & \sum_{i=1}^r G_i \cdot F_i \end{array}$$

Thus $\Phi(t)$ is a linear map: choose bases of domain and codomain and let $M(t)$ be the matrix associated to $\Phi(t)$. Clearly the entries of $M(t)$ are elements of $\mathbb{K}[t_1, \ldots, t_m]$. By hypothesis $\Phi(\bar{t})$ is surjective and hence there exists a maximal minor of $M(t)$, say $M_{I,J}(t)$, such that $\det M_{I,J}(\bar{t}) \neq 0$. The open $(\mathbb{A}^m \backslash V(\det M_{I,J}))$ is contained in $(T \backslash \pi(X))$. This finishes the proof of Theorem 1.6.1. ☐

We will give a few corollaries of Theorem 1.6.1. First, we prove an elementary auxiliary result.

Lemma 1.6.2. *Let $f \colon X \to Y$ be a regular map between quasi-projective varieties. The graph of f*

$$\Gamma_f := \{(x, f(x)) \mid \ p \in X\}$$

is closed in $X \times Y$.

Proof. The map

$$f \times \mathrm{Id}_Y \colon X \times Y \to Y \times Y$$

is regular, and $\Gamma_f = (f \times \mathrm{Id}_X)^{-1}(\Delta_Y)$. Hence Γ_f is closed because Δ_Y is closed in $Y \times Y$. ☐

Proposition 1.6.3. *Let $X \subset \mathbb{P}^n$ be closed, and let Y be a quasi-projective set. A regular map $f \colon X \to Y$ is closed.*

Proof. Since closed subsets of X are projective it suffices to prove that $f(X)$ is closed in Y. Let $\pi \colon X \times Y \to Y$ be the projection map. Then $f(X) = \pi(\Gamma_f)$. By Lemma 1.6.2 and the Main Theorem of elimination theory we get that $f(X)$ is closed. ☐

Corollary 1.6.4. *A locally-closed subset of $\mathbb{P}^n$ is projective if and only if it is closed.*

Proof. Let $X \subset \mathbb{P}^n$ be a locally closed subset. If it is closed, then it is projective by definition. Conversely, suppose that X is projective. Hence there exist a closed subset $Y \subset \mathbb{P}^m$ and an isomorphism $f \colon Y \xrightarrow{\sim} X$. Composing f with the inclusion $X \hookrightarrow \mathbb{P}^n$, we get a regular map $g \colon Y \to \mathbb{P}^n$. Then $X = g(Y)$ is closed by Proposition 1.6.3. ☐

Remark 1.6.5. By way of contrast, notice that it is *not* true that a locally-closed subset of $\mathbb{A}^n$ is affine if and only if it is closed. In fact the complement of a hypersurface $V(f) \subset \mathbb{A}^n$ is affine but not closed.

Corollary 1.6.6. *Let X be a projective variety. A regular map $f\colon X \to \mathbb{K}$ is locally constant.*

Proof. Composing f with the inclusion $j\colon \mathbb{K} \hookrightarrow \mathbb{P}^1$ we get a regular map $\overline{f}\colon X \to \mathbb{P}^1$. By Proposition 1.6.3 $\overline{f}(X)$ is closed. Since $\overline{f}(X) \not\ni [0,1]$ it follows that $\overline{f}(X) = f(X)$ is a finite set. □

1.7 Grassmannians

Let V be a complex vector space of finite dimension, and let $0 \leqslant h \leqslant \dim V$. The *Grassmannian* of h-dimensional vector subpaces of V is the set of (complex) subvector spaces of V of dimension h:

$$\mathrm{Gr}\,(h, V) := \{W \subset V \mid \dim W = h\}\,.$$

Notice that if $h \in \{0, \dim V\}$, then $\mathrm{Gr}\,(h, V)$ is a singleton, that $\mathrm{Gr}\,(1, V) = \mathbb{P}(V)$, and that we have a bijection

$$\begin{array}{ccc} \mathbb{P}(V^\vee) & \longrightarrow & \mathrm{Gr}\,(\dim V - 1, V) \\ [f] & \mapsto & \ker(f) \end{array}$$

We will identify the elements of $\mathrm{Gr}\,(h, V)$ with the points of a projective variety. Consider the *Plücker map*

$$\begin{array}{ccc} \mathrm{Gr}\,(h, V) & \stackrel{\mathscr{P}}{\longrightarrow} & \mathbb{P}\left(\bigwedge^h V\right) \\ W & \mapsto & \bigwedge^h W. \end{array}$$

(this makes sense: $\bigwedge^h W$ is a 1-dimensional subspace of $\bigwedge^h V$ because $\dim W = h$).

Proposition 1.7.1. *Keep notation as above. Then $\mathscr{P}$ is injective, and $\mathrm{Im}\,\mathscr{P}$ is a closed subset of $\mathbb{P}\left(\bigwedge^h V\right)$.*

Before proving Proposition 1.7.1, we prove the result below.

Lemma 1.7.2. *Let $v_1, \ldots, v_a \in V$ be linearly independent, and let $\alpha \in \bigwedge^h V$. Then*

$$v_i \wedge \alpha = 0 \quad \forall i \in \{1, \ldots, a\} \tag{1.7.1}$$

if and only if there exists $\beta \in \bigwedge^{h-a} V$ such that

$$\alpha = v_1 \wedge \ldots \wedge v_a \wedge \beta. \tag{1.7.2}$$

Proof. The non trivial statement is that if (1.7.1) holds, then (1.7.2) holds. Extend $v_1, \ldots, v_s$ to a basis $v_1, \ldots, v_m$ of V. Given a subset $I \subset \{1, \ldots, m\}$ of cardinality s, we let $v_I = v_{i_1} \wedge \ldots \wedge v_{i_s}$, where $I = \{i_1, \ldots, i_s\}$ and $1 \leqslant i_1 < \ldots < i_s \leqslant m$. The collection of the v_I's is a basis of the exterior algebra $\bigwedge^\bullet V$. Hence

$$\alpha = \sum_{|I|=h} c_I v_I,$$

where c_I are complex numbers. Since

$$0 = v_i \wedge \alpha = \sum_{\substack{|I|=h \\ i \notin I}} c_I v_i \wedge v_I,$$

it follows that $\{1, \ldots, a\} \subset I$ for all I such that $c_I \neq 0$. Now, if $\{1, \ldots, a\} \subset I$, then $v_I = v_1 \wedge \ldots \wedge v_a \wedge \gamma$. It follows that (1.7.2) holds.

□

Proof of Proposition 1.7.1. For $\alpha \in \bigwedge^h V$, let m_α be the linear map

$$\begin{array}{ccc} V & \xrightarrow{m_\alpha} & \bigwedge^{h+1} V \\ v & \mapsto & v \wedge \alpha \end{array}$$

It follows from Lemma 1.7.2 that if $\alpha \neq 0$, then the kernel of m_α has dimension at most h, and that $\dim \ker(m_\alpha) = h$ if and only if α is *decomposable*, i.e. $\alpha = w_1 \wedge \ldots \wedge w_h$, where $w_1 \wedge \ldots \wedge w_h \in V$ are linearly independent. Thus

$$\operatorname{Im}(\mathscr{P}) = \left\{ [\alpha] \in \mathbb{P}\left(\bigwedge^h V\right) \mid \dim(\ker m_\alpha) \geqslant h \right\}, \tag{1.7.3}$$

and if $[\alpha] \in \operatorname{Im}(\mathscr{P})$, then $[\alpha] = \bigwedge^h \ker(m_\alpha)$. The latter equality shows that $\mathscr{P}$ is injective. Morover, the equality in (1.7.3) shows that $\operatorname{Im}(\mathscr{P})$ is closed in $\mathbb{P}(\bigwedge^h V)$. In fact, choose a basis $v_1, \ldots, v_m$ of V, and let $x_1, \ldots, x_n$ be the associated dual basis. Notice that the basis of V determines the basis $\ldots, v_I, \ldots$ (where $|II = h$) of $\bigwedge^h V$, and hence projective coordinates $[\ldots, Z_I, \ldots]$ (where $|II = h$) on $\mathbb{P}(\bigwedge^h V)$. Then m_α is described (with respect to the chosen bases) by a matrix of order $\binom{n}{h+1} \times h$ with entries linear functions in $x_1, \ldots, x_n$. Hence the right hand side of (1.7.3) is the set of points where all determinants of minors of order $(n-h+1) \times (n-h+1)$ of m_α vanish. Thus $\operatorname{Im}(\mathscr{P})$ is equal to the common zeroes of homogeneous polynomials (of degree $n-h+1$) in the homogeneous coordinates $[\ldots, Z_I, \ldots]$, it follows that is closed. ☐

Remark 1.7.3. In the proof of Proposition 1.7.1 we exhibited polynomials defining $\mathrm{Gr}\,(h, V)$ which are of high degree. In fact, the ideal of $\mathrm{Gr}\,(h, V) \subset \mathbb{P}(\bigwedge^{h+1} V)$ is generated by quadrics. In the first non-trivial case, i.e. $h \notin \{0, 1, \dim V - 1, \dim V\}$, i.e. $\mathrm{Gr}\,(2, V)$ with $\dim V \geqslant 4$, we can easily describe the Plücker quadrics generating the (homoheneous) ideal of the Grassmannian; in fact $\alpha \in \bigwedge^2 V$ is decomposable if and only if $\alpha \wedge \alpha = 0$.

Remark 1.7.4. We have a bijection between $\mathrm{Gr}\,(k+1, V)$ and the set of linear subspaces of $\mathbb{P}(V)$ of dimension k:

$$\begin{array}{ccc} \mathrm{Gr}\,(k+1, V) & \longrightarrow & \mathrm{Gr}(k, \mathbb{P}(V)) := \{L \subset \mathbb{P}(V) \mid L \text{ linear subspace, } \dim L = k\} \\ W & \mapsto & \mathbb{P}(W). \end{array}$$

Thus by Proposition 1.7.1 we may identify $\mathrm{Gr}\,(k, \mathbb{P}(V))$ with a projective set.

In order to do computations, we will need to write explicitly homogeneous of the Plücker image of elements $W \in \mathrm{Gr}\,(h, V)$. This is done as follows. Let $v_1, \ldots, v_m$ be a basis of V, and let $(\ldots, v_I, \ldots)$ be the associated basis of $\bigwedge^h V$, where I runs through subsets of $\{1, \ldots, m\}$ of cardinality h (notation as in the proof of Lemma 1.7.2). Thus we also have associated homogeneous coordinates $[\ldots, T_I, \ldots]$ on $\mathbb{P}(\bigwedge^h V)$. By associating to linearly independent vectors $w_1, \ldots, w_h \in V$ the matrix with rows the coordinates of the w_i's in the chosen basis, we get a matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{h1} & \cdots & a_{hm} \end{pmatrix}$$

of rank h. Viceversa, every such matrix determines the coordinates of linearly independent vectors $w_1, \ldots, w_h \in V$. Now, the homogeneous coordinates $[\ldots, T_I, \ldots]$ of $\mathscr{P}(\langle w_1, \ldots, w_h \rangle)$ are given by

$$T_I = \det \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_h} \\ \vdots & \ddots & \vdots \\ a_{h,i_1} & \cdots & a_{h,i_h} \end{pmatrix},$$

where, as usual, $I = \{i_1, \ldots, i_h\}$ with $1 \leqslant i_1 < \ldots < i_h \leqslant \dim V$.

Proposition 1.7.5. *The Grassmannian* $\mathrm{Gr}(h,V)$ *has an open covering by pairwise intersecting open subsets isomorphic to an affine space of dimension* $h\cdot(\dim V-h)$.

Proof. We identify $\mathrm{Gr}(h,V)$ with its image by the Plücker map $\mathscr{P}(\mathrm{Gr}(h,V))\subset\mathbb{P}(\bigwedge^h V)$. Let $m:=\dim V$, and let $v_1,\ldots,v_m$ be a basis of V. Keep the notation introduced above. In particular $[\ldots,T_I,\ldots]$ are homogeneous coordinates on $\mathbb{P}(\bigwedge^h V)$, where I runs through subsets of $\{1,\ldots,m\}$ of cardinality h. Thus we have the open covering

$$\mathrm{Gr}(h,V)=\bigcup_{|I|=h}\mathrm{Gr}(h,V)_I,\tag{1.7.4}$$

where, as usual $\mathrm{Gr}(h,V)_I\subset\mathrm{Gr}(h,V)$ is the open subset of points such that $T_I\neq 0$. Let $I=\{1,\ldots,h\}$. The map

$$\begin{array}{ccc}\mathscr{M}_{h,m-h}(\mathbb{K}) & \longrightarrow & \mathrm{Gr}(h,V)_I\\ \begin{pmatrix}a_{1,1} & \cdots & a_{1,m-h}\\ \vdots & \ddots & \vdots\\ a_{h,1} & \cdots & a_{h,m-h}\end{pmatrix} & \mapsto & \langle\ldots,v_i+\sum_{j=1}^{m-h}a_{i,j}v_{h+j},\ldots\rangle_{1\leqslant i\leqslant h}\end{array}\tag{1.7.5}$$

is an isomorphism. We have similar isomorphisms

$$\mathbb{A}^{h(m-k)}\cong\mathscr{M}_{h,m-h}(\mathbb{K})\xrightarrow{\sim}\mathrm{Gr}\,(h,V)_J$$

for any other multiindex J. One easily checks that for all subsets $I,J\subset\{1,\ldots,m\}$ of cardinality h the interesection $\mathrm{Gr}\,(h,V)_I\cap\mathrm{Gr}\,(h,V)_J$ is non empty. $\square$

Corollary 1.7.6. *The Grassmannian* $\mathrm{Gr}(h,V)$ *is irreducible.*

Remark 1.7.7. Let $E\subset\mathrm{Gr}(h,V)\times V$ be the subset of couples (v,W) such that $v\in W$, and let $\pi\colon E\to\mathrm{Gr}(h,V)$ be the defined by $(v,W)\mapsto W$. One easily checks that E is closed, and that π is a regular map. The inverse image $\pi^{-1}(\mathrm{Gr}(h,V)_I)$ is described as follows. For $A\in\mathscr{M}_{h,m-h}(\mathbb{K})$, let $w_i(A)\in V$ for $i\in\{1,\ldots,h\}$ be the vector appearing in (1.7.5). Then (1.7.5) gives an isomorphism

$$\begin{array}{ccc}\mathscr{M}_{h,m-h}(\mathbb{K})\times\mathbb{K}^h & \longrightarrow & \pi^{-1}(\mathrm{Gr}(h,V)_I)\\ (A,t) & \mapsto & (\langle w_1(A),\ldots,w_h(A)\rangle,\,\sum_{i=1}^h t_iw_i(A))\end{array}\tag{1.7.6}$$

where $t=(t_1,\ldots,t_h)\in\mathbb{K}^h$.

1.8 Exercises

Exercise 1.8.1. Let k be a field. Given a finite-dimensional k-vector space V define the formal power series $p_V\in k[[t]]$ as

$$P_V:=\sum_{d=0}^{\infty}(\dim_k\mathrm{Sym}^d V)t^d$$

where $\mathrm{Sym}^d V$ is the symmetric product of V. Thus if $V=k[x_1,\ldots,x_n]_1$ then $S^d(k[x_1,\ldots,x_n]_1)=k[x_1,\ldots,x_n]_d$.

1. Prove that if $V=U\oplus W$ then $P_V=P_U\cdot P_W$.
2. Prove that if $\dim_k V=n$ then $P_V=(1-t)^{-n}$ and hence (1.3.8) holds.

Exercise 1.8.2. The purpose of the present exercise is to give a different proof of the properties of the Veronese map ν_d^n discussed in Example 1.3.6, *valid if* $\mathrm{char}\,\mathbb{K}=0$, *or more generally* $\mathrm{char}\,\mathbb{K}$ *does not divide* $d!$. Let

$$\begin{array}{ccc}\mathbb{P}(\mathbb{K}[T_0,\ldots,T_n]_1) & \xrightarrow{\mu_d^n} & \mathbb{P}(\mathbb{K}[T_0,\ldots,T_n]_d)\\ [L] & \mapsto & [L^d]\end{array}\tag{1.8.7}$$

and let $\mathscr{W}_d^n = \operatorname{Im}(\mu_d^n)$. The above map can be identified with the Veronese map ν_d^n. In fact, writing $L \in \mathbb{K}[T_0, \ldots, T_n]_1$ as $L = \sum_{i=0}^n \alpha_i T_i$, we see that $[\alpha_0, \ldots, \alpha_n]$ are coordinates on $\mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1)$, and they give an identification $\mathbb{P}^n \xrightarrow{\sim} \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1)$. Moreover, let

$$\begin{array}{ccc} \mathbb{P}^{\binom{d+n}{n}-1} & \xrightarrow{\sim} & \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_d), \\ [\ldots, \xi_I, \ldots] & \mapsto & \sum\limits_{\substack{I=(i_0,\ldots,i_n) \\ i_0+\ldots+i_n=d}} \frac{d!}{i_0! \cdot \ldots \cdot i_n!} \xi_I T^I \end{array}$$

where $T^I = T_0^{i_0} \cdot \ldots \cdot T_n^{i_n}$. By Newton's formula $(\sum_{i=0}^n \alpha_i T_i)^d = \sum_I \frac{d!}{i_0! \cdot \ldots \cdot i_n!} \alpha^I T^I$, we see that, modulo the above isomorphisms, the Veronese map ν_d^n is identified with μ_d^n, and hence $\mathscr{V}_d^n$ is identified with $\mathscr{W}_d^n$.

Now let us show that $\mathscr{W}_d^n$ is closed. The key observation is that $[F] \in \mathscr{W}_d^n$ if and only if $\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}$ span a 1-dimensional subspace of $\mathbb{K}[Z_0, \ldots, Z_n]$. This may be proved by induction on $\deg F$ and *Euler's identity*

$$\sum_{j=0}^n Z_j \frac{\partial F}{\partial Z_j} = (\deg F) \cdot F, \tag{1.8.8}$$

valid for F *homogeneous*. Now, the condition that $\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}$ span a 1-dimensional subspace of $\mathbb{K}[Z_0, \ldots, Z_n]$ is equivalent to the vanishing of determinants of all 2×2 minors of the matrix whose entries are the coordinates of $\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}$; thus $\mathscr{W}_d^n$ is closed.

In order to show that μ_d^n is an isomorphism, we notice that if $F = L^d$, where $L \in \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1$ is non zero, then for each $i \in \{0, \ldots, n\}$ the partial derivative $\frac{\partial^{n-1} F}{\partial Z_i^{n-1}}$ is a multiple of L (eventually equal to 0 if $\frac{\partial L}{\partial Z_i} = 0$), and that one at least of such $(n-1)$-th partial derivative is non zero. Thus, the inverse of μ_d^n is the regular map $\theta_d^n \colon \mathscr{W}_d^n \longrightarrow \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1)$ defined by

$$\theta_d^n([F]) := \begin{cases} [\frac{\partial^{n-1} F}{\partial Z_0^{n-1}}] & \text{if } \frac{\partial^{n-1} F}{\partial Z_0^{n-1}} \neq 0, \\ \ldots\ldots & \ldots\ldots \\ [\frac{\partial^{n-1} F}{\partial Z_n^{n-1}}] & \text{if } \frac{\partial^{n-1} F}{\partial Z_n^{n-1}} \neq 0. \end{cases} \tag{1.8.9}$$

Exercise 1.8.3. We recall that if $\phi \colon B \to A$ is a homomorphism of rings, and $I \subset A$, $J \subset B$ are ideals, the *contraction* $I^c \subset B$ and the *extension* $J^e \subset A$ are the ideals defined as follows:

$$I^c := \phi^{-1} I, \quad J^e := \left\{ \sum_{i=1}^r \lambda_i \phi(b_i) \mid \lambda_i \in A,\ b_i \in J\ \forall i = 1, \ldots, r \right\} \tag{1.8.10}$$

(In other words, J^e is the ideal of A generated by $\phi(J)$.)

Let $f \colon X \to Y$ be a regular map between affine varieties and suppose that $f^* \colon \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ is injective.

1. Let $p \in X$. Prove that $\mathfrak{m}_p^c = \mathfrak{m}_{f(p)}$, in particular it is maximal.
2. Let $q \in Y$. Prove that
$$f^{-1}(q) = \{p \in X \mid \mathfrak{m}_p \supset \mathfrak{m}_q^e\},$$
and conclude, by the Nulstellensatz, that $f^{-1}(q)$ is not empty if and only if $\mathfrak{m}_q^e \neq \mathbb{K}[X]$.

Exercise 1.8.4. The left action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{A}^n$ defines a left action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{K}[z_1, \ldots, z_n]$ as follows. Let $\phi \in \mathbb{K}[z_1, \ldots, z_n]$ and $g \in \mathrm{GL}_n(\mathbb{K})$. Let z be the column vector with entries $z_1, \ldots, z_n$: we define $g\phi \in \mathbb{K}[z_1, \ldots, z_n]$ by letting
$$g\phi(X) := \phi(g^{-1} \cdot z).$$
Now let $G < \mathrm{GL}_n(\mathbb{K})$ be a subgroup. The *algebra of G-invariant polynomials* is
$$\mathbb{K}[z_1, \ldots, z_n]^G := \{\phi \mathbb{K}[z_1, \ldots, z_n] \in \mid g\phi = \phi\ \forall g \in G\}.$$
(it is clearly a $\mathbb{K}$-algebra). Now suppose that G is finite. One identifies $\mathbb{A}^n/G$ with an affine variety proceeding as follows.

1. Define the *Reynolds operator* as
$$\begin{array}{ccc} \mathbb{K}[z_1, \ldots, z_n] & \longrightarrow & \mathbb{K}[z_1, \ldots, z_n]^G \\ \phi & \mapsto & \frac{1}{|G|} \sum_{g \in G} g\phi. \end{array}$$
Prove the *Reynolds identity*
$$R(\phi\psi) = \phi R(\psi) \quad \forall \phi \in \mathbb{K}[z_1, \ldots, z_n]^G.$$

2. Let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be the ideal generated by *homogeneous* $\phi \in \mathbb{K}[z_1, \ldots, z_n]^G$ of strictly positive degree (i.e. non-constant). By Hilbert's basis theorem there exists a finite basis $\{\phi_1, \ldots, \phi_d\}$ of I; we may assume that each ϕ_i is homogeneous and G-invariant. Prove that $\mathbb{K}[z_1, \ldots, z_n]^G$ is generated *as* $\mathbb{K}$*-algebra* by $\phi_1, \ldots, \phi_d$. Since $\mathbb{K}[z_1, \ldots, z_n]^G$ is an integral domain with no nilpotents it follows that there exist an affine variety X (well-defined up to isomorphism) such that $\mathbb{K}[X] \xrightarrow{\sim} \mathbb{K}[z_1, \ldots, z_n]^G$. One sets $\mathbb{A}^n/G =: X$.
3. Let $\iota\colon \mathbb{K}[z_1, \ldots, z_n]^G \hookrightarrow \mathbb{K}[z_1, \ldots, z_n]$ be the inclusion map. By Proposition 1.4.6, there exist a unique regular map
$$\mathbb{A}^n \xrightarrow{\pi} X = \mathbb{A}^n/G. \tag{1.8.11}$$
such that $\iota = \pi^*$. Prove that
$$\pi(p) = \pi(q) \quad \text{if and only if} \quad q = gp \text{ for some } g \in G,$$
and that π is surjective. [*Hint:* Let $J \subset \mathbb{K}[z_1, \ldots, z_n]^G$ be an ideal. Show that $J^e \cap \mathbb{K}[z_1, \ldots, z_n]^G = J$ where J^e is the extension relative to the inclusion ι.]

Exercise 1.8.5. Keep notation and hypotheses as in Exercise 1.8.4. Describe explicitly $\mathbb{A}^n/G$ and the quotient map $\pi\colon \mathbb{A}^n \to \mathbb{A}^n/G$ for the following groups $G < \mathrm{GL}_n(\mathbb{K})$:

1. $n = 2$, $G = \{\pm 1_2\}$.
2. $n = 2$, $G = \left\langle \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k^{-1} \end{pmatrix} \right\rangle$ where ω_k is a primitive k-th rooth of 1.
3. $G = \mathcal{S}_n$, the group of permutation of n elements viewed in the obvious way as a subgroup of $\mathrm{GL}_n(\mathbb{K})$ (group of permutations of coordinates).

We introduce definitions that will be discussed more in general later on. Let $\mathrm{Div}(\mathbb{P}^n)$ be the abelain group with generators the irreducible hypersurfaces in $\mathbb{P}^n$. Thus an element of $\mathrm{Div}(\mathbb{P}^n)$ is a *formal* finite sum $\sum_{i \in I} m_i X_i$, where each m_i is an integer, and each X_i id an irreducible hypersurface in $\mathbb{P}^n$.

Let $F \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ be non zero. Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface, and let $I(X) = (G)$. The *multiplicity of* F *along* X is the maximum m such that G^m divides F, and is denoted $\mathrm{mult}_X V(F)$.

Let $F = \prod_{i=1}^{r} F_i^{m_i}$ be the decomposition into prime factors, where for $i \neq j$ the factors F_i and F_j are not associated. The *divisor* of F is the element of $\mathrm{Div}(\mathbb{P}^n)$ defined by
$$\mathrm{div}(F) := \sum_{\substack{X \subset \mathbb{P}^n \\ \text{irred. hypers.}}} \mathrm{mult}_X V(F) = \sum_{i=1}^{r} m_i V(F_i). \tag{1.8.12}$$

Exercise 1.8.6. Let $F \in \mathbb{K}[Z_0, Z_1]_d$.

(a) Notice that unless $F = 0$ the cardinality of $V(F)$ is at most d, and it equals d if and only if $\mathrm{mult}_p(F) \leqslant 1$ for all $p \in \mathbb{P}^1$.

(b) Let $\Delta_d \subset \mathbb{P}(\mathbb{K}[Z_0, Z_1]_d)$ be the subset of $[F]$ such that there exists $p \in \mathbb{P}^1$ for which $\mathrm{mult}_p(F) \geqslant 2$. Prove that Δ_d is a closed irreducible subset of $\mathbb{P}(\mathbb{K}[T_0, T_1]_d)$. (Hint: let $\widetilde{\Delta}_d \subset \mathbb{P}(\mathbb{K}[T_0, T_1]_d) \times \mathbb{P}^1$ be the subset of couples $([F], [Z])$ such that F has a multiple root at Z. Show that $\widetilde{\Delta}_d$ is closed in $\mathbb{P}(\mathbb{K}[T_0, T_1]_d) \times \mathbb{P}^1$, and then project to the first factor.)

(c) Assume that $\mathrm{char}\,\mathbb{K}$ does not divide d. Let $p = [a_0, a_1] \in \mathbb{P}^1$. Prove that $\mathrm{mult}_p(F) \geqslant 2$ if and only if
$$\frac{\partial F(a_0, a_1)}{\partial Z_0} = \frac{\partial F(a_0, a_1)}{\partial Z_1} = 0. \tag{1.8.13}$$
(Hint: use Euler's relation (1.8.8).)

Chapter 2

Rational maps, dimension

2.1 Introduction

A rational function on an irreducible locally closed subset $X \subset \mathbb{P}^n$ is defined by a quotient $\frac{F}{G}$, where $F, G \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ are homogeneous polynomials of the same degree, and G does not vanish at all points of X. The set of rational functions on X, with addition and multiplication defined pointwise, is a field denoted $\mathbb{K}(X)$, finitely generated over the subfield $\mathbb{K}$ of constant functions. One defines the dimension of X as the transcendence degree of $\mathbb{K}(X)$ over $\mathbb{K}$. The dimension is well-behaved (e.g. the dimension of $\mathbb{A}^n$ or $\mathbb{P}^n$ is equal to n), and is invariant under isomorphisms. Two irreducible varieties X, Y are birational if $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ are isomorphic (as extensions of $\mathbb{K}$) - this is equivalent to the existence of isomorphic open dense subsets $U \subset X$ and $V \subset Y$. This relation is weaker than isomorphism; it plays a crucial rôle in algebraic geometry.

Let $f \colon X \dashrightarrow Y$ be a rational map. The degree of f is a number (possibly ∞) related to the cardinality of $f^{-1}(y)$ for y in an open dense subset of Y. If f factors through the inclusion of a proper closed subset $W \subset Y$, then the degree is 0, otherwise f defines by pull-back an inclusion $\mathbb{K}(Y) \subset \mathbb{K}(X)$ and the degree of f is equal to $[\mathbb{K}(X) : \mathbb{K}(Y)]$. Suppose that the degree is finite: if the extension $\mathbb{K}(X) \supset \mathbb{K}(Y)$ is separable, then the result about the cardinality of a generic fiber holds, in general it holds with the degree replaced by the separable degree of $\mathbb{K}(X) \supset \mathbb{K}(Y)$.

Let $X \subset \mathbb{P}^n$ be a closed subset. There exists a positive number d, called the degree of X, with the property that, for a generic linear subspace $\Lambda \subset \mathbb{P}^n$ of dimension $(n - \dim X)$, the cardinality of $\Lambda \cap X$ is d. In order to make sense of the word "generic"(which has a precise meaning despite itself), and to prove this statement, we introduce the Grassmannian parametrizing linear subspaces of a projective space and we identify it with a projective variety. Along the road, we will characterize the dimension of a closed subset of a projective space via its intersection with linear subspaces - this allows us to prove a (highly non trivial) generalization of the well known result from linear algebra: a set of m homogeneous linear equations in n unknowns has a non trivial solution if $m < n$.

2.2 Rational maps

Let X and Y be quasi projective varieties. We define a relation on the set of couples (U, φ) where $U \subset X$ is open dense and $\varphi \colon U \to Y$ is a regular map, as follows: $(U, \varphi) \sim (V, \psi)$ if the restrictions of φ and ψ to $U \cap V$ are equal. One checks easily that $\sim$ is an equivalence relation.

Definition 2.2.1. A *rational map* $f \colon X \dashrightarrow T$ is a $\sim$-equivalence class of couples (U, φ) where $U \subset X$ is open dense and $\varphi \colon U \to Y$ is a regular map. Let $f \colon X \dashrightarrow Y$ be a rational map.

1. The map f is *regular* at $x \in X$ (equivalently x is a *regular point* of f), if there exists (U, φ) in the equivalence class of f such that $x \in U$. We let $\operatorname{Reg}(f) \subset X$ be the set of regular points of f. By definition $\operatorname{Reg}(f)$ is an open subset of X.

2. The *indeterminancy set* of f is $\operatorname{Ind}(f) := X \backslash \operatorname{Reg}(f)$ (notice that $\operatorname{Ind}(f)$ is closed). A point $x \in X$ is a *point of indeterminancy* if it belongs to $\operatorname{Ind}(f)$.

From now on we will consider only rational maps between *irreducible* quasi projective varieties. Let $f: X \dashrightarrow Y$ and $g: Y \dashrightarrow W$ be rational maps between (irreducible) quasi projective varieties. It might happen that for all $x \in \operatorname{Reg}(f)$ the image $f(x)$ does not belong to $\operatorname{Reg}(g)$, and then the composition $g \circ f$ makes no sense. In order to deal with compositions of reational maps, we give the following definition.

Definition 2.2.2. A rational map $f: X \dashrightarrow Y$ between irreducible quasi projective varieties is *dominant* if it is represented by a couple (U, φ) such that $\varphi(U)$ is dense in Y.

Notice that if $f: X \dashrightarrow Y$ is dominant and (V, ψ) is an arbitrary representative of f then $\psi(V)$ is dense in Y.

Definition 2.2.3. Let $f: X \dashrightarrow Y$ be a dominant rational map, and let $g: Y \dashrightarrow W$ be a rational map (X, Y, W are irreducible). Let (U, φ) and (V, ψ) be representatives of f and g respectively. Then $\varphi^{-1}V$ is open dense in X. We let $g \circ f: X \dashrightarrow W$ be the rational map represented by $(\varphi^{-1}V, \psi \circ \varphi)$. (The equivalence class of $(\varphi^{-1}V, \psi \circ \varphi)$ is independent of the representatives (U, φ) and (V, ψ).)

Definition 2.2.4. A dominant rational map $f: X \dashrightarrow Y$ between irreducible quasi projective varieties is *birational* if there exists a dominant rational map $g: Y \dashrightarrow X$ such that $g \circ f = \operatorname{Id}_X$ and $f \circ g = \operatorname{Id}_Y$. An irreducible quasi projective variety X is *rational* if it is birational to $\mathbb{P}^n$ for some n, it is *unirational* if there exists a dominant rational map $f: \mathbb{P}^n \dashrightarrow X$.

Example 2.2.5.

1. Of course isomorphic irreducible quasi projective varieties are birational. On the other a quasi projective (irreducible) variety is birational to any of its dense open subsets. In particular $\mathbb{P}^n$ is birational to $\mathbb{A}^n$, although they are not isomorphic if $n > 0$ (if they were isomorphic, they would be diffeomorphic as C^∞ manifolds, but $\mathbb{P}^n$ is compact, $\mathbb{A}^n$ is not).

2. Let $0 \neq F \in \mathbb{K}[Z_0, \ldots, Z_n]_2$, and let $Q^{n-1} := V(F) \subset \mathbb{P}^n$. Suppose that F is prime, i.e that $\operatorname{rk} F \geqslant 3$, and hence Q^{n-1} is irreducible. We claim that Q^{n-1} is rational. In fact, after a suitable change of coordinates, we may assume that $F = Z_0 Z_n - G$, where $0 \neq G \in \mathbb{K}[Z_1, \ldots, Z_{n-1}]_2$. The rational maps
$$\begin{array}{ccc} Q^{n-1} & \stackrel{f}{\dashrightarrow} & \mathbb{P}^{n-1} \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_{n-1}] \end{array}$$
and
$$\begin{array}{ccc} \mathbb{P}^{n-1} & \stackrel{g}{\dashrightarrow} & Q^{n-1} \\ [T_0, \ldots, T_{n-1}] & \mapsto & [T_0^2, T_0T_1, \ldots, T_0T_{n-1}, G(T_1, \ldots, T_{n-1})] \end{array}$$
are dominant, and they are inverses of each other. Notice that if $n = 2$, then f and g are regular (see Example 1.3.5), while for $n \geqslant 3$, the quadric Q^{n-1} is not isomorphic to $\mathbb{P}^{n-1}$, because the underlying C^∞ manifolds are not homeomorphic.

Proposition 2.2.6. *Irreducible quasi varieties X, Y are birational if and only if there exist open dense subsets $U \subset X$ and $V \subset Y$ that are isomorphic.*

Proof. An isomorphism $\varphi: U \xrightarrow{\sim} V$ clearly defines a birational map $f: X \dashrightarrow Y$. Conversely, suppose that $f: X \dashrightarrow Y$ is birational with inverse $g: Y \dashrightarrow X$. Let (U, φ) represent f and (V, ψ) represent g. Then $\varphi^{-1}V \subset U$ and $\psi^{-1}U \subset V$ are open dense. By hypothesis the composition
$$\psi \circ \left(\varphi_{|\varphi^{-1}V}\right) : \varphi^{-1}V \to U$$
is equal to the identity on an open non-empty subset of $\varphi^{-1}V$. By Proposition 1.5.10, we get that $\psi \circ \left(\varphi_{|\varphi^{-1}V}\right) = \operatorname{Id}_{\varphi^{-1}V}$. In particular $\psi \circ \varphi\left(\varphi^{-1}V\right) \subset U$ i.e. $\varphi\left(\varphi^{-1}V\right) \subset \psi^{-1}U$, and similarly
$$\varphi \circ \left(\psi_{|\psi^{-1}U}\right) = \operatorname{Id}_{\psi^{-1}U}, \quad \psi\left(\psi^{-1}U\right) \subset \varphi^{-1}V.$$
Thus we have isomorphisms $\varphi^{-1}V \xrightarrow{\sim} \psi^{-1}U$ and $\psi^{-1}U \xrightarrow{\sim} \varphi^{-1}V$. □

Many natural invariants of projective varieties do not separate between (projective) birational varieties. This fact gives practical criteria that allow to establish that certain projective varieties are not birational. On the other hand, it leads us to approach the classification of isomorphism classes of projective varieties in two steps: first we classify equivalence classes for birational equivalence, then we distinguish isomorphim classes within each birational equivalence class.

2.3 Blow-up

Blow-up of a projective space

Let $p = [v_0] \in \mathbb{P}(V)$. The blow-up of $\mathbb{P}(V)$ at p is a projective variety obtained from $\mathbb{P}(V)$ by replacing the point p with all the tangent directions at p, i.e. the set Σ_p of lines containing p. In order to define it, we notice that we have an identification

$$\begin{array}{ccc} \Sigma_p = \{\mathbb{P}(U) \mid U \in \mathrm{Gr}(2,V),\ v_0 \in U\} & \longrightarrow & \mathbb{P}(V/[v_0]) \\ U & \mapsto & U/[v_0] \end{array}$$

and hence Σ_p is a projective space whose dimension is one less than the dimension of $\mathbb{P}(V)$. The *blow-up of* $\mathbb{P}(V)$ *at* p is the subset of $\mathbb{P}(V) \times \Sigma_p$ defined by

$$\mathrm{Bl}_p(\mathbb{P}(V)) := \{(x, \Lambda) \in \mathbb{P}(V) \times \Sigma_p \mid x \in \Lambda\}.$$

Claim 2.3.1. $\mathrm{Bl}_p(\mathbb{P}(V))$ *is closed in* $\mathbb{P}(V) \times \Sigma_p$*, and irreducible.*

Proof. Let $[Z_0, \ldots, Z_n]$ be homogeneous coordinates such that $p = [1, 0, \ldots, 0]$. The map

$$\begin{array}{ccc} \mathbb{P}^{n-1} & \xrightarrow{\sim} & \Sigma_p \\ [T_1, \ldots, T_n] & \mapsto & \langle [1, 0, \ldots, 0], [0, T_1, \ldots, T_n] \rangle \end{array} \tag{2.3.1}$$

is an isomorphism. With these identifications

$$\mathrm{Bl}_p(\mathbb{P}(V)) := \{([Z_0, \ldots, Z_n], [T_1, \ldots, T_n]) \mid 0 = Z_i T_j - Z_j T_i \quad \forall 1 \leqslant i < j \leqslant n\}. \tag{2.3.2}$$

Thus $\mathrm{Bl}_p(\mathbb{P}(V))$ is closed in $\mathbb{P}(V) \times \Sigma_p$.

