A Not So Long Introduction to the Weak Theory of Parabolic Problems with Singular Data

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by

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Contents

CHAPTER 1

Motivations for the problem and basic tools

1.1. Motivations

Here we want to study parabolic equations whose simplest model is the so called *(Nonhomogeneous) Heat Equation*

$$u_t - \Delta u = f,$$

subject to suitable initial and boundary conditions. Here $t > 0$ and $x \in \Omega$ which is an open subset of $\mathbb{R}^N$. The unknown is the function $u : \Omega \times (0,T) \mapsto \mathbb{R}$, where T is a positive, possibly infinity, constant, and Δ is the usual *Laplace Operator* with respect to the space variables, that is

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i},$$

while the function $f : \Omega \times (0,T) \mapsto \mathbb{R}$ is a given datum.

Historically, the study of parabolic equations followed a parallel path with respect to the elliptic theory: so many results of the elliptic framework (harmonic properties, maximum principles, representations of solution, ...) turn out to have a (usually more complicated ☹) parabolic counterpart.

However, unfortunately (or by chance...) the statement *Every elliptic problem becomes parabolic just with time* is, in general, false.

On the other hand, a natural question is the reverse one: *is it true that every parabolic problem turns out to become elliptic with time?* We will try to give an answer to this problem at the end of the last part of the final class (If I can...).

In any case, the physical interpretation is much more *clear* and so these type of equations turned out to admit many many applications in a wide variety of fields as, among others, *Thermodynamics* (ça va sans dire...), *Statistics* (Brownian Motion), *Fluid Mechanics* (Navier-Stokes equations... there is a prize about it[1]), *Finance* (Black-Scholes equation... here there is not a prize anymore... ☹), an so on.

The heat equation can be considered as *diffusion equation* and it was firstly studied to describe the evolution in time of the density u of some quantity such as heat or chemical concentration. If $\omega \subset \Omega$ is a smooth

[1] For instance it is still not known if the N-S equation which describes the flow of air around an airplane has a solution ☹

subregion, the rate of change of the total quantity within ω should equal the negative of the flux through $\partial\omega$, that is

$$\frac{d}{dt}\int_\omega u\ dx = -\int_{\partial\omega} F\cdot\nu\ d\sigma,$$

F being the flux density. Thus,

$$u_t = -\mathrm{div}F,$$

as ω is arbitrary. In many applications F turns out to be proportional to ∇u, that leads to $u_t - \lambda\Delta u = 0$, that is the *Heat Equation* for $\lambda = 1$;

$$u_t - \Delta u = 0. \tag{1.1}$$

Let us explicitly remark that the heat equation involves *one* derivative with respect to the time and *two* with respect to x. Consequently, we can easily check that, if u solves (1.1), then so does $u(\lambda x, \lambda^2 t)$, for $\lambda \in \mathbb{R}$; this inhomogeneity suggests that the ratio $\frac{|x|^2}{t}$ is important to study this type of equations and that an explicit radial solution can be searched of the form $u(x,t) = v(\frac{r^2}{t})$, where $r = |x|$, and v is the new unknown.

Let us formally motivate the introduction of the so called *fundamental solution* for (1.1).

It is quicker to search for radial solutions through the invariant scaling $r = |y| = t^{-\frac{1}{2}}|x|$, which yields, after few calculations to derive the radial form of (1.1):

$$\frac{N}{2}v + \frac{1}{2}rv' + v'' + \frac{N-1}{r}v' = 0,$$

which, multiplying the equation by r^{N-1}, turns out to be equivalent to

$$(r^{N-1}v')' + \frac{1}{2}(r^N v)' = 0,$$

that is,

$$r^{N-1}v' + \frac{1}{2}r^N v = a,$$

for some constant a. Now assuming that we look for solutions vanishing at infinity with its derivatives we conclude that $a = 0$. Thus

$$v' = -\frac{1}{2}rv,$$

and so, finally

$$v = be^{-\frac{r^2}{4}}, (b > 0).$$

With a suitable choice of the contants (chosen in such a way that the total mass of $\Phi(x,t)$ on $\mathbb{R}^N$ is equal to 1 for every $t > 0$) we can write the classic fundamental solution for problem (1.1), that is

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4t}} \quad (x \in \mathbb{R}^N, t > 0). \tag{1.2}$$

The fundamental solution can be used to represent solutions for *initial-boundary value problems (Cauchy problems)* of the type

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } \ \mathbb{R}^N \times (0,\infty) \\ u(x,0) = g. \end{cases} \tag{1.3}$$

In fact the following result holds true

THEOREM 1.1. *Let $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then*

$$u(x,t) = \int_{\mathbb{R}^N} \Phi(x-y,t)g(y)\ dy, \tag{1.4}$$

belongs to $C^\infty(\mathbb{R}^N \times (0,\infty))$, solves the equation in (1.3) and

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0), (x_0 \in \mathbb{R}^N).$$

