Elliptic equations with measure data

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Contents

CHAPTER 1

Existence with regular data in the linear case

Before stating and proving the existence theorem for linear elliptic equations, we need some tools.

1. Minimization in Banach spaces

Let E be a Banach space, and let $J : E \to \mathbb{R}$ be a functional.

DEFINITION 1.1. A functional $J : E \to \mathbb{R}$ is said to be *weakly lower semicontinuous* if

$$u_n \rightharpoonup u \quad \Rightarrow \quad J(u) \leq \liminf_{n\to+\infty} J(u_n).$$

DEFINITION 1.2. A functional $J : E \to \mathbb{R}$ is said to be *coercive* if

$$\lim_{\|u\|_E \to +\infty} J(u) = +\infty.$$

EXAMPLE 1.3. If $E = \mathbb{R}$, the function $J(x) = x^2$ is an example of a (weakly) lower semicontinuous and coercive functional. Another example is $J(u) = \|u\|_E$.

THEOREM 1.4. *Let E be a reflexive Banach space, and let $J : E \to \mathbb{R}$ be a coercive and weakly lower semicontinuous functional (not identically equal to $+\infty$). Then J has a minimum on E.*

Proof. Let

$$m = \inf_{v\in E} J(v) < +\infty,$$

and let $\{v_n\}$ in E be a minimizing sequence, i.e., v_n is such that

$$\lim_{n\to+\infty} J(v_n) = m.$$

We begin by proving that $\{v_n\}$ is bounded. Indeed, if it were not, there would be a subsequence $\{v_{n_k}\}$ such that

$$\lim_{k\to+\infty} \|v_{n_k}\| = +\infty.$$

Since J is coercive, we will have

$$m = \lim_{n\to+\infty} J(v_n) = \lim_{k\to+\infty} J(v_{n_k}) = +\infty,$$

which is false. Therefore, $\{v_n\}$ is bounded in E and so, being E reflexive, there exists a subsequence $\{v_{n_k}\}$ and an element v of E such that v_{n_k} weakly converges to v as k diverges. Since J is weakly lower semicontinuous, we have

$$m \leq J(v) \leq \liminf_{k\to+\infty} J(v_{n_k}) = \lim_{n\to+\infty} J(v_n) = m,$$

so that v is a minimum of J. □

2. Hilbert spaces

2.1. *Linear forms and dual space.* We recall that a Hilbert space H is a vector space where a scalar product $(\cdot|\cdot)$ is defined, which is complete with respect to the distance induced by the scalar product by the formula

$$d(x, y) = \sqrt{(x - y|x - y)}.$$

Examples of Hilbert spaces are $\mathbb{R}$ (with $(x|y) = x\,y$), $\mathbb{R}^N$ (with the "standard" scalar product), ℓ^2, and $L^2(\Omega)$ with

$$(f|g) = \int_\Omega f\,g.$$

THEOREM 1.5 (Riesz). *Let H be a separable Hilbert space, and let T be an element of its dual H', i.e., a linear application $T : H \to \mathbb{R}$ such that there exists $C \geq 0$ such that*

$$|\langle T, x\rangle| \leq C\|x\|, \quad \forall x \in H. \tag{1.1}$$

Then there exists a unique y in H such that

$$\langle T, x\rangle = (y|x), \quad \forall x \in H.$$

Proof. Denote by $\{e_h\}$ a complete orthonormal system in H, i.e. a sequence of vectors of H such that $(e_h|e_k) = \delta_{hk}$, and such that, for every x in H, one has

$$x = \sum_{h=1}^{+\infty}(x|e_h)e_h.$$

It is then well known that there exists a bijective isometry $\mathcal{F}$ from H to ℓ^2, defined by $\mathcal{F}(x) = \{(x|e_h)\}$. We claim that $\{\langle T, e_h\rangle\}$ belongs to ℓ^2. Indeed, if

$$y_n = \sum_{h=1}^{n}\langle T, e_h\rangle e_h,$$

we have, by linearity and by (1.1),

$$\sum_{h=1}^{n}(\langle T, e_h\rangle)^2 = \langle T, y_n\rangle \le C\|y_n\| = C\left(\sum_{h=1}^{n}(\langle T, e_h\rangle)^2\right)^{\frac{1}{2}},$$

so that

$$\sum_{h=1}^{n}(\langle T, e_h\rangle)^2 \le C^2,$$

which yields (letting n tend to infinity) that $\{\langle T, e_h\rangle\}$ belongs to ℓ^2. Therefore, one has, again by linearity and by (1.1),

$$\langle T, x\rangle = \sum_{h=1}^{+\infty}(x|e_h)\langle T, e_h\rangle, \quad \forall x \in H.$$

Let now y be the vector of H defined by

$$y = \sum_{h=1}^{+\infty}\langle T, e_h\rangle e_h.$$

Then, since $\langle T, e_h\rangle = (y|e_h)$, one has

$$\langle T, x\rangle = \sum_{h=1}^{+\infty}(x|e_h)(y|e_h), \quad \forall x \in H,$$

and the right hand side is nothing but the scalar product in ℓ^2 of $\mathcal{F}(x)$ and $\mathcal{F}(y)$. Since $\mathcal{F}$ is an isometry, we then have

$$\langle T, x\rangle = (y|x), \quad \forall x \in H,$$

as desired. Uniqueness follows from the fact that $(y|x) = (z|x)$ for every x in H implies $y = z$ (just take $x = y - z$). $\square$

Corollary 1.6. *The map $T \mapsto y$ is a bijective linear isometry between H' and H.*

Proof. Since $\langle T + S, x\rangle = \langle T, x\rangle + \langle S, x\rangle$, and $\langle \lambda T, x\rangle = \lambda\langle T, x\rangle$, it is clear that the map $T \mapsto y$ is linear. In order to prove that it is an isometry, we have

$$|\langle T, x\rangle| = |(y|x)| \le \|y\|\|x\|,$$

which implies $\|T\| \le \|y\|$. Furthermore

$$\|y\|^2 = (y|y) = \langle T, y\rangle \le \|T\|\|y\|,$$

so that $\|y\| \le \|T\|$. The map is clearly injective, and it is surjective since the application $x \mapsto (y|x)$ is linear and continuous on H (by Cauchy-Schwartz inequality). $\square$

2.2. *Bilinear forms.* An application $a : H \times H \to \mathbb{R}$ such that

$$a(\lambda x + \mu y, z) = \lambda a(x, z) + \mu a(y, z),$$

and

$$a(z, \lambda x + \mu y) = \lambda a(z, x) + \mu a(z, x),$$

for every x and y in H, and for every λ and μ in $\mathbb{R}$, is called *bilinear form*. A bilinear form is said to be *continuous* if there exists $\beta \geq 0$ such that

$$|a(x, y)| \leq \beta \|x\| \|y\|, \quad \forall x, y \in H,$$

and is said to be *coercive* if there exists $\alpha > 0$ such that

$$a(x, x) \geq \alpha \|x\|^2, \quad \forall x \in H.$$

An example of bilinear form on H is the scalar product, which is both continuous (with $\beta = 1$, thanks to the Cauchy-Schwartz inequality), and coercive (with $\alpha = 1$, by definition of the norm in H).

THEOREM 1.7. *Let $a : H \times H \to \mathbb{R}$ be a continuous bilinear form. Then there exists a linear and continuous map $A : H \to H$ such that*

$$a(x, y) = (A(x)|y), \quad \forall x, y \in H.$$

Proof. Since a is linear in the second argument and continuous, for every fixed x in H the map $y \mapsto a(x, y)$ is linear and continuous, so that it belongs to H'. By Riesz theorem, there exists a unique vector $A(x)$ in H such that

$$a(x, y) = (A(x)|y), \quad \forall x, y \in H.$$

Since a is linear in the first argument, the map $x \mapsto A(x)$ is linear. Furthermore, by the continuity of a,

$$\|A(x)\|^2 = (A(x)|A(x)) = a(x, A(x)) \leq \beta \|x\| \|A(x)\|,$$

so that $\|A(x)\| \leq \beta \|x\|$, and the map is continuous. □

2.3. *Banach-Caccioppoli and Lax-Milgram theorems.*

THEOREM 1.8 (Banach-Caccioppoli). *Let (X, d) be a complete metric space, and let $S : X \to X$ be a contraction mapping, i.e., a continuous application such that there exists θ in $[0, 1)$ such that*

$$d(S(x), S(y)) \leq \theta \, d(x, y), \quad \forall x, y \in X.$$

Then there exists a unique $\overline{x}$ in X such that $S(\overline{x}) = \overline{x}$.

Proof. Let x_0 in X be fixed, and define $x_1 = S(x_0)$, $x_2 = S(x_1)$, and, in general, $x_n = S(x_{n-1})$. We then have, since S is a contraction mapping,

$$d(x_{n+1}, x_n) = d(S(x_n), S(x_{n-1})) \leq \theta\, d(x_n, x_{n-1}),$$

and iterating we obtain

$$d(x_{n+1}, x_n) \leq \theta^n\, d(x_1, x_0).$$

Therefore, by the triangular inequality,

$$d(x_n, x_m) \leq \sum_{h=m}^{n-1} d(x_{h+1}, x_h) \leq \sum_{h=m}^{n-1} \theta^h\, d(x_1, x_0) = \frac{\theta^m - \theta^n}{1-\theta}.$$

Since $\{\theta^h\}$ is a Cauchy sequence in $\mathbb{R}$ (being convergent to zero), it then follows that $\{x_n\}$ is a Cauchy sequence in (X, d), which is complete. Therefore, there exists $\overline{x}$ in X such that x_n converges to $\overline{x}$. Since S is continuous, on one hand $S(x_n)$ converges to $S(\overline{x})$, and on the other hand $S(x_n) = x_{n+1}$ converges to $\overline{x}$ so that $\overline{x}$ is a fixed point for S. If there exist $\overline{x}$ and $\overline{y}$ such that $S(\overline{x}) = \overline{x}$ and $S(\overline{y}) = \overline{y}$, then, since S is a contraction mapping,

$$d(\overline{x}, \overline{y}) = d(S(\overline{x}), S(\overline{y})) \leq \theta\, d(\overline{x}, \overline{y}),$$

which implies (since $\theta < 1$) $d(\overline{x}, \overline{y}) = 0$ and so $\overline{x} = \overline{y}$. □

THEOREM 1.9 (Lax-Milgram). *Let $a : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form, and let T be an element of H'. Then there exists a unique $\overline{x}$ in H such that*

$$a(\overline{x}, z) = \langle T, z\rangle, \quad \forall z \in H. \tag{1.2}$$

Proof. Using the Riesz theorem and Theorem 1.7, solving the equation (1.2) is equivalent to find $\overline{x}$ such that

$$a(\overline{x}, z) = (A(\overline{x})|z) = (y|z) = \langle T, z\rangle, \quad \forall z \in H,$$

i.e., to solve the equation $A(\overline{x}) = y$. Given $\lambda > 0$, this equation is equivalent to $\overline{x} = \overline{x} - \lambda A(\overline{x}) + \lambda y$, which is a fixed point problem for the function $S(x) = x - \lambda A(x) + \lambda y$. Since, being A linear, one has

$$S(x_1) - S(x_2) = x_1 - x_2 - \lambda A(x_1) + \lambda A(x_2) = x_1 - x_2 - \lambda A(x_1 - x_2),$$

in order to prove that S is a contraction mapping, it is enough to prove that there exists $\lambda > 0$ such that

$$\|x - \lambda A(x)\| \leq \theta \|x\|,$$

for some $\theta < 1$ and for every x in H. We have

$$\|x - \lambda A(x)\|^2 = \|x\|^2 + \lambda^2 \|A(x)\|^2 - 2\lambda (A(x)|x).$$

Recalling Theorem 1.7 and the definition of A, we have

$$\|A(x)\|^2 \leq \beta^2\|x\|^2, \quad (A(x)|x) = a(x,x) \geq \alpha\|x\|^2,$$

so that

$$\|x - \lambda A(x)\|^2 \leq (1 + \lambda^2\beta^2 - 2\lambda\alpha)\|x\|^2.$$

If $0 < \lambda < \frac{2\alpha}{\beta^2}$, we have $\theta^2 = 1 + \lambda^2\beta^2 - 2\lambda\alpha < 1$, so that S is a contraction mapping. □

3. Sobolev spaces

The Banach spaces where we will look for solutions are space of functions in Lebesgue spaces "with derivatives in Lebesgue spaces" (whatever this means).

3.1. *Definition of Sobolev spaces.* Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 1$, and let u be a function in $L^1(\Omega)$. We say that u has a *weak* (or *distributional*) *derivative* in the direction x_i if there exists a function v in $L^1(\Omega)$ such that

$$\int_\Omega u\,\frac{\partial\varphi}{\partial x_i} = -\int_\Omega v\,\varphi, \quad \forall\varphi \in C_0^1(\Omega).$$

In this case we define the weak derivative $\frac{\partial u}{\partial x_i}$ as the function v. If u has weak derivatives in every direction, we define its (weak, or distributional) gradient as the vector

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right).$$

If $p \geq 1$, we define the Sobolev space $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N\right\}.$$

The Sobolev space $W^{1,p}(\Omega)$ becomes a Banach space under the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{(L^p(\Omega))^N},$$

and $W^{1,2}(\Omega)$ is a Hilbert space under the scalar product

$$(u|v)_{W^{1,2}(\Omega)} = \int_\Omega u\,v + \int_\Omega \nabla u \cdot \nabla v.$$

For historical reasons the space $W^{1,2}(\Omega)$ is usually denoted by $H^1(\Omega)$: we will use this notation from now on.

Since we will be dealing with elliptic problems with zero boundary conditions, we need to define functions which somehow are "zero" on the boundary of Ω. Since $\partial\Omega$ has zero Lebesgue measure, and functions in $W^{1,p}(\Omega)$ are only defined up to almost everywhere equivalence, there is no "direct" way of defining the boundary value a function u in some Sobolev space. We then give the following definition.

DEFINITION 1.10. We define $W_0^{1,p}(\Omega)$ as the closure of $C_0^1(\Omega)$ in the norm of $W^{1,p}(\Omega)$. If $p = 2$, we will denote $W_0^{1,2}(\Omega)$ by $H_0^1(\Omega)$, which is a Hilbert space.

From now on we will mainly deal with $W_0^{1,p}(\Omega)$.

3.2. *Properties of Sobolev spaces.* Since a function in $W_0^{1,p}(\Omega)$ is "zero at the boundary" it is possible to control the norm of u in $L^p(\Omega)$ with the norm of its gradient in the same space. This is known as Poincaré inequality.

THEOREM 1.11 (Poincaré inequality). *Let $p \geq 1$; then there exists a constant C, only depending on Ω, N and p, such that*

$$\|u\|_{L^p(\Omega)} \leq C \, \|\nabla u\|_{(L^p(\Omega))^N} , \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.3}$$

Proof. We only give an idea of the proof in dimension 1. Let u belong to $C_0^1((0,1))$. Then

$$u(x) = u(0) + \int_0^x u'(t)\, dt = \int_0^x u'(t)\, dt, \quad \forall x \in (0,1).$$

Thus, by Hölder inequality

$$|u(x)|^p = \left| \int_0^x u'(t)\, dt \right|^p \leq x^{\frac{p}{p'}} \int_0^x |u'(t)|^p \leq \int_0^1 |u'(t)|^p.$$

Integrating this inequality yields the result for $C_0^1((0,1))$ functions. The result for functions in $W_0^{1,p}(\Omega)$ then follows by a density argument. □

As a consequence of Poincaré inequality, we can define on $W_0^{1,p}(\Omega)$ the equivalent norm built after the norm of ∇u in $(L^p(\Omega))^N$. From now on, we define

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{(L^p(\Omega))^N} .$$

Even though functions in $W_0^{1,p}(\Omega)$ should only belong to $L^p(\Omega)$, the assumptions made on the gradient allow to improve the summability of functions belonging to Sobolev spaces. This is what is stated in the following "embedding" theorem.

THEOREM 1.12. *Let $1 \leq p < N$, and let $p^* = \frac{Np}{N-p}$ (p^* is called the Sobolev embedding exponent). Then there exists a constant $\mathcal{S}_p$ (depending only on N and p) such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq \mathcal{S}_p \, \|u\|_{W_0^{1,p}(\Omega)} , \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.4}$$

REMARK 1.13. The fact that p^* is the correct exponent can be easily recovered by a scaling argument. Indeed, if u belongs to $W_0^{1,p}(\mathbb{R}^N)$, then $u(\lambda x)$ belongs to the same space. But then

$$\int_{\mathbb{R}^N} |u(\lambda x)|^q \, dx = \frac{1}{\lambda^N} \int_{\mathbb{R}^N} |u(y)|^q \, dy,$$

and

$$\int_{\mathbb{R}^N} |\nabla u(\lambda x)|^p \, dx = \frac{1}{\lambda^{N-p}} \int_{\mathbb{R}^N} |\nabla u(y)|^p \, dy.$$

Therefore, if (1.4) holds for some constant C (independent on λ) and some exponent q, one should have

$$\frac{N}{q} = \frac{N-p}{p},$$

which implies $q = \frac{Np}{N-p} = p^*$.

By (1.4), the embedding of $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$ is continuous. We recall that a map $T : X \to Y$ (with X and Y Banach spaces) is said to be **compact** if the closure of $T(B)$ is compact in Y for every bounded set B in X. To obtain compactness of the embedding of $W_0^{1,p}(\Omega)$ in Lebesgue spaces, we cannot consider exponents up to p^*.

THEOREM 1.14. *Let $1 \leq p < N$, and let $1 \leq q < p^*$. Then the embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ is compact.*

REMARK 1.15. The fact that the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is not compact is at the basis for several nonexistence results for equations like $-\Delta u = u^q$ if q is "too large". But this is another story...

An important role will be played by the dual of a Sobolev space. We have the following representation theorem.

THEOREM 1.16. *Let $p > 1$, and let T be an element of $(W_0^{1,p}(\Omega))'$. Then there exists F in $(L^{p'}(\Omega))^N$ such that*

$$\langle T, u \rangle = \int_\Omega F \cdot \nabla u, \quad \forall u \in W_0^{1,p}(\Omega).$$

The dual of $W_0^{1,p}(\Omega)$ will be denoted by $W^{-1,p'}(\Omega)$, while the dual of $H_0^1(\Omega)$ is $H^{-1}(\Omega)$.