Let $\rho\colon \mathrm{Bl}_p(\mathbb{P}(V)) \to \Sigma_p$ be the restriction to $\mathrm{Bl}_p(\mathbb{P}(V))$ of the second projection of $\mathbb{P}(V) \times \Sigma_p$:

$$\begin{array}{ccc} \mathrm{Bl}_p(\mathbb{P}(V)) & \xrightarrow{\rho} & \Sigma_p \\ (x, \Lambda) & \mapsto & \Lambda \end{array} \tag{2.3.3}$$

Since $\mathbb{P}^{n-1} = \bigcup_{j=1}^n \mathbb{P}^{n-1}_{T_j}$ we have

$$\mathrm{Bl}_p(\mathbb{P}(V)) = \bigcup_{j=1}^{n} \rho^{-1}(\mathbb{P}^{n-1}_{T_j}). \tag{2.3.4}$$

By (2.3.14) we have an isomorphism

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{A}^{n-1} & \xrightarrow{\sim} & \rho^{-1}(\mathbb{P}^{n-1}_{T_j}) \\ ([Z_0,Z_j],(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)) & \mapsto & ([Z_0,Z_jt_1,\ldots,Z_jt_{j-1},Z_j,Z_jt_{j+1},\ldots,Z_jt_n],[t_1,\ldots,t_{j-1},1,t_{j+1},\ldots,t_n]) \end{array} \tag{2.3.5}$$

Thus (2.3.4) defines an open covering of $\mathrm{Bl}_p(\mathbb{P}(V))$ in which each open set is irreducible. Any two such open sets have non empty intersection: it follows that $\mathrm{Bl}_p(\mathbb{P}(V))$ is irreducible. $\square$

The *blow-down* (or *contraction*) map is

$$\begin{array}{ccc} \mathrm{Bl}_p(\mathbb{P}(V)) & \xrightarrow{\pi} & \mathbb{P}(V) \\ (x, \Lambda) & \mapsto & x \end{array} \tag{2.3.6}$$

Clearly π is regular, and

$$\pi^{-1}(x) = \begin{cases} \langle p, x\rangle & \text{if } x \neq p, \\ \Sigma_p = \mathbb{P}(V/[v_0]) & \text{if } x = p. \end{cases} \tag{2.3.7}$$

Equation (2.3.7) explains the name "blow-up at p": the fiber of π over p is a blown-up (as in photography) version of p. The map π is birational. In fact let

$$\begin{array}{ccc} (\mathbb{P}(V)\backslash\{p\}) & \xrightarrow{\phi} & \mathrm{Bl}_p(\mathbb{P}(V)) \\ x & \longrightarrow & (x, \langle p, x\rangle) \end{array} \tag{2.3.8}$$

Then ϕ is regular, and

$$\pi \circ \phi = \mathrm{Id}_{\mathbb{P}(V)\backslash\{p\}}, \qquad \phi \circ (\pi|_{\pi^{-1}(\mathbb{P}(V)\backslash\{p\})}) = \mathrm{Id}_{\pi^{-1}(\mathbb{P}(V)\backslash\{p\}}.$$

Since $(\pi^{-1}(\mathbb{P}(V)\backslash\{p\})$ is open dense in $\mathrm{Bl}_p(\mathbb{P}(V))$ the (equivalence class of) the map ϕ is the (rational) inverse of π.

Blow-up of a locally-closed subset of a projective space

Let $X \subset \mathbb{P}(V)$ be a locally closed subset and $p \in X$. We assume that $\dim_p X > 0$. Let ϕ be the map given by (2.3.8). The *blow-up of X at p* is the subset of $\mathrm{Bl}_p(\mathbb{P}(V))$ defined as the closure of $\pi^{-1}(X\backslash\{p\})$:

$$\mathrm{Bl}_p(X) := \overline{\pi^{-1}(X\backslash\{p\})}. \tag{2.3.9}$$

Notice that $\mathrm{Bl}_p(X)$ is locally-closed in the projective variety $\mathrm{Bl}_p(\mathbb{P}(V))$, hence it is a quasi-projective variety. If X is closed in $\mathbb{P}(V)$, then $\mathrm{Bl}_p(X)$ is closed in $\mathrm{Bl}_p(\mathbb{P}(V))$, and hence it is projective.

We let $\pi_X\colon \mathrm{Bl}_p(X) \to X$ be the restriction to $\mathrm{Bl}_p(X)$ of the blow-down map $\pi\colon \mathrm{Bl}_p \mathbb{P}(V) \to \mathbb{P}(V)$ (thus $\pi_{\mathbb{P}(V)} = \pi$). Let $\phi_X\colon (X\backslash\{p\}) \to \mathrm{Bl}_p(X)$ be defined by restricting the map ϕ of (2.3.8) to $(X\backslash\{p\})$. Then ϕ_X defines a rational inverse of π_X.

We examine a few examples. Throughout the examples we let $[Z_0, \ldots, Z_n]$ be homogeneous coordinates such that $p = [1, 0, \ldots, 0]$, and $z_i := Z_i/Z_0$.

Example 2.3.2. Let $X = \mathbb{P}^n_{Z_0}$. Thus X is the affine space $\mathbb{A}^n$ and $z_i = Z_i/Z_0$ for $1 \leqslant i \leqslant n$ are affine coordinates on X. For $1 \leqslant j \leqslant n$ let

$$\mathscr{U}_j := \mathrm{Bl}_p(\mathbb{A}^n) \cap \rho^{-1}(\mathbb{P}^{n-1}_{T_j}) \tag{2.3.10}$$

where ρ is the map (2.3.3). Thus $\mathrm{Bl}_p(\mathbb{A}^n)$ is the union of the open sets $\mathscr{U}_j$ for $1 \leqslant j \leqslant n$. Equation (2.3.5) gives an isomorphism

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{\phi_j} & \mathscr{U}_j \\ (z_j,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n) & \mapsto & ((z_jt_1,\ldots,z_jt_{j-1},z_j,z_jt_{j+1},\ldots,z_jt_n),[t_1,\ldots,t_{j-1},1,t_{j+1},\ldots,t_n]) \end{array} \tag{2.3.11}$$

Thus we have

$$\pi_{\mathbb{A}^n} \circ \phi_j(z_j, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) = (z_jt_1, \ldots, z_jt_{j-1}, z_j, z_jt_{j+1}, \ldots, z_jt_n) \tag{2.3.12}$$

and the (rational) inverse of $\pi_{\mathbb{A}^n} \circ \phi_j$ is given by

$$(\pi_{\mathbb{A}^n} \circ \phi_j)^{-1}(z_1, \ldots, z_n) = (\frac{z_1}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \ldots \frac{z_n}{z_j}).$$

Remark 2.3.3. Let $\mathfrak{m}_p \subset \mathbb{K}[z_1, \ldots, z_n]$ be the maximal ideal of polynomials vanishing at p. Equation (2.3.12) gives that the ideal of $\mathbb{K}[z_1, \ldots, z_n]$ *generated* by $(\pi_{\mathbb{A}^n} \circ \phi_j)^*(\mathfrak{m}_p)$ is equal to the *principal* ideal (x_j). This simple fact will be the key to the universal property of blow-up.

Example 2.3.4. Let $X \subset \mathbb{A}^n$ be a hypersurface containing $p = (0, \ldots, 0)$. We will give explicit equations for the intersection of $\mathrm{Bl}_p(X)$ with each of the open affine subsets $\mathscr{U}_j \subset \mathrm{Bl}_p(\mathbb{A}^n)$ described above. Let $I(X) = (f)$. Expand f around the origin:

$$f = f_m + f_{m+1} + \ldots + f_d, \qquad f_s \in \mathbb{K}[z_1, \ldots, z_n]_s, \quad f_m \neq 0.$$

Identify $\mathscr{U}_j$ with $\mathbb{A}^n$ via (2.3.11); then

$$\pi_{\mathbb{A}^n}^{-1}(X) \cap \mathscr{U}_j = V(z_j^m \sum_{s=0}^{d-m} z_j^s f_{m+s}(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_n)).$$

By Remark 2.3.3 it follows that the polynomial

$$\sum_{s=0}^{d-m} z_j^s f_{m+s}(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_n) \tag{2.3.13}$$

vanishes on $\pi_{\mathbb{A}^n}^{-1}(X \backslash \{0\}) \cap \mathscr{U}_j$. Since $\mathrm{Bl}_p(X)$ is the closure of $\pi_{\mathbb{A}^n}^{-1}(X \backslash \{0\})$, it follows that the polynomial in (2.3.13) vanishes on $\mathrm{Bl}_p(X) \cap \mathscr{U}_j$. In fact, we claim that

$$\mathrm{Bl}_p(X) \cap \mathscr{U}_j = V(\sum_{s=0}^{d-m} z_j^s f_{m+s}(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_n)). \tag{2.3.14}$$

In fact, suppose that $\varphi \in \mathbb{K}[x_j, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n]$ vanishes on $\mathrm{Bl}_p(X) \cap \mathscr{U}_j$. Then the rational function $\varphi(\frac{z_1}{z_j}, \ldots \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j})$ vanishes at all points of $V(f) \backslash V(z_j)$. Let $e \in \mathbb{N}_+$ be such that

$$z_j^e \varphi(\frac{z_1}{z_j}, \ldots \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}) \tag{2.3.15}$$

is a polynomial in $z_1, \ldots, z_n$. The polynomial in (2.3.15) vanishes on $V(f)$, and hence it is a multiple of f, i.e. there exists $\psi \in \mathbb{K}[z_1, \ldots, z_n]$ such that

$$z_j^e \varphi(\frac{z_1}{z_j}, \ldots \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}) = \psi \cdot f(z_j t_1, \ldots, z_j t_{j-1}, z_j, z_j t_{j+1}, \ldots, z_j t_n).$$

Divide the above equality by z_j^m, and replace $\frac{z_i}{z_j}$ by t_j for $i \neq j$. We get

$$z_j^e \varphi(t_1, \ldots t_{j-1}, z_j, t_{j+1}, \ldots, t_n) = \psi \cdot \left(\sum_{s=0}^{d-m} z_j^s f_{m+s}(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_n) \right).$$

Since z_j does not divide the second factor of the right hand side, we get that z_j^e divides ψ, and hence φ is a multiple of the polynomial in the right hand side of (2.3.14).

In particular we get that

$$\pi_X^{-1}(p) = \{((\underbrace{0, \ldots, 0}_{n}), [T_1, \ldots, T_n] \in \mathbb{P}^{n-1} \mid f_m(T_1, \ldots, T_n) = 0\}. \tag{2.3.16}$$

Universal property of the blow-up

Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be isomorphic locally closed subsets, and let $\varphi \colon X \xrightarrow{\sim} Y$ be an isomorphism. Choose $p \in X$, and let $q := \varphi(p) \in Y$. We will show that the isomorphism between $X \backslash \{p\}$ and $Y \backslash \{q\}$ defined by φ extends to an isomorphism $\mathrm{Bl}_p(X) \xrightarrow{\sim} \mathrm{Bl}_q(Y)$. This means that we may speak of the blow up of a quasi projective variety at a point without specifying an isomorphism of the variety with a locally closed subset of a projective space.

The key point is to realize that the blow up has a universal property.

Proposition 2.3.5 (Universal property of the blow-up)**.** *Let $Y \subset \mathbb{P}^m$ be a locally closed subset and $q \in Y$. We assume that $\dim_q Y > 0$ so that the blow-up $\mathrm{Bl}_q(Y)$ is defined. Let X be a quasi-projective variety and $F\colon X \to Y$ be a regular map. Let $V \subset Y$ be an open affine subset containing q and $\mathfrak{m}_q \subset \mathbb{K}[V]$ be the (maximal) ideal of functions vanishing at q. Suppose that there exists an open affine covering*

$$F^{-1}V = \bigcup_{j\in J} \mathscr{U}_j$$

(affine *means that each $\mathscr{U}_j$ is an affine variety) such that for every $j \in J$ the ideal of $\mathbb{K}[\mathscr{U}_j]$ generated by $(F|_{\mathscr{U}_j})^*\mathfrak{m}_q$ is principal, generated by a function $\rho_j \in \mathbb{C}[\mathscr{U}_j]$ which is* not a zero-divisor*. Then there exists a lift of F i.e. a regular map $\widetilde{F}\colon X \to\colon \mathrm{Bl}_q(Y)$ fitting into the commutative diagram*

$$\begin{tikzcd} & \mathrm{Bl}_q(Y) \arrow[dd, "\pi_Y"] \\ X \arrow[ur, "\widetilde{F}"] \arrow[dr, "F"'] & \\ & Y \end{tikzcd}$$

Moreover, such a lift is unique.

Proof. Away from $F^{-1}(q)$ we define $\widetilde{F}$ as $\widetilde{F} := \phi_Y \circ F$ where $\phi_Y\colon (Y\backslash\{q\}) \to \mathrm{Bl}_q(Y)$ is the rational inverse of π_Y. Let us show that $\phi_Y \circ F$ extends to a regular map at all points of $F^{-1}(q)$. Let $\psi_1, \ldots, \psi_r \in \mathbb{C}[V]$ be generators of $\mathfrak{m}_q \subset \mathbb{C}[V]$. Choose homogeneous coordinates $[Z_0, \ldots, Z_m]$ on $\mathbb{P}^m$ such that $q = [1, 0, \ldots, 0]$, and let $z_i = Z_i/Z_0$ be affine coordinates on $\mathbb{P}^n_{Z_0}$ - notice that $q \in \mathbb{P}^n_{Z_0}$. Let $\overline{p} \in F^{-1}(q)$. Then there exists $j \in J$ such that $\overline{p} \in \mathscr{U}_j$. Let $F_j := F|_{\mathscr{U}_j}$. By hypothesis

$$F_j^*\psi_s = \lambda_{sj}\rho_j \tag{2.3.17}$$

with λ_{sj} regular for $1 \leqslant s \leqslant r$, and since ρ_j is in the ideal generated by $F_j^*\psi_1, \ldots, F_j^*\psi_r$ there exists $1 \leqslant s_0 \leqslant r$ such that

$$\lambda_{s_0,j}(\overline{p}) \neq 0. \tag{2.3.18}$$

Let

$$\begin{array}{ccc} F^{-1}(Y_{Z_0}) & \xrightarrow{\Phi} & Y_{Z_0} \\ p & \mapsto & F(p) \end{array}$$

be the restriction of F. The equivalence classes of $\psi_1, \ldots, \psi_r$ in $\mathscr{O}_{Y,q}$ generate the maximal ideal $\mathfrak{m}_q \subset \mathscr{O}_{Y,q}$; since the equivalence classes of $z_1, \ldots, z_m$ in $\mathscr{O}_{Y,q}$ belong to $\mathfrak{m}_q$ it follows that there exist $\mu_{ij} \in \mathscr{O}_{X,\overline{p}}$ for $i = 1, \ldots, m$ such that

$$F^*(z_i) = \mu_{ij}\rho_j \text{ in } \mathscr{O}_{X,\overline{p}}. \tag{2.3.19}$$

On the other hand the equivalence classes of $z_1, \ldots, z_m$ in $\mathscr{O}_{Y,q}$ also generate $\mathfrak{m}_q$, thus (2.3.17) and (2.3.18) give that there exists $1 \leqslant i_0 \leqslant m$ such that $\mu_{i_0,j}(\overline{p}) \neq 0$. Shrinking $\overline{\mathscr{U}}_j$ around $\overline{p}$ we may assume that for all $p \in \overline{\mathscr{U}}_j$ we have $\mu_{i_0,j}(p) \neq 0$. By (2.3.19) the restriction of $\phi_Y \circ F$ to $\overline{\mathscr{U}}_j$ is equal to

$$\begin{array}{ccc} \overline{\mathscr{U}}_j & \longrightarrow & Y \times \mathbb{P}^{n-1} \\ p & \mapsto & (F(p), [\mu_{1,j}(p), \ldots, \mu_{m,j}(p)]). \end{array}$$

Since $\mu_{i_0,j}(p) \neq 0$ for all $p \in \overline{\mathscr{U}}_j$ the above map is regular. Now we must check that the local extensions glue together and that the resulting lift is unique. Both statements follow from the hypothesis that the ρ_j's are not zero-divisors: this implies that no irreducible component of X is contained in $F^{-1}(q)$. Now let $\mathscr{U}_1, \mathscr{U}_2 \subset X$ be open subsets such that $F|_{\mathscr{U}_1}$ lifts to $\widetilde{F}_1\colon \mathscr{U}_1 \to \mathrm{Bl}_q(Y)$ and $F|_{\mathscr{U}_2}$ lifts to

$\widetilde{F}_2\colon \mathscr{U}_2 \to \mathrm{Bl}_q(Y)$. The intersection $\mathscr{U}_1 \cap \mathscr{U}_2 \cap (X\backslash F^{-1}(q))$ is dense in $\mathscr{U}_1 \cap \mathscr{U}_2$ because no irreducible component of X is contained in $F^{-1}(q)$. Since $\widetilde{F}_1|_{\mathscr{U}_1\cap\mathscr{U}_2\backslash F^{-1}(q)} = \widetilde{F}_2|_{\mathscr{U}_1\cap\mathscr{U}_2\backslash F^{-1}(q)}$ it follows that $\widetilde{F}_1 = \widetilde{F}_2$. Thus there exists a lift of F. Unicity of the lift follows from unicity of the restriction of a lift to $(Y\backslash\{q\})$ and the fact that no irreducible component of X is contained in $F^{-1}(q)$. □

Corollary 2.3.6. *Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be locally-closed subsets and $F\colon X \xrightarrow{\sim} Y$ be an isomorphism. Let $p \in X$ and $q := F(p) \in Y$. There exists a unique isomorphism $G\colon \mathrm{Bl}_p(X) \xrightarrow{\sim} \mathrm{Bl}_q(Y)$ fitting into the commutative diagram*

$$\begin{tikzcd} \mathrm{Bl}_p(X) \arrow[r, "G"] \arrow[d, "\pi_X"] & \mathrm{Bl}_p(Y) \arrow[d, "\pi_Y"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

Proof. Let us show that $F \circ \pi_X\colon \mathrm{Bl}_p(X) \to Y$ lifts to a regular map $G\colon \mathrm{Bl}_p(X) \to \mathrm{Bl}_q(Y)$. Let $[Z_0, \ldots, Z_m]$ be homogeneous coordinates on $\mathbb{P}^m$ such that $q = [1, 0, \ldots, 0]$ and let $z_i = Z_i/Z_0$ for $i \in \{1, \ldots, m\}$ be affine coordinates on $\mathbb{P}^m_{Z_0}$. The maximal ideal $\mathfrak{m}_q \subset \mathbb{C}[X_{Z_0}]$ is generated by $z_1, \ldots, z_n$.

Let $U := F^{-1}(X_{Z_0})$, and let $\Phi\colon U \to X_{Z_0}$ be the map given by restriction of F. Let $\Psi\colon \pi_X^{-1}(U) \to U$ be the map given by restriction of π_X.

Since F is an isomorphism, U is an open affine subset of X containing p and hence $\Phi^*(z_1), \ldots, \Phi^*(z_n)$ generate the maximal ideal $\mathfrak{m}_p \subset \mathbb{K}[U]$.

On the other hand, let $[W_0, \ldots, W_n]$ be homogeneous coordinates on $\mathbb{P}^n$ such that $p = [1, 0, \ldots, 0]$ and let $w_k = W_k/W_0$ for $k \in \{1, \ldots, n\}$ be affine coordinates on $\mathbb{P}^n_{W_0}$. The maximal ideal $\mathfrak{m}_p \subset \mathbb{C}[X_{W_0}]$ is generated by $w_1, \ldots, w_n$.

Hence, there exists an open affine set $V \subset X_{W_0} \cap U$ containing p such that the ideal in $\mathbb{K}[V]$ generated by the restrictions of $\Phi^*(z_1), \ldots, \Phi^*(z_n)$ to V is equal to the ideal generated by the restrictions of $w_1, \ldots, w_n$ to V.

By Remark 2.3.3 it follows that there exists an open affine covering $\{\mathscr{U}_j\}_{j\in J}$ of $\pi_X^{-1}(V)$ such that for every $j \in J$ the ideal of $\mathbb{C}[\mathscr{U}_j]$ generated by $(\pi_{X|\pi_X^{-1}(V)})^*(\Phi^*(z_1)), \ldots, (\pi_{X|\pi_X^{-1}(V)})^*(\Phi^*(z_n))$ is principal, generated by a function which is not a zero-divisor. Hence $F \circ \pi_X\colon \mathrm{Bl}_p(X) \to Y$ lifts to a regular map $G\colon \mathrm{Bl}_p(X) \to \mathrm{Bl}_q(Y)$ by Proposition 2.3.6.

Symmetrically $F^{-1} \circ \pi_Y$ lifts to a regular map $H\colon \mathrm{Bl}_q(Y) \to \mathrm{Bl}_p(X)$. The composition $H \circ G$ is equal to the identity map because it is the identity on the open dense subset $\pi_X^{-1}(X\backslash\{p\})$. By the same argument, also the composition $G \circ H$ is the identity. □

Let $p_1, \ldots, p_r \in Y$ be a finite collection of distinct points. Since $\mathrm{Bl}_{p_1}(Y) \to Y$ is an isomorphism outside p_1 we may view $p_2, \ldots, p_r$ as points of $\mathrm{Bl}_{p_1}(Y)$, consider $\mathrm{Bl}_{p_2}(\mathrm{Bl}_{p_1}(Y))$ and iterate, blowing up r times. Of course we may repeat this operation with a different ordering of the same points. Let $\widetilde{Y}$ be one of these blow-ups. Then $\widetilde{Y}$ enjoys the following universal property similar to that given by Proposition 2.3.6. Let $U \subset Y$ be an open affine set containing $p_1, \ldots, p_r$ and $F\colon X \to Y$ be a regular map such that locally on $F^{-1}U$ the ideal generated by $(F|_{F^{-1}(U)})^*(\mathfrak{m}_{p_1} \oplus \ldots \oplus \mathfrak{m}_{p_r})$ (here $\mathfrak{m}_{p_i} \subset \mathbb{C}[U]$ is the maximal ideal of p_i) is principal, generated by a function which is not a zero-divisor: then F lifts uniquely to a regular map $\widetilde{F}\colon X \to \widetilde{Y}$. Since this property is *independent* of the ordering of the points it follows that any two such blow-ups are isomorphic. From now on we will denote such a blow-up by $\mathrm{Bl}_{p_1,\ldots,p_r}(Y)$.

Example 2.3.7. Let $p_1, \ldots, p_r \in \mathbb{P}^n$. Then $\mathrm{Bl}_{p_1,\ldots,p_r}\mathbb{P}^n$ has an open cover by *affine* n-dimensional spaces.

2.4 The field of rational functions

If we consider the category whose objects are irreducible quasi projective varieties, and morphisms are dominant rational maps, we get a familiar algebraic category. In order to explain this, we introduce a key definition. Let X be an irreducible quasi projective variety. The *field of rational functions on* X is

$$\mathbb{K}(X) := \{f\colon X \dashrightarrow \mathbb{K} \mid f \text{ is a rational map}\}\,. \tag{2.4.20}$$

Addition and multiplication are defined on representatives. Let $f, g \in \mathbb{K}(X)$ be represented by (U,φ) and (V,ψ) respectively. Then

$$\begin{aligned} f+g &:= [(U\cap V, \varphi_{|U\cap V} + \psi_{|U\cap V})], \\ f\cdot g &:= [(U\cap V, \varphi_{|U\cap V} \cdot \psi_{|U\cap V})]. \end{aligned}$$

Example 2.4.1.

- $\mathbb{K}(\mathbb{P}^n) \cong \mathbb{K}(z_1,\ldots,z_n)$ is the purely transcendental extension of $\mathbb{K}$ of transcendence degree n.
- Let $p \in \mathbb{K}[z]$ be free of square factors (and $\deg p \geqslant 1$). Then $t^2 - p(z)$ is prime and hence $X := V\left(t^2 - p(z)\right) \subset \mathbb{A}^2$ is irreducible. Then $\mathbb{K}(z) \subset \mathbb{K}(X)$ is an extension of degree 2. We may ask whether $\mathbb{K}(X)$ is a purely trascendental extension of $\mathbb{K}$. The answer is *yes* if $\deg p = 1, 2$ (see Example 1.3.5), *no* if $\deg p \geqslant 3$ (this requires new ideas).

Let $f\colon X \dashrightarrow Y$ be a dominant rational map of irreducible quasi projective varieties. We have a well-defined *pull-back*

$$\begin{array}{ccc} \mathbb{K}(Y) & \xrightarrow{\varphi^*} & \mathbb{K}(X) \\ \varphi & \mapsto & \varphi\circ f \end{array}$$

(The composition is well defined because by hypothesis f is dominant.) The map f^* is an inclusion of extensions of $\mathbb{K}$. Suppose that $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow W$ are dominant rational maps of irreducible quasi projective varieties. Then $g\circ f\colon X \dashrightarrow W$ is dominant and

$$f^*\circ g^* = (g\circ f)^*\,. \tag{2.4.21}$$

Of course $\mathrm{Id}_X^*\colon \mathbb{K}(X) \to \mathbb{K}(X)$ is the identity map. We will prove the following result.

Theorem 2.4.2. *By associating to each quasi projective variety its field of fractions, and to each dominant rational map $f\colon X \dashrightarrow Y$ of irreducible quasi projective varieties the pull back, we get an equivalence between the category of irreducible quasi projective varieties with homomorphisms dominant rational maps, and the category of finitely generated field extensions of* $\mathbb{K}$.

What must be proved are the following two statements:

1. An extension of fields $\mathbb{K} \subset E$ is isomorphic to the filed of rational functions $\mathbb{K}(X)$ of a quasi projective variety X if and only it it is finitely generated over $\mathbb{K}$.
2. Let E, F be finitely generated field extensions of $\mathbb{K}$, and let $\alpha\colon E \to F$ be a homomorphism of $\mathbb{K}$ extensions (i.e. an inclusion $E \hookrightarrow F$ which is the identity on $\mathbb{K}$). Let Y, X be irreducible quasi projective varieties such that $\mathbb{K}(Y), \mathbb{K}(X)$ are isomorphic to E and F respectively as extensions of $\mathbb{K}$ (they exist by Item (1)). Then there exists a unique dominant rational map $f\colon X \dashrightarrow Y$ such that $f^* = \alpha$.

Item (1) is proved in Proposition 2.4.4. Item (2) is proved in Proposition 2.4.6.

We start by observing that we may restrict our attention to affine (irreducible) varieties. In fact, let X be an irreducible quasi projective variety, and let $Y \subset X$ be an open dense affine subset (e.g. a prinipal open subset). We have a well-defined restriction map

$$\mathbb{K}(X) \dashrightarrow \mathbb{K}(Y). \tag{2.4.22}$$

In fact, let $f \in \mathbb{K}(X)$, and let (U,φ) be a couple representing an element. Then $U\cap Y$ is an open dense subset of Y, and the couple $(U\cap Y, \varphi_{|U\cap Y})$ represents an element $\overline{f} \in \mathbb{K}(Y)$, which is independnet of the representative of f. The restriction map in (2.4.22) is an isomorphism of $\mathbb{K}$ extensions. Hence, when dealing with the field of fractions of a quasi projective variety, we may assume that the variety is affine.

Let X be an irreducible quasi projective variety. We have an inclusion of $\mathbb{K}$ extensions:

$$\begin{array}{ccc} (\text{field of fractions of } \mathbb{K}[X]) & \hookrightarrow & \mathbb{K}(X) \\ \frac{\alpha}{\beta} & \mapsto & [(X \backslash V(\beta), \frac{\alpha}{\beta})] \end{array} \tag{2.4.23}$$

Claim 2.4.3. *Let X be an affine irreducible variety. Then* (2.4.23) *is an isomorphism.*

Proof. We must prove that the map in (2.4.23) is surjective. Let $f \in \mathbb{K}(X)$, and let (U, φ) represent f. By Remark 1.4.4, there exists $0 \neq \gamma \in \mathbb{K}[X]$ such that the dense principal open subset X_γ is contained in U. Moreover, by Remark 1.4.4 and Theorem 1.4.2, $\mathbb{K}[X_f]$ is generated as $\mathbb{K}$-algebra by $\mathbb{K}[X]$ and γ^{-1}, hence ϕ is represented by $(X_\gamma, \frac{\alpha}{\gamma^m})$ where $\alpha \in \mathbb{K}[X]$. Let $\beta := \gamma$. Since $X_\gamma = X_\beta$, we have proved that f belongs to the image of (2.4.23). $\square$

Proposition 2.4.4. *A field extension of $\mathbb{K}$ is isomorphic to the field of fractions of an irreducible quasi projective variety if and only if it is finitely generated over $\mathbb{K}$.*

Proof. Let X be a quasi projective variety. Let us prove that $\mathbb{K}(X)$ is finitely generated over $\mathbb{K}$. The field $\mathbb{K}(X)$ is isomorphic to the field of fractions of an open dense affine subset of X. Thus we may assume that $X \subset \mathbb{A}^n$ is closed. By Claim 2.4.3, $\mathbb{K}(X)$ is the field of quotients of $\mathbb{K}[X]$, and moreover $\mathbb{K}[X]$ is generated over $\mathbb{K}$ by the restrictions of the coordinate functions $z_1, \ldots, z_n$ by Theorem 1.4.2. Hence the restrictions of the coordinate functions $z_1, \ldots, z_n$ to X generate $\mathbb{K}(X)$ over $\mathbb{K}$.

Now assume that E is a finitely generated field extension of $\mathbb{K}$.

In particular the transcendenece degree of E over $\mathbb{K}$ is finite, say m. By Corollary A.5.7, there exists a prime polynomial $P \in \mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]$ such that E (as extension of $\mathbb{K}$) is isomorphic to the field $\mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]/(P)$. Write

$$P = z_{m+1}^d + c_1 z_{m+1}^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{K}\left(z_1, \ldots, z_m\right).$$

Then, for $i \in \{1, \ldots, d\}$, we have $c_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{K}[z_1, \ldots, z_m]$ and $b_i \neq 0$. Let $\widetilde{P} \in \mathbb{K}[z_1, \ldots, z_{m+1}]$ be obtained from P by clearing denominators, i.e. $\widetilde{P} = (b_1 \cdot \ldots \cdot b_d)P$. Lastly, let $Q \in \mathbb{K}[z_1, \ldots, z_{m+1}]$ be obtained from $\widetilde{P}$ by factoring out the maximum common divisor of the coefficients of $\widetilde{P}$ as polynomial in z_{m+1} (recall that $\mathbb{K}[z_1, \ldots, z_m]$ is a UFD). Notice that Q is irreducible and hence prime. Write

$$Q = e_0 z_{m+1}^d + e_1 z_{m+1}^{d-1} + \cdots + e_d, \qquad e_i \in \mathbb{K}[z_1, \ldots, z_m], \quad e_0 \neq 0.$$

Then $X := V(Q) \subset \mathbb{A}^{m+1}$ is an irreducible hypersurface because Q is prime. Let $\overline{z}_i := z_{i|X}$. We claim that the rational functions on X represented by $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{K}$. In fact, suppose that $R \in \mathbb{K}[t_1, \ldots, t_m]$ and $R(\overline{z}_1, \ldots, \overline{z}_n) = 0$. By the fundamental Theorem of Algebra, for any $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^m \backslash V(e_0))$ there exists $\xi_{m+1} \in \mathbb{K}$ such that $(\xi_1, \ldots, \xi_m, \xi_{m+1}) \in X$. It follows that $R(\xi_1, \ldots, \xi_m) = 0$ for all $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^n \backslash V(e_0))$, and hence $R \cdot e_0$ vanishes identically on $\mathbb{A}^m$. Thus $R \cdot e_0 = 0$, and since $e_0 \neq 0$ it follows that $R = 0$. This proves that $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{K}$. On the other hand $\overline{z}_{m+1}$ is algebraic over $\mathbb{K}(\overline{z}_1, \ldots, \overline{z}_m)$ and its minimal polynomial equals P. Hence the field of fractions of X is isomorphic to $\mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]/(P)$. $\square$

Proposition 2.4.5. *Let X and Y be irreducible quasi projective varieties. Suppose that $\alpha \colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$ is an inclusion of extensions of $\mathbb{K}$. There exists a unique dominant rational map $f \colon X \dashrightarrow Y$ such that $f^* = \alpha$.*

Proof. We may assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are closed. By Claim 2.4.3 $\mathbb{K}(X)$, $\mathbb{K}(Y)$ are the fields of fractions of $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ respectively, and by Theorem 1.4.2, $\mathbb{K}[X] = \mathbb{K}[z_1, \ldots, z_n]/I(X)$ and $\mathbb{K}[Y] = \mathbb{K}[w_1, \ldots, w_m]/I(Y)$. Given $p \in \mathbb{K}[z_1, \ldots, z_n]$ and $q \in \mathbb{K}[w_1, \ldots, w_m]$ we let $\overline{p} := p|_X$ and $\overline{q} := q|_Y$. We have

$$\alpha\left(\overline{w}_i\right) = \frac{\overline{f}_i}{\overline{g}_i}, \quad f_i, g_i \in \mathbb{K}[z_1, \ldots, z_n], \quad \overline{g}_i \neq 0.$$

Let $U := X \backslash (V(g_1) \cup \ldots \cup V(g_m))$. Then U is open and dense in X. Let

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{\phi}} & \mathbb{A}^m \\ a & \mapsto & \left(\frac{f_1(a)}{g_1(a)}, \ldots, \frac{f_m(a)}{g_m(a)}\right) \end{array}$$

We claim that $\widetilde{\phi}(U) \subset Y$. In fact let $h \in I(Y)$. Since α is an inclusion of extensions of $\mathbb{K}$,

$$h(\overline{f}_1/\overline{g}_1, \ldots, \overline{f}_m/\overline{g}_m) = h(\alpha(\overline{w}_1), \ldots, \alpha(\overline{w}_m)) = \alpha(h(\overline{w}_1, \ldots, \overline{w}_m) = \alpha(0) = 0.$$

This proves that if $h \in I(Y)$ then h vanishes on $\widetilde{\phi}(U)$, i.e. $\widetilde{\phi}(U) \subset Y$. Thus $\widetilde{\phi}$ induces a regular map $\phi \colon U \to Y$. Let $f \colon X \dashrightarrow Y$ be the equivalence class of (U, ϕ). Then $f^* = \alpha$.

It is clear by the above construction that f is the unique rational (dominant) map such that $f^* = \alpha$. $\square$

The result below follows at once from what has been proved above.

Corollary 2.4.6. *Irreducible quasi projective varieties are birational if and only if their fields of rational functions are isomorphic as extensions of* $\mathbb{K}$.

The result below follows from the above corollary and the proof of Proposition 2.4.4.

Proposition 2.4.7. *Let X be an irreducible quasi projective variety and let $m := \operatorname{Tr.deg}_{\mathbb{K}} \mathbb{K}(X)$. Then X is birational to an irreducible hypersurface in $\mathbb{A}^{m+1}$.*

2.5 Dimension

Let X be an irreducible quasi projective variety. The *dimension of* X is defined to be the transcendence degree of $\mathbb{K}(X)$ over $\mathbb{K}$. Next, let X be an arbitrary quasi projective variety, and let $X = X_1 \cup \cdots \cup X_r$ be its irreducible decomposition.

1. The *dimension of* X is the maximum of the dimensions of its irreducible components. We say that X has *pure dimension* n if every irreducible component of X has dimension n.

2. Let $p \in X$. The *dimension of X at p* is the maximum of the dimensions of the irreducible components of X containing p.

Example 2.5.1. The dimension of $\mathbb{A}^n$ is equal to n because $\{z_1, \ldots, z_n\}$ is a transcendence basis of $\mathbb{K}(z_1, \ldots, z_n)$ over $\mathbb{K}$.

Remark 2.5.2. (a) The dimension of X is equal to the dimension of any open dense subset $U \subset X$. In fact, by definition it suffices to rove it for irreducible X, and in that case it holds because the fields of rational functions $\mathbb{K}(X)$ and $\mathbb{K}(U)$ are isomorphic extensions of $\mathbb{K}$. Hence the dimension of $\operatorname{Gr}(h, V)$ is equal to $h \cdot (\dim V - h)$, because it is irreducible and it contains an open subset isomorphic to an affine space of dimension $h \cdot (\dim V - h)$, see Proposition 1.7.5.

(b) If $\dim X = 0$, then X is a finite set. It suffices to prove that if X is irreducible and $\mathbb{K}(X) = \mathbb{K}$, then X is a singleton. Let $X \subset \mathbb{P}^n$ be locally closed and irreducible, and suppose that it contains two distinct points x_1, x_2. Then there exist $L, M \in \mathbb{K}[Z_0, \ldots, Z_n]_1$ such that $L(x_1) = 0 \neq L(x_2)$, and $M(x_1) \neq 0 \neq M(x_2)$. Then L/M defines a rational function $f \colon X \dashrightarrow \mathbb{K}$, regular at x_1 and x_2, such that $f(x_1) = 0 \neq f(x_2)$. Thus $\mathbb{K}(X) \neq \mathbb{K}$.

(c) Let $f \colon X \dashrightarrow Y$ be a dominant map of irreducible quasi projective varieties. Then $\dim Y \leqslant \dim X$, because we have the inclusion $f^* \colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$ of field extensions of $\mathbb{K}$.

Proposition 2.5.3. *Let X be an irreducible quasi projective variety and $Y \subset X$ be a proper closed subset. Then* $\dim Y < \dim X$.

Proof. We may assume that Y is irreducible. Since X is covered by open affine varieties, we may assume that X is affine. Thus $X \subset \mathbb{A}^n$ is a closed (irreducible) subset, and so is Y. We may choose a transcendence basis $\{f_1, \dots, f_d\}$ of $\mathbb{K}(Y)$, where each f_i is a regular function on Y (for example a coordinate function).

Let $\tilde{f}_1, \dots, \tilde{f}_d \in \mathbb{K}[X]$ such that $\tilde{f}_{i|}W = f_i$. Since Y is a proper closed subset of X, there exists a non zero $g \in \mathbb{K}[X]$ such that $g_{|Y} = 0$. It suffices to prove that $\tilde{f}_1, \dots, \tilde{f}_d, g$ are algebraically independent over. We argue by contradiction. Suppose that there exists $0 \neq P \in \mathbb{K}[S_1, \dots, S_d, T]$ such that $P(\tilde{f}_1, \dots, \tilde{f}_d, g) = 0$. Since X is irreducible we may assume that P is irreducible. Restricting to Y the equality $P(\tilde{f}_1, \dots, \tilde{f}_d, g) = 0$, we get that $P(f_1, \dots, f_d, 0) = 0$. Thus $P(S_1, \dots, S_d, 0) = 0$, because $f_1, \dots, f_d$ are algebraically independent. This means that T divides P. Since P is irreducible $P = cT$, $c \in \mathbb{K}^*$. Thus $P(\tilde{f}_1, \dots, \tilde{f}_d, g) = 0$ reads $g = 0$, and that is a contradiction. $\square$

Corollary 2.5.4. *A (non empty) closed subset $X \subset \mathbb{A}^{n+1}$ has pure dimension n if and only if it is an irreducible hypersurface. Similarly, a closed subset $X \subset \mathbb{P}^{n+1}$ has pure dimension n if and only if it is an irreducible hypersurface.*

Proof. Let $X \subset \mathbb{A}^{n+1}$ be an irreducible hypersurface. Let $I(X) = (f)$. Reordering the coordinates $(z_1, \dots, z_n, z_{n+1})$ we may assume that

$$f = c_0 z_{n+1}^d + c_1 z_{n+1}^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{K}[z_1, \dots, z_n], \quad c_0 \neq 0, \quad d > 0.$$

In proving Proposition 2.4.7 we showed that the restrictions to X of the z_i's, for $i = 1, \dots, d$ give a transcendence basis of $\mathbb{K}(X)$. Thus $\dim X = n$. Since the irreducible components of a hypersurface are hypersurfaces (if $f = \prod f_i^{m_i}$ is the decomposition of f into prime factors, the irreducible components of $V(f)$ are the hypersurfaces $V(f_i)$), it follows that a hypersurface $X \subset \mathbb{A}^{n+1}$ is of pure dimension n.