PROOF. [**E**], p. 47. □

If we have a nonhomogeneous smooth forcing term f the representation formula is more complicated (but just a little bit...) and involves the so called superposition *Duhamel Principle*. In fact, if we assume for simplicity $f \in C_1^2(\mathbb{R}^N \times [0,\infty))$ (i.e., two continuous derivatives in space and one in time) with compact support, then the representation formula for the solution to problem

$$\begin{cases} u_t - \Delta u = f \quad \text{in } \ \mathbb{R}^N \times (0,\infty) \\ u(x,0) = g, \end{cases} \tag{1.5}$$

will be

$$u(x,t) = \int_{\mathbb{R}^N} \Phi(x-y,t)g(y)\ dy + \int_0^t \int_{\mathbb{R}^N} \Phi(x-y,t-s)f(y,s)\ dyds.$$

REMARK 1.2. In view of Theorem 1.1 we sometimes say that the fundamental solution solves

$$\begin{cases} \Phi_t - \Delta \Phi = 0 \quad \text{in } \ \mathbb{R}^N \times (0,\infty) \\ \Phi(x,0) = \delta_0, \end{cases} \tag{1.6}$$

where δ_0 denotes the *Dirac mass* at 0. Notice moreover that, from (1.4), we derive that for nonnegative data $g \neq 0$ the solution turns out to be strictly positive for all $x \in \mathbb{R}^N$, and $t > 0$. This is a *key feature* for parabolic solutions which have the so called *infinite propagation speed.* If the initial temperature is nonnegative and positive somewhere, then at any positive time t the temperature is positive anywhere. This fact turns out to play an essential difference with other types of evolution equations such, for instance, *Hyperbolic Equations.*

As we said many features of harmonic functions are inherited by solutions of the heat equation. For example, the *parabolic mean value formula.*

THEOREM 1.3 (Mean value formula). *Let Ω be a smooth, connected, bounded open set of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$, $T > 0$. Assume $u \in C_1^2(\Omega \times (0,T]) \cap C(\overline{Q})$ solves the heat equation in Q. Then we have*

$$u(x,t) = \frac{1}{4r^N} \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds$$

for every $E(x,t;r) \subset Q$, where

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^N \times \mathbb{R} : s \le t, \Phi(x-y, s-t) \ge \frac{1}{r^N} \right\},$$

is the so-called **heat ball**.

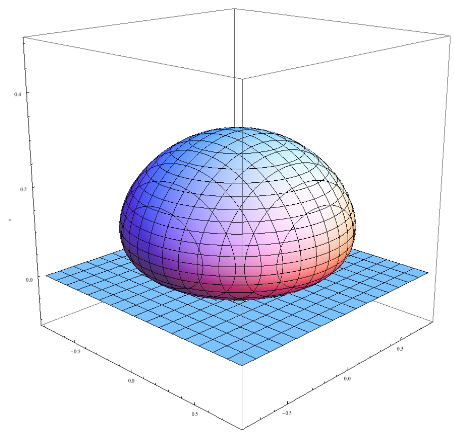


FIGURE 1. The heat ball

PROOF. See [**E**], p. 52. □

As for harmonic functions, from the mean value formula a maximum principle follows.

THEOREM 1.4 (Strong Maximum Principle). *Let Ω be a smooth, connected, bounded open set of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$, $T > 0$. Assume $u \in C_1^2(\Omega \times (0,T]) \cap C(\overline{Q})$ solves the heat equation in Q. Then, if we denote $\Gamma = \overline{Q} \setminus (\Omega \times (0,T])$, we have*

i)
$$\max_{\overline{Q}} u = \max_{\Gamma} u,$$

ii) *If u attains its maximum at $(x_0, t_0) \in Q$, then u is constant in $\overline{\Omega} \times [0, t_0]$.*

PROOF. See [**E**], p. 54. □

Theorem 1.4 has a very suggestive interpretation: with constant data on the boundary, the solution keeps itself constant until *something* happens to change this quiet status (think about a change of boundary conditions from t_0 on). In some sense, a solution behaves in a very intuitive way since *the past turns out to be independent on the future*. This fact is strongly related to the *irreversibility* of the heat equation, that is on the ill-posedness of the *final-boundary value problem*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0, T) \\ u(x, T) = g. \end{cases} \tag{1.7}$$

Looking for solutions to problem (1.7) is, in some sense, equivalent to find out an initial datum such that the corresponding solution of the Cauchy problem coincides with g at time T. However, because of the strong regularization of the solution emphasized by Theorem 1.1, if g is not smooth enough there is no chance to solve (1.7). So this should convince us that the use of the symbol t to denote the last variable of the unknown u is not just a mere chance.

Of course, once the maximum principle is at hand, uniqueness of the solution follows by difference:

THEOREM 1.5 (Uniqueness on bounded domains). *Let $g \in C(\Gamma)$, $f \in C(Q)$. Then there exists at most one solution $u \in C^2_1(Q) \cap C(\overline{Q})$ of the initial boundary problem for the heat equation.*

PROOF. See [**E**], p. 57. $\square$

What happens if the domain is unbounded, for example is the whole $\mathbb{R}^N$? In this case uniqueness is no longer true, but we can recover it (as well as the maximum principle), by adding a suitable control on the solution for large $|x|$.

THEOREM 1.6 (Maximum principle in unbounded domains). *If u is a solution of the heat equation in $\mathbb{R}^N \times (0, T)$ (with initial datum g), and u is such that*

$$u(x, t) \leq A\, \mathrm{e}^{a|x|^2}\,, \tag{1.8}$$

for some constants $A > 0$ and $a > 0$, then

$$\sup_{\mathbb{R}^N \times [0,T]} u(x, t) = \sup_{\mathbb{R}^N} g(x).$$

PROOF. See [**E**], p. 57. $\square$

This maximum principle then implies as before the uniqueness of the solutions of the Cauchy problem in the class of functions whose absolute value satisfies (1.8).