REMARK 1.17. The space $H_0^1(\Omega)$ is a Hilbert space. Therefore, by Theorem 1.5, it is isometrically equivalent to its dual $H^{-1}(\Omega)$. Furthermore, by Poincaré inequality, $H_0^1(\Omega)$ is embedded into $L^2(\Omega)$, which is itself a Hilbert space. Since the embedding is continuous and dense, we also have that the the dual of $L^2(\Omega)$ (which is $L^2(\Omega)$) is embedded into $H^{-1}(\Omega)$. We therefore have

$$H_0^1(\Omega) \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset (H_0^1(\Omega))' = H^{-1}(\Omega).$$

If we identify both $L^2(\Omega)$ and its dual, **and** $H_0^1(\Omega)$ and its dual, we obtain a contradiction (since $H_0^1(\Omega)$ and $L^2(\Omega)$ are different spaces). Therefore, we have to choose which identification to make: which will be that $L^2(\Omega)$ is equivalent to its dual.

Remark 1.18. Since, by Sobolev embedding, $W_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, we have by duality that $(L^{p^*}(\Omega))'$ is continuously embedded in $W^{-1,p'}(\Omega)$. If we define

$$p_* = (p^*)' = \frac{Np}{Np - N + p},$$

we then have

$$L^{p_*}(\Omega) \subset W^{-1,p'}(\Omega).$$

If $p = 2$, we have $2_* = \frac{2N}{N+2}$, and the embedding of $L^{2_*}(\Omega)$ into $H^{-1}(\Omega)$.

The final result on Sobolev spaces will be about composition with regular functions.

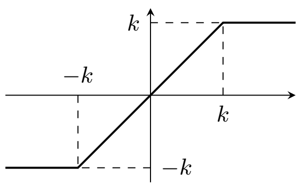
Theorem 1.19 (Stampacchia). *Let $G : \mathbb{R} \to \mathbb{R}$ be a lipschitz continuous functions such that $G(0) = 0$. If u belongs to $W_0^{1,p}(\Omega)$, then $G(u)$ belongs to $W_0^{1,p}(\Omega)$ as well, and*

$$\nabla G(u) = G'(u)\, \nabla u, \quad \textit{almost everywhere in } \Omega. \tag{1.5}$$

Remark 1.20. Recall that a lipschitz continuous function is only almost everywhere differentiable, so that the right-hand side of (1.5) may not be defined. We have however two possible cases: if k is a value such that $G'(k)$ does not exist, either the set $\{u = k\}$ has zero measure (and so, since identity (1.5) only holds almost everywhere, this value does not give any problems), or the set $\{u = k\}$ has positive measure. In this latter case, however, we have both $\nabla u = 0$ and $\nabla G(u) = 0$ almost everywhere, so that (1.5) still holds.

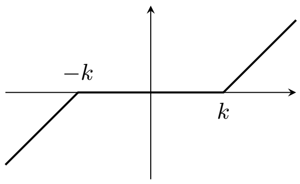
Let $k > 0$; in what follows, we will often use composition of functions in Sobolev spaces with the lipschitz continuous functions

$$T_k(s) = \max(-k, \min(s, k)), \tag{1.6}$$



and

$$G_k(s) = s - T_k(s) = (|s| - k)_+ \mathrm{sgn}(s). \tag{1.7}$$



By Theorem 1.19, we have

$$\nabla T_k(u) = \nabla u \, \chi_{\{|u| \le k\}}, \quad \nabla G_k(u) = \nabla u \, \chi_{\{|u| \ge k\}},$$

almost everywhere in Ω.

4. Weak solutions for elliptic equations

We have now all the tools needed to deal with elliptic equations.

4.1. *Definition of weak solution.* Let $A : \Omega \to \mathbb{R}^{N^2}$ be a matrix-valued measurable function such that there exist $0 < \alpha \le \beta$ such that

$$A(x)\xi \cdot \xi \ge \alpha |\xi|^2, \quad |A(x)| \le \beta, \tag{1.8}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$. We will consider the following uniformly elliptic equation with Dirichlet boundary conditions

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where f is a function defined on Ω which satisfies suitable assumptions. If the matrix A is the identity matrix, problem (1.9) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., the Dirichlet problem for the laplacian operator.

4.2. *Classical solutions and weak solutions.* Suppose that the matrix A and the functions u and f are sufficiently smooth so that one can "classically" compute $-\mathrm{div}(A(x)\nabla u)$. If φ is a function in $C_0^1(\Omega)$, we can then multiply the equation in (1.9) by φ and integrate on Ω. Since

$$-\mathrm{div}(A(x)\nabla u)\,\varphi = -\mathrm{div}(A(x)\nabla u\,\varphi) + A(x)\nabla u\cdot\nabla\varphi,$$

we get

$$\int_\Omega A(x)\nabla u\cdot\nabla\varphi - \int_\Omega \mathrm{div}(A(x)\nabla u\,\varphi) = \int_\Omega f\,\varphi.$$

By Gauss-Green formula, we have (if ν is the exterior normal to Ω)

$$\int_\Omega \mathrm{div}(A(x)\nabla u\,\varphi) = \int_{\partial\Omega} A(x)\nabla u\cdot\nu\,\varphi = 0,$$

since φ has compact support in Ω. Therefore, if u is a classical solution of (1.9), we have

$$\int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f\,v, \quad \forall v\in C_0^1(\Omega).$$

We now remark that there is no need for A, u, φ and f to be smooth in order for the above identity to be well defined. It is indeed enough that A is a bounded matrix, that u and φ belong to $H_0^1(\Omega)$, and that f is in $L^2(\Omega)$ (or in $L^{2_*}(\Omega)$, thanks to Sobolev embedding, see Remark 1.18).

We therefore give the following definition.

Definition 1.21. Let f be a function in $L^{2_*}(\Omega)$. A function u in $H_0^1(\Omega)$ is a *weak solution* of (1.9) if

$$\int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f\,v, \quad \forall v\in H_0^1(\Omega). \tag{1.10}$$

If u is a weak solution of (1.9), and u is sufficiently smooth in order to perform the same calculations as above "going backwards", then it can be proved that u is a "classical" solution of (1.9). The study of the assumptions on f and A such that a weak solution is also a classical solution goes beyond the purpose of this text (also because we are interested in "bad" data!).

4.3. *Existence of solutions (using Lax-Milgram).*

Theorem 1.22. *Let f be a function in $L^{2_*}(\Omega)$. Then there exists a unique solution u of (1.9) in the sense of (1.10).*

Proof. We will use Lax-Milgram theorem. Indeed, if we define the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega A(x)\nabla u \cdot \nabla v,$$

and the linear and continuos (thanks to Sobolev embedding) functional $T : H_0^1(\Omega) \to \mathbb{R}$ by

$$\langle T, v\rangle = \int_\Omega f\, v,$$

solving problem (1.9) in the sense of (1.10) amounts to finding u in $H_0^1(\Omega)$ such that

$$a(u, v) = \langle T, v\rangle, \quad \forall v \in H_0^1(\Omega),$$

which is exactly the result given by Lax-Milgram theorem. In order to apply the theorem, we have to check that a is continuous and coercive (the fact that it is bilinear being evident). We have, by (1.8), and by Hölder inequality,

$$|a(u, v)| \le \int_\Omega |A(x)||\nabla u||\nabla v| \le \beta \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)},$$

so that a is continuous. Furthermore, again by (1.8), we have

$$a(u, u) = \int_\Omega A(x)\nabla u \cdot \nabla u \ge \alpha \int_\Omega |\nabla u|^2 = \alpha \|u\|^2_{H_0^1(\Omega)},$$

so that a is also coercive. □

4.4. *Existence of solutions (using minimization).* If the matrix A satisfies (1.8) and is symmetrical, existence and uniqueness of solutions for (1.9) can be proved using minimization of a suitable functional.

THEOREM 1.23. *Let f be a function in $L^{2_*}(\Omega)$, and let $J : H_0^1(\Omega) \to \mathbb{R}$ be defined by*

$$J(v) = \frac{1}{2}\int_\Omega A(x)\nabla v \cdot \nabla v - \int_\Omega f\, v, \quad \forall v \in H_0^1(\Omega).$$

Then J has a unique minimum u in $H_0^1(\Omega)$, which is the solution of (1.9) in the sense of (1.10).

Proof. We begin by proving that J is coercive and weakly lower semicontinuous on $H_0^1(\Omega)$, so that a minimum will exist by Theorem 1.4. Recalling (1.8) and using Hölder and Sobolev inequalities, we have

$$\begin{aligned} J(v) &\ge \frac{\alpha}{2}\int_\Omega |\nabla v|^2 - \|f\|_{L^{2_*}(\Omega)} \|v\|_{L^{2^*}(\Omega)} \\ &\ge \frac{\alpha}{2}\|v\|^2_{H_0^1(\Omega)} - \mathcal{S}_2 \|f\|_{L^{2_*}(\Omega)} \|v\|_{H_0^1(\Omega)}, \end{aligned}$$

and the right hand side diverges as the norm of u in $H_0^1(\Omega)$ diverges, so that J is coercive. Let now $\{v_n\}$ be a sequence of functions which is weakly convergent to some v in $H_0^1(\Omega)$. Since f belongs to $L^{2_*}(\Omega)$, and v_n converges weakly to v in $L^{2^*}(\Omega)$, we have

$$\lim_{n\to+\infty} \int_\Omega f\, v_n = \int_\Omega f\, v,$$

so that the weak lower semicontinuity of J is equivalent to the weak lower semicontinuity of

$$K(v) = \int_\Omega A(x)\nabla v \cdot \nabla v.$$

By (1.8) we have

$$K(v - v_n) = \int_\Omega A(x)\,\nabla(v - v_n) \cdot \nabla(v - v_n) \geq 0,$$

which, together with the symmetry of A, implies

$$2\int_\Omega A(x)\nabla v \cdot \nabla v_n - \int_\Omega A(x)\nabla v \cdot \nabla v \leq \int_\Omega A(x)\nabla v_n \cdot \nabla v_n. \tag{1.11}$$

Since ∇v_n converges weakly to ∇v in $(L^2(\Omega))^N$, and since $A(x)\nabla v$ is fixed in the same space, we have

$$\lim_{n\to+\infty} \int_\Omega A(x)\nabla v \cdot \nabla v_n = \int_\Omega A(x)\nabla v \cdot \nabla v,$$

so that taking the inferior limit in both sides of (1.11) implies

$$K(v) = \int_\Omega A(x)\nabla v \cdot \nabla v \leq \liminf_{n\to+\infty} \int_\Omega A(x)\nabla v_n \cdot \nabla v_n = \liminf_{n\to+\infty} K(v_n),$$

which means that K is weakly lower semicontinuous on $H_0^1(\Omega)$, as desired.

Let now u be a minimum of J on $H_0^1(\Omega)$. We are going to prove that it is unique. Indeed, if u and v are both minima of J, one has

$$J(u) \leq J\Big(\frac{u+v}{2}\Big), \quad J(v) \leq J\Big(\frac{u+v}{2}\Big),$$

that is,

$$J(u) + J(v) \leq 2J\Big(\frac{u+v}{2}\Big),$$

which becomes (after cancelling equal terms and multiplying by 4)

$$2\int_\Omega A(x)\nabla u \cdot \nabla u + 2\int_\Omega A(x)\nabla u \cdot \nabla u = \int_\Omega A(x)\nabla(u+v) \cdot \nabla(u+v).$$

Using the fact that A is symmetric, expanding the right hand side, and cancelling equal terms, we arrive at

$$\int_\Omega A(x)\nabla u \cdot \nabla u - 2\int_\Omega A(x)\nabla u \cdot \nabla v + \int_\Omega A(x)\nabla v \cdot \nabla v \leq 0,$$

which can be rewritten as

$$\int_\Omega A(x)\nabla(u-v)\cdot\nabla(u-v) \leq 0.$$

Using (1.8) we therefore have

$$\alpha \, \|u-v\|^2_{H^1_0(\Omega)} \leq 0,$$

which implies $u = v$, as desired.

We are now going to prove that the minimum u is a solution of (1.9) in the sense of (1.10). Given v in $H^1_0(\Omega)$ and t in $\mathbb{R}$, we have $J(u) \leq J(u+tv)$, that is

$$\frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega f u \leq \frac{1}{2}\int_\Omega A(x)\nabla(u+tv)\cdot\nabla(u+tv) - \int_\Omega f(u+tv).$$

Expanding the right hand side, cancelling equal terms, and using the fact that A is symmetric, we obtain

$$t\int_\Omega A(x)\nabla u \cdot \nabla v + \frac{t^2}{2}\int_\Omega A(x)\nabla v\cdot\nabla v - t\int_\Omega f\,v \geq 0.$$

If $t > 0$, dividing by t and then letting t tend to zero implies

$$\int_\Omega A(x)\nabla u\cdot\nabla v - \int_\Omega f\,v \geq 0,$$

while if $t < 0$, dividing by t and then letting t tend to zero implies the reverse inequality. It then follows that

$$\int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f\,v, \quad \forall v \in H^1_0(\Omega),$$

and so u solves (1.9) (in the sense of (1.10)). In order to prove that such a solution is unique, we are going to prove that if u solves (1.9), then u is a minimum of J. Indeed, choosing $u - v$ as test function in (1.10), we have

$$\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f(u-v).$$

This implies

$$J(u) + \frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v = J(v) - \frac{1}{2}\int_\Omega A(x)\nabla v\cdot\nabla v,$$

which implies $J(u) \leq J(v)$ since

$$\frac{1}{2}\int_\Omega A(x)\nabla u \cdot \nabla u - \int_\Omega A(x)\nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega A(x)\nabla v \cdot \nabla v$$

is nonnegative by (1.8) since it is equal to

$$\frac{1}{2}\int_\Omega A(x)\nabla(u-v) \cdot \nabla(u-v).$$

□

For Hilbert spaces, Sobolev spaces, and the definition of weak solution for elliptic equations, see the book by H. Brezis ([**6**]), chapters V, VIII (in dimension 1) and IX (in dimension N).

CHAPTER 2

Regularity results

Warning to the reader:
from now, unless explicitly stated, $N \geq 3$.

Thanks to the results of the previous section, we have existence of solutions for data f in $L^{2_*}(\Omega)$. The solution u then belongs to $H_0^1(\Omega)$ and (thanks to Sobolev embedding) to $L^{2^*}(\Omega)$. One then wonders whether an increase on the regularity of f will yield more regular solutions.

1. Examples

We are going to study a model case, in which the solution of (1.9) can be explictly calculated. This example will give us a hint on what happens in the general case.

EXAMPLE 2.1. Let $\Omega = B_{\frac{1}{2}}(0)$, let $N \geq 3$, let $\alpha < N$ and define

$$f(x) = \frac{1}{|x|^{\alpha}\,(-\log(|x|))}.$$

It is well known that f belongs to $L^p(\Omega)$, with $p = \frac{N}{\alpha}$. We are going to study the regularity of the solution u of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

taking advantage of the fact that the solution will be radially symmetric. Recalling the formula for the laplacian in radial coordinates, we have

$$-\frac{1}{\rho^{N-1}}(\rho^{N-1}u'(\rho))' = \frac{1}{\rho^{\alpha}\,(-\log(\rho))}.$$

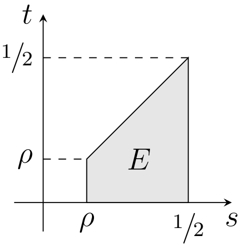
Multiplying by ρ^{N-1} and integrating between 0 and ρ, we obtain

$$\rho^{N-1}\,u'(\rho) = \int_0^{\rho} \frac{t^{N-1-\alpha}}{\log(t)}\,dt.$$

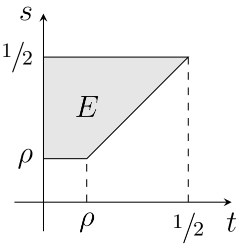
Dividing by ρ^{N-1} and integrating between $\frac{1}{2}$ and ρ we then get (recalling that $u(\frac{1}{2})=0$)

$$u(\rho) = -\int_\rho^{\frac{1}{2}} \frac{1}{s^{N-1}} \left(\int_0^s \frac{t^{N-1-\alpha}}{\log(t)}\, dt \right) ds.$$

We are integrating on the set $E = \{(s,t) \in \mathbb{R}^2 : \rho \le s \le \frac{1}{2},\ 0 \le t \le s\}$,



which, after exchanging t with s, becomes $E = \{(t,s) \in \mathbb{R}^2 : 0 \le t \le \frac{1}{2},\ \max(\rho,t) \le s \le \frac{1}{2}\}$,



Exchanging the integration order, we then have

$$\begin{aligned} u(\rho) &= -\int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} \left(\int_{\max(\rho,t)}^{\frac{1}{2}} \frac{ds}{s^{N-1}} \right) dt \\ &= \frac{1}{N-2} \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} \left[\left(\frac{1}{2}\right)^{2-N} - (\max(\rho,t))^{2-N} \right] dt \\ &= \frac{2^{N-2}}{N-2} \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} dt - \frac{1}{N-2} \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}(\max(\rho,t))^{2-N}}{\log(t)}\, dt. \end{aligned}$$

Since $\alpha < N$, the first integral is bounded, so that it is enough to study the behaviour near zero of the function

$$\begin{aligned} v(\rho) &= \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}(\max(\rho,t))^{2-N}}{\log(t)}\, dt \\ &= \rho^{2-N} \int_0^\rho \frac{t^{N-1-\alpha}}{\log(t)}\, dt + \int_\rho^{\frac{1}{2}} \frac{t^{1-\alpha}}{\log(t)}\, dt \\ &= \rho^{2-N}\, w(\rho) + z(\rho). \end{aligned}$$

It is easy to see (using the de l'Hopital rule), that if $\alpha \neq 2$

$$w(\rho) \approx \frac{\rho^{N-\alpha}}{\log(\rho)}, \quad \text{and} \quad z(\rho) \approx \frac{\rho^{2-\alpha}}{\log(\rho)},$$

as ρ tends to zero, so that, if $\alpha \neq 2$,

$$u(\rho) \approx \frac{\rho^{2-\alpha}}{\log(\rho)},$$

as ρ tends to zero. This implies that u belongs to $L^\infty(\Omega)$ if $\alpha < 2$, while it is in $L^m(\Omega)$, with $m = \frac{N}{\alpha-2}$, if $2 < \alpha < N$. Recalling that f belongs to $L^p(\Omega)$ with $p = \frac{N}{\alpha}$, we therefore have that u belongs to $L^\infty(\Omega)$ if f belongs to $L^p(\Omega)$, and $p > \frac{N}{2}$, while it is in $L^m(\Omega)$, with $m = \frac{Np}{N-2p}$, if f belongs to $L^p(\Omega)$, with $1 < p < \frac{N}{2}$.

If $\alpha = 2$, then

$$w(\rho) \approx \frac{\rho^{N-\alpha}}{\log(\rho)}, \quad \text{and} \quad z(\rho) \approx \log(-\log(\rho)),$$

so that u is in every $L^m(\Omega)$, but not in $L^\infty(\Omega)$, if f belongs to $L^p(\Omega)$ with $p = \frac{N}{2}$. In this case (which we will not study in the following), it can be proved that $\mathrm{e}^{|u|}$ belongs to $L^1(\Omega)$.