In order to prove the converse, let $X \subset \mathbb{A}^{n+1}$ be a closed subset of pure dimension n. Thus every irreducible component of X is a closed subset of $\mathbb{A}^{n+1}$ of dimension n. Since the union of hypersurfaces in $\mathbb{A}^{n+1}$ is a hypersurface in $\mathbb{A}^{n+1}$, it suffices to prove that each irreducible component of $X$is a hypersurface, i.e we may assume that X is irreducible. Since $\dim X = n < \dim \mathbb{A}^{n+1}$, there exists a non zero $f \in I(X) \subset \mathbb{K}[z_1, \dots, z_{n+1}]$. Since X is irreducible, the ideal $I(X)$ is prime, and hence there exists a prime factor g of f which vanishes on X. Thus $X \subset V(g)$, $\dim X = n = \dim V(g)$ (by the result that we just proved), $V(g)$ is irreducible, and X is closed in $V(g)$. By Proposition 2.5.3 we get that $X = V(g)$. This finishes the proof for closed subsets of $\mathbb{A}^{n+1}$.

The result for closed subsets of $\mathbb{P}^{n+1}$ follows by a smilar proof, or by intersecting with standard open affine subsets $\mathbb{P}^n_{Z_i}$. $\square$

Proposition 2.5.5. *Let X, Y be quasi projective varieties. Then* $\dim(X \times Y) = \dim X + \dim Y$.

Proof. We may assume that X and Y are irreducible affine varieties. There exist transcendence bases $\{f_1, \dots, f_d\}$, $\{g_1, \dots, g_e\}$ of $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ respectively given by regular functions. Let $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ be the projections. We claim that $\{\pi_X^*(f_1), \dots, \pi_X^*(f_d), \pi_Y^*(g_1), \dots, \pi_Y^*(g_e)\}$ is a transcendence basis of $\mathbb{K}(X \times Y)$.

First, by Proposition 1.5.1 $\mathbb{K}[X \times Y]$ is algebraic over the subring generated (over $\mathbb{K}$) by $\pi_X^*(f_1), \dots, \pi_Y^*(g_e)$.

Secondly, let us show that $\pi_X^*(f_1), \dots, \pi_Y^*(g_e)$ are algebraically independent. Suppose that there is a polynomial relation

$$\sum_{0 \leqslant m_1, \dots, m_e \leqslant N} P_{m_1, \dots, m_e}(\pi_X^*(f_1), \dots, \pi_X^*(f_d)) \cdot \pi_Y^*(g_1)^{m_1} \cdot \dots \cdot \pi_Y^*(g_e)^{m_e} = 0,$$

where each $P_{m_1, \dots, m_e}$ is a polynomial. Since $g_1, \dots, g_e$ are algebraically independent we get that $P_{m_1, \dots, m_e}(f_1(a), \dots, f_d(a)) = 0$ for every $a \in X$. Since $f_1, \dots, f_d$ are algebraically independent, it follows that $P_{m_1, \dots, m_e} = 0$ for every $0 \leqslant m_1, \dots, m_e \leqslant N$, and hence $P = 0$. This proves that $\pi_X^*(f_1), \dots, \pi_Y^*(g_e)$ are algebraically independent. $\square$

Chapter 3

Projective methods

3.1 Introduction

3.2 Maps of finite degree

Definition 3.2.1. Let $f\colon X \dashrightarrow Y$ be a rational map of irreducible quasi-projective varieties. The *degree of* f is given by

$$\deg f := \begin{cases} 0 & \text{if } f \text{ is } \textit{not} \text{ dominant,} \\ [\mathbb{K}(X) : f^*(\mathbb{K}(Y))] & \text{if } f \text{ is dominant (hence pull-back of rational functions makes sense).} \end{cases}$$

If f is dominant, the pull-back $f^*\colon \mathbb{K}(Y) \to \mathbb{K}(X)$ is an embedding of fields; abusing notation we denote the image by $\mathbb{K}(Y)$. We recall that

$$[\mathbb{K}(X) : \mathbb{K}(Y)] = \dim_{\mathbb{K}(Y)} \mathbb{K}(X).$$

Thus $0 < \deg f < \infty$ if and only if f is dominant and $\dim X = \dim Y$, or equivalently $\mathbb{K}(X)$ is a finite extension of $\mathbb{K}(Y)$.

Example 3.2.2. Let $X \subset \mathbb{A}^{n+1}$ be an irreducible hypersurface such that $I(X) = (P)$, where

$$P = a_0 z_{n+1}^d + a_1 z_{n+1}^{d-1} + \cdots + a_d, \qquad a_i \in \mathbb{K}[z_1, \ldots, z_n], \quad a_0 \neq 0.$$

Let $f\colon X \to \mathbb{A}^n$ be the projection map $f(z) = (z_1, \ldots, z_n)$. Then $\deg f = d$. In fact suppose that $d = 0$. Then $\operatorname{Im} f = V(a_0)$ and hence f is not dominant. If $d > 0$, then $\mathbb{K}(X) = \mathbb{K}(z_1, \ldots, z_n)[z_{n+1}]/(P)$, and hence $[\mathbb{K}(X) : \mathbb{K}(z_1, \ldots, z_n)] = d$.

Definition 3.2.3. Let $f\colon X \dashrightarrow Y$ be a rational map of irreducible quasi-projective varieties, of finite degree (hence $\mathbb{K}(X)$ is a finite extension of $\mathbb{K}(Y)$). The *separable degree of* f is equal to 0 if $\deg f = 0$, and if $\deg f \neq 0$ it is the separable degree $[\mathbb{K}(X) : \mathbb{K}(Y)]_s$, see Theorem A.5.3. We denote it by $\deg_s f$.

Example 3.2.4. Let $X \subset \mathbb{A}^{n+1}$ and $f\colon X \to \mathbb{A}^n$ be as in Example 3.2.2. Suppose that $d \neq 0$, and hence $\deg f = d > 0$. If $\operatorname{char} \mathbb{K} = 0$, then $\mathbb{K}(X)$ is a separable extension of $\mathbb{K}(z_1, \ldots, z_n)$, and hence $\deg_s f = \deg f = d$. If $\operatorname{char} \mathbb{K} = p > 0$, then there is a maximum $r \geqslant 0$ such that $P = Q(z_1, \ldots, z_n, z_{n+1}^{p^r})$, where $Q \in \mathbb{K}[z_1, \ldots, z_n, w]$. Then $\frac{\partial Q}{\partial z_{n+1}} \neq 0$. It follows that the maximal separable extension of $\mathbb{K}(z_1, \ldots, z_n)$ in $\mathbb{K}(X)$ is obtained by adjoining $z_{n+1}^{p^r}$. Since the minimal polynomial of $z_{n+1}^{p^r}$ is $\frac{Q}{a_0}$ (the minimal polynomial is monic), the separable degree $\deg_s f$ is equal to $\deg_w Q = \frac{d}{p^r} = \frac{\deg f}{p^r}$.

Below is the main result of the present section.

Proposition 3.2.5. *Let $f\colon X \to Y$ be a regular map between irreducible quasi-projective varieties. Suppose that* $\deg f < \infty$. *There exists an open dense $Y^0 \subset Y$ such that*

$$|f^{-1}\{q\}| = \deg_s f \quad \forall q \in Y^0.$$

Example 3.2.6. Let us check the statement of Proposition 3.2.5 for the map $f\colon X \to \mathbb{A}^n$ of Example 3.2.2. Let $Q \in \mathbb{K}[z_1, \ldots, z_n, w]$ be as in Example 3.2.4, and let $Y := V(Q) \subset \mathbb{A}^n$. Let $h\colon X \to Y$ be defined by $h(z) = (z_1, \ldots, z_n, z_{n+1}^{p^r})$, and let $g\colon Y \to \mathbb{A}^n$ be the projection $g(z_1, \ldots, z_n, w) = (z_1, \ldots, z_n)$. Then $f = g \circ h$.

The non zero polynomial $\partial Q/\partial w$ has degree in w strictly smaller than the degree in w of Q, hence it is not a multiple of the prime polynomial Q. It follows that $V(Q, \partial Q/\partial w)$ is a proper closed subset of X and hence it has dimension strictly smaller than $\dim X = n$. Therefore, the closure

$$\Delta := \overline{f(V(Q, \partial Q/\partial w))}$$

is a proper closed subset of $\mathbb{A}^n$ and hence $U := (\mathbb{A}^n \backslash \Delta \backslash V(a_0))$ is an open dense subset of $\mathbb{A}^n$. Let $a \in U$. Then $|g^{-1}(a)| = \deg_s f$, because $Q(a, w) \in \mathbb{K}[w]$ is a polynomial of degree $\deg_s f$ with simple roots. Since $f^{-1}(a) = h^{-1}(g^{-1}(a))$, and h is bijective (since $\operatorname{char} \mathbb{K} = p$, every elemnt of $]\mathbb{K}$ has exactly one p-th root), we get that $|f^{-1}(a)| = \deg_s f$.

The proof of Proposition 3.2.5 follows some preliminary results.

Let $f\colon X \to Y$ and $g\colon W \to Y$ be regular dominant maps of irreducible varieties. Suppose that there exists an isomorphism of fields $\varphi\colon \mathbb{K}(W) \xrightarrow{\sim} \mathbb{K}(X)$ which is the identity on $\mathbb{K}(Y)$. Let $h\colon X \dashrightarrow W$ be the birational map such that $h^* = \varphi$ (see Proposition 2.4.6). Since φ is the identity on $\mathbb{K}(Y)$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

Lemma 3.2.7. *Keeping notation and hypotheses as above, there exists a dense open subset $Y^0 \subset Y$ such that, for all $y \in y^0$, the inverse image $f^{-1}(y)$ is contained in* $\operatorname{Reg}(h)$*, and the map h defines a bijection*

$$\begin{array}{ccc} f^{-1}(y) & \longrightarrow & g^{-1}(y) \\ x & \mapsto & h(x). \end{array}$$

Proof. Let $h^{-1}\colon W \dashrightarrow X$ be the inverse of h. By Proposition 2.2.6, there exist open dense subsets $U_X \subset \operatorname{Reg}(h)$ and $U_W \subset \operatorname{Reg}(h^{-1})$ such that the restriction of h defines an isomorphism $U_X \xrightarrow{\sim} U_W$, with inverse the restriction of h^{-1} to U_W (this result is not in the statement of the proposition, but in the proof). Let

$$Y^0 := Y \backslash \overline{f(X \backslash U_X)} \backslash \overline{g(W \backslash U_W)}.$$

Since $X \backslash U_X$, $W \backslash U_W$ are closed proper subsets of the irreducible varieties X, Y, and $\dim X = \dim Y = \dim W$, the open $Y^0 \subset Y$ is non empty, and hence dense (Y is irreducible). One easily checks that the Lemma holds for the Y^0 that we have just defined. $\square$

Next, we consider a more general version of Example 3.2.2. Let Y be an affine variety. Let $P \in \mathbb{K}(Y)[t]$ be an *irreducible* polynomial:

$$P = t^d + a_1 t^{d-1} + \cdots + a_d.$$

Since Y is affine $\mathbb{K}(Y)$ is the field of fractions of $\mathbb{K}[Y]$. Thus there exists $0 \neq b \in \mathbb{K}[Y]$ such that $b \cdot a_i \in f^*(\mathbb{K}[Y])$ for all $1 \leqslant i \leqslant d$. Let $c_0 := b$, $c_i := b \cdot a_i$, $1 \leqslant i \leqslant d$ and

$$Q := c_0 t^d + c_1 t^{d-1} + \cdots + c_d \in \mathbb{K}[Y][t]. \tag{3.2.1}$$

Let $\pi\colon Y \times \mathbb{A}^1 \to Y$ be the projection.

Lemma 3.2.8. *Keep hypotheses and notation as above. Assume moreover that*

$$\frac{dP}{dt} = dt^{d-1} + (d-1)a_1 t^{d-2} + \cdots + a_{d-1} \neq 0.$$

(If $\operatorname{char}\mathbb{K}$ *this holds as soon as* $d > 0$*, if* $\operatorname{char}\mathbb{K} = p$ *it holds if and only if there exists* $i \in \{1, \ldots, d\}$*, which is not a multiple of* p*, such that* $a_i \neq 0$*.)*

(a) *There is one and only one irreducible component* $V(Q)_i$ *of* $V(Q)$ *which* dominates Y*, i.e. such that* $\overline{\pi(V(Q)_i)} = Y$*; call it* V*, and let* $g\colon V \to Y$ *be the map defined by* π*.*

(b) *The extension of fields* $\mathbb{K}(V) \supset \mathbb{K}(Y)$ *given by the dominant map* $g\colon V \to Y$ *is generated by the restriction of the function* t *to* V*, and* P *is the minimal polynomial of* t *over* $\mathbb{K}(Y)$*.*

(c) *There is an open dense* $U \subset Y$ *such that* $|g^{-1}(y)| = d$ *for every* $y \in U$*.*

Proof. (a): We have $\pi(V(Q)) \supset Y\backslash V(c_0)$, and $Y\backslash V(c_0)$ is dense in Y because $c_0 \neq 0$. It follows that there exists at least one irreducible component $V(Q)_0$ of V such that $\overline{\pi(V(Q)_0)} = Y$. Let $g \in I(V(Q)_0)$. We claim that

$$Q|g \text{ in } \mathbb{K}(Y)[t]. \tag{3.2.2}$$

(Notice: we do *not* claim that $Q|g$ in $\mathbb{K}[Y][t]$.) In fact suppose that (3.2.2) does not hold. Then Q and g are coprime (in $\mathbb{K}(Y)[t]$) because Q is prime, and hence there exist $\alpha, \beta \in \mathbb{K}(Y)[t]$ such that $\alpha \cdot Q + \beta \cdot g = 1$. Multiplying by $0 \neq \gamma \in \mathbb{K}[Y][t]$ such that $\alpha \cdot \gamma, \beta \cdot \gamma \in \mathbb{K}[Y][t]$ we get that

$$(\alpha \cdot \gamma)Q + (\beta \cdot \gamma)g = \gamma.$$

It follows that if $q \in V(Q)_0$ then $\gamma(q) = 0$. Since $\gamma \neq 0$ we get that $\overline{\pi(V(Q)_0)} \neq Y$: that is a contradiction. This proves (3.2.2). Let $I(V(Q)_0) = (g_1, \ldots, g_r)$. From (3.2.2) we get that there exist $h_1, \ldots, h_r \in \mathbb{K}[Y][t]$ and $m_1, \ldots, m_r \in \mathbb{K}[Y]$ such that

$$m_i \cdot g_i = Q \cdot h_i, \quad m_i \neq 0, \quad i = 1, \ldots, r. \tag{3.2.3}$$

Set $m = m_1 \cdot \cdots \cdot m_r$. Then $V(Q)_0\backslash V(m) = V(Q)\backslash V(m)$ by (3.2.3), and hence $V(Q)_0$ is the unique irreducible component of $V(Q)$ dominating Y.

(b): This is clear by construction.

(c): Let

$$\frac{dQ}{dt} := dc_0t^{d-1} + (d-1)c_1t^{d-2} + \cdots + c_{d-1} \in \mathbb{K}[Y][t].$$

be the derivative of Q with respect to t. By hypothesis $\frac{dQ}{dt} \neq 0$, and $\deg_t \frac{dQ}{dt} < d = \deg_t Q$. Thus Q and $\frac{dQ}{dt}$ are coprime in $\mathbb{K}(Y)[t]$ and hence there exist $\mu, \nu \in \mathbb{K}(Y)[t]$ such that

$$\mu \cdot Q + \nu \cdot \frac{dQ}{dt} = 1.$$

Arguing as above we get that there exists a *proper* closed $C \subset Y$ such that

$$\pi^{-1}(Y\backslash C) \cap V(Q) \cap V\left(\frac{dQ}{dt}\right) = \varnothing. \tag{3.2.4}$$

Now let $U := (Y\backslash C\backslash V(c_0)\backslash V(m))$: then $|\pi^{-1}(q)| = d$ and $\pi^{-1}(q) \subset V$ for every $q \in U$. $\square$

Remark 3.2.9. If $\mathbb{K}[Y]$ is a UFD we may factor out the $\gcd\{c_0, \ldots, c_d\}$ and hence by renaming the c_i's we may assume that $\gcd\{c_0, \ldots, c_d\} = 1$. It follows that $V(Q)$ is irreducible (the proof is the same as the one for hypersurfaces in $\mathbb{A}^n$). The problem is that in general $\mathbb{K}[Y]$ will not be a UFD (an example: $Y = V(x_1x_2 - x_3x_4) \subset \mathbb{A}^4$ and $Q = x_1y - x_3$), and hence there might be no way of "reducing" the polynomial of (3.2.1) in order to get that $V(Q)$ is irreducible.

Proof of Proposition 3.2.5. Suppose that $\deg f = 0$. Then $\overline{f(X)} \neq Y$ and $Y^0 := Y \backslash \overline{f(X)}$ will do.

Now suppose that $\deg f > 0$. We construct, via Lemma 3.2.8, a dominant map $g\colon W \to Y$ of irreducible varieties such that the extension $\mathbb{K}(W) \supset \mathbb{K}(Y)$ is isomorphic to the extension $\mathbb{K}(X) \supset \mathbb{K}(Y)$, and so that $|g^{-1}(y)| = \deg_s f$ for y in an open dense subset of Y. Then Proposition 3.2.5 follows from Lemma 3.2.7.

Since Y is covered by open affine sets we may assume that Y itself is affine. The extension of fields $\mathbb{K}(X) \supset \mathbb{K}(Y)$ is algebraic and finitely generated. Let $\mathbb{K}(X)^s \supset \mathbb{K}(Y)$ be the maximal separable extension of $\mathbb{K}(Y)$ in $\mathbb{K}(X)$. Let $d := [\mathbb{K}(X)^s : \mathbb{K}(Y)] = \deg_s f$. By Theorem A.5.3, there exists a primitive element ξ of $\mathbb{K}(X)^s$ over $\mathbb{K}(Y)$. Let

$$P = t^d + a_1 t^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}(Y)$$

be the minimal polynomial of ξ. Let $Q \in \mathbb{K}[Y][t]$ be the polynomial in (3.2.1), obtained by clearing denominators of $a_1, \ldots, a_d$. Let $V \subset V(Q)$ be the unique irreducible component dominating Y, and let $g\colon V \to Y$ be the restriction of π, see Lemma 3.2.8. By Item (b) of that lemma, the extension of fields $\mathbb{K}(X)^s \supset \mathbb{K}(Y)$ and $\mathbb{K}(V) \supset \mathbb{K}(Y)$ are isomorphic.

If $\mathbb{K}(X)^s = \mathbb{K}(X)$, let $W := V$.

If $\mathbb{K}(X)^s \neq \mathbb{K}(X)$, then $\operatorname{char} \mathbb{K} = p > 0$, and if $\alpha_1, \ldots, \alpha_m$ are generators of $\mathbb{K}(X)$ over $\mathbb{K}(X)^s$, there exist $\beta_1, \ldots, \beta_m \in \mathbb{K}(X)^s$ and $r_1, \ldots, r_m \in \mathbb{N}$ such that $\alpha_i^{p^{r_i}} = \beta_i$ for $i \in \{1, \ldots, m\}$, see Theorem A.5.3.

We may view $\beta_1, \ldots, \beta_m$ as rational functions on V, because the extension of fields $\mathbb{K}(X)^s \supset \mathbb{K}(Y)$ and $\mathbb{K}(V) \supset \mathbb{K}(Y)$ are isomorphic. Replacing Y by an open dense subset $\mathscr{U}$ and V by $g^{-1}(\mathscr{U})$ we may assume that β_i are regular functions for $i \in \{1, \ldots, m\}$ (recall that $\dim V = \dim Y$). Let $W \subset Y \times \mathbb{A}^1 \times \mathbb{A}^m$ be the subset defined by

$$W := \{(y, t, z_1, \ldots, z_m) \in Y \times \mathbb{A}^1 \times \mathbb{A}^m \mid (y, t) \in W, \quad z_i^{p^{r_i}} = \beta_i\}.$$

In both cases ($\mathbb{K}(X)$ separable or not separable over $\mathbb{K}(Y)$) we let $g\colon W \to Y$ be the projection map.

By construction the extension of fields $\mathbb{K}(X) \supset \mathbb{K}(Y)$ and $\mathbb{K}(W) \supset \mathbb{K}(Y)$ are isomorphic, hence by Lemma 3.2.7 it suffices to prove that there exists an open dense $U \subset Y$ such that $|g^{-1}(y)| = d$ for $y \in U$. This follows from Lemma 3.2.8, because if $\operatorname{char} \mathbb{K} = p$, the equation $z^{p^r} = \beta$ has one solution. $\square$

Definition 3.2.10. We introduce some terminology. Let Y be a quasi-projective set and $\mathscr{P}$ a property that might or might not hold for a given $y \in Y$ (formally $\mathscr{P}$ is a subset of Y). We say that property $\mathscr{P}$ holds for the *generic point of* Y if there exists an *open dense* $Y^0 \subset Y$ such that property $\mathscr{P}$ holds for all $y \in Y^0$.

Example 3.2.11. 1. The generic point of Y is smooth.

2. If $f\colon X \to Y$ is a map of quasi-projective varieties and $\deg f < \infty$ then $|f^{-1}\{q\}| = \deg f$ for the generic $q \in Y$.

3.3 Degree of a closed subset of $\mathbb{P}^n$

Let X be an irreducible quasi-projective variety. The *codimension* of a closed subset $Y \subset X$ is equal to $\dim X - \dim Y$, and is denoted by $\operatorname{cod}(Y, X)$. Below is the main result of the present section.

Theorem 3.3.1. *Let $X \subset \mathbb{P}^n$ be closed, and let $c := \operatorname{cod}(X, \mathbb{P}^n)$.*

1. *If $0 \leqslant k < c$ and $\Lambda \in \operatorname{Gr}(k, \mathbb{P}^n)$ is generic, then Λ does not intersect X.*

2. *If $c \leqslant k \leqslant n$ and $\Lambda \in \operatorname{Gr}(k, \mathbb{P}^n)$, then Λ does intersect X.*

3. *There exists a strictly positive integer $\deg X$ such that for a generic $\Lambda \in \operatorname{Gr}(c, \mathbb{P}^n)$ the intersection $\Lambda \cap X$ has cardinality $\deg X$.*

The proof of the Items in Theorem 3.3.1 will follow from some preliminary results.

Example 3.3.2. Let $X \subset \mathbb{P}^n$ be a hypersurface. Thus $c = \mathrm{cod}(X, \mathbb{P}^n) = 1$. Item (1) of Theorem 3.3.1 is trivially verified, because $\mathbb{P}^n \backslash X$ is an open dense subset of $\mathbb{P}^n$. It is also straightforward to check that Item (2) holds. In fact let $\Lambda = \mathbb{P}(W)$, where W is a vector subspace of dimension at least 2. If $X = V(F)$, then$\Lambda \cap X = V(F_{|W})$, and since $\dim W \geqslant 2$, the non constant homogeneous polynomial $F_{|W}$ has non trvial zeroes, i.e. $\Lambda \cap X$ is not empty. Regarding Item (3): let F be a generator of the homogeneous ideal $I(X)$; thus F is determined up to multiplication by a non zero factor. Then $\deg X = \deg F$ - see Exercise 3.6.1.

Given $0 \leqslant k \leqslant n$ let $\Gamma_X(k) \subset X \times \mathrm{Gr}(k, \mathbb{P}^n)$ be defined by

$$\Gamma_X(k) = \{(p, \Lambda) \in X \times \mathrm{Gr}(k, \mathbb{P}^n) \mid p \in \Lambda\}.$$

Restricting to $\Gamma_X(k)$ the projections of $X \times \mathrm{Gr}(k, \mathbb{P}^n)$, we get regular maps

$$\begin{tikzcd} & \Gamma_X(k) \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ X & & \mathrm{Gr}(k, \mathbb{P}^n) \end{tikzcd} \tag{3.3.1}$$

If $\Lambda \in \mathrm{Gr}(k, \mathbb{P}^n)$, then $\rho^{-1}(\Lambda)$ is identified with $\Lambda \cap X$. Hence Theorem 3.3.1 is a statement about the fibers of the map ρ. Hence we must start by studying $\Gamma_X(k)$. The result below is essentially obtained by considering the fibers of the map π, which are all alike.

Proposition 3.3.3. *Let $X \subset \mathbb{P}^n$ be closed and irreducible. Then $\Gamma_X(k)$ is closed irreducible of dimension*

$$\dim \Gamma_X(k) = \dim X + k(n-k). \tag{3.3.2}$$

Proof. A straightforward computation shows that $\Gamma_X(k)$ is closed. Let $0 \leqslant i \leqslant n$. We identify $\mathbb{P}^n_{Z_i}$ with $\mathbb{K}^n$, as usual. We have an isomorphism

$$\begin{array}{ccc} X_{Z_i} \times \mathrm{Gr}(k, \mathbb{K}^n) & \xrightarrow{\alpha_i} & \Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathrm{Gr}(k, \mathbb{P}^n)) \\ (p, W) & \mapsto & (p, \overline{p+W}) \end{array}$$

Notice that W is a k-dimensional *vector subspace* of $\mathbb{K}^n$. Moreover $\overline{p+W}$ denotes the closure in $\mathbb{P}^n$ of the affine subspace $p + W \subset \mathbb{P}^n_{Z_i} \simeq \mathbb{K}^n$. Omitting those indices i such that $X \subset V(Z_i)$, we get that $\Gamma_X(k)$ is covered by open irreducible subsets of dimension

$$\dim(X_{Z_i} \times \mathrm{Gr}(k, \mathbb{K}^n)) = \dim X + \dim \mathrm{Gr}(k, \mathbb{K}^n) = \dim X + k(n-k).$$

Since X is irreducible $X_{Z_i} \cap X_{Z_j} \neq \varnothing$ for every couple (i,j) of indices such that X_{Z_i} and X_{Z_j} are non empty. It follows that $\Gamma_X(k)$ is irreducible, of dimension given by (3.3.2). □

Corollary 3.3.4. *Let $X \subset \mathbb{P}^n$ be closed. Then $\Gamma_X(k)$ is closed of dimension*

$$\dim \Gamma_X(k) = \dim X + k(n-k). \tag{3.3.3}$$

If $k \leqslant \mathrm{cod}(X, \mathbb{P}^n)$ then

$$\dim \Gamma_X(k) \leqslant \dim \mathrm{Gr}(k, \mathbb{P}^n) \tag{3.3.4}$$

with equality if and only if $k = \mathrm{cod}(X, \mathbb{P}^n)$.

Proof. Let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. Then

$$\Gamma_X(k) = \Gamma_{X_1}(k) \cup \cdots \cup \Gamma_{X_r}(k).$$

Thus (3.3.3) follows from Proposition 3.3.3. Let's prove (3.3.4). Let $c := \mathrm{cod}(X, \mathbb{P}^n)$ and X_i such that $c = n - \dim X_i$. Then

$$\dim \Gamma_{X_i}(c) = n - c + c(n-c) = (c+1)(n-c) = \dim \mathrm{Gr}(c, \mathbb{P}^n).$$

This gives (3.3.4). □

Proof of Item (a) of Theorem 3.3.1. By Corollary 3.3.4, the image of the map ρ in (3.3.1) is a proper closed subset of $\mathrm{Gr}(k, \mathbb{P}^n)$. Hence for generic $\Lambda \in \mathrm{Gr}(k, \mathbb{P}^n)$, the fiber $\rho^{-1}(\Lambda) = \Lambda \cap X$ is empty. □

The result below will be useful in proving Items (b), (c) of Theorem 3.3.1, and also in other circumstances.

Proposition 3.3.5. *Let $X \subset \mathbb{P}^n$ be closed. Suppose that $p \in \mathbb{P}^n \backslash X$ and that $H \subset \mathbb{P}^n \backslash \{p\}$ is a hyperplane. Let*

$$\begin{array}{ccc} (\mathbb{P}^n \backslash \{p\}) & \xrightarrow{\pi} & H \\ x & \mapsto & \langle p, x \rangle \cap H \end{array}$$

be the projection. Then $\pi(X)$ is a closed subset of H and $\dim \pi(X) = \dim X$.

Proof. We may assume that X is irreducible. Since $\pi_{|X}$ is regular and X is projective $\pi(X)$ is closed by Proposition 1.6.3. It remains to prove that $\dim \pi(X) = \dim X$. We may assume that $p = [0, \ldots, 0, 1]$, $H = V(Z_n)$, and X is not contained in $V(Z_0)$. We have

$$\pi([Z_0, \ldots, Z_n]) = [Z_0, \ldots, Z_{n-1}].$$

Let $Y := \pi(X)$. We have an injection of fields $\pi^* \colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$, and we must prove that $[\mathbb{K}(X) : \pi^*(\mathbb{K}(Y))] < \infty$. The field $\mathbb{K}(Y)$ is generated (over $\mathbb{K}$) by

$$(Z_1/Z_0)_{|Y}, \ldots, (Z_{n-1}/Z_0)_{|Y}.$$

On the other hand $\mathbb{K}(X)$ is generated by

$$(Z_1/Z_0)_{|X}, \ldots, (Z_{n-1}/Z_0)_{|X}, (Z_n/Z_0)_{|X}.$$

Since $\pi^*((Z_i/Z_0)_{|Y}) = (Z_i/Z_0)_{|X}$, it suffices to prove that $(Z_n/Z_0)_{|X}$ is algebraic over $(Z_1/Z_0)_{|X}, \ldots, (Z_{n-1}/Z_0)_{|X}$. There exists $F \in I(X)$ such that $F(p) \neq 0$ because $p \notin X$. Since $p = [0, \ldots, 0, 1]$ we get that

$$F = a_0 Z_n^d + a_1 Z_n^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}[Z_0, \ldots, Z_{n-1}]_i, \quad a_0 \neq 0.$$

Dividing by Z_0^d and restricting to X we get that

$$a_0 \cdot ((Z_n/Z_0)_{|X})^d + \overline{a}_1 \cdot ((Z_n/Z_0)_{|X})^{d-1} + \cdots + \overline{a}_d = 0 \tag{3.3.5}$$

where $\overline{a}_j := (a_j/Z_0^j)_{|X}$ for $1 \leqslant j \leqslant d$. Since $a_0 \neq 0$, Equation (3.3.5) shows that $(Z_n/Z_0)_{|X}$ is algebraic over $(Z_1/Z_0)_{|X}, \ldots, (Z_{n-1}/Z_0)_{|X}$. □

Proof of Item (b) of Theorem 3.3.1. The proof is by induction on $\mathrm{cod}(X, \mathbb{P}^n)$. If $\mathrm{cod}(X, \mathbb{P}^n) = 0$ the result is trivial (if you don't like to start from $\mathrm{cod}(X, \mathbb{P}^n) = 0$ you may begin from $\mathrm{cod}(X, \mathbb{P}^n) = 1$, i.e. X a hypersurface). Let's prove the inductive step. Let $p \in \Lambda$. If $p \in X$ there is nothing to prove; thus we may assume that $p \notin X$. Choose a hyperplane $H \subset \mathbb{P}^n$ not containing p, and let π be projection from p, as in (3.3.6). Then $Y := \pi(X) \subset H \simeq \mathbb{P}^{n-1}$ is closed and $\dim Y = \dim X$ by Proposition 3.3.5. Thus $\mathrm{cod}(Y, \mathbb{P}^{n-1}) = (\mathrm{cod}(X, \mathbb{P}^n) - 1)$. Let $\Lambda' := \pi(\Lambda \backslash \{p\})$. Then $\Lambda' \subset H$ is a linear subspace with $\dim \Lambda' = (\dim \Lambda - 1)$, and hence $\dim \Lambda' \geqslant \mathrm{cod}(Y, \mathbb{P}^{n-1})$. By the inductive hypothesis it follows that $\Lambda' \cap Y$ is not empty. Let $y \in \Lambda \cap Y$. Since $y \in \pi(X)$ there exists $x \in X$ such that $\pi(x) = y$. By definition of π, we have $x \in \langle p, y \rangle$. Since $p \in \Lambda$ and $y \in \Lambda$ (because $\Lambda' = \Lambda \cap H$), it follows that $x \in \Lambda$. Thus $x \in \Lambda \cap X$. □

Proof of Item (c) of Theorem 3.3.1. We start by defining the degree of a closed $X \subset \mathbb{P}^n$. First assume that X is irreducible. Let $c := \mathrm{cod}(X, \mathbb{P}^n)$. Let

$$\begin{array}{ccc} \Gamma_X(c) & \xrightarrow{\pi} & \mathrm{Gr}(c, \mathbb{P}^n) \\ (p, \Lambda) & \mapsto & \Lambda \end{array} \tag{3.3.6}$$

Since $\Gamma_X(c)$ and $\mathrm{Gr}(c,\mathbb{P}^n)$ are varieties we have a well-defined $\deg\pi$. By Corollary 3.3.4 we have $\dim\Gamma_X(c) = \dim\mathrm{Gr}(c,\mathbb{P}^n)$: thus $\deg\pi < \infty$. The *degree of* X is defined to be

$$\deg X := \deg(\Gamma_X(c) \xrightarrow{\pi} \mathrm{Gr}(c,\mathbb{P}^n)). \tag{3.3.7}$$

In general let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. The *degree of* X is defined to be

$$\deg X := \sum_{\dim X_i = \dim X} \deg X_i. \tag{3.3.8}$$

If X is irreducible, Item (c) of Theorem 3.3.1 follows from Proposition 3.2.5 applied to the map π of (3.3.6). In general let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. By Item (a) of Theorem 3.3.1, for generic $\Lambda \in \mathrm{Gr}(c,\mathbb{P}^n)$

$$\Lambda \cap X_i = \varnothing \text{ if } \dim X_i < \dim X, \quad \Lambda \cap (X_i \cap X_j) = \varnothing \text{ if } i \neq j.$$

It follows that for Λ generic

$$\Lambda \cap X = \bigsqcup_{\dim X_i = \dim X} \Lambda \cap X_i$$

and hence Item (c) follows from the case X irreducible. □

Remark 3.3.6. Theorem 3.3.1 gives a characterization of the dimension of a closed $X \subset \mathbb{P}^n$ via its intersections with linear subspaces.

The degree of a closed subset of a projctive space may be considered as a first, very rough, measure of its complexity. The (classical) result below gives a lower bound of the degree.

Proposition 3.3.7. *Let $X \subset \mathbb{P}^n$ be closed, irreducible and non degenerate, i.e. spanning the whole projective space. Then*

$$\deg X \geqslant \mathrm{cod}(X,\mathbb{P}^n) + 1. \tag{3.3.9}$$

Proof. By induction on $c := \mathrm{cod}(X,\mathbb{P}^n)$. If $c \neq 0$, then $X = \mathbb{P}^n$, and certainly (3.3.9) holds. If $c = 1$, then X is a hypersurface. Let $I(X) = (F)$. Then $\deg F \geqslant 2$, because X is non degenerate. Since $\deg X = \deg F$ by Exercise 3.6.1, we get that (3.3.9) holds in this case as well.

Let us prove the inductive step. Thus we assume that $c \geqslant 2$. Since the generic $\Lambda \in \mathrm{Gr}(c,\mathbb{P}^n)$ intersects X in $\deg X$ points, it follows that for a *generic* $p \in X$, the *generic* $\Lambda \in \mathrm{Gr}(c,\mathbb{P}^n)$ *containing* p intersects X in $\deg X$ points. The idea is to project X from p to a hyperplane $H \cong \mathbb{P}^{n-1}$ not containing p, and compare the degree of X and the degree of the image, which has codimension $c-1$ in $\mathbb{P}^{n-1}$, and hence satisfies the inequality in (3.3.9).

Explicitly, choose homogeneous coordinates such that $p = [0,\ldots,0,1]$, and $H = V(Z_n)$. The projection of X from p to H is the rational map

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^{n-1} \\ [Z] & \mapsto & [Z_0,\ldots,Z_{n-1}] \end{array}$$

Let $\mathrm{Bl}_p(\mathbb{P}^n)$ be the blow up of $\mathbb{P}^n$ in p, and let $\rho\colon \mathrm{Bl}_p(\mathbb{P}^n) \to \Sigma_p$, where Σ_p is the set of lines containing p, see (2.3.3). We may identify Σ_p with H, by mapping a line in Σ_p with its intersection with H. Let $\widetilde{X} := \mathrm{Bl}_p(X) \subset \mathrm{Bl}_p(\mathbb{P}^n)$, and let $\widetilde{\varphi}\colon \widetilde{X} \to \mathbb{P}^{n-1}$ be the restriction of ρ. We have a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"] \arrow[dr, "\widetilde{\varphi}"] & \\ X \arrow[r, dotted, "\varphi"] & \mathbb{P}^{n-1} \end{tikzcd}$$

Let $Y := \mathrm{Im}(\widetilde{\varphi})$. Then Y is closed and irreducible, because $\widetilde{X}$ is projective and irreducible. We claim that $\dim Y = \dim X$. Let $\psi\colon \widetilde{X} \to Y$ be the map defined by $\widetilde{\varphi}$. Since ψ is surjective, we have an injection $\psi^*(\mathbb{K}(Y)) \subset \mathbb{K}(\widetilde{X})$. By our hypothesis on p, the variety X is not a cone with vertex p, and hence there exists $F \in I(X)$, homogeneous of degree d, which is not an element of $\mathbb{K}[Z_0, \ldots, Z_{n-1}]$, i.e.

$$F = a_e \cdot Z_n^{d-e} + \ldots + a_d, \quad a_i \in \mathbb{K}[Z_0, \ldots, Z_{n-1}]_i, \ , e < d, \ a_e \neq 0.$$

Dividing F by Z_0^d, we get that

$$\left(a_e \left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_{n-1}}{Z_0} \right) \cdot \left(\frac{Z_n}{Z_0} \right)^{d-e} + \ldots + a_d \left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_{n-1}}{Z_0} \right) \right)_{|X} = 0.$$

It follows that the rational function $\pi^* \left(\frac{Z_n}{Z_0}_{|X} \right)$ is algebraic on $\psi^*(\mathbb{K}(Y))$, and hence $\mathbb{K}(\widetilde{X})$ is algebraic over $\psi^*(\mathbb{K}(Y))$. This proves that that $\dim Y = \dim X$. Thus $\operatorname{cod}(Y, \mathbb{P}^{n-1}) = c - 1$, and by the inductive hypothesis $\deg Y \geqslant c$. On the other hand, let Λ be a generic dimension-$(c-1)$ linear subspace of $\mathbb{P}^{n-1}$. Then

$$|\Lambda \cap Y| = \deg Y, \quad |\langle \Lambda, p \rangle \cap X| = \deg X.$$

(The second equality holds because $\langle \Lambda, p \rangle$ is a generic dimension-c linear subspace of $\mathbb{P}^n$ containing p.) Again by genericity, the fiber of $\widetilde{\varphi}$ over each point of $\Lambda \cap Y$ does not intersect the exceptional set of $\pi \colon \widetilde{X} \to X$, i.e. $\pi^{-1}(p)$; it follows that

$$\deg X = |\langle \Lambda, p \rangle \cap X| \geqslant |\Lambda \cap Y| + 1 = \deg Y + 1 \geqslant c + 1.$$

□

3.4 Intersection of closed subsets of a projective space

Theorem 3.3.1 shows that the dimension of a closed subset of a projective space is determined by the intersections of the subset with linear subspaces. Interesting consequences of this fact are proved in the present section.