Notation and remarks. Let us spend a few words on how positive constant will be denoted hereafter. If no otherwise specified, we will write C to denote any positive constant (possibly different) which only depends on the data, that is on quantities that are fixed in the assumptions (N, Ω, Q, p, and so on...). In any case such constants never depend on the different indexes having a limit we often introduce. Finally, for the sake of simplification of the notation we will indicate the time derivative of a function u with u_t, $\frac{du}{dt}$ or u' depending on the context.

For the convenience of the reader in Appendix A we recall some basic results of *measure and integration theory* we will always assume to be known in the following.

1.2. Functional spaces involving time

Since we want to study an equation involving one derivative in time and two in space, the right functional setting should be C^1 in time and C^2 in space, and, in fact, this is the classical setting we mentioned above.

However, as in the elliptic case, we would like to solve problems with less regular data. Due to this fact, we will deal with the *weak theory* of parabolic problems, so that, to our aims, it would be sufficient a functional setting involving zero derivatives in time (*Lebesgue regularity*) and just one in space (*Sobolev regularity*).

Let us just recall that, if E and F are Banach spaces,then the function

$$\begin{array}{rccc} f: & E & \rightarrow & F \\ & x & \mapsto & f(x), \end{array}$$

is said to be *Fréchet differentiable* at $a \in E$, if there exist a linear bounded map D from E to F such that

$$\lim_{\|h\|_E \to 0} \frac{\|f(a+h) - f(a) - D_a(h)\|_F}{\|h\|_E} = 0.$$

Given a real Banach space V, we will denote by $C^\infty(\mathbb{R}; V)$ the space of functions $u : \mathbb{R} \to V$ which are infinitely many times Fréchet differentiable and by $C_0^\infty(\mathbb{R}; V)$ the space of functions in $C^\infty(\mathbb{R}; V)$ having compact support. As we mentioned above, for $a, b \in \mathbb{R}$, $C^\infty([a, b]; V)$ will be the space of the restrictions to $[a, b]$ of functions of $C_0^\infty(\mathbb{R}; V)$, and $C([a, b]; V)$ the space of all continuous functions from $[a, b]$ into V.

We recall that a function $u : [a, b] \to V$ is said to be Lebesgue measurable if there exists a sequence $\{u_n\}$ of step functions (i.e. $u_n = \displaystyle\sum_{j=1}^{k_n} a_j^n \chi_{A_j^n}$ for a finite number k_n of Borel subsets $A_j^n \subset [a, b]$ and with $a_j^n \in V$) converging to u almost everywhere with respect to the Lebesgue measure in $[a, b]$.

Then for $1 \le p < \infty$, $L^p(a,b;V)$ is the space of measurable functions $u : [a,b] \to V$ such that

$$\|u\|_{L^p(a,b;V)} = \left(\int_a^b \|u\|_V^p \, dt \right)^{\frac{1}{p}} < \infty,$$

while $L^\infty(a,b;V)$ is the space of measurable functions such that:

$$\|u\|_{L^\infty(a,b;V)} = \sup_{[a,b]} \|u\|_V < \infty.$$

Of course both spaces are meant to be quotiented, as usual, with respect to the almost everywhere equivalence. The reader can find a presentation of these topics in [**DL**].

Let us recall that, for $1 \le p \le \infty$, $L^p(a,b;V)$ is a Banach space. Moreover, for $1 \le p < \infty$ and if the dual space V' of V is separable, then the dual space of $L^p(a,b;V)$ can be identified with $L^{p'}(a,b;V')$.

For our purpose V will mainly be either the Lebesgue space $L^p(\Omega)$ or the Sobolev space $W_0^{1,p}(\Omega)$, with $1 \le p < \infty$ and Ω will be a bounded open set of $\mathbb{R}^N$. Since, in this case, V is separable, we have that $L^p(a,b;L^p(\Omega)) = L^p((a,b) \times \Omega))$, the ordinary Lebesgue space defined in $(a,b) \times \Omega$. Note that $L^p(a,b;W_0^{1,p}(\Omega))$ consists of all functions $u : [a,b] \times \Omega \to \mathbb{R}$ which belong to $L^p((a,b) \times \Omega))$ and such that $\nabla u = (u_{x_1}, \ldots, u_{x_N})$ belongs to $(L^p((a,b) \times \Omega))^N$. Moreover,

$$\left(\int_a^b \int_\Omega |\nabla u|^p \, dxdt \right)^{\frac{1}{p}}$$

defines an equivalent norm by Poincaré inequality.