Observe that if $\alpha = \frac{N+2}{2}$, so that f belongs to $L^{2_*}(\Omega)$, we get that u belongs to $L^{2^*}(\Omega)$, which is exactly the results we already knew by Sobolev embedding. Also remark that the above example gives informations also if f does not belong to $L^{2_*}(\Omega)$ (i.e., if $\frac{N+2}{2} < \alpha < N$), an assumption under which we do not have any existence results (yet!).

If we want to take $\alpha = N$, we need to change the definition of f. We fix $\beta > 1$ and define

$$f(x) = \frac{1}{|x|^N (-\log(|x|))^\beta},$$

which is a function belonging to $L^1(\Omega)$. Performing the same calculations as above, we obtain

$$u(\rho) = \frac{1}{\beta - 1} \int_\rho^{\frac{1}{2}} \frac{dt}{t^{N-1} (-\log(t))^{\beta-1}},$$

so that

$$u(\rho) \approx \frac{1}{\rho^{N-2} (-\log(\rho))^{\beta-1}},$$

as ρ tends to zero. Observe that in this case f belongs to $L^1(\Omega)$ for every $\beta > 1$, but u belongs to $L^m(\Omega)$, with $m = \frac{N\cdot 1}{N-2\cdot 1} = \frac{N}{N-2}$ if and only if $\beta > 2 - \frac{2}{N}$. If $1 < \beta \leq 2 - \frac{2}{N}$, the solution u belongs "only" to $L^m(\Omega)$, for every $m < \frac{N}{N-2}$.

We leave to the interested reader the study of the case $N = 2$.

2. Stampacchia's theorems

The regularity results we are going to prove now show that the previous example is not just an example. We begin with a real analysis lemma.

LEMMA 2.2 (Stampacchia). *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that*

$$\psi(h) \leq \frac{M\,\psi(k)^\delta}{(h-k)^\gamma}, \quad \forall h > k > 0, \tag{2.12}$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where

$$d^\gamma = M\,\psi(0)^{\delta-1}\,2^{\frac{\delta\gamma}{\delta-1}}.$$

Proof. Let n in $\mathbb{N}$ and define $d_n = d(1 - 2^{-n})$. We claim that

$$\psi(d_n) \leq \psi(0)\,2^{-\frac{n\gamma}{\delta-1}}. \tag{2.13}$$

Indeed, (2.13) is clearly true if $n = 0$; if we suppose that it is true for some n, then, by (2.12),

$$\psi(d_{n+1}) \leq \frac{M\,\psi(d_n)^\delta}{(d_{n+1}-d_n)^\gamma} \leq M\,\psi(0)^\delta\,2^{-\frac{n\gamma\delta}{\delta-1}}\,2^{(n+1)\gamma}\,d^{-\gamma} = \psi(0)\,2^{-\frac{(n+1)\gamma}{\delta-1}},$$

which is (2.13) written for $n+1$. Since (2.13) holds for every n, and since ψ is non increasing, we have

$$0 \leq \psi(d) \leq \liminf_{n\to+\infty} \psi(d_n) \leq \lim_{n\to+\infty} \psi(0)^{\delta-1}\,2^{-\frac{n\gamma}{\delta-1}} = 0,$$

as desired. □

The first result (due to Guido Stampacchia, see [**13**]), deals with bounded solutions for (1.9).

THEOREM 2.3 (Stampacchia). *Let f belong to $L^p(\Omega)$, with $p > \frac{N}{2}$. Then the solution u of (1.9) belongs to $L^\infty(\Omega)$, and there exists a constant C, only depending on N, Ω, p and α, such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^p(\Omega)}\,. \tag{2.14}$$

Proof. Let $k > 0$ and choose $v = G_k(u)$ as test function in (1.9) ($G_k(s)$ has been defined in (1.7)). Defining $A_k = \{x \in \Omega : |u(x)| \geq k\}$ one then has, since $\nabla v = \nabla u\,\chi_{A_k}$ by Theorem 1.19, and using (1.8)

$$\alpha \int_{A_k} |\nabla G_k(u)|^2 \leq \int_\Omega A(x)\nabla u \cdot \nabla u\,\chi_{A_k} = \int_\Omega f\,G_k(u) = \int_{A_k} f\,G_k(u).$$

Using Sobolev inequality (in the left hand side), and Hölder inequality (in the right hand side), one has

$$\frac{\alpha}{\mathcal{S}_2^2}\left(\int_{A_k}|G_k(u)|^{2^*}\right)^{\frac{2}{2^*}} \le \left(\int_{A_k}|f|^{2_*}\right)^{\frac{1}{2_*}}\left(\int_{A_k}|G_k(u)|^{2^*}\right)^{\frac{1}{2^*}}.$$

Simplifying equal terms, we thus have

$$\int_{A_k}|G_k(u)|^{2^*} \le \left(\frac{\mathcal{S}_2^2}{\alpha}\right)^{2^*}\left(\int_{A_k}|f|^{2_*}\right)^{\frac{2^*}{2_*}}.$$

Recalling that f belongs to $L^p(\Omega)$, and that $p > 2_*$ since $p > \frac{N}{2}$, we have (again by Hölder inequality)

$$\int_{A_k}|G_k(u)|^{2^*} \le \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*} m(A_k)^{\frac{2^*}{2_*}-\frac{2^*}{p}}.$$

We now take $h > k$, so that $A_h \subseteq A_k$, and $G_k(u) \ge h-k$ on A_h. Thus,

$$(h-k)^{2^*}\,m(A_h) \le \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*} m(A_k)^{\frac{2^*}{2_*}-\frac{2^*}{p}},$$

which implies

$$m(A_h) \le \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*}\frac{m(A_k)^{\frac{2^*}{2_*}-\frac{2^*}{p}}}{(h-k)^{2^*}}.$$

We define now $\psi(k) = m(A_k)$, so that

$$\psi(h) \le \frac{M\,\psi(k)^\delta}{(h-k)^\gamma},$$

where

$$M = \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*}, \quad \delta = \frac{2^*}{2_*} - \frac{2^*}{p}, \quad \gamma = 2^*.$$

The assumption $p > \frac{N}{2}$ implies $\delta > 1$, so that applying Lemma 2.2, we have that $\psi(d) = 0$, where

$$d^{2^*} = C(\Omega, N, p)\,M.$$

Since $m(A_d) = 0$, we have $|u| \le d$ almost everywhere, which implies

$$\|u\|_{L^\infty(\Omega)} \le d = C(N, \Omega, p, \alpha)\,\|f\|_{L^p(\Omega)},$$

as desired. □

Remark 2.4. Observe that, in order to prove the previous theorem, we did not use two of the properties of the equation: that the matrix A is bounded from above (we only used its ellipticity) and, above all, the fact that the equation was *linear*: in other words, the proof above also holds for every uniformly elliptic operator.

The second results deals with the case of unbounded solutions.

THEOREM 2.5 (Stampacchia). *Let f belong to $L^p(\Omega)$, with $2_* \leq p < \frac{N}{2}$. Then the solution u of (1.9) belongs to $L^m(\Omega)$, with $m = p^{**} = \frac{Np}{N-2p}$, and there exists a constant C, only depending on N, Ω, p and α, such that*

$$\|u\|_{L^{p^{**}}(\Omega)} \leq C \, \|f\|_{L^p(\Omega)} \, . \tag{2.15}$$

Proof. We begin by observing that if $p = 2_*$, then $p^{**} = 2^*$, so that the result is true in this limit case by the Sobolev embedding. Therefore, we only have to deal with the case $p > 2_*$.

The original proof of Stampacchia used a linear interpolation theorem; i.e., it is typical of a linear framework. We are going to give another proof, following [**5**], which makes use of a technique that can also be applied in a nonlinear context.

Let $k > 0$ be fixed, let $\gamma > 1$ and choose $v = |T_k(u)|^{2\gamma-2}\, T_k(u)$ as test function in (1.9) ($T_k(s)$ has been defined in (1.6)). We obtain, by Theorem 1.19,

$$(2\gamma-1)\int_\Omega A(x)\nabla u \cdot \nabla T_k(u)\, |T_k(u)|^{2\gamma-2} = \int_\Omega f\, |T_k(u)|^{2\gamma-2}\, T_k(u).$$

Using (1.8), and observing that $\nabla u = \nabla T_k(u)$ where $\nabla T_k(u) \neq 0$, we then have

$$\alpha\,(2\gamma-1)\int_\Omega |\nabla T_k(u)|^2\, |T_k(u)|^{2\gamma-2} \leq \int_\Omega |f|\, |T_k(u)|^{2\gamma-1}.$$

Since, again by Theorem 1.19, $|\nabla T_k(u)|^2\, |T_k(u)|^{2\gamma-2} = \frac{1}{\gamma^2}\, |\nabla |T_k(u)|^\gamma|^2$, we have

$$\frac{\alpha\,(2\gamma-1)}{\gamma^2}\int_\Omega |\nabla |T_k(u)|^\gamma|^2 \leq \int_\Omega |f|\, |T_k(u)|^{2\gamma-1}.$$

Using Sobolev inequality (in the left hand side), and Hölder inequality (in the right hand one), we obtain

$$\frac{\alpha\,(2\gamma-1)}{\mathcal{S}_2^2\gamma^2}\left(\int_\Omega |T_k(u)|^{\gamma 2^*}\right)^{\frac{2}{2^*}} \leq \|f\|_{L^p(\Omega)}\left(\int_\Omega |T_k(u)|^{(2\gamma-1)p'}\right)^{\frac{1}{p'}}.$$

We now choose γ so that $\gamma 2^* = (2\gamma-1)p'$, that is $\gamma = \frac{p^{**}}{2^*}$ (as it is easily seen). With this choice, $\gamma > 1$ if and only if $p > 2_*$ (which is true). Since $p < \frac{N}{2}$, we also have $\frac{2}{2^*} > \frac{1}{p'}$, and so

$$\left(\int_\Omega |T_k(u)|^{p^{**}} \right)^{\frac{2}{2^*} - \frac{1}{p'}} \leq C(N, \Omega, p, \alpha) \, \|f\|_{L^p(\Omega)} .$$

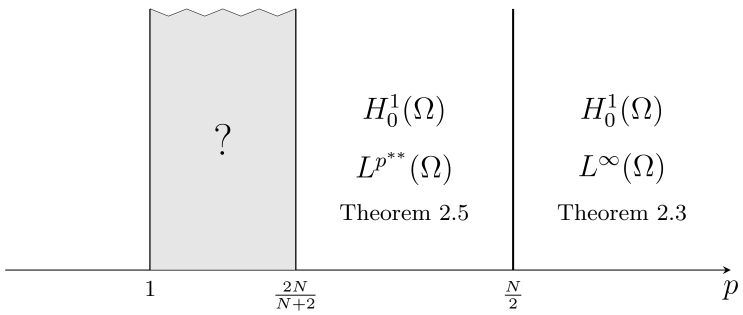
Observing that $\frac{2}{2^*} - \frac{1}{p'} = \frac{1}{p^{**}}$, we have therefore proved that

$$\|T_k(u)\|_{L^{p^{**}}(\Omega)} \leq C(N, \Omega, p, \alpha) \, \|f\|_{L^p(\Omega)} , \quad \forall k > 0.$$

Letting k tend to infinity, and using Fatou lemma (or the monotone convergence theorem), we obtain the result. □

REMARK 2.6. The results of theorems 2.3 and 2.5 are somehow "natural" if we make a mistake... Indeed, let u be the solution of $-\Delta u = f$, with f in $L^p(\Omega)$. Then, if we read the equation, we have that u has two derivatives in $L^p(\Omega)$, so that it belongs to $W_0^{2,p}(\Omega)$. By Sobolev embedding, u then belongs to $W_0^{1,p^*}(\Omega)$ and, again by Sobolev embedding, to $L^{p^{**}}(\Omega)$ (or to $L^\infty(\Omega)$ if $p > \frac{N}{2}$). The "mistake" here is to deduce from the fact that the sum of (some) derivatives of u belongs to $L^p(\Omega)$, the fact that all derivatives are in the same space. Surprisingly, it turns out that, in the case of the laplacian, the fact that $-\Delta u$ belongs to $L^p(\Omega)$ actually implies that u is in $W_0^{2,p}(\Omega)$ (this is the so-called Calderun-Zygmund theory), so that the "mistake" is not an actual one...

Summarizing the results of this chapter, we have the following picture.



We will deal with the "?" part in the forthcoming chapter (actually, in all the forthcoming chapters).

As stated in the chapter, the results proved here can be found in the paper by Stampacchia ([**13**]), also for the proof in the case $2_* \leq p < \frac{N}{2}$, and for $p = \frac{N}{2}$ (for the interested reader), and in the paper by Boccardo and Giachetti ([**5**], for the “nonlinear” proof of Theorem 2.5.

CHAPTER 3

Existence via duality for measure data

We are now going to deal with existence results for data which do not belong to $L^{2_*}(\Omega)$ (i.e., they are not in $H^{-1}(\Omega)$), so that neither Lax-Milgram theorem nor minimization techniques can be applied. Before going on, we need some definitions.

1. Measures

We recall that a *nonnegative measure* on Ω is a set function $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ defined on the σ-algebra $\mathcal{B}(\Omega)$ of Borel sets of Ω (i.e., the smallest σ-algebra containing the open sets) such that $\mu(\emptyset) = 0$ and such that

$$\mu\Big(\bigcup_{n=1}^{+\infty} E_n\Big) = \sum_{n=1}^{+\infty} \mu(E_n),$$

for every sequence $\{E_n\}$ of *disjoint* sets in $\mathcal{B}(\Omega)$. This latter property is called σ-additivity. A σ-additive measure μ is also σ-subadditive, i.e., one has

$$\mu\Big(\bigcup_{n=1}^{+\infty} E_n\Big) \leq \sum_{n=1}^{+\infty} \mu(E_n),$$

for every sequence $\{E_n\}$ of sets in $\mathcal{B}(\Omega)$. A nonnegative measure μ is also *monotone*, i.e., one has that

$$A \subseteq B \quad \text{implies} \quad \mu(A) \leq \mu(B).$$

A measure μ is said to be *regular* if for every E in $\mathcal{B}(\Omega)$ and for every $\varepsilon > 0$ there exist an open set A_ε, and a closed set C_ε, such that

$$C_\varepsilon \subseteq E \subseteq A_\varepsilon, \quad \mu(A_\varepsilon \setminus C_\varepsilon) < \varepsilon.$$

A measure μ is said to be *bounded* if $\mu(\Omega) < +\infty$. The set of nonnegative, regular, bounded measures on Ω will be denoted by $\mathcal{M}^+(\Omega)$. We define the set of *bounded Radon measures* on Ω as

$$\mathcal{M}(\Omega) = \{\mu_1 - \mu_2,\ \mu_i \in \mathcal{M}^+(\Omega)\}.$$

Given a measure μ in $\mathcal{M}(\Omega)$, there exists a unique pair (μ^+, μ^-) in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\Omega)$ such that

$$\mu = \mu^+ - \mu^-,$$

and such there exist E^+ and E^- in $\mathcal{B}(\Omega)$, *disjoint sets*, such that

$$\mu^\pm(E) = \mu(E \cap E^\pm), \quad \forall E \in \mathcal{B}(\Omega).$$

The measures μ^+ and μ^- are the *positive* and *negative* parts of the measure μ. Given a measure μ in $\mathcal{M}(\Omega)$, the measure $|\mu| = \mu^+ + \mu^-$ is said to be the *total variation* of the measure μ. If we define

$$\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega),$$

the vector space $\mathcal{M}(\Omega)$ becomes a Banach space, which turns out to be the dual of $C_0^0(\Omega)$.

A bounded Radon measure μ is said to be *concentrated* on a Borel set E if $\mu(B) = \mu(B \cap E)$ for every Borel set B. In this case, we will write $\mu \llcorner E$. For example, we have $\mu^\pm = \mu \llcorner E^\pm$, with $E^\pm$ as above.

Given two Radon measures μ and ν, we say that μ is *absolutely continuous* with respect to ν if $\nu(E) = 0$ implies $\mu(E) = 0$. In this case we will write $\mu << \nu$. Two Radon measures μ and ν are said to be *orthogonal* if there exists a set E such that $\mu(E) = 0$, and $\nu = \nu \llcorner E$. In this case, we will write $\mu \perp \nu$. For example, given a Radon measure μ, we have $\mu^+ \perp \mu^-$.

THEOREM 3.1. *Let ν be a nonnegative Radon measure. Given a Radon measure μ, there exists a unique pair (μ_0, μ_1) of Radon measures such that*

$$\mu = \mu_0 + \mu_1, \quad \mu_0 << \nu, \quad \mu_1 \perp \nu.$$

Proof. Suppose that μ is nonnegative, and define

$$\mathcal{A} = \{\mu(E) : E \in \mathcal{B}(\Omega),\ \nu(E) = 0\}.$$

Let $\alpha = \sup \mathcal{A}$, and let E_n be a maximizing sequence, i.e., a sequence of Borel sets such that

$$\lim_{n\to+\infty} \mu(E_n) = \alpha, \quad \nu(E_n) = 0.$$

If we define E as the union of the E_n, clearly $\nu(E) = 0$ (since ν is σ-subadditive), and $\mu(E) = \alpha$ (since $\mu(E) \ge \mu(E_n)$ for every n). Define now

$$\mu_1 = \mu \llcorner E, \quad \mu_0 = \mu - \mu_1.$$

Clearly, $\mu_1 \perp \nu$ (since $\nu(E) = 0$, and since μ_1 is concentrated on E by definition). On the other hand, if $\nu(B) = 0$, then $\mu_0(B) = 0$; and indeed, if it were $\mu_0(B) > 0$ for some $B \neq E$, then

$$0 < \mu_0(B) = \mu(B) - \mu(B \cap E) = \mu(B \setminus E),$$

so that $B \cup E$ will be such that $\nu(B \cup E) = 0$, and

$$\mu(B \cup E) = \mu(E) + \mu(B \setminus E) = \alpha + \mu(B \setminus E) > \alpha,$$

thus contradicting the definition of α.

As for uniqueness, if $\mu = \mu_0 + \mu_1 = \mu_0' + \mu_1'$, then $\mu_0 - \mu_0' = \mu_1' - \mu_1$. If $\nu(B) = 0$, we will have $(\mu_1 - \mu_1')(B) = 0$. Since $\mu_1 - \mu_1'$ is also orthogonal with respect to ν, this implies that $(\mu_1 - \mu_1')(E) = 0$ for every Borel set E, so that $\mu_1 = \mu_1'$, hence $\mu_0 = \mu_0'$.