First we define the join of two closed subsets $X, Y \subset \mathbb{P}^N$ such that

$$\langle X \rangle \cap \langle Y \rangle = \varnothing, \tag{3.4.1}$$

where $\langle X \rangle$ and $\langle Y \rangle$ are the linear subspaces generated by X and Y respectively.

Definition 3.4.1. The *join of X and Y* is the subset of $\mathbb{P}^N$ swept out by the lines joining a point of X to a point of Y:

$$J(X, Y) := \bigcup_{x \in X, y \in Y} \langle x, y \rangle. \tag{3.4.2}$$

Lemma 3.4.2. *Let $X, Y \subset \mathbb{P}^N$ be closed subsets such that* (3.4.1) *holds. Then*

1. *$J(Y, W)$ is closed,*
2. *if X and Y are irreducible $J(X, Y)$ is irreducible,*
3. $\dim J(X, Y) = \dim X + \dim Y + 1$.

Proof. Let $m := \dim\langle X \rangle$ and $n := \dim\langle Y \rangle$. There exist homogeneous coordinates

$$[S_0, \ldots, S_m, T_0, \ldots, T_n, U_0, \ldots, U_p]$$

on $\mathbb{P}^N$ such that

$$\langle X \rangle = \{[S_0, \ldots, S_m, 0, \ldots, 0]\}, \quad \langle Y \rangle = \{[0, \ldots, 0, T_0, \ldots, T_n, 0, \ldots, 0]\}.$$

Then

$$J(X,Y) = \{[S_0,\ldots,S_m,T_0,\ldots,T_n,0,\ldots,0] \mid [S] \in X, \quad [T] \in Y\}. \tag{3.4.3}$$

Item (1) follows at once. Let $p \in (J(Y,W)\backslash X\backslash Y)$. By (3.4.1) there is *unique* couple $(\varphi_1(p),\varphi_2(p)) \in X \times Y$ such that $p \in \langle \varphi_1(p),\varphi_2(p)\rangle$, and the map

$$\begin{array}{ccc} (J(X,Y)\backslash X\backslash Y) & \xrightarrow{\varphi} & X \times Y \\ p & \mapsto & (\varphi_1(p),\varphi_2(p)) \end{array} \tag{3.4.4}$$

is regular, with fibers isomorphic to $\mathbb{K}^*$. Moreover for any $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$ the inverse image $\varphi^{-1}(X_{S_i} \times Y_{T_j})$ is isomorphic to $X_{S_i} \times Y_{T_j} \times \mathbb{K}^*$. Items (2) and (3) follow. □

Proposition 3.4.3. *Let $X \subset \mathbb{P}^n$ be closed, irreducible of strictly positive dimension. Let $H \subset \mathbb{P}^n$ a hyperplane* not *containing X. Then $X \cap H$ is not empty and every irreducible component of $X \cap H$ has dimension equal to* $(\dim X - 1)$.

Proof. The intersection is non empty by Theorem 3.3.1. First we will prove a weaker result, namely that

$$\dim X \cap H = \dim X - 1. \tag{3.4.5}$$

Let $c := \operatorname{cod}(X,\mathbb{P}^n)$. Then (3.4.5) is equivalent to $\operatorname{cod}(X \cap H, H) = c$. Since $X \cap H \subsetneq X$ we have $\dim X \cap H < \dim X$ and hence $\operatorname{cod}(X \cap H, H) \geqslant c$. By Theorem 3.3.1 applied to the closed $(X \cap H) \subset H$ it suffices to prove that if $L \subset H$ is an arbitrary linear subspace with $\dim L = c$ then $L \cap (X \cap H) \neq \emptyset$. By Theorem 3.3.1 applied to X we have $L \cap X \neq \emptyset$: since $L \subset H$ we have $L \cap X \subset L \cap (X \cap H)$. This proves (3.4.5). The proposition states a stronger result namely that *every* irreducible component of $X \cap H$ has dimension equal to $(\dim X - 1)$. The proof is by induction on $\operatorname{cod}(X,\mathbb{P}^n)$, the initial case being $\operatorname{cod}(X,\mathbb{P}^n) = 1$ (Notice that if $\operatorname{cod}(X,\mathbb{P}^n) = 0$ the statement of the prosition is trivially true). If $\operatorname{cod}(X,\mathbb{P}^n) = 1$ then X is a hypersurface by Corollary 2.5.4 and hence $X \cap H$ is a hypersurface in H: by Corollary 2.5.4 every irreducible component of $X \cap H$ has coddimension one in H. Let's prove the inductive step. We assume that $\operatorname{cod}(X,\mathbb{P}^n) = c \geqslant 2$. Suppose that W_1 is an irreducible component of $X \cap H$. Pick a point $p \in H\backslash X$ and a hyperplane H' not containing p and *different from* H. Let

$$\begin{array}{ccc} \mathbb{P}^n\backslash\{p\} & \xrightarrow{\pi_p} & H' \\ q & \mapsto & \langle p,q\rangle \cap H' \end{array}$$

be the projection. We will consider $\pi_p(X) \cap \pi_p(H)$. Let $X \cap H = W_1 \cup \cdots \cup W_r$ be the irreducible decomposition of $X \cap H$. Let us prove that there exists p such that

$$\pi_p(W_1) \not\subset \pi_p(W_i) \quad \forall i \in \{2,\ldots,r\}. \tag{3.4.6}$$

In fact, let $q \in W_1\backslash\bigcup_{i=2}^r W_i$, and let $i \in \{2,\ldots,r\}$. Then $J(q,W_i)$ is defined, and by Lemma 3.4.2, it is closed irreducible. Moreover, by Lemma 3.4.2

$$\dim J(q,W_i) = \dim W_i + 1 \tag{3.4.7}$$

Since $H \not\supset X$, $\dim W_i \leqslant \dim X - 1$ and since $\operatorname{cod}(X,\mathbb{P}^n) \geqslant 2$ we have $\dim W_i \leqslant \dim H - 2$. Thus (3.4.7) gives that $J(q,W_i) \neq H$. Hence there exists $p \in H\backslash\bigcup_{i=2}^r J(q,W_i)$. Then $\pi_p(q) \notin \pi_p(W_i)$ for $i \in \{2,\ldots,r\}$, and hence (3.4.6) holds.

Each of $\pi_p(W_1),\ldots,\pi_p(W_r)$ is closed, and

$$\pi_p(X) \cap \pi_p(H) = \pi_p(W_1) \cup \ldots \cup \pi_p(W_r).$$

Moreover, by (3.4.6) it follows that $\pi_p(W_1)$ is an irreducible component of $\pi_p(X) \cap \pi_p(H)$. By Proposition 3.3.5 we have $\dim \pi_p X = \dim X$ and hence $\operatorname{cod}(\pi_p(X),H') = (\operatorname{cod}(X,\mathbb{P}^n) - 1)$. By the inductive hypothesis we get that $\operatorname{cod}(\pi_p(W_1),\pi_p(X)) = 1$. Since $\dim \pi_p(W_1) = \dim W_1$ and $\dim \pi_p(X) = \dim X$ (by Proposition 3.3.5) we get that $\operatorname{cod}(W_1,X) = 1$. □

Corollary 3.4.4. *Let $X \subset \mathbb{P}^n$ be closed of codimension c. Let $\Lambda \in \mathrm{Gr}(c, \mathbb{P}^n)$. Then $X \cap \Lambda$ is not empty and every irreducible component of $X \cap \Lambda$ has dimension at least* $(\dim X - c)$.

The proposition below is a remarkable generalization of the well-known linear algebra result: "a system of n homogeneous linear equations in $(n+1)$ unknowns has at a non-trivial solution".

Proposition 3.4.5. *Let $Y, W \subset \mathbb{P}^n$ be closed and suppose that* $(\dim Y + \dim W) \geqslant n$. *Then $Y \cap W \neq \varnothing$ and moreover every irreducible component of $Y \cap W$ has dimension at least* $(\dim Y + \dim W - n)$.

Proof of Proposition 3.4.5. Let $[s_0, \ldots, s_n, t_0, \ldots, t_n]$ be homogeneous coordinates on $\mathbb{P}^{2n+1}$. We have two embeddings

$$\begin{array}{ccccccc} \mathbb{P}^n & \xrightarrow{i} & \mathbb{P}^{2n+1} & \quad & \mathbb{P}^n & \xrightarrow{j} & \mathbb{P}^{2n+1} \\ [X_0, \ldots, X_n] & \mapsto & [s_0, \ldots, s_n, 0, \ldots, 0] & & [X_0, \ldots, X_n] & \mapsto & [0, \ldots, 0, X_0, \ldots, X_n] \end{array} \tag{3.4.8}$$

Since the images of i and j are disjoint linear subspaces of $\mathbb{P}^{2n+1}$ the join $J(i(Y), j(W))$ is defined. We will intersect $J(i(Y), j(W))$ with the linear subspace of $\mathbb{P}^{2n+1}$ defined by

$$\Lambda := V(s_0 - t_0, \ldots, s_n - t_n).$$

We have an isomorphism

$$\begin{array}{ccc} Y \cap W & \xrightarrow{\sim} & \Lambda \cap J(i(Y), j(W)) \\ [X_0, \ldots, X_n] & \mapsto & [X_0, \ldots, X_n, X_0, \ldots, X_n] \end{array} \tag{3.4.9}$$

By Lemma 3.4.2 the closed $J(i(Y), j(W)) \subset \mathbb{P}^{2n+1}$ has dimension $(\dim Y + \dim W + 1)$. On the other hand Λ is a codimension-$(n+1)$ linear subspace of $\mathbb{P}^{2n+1}$; by Corollary 3.4.4 $\Lambda \cap J(i(Y), j((W))$ is not empty and every irreducible component of $\Lambda \cap J(i(Y), j((W))$ has dimension at least $(\dim Y + \dim W - n)$. Isomorphism (3.4.9) gives that $Y \cap W$ is not empty and every irreducible component of $Y \cap W$ has dimension at least $(\dim Y + \dim W - n)$. □

Example 3.4.6. Let $n \geqslant 2$ and $X \subset \mathbb{P}^n$ be a smooth hypersurface. Then X is irreducible. In fact suppose that $X = Y \cup W$ where Y, W are proper closed subsets of X. Then Y and W are of pure dimension $(n-1)$ and hence $Y \cap W$ is not empty by Proposition 3.4.5. Let $p \in Y \cap W$: as is easily checked X is singular at p, that is a contradiction.

3.5 Dimension of fibers

The following key result is a particular case of *Krull's HauptidealSatz* (valid for arbitrary Noetherian rings).

Theorem 3.5.1. *Let X be an irreducible quasi projective variety and $0 \neq f \in \mathbb{K}[X]$. Every irreducible component of $V(f)$ has dimension* $(\dim X - 1)$.

Proof. We may assume that X is affine. Thus there exists n such that $X \subset \mathbb{A}^n$ is closed. By Theorem 1.4.2 there exists $\tilde{f} \in \mathbb{K}[z_1, \ldots, z_n]$ such that $f = \tilde{f}_{|X}$. We must prove that, if Y is an irreducible component of $V(\tilde{f})$, then $\dim Y = \dim X - 1$. We view $\mathbb{A}^n$ as the open affine set $\mathbb{P}^n_{X_0} \subset \mathbb{P}^n$, and we let $\overline{X}, \overline{V(\tilde{f})}, \overline{Y} \subset \mathbb{P}^n$ be the closures of X, $V(\tilde{f})$ and Y respectively. Let d be the degree of the hypersurface $\overline{V(\tilde{f})} \subset \mathbb{P}^n$. Let $N := (\binom{d+n}{n} - 1)$ and let

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu_d^n} & \mathbb{P}^N \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0^d, Z_0^{d-1} Z_1, \ldots, Z_n^d] \end{array}$$

be the Veronese map. Then ν_d^n defines an isomophism $\overline{X} \xrightarrow{\sim} \nu(\overline{X})$. Since $\overline{V(\tilde{f})}$ is a hypersurface of degree d there exists a hyperplane $H \subset \mathbb{P}^N$ such that $\nu^{-1}(H) = \overline{V(\tilde{f})}$. Thus ν_d^n defines an isomorphism $\overline{X} \cap \overline{V(\tilde{f})} \xrightarrow{\sim} \nu(\overline{X}) \cap H$. It follows that $\nu(\overline{W})$ is an irreducible component of $\nu(\overline{X}) \cap H$. By Proposition 3.4.4 we have

$$\dim W = \dim \overline{W} = \dim \nu(\overline{W}) = \dim \overline{X} - 1 = \dim X - 1.$$

□

Proposition 3.5.2. *Let $f \colon X \to Y$ be a regular map of quasi-projective varieties. Then the following hold:*

(a) If $x_0 \in X$, the dimension at x_0 of every irreducible component of $f^{-1}(f(x_0))$ is at least $\dim_{x_0} X - \dim_{f(x_0)} Y$.

(b) Assume that X, Y are irreducible, f is projective and dominant. There exists an open dense $U \subset Y$ such that for all $y \in U$ the fiber $f^{-1}(y)$ has pure dimension $\dim X - \dim Y$.

(c) Assume that f is projective. The function

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & \mathbb{N} \\ y & \mapsto & \dim f^{-1}(y) \end{array}$$

is upper-semicontinuous, i.e. given $k \in \mathbb{N}$ the set $\{y \in Y \mid \dim f^{-1}(y) \geqslant k\}$ is closed.

Proof. (a): We may assume that Y is affine. Let $\dim_{f(x_0)} Y = m$. Then there exist $\phi_1, \ldots, \phi_m \in \mathbb{K}[Y]$ such that $f(x_0)$ is an irreducible component of $V(\phi_1, \ldots, \phi_m)$ (in fact choose $0 \neq \phi_1 \in I(\{f(x_0)\})$, then choose $\phi_2 \in I(\{f(x_0)\})$ not vanishing on any irreducible component of $V(\phi_1)$ etc.). Thus, by shrinking Y around $f(x_0)$, we may assume that $\{x_0\} = V(\phi_1, \ldots, \phi_m)$, and hence

$$f^{-1}(f(x_0)) = V(f^*\phi_1, \ldots, f^*\phi_m.$$

By repeated application of Theorem 3.5.1 every irreducible component of $V(f^*\phi_1, \ldots, f^*\phi_m)$ has dimension at x_0 at least $(\dim_{x_0} X - m) = (\dim_{x_0} X - \dim_{f(x_0)} Y)$.

(b): By induction on $e := \dim X - \dim Y$. Suppose that $e = 0$, i.e. $\dim X = \dim Y$. By our hypotheses $0 < \deg f < \infty$, hence in this case the assertion holds by Proposition 3.2.5.

Let us prove the inductive step. Suppose that $e > 0$. Since f is projective and dominant, it is surjective. Hence for all $y \in Y$, we have $\dim f^{-1}(y) \geqslant e$ by Item (a) (more precisely, every irreducible component of $f^{-1}(y)$ has dimension at least e). Since f is projective, we may assume that $X \subset \mathbb{P}^N \times Y$ is closed, and $f = \pi_{|X}$, where $\pi \colon \mathbb{P}^N \times Y \to Y$ is the projection map. Let $\rho \colon \mathbb{P}^N \times Y \to \mathbb{P}^N$ be the other projection. Let $y_0 \in Y$, and let $H \subset \mathbb{P}^N$ be a hyperplane that does not contain $f^{-1}(y_0)$. Then $W := H \times Y$ is a proper closed subset of X. Let $g \colon W \to Y$ be the restriction of the projection π (i.e. the restriction of f). The map g is projective, and $\pi(W) = Y$, because, given $y \in Y$, we have $\rho(g^{-1}(y)) = \rho(f^{-1}(y)) \cap H$, and $f^{-1}(y)$ is a closed subset of $\mathbb{P}^N$ of dimension at least $e > 0$. Let $W = W_1 \cup \ldots \cup W_r$ be the decomposition into irreducible components. Each W_i is closed in $\mathbb{P}^N \times Y$, hence $g(W_i)$ is closed for every $i \in \{1, \ldots, r\}$. Let

$$U := Y \backslash \bigcup g(W_i) \neq Y g(W_i).$$

Then U is open dense in Y. Shrinking Y we may assume that $U = Y$, i.e. $g(W_i) = Y$ for all $i \in \{1, \ldots, r\}$. Every W_i has dimension $\dim X - 1$ by Proposition 3.5.1 (H is locally the zero set of single non zero function). Let $g_i \colon W_i \to Y$ be the restriction of g. Then $\dim W_i - \dim Y = e - 1$, and hence by the inductive hypothesis, there exists an open dense $Y_0(i) \subset Y$ such that $g^{-1}(y)$ has pure dimension $e - 1$ for all $y \in Y_0(i)$. Let

$$Y_0 := \bigcap_{i=1}^{r} Y_0(i).$$

Clearly Y_0 is open and dense in Y. Let $y \in Y_0$. Then

$$H \cap \rho(g^{-1}(y)) = \bigcup_{i=1}^{r} \rho(g_i^{-1}(y)).$$

The right hand side is the intersection of the hyperplane H with closed subset of $\mathbb{P}^N$ all of whose components have dimension at least e. The right hand side is a union of irreducible closed subsets of dimension $e-1$. It follows that $g^{-1}(y)$ has pure dimension e.

(c): Let $y \in Y$. Then $\dim f^{-1}(y) \geqslant k$ if and only if $\Lambda \cap f^{-1}(y)$ is not empty for all $\Lambda \in \mathrm{Gr}(N-k, \mathbb{P}^N)$, by Theorem 3.3.1. Hence

$$\{y \in Y \mid \dim f^{-1}(y) \geqslant k\} = \bigcap_{\Lambda \in \mathrm{Gr}(N-k,\mathbb{P}^N)} \pi(\Lambda \times Y \cap X).$$

Hence the left hand side is closed by Elimination Theory ($\Lambda \times Y$ and X are closed in $\mathbb{P}^N \times Y$).

□

Example 3.5.3. The function α of Proposition 3.5.2 is not constant in general. A typical example is provided by the *blow-up of* $\mathbb{P}^n$ *at* $p_0 \in \mathbb{P}^n$, i.e. the set

$$\mathrm{Bl}_{p_0}\mathbb{P}^n := \{(p, \ell) \in \mathbb{P}^n \times \mathrm{Gr}(1, \mathbb{P}^n) : \ell \supset \{p_0, p\}\}.$$

As is easily checked $\mathrm{Bl}_{p_0}\mathbb{P}^n$ is closed in $\mathbb{P}^n \times \mathrm{Gr}(1, \mathbb{P}^n)$. Let

$$f\colon \mathrm{Bl}_{p_0}\mathbb{P}^n \to \mathbb{P}^n, \quad (p, \ell) \mapsto p$$

be projection; then

$$f^{-1}\{p\} = \begin{cases} \langle p_0, p\rangle & \text{if } p \neq p_0, \\ \{\ell \in \mathrm{Gr}(1, \mathbb{P}^n) : p_0 \in \ell\} & \text{if } p = p_0. \end{cases}$$

Thus

$$\dim_p f^{-1}\{p\} = \begin{cases} 0 & \text{if } p \neq p_0, \\ n-1 & \text{if } p = p_0. \end{cases}$$

3.6 Exercises

Exercise 3.6.1. Let $X \subset \mathbb{P}^n$ be a hypersurface, and let $I(X) = (F)$.

(a) Let $\Delta(F) \subset \mathrm{Gr}(1, \mathbb{P}^n)$ be the subset of lines $\mathbb{P}(W)$ such that there exist $p \in \mathbb{P}(W)$ for which $\mathrm{mult}_p(F) \geqslant 2$. - see Exercise 1.8.6. Prove that $\Delta(F)$ is a proper closed subset of $\mathrm{Gr}(1, \mathbb{P}^n)$. (Hint: for the proof that $\Delta(F)$ is closed, see Exercise 1.8.6, for the proof that it is a proper subset, see the proof of Lemma 3.2.8.)

(b) Prove that $\deg X = \deg F$. (Hint: recall Item (b) of Exercise 1.8.6.)

Exercise 3.6.2. Let $X \subset \mathbb{P}^n$ be a hypersurface. Prove that

$$\deg X = \max\{|\Lambda \cap X| \mid \Lambda \in \mathrm{Gr}(1, \mathbb{P}^n) \text{ such that } \Lambda \cap X \text{ is finite}\}. \tag{3.6.1}$$

(An analogous result holds for a closed pure dimensional $X \subset \mathbb{P}^n$ of any codimension, see Proposition **??**.)

Exercise 3.6.3. Let $\Delta_d \subset \mathbb{P}(\mathbb{K}[T_0, T_1]_d)$ be the subset of $[F]$ for which there exist $p \in \mathbb{P}^1$ such that $\mathrm{mult}_p(F) \geqslant 2$ - see Exercise 1.8.6.

(a) Prove that Δ_d is an irreducible hypersurface in $\mathbb{P}(\mathbb{K}[T_0, T_1]_d)$.

(b) Prove that Δ_d has degree $2d-2$.

Exercise 3.6.4. Let R be an integral domain. Let $F \in R[T_0,T_1]_m$ and $G \in R[T_0,T_1]_n$; we assume throughout that m,n are not both 0. The resultant $\mathscr{R}_{m,n}(F,G)$ is the element of R defined as follows. Consider the map of free R-modules

$$\begin{array}{ccc} R[T_0,T_1]_{n-1} \oplus R[T_0,T_1]_{m-1} & \xrightarrow{L_{m,n}(F,G)} & R[T_0,T_1]_{m+n-1} \\ (\Phi,\Psi) & \mapsto & \Phi\cdot F + \Psi\cdot G \end{array} \tag{3.6.2}$$

and let $S_{m,n}(F,G)$ be the matrix of $L_{m,n}(F,G)$ relative to the basis

$$(T_0^{n-1},0),\ (T_0^{n-2}T_1,0),\ \ldots\ ,\ (0,T_0^{m-1}),\ (0,T_0^{m-2}T_1),\ \ldots\ ,\ (0,T_1^{m-1})$$

of the domain and the basis

$$T_0^{m+n-1},\ T_0^{m+n-2}T_1,\ \ldots\ ,\ T_0T_1^{m+n-2},\ T_1^{m+n-1}$$

of the codomain. Then

$$\mathscr{R}_{m,n}(F,G) := \det S_{m,n}(F,G). \tag{3.6.3}$$

Explicitly: if

$$F = \sum_{i=0}^{m} a_i T_0^{m-i}T_1^i, \quad G = \sum_{j=0}^{n} b_j T_0^{n-j}T_1^j \tag{3.6.4}$$

then

$$\mathscr{R}_{m,n}(F,G) = \det \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & a_0 & \vdots & \vdots & \cdots & b_0 \\ a_m & a_{m-1} & \cdots & \vdots & b_n & b_{n-1} & \cdots & \vdots \\ 0 & a_m & \cdots & \vdots & 0 & b_n & \cdots & \vdots \\ 0 & 0 & \cdots & \vdots & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_m & 0 & 0 & \cdots & b_n \end{pmatrix}. \tag{3.6.5}$$

Now let k be a field and let $\mathbb{K}$ be an algebraic closure of k. Let $F \in k[T_0,T_1]_m$ and $G \in k[T_0,T_1]_n$.

(a) Prove that $\mathscr{R}_{m,n}(F,G) = 0$ if and only if there exists $H \in k[T_0,T_1]_d$ with $d > 0$ which divides F and G in $k[T_0,T_1]$.

(b) Prove that $\mathscr{R}_{m,n}(F,G) = 0$ if and only if there exists a common non-trivial root of F and G in $\mathbb{P}^1_{\mathbb{K}}$, i.e. a non zero $(T_0,T_1) \in \mathbb{K}^2$ such that $F(T_0,T_1) = G(T_0,T_1) = 0$.

(c) Suppose that $\operatorname{char}\mathbb{K}$ does not divide d. Give an explicit homogeneous polynomial of degree $(2d-2)$ in the coefficients c_i of

$$F = \sum_{i=0}^{d} c_i T_0^i \cdot T_1^{d-i}$$

which vanishes if and only if there exists $p \in \mathbb{P}^1_{\mathbb{K}}$ such that $\operatorname{mult}_p(F) \geqslant 2$. (Hint: recall Item (d) of Exercise 1.8.6.) Compare to Item (b) of Exercise 3.6.3.

Exercise 3.6.5. Let $\mathcal{C}_n \subset \mathbb{P}^n$ be the the image of the Veronese map $\nu_n\colon \mathbb{P}^1 \to \mathbb{P}^n$ given by $\nu_n([s,t]) = [s^n, s^{n-1}t, \ldots, t^n]$. Prove that $\deg \mathcal{C}_n = n$. Notice that $\mathscr{C}_n$ is irreducible and non degenerate. Thus $\mathscr{C}_n$ has the minimum degree that an irreducible nondegenerate (closed) curve can have according to Proposition 3.3.7. Prove that if $X \subset \mathbb{P}^n$ is closed, irreducible, non degenerate, and $\deg X = n$, then X is projectively equivalent to $\mathscr{C}_n$.

Exercise 3.6.6. We recall that a closed $X \subset \mathbb{P}^n$ is a cone with vertex p, if whenever $x \in (X\backslash\{p\})$, the line $\langle p, x\rangle$ is contained in X. Equivalently, there is a closed $Y \subset H$, where $H \subset \mathbb{P}^n$ is a hyperplane not containing p, such that X is the union of the lines $\langle p, y\rangle$, for $y \in Y$. Suppose that (notation as above), Y is irreducible, and non degenerate in H. Prove that $\deg X = \deg Y$. From this and the previous exercise, deduce that given any $0 \leqslant c < n$, there exists closed irreducible non degenerate $X \subset \mathbb{P}^n$ such that $\deg X = c+1$ (the minimum according to Proposition 3.3.7).

Exercise 3.6.7. Let $A, B \subset \mathbb{P}^{a+b+1}$ be disjoint linear subspaces of dimensions a and b respectively. Let $\mathscr{C}_a \subset A$ and $\mathscr{C}_b \subset B$ be closed irreducible non degenerate curves of degrees a and b respectively (see Exercise 3.6.5). Choose an isomorphism $f\colon \mathscr{C}_a \xrightarrow{\sim} \mathscr{C}_b$ (they are both isomorphic to $\mathbb{P}^1$ according toExercise 3.6.5), and let

$$X_{a,b} := \bigcup_{x\in\mathscr{C}_a} \langle x, f(x)\rangle.$$

Explicitly, up to a change of homogegenous coordinates

$$X_{a,b} := \{[\lambda s^a, \lambda s^{a-1}t, \ldots, \lambda t^a, \mu s^b, \mu s^{b-1}t, \ldots, \mu t^b] \mid [\lambda, \mu] \in \mathbb{P}^1,\ [s,t] \in \mathbb{P}^1\}.$$

(i) Prove that $X_{a,b}$ is closed, irreducible, of dimension 2, non degenerate.

(ii) Prove that $\deg X_{a,b} = a+b$. Thus $X_{a,b}$ has the minimum degree according to Proposition 3.3.7. Show that $X_{a,b}$ is *not* a cone, except in the degenerate case $a = 0$ or $b = 0$.

Let $X \subset \mathbb{P}^n$ be a closed subset. For $k \in \{0, \ldots, n\}$, we let

$$F_k(X) := \{\Lambda \in \mathrm{Gr}(k, \mathbb{P}^n) \mid \Lambda \subset X\}$$

be the set of k dimensional linear spaces contained in X. Thus $F_0(X) = X$. The first interesting case is $F_1(X)$, i.e. the set of lines contained in X. By solving the following exercises, one proves interesting results about $F_k(X)$.

Exercise 3.6.8. Let $X \subset \mathbb{P}^n$ be a closed subset. Prove that $F_k(X)$ is a closed subset of $\mathrm{Gr}(k, \mathbb{P}^n)$, arguing as follows:

1. If $X = \mathbb{P}^n$, then $F_k(\mathbb{P}^n) = \mathrm{Gr}(k, \mathbb{P}^n)$. If X is not $\mathbb{P}^n$, then $X = V(P_1) \cap \ldots \cap V(P_r)$ and $F_k(X) = F_k(V(P_1)) \cap \ldots \cap F_k(V(P_r))$. Hence it suffices to prove the result for $X = V(P) \subset \mathbb{P}^n$ a hypersurface.
2. Since we have the open covering of $\mathrm{Gr}(k, \mathbb{P}^n)$ given by (1.7.4), it suffices to show that the intersection $F_k(V(P)) \cap \mathrm{Gr}(k, \mathbb{P}^n)_I$ is closed for every multiindex $I \subset \{0, \ldots, n\}$ of cardinality $k+1$. Prove by explicit computation that $F_k(V(P)) \cap \mathrm{Gr}(k, \mathbb{P}^n)_I$ is closed.

Exercise 3.6.9. Let $L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d) \subset \mathrm{Gr}(k, \mathbb{P}^n) \times \mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ be

$$L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d) := \{(\Lambda, [P]) \mid \Lambda \subset V(P)\}.$$

Prove that $L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ is closed, arguing as follows:

1. Since we have the open covering

$$\mathrm{Gr}(k, \mathbb{P}^n) \times \mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d) = \bigcup_{|I|=k+1} \mathrm{Gr}(k, \mathbb{P}^n)_I \times \mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d),$$

in order to prove that $L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ is closed it suffices to show that the intersection of $L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ with the open subset indicized by I, call it $L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d)_I$, is closed.

2. Let $I = \{0, \ldots, k\}$. Identify $\mathrm{Gr}(k, \mathbb{P}^n)_I$ with $M_{k+1,n-k}(\mathbb{K})$ via the isomorphism in (1.7.5). Then

$$L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d)_I = \{(A, [P]) \mid C_J(A, P) = 0 \quad \forall J\}. \tag{3.6.6}$$

Since each $C_J(A, P)$ is a polynomial in the entries of A and (the coefficients) of P, homogeneous in P, we get that $L_k(\mathbb{K}[Z_0, \ldots, Z_n]_d)_I$ is closed.

Exercise 3.6.10. Prove that if $k < n$ then $L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)$ is irreducible and

$$\dim L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d) = \dim \mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d) + (k+1)\cdot(n-k) - \binom{d+k}{k}, \tag{3.6.7}$$

arguing as follows:

1. Show that
$$L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)_I \cap L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)_{I'} \neq \emptyset$$
for any two subsets $I, I' \subset \{0,\ldots,n\}$.
2. Since each $L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)_I$ is open, and any two have non empty intersection by the previous item, it will suffice to show that each $L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)_I$ is irreducible of dimension given by (3.6.7). For $P \in \mathbb{K}[Z_0,\ldots,Z_n]_d$, let
$$P = \sum_{\deg K = d} P_K Z^K,$$
where K runs through multiindices $K = (k_0,\ldots,k_n)$ of degree d. Rewrite (3.6.6) as
$$L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)_I = \{(A,[P]) \mid \sum_{\deg K = d} D_{J,K}(A) P_K = 0 \quad \forall J\},$$
where $D_{J,K}(A)$ is a polynomial in the entries of the matrix A.
3. By the previous item, $L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)_I$ is the set of couples $(A,[P])$, where P is any non trivial solution of $\binom{d+k}{k}$ homogeneous linear equations. Show the system of linear equations has maximum rank for each A by observing that the restriction map
$$\begin{array}{ccc} \mathbb{K}[Z_0,\ldots,Z_n]_d & \longrightarrow & \mathbb{K}[\lambda_0,\ldots,\lambda_k]_d \\ P & \mapsto & P(\lambda_0 w_0(A) + \ldots + \lambda_k w_k(A)) \end{array}$$
is surjective, where $w_i(A) := v_i + \sum_{j=1}^{m-h} a_{i,j} v_{h+j}$ for $i \in \{0,\ldots,k\}$, so that Λ_A (the linear subspace corresponding to A) is the span of $[w_0(A)],\ldots,[w_k(A)]$.
4. Given A, by the previous item there exists a $\binom{d+k}{k} \times \binom{d+k}{k}$ minor of the matrix $(D_{J,k}(A))_{J,K}$, call it $m(A)$ with non zero determinant. Let $M_{k+1,n-k}(\mathbb{K})_m \subset M_{k+1,n-k}(\mathbb{K})$ be the open subset of points such the minor $m(A)$ has non zero determinant. Show that the open subset
$$L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)_I \cap \{A \in M_{k+1,n-k}(\mathbb{K}) \mid \det m(A) \neq 0\} \times \mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d) \tag{3.6.8}$$
is isomorphic to $\{A \in M_{k+1,n-k}(\mathbb{K}) \mid \det m(A) \neq 0\} \times \mathbb{P}^r$, where
$$r = \mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d) - \binom{d+k}{k}.$$

   Conclude from this that $L_k(\mathbb{K}[Z_0,\ldots,Z_n]_d)$ is irreducible, of dimension given by (3.6.7).

Exercise 3.6.11. Let $k < n$. Prove that the subset of $\mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d)$ defined by

$$\{[P] \in \mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d) \mid F_k(X) \neq \emptyset\} \tag{3.6.9}$$

is closed, irreducible, of dimension at most equal to

$$\dim \mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d) + (k+1)\cdot(n-k) - \binom{d+k}{k}.$$

In particular, show that for all $d \geqslant 2n-2$ there exist hypersurfaces $V(P) \subset \mathbb{P}^n$ defined by a degree d homogeneous P which do not contain a line.

Chapter 4

Tangent space, smooth points

4.1 Introduction

One definition of tangent space of a C^∞ manifold M at a point $x \in M$ is as the real vector space of derivations of the space $\mathscr{E}_{M,x}$ of germs of C^∞ functions at x. We will give an analogous definition of the Zariski tangent space of a quasi projective variety. The advantage of such an abstract definition is that it is intrinsic by definition. On the other hand, we will identify the Zariski tangent space at a point a of a closed subset $X \subset \mathbb{A}^n$ with the classical embedded tangent space, defined by the common zeroes of the linear approximations at a of polynomials in a basis of the ideal $I(X)$.

A fundamental difference between quasi projective varieties and smooth manifolds is that the dimension of the tangent space at a point might depend on the point, even for an irreducible variety. The points where the dimension has a local minimum are the so-called smooth points of the variety. If the field $\mathbb{K}$ is $\mathbb{C}$, in a neighborhood of a smooth point the variety is naturally a complex manifold.

4.2 The local ring of a variety at a point

Let X be a quasi projective variety. We start by defining the ring of germs of regular functions at $x \in X$.

Definition 4.2.1. Let X be a quasi projective variety, and let $x \in X$. Let (U, ϕ) and (V, ψ) be couples where U, V are open subsets of X containing x, and $\phi \in \mathbb{K}[U]$, $\psi \in \mathbb{K}[V]$. Then $(U, \phi) \sim (V, \psi)$ if there exists an open subset $W \subset X$ containing x such that $W \subset U \cap V$ and $\phi_{|W} = \psi_{|W}$.

One checks easily that $\sim$ is an equivalence relation: an equivalence class for the realtion $\sim$ is a *germ of regular function of X at x*. We may define a sum and a product on the set of germs of regular functions of X at x by setting

$$[(U, \phi)] + [(V, \psi)] := [\big(U \cap V, \phi_{|U \cap V} + \psi_{|U \cap V}\big)], \tag{4.2.1}$$

and

$$[(U, \phi)] \cdot [(V, \psi)] := [\big(U \cap V, \phi_{|U \cap V} \cdot \psi_{|U \cap V}\big)]. \tag{4.2.2}$$

Of course one has to check that the equivalence class of the sum and product is independent of the choice of representatives: this is easy, we leave details to the reader. With these operations, the set of germs of regular functions of X at x is a ring.

Definition 4.2.2. Let X be a quasi projective variety, and let $x \in X$. The *local ring of X at x* is the ring of germs of regular functions of X at x, and is denoted $\mathscr{O}_{X,x}$.

We have a natural homomorphism of rings

$$\begin{array}{ccc} \mathbb{K}[X] & \xrightarrow{\rho} & \mathscr{O}_{X,x} \\ f & \mapsto & [(X, f)] \end{array} \tag{4.2.3}$$

Lemma 4.2.3. *Suppose that X is an affine variety, and let $x \in X$. If $\varphi \in \mathscr{O}_{X,x}$ then there exist $f, g \in \mathbb{K}[X]$, with $g(x) \neq 0$, such that $\varphi = \frac{\rho(f)}{\rho(g)}$.*

Proof. Let φ be represented by (U, h), where $U \subset X$ is open, and $x \in U$. Since the principal open affine subsets of X form a basis of the Zariski topology, there exists $\alpha \in \mathbb{K}[X]$ such that $X_\alpha \subset U$ and $x \in X_\alpha$ (see Remark 1.4.4). Thus $\varphi = [(X_\alpha, h_{|X_\alpha})]$. By Remark 1.4.4, there exist $f \in \mathbb{K}[X]$ and $m \in \mathbb{N}$ such that h is the restriction to X_α of $\frac{f}{\alpha^m}$. Then $\varphi = \frac{\rho(f)}{\rho(\alpha^m)}$. □

There is a well-defined surjective homomorphism

$$\begin{array}{ccc} \mathscr{O}_{X,x} & \longrightarrow & \mathbb{K} \\ [(U,\phi)] & \mapsto & \phi(a) \end{array} \tag{4.2.4}$$

The kernel

$$\mathfrak{m}_x := \{[(U,\phi)] \mid \phi(x) = 0\}$$

of (4.2.4) is a maximal ideal, because (4.2.4) is a surjection to a field.

Proposition 4.2.4. *With notationas above, $\mathfrak{m}_x$ is the unique maximal ideal of $\mathscr{O}_{X,x}$, and hence $\mathscr{O}_{X,x}$ is a local ring. Moreover, $\mathscr{O}_{X,x}$ is Noetherian.*

Proof. Let $f = [(U,\phi)] \in (\mathscr{O}_{X,x} \backslash \mathfrak{m}_x)$. Then $W := (U \backslash V(\phi))$ is an open subset of X containing x and hence $g := [(W, (\phi|_W)^{-1}]$ belongs to $\mathscr{O}_{X,x}$. Since $gf = 1$ we get that f is invertible. It follows that $\mathfrak{m}_x$ contains any proper ideal of $\mathscr{O}_{X,x}$ and hence is the unique maximal ideal of $\mathscr{O}_{X,x}$.