Given a function u in $L^p(a,b;V)$ it is possible to define a time derivative of u in the space of vector valued distributions $\mathcal{D}'(a,b;V)$ which is the space of linear continuous functions from $C_0^\infty(a,b)$ into V (see [**Sc**] for further details). In fact, the definition is the following:

$$\langle u_t, \psi \rangle = - \int_a^b u \, \psi_t \, dt \,, \qquad \forall \;\; \psi \in C_0^\infty(a,b),$$

where the equality is meant in V. If $u \in C^1(a,b;V)$ this definition clearly coincides with the Fréchet derivative of u. In the following, u_t is said to belong to a space $L^q(a,b;\tilde{V})$ ($\tilde{V}$ being a Banach space) if there exists a function $z \in L^q(a,b;\tilde{V}) \cap \mathcal{D}'(a,b;V)$ such that:

$$\langle u_t, \psi \rangle = - \int_a^b u \, \psi_t \, dt = \langle z, \psi \rangle \,, \qquad \forall \;\; \psi \in C_0^\infty(a,b).$$

In the following, we will also use sometimes the notation $\dfrac{\partial u}{\partial t}$ instead of u_t. We recall the following classical embedding result

THEOREM 1.7. *Let H be an Hilbert space such that:*

$$V \underset{dense}{\hookrightarrow} H \hookrightarrow V' .$$

x Let $u \in L^p(a,b;V)$ be such that u_t, defined as above in the distributional sense, belongs to $L^{p'}(a,b;V')$. Then u belongs to $C([a,b];H)$.

SKETCH OF THE PROOF. We give a sketch of the proof of this result in the particular case $p = 2$ and $V = H^1_0(\Omega)$ (in this case the *pivot space* H will be $L^2(\Omega)$). A complete proof of Theorem 1.7 can be found in [**DL**]. for simplicity we also choose $a = 0$, and $b = T$.

Extend u to the larger interval $[-\sigma, T+\sigma]$, for $\sigma > 0$, and define the regularizations $u^\varepsilon = \eta^\varepsilon * u$, where η^ε is a mollifier on $\mathbb{R}$. One can easily check that,

$$\begin{cases} u^\varepsilon \to u \ \text{ in } L^2(0,T;H^1_0(\Omega)), \\ u^\varepsilon_t \to u_t \ \text{ in } L^2(0,T;H^{-1}(\Omega)). \end{cases} \tag{1.9}$$

Then, for $\varepsilon, \delta > 0$,

$$\frac{d}{dt}\|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(\Omega)} = 2\langle u^\varepsilon_t(t) - u^\delta_t(t), u^\varepsilon(t) - u^\delta(t)\rangle_{L^2(\Omega)}.$$

Thus, integrating between s and t we have

$$\begin{aligned} \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(\Omega)} &= \|u^\varepsilon(s) - u^\delta(s)\|^2_{L^2(\Omega)} \\ &+2\int_s^t \langle u^\varepsilon_t(\tau) - u^\delta_t(\tau), u^\varepsilon(\tau) - u^\delta(\tau)\rangle_{L^2(\Omega)}\ d\tau , \end{aligned} \tag{1.10}$$

for all $0 \le s,t \le T$. Now, as a consequence of (1.9), for a.e. $s \in (0,T)$, we have

$$u^\varepsilon(s) \longrightarrow u(s) \quad \text{in } L^2(\Omega).$$

So that, for these s, from (1.10), using both *Cauchy-Schwartz* and *Young's* inequality, we can write

$$\begin{aligned} \sup_{0\le t\le T} \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(\Omega)} &\le \|u^\varepsilon(s) - u^\delta(s)\|^2_{L^2(\Omega)} \\ &+ \int_0^T \|u^\varepsilon_t(\tau) - u^\delta_t(\tau)\|^2_{H^{-1}(\Omega)} + \|u^\varepsilon(\tau) - u^\delta(\tau)\|^2_{H^1_0(\Omega)}\ d\tau = \omega(\varepsilon,\delta), \end{aligned}$$

thanks to (1.9) (here $\omega(\varepsilon,\delta)$ tends to zero as ε and δ tend to zero).

So u^ε converges to a function v in $C([0,T];L^2(\Omega))$; since we know that $u^\varepsilon(t) \to u(t)$ for a.e. t, we deduce that $v = u$ a.e.

□

This result also allows us to deduce, for functions u and v enjoying these properties, the integration by parts formula:

$$\int_a^b \langle v, u_t\rangle\ dt + \int_a^b \langle u, v_t\rangle\ dt = (u(b),v(b)) - (u(a),v(a))\,, \tag{1.11}$$

where $\langle\cdot,\cdot\rangle$ is the duality between V and V' and $(\cdot,\cdot)$ the scalar product in H. Notice that the terms appearing in (1.11) make sense thanks to Theorem 1.7. The proof of (1.11) relies on the fact that $C_0^\infty(a,b;V)$ is dense in the space of functions $u\in L^p(a,b;V)$ such that $u_t\in L^{p'}(a,b;V')$ endowed with the norm $\|u\|=\|u\|_{L^p(a,b;V)}+\|u_t\|_{L^{p'}(a,b;V')}$, together with the fact that (1.11) is true for u, $v\in C_0^\infty(a,b;V)$ by the theory of integration and derivation in Banach spaces. Note however that in this context (1.11) is subject to the satisfication of the hypotheses of Theorem 1.7; if, for instance, $V=W_0^{1,p}(\Omega)$, then

$$W_0^{1,p}(\Omega)\underset{dense}{\hookrightarrow}L^2(\Omega)\hookrightarrow W^{-1,p'}(\Omega)$$

only if $p\geq\frac{2N}{N+2}$; for the sake of simplicity we will often work under this bound, that actually turns out to be only technical to our purposes.