If the measure μ has a sign, it is enough to apply the result to μ^+ and μ^-. □

Examples of bounded Radon measures are the Lebesgue measure $\mathcal{L}^N$ concentrated on a bounded set of $\mathbb{R}^N$, or the measure defined by

$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E, \\ 0 & \text{if } x_0 \notin E, \end{cases}$$

which is called the *Dirac's delta* concentrated at x_0. We clearly have $\delta_{x_0} \perp \mathcal{L}^N$. Another example of Radon measure is the measure defined by

$$\mu(E) = \int_E f(x)\, dx,$$

with f a function in $L^1(\Omega)$. In this case $\mu << \mathcal{L}^N$, and

$$\mu^{\pm}(E) = \int_E f^{\pm}(x)\, dx, \quad |\mu|(E) = \int_E |f(x)|\, dx.$$

Therefore, $L^1(\Omega) \subset \mathcal{M}(\Omega)$. For sequences of measures, we have two notions of convergence: the weak*:

$$\int_\Omega \varphi\, d\mu_n \to \int_\Omega \varphi\, d\mu, \quad \forall \varphi \in C_0^0(\Omega),$$

and the *narrow convergence*:

$$\int_\Omega \varphi\, d\mu_n \to \int_\Omega \varphi\, d\mu, \quad \forall \varphi \in C_{\mathrm{b}}^0(\Omega).$$

For positive measures, narrow convergence is equivalent to weak* convergence and convergence of the "masses" (i.e., $\mu_n(\Omega)$ converges to $\mu(\Omega)$). If x_n is a sequence in Ω which converges to a point x_0 on $\partial\Omega$, then δ_{x_n} converges to zero for the weak* convergence (since the measure δ_{x_0} is indeed the zero measure in Ω), but not for the narrow convergence.

Measures can be approximated (in either convergence) by sequences of bounded functions.

Before dealing with existence results for elliptic equations with measure data, we will begin with a particular case.

2. Duality solutions for L^1 data

Let f and g be two functions in $L^\infty(\Omega)$, and let u and v be the solutions of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\mathrm{div}(A^*(x)\,\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

where A^* is the transposed matrix of A (note that A^* satisfies (1.8) with the same constants as A). Since both u and v belong to $H^1_0(\Omega)$, u can be chosen as test function in the formulation of weak solution for v, and *vice versa*. One obtains

$$\int_\Omega u\,g = \int_\Omega A^*(x)\nabla v \cdot \nabla u = \int_\Omega A(x)\nabla u \cdot \nabla v = \int_\Omega f\,v.$$

In other words, one has

$$\int_\Omega u\,g = \int_\Omega f\,v,$$

for every f and g in $L^\infty(\Omega)$, where u and v solve the corresponding problems with data f and g respectively. Clearly, both u and v belong to $L^\infty(\Omega)$ by Theorem 2.3, but we remark that the two integrals are well-defined also if f only belongs to $L^1(\Omega)$, and u only belongs to $L^1(\Omega)$ (always maintaining the assumption that g — and so v — is a bounded function). This fact inspired to Guido Stampacchia the following definition of solution for (1.9) if the datum is in $L^1(\Omega)$.

DEFINITION 3.2. Let f belong to $L^1(\Omega)$. A function u in $L^1(\Omega)$ is a *duality solution* of (1.8) with datum f if one has

$$\int_\Omega u\,g = \int_\Omega f\,v,$$

for every g in $L^\infty(\Omega)$, where v is the solution of

$$\begin{cases} -\mathrm{div}(A^*(x)\,\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

THEOREM 3.3 (Stampacchia). *Let f belong to $L^1(\Omega)$. Then there exists a unique duality solution of (1.8) with datum f. Furthermore, u belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-2}$.*

Proof. Let $p > \frac{N}{2}$ and define the linear functional $T : L^p(\Omega) \to \mathbb{R}$ by

$$\langle T, g\rangle = \int_\Omega f\, v.$$

By Theorem 2.3, the functional is well-defined; furthermore, since (2.14) holds, there exists $C > 0$ such that

$$|\langle T, g\rangle| \leq \int_\Omega |f|\,|v| \leq \|f\|_{L^1(\Omega)}\,\|v\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^1(\Omega)}\,\|g\|_{L^p(\Omega)}\,,$$

so that T is continuous on $L^p(\Omega)$. By Riesz representation Theorem for L^p spaces, there exists a unique function u_p in $L^{p'}(\Omega)$ such that

$$\langle T, g\rangle = \int_\Omega u_p\, g, \quad \forall g \in L^p(\Omega).$$

Since $L^\infty(\Omega) \subset L^p(\Omega)$, we have

$$\int_\Omega u_p\, g = \langle T, g\rangle = \int_\Omega f\, v, \quad \forall g \in L^\infty(\Omega),$$

so that u_p is a duality solution of (1.9), as desired. We claim that u_p does not depend on p; indeed, if for example $p > q > \frac{N}{2}$, we have

$$\int_\Omega u_p\, g = \int_\Omega f\, v = \int_\Omega u_q\, g, \quad \forall g \in L^\infty(\Omega),$$

so that $u_p = u_q$ in $L^1(\Omega)$ (and so they are almost everywhere the same function). Therefore, there exists a unique function u which is a duality solution of (1.9), and it belongs to $L^{p'}(\Omega)$ for every $p > \frac{N}{2}$; i.e., u belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-2}$, as desired. $\square$

Remark that the fact that u belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-2}$ is consistent with the results of the last part of Example 2.1 (the case $\alpha = N$).

3. Duality solutions for measure data

The case of $L^1(\Omega)$ data is only a particular one, since $L^1(\Omega)$ is a subset of $\mathcal{M}(\Omega)$. However, recalling that $\mathcal{M}(\Omega)$ is the dual of $C^0(\overline{\Omega})$, the proof of Theorem 3.3 could be performed in exactly the same way if one knew that the solution of (1.9) were not only bounded, but also *continuous* on Ω if the datum is in $L^p(\Omega)$ with $p > \frac{N}{2}$. This is exactly the case if the boundary of Ω is sufficiently regular.

THEOREM 3.4 (De Giorgi). *Let Ω be of class C^1, and let f be in $L^p(\Omega)$, with $p > \frac{N}{2}$. Then the solution u of (1.9) with datum f belongs to $C^0(\overline{\Omega})$, and there exists a constant C_p such that*

$$\|u\|_{C^0(\overline{\Omega})} \leq C_p \, \|f\|_{L^p(\Omega)} \, .$$

Thanks to the previous result, we thus have the following existence result.

THEOREM 3.5. *Let μ be a measure in $\mathcal{M}(\Omega)$. Then there exists a unique duality solution of (1.8) with datum μ, i.e., a function u in $L^1(\Omega)$ such that*

$$\int_\Omega u\, g = \int_\Omega v \, d\mu, \quad \forall g \in L^\infty(\Omega),$$

where v is the solution of (1.9) with datum g and matrix A^*. Furthermore, u belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-2}$.

4. Regularity of duality solutions

If the datum f belongs to $L^p(\Omega)$, with $1 < p < 2_*$, then the duality solution of (1.9) is more regular.

THEOREM 3.6. *Let f belong to $L^p(\Omega)$, $1 < p < 2_*$. Then the duality solution u of (1.8) belongs to $L^{p^{**}}(\Omega)$, $p^{**} = \frac{Np}{N-2p}$.*

Proof. Let $q = \frac{Np}{Np-N+2p}$, and define $T : L^q(\Omega) \to \mathbb{R}$ as in the proof of Theorem 3.3. We then have

$$|\langle T, g\rangle| \leq \int_\Omega |f| \, |v| \leq \|f\|_{L^p(\Omega)} \, \|v\|_{L^{p'}(\Omega)} \, .$$

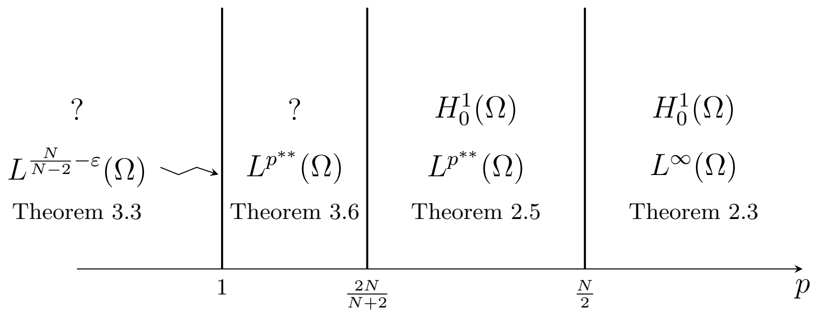
By Theorem 2.5, the norm of v in $L^r(\Omega)$ is controlled by a constant times the norm of g in $L^s(\Omega)$, with $r = s^{**}$. Taking $r = p'$, this gives $s = q$; hence,

$$|\langle T, g\rangle| \leq C \, \|f\|_{L^p(\Omega)} \, \|g\|_{L^q(\Omega)} \, ,$$

so that the function u which represents T belongs to $L^{q'}(\Omega)$; since we have $q' = \frac{Np}{N-2p}$, the result is proved. $\square$

Once again, the fact that u belongs to $L^{p^{**}}(\Omega)$ is consistent with the results of Example 2.1 (the case $\frac{N+2}{2} < \alpha < N$).

The picture at the end of Chapter 2 can now be improved as follows.



Once again we refer the reader to the paper by Stampacchia ([**13**]), where not only the existence and uniqueness of duality solutions is stated and proved, but also a representation formula of the kind

$$u(x) = \int_{\Omega} G(x, y)\, d\mu(y)\,,$$

is given; here $G(x, y)$ is the duality solution of the adjoint equation

$$-\mathrm{div}(A^*(y)\nabla G(x, y)) = \delta_x\,,$$

with homogeneous Dirichlet boundary conditions. The Hölder regularity paper by De Giorgi is [**8**].

CHAPTER 4

Existence via approximation for measure data

The result of Theorem 3.5 is somewhat unsatisfactory: even though it proves that there exists a unique solution by duality of (1.9) if the datum belongs to $\mathcal{M}(\Omega)$, it only states that the solution belongs to some Lebesgue space, and does not say anything about the gradient of such a solution. In order to prove gradient estimates on the duality solution we have to proceed in a different way.

THEOREM 4.1. *Let μ belong to $\mathcal{M}(\Omega)$. Then the unique duality solution of (1.8) with datum f belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.*

Proof. Let f_n be a sequence of $L^\infty(\Omega)$ functions which converges to μ in $\mathcal{M}(\Omega)$, with the property that $\|f_n\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}(\Omega)}$, and let u_n be the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ \qquad u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k > 0$ and choose $v = T_k(u_n)$ as test function of the weak formulation for u_n. We obtain, recalling that $\nabla u_n = \nabla T_k(u_n)$ where $\nabla T_k(u_n) \neq 0$, and using (1.8),

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq \int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u_n) = \int_\Omega f_n T_k(u_n) \leq k\,\|\mu\|_{\mathcal{M}(\Omega)}\,,$$

where in the last passage we have used that $|T_k(u_n)| \leq k$. Using Sobolev embedding in the left hand side, we have

$$\frac{\alpha}{\mathcal{S}_2^2}\left(\int_\Omega |T_k(u_n)|^{2^*}\right)^{\frac{2}{2^*}} \leq k\,\|\mu\|_{\mathcal{M}(\Omega)}\,.$$

Observing that $|T_k(u_n)| = k$ on the set $A_{n,k} = \{x \in \Omega : |u_n(x)| \geq k\}$, we have

$$\frac{\alpha}{\mathcal{S}_2^2}\,k^2\,(m(A_{n,k}))^{\frac{2}{2^*}} \leq k\,\|\mu\|_{\mathcal{M}(\Omega)}\,,$$

which implies

$$m(A_{n,k}) \leq C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{k}\Big)^{\frac{N}{N-2}},$$

with C depending only on N and α. Now we fix $\lambda > 0$, and we have

$$\{|\nabla u_n| \geq \lambda\} = \{|\nabla u_n| \geq \lambda, |u_n| < k\} \cup \{|\nabla u_n| \geq \lambda, |u_n| \geq k\},$$

so that

$$\{|\nabla u_n| \geq \lambda\} \subset \{|\nabla u_n| \geq \lambda, |u_n| < k\} \cup A_{n,k}.$$

Since

$$m(\{|\nabla u_n| \geq \lambda, |u_n| < k\}) \leq \frac{1}{\lambda^2} \int_\Omega |\nabla T_k(u_n)|^2 \leq \frac{k \; \|\mu\|_{\mathcal{M}(\Omega)}}{\lambda^2},$$

we have

$$m(\{|\nabla u_n| \geq \lambda\}) \leq \frac{k \; \|\mu\|_{\mathcal{M}(\Omega)}}{\lambda^2} + C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{k}\Big)^{\frac{N}{N-2}},$$

for every $k > 0$. If we choose $k = \lambda^{\frac{N-2}{N-1}} \; \|\mu\|_{\mathcal{M}(\Omega)}^{\frac{1}{N-1}}$, the above inequality becomes

$$m(\{|\nabla u_n| \geq \lambda\}) \leq C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{\lambda}\Big)^{\frac{N}{N-1}}.$$

Let $q < \frac{N}{N-1}$ be fixed, and let $t > 0$. Then

$$\begin{aligned}
\int_\Omega |\nabla u_n|^q &= \int_{\{|\nabla u_n| < t\}} |\nabla u_n|^q + \int_{\{|\nabla u_n| \geq t\}} |\nabla u_n|^q \\
&\leq t^q \, m(\Omega) + (q-1) \int_t^{+\infty} \lambda^{q-1} \, m(\{|\nabla u_n| \geq \lambda\}) \, d\lambda \\
&\leq t^q \, m(\Omega) + C(q-1) \, \|f\|_{L^1(\Omega)}^{\frac{N}{N-1}} \int_t^{+\infty} \lambda^{q-1-\frac{N}{N-1}} \, d\lambda \\
&= t^q \, m(\Omega) + \frac{C(q-1)}{\frac{N}{N-1} - q} \frac{\|\mu\|_{\mathcal{M}(\Omega)}^{\frac{N}{N-1}}}{t^{\frac{N}{N-1}-q}}.
\end{aligned}$$

Choosing $t = \|\mu\|_{\mathcal{M}(\Omega)}$, we obtain

$$\int_\Omega |\nabla u_n|^q \leq C_q \; \|\mu\|_{\mathcal{M}(\Omega)}^q \,, \tag{4.16}$$

so that u_n is bounded in $W_0^{1,q}(\Omega)$, with $q < \frac{N}{N-1}$. Note that C_q diverges as q tends to $\frac{N}{N-1}$. Therefore, up to subsequences, u_n converges to some function u_q weakly in $W_0^{1,q}(\Omega)$ and strongly in $L^1(\Omega)$. Since u_n, being a weak solution, is such that

$$\int_\Omega u_n \, g = \int_\Omega f_n \, v, \quad \forall g \in L^\infty(\Omega), \; \forall n \in \mathbb{N},$$

we can pass to the limit as n tends to infinity to have

$$\int_\Omega u_q \, g = \int_\Omega v \, d\mu, \quad \forall g \in L^\infty(\Omega),$$

so that u_q (which belongs to $W_0^{1,q}(\Omega)$ for *some* $q < \frac{N}{N-1}$) is **the** duality solution of (1.9) with datum μ. This fact is true for *every* $q < \frac{N}{N-1}$, so that u_q does not depend on q. It then follows that the duality solution u of (1.9) belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. □

REMARK 4.2. If $\mu = f$ is a function in $L^1(\Omega)$, and f_n converges to f strongly in $L^1(\Omega)$, we have that f_n is a Cauchy sequence in $L^1(\Omega)$. Thus, if we repeat the proof of the previous theorem working with $u_n - u_m$, using the linearity of the operator, and "keeping track" of $f_n - f_m$, we find that (4.16) becomes

$$\int_\Omega |\nabla u_n - u_m|^q \leq C_q \, \|f_n - f_m\|^q_{L^1(\Omega)} \, ,$$

for every $q < \frac{N}{N-1}$. Since $\{f_n\}$ is a Cauchy sequence in $L^1(\Omega)$, it then follows that u_n is a Cauchy sequence in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. This implies that u_n *strongly* converges to the solution u in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, so that (up to subsequences) ∇u_n converges to ∇u almost everywhere in Ω.

REMARK 4.3. If $\mu = f$ is a function in $L^1(\Omega)$, and we repeat the proof of the previous theorem working with $u_n - v_n$, where v_n is the solution of (1.9) with a datum g_n which converges to f in $L^1(\Omega)$, we find as before that

$$\int_\Omega |\nabla (u_n - v_n)|^q \leq C \, \|f_n - g_n\|^q_{L^1(\Omega)} \, , \tag{4.17}$$

for every $q < \frac{N}{N-1}$. Since $\{f_n - g_n\}$ tends to zero in $L^1(\Omega)$, it then follows that $u_n - v_n$ tends to zero in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. In other words, the solution u found by approximation does not depend on the sequence we choose to approximate the datum f. We already knew this fact (since every approximating sequence converges to the duality solution which is unique), but this different proof may be useful if, for example, the differential operator is not linear, but allows to prove (4.17) in some way, so that the concept of duality solution is not available.

If the datum f is "more regular", one expects solutions with an increased regularity. We already know, from Theorem 3.6, that the summability of u increases with the summability of f, but what happens to the gradient? Recall that if the datum f is "regular" (i.e., if it belongs to $L^{2_*}(\Omega)$), the summability of u increases with that of f, but the gradient of u always belongs to $(L^2(\Omega))^N$. Surprisingly, this is not the case for "bad" solutions, as the following theorem shows.