In order to prove that $\mathscr{O}_{X,x}$ is Noetherian, we notice that if $U \subset X$ is Zariski open and contains x, then the natural homomorphism $\mathscr{O}_{U,x} \to \mathscr{O}_{X,x}$ is an isomorphism. Since X is covered by open affine subsets, it follows that we may assume that X is affine. Let $I \subset \mathscr{O}_{X,x}$ be an ideal. Let ρ be the homomorphism in (4.2.3). Then $\rho^{-1}(I)$ is a finitely generated ideal, because $\mathbb{K}[X]$ is Noetherian. Let $f_1, \ldots, f_r$ be generators of $\rho^{-1}(I)$. Then $\rho(f_1), \ldots, \rho(f_r)$ generate I. In fact let $\varphi \in I$. By Lemma 4.2.3, there exist $f, g \in \mathbb{K}[X]$, with $g(x) \neq 0$, such that $\varphi = \frac{\rho(f)}{\rho(g)}$. We have $f = \sum_{i=1}^r a_i f_i$, and hence $\varphi = \sum_{i=1}^r \frac{\rho(a_i)}{\rho(g)} \rho(f_i)$. □

4.3 The Zariski tangent space

The homomorphism (4.2.4) equips $\mathbb{K}$ with a structure of $\mathscr{O}_{X,x}$-module. Moreover $\mathscr{O}_{X,x}$ is a $\mathbb{K}$-algebra. Thus it makes sense to speak of $\mathbb{K}$-derivations of $\mathscr{O}_{X,x}$ to $\mathbb{K}$.

Definition 4.3.1. Let X be a quasi projective variety, and let $x \in X$. The *Zariski tangent space to X at x* is $\mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X,x}, \mathbb{K})$, and will be denoted by $\Theta_x X$. Thus $\Theta_x X$ is an $\mathscr{O}_{X,x}$-module (see Section **??**), and since $\mathfrak{m}_x$ annihilates every derivation $\mathscr{O}_{X,x} \to \mathbb{K}$, it is a complex vector space.

Lemma 4.3.2. *Let $a \in \mathbb{A}^n$. The complex linear map*

$$\begin{array}{ccc} \Theta_a \mathbb{A}^n & \longrightarrow & \mathbb{K}^n \\ D & \mapsto & (D(z_1), \ldots, D(z_n)) \end{array} \tag{4.3.5}$$

is an isomorphism.

Proof. The formal partial derivative $\frac{\partial}{\partial z_m}$ defined by (A.6.1) defines an element of $\Theta_a \mathbb{A}^n$ by the familiar formula

$$\frac{\partial}{\partial z_m}\left(\frac{f}{g}\right)(a) := \frac{\frac{\partial f}{\partial z_m}(a) \cdot g(a) - f(a) \cdot \frac{\partial g}{\partial z_m}(a)}{g(a)^2}.$$

(See Example A.6.3.) Since $\frac{\partial}{\partial z_m}(z_j) = \delta_{mj}$, the map in (4.3.5) is surjective.

Let's prove that the map in (4.3.5) is injective. Assume that $D \in \Theta_{X,x}$ is mapped to 0 by the map in (4.3.5), i.e. $D(x_j) = 0$ for $j \in \{1, \ldots, n\}$. Let $f, g \in \mathbb{K}[z_1, \ldots, z_n]$, with $g(a) \neq 0$. Then

$$D\left(\frac{f}{g}\right) = \frac{D(f) \cdot g(a) - f(a) \cdot D(g)}{g(a)^2}.$$

(See Example A.6.3.) Hence it suffices to show that $D(f) = 0$ for every $f \in \mathbb{K}[z_1, \ldots, z_n]$. Consider the first-order expansion of f around a i.e. write

$$f = f(a) + \sum_{i=1}^{n} c_i(z_i - a) + R, \qquad R \in \mathfrak{m}_a^2. \tag{4.3.6}$$

Since D is zero on constants (because D is a $\mathbb{K}$-derivation) and $D(z_j) = 0$ for all j it follows that $D(f) = D(R)$, and the latter vanishes by Leibniz' rule and the hypothesis $D(z_j) = 0$ for all j. □

The differential of a regular map at a point of the domain is defined by the usual procedure. Explicitly, let $f \colon X \to Y$ be a regular map of quasi projective varieties, let $x \in X$ and $y := f(x)$. There is a well-defined pull-back homomorphism

$$\begin{array}{ccc} \mathscr{O}_{Y,y} & \xrightarrow{f^*} & \mathscr{O}_{X,x} \\ [(U, \phi)] & \mapsto & [(f^{-1}U, \phi \circ (f_{|f^{-1}U}))] \end{array} \tag{4.3.7}$$

The *differential of f at x* is the linear map of complex vector spaces

$$\begin{array}{ccc} T_x X & \xrightarrow{df(x)} & T_y Y \\ D & \mapsto & (\phi \mapsto D(f^*\phi)) \end{array} \tag{4.3.8}$$

The differential has the customary functorial properties. Explicitly, suppose that we have

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 , \quad x_1 \in X_1, \quad x_2 = f_1(x_1).$$

Since $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$ we have

$$d(f_2 \circ f_1)(x_1) = df_2(x_2) \circ df_1(x_1). \tag{4.3.9}$$

Moreover $d\,\mathrm{Id}_X(x) = \mathrm{Id}_{T_x X}$ for $x \in X$.

Remark 4.3.3. It follows from the above that if f is an isomorphism, then $df(x) \colon T_x X \to T_{f(x)} Y$ is an isomorphism, in particular $\dim T_x X = \dim T_y Y$.

The next result shows how to compute the Zariski tangent space of a closed subset of $\mathbb{A}^n$. Since every point x of a quasi projective variety X is contained in an open affine subset U, and $\Theta_x X = \Theta_x U$ (because restriction defines an identification $\mathscr{O}_{X,x} = \mathscr{O}_{U,x}$), the result will allow to compute the Zariski tangent space in general.

Proposition 4.3.4. *Let $\iota \colon X \hookrightarrow \mathbb{A}^n$ be the inclusion of a closed subset and $a \in X$. The differential $d\iota(a) \colon \Theta_a X \to \Theta_a \mathbb{A}^n$ is injective and, identifying $\Theta_a \mathbb{A}^n$ with $\mathbb{K}^n$ via* (4.3.5)*, we have*

$$\operatorname{Im} dj(a) = \left\{ v = (v_1, \ldots, v_n) \in \mathbb{K}^n \mid \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) \cdot v_i = 0 \quad \forall f \in I(X) \right\}. \tag{4.3.10}$$

Proof. The differential $d\iota(a)$ is injective because the pull-back $\iota^* \colon \mathscr{O}_{\mathbb{A}^n,a} \to \mathscr{O}_{X,a}$ is surjective. Let $D \in \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X,a}, \mathbb{K})$. If $f \in I(X) \subset \mathbb{K}[z_1, \ldots, z_n]$, then $d\iota(D)(f) = D(\iota^* f) = D(0) = 0$. Hence $\operatorname{Im} d\iota(a)$ is contained in the right-hand side of (4.3.10). Let's prove that $\operatorname{Im} d\iota(a)$ contains the right-hand side of (4.3.10). Let $\widetilde{D} \in \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{\mathbb{A}^n,a}, \mathbb{K})$ belong to the right hand side of (4.3.10), i.e. $\widetilde{D}(f) = 0$ for all $f \in I(X)$. By Item (3) of Example A.6.3 it follows that $\widetilde{D}(\frac{f}{g}) = 0$ whenever $f, g \in \mathbb{K}[z_1, \ldots, z_n]$ and $f \in I(X)$ (of course we assume that $g(a) \neq 0$). Thus $\widetilde{D}$ descends to a $\mathbb{K}$-derivation $D \in \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X,a}, \mathbb{K})$, and $\widetilde{D} = d\iota_*(a)(D)$. □

Remark 4.3.5. With the hypotheses of Proposition 4.3.5, suppose that $I(X)$ is generated by $f_1, \ldots, f_r$. Then

$$\operatorname{Im} dj(a) = \left\{ v = (v_1, \ldots, v_n) \in \mathbb{K}^n \mid \sum_{i=1}^{n} \frac{\partial f_k}{\partial z_i}(a) \cdot v_i = 0 \quad k \in \{1, \ldots, r\} \right\}.$$

In fact, the right hand side of the above equation is equal to the right hand side of (4.3.10), because if $f = \sum_{j=1}^{r} g_j f_j$, then $\frac{\partial f}{\partial z_i}(a) = \sum_{j=1}^{r} g_j(a) \frac{\partial f_j(a)}{\partial z_i}$.

Example 4.3.6. Let $f \in \mathbb{K}[z_1, \ldots, z_n]$ be a polynomial without multiple factors, i.e. such that $\sqrt{(f)} = (f)$, and let $X = V(f)$. Let $a \in X$; by Remark 4.3.5 Zariski's tangent space to X is the subspace of $\mathbb{K}^n$ defined by

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) \cdot v_i = 0.$$

Hence

$$\dim \Theta_a X = \begin{cases} n-1 & \text{if } (\frac{\partial f}{\partial z_1}(a), \ldots, \frac{\partial f}{\partial z_n}(a)) \neq 0, \\ n & \text{if } (\frac{\partial f}{\partial z_1}(a), \ldots, \frac{\partial f}{\partial z_n}(a)) = 0. \end{cases}$$

Let us show that

$$X \backslash V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) \tag{4.3.11}$$

is an open *dense* subset of X (it is obviously open, the point is that it is dense), i.e. $\dim \Theta_a X = n-1$ for a in an open dense subset of X.

First assume that f is irreducible. First we notice that there exists $i \in \{1, \ldots, n\}$ such that

$$\frac{\partial f}{z_i} \neq 0. \tag{4.3.12}$$

In fact assume the contrary. It follows that $\operatorname{char} \mathbb{K} = p > 0$, and that there exists a polynomial $g \in \mathbb{K}[z_1, \ldots, z_n]$ such that $f = g(z_1^p, \ldots, z_n^p)$. Let $g = \sum_I a_I z^I$, where I runs through a (finite) collection of multiindices. Since $\mathbb{K}$ is algebraically closed, there exists a (unique) p-th root $a_I^{1/p}$. Let $h = \sum_I a_I^{1/p} z^I$. Then $f = h(z_1, \ldots, z_n)^p$ (recall that $(a+b)^p = a^p + b^p$), and this is a contradiction because f is irreducible. This proves that there exists $i \in \{1, \ldots, n\}$ such that (4.3.12) holds.

Reordering the coordinates, we may assume that $i = n$. hence

$$f = a_0 z_n^d + a_1 z_n^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}[z_1, \ldots, z_{n-1}], \quad a_0 \neq 0, \quad d > 0.$$

Thus

$$\frac{\partial f}{z_n} = d a_0 z_n^{d-1} + (d-1) a_1 z_n^{d-2} + \cdots + a_{d-1} \neq 0.$$

The degree in z_n of f is d, i.e. f has degree d as element of $\mathbb{K}[z_1, \ldots, z_{n-1}][z_n]$. On the other hand, $\frac{\partial f}{z_n}$ is non zero and its degree in z_n is strictly smaller than d. Thus $f \nmid \frac{\partial f}{z_n}$, and hence the set in (4.3.11) is dense in X (recall that f is irreducible).

In general, let $f = f_1 \cdot \cdots \cdot f_r$ be the decomposition of f as product of prime factors. Let $X_i = V(f_i)$. Then

$$X = X_1 \cup \cdots \cup X_r$$

is the irreducible decomposition of X. As shown above, for each $i \in \{1, \ldots, r\}$

$$X_j \backslash V\left(\frac{\partial f_j}{z_1}, \ldots, \frac{\partial f_j}{z_n}\right) \neq \varnothing.$$

Hence there exists $a \in X_j$ such that $\frac{\partial f_j}{z_h}(a) \neq 0$ for a certain $1 \leqslant h \leqslant n$. We may assume in addition that a does not belong to any other irreducible component of X. It follows that

$$\frac{\partial f}{z_h}(a) = \frac{\partial f_j}{z_h}(a) \cdot \prod_{k \neq j} f_k(a) \neq 0.$$

This proves that the open set in(4.3.11) has non empty intersection with every irreducible component of X, and hence is dense in X.

Notice also that if a belongs to more than one irreducible component of X, then all partial derivatives of f vanish at a. In other words, any point in the open dense subset of points a such that $\dim \Theta_a = n-1$ belongs to a single irreducible component of X.

The result below shows that the behaviour of the tangent space examined in the above example is typical of what happens in general.

Proposition 4.3.7. *Let X be a quasi projective variety. The function*

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{N} \\ x & \mapsto & \dim \Theta_x X \end{array} \tag{4.3.13}$$

is Zariski upper-semicontinuous, i.e. for every $k \in \mathbb{N}$

$$X_k := \{x \in X \mid \dim \Theta_x X \geqslant k\}$$

is closed in X.

Proof. Since X has an open affine covering, we may suppose that $X \subset \mathbb{A}^n$ is closed. Let $I(X) = (f_1, \ldots, f_r)$. For $x \in \mathbb{A}^n$ let

$$J(f_1, \ldots, f_s)(x) := \begin{pmatrix} \frac{\partial f_1}{z_1}(x) & \cdots & \frac{\partial f_1}{z_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{z_1}(x) & \cdots & \frac{\partial f_r}{z_n}(x) \end{pmatrix}$$

be the Jacobian matrix of $(f_1, \ldots, f_s)$ at x. By Proposition 4.3.5 we have that

$$X_k = \{x \in X \mid \operatorname{rk} J(f_1, \ldots, f_r)(x) \leqslant n-k\}. \tag{4.3.14}$$

Given multi-indices $I = \{1 \leqslant i_1 < \ldots < i_m \leqslant s\}$ and $J = \{1 \leqslant j_1 < \ldots < j_m \leqslant n\}$ let $J(f_1, \ldots, f_s)(x)_{I,J}$ be the $m \times m$ minor of $J(f_1, \ldots, f_r)(x)$ with rows corresponding to I and columns corresponding to J (if $m > \min\{r, n\}$ we set $J(f_1, \ldots, f_s)(x)_{I,J} = 0$). We may rewrite (4.3.14) as

$$X_k = X \cap V\left(\ldots, \det J(f_1, \ldots, f_r)(x)_{I,J}, \ldots\right)_{|I|=|J|=n-k+1}.$$

It follows that X_k is closed. □

4.4 Cotangent space

Let X be a quasi projective variety, and let $x \in X$. The *cotangent space to X at x* is the dual complex vector space of the tangent space $\Theta_x X$, and is denoted $\Omega_X(x)$:

$$\Omega_X(x) := (\Theta_x X)^\vee. \tag{4.4.1}$$

We define a map

$$\mathscr{O}_{X,x} \xrightarrow{d} \Omega_X(x) \tag{4.4.2}$$

as follows. Let $f \in \mathscr{O}_{X,x}$ be represented by (U, ϕ). The codomain of the differential $d\phi(x) \colon \Theta_x U \to \Theta_{\phi(x)}\mathbb{K}$ is identified with with $\mathbb{K}$, because of the isomorphism in (4.3.5), and hence $d\phi(x) \in (\Theta_x U)^\vee$.

Since $U \subset Z$ is an open subset containing x, the differential at x of the inclusion map defines an identification $\Theta_x U \xrightarrow{\sim} \Theta_x X$. Thus $d\phi(x) \in (\Theta_x X)^\vee = \Omega_X(x)$. One checks immediately that if (V, ψ) is another representative of f then $d\psi(x) = d\phi(x)$. We let

$$df(x) := d\phi(x), \qquad (U, \phi) \text{ any representative of } f.$$

Remark 4.4.1. We equip $\Omega_X(x)$ with a structure of $\mathscr{O}_{X,x}$-module by composing the evaluation map $\mathscr{O}_{X,x} \to \mathbb{K}$ given by (4.2.4) and scalar multiplication of the complex vector-space $\Omega_Z(a)$. With this structure (4.4.2) is a derivation over $\mathbb{K}$.

Remark 4.4.2. Let $f \in \mathbb{K}[z_1, \ldots, z_n]$ and $a \in \mathbb{A}^n$. Then the familiar formula

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) dz_i(a)$$

holds. In fact this follows from the first-order Taylor expansion of f at a:

$$f = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a)(z_i - a_i) + \sum_{1 \leqslant i,j \leqslant n} m_{ij}(z_i - a_i)(z_j - a_j), \qquad m_{ij} \in \mathbb{K}[z_1, \ldots, z_n]. \tag{4.4.3}$$

Remark 4.4.3. Let $X \subset \mathbb{A}^n$ be closed, and let $a \in X$. Identify $\Theta_a \mathbb{A}^n$ with $\mathbb{K}^n$ via Lemma 4.3.2. By Remark 4.4.2 we have the identification

$$T_a X = \operatorname{Ann}\{df(a) \mid f \in I(X)\}.$$

Let X be a quasi projective variety, and let $x \in X$. Let $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the maximal ideal. By Leibiniz' rule $d\phi(x) = 0$ if $\phi \in \mathfrak{m}_x^2$ (recall that $d\colon \mathscr{O}_{X,x} \to \Omega_X(x)$ is a derivation over $\mathbb{K}$). Thus we have an induced $\mathbb{K}$-linear map

$$\begin{array}{ccc} \mathfrak{m}_x/\mathfrak{m}_x^2 & \xrightarrow{\delta(x)} & \Omega_X(x) \\ [\phi] & \mapsto & d\phi(a) \end{array} \tag{4.4.4}$$

Proposition 4.4.4. *Keep notation as above. Then $\delta(x)$ is an isomorphism of complex vector spaces.*

Proof. First we prove that $\delta(x)$ is surjective. If $X = \mathbb{A}^n$, surjectivity follows at once from Lemma 4.3.2. In general, we may assume that X is a closed subset of $\mathbb{A}^n$, and surjectivity follows from Proposition 4.3.5.

In order to prove injectivity of $\delta(x)$, we must show that if $\phi \in \mathfrak{m}_x$ is such that $d\phi(x)(D) = 0$ for all $D \in \Theta_x X$, then $\phi \in \mathfrak{m}_x^2$. We may suppose that X is a closed subset of $\mathbb{A}^n$. In order to avoid confusion, we let $x = a = (a_1, \ldots, a_n)$. Let $(U, f/g)$ be a representative of ϕ, where $f, g \in \mathbb{K}[X]$, and $f(a) = 0$, $g(a) \neq 0$. It will suffice to prove that $f \in \mathfrak{m}_a^2$. Since $0 = d\phi(a) = g(a)^{-1} df(a)$ we have $df(a) = 0$. By Theorem 1.4.2 there exists $\tilde{f} \in \mathbb{K}[z_1, \ldots, z_n]$ such that $\tilde{f}_{|X} = f$. By Proposition 4.3.5 we may identify $\Theta_a X$ with the subspace of $T_a \mathbb{K}^n = \mathbb{K}^n$ given by (4.3.10). By hypothesis $d\tilde{f}(a)(D) = 0$ for all $D \in \Theta_a X$, i.e.

$$d\tilde{f}(a) \in \operatorname{Ann}(\Theta_a X) \subset \Omega_{\mathbb{A}^n}(x).$$

By (4.3.10) there exists $h \in I(X)$ such that $d\tilde{f}(a) = dh(a)$. Then $(\tilde{f} - h)_{|X} = f$ and $d(\tilde{f} - h)(a) = 0$. Thus $(\tilde{f} - h) \in \mathbb{K}[z_1, \ldots, z_n]$ has vanishing value and differential at a. It follows (first-order Taylor expansion of $\tilde{f} - h$ at a) that

$$(\tilde{f} - h) \in (z_1 - a_1, \ldots, z_n - a_n)^2.$$

Since $h \in I(X)$ we get that $f \in \mathfrak{m}_a^2$. $\square$

The following result is an immediate consequence of Corollary A.7.2.

Corollary 4.4.5. *Let X be a quasi-projective variety and $p \in X$. Let $f_1, \ldots, f_n \in \mathscr{O}_{X,p}$ be germs vanishing at p i.e. belonging to the maximal ideal $\mathfrak{m}_p \subset \mathscr{O}_{X,p}$, and suppose that $\delta(f_1), \ldots, \delta(f_n)$ generate $\Omega_X(p)$. Then $f_1, \ldots, f_n$ generate the maximal ideal $\mathfrak{m}_p \subset \mathscr{O}_{Z,p}$.*

4.5 Smooth points

Definition 4.5.1. Let X be a quasi projective variety, and let $x \in X$. Then X is *smooth at x* if $\dim \Theta_x X = \dim_x X$, it is *singular* at x otherwise. The set of smooth points of X is denoted by X^{sm}. The set of singular points of X is denoted by $\operatorname{sing} X$.

Example 4.5.2. Let $X \subset \mathbb{A}^n$ be a hypersurface. By Corollary 2.5.4, the dimension of X is equal to $n-1$, and hence the set of smooth points of X is an open dense subset of X by Example 4.3.6. By the last sentence in Example 4.3.6, X is locally irreducible at any of its smooth points.

The main result of the present section extends the picture for hypersurfaces to the general case.

Theorem 4.5.3. *Let X be a quasi projective variety. Then the following hold:*

1. *The set X^{sm} of smooth points of X is an open dense subset of X.*
2. *For $x \in X$ we have $\dim \Theta_x X \geqslant \dim_x X$.*
3. *X is locally irreducible at any of its smooth points, i.e. if X is smooth at a, there is a single irreducible component of X containing a.*

We will prove Theorem 4.5.3 at the end of the section. First we go through some preliminary results.

Our first result proves a weaker version of Item (1) of Theorem 4.5.3, and proves Item (2) of the same theorem.

Proposition 4.5.4. *Let X be a quasi projective variety. Then the following hold:*

1. *The set of smooth points of X contains an open dense subset of X.*
2. *For $x \in X$ we have $\dim \Theta_x X \geqslant \dim_x X$.*

Proof. Suppose that X is irreducible of dimension d. By Proposition 2.4.7 there is a birational map $g\colon X \dashrightarrow Y$, where $Y \subset \mathbb{A}^{d+1}$ is a hypersurface. By Proposition 2.2.6 there exist open dense subsets $U \subset X$ and $V \subset Y$ such that g is regular on U, and it defines an isomorphism $f\colon U \xrightarrow{\sim} V$. By Example 4.5.2, the set of smooth points Y^{sm} of Y is open and dense in Y. Since V is open and dense in Y the intersection $Y^{\mathrm{sm}} \cap V$ is open and dense dense in Y and hence $f^{-1}(Y^{\mathrm{sm}} \cap V)$ is an open dense subset of X. Since $f^{-1}(Y^{\mathrm{sm}} \cap V)$ is contained in U^{sm}, we have proved that the set of smooth points of X contains an open dense subset of X. We have proved that Item (1) holds if X is irreducible. In general, let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. Let

$$X_j^0 := (X \backslash \bigcup_{i \neq j} X_i) = (X_j \backslash \bigcup_{i \neq j} X_i)$$

By the result that was just proved, $(X_j^0)^{\mathrm{sm}}$ contains an open dense subset of smooth points. Every smooth point of X_j^0 is a smooth point of X, because X_j^0 is open in X. Thus $\bigcup_i (X_i^0)^{\mathrm{sm}}$ is an open dense subset of X, containing an open dense subset of X. This proves Item (1).

Let us prove Item (2). Let $x_0 \in X$, and let X_0 be an irreducible component of X containing x_0 such that $\dim X_0 = \dim_{x_0} X$. By Item (1) X_0^{sm} contains an open dense subset of points x such that $\dim \Theta_x X_0 = \dim_x X_0$, and hence by Proposition 4.3.7 we have $\dim \Theta_x X_0 \geqslant \dim_x X_0$ for *all* $x \in X$. In particular $\dim \Theta_{x_0} X_0 \geqslant \dim_{x_0} X_0 = \dim_{x_0} X$. Since $\Theta_{x_0} X_0 \subset \Theta_{x_0} X$, it follows that $\dim \Theta_{x_0} X \geqslant \dim_{x_0} X$. □

The next result involves more machinery. We will give an algebraic version of the (analytic) Implicit Function Theorem. The algebraic replacement for the ring of analytic functions defined in a neighborhood of $0 \in \mathbb{A}^n$ is the ring $\mathbb{K}[[z_1, \ldots, z_n]]$ of formal power series in $z_1, \ldots, z_n$ with complex coefficients. We have inclusions

$$\mathbb{K}[z_1, \ldots, z_n] \subset \mathscr{O}_{\mathbb{A}^n,0} \subset \mathbb{K}[[z_1, \ldots, z_n]]. \tag{4.5.1}$$

(The second inclusion is obtained by developing $\frac{f}{g}$ as convergent power series centered at 0, where $f, g \in \mathbb{K}[z_1, \ldots, z_n]$ and $g(0) \neq 0$.) We will need the following elementary results.

Lemma 4.5.5. *Let $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$, $\mathfrak{m}' \subset \mathscr{O}_{\mathbb{A}^n,0}$ and $\mathfrak{m}'' \subset \mathbb{K}[[z_1, \ldots, z_n]]$ be the ideals generated by $z_1, \ldots, z_n$ in the corresponding ring. Then for every $i \geqslant 0$ we have $(\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^i$, and $(\mathfrak{m}')^i \cap \mathbb{K}[z_1, \ldots, z_n] = \mathfrak{m}^i$.*

Proof. By induction on i. For $i = 0$ the statement is trivially true. The proof of the inductive step is the same in both cases. For definiteness let us show that $(\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^{i+1}$, assuming that $(\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^i$. The non trivial inclusion is $(\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0} \subset (\mathfrak{m}')^{i+1}$. Assume that $f \in (\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0}$. Then $f \in (\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0}$, and hence $f \in (\mathfrak{m}')^i$ by the inductive hypothesis. Thus we may write

$$f = \sum_{|I|} \alpha_J z^J,$$

where the sum is over all multiindices $J = (j_1, \ldots, j_n)$ of weight $|J| = \sum_{s=1}^n j_s = i$, and $\alpha_J \in \mathscr{O}_{\mathbb{A}^n,0}$ for all J. Since $f \in (\mathfrak{m}'')^{i+1}$, we have $\alpha_J(0) = 0$ for all J. It follows that $\alpha_J \in \mathfrak{m}'$ for all J, and hence $f \in (\mathfrak{m}')^{i+1}$. □

Proposition 4.5.6 (Formal Implicit Function Theorem)**.** *Let $\varphi \in \mathbb{K}[[z_1, \ldots, z_n]]$, and suppose that*

$$\varphi = z_1 + \varphi_2 + \ldots + \varphi_d + \ldots, \quad \varphi_d \in \mathbb{K}[z_1, \ldots, z_n]_d. \tag{4.5.2}$$

Given $\alpha \in \mathbb{K}[[z_1, \ldots, z_n]]$, there exists a unique $\beta \in \mathbb{K}[[z_1, \ldots, z_n]]$ such that

$$(\alpha - \beta \cdot \varphi) \in \mathbb{K}[[z_2, \ldots, z_n]]. \tag{4.5.3}$$

Proof. Write $\beta = \beta_0 + \beta_1 + \ldots + \beta_d + \ldots$, where $\beta_d \in \mathbb{K}[z_1, \ldots, z_n]_d$, and the β_d's are the indeterminates. Expand the product $\beta \cdot \varphi$, and solve for β_0 by requiring that $\beta \cdot \varphi$ have the same linear term modulo $z_2, \ldots, z_n$ as α, then solve for β_1 by requiring that $\beta \cdot \varphi$ have the same quadratic term modulo $(z_2, \ldots, z_n)^2$ as α , etc. By (4.5.2) there is one and only one solution at each stage. □

Corollary 4.5.7. *With hypotheses as in Proposition 4.5.7, the natural map $\mathbb{K}[[z_2, \ldots, z_n]] \to \mathbb{K}[[z_1, \ldots, z_n]]/(\varphi)$ is an isomorphism.*

Proposition 4.5.8. *Let $f_1, \ldots, f_k \in \mathbb{K}[z_1, \ldots, z_n]$ and $a \in \mathbb{A}^n$. Suppose that*

(i) each f_i vanishes at a, and

(ii) the differentials $df_1(a), \ldots, df_k(a)$ are linearly independent.

Then $V(f_1, \ldots, f_k) = X \cup Y$, where

1. *X, Y are closed in $\mathbb{A}^n$, $a \in X$, while Y does not contain a;*
2. *X is irreducible of dimension $n-k$, it is smooth at a, and $T_a(X) = \operatorname{Ann}(\langle df_1(a), \ldots, df_k(a)\rangle)$ (as subspace of $T_a\mathbb{A}^n$).*

Moreover, there exists a principal open affine set $\mathbb{A}^n_g$ containing a such that $f_{1|\mathbb{A}^n_g}, \ldots, f_{k|\mathbb{A}^n_g}$ generate the ideal of $X \cap \mathbb{A}^n_g$.

Proof. By changing affine coordinates, if necessary, we may assume that $a = 0$, and that $df_i(0) = z_i$ for $i \in \{1, \ldots, k\}$. Let $J' \subset \mathscr{O}_{\mathbb{A}^n,0}$ be the ideal generated by $f_1, \ldots, f_k$ (to be consistent with our notation, we should write $J' = (\varphi(f_1), \ldots, \varphi(f_k)))$, let $J := J' \cap \mathbb{K}[z_1, \ldots, z_n]$, and let $J'' \subset \mathbb{K}[[z_1, \ldots, z_n]]$ be the ideal generated by $f_1, \ldots, f_k$. Lastly, let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be the ideal generated by $f_1, \ldots, f_k$. We claim that

$$J \cdot g \subset I \subset J. \tag{4.5.4}$$

for a suitable $g \in \mathbb{K}[z_1, \ldots, z_n]$ with $g(0) \neq 0$. In fact, the second inclusion is trivially true. In order to prove the first inclusion, let $h_1, \ldots, h_r$ be generators of the ideal $J \subset \mathbb{K}[z_1, \ldots, z_n]$. By definition of J, there exist $a_i, g_i \in \mathbb{K}[z_1, \ldots, z_n]$, for $i \in \{1, \ldots, r\}$, such that $a_i \in I$, $g_i(0) \neq 0$, and $h_i = \frac{a_i}{g_i}$.

Hence the second inclusion in (4.5.4) holds with $g = g_1 \cdot \ldots \cdot g_r$. This proves (4.5.4), and hence we have $V(J) \subset V(I) \subset (V(J) \cup V(g))$. It follows that, letting $X := V(J)$, there exists a closed $Y \subset V(g)$ such that

$$V(f_1, \ldots, f_k) = X \cup Y, \quad 0 \notin Y. \tag{4.5.5}$$

Let us prove that J is a prime ideal, so that in particular X is irreducible. First, we claim that

$$J'' \cap \mathscr{O}_{\mathbb{A}^n,0} = J'. \tag{4.5.6}$$

The non trivial inclusion to be proved is $J'' \cap \mathscr{O}_{\mathbb{A}^n,0} \subset J'$. Let $f \in J'' \cap \mathscr{O}_{\mathbb{A}^n,0}$. Then there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{K}[[z_1, \ldots, z_n]]$ such that $f = \sum_{j=1}^k \alpha_j f_j$. Given $s \in \mathbb{N}$, let α_j^s be the MacLaurin polynomial of α_j of degree s, i.e. such that $(\alpha_j - \alpha_j^s) \in (\mathfrak{m}'')^{s+1}$, where $\mathfrak{m}''$ is as in Lemma 4.5.5. Then

$$f = \sum_{j=1}^k \alpha_j^{(s)} f_j + \sum_{j=1}^k (\alpha_j - \alpha_j^s) f_j.$$

Both addends are in $\mathscr{O}_{\mathbb{A}^n,0}$. In addition, the first addend belongs to J', and the second one belongs to $(\mathfrak{m}'')^{s+1}$. By Lemma 4.5.5, it follows that the second one belongs to $(\mathfrak{m}')^{s+1}$. Hence $f \in \bigcap_{s=0}^\infty (I' + (\mathfrak{m}')^{s+1})$. By Corollary A.8.2, it follows that $f \in I'$. This proves (4.5.6). By (4.5.6) and the definition of J, we have an inclusion

$$\mathbb{K}[z_1, \ldots, z_n]/J \subset \mathbb{K}[[z_1, \ldots, z_n]]/J''.$$

Hence, in order to prove that J is prime, it suffices to show that $\mathbb{K}[[z_1, \ldots, z_n]]/J''$ is an integral domain. In fact we will see that the natural map

$$\mathbb{K}[z_{k+1}, \ldots, z_n] \longrightarrow \mathbb{K}[[z_1, \ldots, z_n]]/J'' \tag{4.5.7}$$

is an isomorphism of rings. This follows from the algebraic version of the Implicit Function Theorem, i.e. Proposition 4.5.7. In fact, by Proposition 4.5.7, the natural map $\mathbb{K}[[z_2, \ldots, z_n]] \to \mathbb{K}[[z_1, \ldots, z_n]]/(f_1)$ is an isomorphism. Let $i \in \{2, \ldots, k\}$. Given the identification $\mathbb{K}[[z_1, \ldots, z_n]]/(f_1) = \mathbb{K}[[z_2, \ldots, z_n]]$, the image of f_i under the quotient map $\mathbb{K}[[z_1, \ldots, z_n]] \to \mathbb{K}[[z_1, \ldots, z_n]]/(f_1)$ is an element $z_i + f_i'$, where $f_i' \in (\mathfrak{m}'')^2$ (notation as in Lemma 4.5.5). Iterating, we get that the map in (4.5.7) is an isomorphism of rings. As explained above, this proves that J is a prime ideal. In particular X is irreducible. Moreover, since $z_{k+1}, \ldots, z_n \in \mathbb{K}[X]$, the isomorphism in (4.5.7) shows that $\mathbb{K}(X)$ has transcendence degree $n-k$, i.e. X has dimension $n-k$. Since $f_1, \ldots, f_k$ vanish on X, and their differentials are linearly independent, it follows that $\dim \Theta_0(X) \leqslant (n-k) = \dim_0 X$. Hence $\dim \Theta_0(X) = (n-k) = \dim_0 X$, by Item (2) of Proposition 4.5.4, i.e. X is smooth at 0, and $\Theta_0(X) \subset \Theta_0 \mathbb{A}^n$ is the annihilator of $df_1(0), \ldots, df_k(0)$. This proves Items (1) and (2). The last statement in the proposition holds with the polynomial g appearing in (4.5.4). □

Corollary 4.5.9. *Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Let a be a smooth point of X, and let $k = n - \dim_a X$. Then following hold:*

1. *there exist $f_1, \ldots, f_k \in \mathbb{K}[z_1, \ldots, z_n]$ with linerly independent differentials $df_1(a), \ldots, df_k(a)$, and a Zariski open affine subset $U \subset \mathbb{A}^n$ containing a, such that $I(X \cap U) = (f_{1|U}, \ldots, f_{k|U})$;*

2. *there is a unique irreducible component of X containing a.*

Proof. Since X is smooth at a, and $\dim_a X = n-k$, there exist $f_1, \ldots, f_k \in I(X)$ such that $df_1(a), \ldots, df_k(a)$ are linearly independent. Of course $X \subset V(f_1, \ldots, f_k)$. By Proposition 4.5.9 there is a unique irreducible component of $V(f_1, \ldots, f_k)$ containing a, call it Y, and $\dim Y = n-k$. Every irreducible component of X containing a is contained in Y. Since $\dim_a X = n-k$, there exists (at least) one irreducible component of X containing a of dimension $n-k$. Let X' be such an irreducible component; by Proposition 2.5.3, $X' = Y$. It follows that there is a single component of X containing a, and it is equal to the unique irreducible component of $V(f_1, \ldots, f_k)$ containing a. Hence the corollary follows from Proposition 4.5.9. □

Proof of Theorem 4.5.3. Item (2) has been proved in Proposition 4.5.4. Item (3) follows at once from Corollary 4.5.9, because X is covered by open affine subset.

In order to prove Item (1), let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. Since X is covered by open affine subset, Corollary 4.5.9 gives that

$$X^{\mathrm{sm}} \subset X \backslash \bigcup_{\substack{i,j \in I \\ i \neq j}} (X_i \cap X_j). \tag{4.5.8}$$

The right hand side of (4.5.8) is an open dense subset of X. Let X_i^0 be an irreducible component of the right hand side of (4.5.8). Thus $X_i^0 \subset X_i$ is the complement of the intersection of X_i with the other irreducible componets of X. The set of smooth points of X_i^0 is non empty by Proposition 4.5.4, and it is open by upper semicontinuity of the dimension of $\Theta_x X$ (Proposition 4.3.7), because $\dim_x X$ is independent of $x \in X_i^0$. Hence X^{sm} is an open dense subset of the open dense subset of X given by the right hand side of (4.5.8), and hence is open and dense in X. □

4.6 Rational maps on smooth curves

A *curve* is a quasi-projective variety of pure dimension 1. Below is the main result of the present section.

Proposition 4.6.1. *Let X be a smooth curve, and Y be a projective variety. A rational map $f \colon X \dashrightarrow Y$ is regular.*

We start with a preliminary result.

Lemma 4.6.2. *Let X be a smooth curve, and $p \in X$. Let $t \in \mathscr{O}_{X,p}$ be a germ vanishing at p, with non zero differential at p (*a local parameter at p*). If $f \in \mathscr{O}_{X,p}$ is non zero, there exist a unit $u \in \mathscr{O}_{X,p}$ and an exponent $e \in \mathbb{N}$ such that $f = u \cdot t^e$.*

Proof. Since X is a smooth curve, the cotangent space $\Omega_p(X)$ has dimension 1. By Corollary 4.4.5, the germ t generates the maximal ideal $\mathfrak{m}_p$, i.e. $\mathfrak{m}_p = (t)$. Thus $\mathfrak{m}_p^i = (t^i)$ for every $i \in \mathbb{N}$. By Krull's Theorem A.8.2, there exists $e \in \mathbb{N}$ such that $f \in \mathfrak{m}_p^e$ and $f \notin \mathfrak{m}_p^{e+1}$. Then $f = u \cdot t^e$, where $u(p) \neq 0$, and hence u is a unit. □

Proof of Proposition 4.6.1. Since X is smooth, it is locally irreducible by Theorem 4.5.3. Hence we may assume that X is irreducible. Since every quasi-projective variety is a union of open affine varieties we may assume in addition that $X \subset \mathbb{A}^m$ is closed. By hypothesis $Y \subset \mathbb{P}^n$ is closed. Let $g \colon X \dashrightarrow \mathbb{P}^n$ be the composition of f and the inclusion map $Y \hookrightarrow \mathbb{P}^n$. The key point is to show that g is regular.