1.2.1. Further useful results. Here we give some further results that will be very useful in what follows; the first one contains a generalization of the integration by parts formula (1.11) where the time derivative of a function is less regular than there; its proof can be found in [**DP**] (see also [**CW**]).

LEMMA 1.8. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous piecewise C^1 function such that $f(0)=0$ and f' is compactly supported on $\mathbb{R}$; let us denote $F(s)=\int_0^s f(r)dr$. If $u\in L^p(0,T;W_0^{1,p}(\Omega))$ is such that $u_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$ and if $\psi\in C^\infty(\overline{Q})$, then we have*

$$\begin{aligned}\int_0^T\langle u_t,f(u)\psi\rangle\ dt&=\int_\Omega F(u(T))\psi(T)\ dx\\ &-\int_\Omega F(u(0))\psi(0)\ dx-\int_Q\psi_t\ F(u)\ dxdt.\end{aligned}\tag{1.12}$$

Now we state three embedding theorems that will play a central role in our work; the first one is the well-known *Gagliardo-Nirenberg embedding theorem* followed by an important consequence of it for the evolution case, while the second one is an Aubin-Simon type result that we state in a form general enough to our purpose; the third one is a useful generalization of Theorem 1.7.

THEOREM 1.9 (Gagliardo-Nirenberg). *Let v be a function in $W_0^{1,q}(\Omega)\cap L^\rho(\Omega)$ with $q\geq1$, $\rho\geq1$. Then there exists a positive constant C, depending on N, q and ρ, such that*

$$\|v\|_{L^\gamma(\Omega)}\leq C\|\nabla v\|^\theta_{(L^q(\Omega))^N}\|v\|^{1-\theta}_{L^\rho(\Omega)}\,,$$

for every θ and γ satisfying

$$0\leq\theta\leq1,\quad 1\leq\gamma\leq+\infty,\quad \frac{1}{\gamma}=\theta\left(\frac{1}{q}-\frac{1}{N}\right)+\frac{1-\theta}{\rho}\,.$$

PROOF. See [**N**], Lecture II. □

A consequence of the previous result is the following embedding result:

COROLLARY 1.10. *Let* $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, *with* $q \geq 1$, $\rho \geq 1$. *Then* $v \in L^\sigma(Q)$ *with* $\sigma = q\frac{N+\rho}{N}$ *and*

$$\int_Q |v|^\sigma \, dxdt \leq C\|v\|_{L^\infty(0,T;L^\rho(\Omega))}^{\frac{\rho q}{N}} \int_Q |\nabla v|^q \, dxdt \,. \tag{1.13}$$

PROOF. By virtue of Theorem 1.9, we can write

$$\int_\Omega |v|^\sigma \leq C \, \|\nabla v\|_{L^q(\Omega)}^{\vartheta\sigma} \, \|v\|_{L^\rho(\Omega)}^{(1-\vartheta)\sigma} \,,$$

that is, integrating between 0 and T

$$\int_0^T \int_\Omega |v|^\sigma \leq C \int_0^T \, \|\nabla v(t)\|_{L^q(\Omega)}^{\vartheta\sigma} \, \|v(t)\|_{L^\rho(\Omega)}^{(1-\vartheta)\sigma} \, dt, \tag{1.14}$$

now, since $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, we have

$$\int_0^T \int_\Omega |v|^\sigma \leq C \, \|v\|_{L^\infty((0,T);L^\rho(\Omega))}^{(1-\vartheta)\sigma} \int_0^T \, \|\nabla v(t)\|_{L^q(\Omega)}^{\sigma\vartheta} \, dt.$$

Now we choose

$$\vartheta = \frac{q}{\sigma} = \frac{N}{N+\rho}$$

so that

$$\sigma\vartheta = q, \quad (1-\vartheta)\sigma = \frac{q\rho}{N},$$

and (1.14) becomes

$$\int_0^T \int_\Omega |v|^\sigma \leq C \, \|v\|_{L^\infty((0,T);L^\rho(\Omega))}^{\frac{q\rho}{N}} \int_0^T \, \|\nabla v(t)\|_{L^q(\Omega)}^{q} \, dt,$$

that is

$$\int_Q |v|^\sigma \leq C \, \|v\|_{L^\infty((0,T);L^\rho(\Omega))}^{\frac{q\rho}{N}} \int_Q |\nabla v|^q \,.$$

□

REMARK 1.11. Let us explicitly remark that Corollary 1.10 gives us a little gain on the *a priori* summability of the involved function (actually this is a consequence of the Gagliardo-Nirenberg inequality, not a consequence of a *Petitta's inequality* ☺). As an example, let us think about a function $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$; in this case the solution turns out to belong to $L^{2+\frac{4}{N}}(Q)$.

THEOREM 1.12. *Let* u^n *be a sequence bounded in* $L^q(0,T;W_0^{1,q}(\Omega))$ *such that* u_t^n *is bounded in* $L^1(Q) + L^{s'}(0,T;W^{-1,s'}(\Omega))$ *with* $q, s > 1$, *then* u^n *is relatively strongly compact in* $L^1(Q)$, *that is, up to subsequences,* u^n *strongly converges in* $L^1(Q)$ *to some function* u.