THEOREM 4.4. *Let f be a function in $L^m(\Omega)$, $1 < m < 2_*$. Then the duality solution of (1.9) belongs to $W_0^{1,m^*}(\Omega)$, $m^* = \frac{Nm}{N-m}$.*

Proof. Let $f_n = T_n(f)$, and let u_n be the unique solution of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since we already know that u_n will converge to the duality solution of (1.9), it is clear that in order to prove the result it will be enough to prove an *a priori* estimate on u_n in $W_0^{1,m^*}(\Omega)$. In order to do that, we fix $h > 0$ and choose $\varphi_h(u_n) = T_1(G_h(u_n))$ as test function in the weak formulation for u_n. If we define $B_h = \{x \in \Omega : h \leq |u_n| \leq h+1\}$, and $A_h = \{x \in \Omega : |u_n| \geq h\}$ (for the sake of simplicity, we omit the dependence on n on the sets), we obtain, recalling (1.8),

$$\alpha \int_{B_h} |\nabla u_n|^2 \leq \int_\Omega A(x)\nabla u_n \cdot \nabla \varphi_h(u_n) = \int_\Omega f_n\, \varphi_h(u_n) \leq \int_{A_k} |f|.$$

Let now $0 < \lambda < 1$; we can then write

$$\begin{aligned}
\int_\Omega \frac{|\nabla u_n|^2}{(1+|u|)^\lambda} &= \sum_{h=0}^{+\infty} \int_{B_h} \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} \leq \sum_{h=0}^{+\infty} \frac{1}{(1+h)^\lambda} \int_{B_h} |\nabla u_n|^2 \\
&\leq \sum_{h=0}^{+\infty} \frac{1}{\alpha(1+h)^\lambda} \int_{A_h} |f| = \sum_{h=0}^{+\infty} \frac{1}{\alpha(1+h)^\lambda} \sum_{k=h}^{+\infty} \int_{B_k} |f| \\
&= \sum_{k=0}^{+\infty} \int_{B_k} |f| \sum_{h=0}^{k} \frac{1}{\alpha(1+h)^\lambda} \\
&\leq C \sum_{k=0}^{+\infty} \int_{B_k} |f|\,(1+k)^{1-\lambda} \leq C \int_\Omega |f|(1+|u_n|)^{1-\lambda} \\
&\leq C\, \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{(1-\lambda)m'} \right)^{\frac{1}{m'}}.
\end{aligned}$$

Let now $q > 1$ be fixed. Then, by Sobolev and Hölder inequality,

$$\begin{aligned}
\frac{1}{\mathcal{S}_q^q} \left(\int_\Omega |u_n|^{q^*} \right)^{\frac{q}{q^*}} &\leq \int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{\lambda\frac{q}{2}}} (1+|u_n|)^{\lambda\frac{q}{2}} \\
&\leq \left(\int_\Omega \frac{|\nabla u_n|^2}{(1+|u|)^\lambda} \right)^{\frac{q}{2}} \left(\int_\Omega (1+|u_n|)^{\frac{\lambda q}{2-q}} \right)^{1-\frac{q}{2}} \\
&\leq C\, \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{(1-\lambda)m'} \right)^{\frac{q}{2m'}} \\
&\quad \times \left(\int_\Omega (1+|u_n|)^{\frac{\lambda q}{2-q}} \right)^{1-\frac{q}{2}}.
\end{aligned}$$

We now choose λ and q in such a way that

$$(1-\lambda)m' = q^* = \frac{\lambda q}{2-q}.$$

This implies

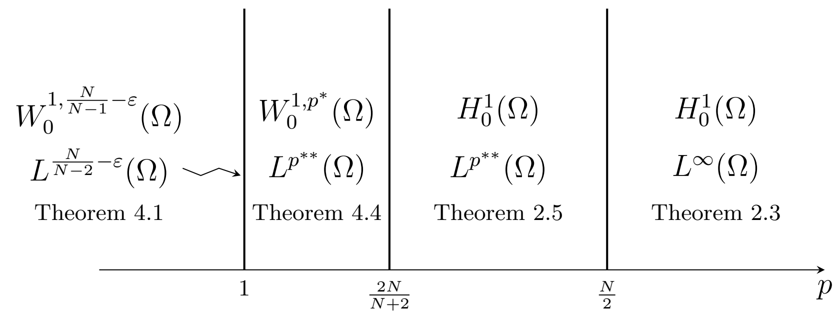
$$\lambda = \frac{N(2-q)}{N-q}, \quad q = m^* = \frac{Nm}{N-m}.$$

It is easy to see that $1 < m < 2_*$ implies $0 < \lambda < 1$, as desired. We thus have

$$\left(\int_\Omega |u_n|^{q^*}\right)^{\frac{q}{q^*}} \leq C \int_\Omega |\nabla u_n|^q \leq C\, \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{q^*}\right)^{1-\frac{q}{2m}}.$$

Since $\frac{q}{q^*} > 1 - \frac{q}{2m}$ is true (being equivalent to $m < \frac{N}{2}$), we obtain from the first and third term that u_n is bounded in $L^{q^*}(\Omega)$ (which is again $L^{m^{**}}(\Omega)$, see Theorem 2.5) by a constant depending (among other quantities) on the norm of f in $L^m(\Omega)$. Once u_n is bounded, the boundedness of $|\nabla u_n|$ in $L^q(\Omega)$ (with $q = m^*$) then follows comparing the second and the third term. $\square$

We can now draw the complete picture.



The proof by approximation of the existence can be found in [**2**] (or [**3**]), and the specific technique of obtaining first an estimate in Marcinkiewicz spaces on u_n, and then on ∇u_n, can be found in [**1**], a paper worth reading for the definition of entropy solutions — and the proof of their uniqueness for $L^1(\Omega)$ data (the extension of this result to (some) measure data can be found in [**4**]).

CHAPTER 5

Nonuniqueness for distributional solutions

If the datum μ is a measure, we have proved in Theorem 4.1 that the sequence u_n of approximating solutions is bounded in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Therefore, and up to subsequences, u_n weakly converges to the solution u in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Choosing a $C_0^1(\Omega)$ test function φ in the formulation (1.10) for u_n, we obtain

$$\int_\Omega A(x)\nabla u_n \cdot \nabla\varphi = \int_\Omega f_n\,\varphi,$$

which, passing to the limit, yields

$$\int_\Omega A(x)\nabla u \cdot \nabla\varphi = \int_\Omega \varphi\,d\mu \quad \forall\varphi \in C_0^1(\Omega),$$

so that u is a solution *in the sense of distributions* of (1.9). Since the definition of solution in the sense of distributions can always be given (even when the notion of duality solution is unavailable due for example to the operator being nonlinear), one may wonder whether there is a way of proving uniqueness of distributional solutions (not passing through duality solutions).

The following example is due to J. Serrin (see [**12**]). Let $\varepsilon > 0$ and $A^\varepsilon(x)$ be the symmetric matrix defined by

$$a_{ij}^\varepsilon(x) = \delta_{ij} + (a_\varepsilon - 1)\frac{x_i\,x_j}{|x|^2}.$$

If $a_\varepsilon = \frac{N-1}{\varepsilon(N-2+\varepsilon)}$, then the function

$$w^\varepsilon(x) = x_1\,|x|^{1-N-\varepsilon}$$

is a solution in the sense of distributions of

$$-\mathrm{div}(A^\varepsilon(x)\,\nabla w^\varepsilon) = 0, \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5.18}$$

Indeed, if we rewrite $w(x) = x_1|x|^\alpha$ and

$$a_{ij}(x) = \delta_{ij} + \beta\frac{x_i\,x_j}{|x|^2},$$

simple (but tedious) calculations imply

$$w_{x_1}(x) = |x|^\alpha + \alpha x_1^2|x|^{\alpha-2}, \quad w_{x_i}(x) = \alpha x_1 x_i|x|^{\alpha-2},$$

so that

$$\sum_{i=1}^{N} a_{ij}(x)\, w_{x_i}(x) = \delta_{1j}|x|^{\alpha} + (\alpha\beta + \alpha + \beta)x_1 x_j |x|^{\alpha-2}.$$

Therefore,

$$(A(x)\,\nabla w)_{x_1} = \alpha x_1|x|^{\alpha-2} + (\alpha\beta+\alpha+\beta)[2x_1|x|^{\alpha-2} + (\alpha-2)x_1^3|x|^{\alpha-4}],$$

and

$$(A(x)\,\nabla w)_{x_j} = (\alpha\beta+\alpha+\beta)[x_1|x|^{\alpha-2} + (\alpha-2)x_1x_j^2|x|^{\alpha-4}],$$

so that

$$\operatorname{div}(A(x)\nabla w) = x_1|x|^{\alpha-2}[\alpha + (N-1+\alpha)(\alpha\beta+\alpha+\beta)].$$

Given $0<\varepsilon<1$, if we choose $\alpha = 1-N-\varepsilon$, and $\beta = \frac{N-1}{\varepsilon(N-2+\varepsilon)}+1$, we have

$$\alpha + (N-1+\alpha)(\alpha\beta+\alpha+\beta) = 0,$$

so that w is a solution of (5.18) if $x \neq 0$. To prove that w^ε is a solution in the sense of distributions in the whole $\mathbb{R}^N$, let φ be a function in $C_0^1(\Omega)$, and observe that since $|A^\varepsilon(x)\nabla w^\varepsilon|$ belongs to $L^1(\Omega)$, we have

$$\int_{\mathbb{R}^N} A^\varepsilon(x)\nabla w^\varepsilon\cdot\nabla\varphi = \lim_{r\to0^+}\int_{\mathbb{R}^N\setminus B_r(0)} A^\varepsilon(x)\nabla w^\varepsilon\cdot\nabla\varphi.$$

Using Gauss-Green formula, and recalling that w^ε is a solution of the equation outside the origin, we have

$$\int_{\mathbb{R}^N} A^\varepsilon(x)\nabla w^\varepsilon\cdot\nabla\varphi = -\lim_{r\to0^+}\int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon\cdot\nu\, d\sigma,$$

where ν is the exterior normal to $B_r(0)$, i.e., $\nu = \frac{x}{r}$. By a direct computation,

$$A^\varepsilon(x)\nabla w^\varepsilon\cdot\frac{x}{r} = Qx_1|r|^{\alpha-1},$$

with $Q = 1+\alpha\beta+\alpha+\beta = -\frac{N-1}{\varepsilon}$. Therefore, recalling the value of α, and rescaling to the unit sphere,

$$-\int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon\cdot\nu\, d\sigma = \frac{N-1}{\varepsilon}\frac{1}{r^\varepsilon}\int_{\partial B_1(0)} \varphi(ry)x_1\, d\sigma.$$

Using again the Gauss-Green formula, we have

$$\int_{\partial B_1(0)} \varphi(ry)x_1\, d\sigma = r\int_{B_1(0)} e_1\cdot\nabla\varphi(rx)\, dx,$$

where $e_1 = (1,0,\dots,0)$. Therefore, since $0<\varepsilon<1$, we have

$$\lim_{r\to0^+}\int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon\cdot\nu\, d\sigma = \lim_{r\to0^+} r^{1-\varepsilon}\int_{B_1(0)} e_1\cdot\nabla\varphi(rx)\, dx = 0,$$

so that w^ε is a solution in the sense of distributions of $-\mathrm{div}(A^\varepsilon \nabla w^\varepsilon) = 0$ in the whole $\mathbb{R}^N$.

Let now $\Omega = B_1(0)$ be the unit ball, and let v_ε be the unique solution of

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\,\nabla v^\varepsilon) = \mathrm{div}(A^\varepsilon(x)\,\nabla x_1) & \text{in } \Omega, \\ v^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists since $\mathrm{div}(A^\varepsilon(x)\,\nabla x_1)$ is a regular function belonging to $H^{-1}(\Omega)$ (as can be easily seen). Therefore, the function $z^\varepsilon = v^\varepsilon + x_1$ is the unique solution in $H^1(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\,\nabla z^\varepsilon) = 0 & \text{in } \Omega, \\ z^\varepsilon = x_1 & \text{on } \partial\Omega, \end{cases}$$

so that the function $u^\varepsilon = w^\varepsilon - z^\varepsilon$ is a solution in the sense of distributions of

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\,\nabla u^\varepsilon) = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which is not identically zero since z^ε belongs to $H^1(\Omega)$, while w^ε belongs to $W_0^{1,q}(\Omega)$ for every $q < q_\varepsilon = \frac{N}{N-1+\varepsilon}$. Hence, the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has infinitely many solutions in the sense of distributions, which can be written as $u = \overline{u} + t\,u^\varepsilon$, t in $\mathbb{R}$, where $\overline{u}$ is the duality solution.

One may observe that the solution found by approximation belongs to $W_0^{1,q}(\Omega)$ for *every* $q < \frac{N}{N-1}$, while the solution of the above example belongs to $W_0^{1,q}(\Omega)$ for *some* $q < \frac{N}{N-1}$, and that we are not allowed to take $\varepsilon = 0$ since in this case a_ε diverges. Thus one may hope that there is still uniqueness of the solution obtained by approximation. However, it is possible to modify Serrin's example in dimension $N \geq 3$ (see [**11**]) to find a nonzero solution in the sense of distributions for

$$\begin{cases} -\mathrm{div}(B^\varepsilon(x)\,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Here

$$B^\varepsilon(x) = \begin{pmatrix} 1 + (a_\varepsilon - 1)\frac{x_1^2}{x_1^2+x_2^2} & (a_\varepsilon - 1)\frac{x_1\,x_2}{x_1^2+x_2^2} & 0 \\ (a_\varepsilon - 1)\frac{x_1\,x_2}{x_1^2+x_2^2} & 1 + (a_\varepsilon - 1)\frac{x_2^2}{x_1^2+x_2^2} & 0 \\ 0 & 0 & I \end{pmatrix},$$

where I is the identity matrix in $\mathbb{R}^{N-2}$, and a_ε is as above, with ε fixed so that $w^\varepsilon(x) = x_1\,(\sqrt{x_1^2 + x_2^2})^{\varepsilon-1}$ belongs to $W^{1,q}(\mathbb{R}^2)$ for every $q < 2$.

On the other hand, in dimension $N = 2$ there is a unique solution in the sense of distributions belonging to $W_0^{1,q}(\Omega)$, for every $q < 2$. The proof of this fact uses Meyers' regularity theorem for linear equations with regular data.

THEOREM 5.1 (Meyers). *Let A be a matrix which satisfies (1.8). Then there exists $p > 2$ (p depends on the ratio $\frac{\alpha}{\beta}$ and becomes larger as $\frac{\alpha}{\beta}$ tends to 1) such that if u is a solution of (1.9) with datum f belonging to $L^\infty(\Omega)$, then u belongs to $W_0^{1,p}(\Omega)$.*

THEOREM 5.2. *Let $N = 2$. Then there exists a unique solution in the sense of distributions of (1.9) such that u belongs to $W_0^{1,q}(\Omega)$, for every $q < 2$.*

Proof. Since the equation is linear, it is enough to prove that if u is such that

$$\int_\Omega A(x)\,\nabla u \cdot \nabla\varphi = 0, \quad \forall \varphi \in C_0^1(\Omega),$$

then $u = 0$. Since u belongs to $W_0^{1,q}(\Omega)$, for every $q < 2$, it is enough to prove that

$$\int_\Omega A(x)\,\nabla u \cdot \nabla\varphi = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega),$$

for some $p > 2$, implies $u = 0$. Let B be a subset of Ω, and let v_B be the solution of

$$\begin{cases} -\mathrm{div}(A^*(x)\,\nabla v_B) = \chi_B & \text{in } \Omega, \\ \qquad v = 0 & \text{on } \partial\Omega. \end{cases}$$

By Meyers' theorem, v_B belongs to $W_0^{1,p}(\Omega)$, for some $p > 2$. Hence

$$\int_\Omega A(x)\,\nabla u \cdot \nabla v_B = 0,$$

while, choosing u as test function in the weak formulation for v_B (which can be done using a density argument and the regularity of ∇v_B), we have

$$\int_\Omega A^*(x)\,\nabla v_B \cdot \nabla u = \int_B u.$$

Therefore,

$$\int_B u = 0, \quad \forall B \subseteq \Omega,$$

and this implies $u \equiv 0$. $\square$

Apart from the papers of Serrin and Prignet already quoted in the chapter, Meyers' regularity result can be found in [**9**].

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Un'equazione non lineare

Supponiamo ora di avere una funzione $a : \mathbb{R} \to \mathbb{R}$ continua e tale che

$$\alpha \le a(s) \le \beta\,, \quad \forall s \in \mathbb{R}\,, \tag{1}$$

con $0 < \alpha \le \beta$ costanti reali. Ci chiediamo se, data f in $L^2(\Omega)$, esista una soluzione debole dell'equazione

$$\begin{cases} -\mathrm{div}(a(u)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega, \end{cases} \tag{2}$$

ovvero una funzione u di $H_0^1(\Omega)$ tale che

$$\int_\Omega a(u)\,\nabla u \cdot \nabla v = \int_\Omega f\,v\,, \quad \forall v \in H_0^1(\Omega)\,.$$

Si noti che, essendo $a(u)$ in $L^\infty(\Omega)$ per ogni u misurabile, l'integrale a primo membro è ben definito.

Una prima risposta, positiva, segue da un semplice cambio di variabile: detta

$$A(s) = \int_0^s a(t)\,dt\,,$$

e posta $v = A(u)$, allora $\nabla v = A'(u)\,\nabla u = a(u)\,\nabla u$, cosicché u risolve (2) se e solo se v risolve in $H_0^1(\Omega)$ l'equazione $-\Delta v = f$. Siccome quest'ultima equazione ha una ed una sola soluzione, ed A è invertibile (essendo iniettiva), $u = A^{-1}(R(f))$ risolve (2).

Sfortunatamente, il "trucco" del cambio di variabile non è più applicabile nel caso in cui la funzione a dipenda anche da x, il che vuol dire che nel caso generale è necessario seguire un'altra strada.

Una strada sbagliata

Un primo tentativo possibile consiste, per analogia con il caso dell'operatore lineare, nel considerare il funzionale

$$J(u) = \frac{1}{2}\int_\Omega a(u)\,|\nabla u|^2 - \int_\Omega f\,u\,, \quad u \in H_0^1(\Omega)\,,$$

e vedere se ad esso si può applicare il teorema di Weierstrass. Essendo $a(s) \ge \alpha > 0$ si vede facilmente che se $\|u\|_{H_0^1(\Omega)}$ diverge, allora $J(u)$ diverge. D'altra parte, se u_n converge debolmente ad u in $H_0^1(\Omega)$, è facile dimostrare la debole semicontinuità inferiore di J passando al limite nell'identità

$$0 \le \int_\Omega a(u_n)\,|\nabla u_n|^2 + \int_\Omega a(u_n)\,|\nabla u|^2 - 2\int_\Omega a(u_n)\,\nabla u_n \cdot \nabla u\,,$$

ed usando la continuità (per il teorema di Rellich-Kondrachov) del termine $\int f\,u$.