There exists an open dense $U \subset X$ such that g is regular on U, and there exist $\phi_0, \ldots, \phi_n \in \mathbb{C}[U]$ such that

$$g(x) = [\varphi_0(x), \ldots, \varphi_n(x)] \qquad \forall x \in U. \tag{4.6.1}$$

For $i \in \{0, \ldots, n\}$ locally we have

$$\varphi_i = \frac{\alpha_i}{\beta_i}, \quad \alpha_i, \beta_i \in \mathbb{C}[X] \tag{4.6.2}$$

and $\beta_i(x) \neq 0$ for all $x \in U$. By shrinking U if necessary, we may assume that (4.6.2) holds on all of U (recall that X is irreducible).

The complement $X \backslash U$ is a finite set. In order to prove that g is regular, we must show that for each $p \in (X \backslash U)$ there exist an open $\mathscr{U} \subset (U \cup \{p\})$ containing p and a regular $G \colon \mathscr{U} \to \mathbb{P}^n$ such that $G_{|(\mathscr{U} \backslash \{p\})} = g_{|(\mathscr{U} \backslash \{p\})}$.

Let $i \in \{0, \ldots, n\}$ be such that $\varphi_i \neq 0$, i.e. $\alpha_i \neq 0$. Applying Lemma 4.6.2 to α_i and β_i, we get that there exist an open $\mathscr{U}_i \subset (U \cup \{p\})$ containing p such that on $\mathscr{U}_i \backslash \{p\}$ we have $\varphi_i = u_i \cdot t^{e_i}$, where u_i is everwhere non zero and $e_i \in \mathbb{Z}$. Let $\mathscr{U}$ be the intersection of the $\mathscr{U}_i$'s. On $\mathscr{U} \backslash \{p\}$ we have

$$g(x) = [\ldots, u_i \cdot t^{e_i}, \ldots] \qquad \forall x \in (\mathscr{U} \backslash \{p\}). \tag{4.6.3}$$

(Some of the φ_i's might be zero.) Let $e := \min\{e_i \mid \varphi_i \neq 0\}$. The map

$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{G} & \mathbb{P}^n \\ x & \mapsto & [,\ldots,t^{-e}\varphi_i(p),\ldots] \end{array}$$

is regular on $\mathscr{U}$, and its restriction to $\mathscr{U}\backslash\{p\}$ is equal to the restriction of g. This proves that g is regular. We also see that for each $p \in (X\backslash U)$ the image $g(p)$ is in the closure of Y. Since Y is closed, the map g restricts to a regular map $g\colon X \to Y$ which is equal to original map f on U. □

Corollary 4.6.3. *Let X, Y be smooth projective curves. A birational map $f\colon X \dashrightarrow Y$ is an isomorphism.*

4.7 Birational models of curves

Desingularization

Definition 4.7.1. A regular map $f\colon X \to Y$ of quasi-projective varieties is

(a) a *closed immersion* if $f(X)$ is closed in Y and f defines an isomorphism between X and $f(X)$.

(b) *projective* if there exists a decomposition $f = \pi \circ j$, where $j\colon X \hookrightarrow \mathbb{P}^n \times Y$ is a closed immersion, and $\pi\colon \mathbb{P}^n \times Y \to Y$ is the projection.

Remark 4.7.2. (1) If X is *projective* then a regular map $f\colon X \to Y$ is projective. In fact assume that $X \subset \mathbb{P}^n$ is closed. Let

$$\begin{array}{ccc} X & \xrightarrow{j} & \mathbb{P}^n \times Y \\ x & \mapsto & (x, f(x)) \end{array}$$

and let $\pi\colon \mathbb{P}^n \times Y \to Y$ be the projection. Then $f = \pi \circ j$, and f is projective because the graph $\Gamma_f \subset X \times Y$ is closed by Lemma 1.6.2, and $X \times Y$ is closed in $\mathbb{P}^n \times Y$.

(2) If $f\colon X \to Y$ is projective and Y is projective, then X is projective. In fact by hypothesis there exists a closed immersion $j\colon X \hookrightarrow \mathbb{P}^n \times Y$. Since Y is projective so is $\mathbb{P}^n \times Y$ and hence X is isomorphic to the projective set $j(X) \subset \mathbb{P}^n \times Y$.

(3) Let $f\colon X \to Y$ be projective and let $W \subset Y$ be locally closed (and hence a quasi-projective variety). The restriction of f to $f^{-1}(W)$ defines a projective map $f^{-1}(W) \to Y$.

Definition 4.7.3. Let X be a quasi-projective variety. A regular map $f\colon \widetilde{X} \to X$ is a *desingularization of X* if the following hold:

1. $\widetilde{X}$ is smooth and $f^{-1}(X^{\mathrm{sm}})$ is dense in $\widetilde{X}$.
2. The restriction of f to $f^{-1}(X^{\mathrm{sm}})$ defines an isomorphism $f^{-1}(X^{\mathrm{sm}}) \xrightarrow{\sim} X^{\mathrm{sm}}$.
3. The map f is projective.

Example 4.7.4. If X is smooth, the identity $\mathrm{Id}_X\colon X \to X$ is a desingularization. Of course a desingularization of X is interesting only if X is not smooth.

A slightly less trivial example is provided by a quasi-projective variety X whose irreducible components, say $X_1, \ldots, X_r$, are smooth. Let $X_1 \sqcup \ldots \sqcup X_r$ be the *disjoint union* of the X_i's (make sense of this), and let $f\colon (X_1 \sqcup \ldots \sqcup X_r) \to X$ be the tautological map. Then f is a desingularization of X.

Suppose that $\mathrm{char}\,\mathbb{K} \notin \{2,3\}$, and let $X \subset \mathbb{P}^2$ be the curve $X := V(Z_0Z_1^2 - Z_0Z_2^2 + Z_1^3)$. A straightforward computation gives that X is irreducible and $\mathrm{sing}\,X = [1,0,0]$ (either you compute the intersection of $\mathrm{sing}\,X$ with each standard affine space $\mathbb{P}^2_{Z_i}$, or you apply Exercise 5.5.2). The map

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{g} & X \\ [s,t] & \mapsto & [s^3, s(t^2 - s^2), t(t^2 - s^2)] \end{array} \tag{4.7.1}$$

is a desingularization of X. Item (1) of Definition 4.7.3 holds because the domain is smooth and irreducible. In order to check that Item (2) of Definition 4.7.3 holds we and that $f^{-1}([1,0,0]) = \{[1,1],[1,-1]\}$. The map

$$\begin{array}{ccc} X\backslash\{[1,0,0]\} & \xrightarrow{h} & \mathbb{P}^1 \\ [Z_0,Z_1,Z_2] & \mapsto & [Z_1,Z_2]. \end{array} \tag{4.7.2}$$

has image contained in $(\mathbb{P}^1\backslash\{[1,1],[1,-1]\}$ and one checks at once that it is the inverse of the restriction of f to $\mathbb{P}^1\backslash f^{-1}([1,0,0])$: thus Item (2) of Definition 4.7.3 holds. Lastly, the map f is projective because the domain is projective.

Remark 4.7.5. Let $f\colon \widetilde{X} \to X$ be a desingularization. Since f is projective the image $f(X)$ is closed in X by the Main Theorem of Elimination Theory 1.6.1, and hence is equal to X because it contains the open dense subset of smooth points. This explains in part why we require that the desingularization map is projective (Items (1) and (2) of Definition 4.7.3 hold for the inclusion map $X^{\mathrm{sm}} \hookrightarrow X$).

Remark 4.7.6. A desingularization is, in general, not unique.

Desingularization of plane curves

The main result of the present subsection is the proof that there exists a resolution of singularities for plane projective curves.

Proposition 4.7.7. *Let $X \subset \mathbb{P}^2$ be a hypersurface. There exists a desingularization $f\colon \widetilde{X} \to X$.*

The formal proof of Proposition 4.7.7 will be given at the end of the present subsection. Let us start by outlining the algorithm that gives a desingularization of X:

Step 1 If X is smooth let $\widetilde{X} = X$ and $f := \mathrm{Id}_X$, otherwise go to Step 2.

Step 2 The singular set of X is finite because $\dim X = 1$. Let $\operatorname{sing} X = \{p_1,\ldots,p_r\}$ and $X_1 := \mathrm{Bl}_{p_1,\ldots,p_r} X$. If X_1 is smooth let $\widetilde{X} = X_1$ and $f := \pi_X$, otherwise iterate.

What must be proved is that the algorithm terminates i.e. that we eventually reach a blow-up X_n which is smooth. In order to accomplish this we will need a measure of how singular a curve X is at a point p. One such measure is the multiplicity of X at p.

Definition 4.7.8. Let X be a quasi-projective variety and $p \in X^{\mathrm{sm}}$ be a *smooth* point of X. Let $f \in \mathscr{O}_{X,p}$. The *multiplicity of (vanishing of) f at p* is equal to the sup of the set of $l \in \mathbb{N}$ such that $f \in \mathfrak{m}_p^l$ - we denote it by $\mathrm{mult}_p f$. Let $Y \subset X$ be a proper closed subset and suppose that there exists an affine open set $U \subset X$ containing p such that $I(Y \cap U) \subset \mathbb{C}[U]$ is a principal ideal generated by f: the *multiplicity of vanishing of Y at p* is equal to $\mathrm{mult}_p f$ - we will denote it $\mathrm{mult}_p Y$, thus dropping X from the notation. (One has to check that this definition is independent of the open affine U, we leave details to the reader[1].)

Example 4.7.9. Let $0 \neq f \in \mathbb{K}[z_1,\ldots,z_n]$. Then $\mathrm{mult}_0 f = m$ if and only if

$$f = f_m(z_1,\ldots,z_n) + \ldots + f_d(z_1,\ldots,z_n), \quad f_s \in \mathbb{C}[w_1,\ldots,w_n]_s, \quad f_m \neq 0, \tag{4.7.3}$$

i.e. it equals the degree of the first non-zero term in the MacLaurin expansion of f.

Remark 4.7.10. Let X be a quasi-projective variety and $p \in X^{\mathrm{sm}}$ be a smooth point of X. Let $Y \subset X$ be proper a closed subset and suppose that there exists an affine open set $U \subset X$ containing p such that $I(X \cap U) \subset \mathbb{C}[U]$ is a principal ideal generated by f. Then

1. $p \in Y$ if and only if $\mathrm{mult}_p Y > 0$, and
2. p is a singular point of Y if and only if $\mathrm{mult}_p f > 1$.

[1]As a matter of fact $\mathrm{mult}_p Y$ is *independent* of the embedding $Y \subset X$; we will not need this result

Let $X \subset \mathbb{A}^2$ be a hypersurface containing 0 and $\pi_X \colon \mathrm{Bl}_0 X \to X$ be the blow-down map. Thus $\mathrm{Bl}_0(X)$ is a closed subset of $\mathbb{A}^2 \times \Sigma_0$ (notation as in Subsection **??**). We make the identification (see (2.3.1))

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\sim} & \Sigma_0 \\ [T_1, T_2] & \mapsto & \langle (0,0), (T_1, T_2) \rangle \end{array} \tag{4.7.4}$$

Let f be a generator of $I(X)$ and let f_m be as in (4.7.3), with $a = (0,0)$. Then (see (2.3.16))

$$\pi_X^{-1}(0) = \{(0, [T_1, T_2]) \mid f_m(T_1, T_2) = 0\}. \tag{4.7.5}$$

(This makes sense because of Identification (4.7.4).) We have $\mathrm{Bl}_0(X) \subset \mathrm{Bl}_0 \mathbb{A}^2$ and $\mathrm{Bl}_0 \mathbb{A}^2$ is the union of the two open affine planes $\mathscr{U}_1$, $\mathscr{U}_2$ given by (2.3.10). Moreover, as shown in Example 2.3.4,

$$I(\mathrm{Bl}_0(X) \cap \mathscr{U}_1) = f_m(1, t_2) + z_1 f_{m+1}(1, t_2) + \ldots + z_1^{d-m} f_d(1, t_2), \tag{4.7.6}$$
$$I(\mathrm{Bl}_0(X) \cap \mathscr{U}_2) = f_m(t_1, 1) + z_2 f_{m+1}(t_1, 1) + \ldots + z_2^{d-m} f_d(t_1, 1). \tag{4.7.7}$$

where $t_1 := T_1/T_2$ and $t_2 := T_2/T_1$ are the standard affine coordinates on $\mathbb{P}^1_{T_2}$ and $\mathbb{P}^1_{T_1}$ respectively.

In particular $\mathrm{Bl}_0(X)$ is locally a hypersurface in the smooth surface $\mathrm{Bl}_0(\mathbb{A}^2)$ (i.e. there is an open affine covering of $\mathrm{Bl}_0(\mathbb{A}^2)$ such that the ideal of the intersection of $\mathrm{Bl}_0(X)$ with each open set is principal) and hence the multiplicity of vanishing of $\mathrm{Bl}_0(X)$ at an arbitrary $q \in \mathrm{Bl}_0(\mathbb{A}^2)$ makes sense. We chose $0 \in X$ for conveniece but it is clear that similar descriptions apply to $\mathrm{Bl}_a(X)$ for an arbitrary $a \in X$. In particular the multiplicity of vanishing of $\mathrm{Bl}_a(X)$ at an arbitrary $q \in \mathrm{Bl}_a \mathbb{A}^2$ makes sense.

Lemma 4.7.11. *Let $X \subset \mathbb{A}^2$ be a hypersurface and suppose that $0 \in X$. Let $m := \mathrm{mult}_o X$. For all $q \in \pi_X^{-1}(0)$ we have*

$$\mathrm{mult}_q \mathrm{Bl}_0(X) \leqslant m = \mathrm{mult}_0 X. \tag{4.7.8}$$

If there exists $q \in \pi_X^{-1}(0)$ such that (4.7.8) is an equality then

$$I(X) = (l(z_1, z_2)^m + f_{m+1}(z_1, z_2) + \ldots + f_d(z_1, z_2)), \tag{4.7.9}$$

where $l \in \mathbb{K}[z_1, z_2]_1$ is non zero, and $f_s \in \mathbb{K}[z_1, z_2]_s$ for $s \in \{(m+1), \ldots, d\}$.

Proof. Expand f in series of MacLaurin, as in (4.7.3); then $f_m \neq 0$ and hence there exist *non-zero* $l_1, \ldots, l_m \in \mathbb{C}[z_1, z_2]_1$ such that

$$f_m = l_1 \cdot \ldots \cdot l_m.$$

By (4.7.5) $q = (0, V(l_i)) \subset \{0\} \times \mathbb{P}^1$ for a certain $i \in \{1, \ldots, m\}$. After a homogeneous change of affine coordinates we may assume that $l_i = z_2$ and hence

$$0 \neq f_m = a_1 z_1^{m-1} z_2 + \ldots + a_m z_2^m. \tag{4.7.10}$$

We have $q \in \mathscr{U}_1$ and the ideal of $\mathrm{Bl}_0(X) \cap \mathscr{U}_1$ is generated by the polynomial in the right hand side of (4.7.6). By (4.7.10) we get that

$$I(\mathrm{Bl}_0(X) \cap \mathscr{U}_1) = (a_1 t_2 + a_2 t_2^2 + \ldots + a_m t_2^m + z_1 f_{m+1}(1, t_2) + \ldots + z_1^{d-m} f_d(1, t_2)).$$

The lemma follows because q is the point with (z_1, t_2)-coordinates equal to 0 $\square$

Proof of Proposition 4.7.7. The proof is by contradiction. Let X be singular. We will assume that the curves $X_1, X_2, \ldots, X_i, \ldots$ described at the beginning of the present subsection are singular for all $i \in \mathbb{N}$. We recall that X_1 is the blow-up of X at $\mathrm{sing}\, X$ and that X_i is the blow-up of X_{i-1} at $\mathrm{sing}\, X_{i-1}$ for $i \geqslant 2$. Now notice that $X_1 \subset \mathrm{Bl}_{\mathrm{sing}\, X} \mathbb{P}^2$, $X_2 \subset \mathrm{Bl}_{\mathrm{sing}\, X_1}(\mathrm{Bl}_{\mathrm{sing}\, X} \mathbb{P}^2)$ and and so on. Let $A_1 := \mathrm{Bl}_{\mathrm{sing}\, X} \mathbb{P}^2$, $A_2 := \mathrm{Bl}_{\mathrm{sing}\, X_1}(\mathrm{Bl}_{\mathrm{sing}\, X} \mathbb{P}^2)$ and so on. Then A_i is a projective surface which has an open cover by affine planes (this is analogous to Example 2.3.7) $\mathscr{W}_{ij}$ such that the ideal $I(X_i \cap \mathscr{W}_{ij}) \subset \mathbb{C}[\mathscr{W}_{ij}]$ is principal with generator computed inductively by applying the procedure of Example 2.3.2. In particular $\mathrm{mult}_q X_i$ is defined for any $q \in A_i$. For $i \geqslant 1$ (we set $X_0 = X$) let $\pi_{X_{i-1}} \colon X_i \to X_{i-1}$ be the blow-down map.

Then $\operatorname{sing} X_i \subset \pi_{X_{i-1}}^{-1}(\operatorname{sing} X_{i-1})$. The hypothesis that the curves $X_1, X_2, \ldots, X_i, \ldots$ are all singular and Lemma 4.7.8 give that we may choose $p_i \in \operatorname{sing} X_i$ for $i = 1, 2, \ldots$ such that $\pi_{X_i}(p_{i+1}) = p_i$. Let $\mathscr{W}_{i,j(i)} \subset A_i$ be an open affine plan as above containing p_i. Let

$$\pi_{X_i}^{-1}(X_i \cap \mathscr{W}_{i,j(i)}) \to X_i \cap \mathscr{W}_{i,j(i)} \tag{4.7.11}$$

be the restriction of the blow-down map π_{X_i}; applying Lemma 4.7.8 with $q = p_{i+1}$ and $a = p_i$ we get that $\operatorname{mult}_{p_{i+1}} X_{i+1} \leqslant \operatorname{mult}_{p_i} X_i$. On the other hand $\operatorname{mult}_{p_i} X_i \geqslant 2$ for all i by Remark 5.5.1. It follows that there exists $i \in \mathbb{N}$ such that

$$2 \leqslant m = \operatorname{mult}_{p_i} X_i = \operatorname{mult}_{p_{i+1}} X_{i+1} = \ldots = \operatorname{mult}_{p_{i+r}} X_{i+r} = \ldots \tag{4.7.12}$$

By Lemma 4.7.8 there exist affine coordinates (z_1, z_2) on $\mathscr{W}_{i,j(i)}$ (notation as above) such that p_i has coordinates $(0,0)$ and

$$I(X_i \cap \mathscr{W}_i) = (f), \qquad f = z_2^m + f_{m+1} + \ldots + f_d, \quad f_s \in \mathbb{C}[z_1, z_2]_s. \tag{4.7.13}$$

Now notice that the restriction of f to $V(z_2)$ does not vanish because X_i is irreducible and $m \geqslant 2$. Thus

$$\operatorname{mult}_0(f|_{V(z_2)}) = \min\{(m+1) \leqslant s \leqslant d \mid f_s(1,0) \neq 0\} < \infty. \tag{4.7.14}$$

We have $\pi_{X_i}^{-1}(p_i) = \{(0, [T_1, T_2]) \mid T_2^m = 0\}$ and hence $p_{i+1} = (0, [1, 0])$. Moreover (see (4.7.6)) the ideal $I(X_{i+1} \cap \mathscr{W}_{i+1,j(i+1)}) \subset \mathbb{C}[\mathscr{W}_{i+1,j(i+1)}]$ is generated by

$$g := t_2^m + z_1 f_{m+1}(1, t_2) + \ldots + z_1^{d-m} f_d(1, t_2). \tag{4.7.15}$$

It follows that

$$\operatorname{mult}_0(g|_{V(t_2)}) = \min\{(m+1) \leqslant s \leqslant d \mid f_s(1,0) \neq 0\} - m = \operatorname{mult}_0(f|_{V(x_2)}) - m. \tag{4.7.16}$$

Iterating this procedure we get a contradiction because the multiplicity of vanishing of a function at a point of a smooth variety is a non-negative integer. $\square$

Smooth projective representative of a birational class

Chapter 5

Smooth points: deeper properties

5.1 Local invertibility of regular maps

In the present subsection we prove the following analogue, in the category of quasi-projective varieties, of the local invertibility results valid for C^∞ or holomorphic maps.

Theorem 5.1.1. *Let $f\colon X \to Y$ be a projective map of quasi-projective sets. Let $p \in X$ and suppose that the following hold:*

1. $f^{-1}(f(p)) = \{p\}$.
2. $df(p)\colon \Theta_p X \to \Theta_{f(p)} Y$ *is injective.*

Then there exists an open $U \subset Y$ containing $f(p)$ such that the restriction of f to $f^{-1}(U)$ is an isomorphism to a closed subset of U.

Before proving Theorem 5.1.1 we give some preliminary result. Let $\varphi\colon A \to B$ be a homomorphism of rings. By setting $a \cdot b := \varphi(a)b$ we equip B with a structure of A-module: we say that B is *finite over* A if it is a finitely generated A-module. Let X, Y be affine varieties, and let $f\colon X \to Y$ be a regular map; the pull back $f^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is a homomorphism of rings, hence (with f understood) it makes sense to state that $\mathbb{K}[X]$ is finite over $\mathbb{K}[Y]$.

Lemma 5.1.2. *Let $f\colon X \to Y$ be a projective map of quasi projective varieties. Let $y_0 \in Y$ and suppose that $f^{-1}(y_0)$ is finite. There exists an open affine $Y_0 \subset Y$ containing y_0 such that $X_0 := f^{-1}(Y_0)$ is affine and $\mathbb{K}[X_0]$ is finite over $\mathbb{K}[Y_0]$.*

Proof. By Definition 4.7.1 we may assume that $X \subset \mathbb{P}^n \times Y$ is closed and f is the restriction of the projection $\pi\colon \mathbb{P}^n \times Y \to Y$. Since $X \cap (\mathbb{P}^n \times y_0)$ is finite there exists homogeneous coordinates $[Z_0, \ldots, Z_n]$ on $\mathbb{P}^n$ such that $X \cap (V(Z_0) \times \{y_0\}) = \emptyset$. The intersection $X \cap (V(Z_0) \times Y)$ is a closed subset of $\mathbb{P}^n \times Y$. By Elimination Theory (i.e. Theorem 1.6.1) $C := \pi(X \cap (V(Z_0) \times Y))$ is closed in Y. Hence $(Y \backslash C)$ is an open subset of Y containing y_0. Let $Y_* \subset (Y \backslash C)$ be an open affine subset containing y_0. Then $X_* := X \cap (\mathbb{P}^n \times Y_*) = f^{-1}(Y_*)$ is a closed subset of the affine set $\mathbb{P}^n_{Z_0} \times Y_*$ and hence is affine. It remains to prove that $\mathbb{K}[X_*]$ is finite over $\mathbb{K}[Y_*]$. The proof is by induction on n. If $n = 0$ then $\mathbb{K}[X_*] = \mathbb{K}[Y_*]$ and there is nothing to prove. Let's prove the inductive step. Since X_* is closed in $\mathbb{P}^n \times Y_*$ there exist $F_i \in \mathbb{K}[X_*][Z_0, \ldots, Z_n]_{d_i}$ for $i = 1, \ldots, r$ such that

$$X_* = V(F_1, \ldots, F_r).$$

(See Claim 1.5.5.) Since $X_* \cap (V(Z_0) \times \{y_0\})$ is empty we have

$$V(F_1(y_0)(0, Z_1, \ldots, Z_n), \ldots, F_r(y_0)(0, Z_1, \ldots, Z_n)) = \emptyset.$$

By Hilbert's Nullstellensatz, there exists $M > 0$ such that

$$(Z_1, \ldots, Z_n)^M \subset (F_1(y_0)(0, Z_1, \ldots, Z_n), \ldots, F_r(y_0)(0, Z_1, \ldots, Z_n)).$$

It follows (see the proof of Theorem 1.6.1) that, shrinking Y_* around y_0, we may assume that

$$Z_1^M, \ldots, Z_n^M \in (F_1(0, Z_1, \ldots, Z_n), \ldots, F_r(0, Z_1, \ldots, Z_n)). \tag{5.1.1}$$

(Actually we may arrange so that (5.1.1) holds for the original Y_* - but we do not need this). Equation (5.1.1) gives that there exists

$$G = (Z_n^M + A_1 Z_n^{M-1} + \ldots + A_M) \in (F_1, \ldots, F_r), \qquad A_i \in \mathbb{K}[Y_*][Z_0, \ldots, Z_{n-1}]_i.$$

Thus $G|_{X_*} = 0$: dividing by Z_0^M and setting $z_i := Z_i/Z_0$, $a_i = A_i/Z_0^i \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ we get that

$$(z_n^M + a_1 z_n^{M-1} + \ldots + a_M)|_{X_*} = 0. \tag{5.1.2}$$

Let $Q := [0, \ldots, 0, 1] \in \mathbb{P}^n$. The product of projection from Q and Id_{Y_*}

$$\begin{array}{ccc} (\mathbb{P}^n \backslash \{P\}) \times Y_* & \xrightarrow{\rho} & \mathbb{P}^{n-1} \times Y_* \\ ([Z_0, \ldots, Z_n], p) & \mapsto & ([Z_0, \ldots, Z_{n-1}], p) \end{array}$$

is *not* projective but the restriction of ρ to X_* is projective. In fact locally over open sets of a covering $\bigcup_{j \in J} U_j$ of Y_* we may embed X_* as a closed subset of $\mathbb{P}^1 \times U_j$ so that ρ is the restriction of the projection $(\mathbb{P}^1 \times U_j) \to U_j$. Thus the image $\rho(X_*)$ is a closed subset of $\mathbb{P}^{n-1} \times Z_*$. Since the fiber of $\rho(X_*) \to Y_*$ over y_0 is finite we may assume (possibly after shrinking Y_* and X_*) that $\rho(X_*)$ is affine (we just proved it). The ring $\mathbb{K}[X_*]$ is obtained from $\mathbb{K}[\rho(X_*]$ by adding z_n. Equation (5.1.2) gives that $\mathbb{K}[X_*]$ is finite over $\mathbb{K}[\rho(X_*]$. By the inductive hypothesis $\mathbb{K}[\rho(X_*]$ is finite over $\mathbb{K}[Y_*]$ (possibly after shrinking $\mathbb{K}[Y_*]$): it follows that $\mathbb{K}[X_*]$ is finite over $\mathbb{K}[Y_*]$. □

Proof of Theorem 5.1.1. Since f is projective it has closed image: thus we may assume that f is surjective. By Lemma 5.1.2 we may assume that X and Y are affine and that $\mathbb{K}[X]$ is finite over $\mathbb{K}[Y]$. By surjectivity of f the pull-back defines an inclusion $f^* \colon \mathbb{K}[Y] \hookrightarrow \mathbb{K}[X]$. We will prove that there exists an open affine $\mathscr{U} \subset Y$ containing q such that $f^*|_{\mathscr{U}} \colon \mathbb{K}[\mathscr{U}] \hookrightarrow \mathbb{K}[f^{-1}\mathscr{U}]$ is surjective: that will give that $f|_{\mathscr{U}} \colon f^{-1}\mathscr{U} \to \mathscr{U}$ is an isomorphism. Let $q := f(p)$. By Item (1) and the Nullstellensatz we have

$$\mathfrak{m}_p = \sqrt{f^* \mathfrak{m}_q \mathbb{K}[X]}. \tag{5.1.3}$$

Here $f^* \mathfrak{m}_q \mathbb{K}[X]$ is the ideal of $\mathbb{K}[X]$ generated by $f^*\phi$ for $\psi \in \mathfrak{m}_q$ (we will use similar notation in the course of the proof). Let $\mathfrak{m}_p = (\phi_1, \ldots, \phi_n)$. Item (2) gives that for each $1 \leqslant i \leqslant n$ there exist an affine open U_i containing p and $\psi_i \in \mathbb{K}[Y]$ such that $(\phi_i - f^*\psi_i)|_{U_i} \in \mathfrak{m}_p^2 \mathbb{K}[U_i]$. Since f is closed it follows that there exists a principal open affine Y_h neighborhood of q (thus $h \in \mathbb{K}[Y]$ with $h(q) \neq 0$) such that

$$(\phi_i - f^*\psi_i)|_{f^{-1}(Y_h)} \in \mathfrak{m}_p^2 \mathbb{K}[f^{-1}(Y_h)] \quad \forall 1 \leqslant i \leqslant n. \tag{5.1.4}$$

Let's prove by "descending induction" on k that

$$\mathfrak{m}_p^k \mathbb{K}[f^{-1}(Y_h)] \subset f^* \mathfrak{m}_q \mathbb{K}[f^{-1}(Y_h)] \quad \forall 1 \leqslant k. \tag{5.1.5}$$

By (5.1.3) there exists $N > 0$ such that (5.1.5) holds for $k \geqslant N$. Let's prove the "inductive step": we assume that (5.1.5) holds with $k \geqslant 2$ and we prove that it holds with k replaced by $(k-1)$. Let

$$\varphi = \sum_{|L|=k-1} c_L \phi_1^{l_1} \ldots \phi_n^{l_n} \in \mathfrak{m}_p^{k-1} \mathbb{K}[f^{-1}(Y_h)]. \tag{5.1.6}$$

By (5.1.4) we may write $\phi_i = f^*\psi_i + \epsilon_i$ where $\epsilon_i \in \mathfrak{m}_p^2 \mathbb{K}[f^{-1}(Y_h)]$ for $i = 1, \ldots, n$: substituting in (5.1.6) and invoking the inductive hypothesis we get that $\varphi \in f^* \mathfrak{m}_q \mathbb{K}[f^{-1}(Y_h)]$. We have proved (5.1.5). Since

$\mathbb{K}[f^{-1}(Y_h)] = \mathbb{K}[Y]_{(f^*h^s)}$ (the localization of $\mathbb{K}[Y]$ with respect to the multiplicative system of powers of f^*h) we get that

$$I_p := \{\varphi \in \mathbb{K}[f^{-1}(Y_h)] \mid \varphi(p) = 0\} = f^*\mathfrak{m}_q\mathbb{K}[f^{-1}(Y_h)]. \tag{5.1.7}$$

Now notice that $\mathbb{K}[f^{-1}(Y_h)]$ is a finite $\mathbb{K}[Y_h]$-module because $\mathbb{K}[f^{-1}Y]$ is a finite $\mathbb{K}[Y]$-module. We will apply Nakayama's Lemma to the finitely generated $\mathbb{K}[Y_h]$-module

$$M := \mathbb{K}[f^{-1}(Y_h)]/f^*\mathbb{K}[Y_h]$$

and the ideal $\mathfrak{m}_q$. We claim that $M \subset \mathfrak{m}_q M$. In fact since $\mathbb{K} \subset f^*\mathbb{K}[Y_h]$ every element of M is represented by $\alpha \in I_p$ (notation as in (5.1.7)) and $\overline{\alpha} \in \mathfrak{m}_q M$ by (5.1.5). By Lemma A.7.2 there exists $\varphi \in \mathfrak{m}_q$ such that

$$(1+\varphi)\mathbb{K}[f^{-1}Y_h] \subset f^*\mathbb{K}[Y_h]. \tag{5.1.8}$$

The open affine $Y_{h(1+\varphi)} \subset Y$ contains q (because $\varphi(q) = 0$). By (5.1.8) we get that

$$\mathbb{K}[f^{-1}Y_{h(1+\varphi)}] = f^*\mathbb{K}[Y_{h(1+\varphi)}].$$

□

Example 5.1.3. Suppose that $X \subset \mathbb{P}^n$ is closed irreducible and $r \in (\mathbb{P}^n \backslash X)$. Let $H \subset \mathbb{P}^n$ be a hyperplane not containing r. Projection

$$\begin{array}{ccc} X & \xrightarrow{\pi} & H \\ p & \mapsto & \langle p, r\rangle \cap H \end{array}$$

is a projective map with finite fibers. Let $p \in X$ and suppose that the projective tangent space $\mathbf{T}_p X$ does *not* contain the line $\langle r, p\rangle$: then $df(p)$ is injective. Suppose in addition that $\pi^{-1}(\pi(p)) = \{p\}$: by Theorem 5.1.1 we get that π is birational onto its image. As long as $\dim \Theta_p(X) < n$, and X has codimension at least 2, there exists a point r such that the two conditions above hold. Iterating we get that if $\dim X = m$ we can choose a projection from a linear space of dimension $(n-m-2)$ giving a birational map from $\varphi\colon X \to Y$ where $Y \subset \mathbb{P}^{m+1}$ is a hypersurface, and such that φ restricts to an isomorphism from a neighborood of p to a neighborhood of $\varphi(p)$.

5.2 Local factoriality

The result below is of fundamental importance.

Theorem 5.2.1. *Let X be a smooth quasi projective variety. Let $D \subset X$ be a closed subset of pure codimension* 1*, and let $a \in D$. There exists an open affine subset $U \subset X$ containing a such that the ideal $I(D \cap U) \subset \mathbb{K}[U]$ is principal.*

Remark 5.2.2. If we assume that D is smooth at a, then Theorem 5.2.1 follows from Proposition 4.5.9 and Corollary 4.5.9. In fact, replacing X by a suitable open affine subset containing a, we may assume that X is affine. Hence there exists an embedding $X \subset \mathbb{A}^n$ as closed subset. Thus $D \subset \mathbb{A}^n$ is also closed. Applying Proposition 4.5.9 and Corollary 4.5.9 to X and D, we get that there exist an open affine subset $U \subset \mathbb{A}^n$ containing a, and functions $f_1, \ldots, f_{k+1} \in \mathbb{K}[U]$, such that

$$I(X \cap U) = (f_1, \ldots, f_k), \qquad I(D \cap U) = (f_1, \ldots, f_{k+1}).$$

Since principal open affine sets form a basis for thge Zariski topology, we may assume that U is a principal open set, say $U = \mathbb{A}^n \backslash V(\varphi)$. Hence also $U \cap X$ is an open principal set, in particular it is affine. Moreover the image of f_{k+1} in $\mathbb{K}[X \cap U]$ is a generator of the ideal of $D \cap X \cap U$.

Proof of Theorem 5.2.1. □

The statement of Theorem 5.2.1 is summarized by stating that a smooth quasi projective variety is locally factorial.

EXPLAIN

The result below follows from Theorem 5.2.1, actually the weak version in Remark 5.2.2 suffices.

Proposition 5.2.3. *Let X be a smooth quasi projective variety, and let $f\colon X \dashrightarrow \mathbb{P}^n$ be a rational map. The indeterminacy set* $\mathrm{Ind}(f)$ *has codimension at least* 2 *in* X.

First we prove the following.

Proof of Proposition 5.2.3. The indeterminacy set $\mathrm{Ind}(f)$ is a proper closed subset of X. We argue by contradiction. Suppose that D is a codimension 1 irreducible closed subset of X contained in $\mathrm{Ind}(f)$. Let a be a smooth point of D. By Lemma 5.2.2 there exist an open affine subset $U \subset X$ containing a and $\varphi \in \mathbb{K}[U]$ such that $I(D \cap U) \subset \mathbb{K}[U]$ is generated by φ. Since X is smooth at a, there is a unique irreducible component of X containing a, hence we may assume that U is irreducible. There exist $f_0, \ldots, f_n \in \mathbb{K}[U]$ such that $V(f_0, \ldots, f_n)$ is a proper subset of U, and

$$f(x) = [f_0(x), \ldots, f_n(x)] \quad \forall x \in (U \backslash V(f_0, \ldots, f_n)).$$

□

5.3 Smooth points of maps

Let $f\colon X \to Y$ be a regular map of quasi projective varieties.

Definition 5.3.1. Let $x \in X$. The map f is *smooth at x* if

1. x is a smooth point of X, $y := f(x)$ is a smooth point of Y,
2. and the differential $df(x)\colon \Theta_x X \to \Theta_y Y$ is surjective.

The following result explains why we might be interested in the points at which a regular map is smooth.

Proposition 5.3.2. *Let $f\colon X \to Y$ be a regular map of quasi-projective varieties. Suppose that $x \in X$ and that f is smooth at x. Then $f^{-1}\{f(x)\}$ is smooth at x and*

$$\dim_x f^{-1}(f(x)) = \dim_x X - \dim_{f(x)} Y.$$

Proof. We may assume that X and Y are affine. Let $n := \dim Y$, and let $y := f(x)$. There exists r such that $Y \subset \mathbb{A}^{n+r}$ is closed. By Corollary 4.5.9 there exist a Zariski open $U \subset \mathbb{A}^{n+r}$ containing y and $\phi_1, \ldots, \phi_r \in \mathbb{K}[z_1, \ldots, z_{n+r}]$ such that

1. $d\phi_1(y), \ldots, d\phi_r(y)$ are linearly independent, and
2. $V(\phi_1, \ldots, \phi_r) \cap U = Y \cap U$.

Let $\psi_1, \ldots, \psi_n \in \mathbb{K}[z_1, \ldots, z_{n+r}]$ be such that $0 = \psi_1(y) = \ldots = \psi_n(y)$ and

$$\{d\phi_1(y), \ldots, d\phi_r(y), d\psi_1(y), \ldots, d\psi_n(y)\}$$

is a basis of the cotangent space of $\mathbb{A}^{n+r}$ at y (we may choose the ψ_i's to be coordinate functions if we wish). By Proposition 4.5.9 $V(\phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_n)$ has dimension zero at y. Thus shrinking the open set U above, if necessary, we may assume that

$$V(\psi_1, \ldots, \psi_n, \phi_1, \ldots, \phi_r) \cap U = \{y\}. \tag{5.3.1}$$

Let $\overline{\psi}_i := \psi_{i|Y}$. By (5.3.1) we have that

$$f^{-1}(y) = V(f^*(\overline{\psi}_1), \ldots, f^*(\overline{\psi}_n)). \tag{5.3.2}$$

We have $d(f^*(\overline{\psi}_i))(x) = f^* d\overline{\psi}_i(y)$. By hypothesis $df(x)$ is surjective, i.e. the transpose

$$\Omega_Y(y) \xrightarrow{df(x)^t} \Omega_X(x)$$

is injective. Since $d\overline{\psi}_1(y), \ldots, d\overline{\psi}_n(y)$ are linearly independent, it follows that $d(f^*\overline{\psi}_1)(x), \ldots, d(f^*\overline{\psi}_n)(x)$ are linearly independent. Let $m := \dim_x X$. Since X is affine, there exists s such that that $X \subset \mathbb{A}^{m+s}$ is closed. By hypothesis X is smooth at x, and hence by Corollary 4.5.9 there exist a Zariski open $\mathscr{U} \subset \mathbb{A}^{m+s}$ containing x and $\psi_1, \ldots, \psi_s \in \mathbb{K}[z_1, \ldots, z_{m+s}]$ such that

1. $V(\psi_1, \ldots, \psi_s) \cap \mathscr{U} = X \cap \mathscr{U}$, and
2. $d\psi_1(x), \ldots, d\psi_s(x)$ are linearly independent.