PROOF. See [**Si**], Corollary 4. □

Let us define, for every $p > 1$, the space S^p as

$$S^p = \{u \in L^p(0,T;W_0^{1,p}(\Omega)); u_t \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))\}, \tag{1.15}$$

endowed with its natural norm

$$\|u\|_{S^p} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}.$$

We have the following trace result:

THEOREM 1.13. *Let $p > 1$, then we have the following continuous injection*

$$S^p \hookrightarrow C(0,T;L^1(\Omega)).$$

PROOF. See [**Po**], Theorem 1.1. □

CHAPTER 2

Weak solutions

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, $t > 0$; we denote by Q_t the cylinder $\Omega \times (0,t)$. If $t = T$ we will often write Q for Q_T. In this chapter we are interested in the study existence, uniqueness, and regularity of the solution of the linear parabolic problem

$$\begin{cases} u_t + L(u) = f & \text{in } \Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.1}$$

where

$$L(u) = -\mathrm{div}(A(x,t)\nabla u),$$

and A is a matrix with bounded, measurable entries, such that

$$|A(x,t)\xi| \leq \beta|\xi|, \tag{2.2}$$

for any $\xi \in \mathbb{R}^N$, with $\beta > 0$, and

$$A(x,t)\xi \cdot \xi \geq \alpha|\xi|^2, \tag{2.3}$$

for any $\xi \in \mathbb{R}^N$, with $\alpha > 0$. As we will see such results strongly depend on the regularity of the data f, u_0 and A.

We first deal existence, uniqueness and (weak) regularity for linear problems in the framework of Hilbert spaces, that is $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in H_0^1(\Omega)$. For such data the solution is supposed to be in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, with $u_t \in L^2(0,T;H^{-1}(\Omega))$. Moreover, we expect the solutions to belong to $C(0,T;L^2(\Omega))$ to give sense at the initial value u_0.

Indeed if we formally multiply the equation in (2.1) by u and using (2.3), integrating on Ω and between 0 and t (here $0 < t \leq T$), we obtain, using also Young's inequality,

$$\begin{aligned} \int_0^t \langle u_t, u\rangle + \alpha \int_Q |\nabla u|^2 &\leq \int_0^t \|f\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)} \\ &\leq \|f\|_{L^2(0,T;H^{-1}(\Omega))} \|u\|_{L^2(0,T;H_0^1(\Omega))} \end{aligned}$$

$$\leq \frac{1}{2\alpha}\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \frac{\alpha}{2}\|u\|^2_{L^2(0,T;H_0^1(\Omega))}.$$

Which, thanks to the fact that

$$\int_0^t \langle u_t, u\rangle = \int_0^t \frac{1}{2}\frac{d}{dt}u^2,$$

using (1.11) yields

$$\frac{1}{2}\int_\Omega u^2(t) + \frac{\alpha}{2}\int_{Q_t} |\nabla u|^2 \le C(\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u_0\|^2_{L^2(\Omega)}).$$

Now, since the right hand side does not depend on t we easily deduce that the same inequality holds true for any $0 \le t \le T$, and so

$$\begin{aligned}(2.4)\qquad &\frac{1}{2}\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\alpha}{2}\|u\|^2_{L^2(0,T;H^1_0(\Omega))}\\ &\le C(\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u_0\|^2_{L^2(\Omega)}),\end{aligned}$$

which thanks to Theorem 1.7 it gives the desired regularity result.

The energy estimates can also be useful to prove the so-called *backward uniqueness*.

THEOREM 2.1 (Backward uniqueness). *Let u and $\tilde u$ be two solutions in Q such that $u(x,T) = \tilde u(x,T)$. Then $u = \tilde u$ in Q.*

PROOF. See [**E**], p. 64. □

REMARK 2.2. Let us stress the fact that, as a difference with the elliptic case, here it is not so easy to face the problem with a *Lax-Milgram* type approach (or with minimizing some functional) because of the features of the involved functional spaces and of the operator itself. Indeed, roughly speaking, the supposed involved *bilinear form* would turn out to be, for instance, not continuous on $L^2(0,T;H^1_0(\Omega))$, not coercive on

$$W = \{u \in L^2(0,T;H^1_0(\Omega)), u_t \in L^2(0,T;H^{-1}(\Omega))\},$$

and $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ is not an Hilbert space.

2.1. Galerkin Method: Existence and uniqueness of a weak solution

Let us first give our definition for weak solutions to problem (2.1)

DEFINITION 2.3. We say that a function $u \in L^2(0,T;H^1_0(\Omega))$, such that $u_t \in L^2(0,T;H^{-1}(\Omega))$ is a *weak solution* for problem (2.1) if

(2.5)

$$\int_0^T \langle u', \varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_Q A(x,t)\nabla u\cdot\nabla\varphi = \langle f,\varphi\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H^1_0(\Omega))},$$

for all $\varphi \in L^2(0,T;H^1_0(\Omega))$ such that $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$, $\varphi(T) = 0$, and $u(x,0) = u_0$ in the sense of $L^2(\Omega)$.

REMARK 2.4. Observe that, taking into account Theorem 1.7, we can easily see that all terms in Definition 2.3 turn out to make sense.