Dal teorema di Weierstrass segue allora l'esistenza del minimo u di J su $H_0^1(\Omega)$:

$$J(u) \leq J(v)\,, \quad \forall v \in H_0^1(\Omega)\,.$$

Fin qui tutto bene; quale equazione possiamo però scrivere (in forma debole) per u? Partendo dalla disuguaglianza $J(u) \leq J(u+tv)$ si arriva, dopo alcuni passaggi, a

$$\begin{aligned} 0 \leq \frac{1}{2} & \int_\Omega [a(u+tv) - a(u)]\,|\nabla u|^2 \\ & + t\int_\Omega a(u+tv)\,\nabla u \cdot \nabla v + \frac{t^2}{2}\int_\Omega a(u+tv)\,|\nabla v|^2 - t\int_\Omega f\,v\,. \end{aligned}$$

Dividendo per $t > 0$ e passando al limite per t tendente a zero, si troverebbe (se ogni passaggio fosse lecito)

$$0 \leq \frac{1}{2}\int_\Omega a'(u)\,|\nabla u|^2\,v + \int_\Omega a(u)\,\nabla u \cdot \nabla v - \int_\Omega f\,v\,,$$

e la disuguaglianza opposta dividendo per $t < 0$. Dunque, u in $H_0^1(\Omega)$ sarebbe tale che

$$\int_\Omega a(u)\,\nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega a'(u)\,|\nabla u|^2\,v = \int_\Omega f\,v\,, \quad \forall v \in H_0^1(\Omega)\,.$$

Il problema, nell'identità appena scritta, è duplice. Innanzitutto la funzione a è solo continua, per cui $a'(s)$ non esiste. Poco male, aggiungiamo l'ipotesi che a sia derivabile con derivata continua (si noti, però, che tale ipotesi è superflua per ottenere l'esistenza del minimo). Successivamente, ed anche se a è derivabile, il termine

$$\int_\Omega a'(u)\,|\nabla u|^2\,v$$

non è detto sia definito per ogni v in $H_0^1(\Omega)$: già il termine $|\nabla u|^2$ da solo è solamente in $L^1(\Omega)$. Il che vuol dire che dobbiamo "restringere" la classe delle funzioni test, passando da $H_0^1(\Omega)$ a $H_0^1(\Omega) \cap L^\infty(\Omega)$; e non basta: dobbiamo aggiungere anche l'ipotesi $a'(s)$ limitata (il fatto che a sia limitata non implica che lo sia la sua derivata...).

Fatte tutte queste ipotesi, ogni minimo u di J è tale che

$$\int_\Omega a(u)\,\nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega a'(u)\,|\nabla u|^2\,v = \int_\Omega f\,v\,, \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega)\,,$$

ed è quindi soluzione debole dell'equazione

$$\begin{cases} -\mathrm{div}(a(u)\,\nabla u) + \frac{1}{2}\,a'(u)\,|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega, \end{cases}$$

la quale si guarda però bene dall'essere (2): abbiamo sbagliato strada, sostanzialmente perché la derivata del prodotto non è il prodotto delle derivate!

Osservazione 1. Non trovate strano il fatto che il minimo u sia in $H_0^1(\Omega)$ nonostante il termine $a'(u)\,|\nabla u|^2$ sia solo in $L^1(\Omega)$? Non c'è contraddizione con quello che abbiamo visto studiando il problema con dati $L^1(\Omega)$?

La strada giusta

Fissiamo v in $L^2(\Omega)$. Allora, essendo $a(v)$ una funzione limitata e strettamente positiva, la matrice $A(x) = a(v(x))\,I$ è uniformemente ellittica e simmetrica ed esiste quindi unica la soluzione u in $H_0^1(\Omega)$ del problema

$$\begin{cases} -\mathrm{div}(a(v)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases} \tag{3}$$

In altre parole, è ben definita l'applicazione $S : L^2(\Omega) \to H_0^1(\Omega)$ data da $S(v) = u$. Inoltre, dato che $H_0^1(\Omega)$ si immerge in $L^2(\Omega)$, S può essere vista come un'applicazione da $L^2(\Omega)$ in sé. È a questo punto chiaro che una soluzione di (2) è un punto fisso per S. Proviamo ad applicare il teorema delle contrazioni?

Siano v e w in $L^2(\Omega)$, e siano $u = S(v)$ e $z = S(w)$. Scegliendo $u-z$ come funzione test nelle formulazioni deboli per u e z e sottraendo, si ottiene

$$\int_\Omega a(v)\,\nabla u\cdot\nabla(u-z) - \int_\Omega a(w)\,\nabla z\cdot\nabla(u-z) = 0\,,$$

da cui

$$\int_\Omega a(v)\,\nabla(u-z)\cdot\nabla(u-z) = \int_\Omega [a(v)-a(w)]\,\nabla z\,\cdot\nabla(u-z)$$

Ricordando l'ellitticità di a, ed applicando la disuguaglianza di Hölder, si ha

$$\alpha\,\|u-z\|^2_{H_0^1(\Omega)} \le \|a(v)-a(w)\|_{L^\infty(\Omega)}\,\|z\|_{H_0^1(\Omega)}\,\|u-z\|_{H_0^1(\Omega)}\,.$$

Ricordando che $\|z\|_{H_0^1(\Omega)} \le R$, si ottiene allora

$$\alpha\,\|u-z\|_{H_0^1(\Omega)} \le R\,\|a(v)-a(w)\|_{L^\infty(\Omega)}\,,$$

Usando la disuguaglianza di Poincaré si ottiene allora

$$\|S(v)-S(w)\|_{L^2(\Omega)} = \|u-z\|_{L^2(\Omega)} \le B\,\|a(v)-a(w)\|_{L^\infty(\Omega)}\,.$$

Possiamo andare avanti da qui senza ulteriori ipotesi? A destra avremmo bisogno di controllare $a(v) - a(w)$ con $v - w$ e senza supporre che a sia lipschitziana non ci riusciamo. Poco male, aggiungiamo l'ipotesi di lipschitzianità su a. Otteniamo allora

$$\|S(v) - S(w)\|_{L^2(\Omega)} \leq B\,L\,\|v - w\|_{L^\infty(\Omega)} .$$

Siamo contenti? Di nuovo, no: non c'è modo di controllare la norma di $v - w$ in $L^\infty(\Omega)$ con la norma di $v - w$ in $L^2(\Omega)$ (vale infatti la disuguaglianza opposta). Si può sostituire la norma in $L^2(\Omega)$ a sinistra con la norma in $L^\infty(\Omega)$? La dimostrazione del teorema di Stampacchia funziona? Innanzitutto dobbiamo prendere f più regolare di $L^2(\Omega)$ per avere $S(v)$ in $L^\infty(\Omega)$ (serve f in $L^p(\Omega)$, con $p > \frac{N}{2}$). Poi, scegliendo $G_k(u-z)$ come funzione test nelle due equazioni e sottraendo si ottiene

$$\int_\Omega a(v)\,\nabla(u-z)\cdot\nabla G_k(u-z) = \int_\Omega [a(v) - a(w)]\,\nabla z\,\cdot\nabla G_k(u-z)\,.$$

Usando l'ellitticità da una parte e la disuguaglianza di Hölder dall'altra, si arriva a

$$\alpha\int_\Omega |\nabla G_k(u-z)|^2 \leq \left(\int_{A_k}|a(v)-a(w)|^2\,|\nabla z|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla G_k(u-z)|^2\right)^{\frac{1}{2}},$$

dove $A_k = \{|u-z| > k\}$. Semplificando ed elevando al quadrato, si ottiene

$$\int_\Omega |\nabla G_k(u-z)|^2 \leq C\int_{A_k}|a(v)-a(w)|^2\,|\nabla z|^2\,.$$

Sfortunatamente a destra c'è una funzione che, anche supponendo a lipschitziana, è solo in $L^1(\Omega)$; in altre parole, non possiamo applicare la disuguaglianza di Hölder una seconda volta per ottenere una potenza della misura di A_k (che era la chiave per far funzionare il metodo di Stampacchia e la stima in $L^\infty(\Omega)$).

In sostanza, il metodo delle contrazioni non funziona: non c'è modo di stimare una norma delle soluzioni con la stessa norma dei "dati".

Fortunatamente, il teorema delle contrazioni non è l'unico teorema di punto fisso esistente...

Teorema 2 (Schauder). *Sia K un convesso chiuso e limitato di uno spazio di Banach e sia $S : K \to K$ un'applicazione continua tale che $\overline{S(K)}$ sia compatto. Allora esiste almeno un punto fisso di S.*

Per applicare il teorema di Schauder nel nostro caso, osserviamo innanzitutto che esiste $R > 0$ tale che $\|S(v)\|_{L^2(\Omega)} \leq R$ per ogni

v in $L^2(\Omega)$. Scegliendo infatti $u = S(v)$ come funzione test nella formulazione debole di (3), ed usando l'ellitticità, si ha

$$(4) \qquad \alpha \int_\Omega |\nabla u|^2 \le \int_\Omega a(v)\, |\nabla u|^2 = \int_\Omega f\, u \le \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} .$$

Ricordando la disuguaglianza di Poincaré, si ottiene

$$\|u\|^2_{L^2(\Omega)} \le C \, \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} ,$$

e quindi la tesi con $R = C\,\|f\|_{L^2(\Omega)}$. In questa maniera, la sfera di $L^2(\Omega)$ raggio R è un convesso chiuso e limitato invariante per S. Mostriamo ora che S è continua. Sia $\{v_n\}$ una successione convergente a v in $L^2(\Omega)$, e siano $u_n = S(v_n)$ le corrispondenti soluzioni di (3). Dalla (4) si ottiene, ricordando che la norma di u_n in $L^2(\Omega)$ è minore di R,

$$\alpha \int_\Omega |\nabla u_n|^2 \le R\, \|f\|_{L^2(\Omega)} ,$$

da cui segue la limitatezza di u_n in $H^1_0(\Omega)$. Possiamo dunque estrarre sottosuccessioni da v_n ed u_n in modo tale che v_n converga a v quasi ovunque e che u_n converga ad u debolmente in $H^1_0(\Omega)$ e fortemente in $L^2(\Omega)$. A questo punto è possibile passare al limite su n nelle identità

$$\int_\Omega a(v_n)\, \nabla u_n \cdot \nabla z = \int_\Omega f\, z\,, \quad \forall z \in H^1_0(\Omega)\,,$$

per dimostrare che u è soluzione di (3) con $a(v)$, ovvero che $u = S(v)$ (per l'unicità della soluzione di (3)). Dal momento che il limite u non dipende dalle sottosuccessioni estratte, tutta la successione $u_n = S(v_n)$ converge ad $u = S(v)$, e quindi S è continua.

Rimane da dimostrare la compattezza di $\overline{S(K)}$. Abbiamo però appena dimostrato che $S(K)$ è limitato in $H^1_0(\Omega)$: per il teorema di Rellich-Kondrachov, $S(K)$ è precompatto in $L^2(\Omega)$, come volevasi dimostrare.

In definitiva, applicando il teorema di Schauder all'operatore S, si dimostra che esiste almeno una soluzione di (2).

Problemi agli autovalori

Sappiamo già che se Ω è un aperto limitato di $\mathbb{R}^N$ ($N \geq 2$), e se f appartiene a $L^2(\Omega)$ (o, più in generale, a $L^{\frac{2N}{N+2}}(\Omega)$), allora esiste un'unica soluzione u in $H_0^1(\Omega)$ dell'equazione ellittica

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega, \end{cases}$$

nel senso che

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f\, v\,, \qquad \forall v \in H_0^1(\Omega).$$

Sappiamo inoltre che tale soluzione è l'unico punto di minimo in $H_0^1(\Omega)$ del funzionale di Dirichlet

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f\, u,$$

punto di minimo che esiste dal momento che il funzionale è *coercitivo*:

$$\lim_{\|u\|_{H_0^1(\Omega)} \to +\infty} I(u) = +\infty,$$

e *debolmente semicontinuo inferiormente*:

$$u_n \rightharpoonup u \quad \Longrightarrow \quad I(u) \leq \liminf_{n\to+\infty} I(u_n).$$

Ci chiediamo ora se la stessa tecnica di minimizzazione funzioni per dimostrare l'esistenza di soluzioni per il problema

$$(1) \qquad \begin{cases} -\Delta u = \lambda u + f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega, \end{cases}$$

con $\lambda \in \mathbb{R}$. La risposta è evidentemente affermativa se $\lambda < 0$: considerando il funzionale

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega f\, u,$$

esso è evidentemente coercitivo (dato che, essendo $\lambda < 0$, $I_\lambda \geq I$) e debolmente semicontinuo inferiormente dato che

$$u_n \rightharpoonup u \quad \Longrightarrow \quad \int_\Omega u_n^2 \to \int_\Omega u^2,$$

essendo compatta per il Teorema di Rellich l'immersione di $H_0^1(\Omega)$ in $L^2(\Omega)$.

I problemi nascono nel caso $\lambda > 0$: il funzionale è illimitato inferiormente se λ è troppo "grande". Sia infatti v in $H_0^1(\Omega)$, $v \neq 0$, e sia $\Lambda > 0$ tale che

$$\Lambda \int_\Omega v^2 \geq 2 \int_\Omega |\nabla v|^2.$$

Chiaramente, un tale valore di Λ esiste sempre se $v \neq 0$. Se calcoliamo $I_\Lambda(tv)$, con $t \in \mathbb{R}$, abbiamo

$$I_\Lambda(tv) = \frac{t^2}{2} \int_\Omega |\nabla v|^2 - \frac{\Lambda t^2}{2} \int_\Omega v^2 - t \int_\Omega f\, v \leq -\frac{\Lambda\, t^2}{4} \int_\Omega v^2 - t \int_\Omega f\, v,$$

e dunque

$$\lim_{t \to +\infty} I_\Lambda(tv) = -\infty.$$

D'altro canto, ricordando la disuguglianza di Poincaré:

$$\mathcal{P} \int_\Omega v^2 \leq \int_\Omega |\nabla v|^2, \qquad \forall v \in H_0^1(\Omega),$$

è chiaro che se $0 < \lambda < \mathcal{P}$ il funzionale I_λ è sia coercitivo che debolmente semicontinuo inferiormente, e quindi esiste un minimo in $H_0^1(\Omega)$.

Per capire cosa accade nel caso generale, affrontiamo innanzitutto il problema con $f \equiv 0$, ovvero l'equazione

$$\begin{cases} -\Delta u = \lambda\, u & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases} \tag{2}$$

Chiaramente $u \equiv 0$ è una soluzione del problema, ma un confronto con la situazione nel caso unidimensionale ci fa capire che esistono altre possibilità. Se, infatti, consideriamo l'equazione

$$\begin{cases} -u'' = \lambda\, u & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases}$$

troviamo la soluzione nulla se λ non è un quadrato di un numero intero, mentre se $\lambda = n^2$ per qualche n in $\mathbb{N}$ abbiamo le infinite soluzioni $A \operatorname{sen}(nx)$, al variare di A in $\mathbb{R}$.

Per capire cosa accade in dimensione qualsiasi, modifichiamo il problema di minimizzazione, trasformandolo da problema "libero" a problema "vincolato". Sia allora

$$m = \inf\left\{ \int_\Omega |\nabla u|^2,\ u \in H_0^1(\Omega),\ \int_\Omega u^2 = 1 \right\}.$$

Osserviamo innanzitutto che $0 \leq m < +\infty$; sia poi u_n tale che

$$\int_\Omega |\nabla u_n|^2 \to m, \quad \int_\Omega u_n^2 = 1.$$

Chiaramente u_n è limitata in $H_0^1(\Omega)$, cosicché possiamo estrarne una sottosuccessione (che chiameremo ancora u_n) debolmente convergente in $H_0^1(\Omega)$ ad una funzione u. Dal momento che l'immersione di $H_0^1(\Omega)$ in $L^2(\Omega)$ è compatta, l'avere u_n norma 1 in $L^2(\Omega)$ implica che anche u ha norma 1 in $L^2(\Omega)$. In altre parole, il limite debole di u_n soddisfa ancora il vincolo. Si ha allora, ricordando la debole semicontinuità inferiore debole della norma,

$$m \leq \int_\Omega |\nabla u|^2 \leq \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^2 = m,$$

da cui segue che m è un minimo, e si ha

$$m = \int_\Omega |\nabla u|^2.$$

Essendo $u \neq 0$ (dato che ha norma 1 in $L^2(\Omega)$), si ha che $m > 0$. Si può poi dimostrare che il minimo, a meno del segno, è unico. Osserviamo ora che se v è in $H_0^1(\Omega)$, con $v \neq 0$, allora $\frac{v}{\|v\|_{L^2(\Omega)}}$ ha norma 1 in $L^2(\Omega)$, e quindi

$$m \leq \int_\Omega \frac{|\nabla v|^2}{\|v\|^2_{L^2(\Omega)}},$$

da cui si ottiene

$$m \int_\Omega v^2 \leq \int_\Omega |\nabla v|^2.$$

Dal momento che tale disuguaglianza è evidentemente vera anche per $v \equiv 0$, si ha

$$m \int_\Omega v^2 \leq \int_\Omega |\nabla v|^2, \qquad \forall v \in H_0^1(\Omega). \tag{3}$$

In altre parole, $m = \mathcal{P}$, la costante di Poincaré.

Sia ora u una delle due funzioni di norma 1 in $L^2(\Omega)$ che realizza l'uguaglianza in (3); scriviamo $u = u_+ - u_-$, definiamo $v(t) = u_+ - t\,u_-$, e calcoliamo

$$F(t) = \int_\Omega |\nabla v(t)|^2 - m \int_\Omega v^2(t).$$

Chiaramente $F(t) \geq 0$ per ogni t e $F(1) = 0$. Dal momento che

$$\int_\Omega \nabla u_+ \cdot \nabla u_- = 0 = \int_\Omega u_+\, u_-,$$

si ha

$$F(t) = \int_\Omega |\nabla u_+|^2 - m \int_\Omega u_+^2 + t^2 \left(\int_\Omega |\nabla u_-|^2 - m \int_\Omega u_-^2 \right). \tag{4}$$

Essendo
$$\int_\Omega |\nabla u_-|^2 - m \int_\Omega u_-^2 \geq 0$$
per la (3), da (4) si vede facilmente che
$$\min_{t\in\mathbb{R}} F(t) = F(0) = \int_\Omega |\nabla u_+|^2 - m \int_\Omega u_+^2,$$
a meno che
$$\int_\Omega |\nabla u_-|^2 - m \int_\Omega u_-^2 = 0,$$
nel qual caso F è costante. Essendo $F(1) = 0$ un altro punto di minimo, ne segue che $F(t)$ deve essere costantemente nulla. Pertanto
$$\int_\Omega |\nabla u_-|^2 - m \int_\Omega u_-^2 = 0,$$
da cui segue
$$F(t) = \int_\Omega |\nabla u_+|^2 - m \int_\Omega u_+^2 = 0.$$
In altre parole, se u ha norma 1 in $L^2(\Omega)$ e realizza l'uguaglianza in (3), allora anche u_+ e u_- realizzano l'uguaglianza. Se $u_+ \neq 0$, allora normalizzando u_+ in $L^2(\Omega)$ si ottiene una funzione di norma 1 che realizza l'uguaglianza in (3) e quindi (per l'unicità a meno del segno di u) si ha $u_+ = \pm u$: ne segue che u ha segno costante in Ω, che supporremo quindi positivo.