Since X is closed in $\mathbb{A}^{m+s}$ there exist $\varphi_1, \ldots, \varphi_n \in \mathbb{K}[z_1, \ldots, z_{m+s}]$ such that $\varphi_i|_X = f^*\overline{\phi}_i$. By (5.3.2) we have that

$$f^{-1}\{y\} \cap \mathscr{U} = V(\psi_1, \ldots, \psi_s, \varphi_1, \ldots, \varphi_n) \cap \mathscr{U}.$$

Applying Proposition 4.5.9 we get that $V(\psi_1, \ldots, \psi_s, \varphi_1, \ldots, \varphi_n)$ is smooth at x of dimension $m - n = \dim_x X - \dim_x Y$. □

The result below is elementary.

Claim 5.3.3. *Let $f\colon X \to Y$ be a regular map of quasi-projective sets. The set of smooth points of f is open in X.*

Proof. The set of points $x \in X$ such that (1) of Definition 5.3.1 holds is equal to $X^{\mathrm{sm}} \cap f^{-1}(Y^{\mathrm{sm}})$ and hence is open by Theorem 4.5.3. Thus it remains to prove that the set of $x \in X^{\mathrm{sm}} \cap f^{-1}(Y^{\mathrm{sm}})$ such that $df(x)$ is not surjective is closed in $X^{\mathrm{sm}} \cap f^{-1}(Y^{\mathrm{sm}})$. It suffices to prove it for X and Y affine, smooth. By Corollary 4.5.9 we may assume that Y is irreducible of dimension d. Thus we must check that the set

$$\{x \in X \mid \operatorname{rk} df(x) \leqslant (d-1)\} \tag{5.3.3}$$

is closed in X. By hypothesis $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are closed. Via the identification provided by Proposition 4.3.5 the differential $df(x)$ gets identified with the Jacobian matrix $Jf(x)$. It follows that (5.3.3) is the set of zeroes of determinants of $d \times d$ minors of $Jf(x)$, and hence is closed. □

A point that is smooth for a regular map $f\colon X \to Y$ is relative version of smooth point of a variety, because X is smooth at a point x if and only if the constant map $X \to \{y_0\}$ is smooth at x. We have proved that the set of smooth points of a quasi projective variety is an open dense subset.

By analogy, one might expect density of the set of points at which a dominant map of irreducible quasi projective varietes is smooth. (We must assume that the map is dominant, otherwise the differential is never a surjection for trivial reasons, and if the domain has more than one irreducible component, then again the set of smooth points of the map can be non dense for trivial reaons.) It turns out that even under the above hypotheses, the set of smooth points of a map might be empty. The Frobenius map is the archetypical example. Suppose that $\operatorname{char} \mathbb{K} = p$, and let $\mathbb{A}^1 \xrightarrow{F} \mathbb{A}^1$ be given by $F(z) = z^p$; then F is dominant, but the differential is zero everywhere.

The result below provides the hypothesis that guarantee density of the set of smooth points of a dominant map between irreducible varieties - in particular the hypothesis is satisfied if $\operatorname{char} \mathbb{K} = 0$.

Proposition 5.3.4. *Let $f\colon X \to Y$ be a regular dominant map of irreducible quasi-projective varieties - thus $f^*\colon \mathbb{K}(Y) \to \mathbb{K}(X)$ is an embedding of field extensions of $\mathbb{K}$. Suppose that $\mathbb{K}(X)$ is a separably generated extension of $\mathbb{K}(Y)$. Then the set of smooth points of f is an open* dense *subset of X.*

Before proving the above result we associate a geometric object to a derivation $D \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}(X), \mathbb{K}(X))$, where X is an irreducible quasi projective variety. This allows to derive Proposition 5.3.4 from Proposition A.6.4. Let $x \in X$. We recall that $\mathscr{O}_{X,x} \subset \mathbb{K}(X)$. Suppose that

$$D(\mathscr{O}_{X,x}) \subset \mathscr{O}_{X,x}. \tag{5.3.4}$$

Then we may define a tangent vector $D(x) \in \Theta_x X$ by setting

$$\begin{array}{ccc} \mathscr{O}_{X,x} & \overset{D(x)}{\longrightarrow} & \mathbb{K} \\ \phi & \mapsto & D(\phi)(x). \end{array} \tag{5.3.5}$$

The result below shows that an element of $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}(X), \mathbb{K}(X))$ may be thought of as a vector field on an open dense subset $U \subset X$ (the open U depends on D).

Claim 5.3.5. *Let X be an irreducible quasi-projective variety and $D \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}(X), \mathbb{K}(X))$. There exists an open dense $U \subset X$ such that for all $x \in U$ Equation* (5.3.4) *holds and hence the tangent vector $D(x) \in \Theta_x X$ is defined.*

Proof. We may assume that X is affine. Thus $\mathbb{K}(X)$ is the fraction field of $\mathbb{K}[X]$. Let $f_1, \ldots, f_r$ be generators of the $\mathbb{K}$-algebra $\mathbb{K}[X]$. There exists $0 \neq g \in \mathbb{K}[X]$ such that $g \cdot D(f_i) \in \mathbb{K}[X]$ for $i = 1, \ldots, r$. Let $U := X_g = (X \backslash V(g))$. Then X_g is an affine open dense subset of X, and its ring of regular functions is the subring of $\mathbb{K}(X)$ given by

$$\mathbb{K}[U] = \{h/g^k \mid h \in \mathbb{K}[X],\ k \geqslant 0\}.$$

Thus (A.6.4) gives that $D(\mathbb{K}[U]) \subset \mathbb{K}[U]$. Applying (A.6.4) again it follows that (5.3.4) holds for $x \in U$. $\square$

Proof of Proposition 5.3.4. We know that the set of smooth points of f is open, we must prove that it is non-empty. We are free to replace X and Y by open dense subsets X^0 and Y^0 respectively (of course we require that $f(X^0) \subset Y^0$): in the course of the proof we will rename X^0 and Y^0 by X and Y respectively. In particular we may assume that X and Y are smooth. By Theorem A.5.6, there exists a separating transcendence basis $\phi_1, \ldots, \phi_m$ of $\mathbb{K}(Y)$ over $\mathbb{K}$. Replacing Y by the open dense subset Y^0 of points where each of $\phi_1, \ldots, \phi_m$ is regular we may assume that $\phi_1, \ldots, \phi_m$ are regular (of course we replace X by $f^{-1}Y^0$). Since f is dominant $f^*\phi_1, \ldots, f^*\phi_m$ are algebraically independent in $\mathbb{K}(X)$. Let $\psi_1, \ldots, \psi_n$ be a separating transcendence basis of $\mathbb{K}(X)$ over $\mathbb{K}(Y)$ (it exeists by hypothesis). Then $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n$ is a separating transcendence basis of $\mathbb{K}(X)$ over $\mathbb{K}$, and hence by Proposition A.6.4 there exist $D_j \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}(X), \mathbb{K}(X))$ for $j = 1, \ldots, m$ such that

$$D_j(f^*\phi_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

By Claim 5.3.5, we may assume that $D_j(\mathscr{O}_{X,x}) \subset \mathscr{O}_{X,x}$ for every $x \in X$ (after shrinking X). Then D_j defines a tangent vector $D_j(x) \in \Theta_x X$ for each $x \in X$. Let $x \in X$: we claim that $df(x)$ is surjective. In fact let $y := f(x)$. Then $df^*\phi_1(y), \ldots, df^*\phi_m(y)$ are linearly independent because

$$\langle D_j(x), f^*\phi_i \rangle = \delta_{ij}. \tag{5.3.6}$$

In particular $d\phi_1(y), \ldots, d\phi_m(y)$ are linearly independent. Since $m = \dim Y$ and Y is smooth it follows that $\{d\phi_1(y), \ldots, d\phi_m(y)\}$ is a basis of $\Omega_Y(y)$. This proves that the transpose of $df(x)$ is injective and hence $df(x)$ is surjective. $\square$

5.4 Regular values

Definition 5.4.1. Let $f\colon X \to Y$ be a regular map between quasi-projective sets. A point $y \in Y$ is a *regular value of* f if f is smooth at *each* $x \in f^{-1}\{y\}$.

Theorem 5.4.2 (Sard's theorem for quasi-projective varieties)**.** *Let $f\colon X \to Y$ be a regular map of quasi-projective varieties over a field $\mathbb{K}$ of characteristic 0. Suppose that X is smooth. Then the set of regular values of f contains an open dense subset of Y.*

Proof. One checks easily that it suffices to prove the theorem for X and Y irreducible. If f is not dominant then every point of the open dense set $(Y\backslash\overline{f(X)})$ is a regular value of f. Now suppose that $f\colon X \to Y$ is dominant. By Proposition 5.3.4 the open set

$$X^0 := \{x \in X \mid df(x) \text{ is surjective}\}$$

is dense in X. Let C be an irreducible component of $X\backslash X^0$; we claim that $\overline{f(C)} \neq Y$. In fact suppose the contrary. Applying Proposition 5.3.4 to $f_{|C}$ we get that there exists an open dense $C^0 \subset C$ sm such that

$$df(x)_{|\Theta_x C}\colon \Theta_x C \to \Theta_{f(x)} Y$$

is surjective. That contradicts the definition of X^0. This proves that $\overline{f(C)} \neq Y$. It follows that

$$\overline{f(X\backslash X^0)} \neq Y.$$

Thus $Y\backslash\overline{f(X\backslash X^0)}$ is an open dense subset of regular values of Y. □

The following result shows, at least in the case of maps of finite degree, that if a map is projective then the fibers of "nearby" regular values all look alike.

Proposition 5.4.3. *Let $f\colon X \to Y$ be a regular projective map between irreducible quasi projective varieties of the same dimension. If $y \in Y$ is a regular value of f, the cardinality of $f^{-1}(y)$ is equal to $\deg f$. In particular, if there exists a regular value of f, then $\mathbb{K}(X)$ is an algebaric separable extension of $\mathbb{K}(Y)$.*

Before proving Proposition 5.4.3, we examine an example that was discussed in Section 3.2. It will convince the reader that Proposition 5.4.3 should be true.

Example 5.4.4. Let Y be an affine variety, and let $P \in \mathbb{K}[Y][t]$ be an *irreducible* polynomial:

$$P = t^d + a_1 t^{d-1} + \cdots + a_d.$$

Let $X := V(P) \subset Y \times \mathbb{A}^1$, and let $f\colon X \to Y$ be the projection, given by $f(y,t) = y$. The closure of X in $Y \times \mathbb{P}^1$ is equal to X, because the leading coefficient of P (in t) is equal to 1. Hence the map f is projective. Clearly $\dim X = \dim Y$, and $\deg f = d$.

Next, we notice that $y_0 \in Y$ is a regular value of f if and only if Y is smooth at y_0, and $\frac{dP}{dt}(y_0,\xi) \neq 0$ for all ξ which are solutions of the degree d polynomial $P(y_0,t) = 0$. Hence, if y_0 is a regular value of f, then all solutions of the equation $P(y_0,t) = 0$ have multiplicity 1, and therefore there are $d = \deg f$ of them.

Proof of Proposition 5.4.3. Since f is projective, we may assume that $X \subset Y \times \mathbb{P}^N$ is closed, and f is the projection map given by $f(y,t) = y$. The proof is by induction on N. If $N = 0$, the statement is trivially true. The inductive step starts from $N = 2$, hence the case $N = 1$ must be examined separately. □

5.5 Exercises

Exercise 5.5.1. Let $n \geqslant 2$, and let $X \subset \mathbb{P}^n$ be a smooth hypersurface. Prove that X is irreducible. Notice that this property is peculiar to hypersurfaces in $\mathbb{P}^n$. If Y is a quasi-projective variety of pure dimension, we may define a *hypersurface in* Y to be a closed $X \subset Y$ of pure dimension equal to $\dim Y - 1$. Give examples of projective smooth Y of dimension at leats 2 and a reducible smooth hypersurfaces $X \subset Y$.

Exercise 5.5.2. Let $X \subset \mathbb{P}^n$ be a hypersurface, and let $I(X) = (F)$.

1. Prove that if $[a]$ is a singular point of X, then all the partial derivatives $\frac{\partial F(a)}{\partial Z_i}$ vanish. (Hint: use Euler's equality (1.8.8).
2. Show that if char $\mathbb{K}$ does not divide the degree of X, then the converse holds, i.e. if all the partial derivatives $\frac{\partial F(a)}{\partial Z_i}$ vanish, then $[a]$ is a singular point of X. (Recall that $\deg X = \deg F$.)

Chapter 6

Some classical results

6.1 Bèzout's Theorem

Definition 6.1.1. Let X be a quasi-projective variety. Let $Y, W \subset X$ be locally-closed. Let $p \in Y \cap W$. We say that Y and W *intersect transversely at* p (or are *transverse at* p) if the following hold:

1. X, Y, and W are smooth at p.

2. The natural map $T_pY \oplus T_pW \to T_pX$ is surjective.

We say that Y and W *intersect transversely* (in symbols $Y \pitchfork W$) if given any irreducible component V of $Y \cap W$ there exists $p \in V$ such that Y and W are transverse at p.

Example 6.1.2. Let (x, y) be affine coordinates on $\mathbb{A}^2_{\mathbb{C}}$. Then $V(y)$ and $V(y - x^2(x-1))$ do *not* intersect transversely at $(0, 0)$, they intersect transversely at $(1, 0)$.

Remark 6.1.3.

1. Suppose that $Y, W \subset X$ are locally closed, $p \in Y \cap W$ and Y, W are transverse at p. Then $(\dim Y + \dim W - \dim X) \geqslant 0$ (obvious) and by ?? there is a unique irreducible component of $Y \cap W$ containing p, call it V. Moreover V is smooth at p of dimension equal to $(\dim Y + \dim W - \dim Z)$.

2. Suppose that $Y, W \subset X$ intersect transversely. Let V be an irreducible component of $Y \cap W$. Since the set of $p \in V$ such that such that (1) and (2) of Definition 6.1.1 holds is an open subset of V (that is easily checked) it follows that there is an open *dense* subset of $p \in V$ such that such that (1) and (2) of Definition 6.1.1 holds.

Theorem 6.1.4 (Transverse Bézout's theorem)**.** *Let $X, Y \subset \mathbb{P}^n$ be closed subsets which intersect* transversely*. Then*

$$\deg X \cap Y = \deg X \cdot \deg Y$$

unless $\dim X + \dim Y < n$ *(in that case $X \cap Y = \varnothing$ by Remark 6.1.3).*

The key element in the proof (that we will give) of Theorem **??** is the following degree computation.

Proposition 6.1.5. *Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subsets such that* (3.4.1) *holds. Then* $\deg J(X, Y) = \deg X \cdot \deg Y$.

Proof. Since $\langle J(X, Y)\rangle = \langle X, Y\rangle$ we might as well assume that $\langle X, Y\rangle = \mathbb{P}^N$. Let

$$c_X := \operatorname{cod}(X, \langle X\rangle), \quad c_Y := \operatorname{cod}(Y, \langle Y\rangle), \quad c_J := \operatorname{cod}(J(X, Y), \mathbb{P}^N).$$

We have

$$n = \dim\langle X\rangle + \dim\langle Y\rangle + 1, \quad \dim J(X, Y) = \dim X + \dim Y + 1.$$

Subtracting we get that $c_J = c_X + c_Z$. Let $\Lambda \subset \langle X\rangle$ and $\Gamma \subset \langle Z\rangle$ be linear subspaces such that

$$\dim \Lambda = c_X, \quad \dim \Gamma = c_Z, \quad \Lambda \pitchfork X, \quad \Gamma \pitchfork Z.$$

Then

$$|\Lambda \cap X| = \deg X, \qquad |\Gamma \cap Y| = \deg Y, \qquad \dim\langle \Lambda, \Gamma\rangle = c_X + c_Y + 1.$$

The intersection of $\langle \Lambda, \Gamma\rangle$ and $J(X,Y)$ is transverse and it equals

$$J(\Lambda \cap X, \Gamma \cap Y) = \bigcup_{\substack{p \in \Lambda \cap X \\ q \in \Gamma \cap Y}} \langle p, q\rangle. \tag{6.1.1}$$

Let $H \subset \mathbb{P}^N$ be a hyperplane transverse to each of the finite lines appearing in the right-hand side of (6.1.1) and such that $H \cap \Lambda \cap X = H \cap \Gamma \cap Y = \varnothing$. Then

$$\dim H \cap \langle \Lambda, \Gamma\rangle = c_X + c_Y.$$

Moreover the linear space $H \cap \langle \Lambda, \Gamma\rangle$ intersects transversely $J(X,Y)$. It follows that

$$\deg J(X,Y) = |(H \cap \langle \Lambda, \Gamma\rangle) \cap J(X,Y)| = \deg X \cdot \deg Y.$$

□

Proof of Theorem **??**. If $\dim X + \dim Y < n$ then $X \cap Y = \varnothing$, and there is nothing to prove. Thus we may assume that

$$e := \dim X + \dim Y - n \geqslant 0. \tag{6.1.2}$$

As the reader will easily check we may assume that X and Y are irreducible. Then e is the dimension of every irreducible component of $X \cap Y$ - see Item (1) of Remark 6.1.3. Let $i, j \colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{2n+1}$ be as in (3.4.8). Let $\Lambda \subset \mathbb{P}^{2n+1}$ be given by

$$\Lambda := V(W_0 - Z_0, \ldots, W_n - Z_n).$$

We recall that we have an isomorphism

$$\begin{array}{ccc} X \cap Y & \xrightarrow{\sim} & \Lambda \cap J(i(X), j(Y)) \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_n, Z_0, \ldots, Z_n] \end{array} \tag{6.1.3}$$

Since $X \pitchfork Y$ the linear space Λ intersects transversely $J(i(X), j(Y))$ (check it). Now let $\Gamma \subset \mathbb{P}^n$ be a linear space transverse to $X \cap Y$ (such a Γ exists by ??). Thus

$$|\Gamma \cap X \cap Y| = \deg(X \cap Y) \tag{6.1.4}$$

by ??. On the other hand $\widetilde{\Gamma} := J(i(\Gamma), j(\mathbb{P}^n))$ is a linear subspace of $\mathbb{P}^{2n+1}$, the linear subspace $\widetilde{\Gamma} \cap \Lambda$ has codimension $(n+1+e)$ (in $\mathbb{P}^{2n+1}$) and it intersects transversely (check it) the closed $J(i(X), j(Y)) \subset \mathbb{P}^{2n+1}$ of dimension $(n+1+e)$. Thus

$$|\widetilde{\Gamma} \cap \Lambda \cap J(i(X), j(Y))| = \deg J(i(X), j(Y)) = \deg X \cdot \deg Y. \tag{6.1.5}$$

(The second equality follows from Proposition 6.1.5.) Isomorphism (6.1.3) defines a bijective correspondence between $\Gamma \cap X \cap Y$ and $\widetilde{\Gamma} \cap \Lambda \cap J(i(X), j(Y))$: thus (6.1.4) and (6.1.5) give that $\deg(X \cap Y) = \deg X \cdot \deg Y$. □

We will apply Bèzout's Theorem in order to compute of the number of flexes of a *plane curve* i.e. a hypersurface $C \subset \mathbb{P}^2$. First we go through a couple of definitions. Let $X \subset \mathbb{A}^n$ be a hypersurface: thus

$$I(X) = (f), \qquad f \in \mathbb{K}[z_1, \ldots, z_n].$$

Let $p \in \mathbb{A}^n$ and $L \subset \mathbb{A}^n$ be a line containing p.

Definition 6.1.6. The *multiplicity of intersection of X and L at p*, denoted $\mathrm{mult}_p(L \cap X)$ is the multiplicity with which $f|_L$ vanishes at p.

Thus $\mathrm{mult}_p(L \cap X) \geqslant 1$ if and only if $p \in X$, $\mathrm{mult}_p(L \cap X) \geqslant 2$ if and only if L belongs to the embedded affine tangent space $p + T_pX$. Now let $X \subset \mathbb{P}^n$ be a hypersurface, $p \in \mathbb{P}^n$ and $L \subset \mathbb{P}^n$ be a line containing p. One defines the multiplicity of intersection of X and L at p by choosing a standard open affine space $\mathbb{P}^n_\Phi$ containing p and setting $\mathrm{mult}_p(L \cap X) := \mathrm{mult}_p(L_\Phi \cap X_\Phi)$ - this makes sense because $\mathrm{mult}_p(L_\Phi \cap X_\Phi)$ is independent of Φ.

Definition 6.1.7. A curve $C \subset \mathbb{P}^2$ *has an inflection point at* p (or p is a *flex* of C) if there exists a line $L \subset \mathbb{P}^2$ such that $\mathrm{mult}_p(L \cap C) \geqslant 3$.

Remark 6.1.8. An easy local computation gives that $p \in C$ is a flex of C if and only if one of the following holds:

1. C is smooth at p and $\mathrm{mult}_p((p + T_pC) \cap C \geqslant 3$.
2. C is singular at p.

Definition 6.1.9. A *flex* p of a curve $C \subset \mathbb{P}^2$ is *ordinary* if C is smooth at p and moreover $\mathrm{mult}_p((p + T_pC) \cap C = 3$.

Remark 6.1.10. Let $C_d := V(Z_0^d - Z_1^d - Z_2^d) \subset \mathbb{P}^2_{\mathbb{C}}$. Let $p = (1, 1, 0)$. Then p is a flex of C_d if and only if $d \geqslant 3$: it is ordinary if and only if $d = 3$.

Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a curve. Let $I(C) = (F)$. We let

$$H_F := \det \begin{pmatrix} \frac{\partial F}{\partial Z_0^2} & \frac{\partial F}{\partial Z_0 \partial Z_1} & \frac{\partial F}{\partial Z_0 \partial Z_2} \\ \frac{\partial F}{\partial Z_1 \partial Z_0} & \frac{\partial F}{\partial Z_1^2} & \frac{\partial F}{\partial Z_1 \partial Z_2} \\ \frac{\partial F}{\partial Z_2 \partial Z_0} & \frac{\partial F}{\partial Z_2 \partial Z_1} & \frac{\partial F}{\partial Z_2^2} \end{pmatrix}$$

(H_F is the Hessian curve of F.) Let $d := \deg F$; then $H_F \in \mathbb{C}[Z_0, Z_1, Z_2]_{3(d-2)}$. One should notice that the locus $V(H_F)$ depends on C and *not* on the homogeneous coordinates (needed to make snese of the partial derivatives).

Proposition 6.1.11. *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a curve and let F be generator of the homogeneous ideal $I(C)$. The following hold:*

1. *The set of flexes of C is equal to $C \cap V(H_F)$.*
2. *The set of flexes of C is finite unless C contains a line.*
3. *If all flexes of C are ordinary then the number of flexes is equal to* $3 \deg C \cdot (\deg C - 2)$.

Proof. (1): Let $p \in C$. We must prove that $p \in V(H_F)$ if and only if p is a flex of C. We may assume that $p = [1, 0, 0]$. Since $p \in C$ we have

$$F = Z_0^{d-1} f_1 + Z_0^{d-2} f_2 + \ldots + f_d, \qquad f_i \in \mathbb{C}[Z_1, Z_2]_i. \tag{6.1.6}$$

It follows that

$$H_F(1, x_1, x_2) = \det \begin{pmatrix} \sum_{i=1}^d (d-i)(d-i-1) f_i & \sum_{i=1}^d (d-i) \frac{\partial f_i}{\partial x_1} & \sum_{i=1}^d (d-i) \frac{\partial f_i}{\partial x_2} \\ \sum_{i=1}^d (d-i) \frac{\partial f_i}{\partial x_1} & \sum_{i=1}^d \frac{\partial^2 f_i}{\partial x_1^2} & \sum_{i=1}^d \frac{\partial^2 f_i}{\partial x_1 \partial x_2} \\ \sum_{i=1}^d (d-i) \frac{\partial f_i}{\partial x_2} & \sum_{i=1}^d \frac{\partial^2 f_i}{\partial x_1 \partial x_2} & \sum_{i=1}^d \frac{\partial^2 f_i}{\partial x_2^2} \end{pmatrix} \tag{6.1.7}$$

Suppose that $p \in \operatorname{sing} C$. Then the first row (and first column) of $H_F(1,0,0)$ vanish and hence $p \in V(H_F)$. Next suppose that C is smooth at p. We may assume that $f_1 = Z_2$. It follows that $H_F(1,0,0) = \partial^2 f_2(0,0)/\partial Z_1^2$. Thus $H_F(1,0,0) = 0$ if and only if p is a flex of C. (2): It is clear that if C contains a line then every point of the line is a flex of C. Now suppose that the set of flexes of C is infinite. By Item (1) the set of flexes of C is a closed subset of C: it follows that it contains an irreducible component C_0 of C. The generic point of C_0 is smooth and hence locally (away from the finite set $C_0 \cap \operatorname{sing} C$) there exist affine coordinates (x,y) such that C_0 is the graph of a holomorphic function $\varphi(x)$. Since every point of C_0 is a flex we get that $\varphi'' = 0$ and hence C_0 is a line. (3): Let $p \in C$ be a flex. We may assume that $p = [1,0,0]$ and hence (6.1.6) holds. The curve C is smooth at p because the flex is ordinary: thus we may assume that $f_1 = Z_2$. Since p is an ordinary flex of C we have

$$f_2 = a_1 Z_1 Z_2 + a_2 Z_2^2, \qquad f_3 = b_0 Z_1^3 + b_1 Z_1^2 Z_2 + b_2 Z_1 Z_2^2 + b_3 Z_2^3, \quad b_0 \neq 0.$$

By (6.1.7) we get that

$$H_F(1,x_1,0) \equiv \det \begin{pmatrix} 0 & 0 & (d-1)+(d-2)a_1x_1 \\ 0 & 6b_0x_1 & * \\ (d-1)+(d-2)a_1x_1 & * & * \end{pmatrix} \pmod{x_1^2}.$$

Thus $H_F(1,x_1,0) \equiv 6(d-1)^2 b_0 x_1 \pmod{x_1^2}$. It follows that H_F has no multiple factors and hence $V(H_F)$ is a curve of degree equal to $\deg H_F = 3(\deg C - 2)$ and the intersection of the curves C and $V(H_F)$ is transverse. By Bèzout we get that

$$|\{P \in C \mid p \text{ is a flex of } C\}| = 3\deg C \cdot (\deg C - 2).$$

□

Corollary 6.1.12. *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth cubic curve. Then C has exactly 9 flexes.*

Proof. By Proposition 6.1.12 it suffices to show that every flex of C is ordinary. Let p be a flex of C. Then C is smooth at p by hypothesis. Let $L := \mathbf{T}_p C$: then $\operatorname{mult}_p(L \cap C) \geqslant 3$ because p is a flex of C. Let $I(C) = (F)$: thus $0 \neq F \in \mathbb{C}[Z_0,Z_1,Z_2]_3$. We have $L = \mathbb{P}(U)$ where $U \subset \mathbb{C}^3$ is a vector subspace of dimension 2. The restriction $F|_U$ is a degree-3 polynomial function which is non-zero because the curve C is irreducible (see Example 3.4.6). Thus $\operatorname{mult}_p(L \cap C) \leqslant 3$; since $\operatorname{mult}_p(L \cap C) \geqslant 3$ we get that $\operatorname{mult}_p(L \cap C) = 3$ i.e. p is an ordinary flex. □

6.2 Cubic surfaces

We will prove the following classical result.

Theorem 6.2.1. *A* smooth *cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ contains 27 lines.*

The proof will be given after a series of preliminary results. Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be a surface: we let

$$F_1(S) := \left\{ L \in \operatorname{Gr}(1, \mathbb{P}^3_{\mathbb{C}}) \mid L \subset S \right\}.$$

Proposition 6.2.2. *Let $G \in \mathbb{K}[Z_0,Z_1,Z_2]_3$ be such that $V(G)$ is a smooth cubic curve. The cubic surface $S := V(Z_3^3 - G) \subset \mathbb{P}^3$ is smooth*[1] *and it contains 27 lines.*

Proof. Let $F := (z_3^3 - G)$. Then $V(\partial F/\partial Z_0, \ldots, \partial F/\partial Z_3) = \varnothing$ because $V(G)$ is a smooth cubic curve. It follows that S is smooth. Since $p_0 := [0,0,0,1]$ does not belong to S the projection from p_0 defines a regular map

$$\begin{array}{ccc} S & \xrightarrow{\pi} & V(Z_3) \\ [Z_0,Z_1,Z_2,Z_3] & \mapsto & [Z_0,Z_1,Z_2] \end{array}$$

[1] Here char $\mathbb{K} \neq 3$.

(We drop the last (zero) coordinate of points of $V(Z_3)$.) Suppose that $L \subset S$ is a line: we claim that $\pi(L)$ is an inflexional tangent of the cubic curve $V(G)$. The key point is the following: S has an automorphism of order 3 namely

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S \\ [Z_0, Z_1, Z_2, Z_3] & \mapsto & [Z_0, Z_1, Z_2, \omega Z_3] \end{array}$$

where $\omega = \exp(2\pi\sqrt{-1}/3)$. Notice that $\pi \circ \varphi = \pi$ and that the cubic curve $V(G)$ is the fixed locus of φ. Let $L \subset S$ be a line. The cubic curve $V(G)$ is irreducible becuse it is smooth - see Example 3.4.6 - and hence L is not contained in $V(Z_3)$. It follows that $L \cap V(Z_3)$ contains a single point, call it q. Let's show that

$$\pi(L) \cap V(G) = \{q\}. \tag{6.2.1}$$

Let $\Lambda_L = \langle p_0, L\rangle$ be the plane spanned by p_0 and L: thus $\pi(L) = \Lambda_L \cap V(Z_3)$. The lines $\varphi(L)$ and $\varphi^2(L)$ are contained in Λ_L and $L, \varphi(L), \varphi^2(L)$ are pairwise distinct because $L \not\subset V(Z_3)$. It follows that $\Lambda_L \cap S = L \cup \varphi(L) \cup \varphi^2(L)$: since $\{q\} = \varphi(L) \cap S = \varphi^2(L) \cap S$ we get that (6.2.1) holds. Equation (6.2.1) proves that $\pi(L)$ is an inflexional tangent of the cubic curve $V(G)$. Thus we have a map

$$\begin{array}{ccc} F_1(S) & \xrightarrow{\rho} & \{R \subset V(Z_3) \mid R \text{ an inflexional tangent of } V(G)\} \\ L & \mapsto & \pi(L) \end{array}$$

By Corollary 6.1.12 we know that $V(G)$ has 9 inflexional tangents and hence in order to finish the proof it will suffice to show that ρ is surjective and each fiber has cardinality 3. Let R an inflexional tangent of $V(G)$. Let q be the inflexion point of $V(G)$ on R: thus $R \cap V(G) = \{q\}$. Let us extend Z_3 (on Λ_R) to homogeneous coordinates Y_0, Y_1, Z_3 on Λ_R such that $q = [1,0,0]$. The cubic curve $S \cap \Lambda_R$ has equation $\alpha Z_3^3 - \beta Y_1^3$ for a certain $(0,0) \neq (\alpha, \beta) \in \mathbb{C}^2$. We have $\alpha \neq 0$ because S does not contain the point $[0,0,0,1]$ and we have $\beta \neq 0$ because $V(G)$ contains no lines: it follows that $S \cap \Lambda_R$ consists of 3 distinct lines as claimed. □

Next let $\mathscr{U}_d \subset \mathbb{P}(\mathbb{K}[Z_0, Z_1, Z_2, Z_3]_d)$ be the subset of $[F]$ where F has no multiple factors: as is easily checked $\mathscr{U}_d$ is a dense open subset of $\mathbb{P}(\mathbb{K}[Z_0, Z_1, Z_2, Z_3]_d)$. Clearly $\mathscr{U}_d$ is the parameter space for degree-d surfaces in $\mathbb{P}^3$. Let

$$\mathscr{R}_d := \{(L, S) \in \mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}}) \times \mathscr{U}_d \mid L \subset S\}.$$

We have two projections

$$\begin{tikzcd} & \mathscr{R}_d \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ \mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}}) & & \mathscr{U}_d \end{tikzcd}$$

Notice that

$$\pi(\mathscr{R}_d) = \{S \in \mathscr{U}_d \mid S \text{ contains a line}\}. \tag{6.2.2}$$

Claim 6.2.3. *Keep notation as above. Then $\mathscr{R}_d$ is closed in $\mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}}) \times \mathscr{U}_d$, moreover it is smooth irreducible of codimension $(d+1)$.*

Proof. Let

$$\mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}})_{I_0} := \{L \in \mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}}) \mid L \cap V(Z_0, Z_1) = \varnothing\}.$$

Then $\mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}})_{I_0}$ is one of the principal open subsets (isomorphic to $\mathbb{A}^4$) that cover $\mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}})$ - see Section ??. We will describe $\rho^{-1}\mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}})_{I_0}$. First we recall how to describe lines that belong to $\mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}})_{I_0}$. Let $r = (r_1, \ldots, r_4) \in \mathbb{A}^4_{\mathbb{C}}$: we let

$$L_r := \mathbb{P}(\langle (1, 0, r_1, r_2), (0, 1, r_3, r_4)\rangle).$$

Then $\mathrm{Gr}(1,\mathbb{P}^3)_{I_0}$ is the set whose elements are the lines L_r. Given $F \in \mathbb{K}[Z_0, Z_1, Z_2, Z_3]_d$ we may write

$$F(\lambda, \mu, \lambda r_1 + \mu r_3, \lambda r_2 + \mu r_4) = \sum_{i=0}^{d} A_i(r, F)\lambda^{d-i}\mu^i.$$

Clearly

$$\rho^{-1}\mathrm{Gr}(1,\mathbb{P}^3)_{I_0} = \{(L_r, V(F)) \in \mathrm{Gr}(1,\mathbb{P}^3)_{I_0} \times \mathscr{U}_d \mid 0 = A_0(r,F) = A_1(r,F) = \ldots = A_d(r,F)\}.$$

Let $A_i \colon \mathbb{A}^4 \times \mathbb{K}[Z_0, Z_1, Z_2, Z_3]_d \to \mathbb{K}$ be the function with value $A_i(r,F)$ at (r,F). Then A_i is a polynomial function and hence $\rho^{-1}\mathrm{Gr}(1,\mathbb{P}^3)_{I_0}$ is a closed subset of $\mathrm{Gr}(1,\mathbb{P}^3)_{I_0} \times \mathscr{U}_d$. Let $f_0, \ldots, f_N$ be the coefficients of F (here $N+1 = (d+3)!/3!d!$): then $A_i \in \mathbb{K}[r_1, \ldots, r_4][f_0, \ldots, f_N]_1$ i.e. A_i is homogeneous of degree 1 in $f_0, \ldots, f_N$. When we fix r the $(d+1)$ homogeneous equations in $f_0, \ldots, f_N$ are linearly independent - this is easily checked. It follows that there is an open cover $\{\mathscr{A}_{I_0,j}\}_{j \in J_0}$ of $\mathrm{Gr}(1,\mathbb{P}^3)_{I_0}$ such that

$$\rho^{-1}\mathscr{A}_{I_0,j} \cong \mathscr{A}_{I_0,j} \times \mathbb{P}^{N-d-1}_{\mathbb{C}}$$

for all $j \in J_0$. A similar picture holds for all principal open subsets of the covering of $\mathrm{Gr}(1,\mathbb{P}^3_{\mathbb{C}})$ defined in Section??. The claim follows. □

Proposition 6.2.4. *Keep notation as above and assume that $d \geqslant 3$. Let $(L,S) \in \mathscr{R}_d$ and suppose that S is smooth at* all *points of L. Then $d\pi(L,S) \colon \Theta_{(L,S)}\mathscr{R}_d \longrightarrow \Theta_S\mathscr{U}_d$ is* injective.

Proof. Choose homogeneous coordinates $Z_0, \ldots, Z_4$ such that $L = V(Z_2, Z_3)$. Let $I(S) = (F)$: thus $0 \neq F \in \mathbb{K}[Z_0, \ldots, Z_4]_d$. Throughout the present proof we will adopt the notation introduced in the proof of Claim 6.2.4. We have $L = L_0$. The proof of Claim 6.2.4 gives that $\Theta_{(L,F)} \subset \Theta_{(L,F)}\mathbb{A}^4_{\mathbb{C}} \times \mathscr{U}_d$ is given by

$$\mathrm{Ann}\langle dA_0(0,F), \ldots, dA_d(0,F)\rangle. \tag{6.2.3}$$

(This is because the differentials $dA_0(0,F), \ldots, dA_d(0,F)$ are linearly independent.) Let $v \in \ker d\pi(L,S)$. Then $v = (v_1, \ldots, v_4) \in \Theta_0\mathbb{A}^4$: since $\mathbb{A}^4$ is an affine space there exists a regular (affine if we wish so) map $\gamma \colon \mathbb{A}^1 \to \mathbb{A}^4$ such that $\gamma(0) = 0$ and $\gamma'(0) = v$. By (6.2.3) we get that

$$\begin{aligned} 0 = \frac{d}{dt}_{|t=0} & F(\lambda, \mu, \lambda\gamma_1 + \mu\gamma_3, \lambda\gamma_2 + \mu\gamma_4) = \\ & = \frac{\partial F}{\partial Z_2}(\lambda, \mu, 0, 0)(\lambda v_1 + \mu v_3) + \frac{\partial F}{\partial Z_3}(\lambda, \mu, 0, 0)(\lambda v_2 + \mu v_4). \end{aligned} \tag{6.2.4}$$

Now suppose that $v \neq 0$: we will arrive at a contradiction. Both $\frac{\partial F}{\partial Z_2}(\lambda, \mu, 0, 0)$ and $\frac{\partial F}{\partial Z_2}(\lambda, \mu, 0, 0)$ are homogeneous polynomials of degree $(d-1)$: equation (6.2.4) gives that they have a common factor because $d = \deg F \geqslant 3$. It follows that there exists $0 \neq (\lambda_0, \mu_0) \in \mathbb{C}^2$ such that

$$0 = \frac{\partial F}{\partial Z_2}(\lambda_0, \mu_0, 0, 0) = \frac{\partial F}{\partial Z_3}(\lambda_0, \mu_0, 0, 0).$$

On the other hand since $L = V(Z_2, Z_3) \subset V(F)$ there exists $G, H \in \mathbb{K}[Z_0, \ldots, Z_3]_{d-1}$ such that $F = GZ_2 + HZ_3$: it follows that

$$0 = \frac{\partial F}{\partial Z_1}(\lambda_0, \mu_0, 0, 0) = \frac{\partial F}{\partial Z_2}(\lambda_0, \mu_0, 0, 0).$$

Since F generates $I(S)$ it follows that S is singular at $[\lambda_0, \mu_0, 0, 0] \in L$: that contradicts our hypothesis. □

Proof of Theorem 6.2.1. Let $\mathscr{U}_3^0 \subset \mathscr{U}_3$ be the open subset whose elements are smooth cubic surfaces. Let $\mathscr{R}_3^0 := \pi^{-1}\mathscr{U}_3^0$ and let $\pi^0 \colon \mathscr{R}_3^0 \to \mathscr{U}_3^0$ be the restriction of π. Then $\operatorname{Im}\pi^0$ is the set of smooth cubic surfaces which contain a line: it is non-empty by Proposition 6.2.2. Since π^0 is a projective map $\operatorname{Im}\pi^0$ is a closed subset of $\mathscr{U}_3^0$. Let S be a smooth point of $\operatorname{Im}\pi^0$ and $(S, L) \in (\pi^0)^{-1}(S)$. Since $\mathscr{R}_3$ is smooth of dimension equal to $\dim \mathbb{K}[Z_0, \ldots, Z_3]_3$ (see Claim 6.2.4) and injectivity of the differential $d\pi(L, S)$ (see Proposition 6.2.4) we get that

$$\dim \mathbb{K}[Z_0, \ldots, Z_3]_3 \leqslant \dim_S \operatorname{Im}\pi^0.$$

It follows that $\operatorname{Im}\pi^0 = \mathscr{U}_3^0$ i.e. every smooth cubic surface contains a line. The map π^0 is a proper (when we consider the euclidean topology) map of smooth varieties because it is projective and it is a local homeomorphism by Proposition 6.2.4. Thus Proposition ?? gives that the number of lines on a smooth cubic surface is independent of the surface: by Proposition 6.2.2 we get that every smooth cubic contains 27 lines. □

Theorem 6.2.5. *Let $S \subset \mathbb{P}^3$ be an irreducible cubic surface; then S is rational unless possibly if it is the cone over a* smooth *cubic (plane) curve.*

Proof. Suppose that S is smooth. By Theorem 6.2.1, there exist skew lines $L, M \subset S$. We define a rational map

$$f \colon S \dashrightarrow L \times M \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

as follows. Let $p \in S \backslash (L \cup M)$: there exists a unique line R_p containing p and intersecting L *and* M. We set

$$f(p) := (R_p \cap L, R_p \cap M).$$

Let $(S \backslash (L \cup M), \phi)$ represent f. Since $F_1(S)$ is finite the generic fiber of ϕ is a single point and it follows that $\dim \phi(S \backslash (L \cup M)) = 2$; since $L \times M$ is irreducible 2-dimensional we get that f is dominant. Moreover since the generic fiber of ϕ consists of one point f is *birational.* Since $\mathbb{P}^1 \times \mathbb{P}^1$ is rational we have proved that a smooth cubic is rational. Now suppose that S is singular but is *not* a cone over a smooth cubic curve; then there exists $p_0 \in S$ of multiplicity 2. Projection from p_0 defines a birational map $\pi \colon S \dashrightarrow \mathbb{P}^2$. □

Exercises

Appendix A

Algebra à la carte

A.1 Introduction

In what follows, rings are always commutative with 1. The proofs of the results below are contained in most Algebra textbooks (e.g. Lang [**?**]).