Moreover, as shown in [**E**] (actually it is not so difficult to check), if $f \in L^2(Q)$, u is a weak solution for problem (2.1) if and only if

$$\langle u'(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega A(x,t)\nabla u(t)\nabla v = \int_\Omega f(t)v, \tag{2.6}$$

for any $v \in H_0^1(\Omega)$ and a.e. in $0 \le t \le T$, with $u(0) = u_0$. We will often denote by $\langle\cdot,\cdot\rangle$ the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, while $(\cdot,\cdot)$ will be occasionally used to indicate the inner product in $L^2(\Omega)$.

Now we state our first existence and uniqueness result. Here, for the sake of simplicity, we choose $f \in L^2(Q)$. We will use the so called *Galerkin Method* which relies on the approximation of our problem by mean of finite dimensional problems. For another possible approach, based on the Hille-Yosida theorem and on the fact that the Laplace operator is maximal monotone, see [**B**], Chapters 7 and 10.

THEOREM 2.5. *Let $f \in L^2(Q)$, and $u_0 \in L^2(\Omega)$. Then there exists a unique weak solution for problem (2.1).*

PROOF. We will consider a sequence of functions $w_k(x)$, $(k = 1, \dots)$ which satisfy

i) $\{w_k\}$ is an *orthogonal* basis of $H_0^1(\Omega)$,
ii) $\{w_k\}$ is an *orthonormal* basis of $L^2(\Omega)$.

Observe that the construction of such a sequence is always possible; as an example (see [**E**]) we can take $w_k(x)$ as a sequence for $-\Delta$ in $H_0^1(\Omega)$ (after a suitable normalization).

Fix now an integer m. We will look for a function $u_m : [0,T] \mapsto H_0^1(\Omega)$ of the form

$$u_m(t) = \sum_{k=1}^m d_m^k(t) w_k, \tag{2.7}$$

and we want to select the coefficients such that

$$d_m^k(0) = (u_0, w_k) \quad (k = 1, \dots, m) \tag{2.8}$$

and

$$(u_m', w_k) + \int_\Omega A(x,t)\nabla u_m \cdot \nabla w_k = (f, w_k), \tag{2.9}$$

a.e. on $0 \le t \le T$, $k = 1, \dots, m$. In other words, we look for the solutions of the *projections* of problem (2.1) to the finite dimensional subspaces of $H_0^1(\Omega)$ spanned by $\{w_k\}(k = 1, \dots, m)$.

During this proof, for the convenience of the reader we will use the following notation

$$a(\psi, \varphi, t) \equiv \int_\Omega A(x,t)\nabla\psi \cdot \nabla\varphi.$$

We split the proof of this result in four steps.

Step 1. Construction of approximate solutions.

Assume that u_m has the structure (2.7); since w_k is orthonormal in $L^2(\Omega)$, then

$$(u'_m, w_k) = \frac{d}{dt} d^k_m(t),$$

and

$$a(u_m, w_k, t) = \sum_{l=1}^{m} e^{kl}(t) d^l_m(t),$$

where $e^{kl}(t) = a(w_l, w_k, t)(k, l = 1, \ldots, m)$. Finally, if $f^k = (f(t), w_k)$ is the projection of the datum f, then (2.9) becomes the linear system of ODE

$$\frac{d}{dt} d^k_m(t) + \sum_{l=1}^{m} e^{kl}(t) d^l_m(t) = f^k(t),$$

for $k = 1, \ldots, m$, subject to the initial condition (2.8). According to the standard existence theory for ordinary differential equations, there exists a unique absolutely continuous function $d_m(t) = (d^l_m, \ldots, d^l_m)$ satisfying (2.8) and the ODE for a.e. $0 \le t \le T$. Then u_m is the desired approximate solution since it turns out to solve (2.9).

Step 2. Energy Estimates.

Multiply the equation (2.9) by $d^k_m(t)$ and sum over k between 1 and m. Then, integrating between 0 and T, we find, recalling that $(u'_m, u_m) = \frac{1}{2}\frac{d}{dt}u^2_m$, and using (2.3)

$$\frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega u_m^2 + \alpha \int_Q |\nabla u_m|^2 \le \int_Q f u_m.$$

Thus, reasoning as in the proof of (2.4) we can check that

$$\|u_m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_m\|^2_{L^2(0,T;H^1_0(\Omega))} \le C(\|f\|^2_{L^2(Q)} + \|u_0\|^2_{L^2(\Omega)}), \tag{2.10}$$

(here we also used that $\|u_m(0)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}$).

Finally, using the fact that w_k is an orthogonal basis in $H^1_0(\Omega)$, we can fix any $v \in H^1_0(\Omega)$ such that $\|v\|_{H^1_0(\Omega)} \le 1$, and deduce from the equation (2.9), after a few easy calculations

$$|\langle u'_m, v\rangle| \le C(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1_0(\Omega)}),$$

that is

$$\|u'_m\|_{H^{-1}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1_0(\Omega)}),$$

whose square integrated between 0 and T, gathered together with (2.10), yields

$$\|u'_m\|^2_{L^2(0,T;H^{-1}(\Omega))} \le C(\|f\|^2_{L^2(Q)} + \|u_0\|^2_{L^2(\Omega)}). \tag{2.11}$$

Step 3. Existence of a solution.