Che equazione risolve il minimo u? Dal momento che si tratta di minimizzazione vincolata, appariranno dei moltiplicatori di Lagrange. Eseguendo i calcoli si trova che u è soluzione di
$$\begin{cases} -\Delta u = m\, u & \text{in } \Omega, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega, \\ \quad u = 0 & \text{su } \partial\Omega. \end{cases}$$
Per motivi "storici", definiamo $\lambda_1 = m$ e $\varphi_1 = u$, che chiamiamo rispettivamente *primo autovalore* e *prima autofunzione* del laplaciano in Ω. Pertanto,
$$\begin{cases} -\Delta \varphi_1 = \lambda_1\, \varphi_1 & \text{in } \Omega, \\ \varphi_1 \geq 0,\ \varphi_1 \not\equiv 0 & \text{in } \Omega, \\ \quad \varphi_1 = 0 & \text{su } \partial\Omega, \\ \|\varphi_1\|_{L^2(\Omega)} = 1. \end{cases} \tag{5}$$
Se lasciamo cadere la normalizzazione in $L^2(\Omega)$, allora anche $t\,\varphi_1$ risolve la stessa equazione, qualsiasi sia t in $\mathbb{R}$: in altre parole, le soluzioni

del problema formano uno spazio vettoriale di dimensione almeno 1, che chiameremo E_1. Per quanto detto prima, la dimensione di E_1 è esattamente pari ad 1 (si noti il parallelo con il caso $N = 1$).

Ricordiamo ora che un teorema dovuto a Stampacchia afferma che se v risolve l'equazione

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{su } \partial\Omega, \end{cases}$$

allora v è in $L^\infty(\Omega)$ se f appartiene a $L^q(\Omega)$ con $q > \frac{N}{2}$, mentre appartiene a $L^{q^{**}}(\Omega)$, con $q^{**} = \frac{Nq}{N-2q}$, se f appartiene a $L^q(\Omega)$ con $\frac{2N}{N+2} \leq q < \frac{N}{2}$. Supponiamo ora $N = 3$: dal momento che φ_1 è in $H_0^1(\Omega)$, per l'immersione di Sobolev si ha φ_1 in $L^{2^*}(\Omega)$, con $2^* = \frac{2N}{N-2} = 6$. Dal momento che $6 > \frac{3}{2} = \frac{N}{2}$, dal teorema di Stampacchia applicato a (5) con $f = \lambda_1 \varphi_1$ si ha subito che φ_1 è in $L^\infty(\Omega)$. Se $N = 4$, allora φ_1 è in $L^4(\Omega)$, ed essendo $4 > 2 = \frac{N}{2}$ si ha nuovamente φ_1 in $L^\infty(\Omega)$, ed analogamente per $N = 5$. Se $N = 6$, invece, φ_1 è in $L^3(\Omega)$, e $3 = \frac{N}{2}$. Per il secondo risultato di Stampacchia (sempre applicato con $f = \lambda_1 \varphi_1$), però, essendo φ_1 appartenente ad esempio a $L^2(\Omega)$ si ha che φ_1 appartiene a $L^{2^{**}}$, con $2^{**} = 6 > 3 = \frac{N}{2}$. Applicando ora il primo risultato di Stampacchia, si trova che φ_1 appartiene a $L^\infty(\Omega)$ anche se $N = 6$. In dimensione maggiore si può ripetere lo stesso procedimento (più è alta la dimensione, più volte lo si dovrà ripetere), trovando che, in dimensione qualsiasi, φ_1 è in $L^\infty(\Omega)$. A questo punto si usa il teorema di De Giorgi per provare che φ_1 è hölderiana e, da qui, i risultati classici per dimostrare che φ_1 è $C^{2,\alpha}(\Omega)$ e poi $C^{4,\alpha}(\Omega)$ e poi ... e poi $C^\infty(\Omega)$ e poi analitica.

Questo metodo, detto di *bootstrap*, funziona tutte le volte (o quasi...) in cui ci si trova a che fare con un'equazione ellittica con la soluzione "da una parte e dall'altra" dell'uguale.

Sappiamo allora risolvere (1) per $\lambda < \lambda_1$ (qualsiasi sia f) e per $\lambda = \lambda_1$ (se $f \equiv 0$). Cosa possiamo dire se $\lambda = \lambda_1$ e $f \not\equiv 0$? Supponiamo che esista una soluzione u: scegliendo φ_1 come funzione test nell'equazione per u e u come funzione test nell'equazione per φ_1, abbiamo

$$\int_\Omega \nabla u \cdot \nabla \varphi_1 = \lambda_1 \int_\Omega u\, \varphi_1 + \int_\Omega f\, \varphi_1,$$

e

$$\int_\Omega \nabla u \cdot \nabla \varphi_1 = \lambda_1 \int_\Omega u\, \varphi_1,$$

da cui si ricava che deve essere

$$\int_\Omega f\,\varphi_1 = 0.$$

Tale condizione (di ortogonalità in $L^2(\Omega)$) è quindi necessaria per l'esistenza di una soluzione. Per mostrare che è anche sufficiente, consideriamo innanzitutto

$$\lambda_2 = \inf\left\{\int_\Omega |\nabla u|^2,\ u \in H_0^1(\Omega),\ \int_\Omega u^2 = 1,\ \int_\Omega u\,\varphi_1 = 0\right\}.$$

Si può dimostrare (come prima) che λ_2 è un minimo e che si ha $\lambda_2 > \lambda_1$. Scrivendo l'equazione per il minimo si trova una soluzione φ_2 del problema

$$\begin{cases} -\Delta\varphi_2 = \lambda_2\,\varphi_2 & \text{in } \Omega, \\ \varphi_2 \not\equiv 0 & \text{in } \Omega, \\ \varphi_2 = 0 & \text{su } \partial\Omega, \end{cases} \tag{6}$$

soluzione che possiamo pensare normalizzata in $L^2(\Omega)$. Chiaramente perdiamo la positività (φ_2 dovendo essere ortogonale a φ_1 deve per forza cambiare segno), ma non la limitatezza (la tecnica di *bootstrap* è ancora applicabile). Lo spazio E_2 di tutte le soluzioni di (6) si dimostra avere dimensione finita (ma, in generale, maggiore o uguale ad 1).

Consideriamo ora il funzionale

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_1}{2}\int_\Omega u^2 - \int_\Omega f\,u, \quad u \in H_0^1(\Omega),\ \int_\Omega u\,\varphi_1 = 0.$$

Si verifica facilmente che J è debolmente semicontinuo inferiormente e che è coercitivo su $E_1^\perp$, dato che

$$\int_\Omega |\nabla u|^2 \geq \lambda_2 \int_\Omega u^2, \quad \forall u \in E_1^\perp,$$

e $\lambda_2 > \lambda_1$. Se ne deduce che J ammette minimo su $E_1^\perp$, e che tale minimo v risolve

$$\begin{cases} -\Delta v = \lambda_1\,v + f & \text{in } \Omega, \\ v = 0 & \text{su } \partial\Omega, \end{cases}$$

in $E_1^\perp$, nel senso che v appartiene a $E_1^\perp$ e

$$\int_\Omega \nabla v\cdot\nabla w = \lambda_1\int_\Omega v\,w + \int_\Omega f\,w, \quad \forall w \in E_1^\perp.$$

Sia ora z in $H_0^1(\Omega)$. Possiamo sempre scrivere $z = t\,\varphi_1 + w$, con w in $E_1^\perp$ e t in $\mathbb{R}$. Scegliendo allora $w = z - t\,\varphi_1$ come funzione test si trova

$$\int_\Omega \nabla v\cdot\nabla(z - t\,\varphi_1) = \lambda_1\int_\Omega v\,(z - t\,\varphi_1) + \int_\Omega f\,(z - t\,\varphi_1).$$

Essendo v ed f in $E_1^\perp$, si ha

$$\int_\Omega \nabla v\, \nabla \varphi_1 = \int_\Omega v\, \varphi_1 = \int_\Omega f\, \varphi_1 = 0,$$

e quindi

$$\int_\Omega \nabla v \cdot \nabla z = \lambda_1 \int_\Omega v\, z + \int_\Omega f\, z, \quad \forall z \in H_0^1(\Omega),$$

cosicché v risolve (1) con $\lambda = \lambda_1$. Ovviamente anche $v + s\,\varphi_1$ risolve la stessa equazione, che ha quindi infinite soluzioni.

Il procedimento può allora continuare, definendo λ_3 come

$$\lambda_3 = \min \left\{ \int_\Omega |\nabla u|^2,\ u \in (E_1 \oplus E_2)^\perp,\ \int_\Omega u^2 = 1 \right\},$$

e determinando φ_3 ed E_3. Alla fine si ha il seguente risultato.

Teorema 1. *Esiste una successione λ_n (autovalori) crescente e divergente di numeri reali positivi tale che:*

- *Per ogni n in $\mathbb{N}$ esiste almeno una soluzione non identicamente nulla (autofunzione) di*

$$\begin{cases} -\Delta \varphi_n = \lambda_n\, \varphi_n & \text{in } \Omega, \\ \varphi_n = 0 & \text{su } \partial\Omega. \end{cases}$$

- *λ_1 è la costante di Poincaré in Ω.*
- *L'insieme delle soluzioni della precedente equazione è uno spazio vettoriale E_n (autospazio) di dimensione finita, fatto di funzioni in $L^\infty(\Omega)$. Lo spazio E_1 ha dimensione 1 e ogni funzione di E_1 ha segno costante in Ω.*
- *Se φ appartiene ad E_n e ψ appartiene ad E_m (con $n \neq m$), allora φ e ψ sono ortogonali.*
- *Si ha*

$$H_0^1(\Omega) = \bigoplus_{n=1}^{+\infty} E_n.$$

- *Se $\lambda \neq \lambda_n$ per ogni n in $\mathbb{N}$, il problema (1) ha una ed una sola soluzione per ogni f in $L^2(\Omega)$.*
- *Se $\lambda = \lambda_n$ per qualche n in $\mathbb{N}$, il problema (1) ha soluzione se e solo se f in $L^2(\Omega)$ appartiene a $E_n^\perp$; in tal caso, esiste un'unica soluzione di (1) in $E_n^\perp$.*

Una volta dimostrato che $H_0^1(\Omega)$ si scrive come somma diretta degli E_n, le ultime due affermazioni del teorema sono facili da dimostrare.

Sia infatti f in $L^2(\Omega)$, che scriviamo

$$f = \sum_{n=1}^{+\infty} \beta_n \, \varphi_n, \quad \beta_n = \int_\Omega f \, \varphi_n.$$

Cercando una soluzione di (1) della forma

$$u = \sum_{n=1}^{+\infty} \alpha_n \, \varphi_n,$$

e ricordando che $-\Delta \varphi_n = \lambda_n \, \varphi_n$ per ogni n, si arriva a

$$\sum_{n=1}^{+\infty} \left((\lambda_n - \lambda) \, \alpha_n - \beta_n \right) \, \varphi_n = 0,$$

da cui (se λ non è un autovalore) si ricava

$$\alpha_n = \frac{\beta_n}{\lambda_n - \lambda},$$

mentre se $\lambda = \lambda_n$ è un autovalore si ha soluzione scegliendo $\alpha_n = 0$ se e solo se $\beta_n = 0$ (e quindi f è in $E_n^\perp$).

Principio di massimo

Sappiamo già che se $\lambda = 0$ e se $f \geq 0$, allora la soluzione di (1) è non negativa. Si dimostra facilmente che se tale proprietà è ancora vera se $\lambda < 0$. Cosa accade se $\lambda > 0$? Iniziamo con il suppore $\lambda \neq \lambda_1$, dato che se $\lambda = \lambda_1$ nessuna funzione $f \geq 0$ può essere ortogonale a φ_1 (che è positiva anch'essa).

Come primo caso, consideriamo $\lambda < \lambda_1$, e sia u l'unica soluzione di (1). Scegliendo u_- come funzione test, otteniamo

$$-\int_\Omega \nabla u_- \cdot \nabla u_- = -\lambda \int_\Omega u_-^2 + \int_\Omega f \, u_-.$$

Dal momento che (per definizione di λ_1)

$$\int_\Omega \nabla u_- \cdot \nabla u_- \geq \lambda_1 \int_\Omega u_-^2,$$

si ha

$$0 \leq \int_\Omega f \, u_- \leq (\lambda - \lambda_1) \int_\Omega u_-^2 \leq 0,$$

da cui si ottiene $u_- = 0$, e quindi $u \geq 0$. Pertanto, il principio di massimo vale se $\lambda < \lambda_1$.

Ebbene, questo è l'unico caso in cui il principio di massimo è valido in generale. Se $\lambda > \lambda_1$ si può dimostrare che non vale, ed anzi, se λ è maggiore di λ_1 ma è vicino a λ_1, vale il cosiddetto *anti-principio* *di massimo*: se $f \geq 0$, allora la soluzione u di (1) è negativa. Che tale fatto possa essere vero si vede facilmente scegliendo $f = \varphi_1$, cui corrisponde la soluzione $u = \varphi_1/(\lambda_1 - \lambda)$, che è negativa se $\lambda > \lambda_1$.

Data l'importanza di λ_1 ai fini del principio del massimo, e dal momento che λ_1 dipende da Ω (a differenza, ad esempio, della costante di Sobolev, che dipende solo dalla dimensione), può essere interessante chiedersi come il primo autovalore dipenda dalla "taglia" di Ω.

Per capire cosa succede, consideriamo l'esempio delle sfere di raggio r. Chiamiamo λ_r il primo autovalore in $B_r(0)$ e $\varphi_r \geq 0$ la corrispondente prima autofunzione:

$$\begin{cases} -\Delta\varphi_r = \lambda_r\,\varphi_r & \text{in } B_r(0), \\ \quad\varphi_r = 0 & \text{su } \partial B_r(0). \end{cases}$$

Definendo $v(x) = \varphi_r(r\,x)$ per x in $B_1(0)$, si ha

$$-\Delta v(x) = -r^2\Delta\varphi_r(rx) = r^2\,\lambda_r\,\varphi_r(rx) = r^2\,\lambda_r\,v(x).$$

Se ne deduce che $v(x)$ è un'autofunzione del Laplaciano in $B_1(0)$, con autovalore $\lambda_? = r^2\,\lambda_r$. Dal momento che v è non negativa, e che solo la prima autofunzione è non negativa, si ha che $\lambda_? = \lambda_1$, e che $v = \alpha\varphi_1$, con $\alpha > 0$ e φ_1 la prima autofunzione del Laplaciano in $B_1(0)$. Si ha allora che $\lambda_1 = r^2\,\lambda_r$, e quindi che

$$\lambda_r = \frac{\lambda_1}{r^2}.$$

Pertanto, più piccolo è il raggio, più grande è il primo autovalore (e quindi è maggiore l'intervallo dei valori di λ per i quali vale il principio del massimo).

In generale, se $\Omega_1 \subset \Omega_2$ sono due aperti, allora $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$. Si ha infatti che la prima autofunzione φ_{1,Ω_1} di Ω_1, prolungata a zero in $\Omega_2\backslash\Omega_1$, appartiene a $H_0^1(\Omega_2)$ ed ha norma 1 in $L^2(\Omega_2)$. Pertanto

$$\lambda_1(\Omega_1) = \int_{\Omega_1} |\nabla\varphi_{1,\Omega_1}|^2 = \int_{\Omega_2} |\nabla\varphi_{1,\Omega_1}|^2 \geq \lambda_1(\Omega_2).$$

Da questi risultati segue che se $\Omega \subset B_r(0)$, allora $\lambda_1(\Omega) \geq \lambda_r = \frac{\lambda_1}{r^2}$. Se r è piccolo, allora il primo autovalore di Ω è grande ed è "più facile" che valga il principio di massimo.

Simmetria delle soluzioni

Il fatto che se il dominio è piccolo allora vale il principio di massimo è il primo caso incontrato in cui la "taglia" di Ω influisce sulle proprietà qualitative delle soluzioni. Il secondo — ben più famoso — è un risultato di simmetria delle soluzioni dovuto a Gidas, Ni e Nirenberg.

Il teorema è il seguente.

Teorema 2. *Sia* $\Omega = B_1(0)$ *e sia* $f : \mathbb{R} \to \mathbb{R}$ *una funzione lipschitziana. Sia* u *una soluzione (classica) di*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases}$$

Allora $u(x) = u(|x|)$ *ed* u *è decrescente (come funzione di* $|x|$*).*

Dimostrazione. Per semplicità, supponiamo di essere in $\mathbb{R}^2$ (la dimostrazione è analoga in dimensione maggiore di 2). Dette (x, y) le varabili, sia $-1 < \lambda \le 0$ e sia r_λ la retta $x = \lambda$. Definiamo

$$\Omega_\lambda = \{(x, y) \in \mathbb{R}^2 : x \ge \lambda,\ (2\lambda - x, y) \in \Omega\},$$

e definiamo

$$\tilde{u}(x, y) = u(2\lambda - x, y), \quad (x, y) \in \Omega_\lambda.$$

Considerando u e $\tilde{u}$ su $\overline{\Omega}_\lambda$, abbiamo $u = \tilde{u}$ se $x = \lambda$, mentre sulla

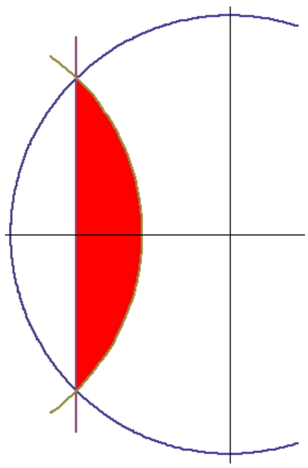

Figura 1. Ω_λ è la zona colorata di rosso

parte di frontiera di Ω_λ "proveniente" dalla frontiera di Ω abbiamo $u > 0$ (per ipotesi) e $\tilde{u} = 0$ (dato che u è zero sulla frontiera di Ω). Siccome derivando due volte $\tilde{u}$ rispetto ad x il segno cambia due volte, abbiamo

$$-\Delta u = f(u) \text{ in } \Omega_\lambda, \quad -\Delta \tilde{u} = f(\tilde{u}) \text{ in } \Omega_\lambda,$$

e quindi, detta $w_\lambda = u - \tilde{u}$, si ha

$$-\Delta w_\lambda = f(u) - f(\tilde{u}) = \frac{f(u) - f(\tilde{u})}{u - \tilde{u}}\, w_\lambda = g\, w_\lambda, \text{ in } \Omega_\lambda,$$

dove

$$g(x,y) = \frac{f(u(x,y)) - f(\tilde{u}(x,y))}{u(x,y) - \tilde{u}(x,y)}$$

è una funzione limitata essendo f lispchitziana. In sostanza si ha

$$\begin{cases} -\Delta w_\lambda = g\, w_\lambda & \text{in } \Omega_\lambda, \\ \quad w_\lambda \geq 0 & \text{su } \partial\Omega_\lambda. \end{cases}$$

Se λ è sufficientemente vicino a -1, il primo autovalore di Ω_λ è maggiore della costante di lipschitzianità di f (Ω_λ si può infatti racchiudere in una sfera di raggio r sufficientemente piccolo), cosicché il principio di massimo è valido in Ω_λ. Se ne deduce che $w_\lambda > 0$ in Ω_λ se λ è "vicino" a -1, e quindi

$$u(x,y) > \tilde{u}(x,y) = u(2\lambda - x, y), \quad (x,y) \in \Omega_\lambda.$$

Definiamo allora

$$E = \{-1 < \lambda \leq 0 : w_\mu(x,y) > 0 \text{ in } \Omega_\mu \text{ per ogni } -1 < \mu \leq \lambda\}.$$

Grazie a quanto abbiamo appena detto, E è non vuoto, e quindi

$$-1 < \Lambda = \sup\, E \leq 0\,.$$

Se $\Lambda < 0$, usando il fatto che $w_\mu > 0$ per ogni $\mu < \Lambda$, si dimostra (usando nuovamente il principio di massimo) che $w_{\Lambda+\varepsilon} > 0$ per ogni ε tale che $\Lambda + \varepsilon \leq 0$, arrivando così ad un assurdo. Se ne deduce che $\Lambda = 0$ (lo stesso ragionamento non si può ripetere perché se $\lambda > 0$ si ha che Ω_λ non è più contenuto in Ω), e quindi che

$$u(x,y) > u(2\lambda - x, y) \text{ in } \Omega_\lambda \quad \forall \lambda < 0.$$

Facendo tendere λ a zero, si ottiene $u(x,y) \geq u(-x,y)$ per ogni (x,y) in Ω_0 (e Ω_0 è il semicerchio).