A.2 Unique factorization

Theorem A.2.1. *Let R be a UFD. Then $R[t]$ is a UFD. Moreover a polynomial $p = a_0 t^d + a_1 t^{d-1} + \ldots + a_d$ is prime if and only if*

1. *p is prime when viewed as element of $K[t]$, where K is the field of fractions of R,*
2. *and the greatest common divisor of $a_0, a_1, \ldots, a_d$ is 1.*

Corollary A.2.2. *The ring $\mathbb{K}[x_1, \ldots, x_n]$ is a unique factorization domain.*

Proof. By induction on n. If $n = 0$, the ring is a field, and hence it is trivially a UFD. The inductive step follows from Theorem A.2.2, because $\mathbb{K}[x_1, \ldots, x_n] \cong \mathbb{K}[x_1, \ldots, x_{n-1}][t]$. □

A.3 Noetherian rings

Definition A.3.1. A (commutative unitary) ring R is *Noetherian* if every ideal of R is finitely generated.

Example A.3.2. A field K is Noetherian, because the only ideals are $\{0\} = (0)$ and $K = (1)$. The ring $\mathbb{Z}$ is Noetherian, because every ideal has a single generator.

Lemma A.3.3. *A (commutative unitary) ring R is Noetherian if and only if every ascending chain*

$$I_0 \subset I_1 \subset \ldots \subset I_m \subset \ldots$$

of ideals of R (here I_m is defiend for all $m \in \mathbb{N}$, and $I_m \subset I_{m+1}$ for all $m \in \mathbb{N}$) is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such hat $I_m = I_{m_0}$ for $m \geqslant m_0$.

Proof. Suppose that R is Noetherian. The union $I := \bigcup_{m \in \mathbb{N}} I_m$ is an ideal because the $\{I_m\}$ form a chain. By Noetherianity I is finitely generated, say $I = (a_1, \ldots, a_r)$. There exists m_0 such that $a_j \in I_{m_0}$ for $j \in \{1, \ldots, r\}$, and hence $I = I_{m_0}$. Let $m \geqslant m_0$; then $I_m \subset I$ and $I \subset I_m$, hence $I = I_m$. Thus $I_{m_0} = I_m$ for $m \geqslant m_0$.

Now suppose that every ascending chain of ideals of R is stationary. Let $I \subset R$ be an ideal. Suppose that I is *not* finitely generated. Let $a_1 \in I$. Then $(a_1) \subsetneq I$ because I is not finitely generated; let

$a_2 \in (I \backslash (a_1))$. Then $(a_1, a_2) \subsetneq I$ because I is not finitely generated. Iterating, we get a non stationary chain of ideals (contained in I)

$$(a_1) \subsetneq (a_1, a_2) \subsetneq \ldots \subsetneq (a_1, \ldots, a_m) \subsetneq$$

This is a contradiction. □

Example A.3.4. The ring $\mathrm{Hol}(\mathbb{K})$ of entire functions of one variable is *not* Noetherian. In fact let $f_m \in \mathrm{Hol}(\mathbb{K})$ be defined by

$$f_m(z) := \prod_{n=m}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \qquad m \geqslant 1.$$

Then $(f_m) \subsetneq (f_{m+1})$. Thus $(f_1) \subset (f_2) \subset \ldots \subset (f_m) \subset \ldots$ is a non-stationary ascending chain of ideals, and hence $\mathrm{Hol}(\mathbb{K})$ is not Noetherian by Lemma A.3.3.

Theorem A.3.5. *Let R be a Noetherian commutative ring. Then $R[t]$ is Noetherian.*

Proof. For a non zero $f \in R[t]$, we let $\ell(f)$ be the *leading coefficient of* f, i.e. if $f = \sum_{i=0}^{m} c_i t^i$ with $c_m \neq 0$, then $\ell(f) = c_m$.

Let $I \subset R[t]$. We must prove that I is finitely generated. If $I = (0)$ there is nothing to prove and hence we may assume $I \neq (0)$. Thus the set

$$\ell(I) := \{\ell(f) \mid 0 \neq f \in I\}$$

is non-empty and it makes sense to define

$$J := \langle \ell(I) \rangle \subset R$$

as the ideal of R generated by $\ell(I)$. By hypothesis J is finitely generated: $J = (c_1, \ldots, c_s)$. Since J is generated by $\ell(I)$ we may assume that each generator is the leading coefficient of a polynomial in I, i.e. for each $1 \leqslant i \leqslant s$ there exists $f_i \in I$ such that $\ell(f_i) = c_i$. Let

$$d := \max_{1 \leqslant i \leqslant s} \{\deg f_i\}.$$

Let $H := I \cap \{f \in R[t] \mid \deg f \leqslant d\}$. Then H is a submodule of $\{f \in R[t] \mid \deg f \leqslant d\} \simeq R^{d+1}$ (as R-modules). Since R is Noetherian every submodule of R^{d+1} is finitely generated (argue by induction on d; if $d = 0$ it holds by definition of Noetherian ring, if $d > 0$ consider the projection $R^{d+1} \to R$) and hence

$$H = (g_1, \ldots, g_t).$$

Let us prove that

$$I = (f_1, \ldots, f_s, g_1, \ldots, g_t).$$

In fact let $f \in I$. If $\deg f \leqslant d$ then $f \in H$ and hence $f \in (g_1, \ldots, g_t) \subset (f_1, \ldots, f_s, g_1, \ldots, g_t)$. Now suppose that $\deg f > d$. Then $\ell(f) = \sum_{i=1}^{s} a_i c_i$. Let

$$h := f - \sum_{i=1}^{s} a_i t^{\deg f - \deg f_i} f_i.$$

Then $\deg h < \deg f$. Since $\sum_{i=1}^{s} a_i t^{\deg f - \deg f_i} f_i \in (f_1, \ldots, f_s, g_1, \ldots, g_t)$ it suffices to prove that $h \in I$. If $\deg h \leqslant d$ we are done, otherwise we iterate until we get down to a polynomial of degree less or equal to d. □

Theorem A.3.6 (Hilbert's basis Theorem)**.** *Every ideal of $\mathbb{K}[x_1, \ldots, x_n]$ is finitely generated.*

Proof. By induction on n. If $n = 0$, the ring is a field, and hence is Noetherian. The inductive step follows from Theorem A.3.5, because $\mathbb{K}[x_1, \ldots, x_n] \cong \mathbb{K}[x_1, \ldots, x_{n-1}][t]$. □

A.4 The Nullstellensatz

If $Y \subset \mathbb{A}^n$ is a subset, we let $I(Y) := \{f \in \mathbb{K}[z_1, \ldots, z_n] \mid f|_Y = 0\}$. We recall that the *radical* of an ideal I ina ring R, is the set of elements $a \in R$ such that $a^m \in I$ for some $m \in \mathbb{N}$. As is easily checked, the radical is an ideal; it is denoted by $\sqrt{I}$,

Theorem A.4.1 (Hilbert's Nullstellensatz, Chapter X of [?]). *Let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be an ideal. Then $I(V(I)) = \sqrt{I}$.*

Before discussing the proof of the Nullstellensatz, we introduce some notation. For $(a_1, \ldots, a_n) \in \mathbb{A}^n$, let

$$\mathfrak{m}_a := (z_1 - a_1, \ldots, z_n - a_n) = \{f \in \mathbb{K}[z_1, \ldots, z_n] \mid f(a_1, \ldots, a_n) = 0\}. \tag{A.4.1}$$

Notice that $\mathfrak{m}_a$ is the kernel of the surjective homomorphism

$$\begin{array}{ccc} \mathbb{K}[z_1, \ldots, z_n] & \xrightarrow{\phi} & \mathbb{K} \\ f & \mapsto & f(a_1, \ldots, a_n), \end{array}$$

and hence is a maximal ideal. The Nullstellensatz is a consequence of the following result.

Proposition A.4.2. *An ideal $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$ is maximal if and only if there exists $(a_1, \ldots, a_n) \in \mathbb{A}^n$ such that $\mathfrak{m} = \mathfrak{m}_a$.*

Proof for uncountable $\mathbb{K}$. We know that $\mathfrak{m}_a$ is maximal. Now suppose that $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$ is a maximal ideal. Let

$$\mathbb{K}[z_1, \ldots, z_n] \xrightarrow{\phi} \mathbb{K}[z_1, \ldots, z_n]/\mathfrak{m} =: E$$

be the quotient map. Notice that $\mathfrak{m} \cap \mathbb{K} = \{0\}$ because $\mathfrak{m} \neq (1)$. Thus $\phi(\mathbb{K})$ is a copy of $\mathbb{K}$ and hence E is a field extension of $\mathbb{K}$. For $i \in \{1, \ldots, n\}$ let $\overline{z}_i := \phi(z_i)$. We claim that

$$\text{for all } i \in \{1, \ldots, n\} \text{ there exists } a_i \in \mathbb{K} \text{ (meaning } a_i \in \phi(\mathbb{K})\text{) such that } \overline{z}_i = a_i. \tag{A.4.2}$$

In fact suppose that $\overline{z}_i \notin \mathbb{K}$. Let $c \in \mathbb{K}$; since $\overline{z}_i \neq c$ and E is a field $(\overline{z}_i - c)^{-1}$ exists. The field E is a quotient of $\mathbb{K}[z_1, \ldots, z_n]$ - a $\mathbb{K}$-vector space of countable dimension - thus E as vector space over $\mathbb{K}$ has a countable basis. Since $\mathbb{K}$ is uncountable we get that $\{(\overline{x}_i - c)^{-1}\}_{c \in \mathbb{K}}$ is a set of linearly dependent elements, and hence there exist pairwise distinct complex numbers $c_1, \ldots, c_s \in \mathbb{K}$ and $\lambda_1, \ldots, \lambda_s \in \mathbb{K}^*$ such that

$$\sum_{h=1}^{s} \lambda_h (\overline{z}_i - c_h)^{-1} = 0. \tag{A.4.3}$$

Multiplying both sides by $\prod_{j=1}^{s} (\overline{z}_i - c_j)$ we get that

$$\sum_{h=1}^{s} \lambda_h \prod_{j \neq h}^{s} (\overline{z}_i - c_j) = 0. \tag{A.4.4}$$

The polynomial $\varphi \in \mathbb{K}[t]$ defined by

$$\varphi := \sum_{h=1}^{s} \lambda_h \prod_{j \neq h}^{s} (t - c_j)$$

is non-zero. In fact $\varphi(c_1) = \lambda_1 \prod_{1 < j \leqslant s}^{s} (c_1 - c_j) \neq 0$. B (A.4.4) we have $\varphi(\overline{z}_i) = 0$; since $\varphi \neq 0$, $\overline{z}_i$ is algebraic over $\mathbb{K}$, and hence $\overline{z}_i \in \mathbb{K}$ because $\mathbb{K}$ is algebraically closed. This is a contradiction, and hence (A.4.2) holds. Thus

$$(z_i - a_i) \in \ker \phi = \mathfrak{m}, \qquad i = 1, \ldots, n.$$

Since $\mathfrak{m}_a$ is generated by $(z_1 - a_1), \ldots, (z_n - a_n)$ it follows that $\mathfrak{m}_a \subset \mathfrak{m}$. The ideal $\mathfrak{m}_a$ is maximal and so is $\mathfrak{m}$: this implies that $\mathfrak{m} = \mathfrak{m}_a$. $\square$

Corollary A.4.3 (Weak Nullstellensatz)**.** *Let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be an ideal. Then $V(I) = \varnothing$ if and only if $I = (1)$.*

Proof. If $I = (1)$, then $V(I) = \varnothing$. Assume that $V(I) = \varnothing$. Suppose that $I \neq (1)$. Then there exists a maximal ideal $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$ containing I. Since $I \subset \mathfrak{m}$, $V(I) \supset V(\mathfrak{m})$. By Proposition A.4.2 there exists $a \in \mathbb{K}^n$ such that $\mathfrak{m} = \mathfrak{m}_a$ and hence $V(\mathfrak{m}) = V(\mathfrak{m}_a) = \{(a_1, \ldots, a_n)\}$. Thus $a \in V(I)$ and hence $V(I) \neq \varnothing$. This is a contradiction, and hence $I = (1)$. □

Proof of Hilbert's Nullsetellensatz (Rabinowitz's trick). Let $f \in I(V(I))$. By Hilbert's basis theorem $I = (g_1, \ldots, g_s)$ for $g_1, \ldots, g_s \in \mathbb{K}[z_1, \ldots, z_n]$. Let $J \subset \mathbb{K}[z_1, \ldots, z_n, w]$ be the ideal

$$J := (g_1, \ldots, g_s, f \cdot w - 1).$$

Since $f \in I(V(I))$ we have $V(J) = \varnothing$ and hence by the Weak Nullstellensatz $J = (1)$. Thus there exist $h_1, \ldots, h_s, h \in \mathbb{K}[x_1, \ldots, x_n, y]$ such that

$$\sum_{i=1}^{s} h_i g_i + h\,(f \cdot w - 1) = 1.$$

Replacing w by $1/f(z)$ in the above equality we get

$$\sum_{i=1}^{s} h_i \left(z, \frac{1}{f(z)}\right) g_i(z) = 1. \tag{A.4.5}$$

Let $d >> 0$: multiplying both sides of (A.4.5) by f^d we get that

$$\sum_{i=1}^{s} \overline{h}_i\,(z)\, g_i(z) = f^d(z), \quad \overline{h}_i \in \mathbb{K}[z_1, \ldots, z_n].$$

Thus $f \in \sqrt{I}$. □

If $\mathbb{K}$ is not algebariaclly closed, then the statement of Theorem A.4.1 is no longer true. For example, if $\mathbb{K} = \mathbb{R}$ and $I := (x^2 + 1) \subset \mathbb{R}[x]$, then $V(I) = \varnothing$ but $I \neq (1)$. There is a modified version of Proposition A.4.2 which holds for an arbitrary field k: it states that if $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ is a maximal ideal then $k[x_1, \ldots, x_n]/\mathfrak{m}$ is an algebraic extension of k, see Chapter X of [**?**].

A.5 Extensions of fields

An extension of fields $F \subset E$ is *algebraic* if every $\alpha \in E$ is the root of a non zero polynomial $\psi \in F[z]$. If this is the case, the set of polynomials vanishing on α is a non zero ideal $F[z]$, and hence it is generated by a unique monic poylnomial φ, which is the *minimal polynomial of* α *over* F. Of course φ is irreducible, hence prime. The subfield of F generated by F and α is isomorphic to the quotient $F[z]/(\varphi)$.

An extension is an *algebraic closure of* F, if it is algebraic over F, and every polynomial in $F[z]$ has a root in E.

Theorem A.5.1 (Chapter VII in [**?**])**.** *An algebraic closure exists, and is unique up to isomorphism, i.e. if E_1, E_2 are two algebriac closures, there exists an isomorphism $E_1 \xrightarrow{\sim} E_2$ which is the identity on F.*

One denotes "the" algebraic closure of F by F^a, or by $\overline{F}$. Notice that a non costant polynomial in $F[z]$ decomposes in $\overline{F}$ as a product of polynomials of degree 1 (it has a root, hence it is divisible by a linear term, if the quotient is not constant it has a root hence it is divisible...)

Let $[E : F]$ be the dimension of E as vector space over F - the *degree of* E *over* F. Notice that if $[E : F]$ is finite, then E is an algebraic extension of F. Suppose that E is algebraic over F. One defines another degree of E over F as follows. Let $\sigma\colon F \hookrightarrow L$ be an embedding into a field which is an algebraic closure of $\sigma(F)$. An extension of σ to E is an embedding $\widetilde{\sigma}\colon E \hookrightarrow L$ such that $\widetilde{\sigma}_{|F} = \sigma$. The number of such extensions is independent of the embedding $\sigma\colon F \hookrightarrow L$, and is the *separable degree of E over F* - one denotes it by $[E : F]_s$.

Example A.5.2. Let $\varphi \in F[z]$ be an irreducible monic polynomial, and let $E = F[z]/(\varphi)$. Let $\alpha \in E$ be the class of z: by construction the minimal polynomial of α is equal to φ.

Let $\sigma\colon F \hookrightarrow L$ be an embedding into a field which is an algebraic closure of $\sigma(F)$. An extension of σ to E is determined by its value on α, and moreover such value can be chosen to be any root of φ in L. Hence the separable degree of E over F is the number of roots of φ in $\overline{F}$ (*not* counted with multiplicity).

If the formal derivative $\frac{d\varphi}{dz}$ is not the zero polynomial, then since its degree is strictly smaller than $\deg\varphi$, and φ is prime, the ideal $(\varphi, \frac{d\varphi}{dz})$ is equal to $F[z]$, and thus $\varphi, \frac{d\varphi}{dz}$ have no common roots. It follows that all the roots of φ have multiplicity 1, and the separable degree of E over F is equal to $\deg\varphi$, which is also the degree of E over F. Hence in this case $[E : F] = [E : F]_s$.

The formal derivative $\frac{d\varphi}{dz}$ is the zero polynomial only if $\operatorname{char} F = p > 0$, and $\varphi = \psi(z^p)$, where $\psi \in F[z]$, i.e. all monomials appearing in f have exponent a multiple of p. Iterating, we may write $\varphi = \rho(z^{p^r})$, where $\rho \in F[z]$ is such that $\frac{d\rho}{dz}$ is not the zero polynomial. Hence the numer of roots of φ is equal to the degree of $h\rho$, and thus $[E : F]_s = \deg\rho$.

Since $[E : F] = \deg\varphi = p^r \cdot \deg\rho = [E : F]_s$, we see (at least in this case) that the separable degree divides the degree. Moreover, let $\beta = \alpha^{p^r}$. Then $E^s := F[\beta]$ is a separable extension of F such that $[E^s : F] = [E : F]_s$, and the extension $E \supset E^s$ is obtained by adjoining p-th roots, and iterating.

The result below states that the example given above is typical.

Theorem A.5.3 (Chapter VII in [**?**]). *Let $E \supset F$ be a finite extension of fields, i.e. $[E : F]$ is finite. There exists a maximal separable extension $E^s \supset F$, containing all subfields of E over F which are separable. The separable degree $[E : F]_s$ is equal to the degree of the extension $E^s \supset F$. The extension $E^s \supset F$ has a primitive element, i.e. there exists $\beta \in E^s$ generating E^s over F. Suppose that $E^s \neq E$; then $\operatorname{char} F = p > 0$, and if $\alpha \in E$, the minimal polynomial of α over E^s is equal to $z^{p^r} - \gamma$ for some $r \geqslant 0$, and $\gamma \in E^s$.*

Example A.5.4. Let $E = \mathbb{F}_p(w, z)$, and let $F = \mathbb{F}_p(w^p, z^p)$. Then $E^s = F$ (in this case one says that $E \supset F$ is a *purely inseparable* extension, and there is no primitive elemnt of E over F.

Elements $\alpha_1, \ldots, \alpha_n \in E$ are *algebraically dependent* over F is there exists a non zero polynomial $\Phi \in F[z_1, \ldots, z_n]$ such that $\Phi(\alpha_1, \ldots, \alpha_n) = 0$ (strictly speaking, we should say that the set $\{\alpha_1, \ldots, \alpha_n\}$ is algebraically dependent over F). A collection $\{\alpha_i\}_{i \in I}$ of elements of E is *algebraically independent* over F if there does *not* exist a non empty finite $\{i_1, \ldots, i_n\} \subset I$ such that $\alpha_{i_1}, \ldots, \alpha_{i_n}$ are algebraically dependent (with the usual abuse of language, we also say that the α_i's are algebraically independent). A *transcendence basis* of E over F is a maximal set of algebraically independent elements of E over F. There always exists a transcendence basis, by Zorn's Lemma. One proves that any two transcendence bases have the same cardinality, which is the *transcendence degree* of E over F; we denote it by $\operatorname{Tr.deg}_F(E)$. An extension is algebraic if and only if its transcendence degree is zero.

Every finitely generated extension $E \supset F$ can be obtained as a composition of extensions $F \subset K$ and $K \subset E$, where $F \subset K$ is a *purely transcendental extension*, i.e. there exists a transcendence basis $\{\alpha_1, \ldots, \alpha_n\}$ of K over F such that $K = F(\alpha_1, \ldots, \alpha_n)$ (thus $F(\alpha_1, \ldots, \alpha_n)$ is isomorphic to the field of rational functions in n indeterminates with coefficients in F), and $F \subset K$ is a finitely generated algebraic extension.

Definition A.5.5. Let $E \supset F$ be an extension of fields. A transcendence basis $\{\alpha_1, \ldots, \alpha_n\}$ of E over F is *separating* if E is a separable extension of the subfield $F(\alpha_1, \ldots, \alpha_n)$. The extension $E \supset F$ is *separably generated* if there exists a separating transcendence basis of E over F.

Theorem A.5.6 (Thm 26.2 in [**?**]). *If $\mathbb{K}$ is an algebrically closed field, any finitely generated extension $E \supset \mathbb{K}$ is separably generated.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be a transcendence basis of E over $\mathbb{K}$. Hence the field $F := \mathbb{K}(\alpha_1, \ldots, \alpha_n)$ is isomorphic to the field of rational functions in n indeterminates, and $E \supset F$ is a finite extension. Let $\beta_1, \ldots, \beta_r$ be elements of E algebraic over F, which generate E over F. If all such β_i's are separable over F (i.e. the subfield of E generated by F and β_i is separable over F), then E is separable over F (see Chapter VII in [**?**]).

Suppose that one of the β_i's is not separable over F. Then $\operatorname{char} F = \operatorname{char} \mathbb{K} = p > 0$. We may reorder the β_i's so that each of $\beta_1, \ldots, \beta_s$ is separable over F, and each of the $\beta_{s+1}, \ldots, \beta_r$ is not separable over F. We find suitable replacements of the α_j's so that E is a separable extension of the subfield generated by the new transcendence basis. Since β_{s+1} is algebraic over F, there exists a polynomial $\Phi \in \mathbb{K}[z_1, \ldots, z_{n+1}]$ such that

$$\Phi(\alpha_1, \ldots, \alpha_n, \beta_{s+1}) = 0.$$

We may, and will, assume that Φ is irreducible. We claim that there exists $i \in \{1, \ldots, n\}$ such that $\frac{\partial \Phi}{\partial z_i} \neq 0$. In fact, suppose the contrary. Then all partial derivatives of Φ are zero, because β_{s+1} is not separable over F (see Example A.5.2). Write

$$\Phi = \sum_{I \in \mathscr{I}} a_I z^I,$$

where $\mathscr{I}$ is a set of multiindices, and we assume that $a_I \neq 0$ for every $I \in \mathscr{I}$. Since $\frac{\partial \Phi}{\partial z_i} \neq 0$ for all $i \in \{1, \ldots, n+1\}$, it follows that each $I \in \mathscr{I}$ is equal to pJ, for a multiindex J. On the other hand there exists a (unique) p-th root of a_I, because $\mathbb{K}$ is algebraically closed. It follows that $\Phi = \Psi^p$. This is a contradiction because Φ is irreducible, and hence we have proved that there exists $i \in \{1, \ldots, n\}$ such that $\frac{\partial \Phi}{\partial z_i} \neq 0$. Then α_i is algebraic and separable over $F' := \mathbb{K}(\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_n, \beta_{s+1})$. Thus $\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_n, \beta_{s+1}$ is a new transcendence basis of E over $\mathbb{K}$, and E is generated over F by $\beta_1, \ldots, \beta_s, \alpha_i, \beta_{s+2}, \ldots, \beta_r$. Moreover, each of $\beta_1, \ldots, \beta_s, \alpha_i$ is separable over F'. Iterating, we get the Theorem. □

Corollary A.5.7. *Let $E \supset \mathbb{K}$ be a finitely generated extension of fields, and suppose that $\mathbb{K}$ is algebraically closed. Let m be the transcendence degree of E over $\mathbb{K}$. Then there exists a prime polynomial $P \in \mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]$ such that E (as extension of $\mathbb{K}$) is isomorphic to the field $\mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]/(P)$.*

A.6 Derivations

Let R be a ring (commutative with unit), and let M be an R-module.

Definition A.6.1. A *derivation from R to M* is a map $D\colon R \to M$ such that additivity and Leibinitz' rule hold, i.e. for all $a, b \in R$,

$$D(a+b) = D(a) + D(b), \quad D(ab) = bD(a) + aD(b).$$

If k is a field and R is a k-algebra a *k-derivation* (or *derivation over k*) $D\colon R \to M$ is a derivation such that $D(c) = 0$ for all $c \in k$. We let $\operatorname{Der}(R, M)$ be the set of derivations from R to M. If R is a k-algebra we let $\operatorname{Der}_k(R, M) \subset \operatorname{Der}(R, M)$ be the subset of k-derivations.

Example A.6.2. Let k be a field, and let $f = \sum_I a_I z^I$ be a polynomial in $k[z_1, \ldots, z_n]$, where the summation is over multiindices I, $a_I \in \mathbb{K}$ for every I, and a_I is almost always zero. The formal derivative of f with respect to z_m is defined by the familar formula

$$\frac{\partial f}{\partial z_m} = \sum_{I \text{ s.t. } i_m > 0} i_h a_I z_1^{i_1} \cdot \ldots \cdot z_{m-1}^{i_{m-1}} \cdot z_m^{i_m - 1} \cdot z_{m+1}^{i_{m+1}} \cdot \ldots z_n^{i_n}. \tag{A.6.1}$$

The map

$$\begin{array}{ccc} k[z_1, \ldots, z_n] & \xrightarrow{\frac{\partial}{\partial z_m}} & k[z_1, \ldots, z_n] \\ f & \mapsto & \frac{\partial f}{\partial z_m} \end{array} \tag{A.6.2}$$

is a k-derivation of the k algebra to istelf. We claim that $\mathrm{Der}_k(k[z_1,\ldots,z_n],k[z_1,\ldots,z_n])$ is freely generated (as $k[z_1,\ldots,z_n]$ module) by $\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}$. In fact there is no relation between $\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}$ because $\frac{\partial z_j}{\partial z_m}=\delta_{jm}$, and moreover, given a k derivation

$$D\colon k[z_1,\ldots,z_n]\to k[z_1,\ldots,z_n]$$

we have $D=\sum_{m=1}^n \alpha_m\frac{\partial}{\partial z_m}$, where $\alpha_m:=D(z_m)$.

Example A.6.3. Let $D\colon R\to M$ be a derivation.

1. By Leibniz we have $D(1)=D(1\cdot 1)=D(1)+D(1)$ and hence $D(1)=0$.
2. Suppose that $g\in R$ is invertible. Then
$$0=D(1)=D(g\cdot g^{-1})=g^{-1}Dg+fD(g^{-1}) \tag{A.6.3}$$
and hence $D(g^{-1})=-g^{-2}D(f)$.
3. Suppose that $f,g\in R$ and that g is invertible. By Item (2) we get that the following familiar formula holds:
$$D(f\cdot g^{-1})=g^{-2}(D(f)\cdot g-f\cdot D(g)). \tag{A.6.4}$$

Let $D,D'\in\mathrm{Der}(R,M)$ and $z\in R$ we let

$$\begin{array}{ccc} R & \xrightarrow{D+D'} & M \\ a & \mapsto & D(a)+D'(a) \end{array} \tag{A.6.5}$$

and

$$\begin{array}{ccc} R & \xrightarrow{zD} & M \\ a & \mapsto & zD(a) \end{array} \tag{A.6.6}$$

Both $D+D'$ and zD are derivations and with these operations $\mathrm{Der}(R,M)$ is an R-module. If R is a k-algebra then $\mathrm{Der}_k(R,M)$ is an R-submodule of $\mathrm{Der}(R,M)$.

Next we suppose that $E\supset F$ is an extension of fields, and we consider $\mathrm{Der}_F(E,E)$. Notice that $\mathrm{Der}_F(E,E)$ is a vector space over F.

Proposition A.6.4. *Suppose that $E\supset F$ is a finitely and separably generated extension of fields. Let $\alpha_1,\ldots,\alpha_n$ be a separating transcendence basis of E over F. Then the map of E-vector spaces*

$$\begin{array}{ccc} \mathrm{Der}_F(E,E) & \longrightarrow & E^n \\ D & \mapsto & (D(\alpha_1),\ldots,D(\alpha_n)) \end{array} \tag{A.6.7}$$

is an isomorphism.

Proof. Let $K:=F(\alpha_1,\ldots,\alpha_n)\subset E$. Since $\alpha_1,\ldots,\alpha_n$ is a separating transcendence basis of E over F, and E is finitely generated (over F), there exists an element $\beta\in E$ primitive over K. Let $P\in K[z]$ be the minimal polynomial of β. In particular

$$P(\beta)=0,\quad \frac{dP}{dz}(\beta)\neq 0. \tag{A.6.8}$$

(The inequality holds because E is a separable extension of K.)

Since K is a purely transcendental extension of F we have an isomorphism of E-vector spaces

$$\begin{array}{ccc} \mathrm{Der}_F(K,E) & \xrightarrow{\sim} & E^n \\ D & \mapsto & (D(\alpha_1),\ldots,D(\alpha_n)). \end{array}$$

Equivalently every $D \in \mathrm{Der}_F(K, E)$ is given by

$$D(\phi) = \sum_{i=1}^{n} c_i \frac{\partial \phi}{\partial \alpha_i}, \quad \alpha_i \in E,$$

and the c_i's may be chosen arbitrarily. Thus we must show that the restriction map

$$\begin{array}{ccc} \mathrm{Der}_F(E,E) & \longrightarrow & \mathrm{Der}_F(K,E) \\ D & \mapsto & D_{|K} \end{array} \tag{A.6.9}$$

defines an isomorphism of E-vector spaces.

Let us prove that the restriction map is injective. Let $P = \sum_{i=0}^{d} a_i z^{d-i}$, where $a_0 = 1$ (recall that P is the minimal polynomila of β over K). Suppose that $D \in \mathrm{Der}_F(E, E)$; by the equality in (A.6.8) we get that

$$0 = D(P(\beta)) = \sum_{i=0}^{d} D(a_i)\beta^{d-i} + \sum_{i=0}^{d-1} D(\beta)a_i(d-i)\beta^{d-i-1} = \sum_{i=0}^{d} D(a_i)\beta^{d-i} + D(\beta)\frac{dP}{dz}(\beta).$$

By the inequality in (A.6.8), we can divide and we get

$$D(\beta) = -\left(\sum_{i=1}^{m} D(a_i)\beta^{m-i}\right) \cdot \frac{dP}{dz}(\beta)^{-1}. \tag{A.6.10}$$

This proves that the map in (A.6.9) is injective.

In order to prove surjectivity, we extend a derivation $D \in \mathrm{Der}_F(K, E)$ to a derivation in $\mathrm{Der}_F(E, E)$ by *defining* its value on β via (A.6.10). □

Corollary A.6.5. *Keep hypotheses and notation as above. Then* $\mathrm{Tr}\deg_k K = \dim_K \mathrm{Der}_k(K, K)$.

A.7 Nakayama's Lemma

Let R be a ring, M be an R-module, and $I \subset R$ be an ideal. We let $IM \subset M$ be the submodule of finite sums $\sum_{k \in K} f_k m_k$, where $f_k \in I$ and $m_k \in M$ for every $k \in K$.

Lemma A.7.1 (Nakayama's Lemma)**.** *Let R be a ring and M a finitely generated R-module. Let $I \subset R$ be an ideal and suppose that $M \subset IM$ (i.e. $M = IM$). Then there exists $\varphi \in I$ such that $(1 + \varphi)M = 0$ i.e. $(1 + \varphi)m = 0$ for all $m \in M$.*

Proof. Let $m_1, \ldots, m_r$ be generators of M. By hypothesis there exist $a_{ij} \in I$ for $1 \leqslant i, j \leqslant r$ such that

$$m_i = \sum_{j=1}^{r} a_{ij} m_j.$$

Let A be the $r \times r$-matrix with entries in R given by $A := (\delta_{ij} - a_{ij})$, where δ_{ij} is the Kronecker symbol i.e. $\delta_{ij} = 1$ if $i = j$ and is 0 otherwise. Let B be the $r \times 1$-matrix with entries $m_1, \ldots, m_r$. Then $A \cdot B = 0$: multiplying by the matrix of cofactors A^c we get that $\det A \cdot m_i = 0$ for $i = 1, \ldots, r$. Expanding $\det A$ we get that $\det A = 1 + \varphi$ where $\varphi \in I$. □

Corollary A.7.2. *Let R be a* local *ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module. Suppose that the quotient module $M/\mathfrak{m}M$ is generated by the classes of $m_1, \ldots, m_r \in M$. Then M is generated by $m_1, \ldots, m_r$.*

Proof. Let $N \subset M$ be the submodule generated by $m_1, \ldots, m_r$ and $P := M/N$ be the quotient module. We must prove that $P = 0$. The module P is finitely generated over R because M is, and moreover $P \subset \mathfrak{m}P$ by hypothesis. By Nakayama's Lemma there exists $\varphi \in \mathfrak{m}$ such that $(1 + \varphi)P = 0$. Since $(1 + \varphi)$ does not belong to $\mathfrak{m}$ it is invertible (it generates all of R because $\mathfrak{m}$ contains all non-trivial ideals of R) and hence it follows that $P = 0$. □

A.8 Order of vanishing

The prototype of a Noetherian local ring $(R, \mathfrak{m})$ is the ring $\mathscr{O}_{X,x}$ of germs of regular functions of a quasi projective variety X at a point $x \in X$, with maximal ideal $\mathfrak{m}_x$, see Proposition 4.2.4. The following result of Krull can be interpreted as stating that a non zero element of $\mathscr{O}_{X,x}$ can not vanish to arbitrary high order at x. In other words, elements of $\mathscr{O}_{X,x}$ behave like analytic functions (as opposed to C^∞ functions).

Theorem A.8.1 (Krull)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$\bigcap_{i \geqslant 0} \mathfrak{m}^i = \{0\}.$$

Proof. Since R is Noetherian the ideal $\mathfrak{m}$ is finitely generated; say $\mathfrak{m} = (a_1, \ldots, a_n)$. Let $b \in \bigcap_{i \geqslant 0} \mathfrak{m}^i$. Let $i \geqslant 0$; since $b \in \mathfrak{m}^i$ there exists $P_i \in R[X_1, \ldots, X_n]_i$ such that $P_i(a_1, \ldots, a_n) = b$. Let $J \subset R[X_1, \ldots, X_n]$ be the ideal generated by the P_i's. Since R is Noetherian so is $R[X_1, \ldots, X_n]$. Thus J is finitely generated and hence there exists $N > 0$ such that $J = (P_0, \ldots, P_N)$. Thus there exists $Q_{N+1-i} \in R[X_1, \ldots, X_n]_{N+1-i}$ for $i = 0, \ldots, N$ such that $P_{N+1} = \sum_{i=0}^N Q_{N+1-i} P_i$. It follows that

$$b = P_{N+1}(a_1, \ldots, a_n) = \sum_{i=0}^N Q_{N+1-i}(a_1, \ldots, a_n) P_i(a_1, \ldots, a_n) = b \sum_{i=0}^N Q_{N+1-i}(a_1, \ldots, a_n). \quad \text{(A.8.11)}$$

Now $Q_{N+1-i}(a_1, \ldots, a_n) \in \mathfrak{m}$ for $i = 0, \ldots, N$ and hence $\epsilon := \sum_{i=0}^N Q_{N+1-i}(a_1, \ldots, a_n) \in \mathfrak{m}$. Equality (A.8.11) gives that $(1 - \epsilon) b = 0$: since $\epsilon \in \mathfrak{m}$ the element $(1 - \epsilon)$ is invertible and hence $b = 0$. $\square$

Corollary A.8.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\mathfrak{I} \subset R$ be an ideal. Then*

$$\bigcap_{i \geqslant 0} (\mathfrak{I} + \mathfrak{m}^i) = \{0\}.$$

Proof. Let $S := R/\mathfrak{I}$. Then S is a Noetherian local ring, with maximal ideal $\mathfrak{m}_S := \mathfrak{I} + \mathfrak{m}$. The corollary follows by applying Theorem A.8.2 to $(S, \mathfrak{m}_S)$. $\square$