Ripetendo il discorso con $\lambda > 0$ (e riflettendo Ω a sinistra della retta $x = \lambda$) si ottiene $u(x,y) \leq u(-x,y)$ per ogni (x,y) in Ω_0, e quindi $u(x,y) = u(-x,y)$ per ogni (x,y) in Ω.

La stessa tecnica si può ripetere considerando una qualsiasi direzione del piano: se r è la retta passante per l'origine avente tale direzione, la u è simmetrica a sinistra ed a destra di tale retta (dato che Ω viene diviso in due metà speculari).

Siano ora $P = (0,r)$ (con $0 < r < 1$), e sia Q un altro punto posto a distanza r dall'origine. Considerando la bisettrice dell'angolo OPQ, P e Q sono simmetrici rispetto ad essa, e quindi $u(P) = u(Q)$.

Abbiamo così dimostrato che se $x^2+y^2=r^2$, allora $u(x,y)=u(0,r)$, e quindi u è radiale. Il fatto che sia decrescente segue dal fatto che se $r_1 > r_2$ allora $(0,r_1)$ e $(0,r_2)$ sono simmetrici rispetto alla retta $y=\frac{r_1+r_2}{2}$ e quindi $u(0,r_2) > u(0,r_1)$. □

Si noti che il metodo, detto del *moving plane*, di far muovere la retta $x=\lambda$ verso destra funziona finché l'insieme Ω_λ è contenuto in Ω, dato che in questo caso è possibile confrontare u con la sua riflessa. Se, quindi, Ω non è una sfera, ma ad esempio un'ellisse, si può spostare la retta $x=\lambda$ fino a $\lambda=0$, e la retta $y=\lambda$ fino a $\lambda=0$, ma non altre rette, dato che l'ellisse manca di simmetria in altre direzioni. In ogni caso, si riesce a dimostrare che u è simmetrica rispetto agli assi, ovvero che la soluzione eredita la simmetria dell'insieme.

L'importanza di tale teorema è chiara: se $\Omega = B_1(0)$, il problema di risolvere l'equazione

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{su } \partial\Omega, \end{cases}$$

viene ricondotto al problema di risolvere l'equazione differenziale ordinaria

$$-\frac{1}{r^{N-1}}\,(r^{N-1}\,u'(r))' = f(u(r)),\ \ u(1)=0,\ \ u'(0)=0.$$

Equazioni sublineari

In questo e nel prossimo paragrafo studieremo l'esistenza di soluzioni positive per equazioni ellittiche della forma

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{su } \partial\Omega, \end{cases}$$

nei casi particolari in cui $g(s)=s^\theta$ (con $0<\theta<1$) e $g(s)=s^p$ (con $p>1$). Prima di affrontare lo studio del primo dei due casi, cerchiamo di capire perché è ragionevole aspettarsi soluzioni. O, meglio, perché per determinate funzioni g (ad esempio $g(s)=\lambda\,\mathrm{e}^s$ con λ grande) il problema possa non avere soluzione.

Esattamente come nel caso $g(s)=\lambda_1\,s+f(x)$, scegliamo φ_1 come funzione test nel problema risolto da u e viceversa. Otteniamo

$$\int_\Omega g(u)\,\varphi_1 = \int_\Omega \nabla u\cdot\nabla\varphi_1 = \int_\Omega \nabla\varphi_1\cdot\nabla u = \lambda_1\int_\Omega u\,\varphi_1,$$

da cui si deduce

$$\int_\Omega [g(u) - \lambda_1 u]\, \varphi_1 = 0.$$

Se abbiamo $g(s) \geq \lambda_1(s)$ per ogni $s \geq 0$ (o, equivalentemente, $g(s) \leq \lambda_1(s)$ per ogni $s \geq 0$), è allora chiaro che l'identità precedente non può essere soddisfatta: il problema non ha alcuna soluzione positiva.

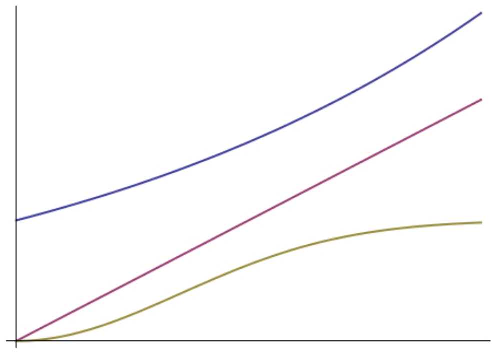

Figura 2. Due funzioni per le quali non c'è esistenza

Nel caso in cui $g(s)$ sia s^θ (ovvero s^p), la retta $\lambda_1\, s$ taglia il grafico di g.

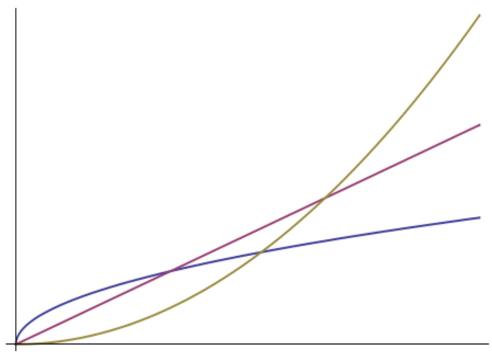

Figura 3. Due funzioni per le quali potrebbe esserci esistenza

Questo fatto non vuol dire che esistono di sicuro soluzioni del problema, anche se è spesso vero che il numero delle volte in cui la retta $\lambda_1\, s$ taglia il grafico di g dà un'informazione sul numero di soluzioni positive del corrispondente problema.

Questo è il caso per l'equazione

$$(7)\qquad \begin{cases} -\Delta u = u^\theta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega, \end{cases}$$

con $0 < \theta < 1$. Per dimostrare che (7) ammette almeno una soluzione, consideriamo il seguente funzionale

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{\theta+1}\int_\Omega u_+^{\theta+1}, \qquad u \in H_0^1(\Omega).$$

Osserviamo che I è ben definito (essendo $\theta+1 < 2$) e che è debolmente semicontinuo inferiormente (sempre perché, essendo $\theta+1 < 2$, l'immersione di $H_0^1(\Omega)$ in $L^{\theta+1}(\Omega)$ è compatta). Si ha poi, per le disuguaglianze di Hölder e Poincaré,

$$I(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{|\Omega|^{1-\frac{\theta+1}{2}}}{\theta+1}\left(\int_\Omega u_+^2\right)^{\frac{\theta+1}{2}} \geq C_1\,\|u\|^2_{H_0^1(\Omega)} - C_2\,\|u\|^{\theta+1}_{H_0^1(\Omega)}.$$

Essendo $\theta + 1 < 2$, $I(u)$ diverge positivamente se la norma di u in $H_0^1(\Omega)$ diverge, e quindi I è coercitivo. Si ottiene così l'esistenza di un minimo u per I, minimo che risolve

$$\begin{cases} -\Delta u = u_+^\theta & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases}$$

Dal momento che u_+^θ è non negativa, il principio del massimo implica che u è non negativa. Pertanto, essendo $u_+ \equiv u$, si ha

$$\begin{cases} -\Delta u = u^\theta & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases}$$

Si noti che per dimostrare che I è ben definito, che I è debolmente semicontinuo inferiormente, e che I è coercitivo, abbiamo usato solo il fatto che $s^{\theta+1}$ cresce all'infinito meno di s^2 (volendo, che s^θ è minore di $\lambda_1\, s$ all'infinito).

Abbiamo trovato la nostra soluzione? Non necessariamente: la funzione $u \equiv 0$ risolve lo stesso problema, e noi stiamo cercando soluzioni non banali. Come possiamo essere sicuri che il punto di minimo u di I non sia la funzione nulla? Calcoliamo $I(t\varphi_1)$: otteniamo

$$I(t\varphi_1) = \frac{t^2}{2}\int_\Omega |\nabla\varphi_1|^2 - \frac{t^{\theta+1}}{\theta+1}\int_\Omega \varphi_1^{\theta+1} = C_1\, t^2 - C_2\, t^{\theta+1}.$$

Dal momento che $\theta + 1 < 2$, si ha $C_1\, t^2 - C_2\, t^{\theta+1} < 0$ per t sufficientemente vicino a zero. Pertanto, il valore minimo di I è strettamente negativo. Essendo $I(0) = 0$, $u \equiv 0$ non può essere il minimo, e quindi la soluzione trovata non è identicamente nulla.

Si noti che per dimostrare che il minimo non è banale abbiamo usato il fatto che $s^{\theta+1}$ è maggiore di s^2 vicino a zero (volendo, che s^θ è maggiore di $\lambda_1\, s$ per s tendente a zero).

Equazioni superlineari

Molto differente dal caso $g(s) = s^\theta$ (con $0 < \theta < 1$) è il caso in cui $g(s) = s^p$ (con $p > 1$). Anche in questo caso la retta $\lambda_1\, s$ interseca il grafico di g per cui sembrerebbe che ci si possa aspettare l'esistenza di soluzioni.

Come nel caso sublineare, proviamo l'approccio funzionale e definiamo

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega u_+^{p+1}, \qquad u \in H_0^1(\Omega).$$

Chiaramente I non è ben definito per ogni valore di p: deve essere, infatti, $p + 1 \leq 2^* = \frac{2N}{N-2}$, ovvero $p \leq \frac{N+2}{N-2}$. Se p è più grande, non è detto che una funzione u in $H_0^1(\Omega)$ appartenga ad $L^{p+1}(\Omega)$. Fatta questa restrizione, ci chiediamo se I sia debolmente semicontinuo inferiormente. La risposta è affermativa, ma p non può essere uguale a $\frac{N+2}{N-2}$, dato che l'immersione di $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$ non è compatta. Per cui I è debolmente semicontinuo inferiormente se e solo se $p < \frac{N+2}{N-2}$. Per tali valori di p, I è coercitivo? No: se, infatti, calcoliamo $I(t\varphi_1)$ per $t > 0$, abbiamo

$$I(t\varphi_1) = \frac{t^2}{2} \int_\Omega |\nabla \varphi_1|^2 - \frac{t^{p+1}}{p+1} \int_\Omega \varphi_1^{p+1} = C_1\, t^2 - C_2\, t^{p+1}.$$

Essendo $p + 1 > 2$, si ha

$$\lim_{t \to +\infty} I(t\varphi_1) = -\infty,$$

cosicché I non solo non è coercitivo, ma è anche illimitato inferiormente. D'altra parte, se $t < 0$ si ha

$$I(t\varphi_1) = \frac{t^2}{2} \int_\Omega |\nabla \varphi_1|^2,$$

e quindi I è illimitato superiormente. Cosa fare?

Come già nel caso lineare, tentiamo la minimizzazione vincolata: definiamo

$$m = \inf \left\{ \int_\Omega |\nabla u|^2,\ u \in H_0^1(\Omega),\ \int_\Omega u^{p+1} = 1 \right\}.$$

Se $p < \frac{N+2}{N-2}$ si vede facilmente (usando la compattezza dell'immersione di $H_0^1(\Omega)$ in $L^{p+1}(\Omega)$) che m è un minimo, raggiunto in corrispondenza di una funzione $v \neq 0$. Scrivendo l'equazione risolta da v, otteniamo

$$\begin{cases} -\Delta v = m\,|v|^{p-1}\,v & \text{in } \Omega, \\ v = 0 & \text{su } \partial\Omega. \end{cases}$$

Ponendo $u = m^{\frac{1}{p-1}}\,v$, si verifica facilmente che u soddisfa l'equazione

$$\begin{cases} -\Delta u = |u|^{p-1}\,u & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega, \end{cases}$$

e $u \neq 0$. Ragionando come nel caso lineare, si vede che si può scegliere u di segno costante, ottenendo così una soluzione di

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases} \tag{8}$$

Cosa succede nel caso $p \geq \frac{N+2}{N-2}$? Per dimostrare che in generale non esiste soluzione abbiamo bisogno di un risultato preliminare (importante di per sé).

Teorema 3 (Identità di Pohozaev). *Sia u una soluzione di*

$$\begin{cases} -\Delta u = g(u) & \textit{in } \Omega, \\ u \geq 0 & \textit{in } \Omega, \\ u = 0 & \textit{su } \partial\Omega. \end{cases}$$

Allora, detta $G(s)$ la primitiva di g nulla in zero, si ha

$$\int_\Omega \left[\left(1 - \frac{N}{2}\right) g(u)\,u + N G(u) \right] = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2\,(x \cdot \nu)\,d\sigma, \tag{9}$$

Dimostrazione. Moltiplichiamo l'equazione per $x \cdot \nabla u$ ed integriamo (si noti che tale funzione non è nulla sulla frontiera di Ω, per cui compariranno dei termini di bordo). Si ha

$$(A) = \int_\Omega -\mathrm{div}(\nabla u)\,x \cdot \nabla u = \int_\Omega g(u)\,x \cdot \nabla u = (B).$$

Si ha, usando il teorema della divergenza,

$$\begin{aligned} (B) &= \int_\Omega x \cdot \nabla G(u) = \int_\Omega [\mathrm{div}(x\,G(u)) - \mathrm{div}(x)\,G(u)] \\ &= \int_{\partial\Omega} G(u)\,x \cdot \nu\,d\sigma - N \int_\Omega G(u) = -N \int_\Omega G(u), \end{aligned}$$

essendo $G(u) = 0$ su $\partial\Omega$ e $\operatorname{div}(x) = N$. Prima di lavorare con (A), eseguiamo alcuni calcoli:

$$\operatorname{div}(\nabla u)\, x \cdot \nabla u = \operatorname{div}((x \cdot \nabla u)\, \nabla u) - \nabla(x \cdot \nabla u) \cdot \nabla u.$$

Ora

$$\begin{aligned}
\nabla(x \cdot \nabla u) \cdot \nabla u &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} x_j \frac{\partial u}{\partial x_j} \right) \frac{\partial u}{\partial x_i} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\delta_{ij} \frac{\partial u}{\partial x_j} + x_j \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \frac{\partial u}{\partial x_i} \\
&= \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2 + \sum_{j=1}^{N} x_j \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \\
&= \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2 + \frac{1}{2} \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2 \\
&= |\nabla u|^2 + \frac{1}{2}\, x \cdot \nabla(|\nabla u|^2) \\
&= |\nabla u|^2 + \frac{1}{2} \operatorname{div}(x\, |\nabla u|^2) - \frac{1}{2} \operatorname{div}(x)\, |\nabla u|^2 \\
&= \left(1 - \frac{N}{2} \right) |\nabla u|^2 + \frac{1}{2} \operatorname{div}(x\, |\nabla u|^2).
\end{aligned}$$

Pertanto,

$$\operatorname{div}(\nabla u)\, x \cdot \nabla u = \operatorname{div}((x \cdot \nabla u)\, \nabla u) - \left(1 - \frac{N}{2} \right) |\nabla u|^2 - \frac{1}{2} \operatorname{div}(x\, |\nabla u|^2),$$

da cui

$$(A) = -\int_{\partial\Omega} (x \cdot \nabla u)\, (\nabla u \cdot \nu)\, d\sigma + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2\, (x \cdot \nu)\, d\sigma + \left(1 - \frac{N}{2} \right) \int_{\Omega} |\nabla u|^2.$$

Su $\partial\Omega$ possiamo scrivere

$$\nabla u = (\nabla_\tau u, \nabla_\nu u),$$

dove con ∇_τ abbiamo indicato le derivate sul piano tangente a $\partial\Omega$ e con ∇_ν la derivata normale. Essendo $u = 0$ su $\partial\Omega$, tutte le sue derivate tangenziali sono nulle, e quindi

$$\nabla u = (\nabla u \cdot \nu)\, \nu \quad \text{su } \partial\Omega.$$

Pertanto, sempre su $\partial\Omega$,

$$|\nabla u|^2\, (x \cdot \nu) = |\nabla u \cdot \nu|^2 (x \cdot \nu) = (x \cdot \nabla u)\, (\nabla u \cdot \nu),$$

e quindi

$$(A) = -\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2\,(x\cdot\nu)\,d\sigma + \left(1-\frac{N}{2}\right)\int_\Omega |\nabla u|^2.$$

Essendo

$$\int_\Omega |\nabla u|^2 = \int_\Omega g(u)\,u,$$

si ha allora

$$\int_\Omega \left[\left(1-\frac{N}{2}\right) g(u)\,u + NG(u)\right] = \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2\,(x\cdot\nu)\,d\sigma,$$

come volevasi dimostrare. □

Supponiamo ora che Ω sia stellato rispetto all'origine (e quindi $x\cdot\nu \geq 0$ per ogni x in $\partial\Omega$) e prendiamo $g(s) = s^p$. Essendo $G(s) = s^{p+1}/(p+1)$, l'identità di Pohozaev implica

$$\int_\Omega \left(1-\frac{N}{2}+\frac{N}{p+1}\right) u^{p+1} = \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2\,(x\cdot\nu)\,d\sigma \geq 0.$$

Se $p+1 > \frac{2N}{N-2}$, la quantità $1-\frac{N}{2}+\frac{N}{p+1}$ è negativa, e quindi $u\equiv 0$: non esistono soluzioni positive per (8) se $p > \frac{N+2}{N-2}$. Se $p+1 = \frac{2N}{N-2}$ allora la quantità $1-\frac{N}{2}+\frac{N}{p+1}$ è nulla, e quindi $|\nabla u|\equiv 0$ su $\partial\Omega$. Questo fatto (assieme all'essere u soluzione di (8)), implica nuovamente che $u\equiv 0$: non esistono soluzioni positive per (8) se $p = \frac{N+2}{N-2}$.