From (2.10) we deduce that there exists a function $u \in L^2(0, T; H^1_0(\Omega))$, such that u_m converges weakly to u in $L^2(0, T; H^1_0(\Omega))$; moreover u'_m weakly converges to some function η in $L^2(0,T;H^{-1}(\Omega))$ (one can easily check by using its definition that $\eta = u'$). Then, we can pass to the limit in the weak formulation of u_m, that is in

(2.12)

$$\int_0^T \langle u'_m, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_Q A(x,t)\nabla u_m \cdot \nabla\varphi = \langle f, \varphi\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))}.$$

for any $\varphi \in L^2(0,T;H_0^1(\Omega))$, such that $\varphi' \in L^2(0,T;H^{-1}(\Omega))$, with $\varphi(T) = 0$, to obtain (2.5).

To check that the initial value is achieved we use (1.11) in (2.5) and (2.12), obtaining respectively

$$-\int_0^T \langle \varphi', u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega u(0)\varphi(0) + \int_Q A(x,t)\nabla u \cdot \nabla\varphi = \int_Q f\varphi,$$

and

$$-\int_0^T \langle \varphi', u_m\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega u_m(0)\varphi(0) + \int_Q A(x,t)\nabla u_m \cdot \nabla\varphi = \int_Q f\varphi.$$

Now, since $u_m(0) \to u_0$ in $L^2(\Omega)$, and $\varphi(0)$ is arbitrary we conclude that $u_0 = u(0)$.

Step 4. Uniqueness of the solution.

Let u and v be two solutions of problem (2.1) in the sense of Definition 2.3; if we take $u - v$ as test function in the weak formulation for both u and v (by a density argument we can see that this function can be chosen as test in (2.5) even if it does not satisfy, a priori, $(u-v)(T) = 0$). By subtracting, using that $(u-v)(0) = 0$, we obtain

$$\frac{1}{2}\int_\Omega |u-v|^2(T) + \int_Q |\nabla(u-v)|^2 \le 0,$$

that implies $u = v$ a.e. in Q.

□

CHAPTER 3

Regularity

3.1. Regularity for finite energy solutions

In this section, we will be concerned with regularity and existence results for solutions of parabolic problem (2.1).

Let us state a first improvement on the regularity of such solutions.

THEOREM 3.1 (Improved regularity). *Let $u_0 \in H_0^1(\Omega)$ and $f \in L^2(Q)$. Then the weak solution u of (2.1) satisfies*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), \quad u' \in L^2(Q),$$

with continuous estimates with respect to the data.

PROOF. [**E**], Theorem 5, pag. 360. $\square$

The previous statement gives a first standard regularity result in the Hilbertian case, but what happens if we know something more (or something less) on the datum f? For instance, what is the *best Lebesgue space* which the solution turns out to belong to?

We will prove the following

THEOREM 3.2. *Assume (2.2), (2.3), $u_0 \in L^2(\Omega)$, and let f belong to $L^r(0,T;L^q(\Omega))$ with r and q belonging to $[1,+\infty]$ and such that*

$$\frac{1}{r} + \frac{N}{2q} < 1\,. \tag{3.13}$$

Then there exists a weak solution of (2.1) belonging to $L^\infty(Q)$. Moreover there exists a positive constant d, depending only from the data (and hence independent on u), such that

$$\|u\|_{L^\infty(Q)} \le d. \tag{3.14}$$

Notice that assumption (3.13) implies that $r \in (1,+\infty]$ and $q \in \left(\frac{N}{2},+\infty\right]$.

To give an idea, let us represent the summability of the datum $f \in L^r(0,T;L^q(\Omega))$ in a diagram with axes $\dfrac{1}{q}$ and $\dfrac{1}{r}$. Since r, $q \in [1,+\infty]$, then all the possible cases of summability are inside of the square $[0,1] \times [0,1]$ (we use the notation $\dfrac{1}{\infty} = 0$).

If f belongs to $L^r(0,T;L^q(\Omega))$ where r and q are large enough, that is, if

$$\frac{1}{r}+\frac{N}{2q}<1 \quad \text{(zone 1 in Figure 1 below)}, \tag{3.15}$$

then every weak solution u belonging to

$$V_2(Q)\equiv L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$$

belongs also to $L^\infty(Q)$ (see Theorem 3.2 above). This fact was proved by Aronson and Serrin in the nonlinear case (see [**AS**]), while can be found in the linear setting in some earlier papers as, among the others, [**LU**] and [**A**].

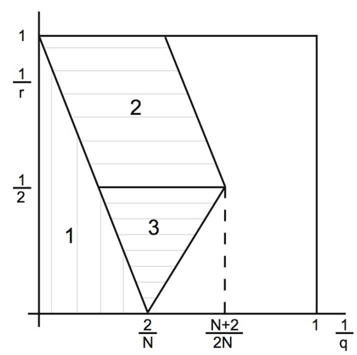


FIGURE 1. Classical regularity results.

In order to prove that the solutions of (2.1) are bounded when the summability exponents of f are in the zone 1 in the figure 1, we enunciate a very well known lemma due to Guido Stampacchia ([**S**]).

LEMMA 3.3. *Let us suppose that φ is a real, nonnegative and nonincreasing function satisfying*

$$\varphi(h)\le\frac{C}{(h-k)^\delta}\,[\varphi(k)]^\nu \quad \forall\ h>k>k_0, \tag{3.16}$$

where C and δ are positive constants and $\nu>1$. Then there exists a positive constant d such that

$$\varphi(k_0+d)=0.$$

Let us give the proof of Theorem 3.2

PROOF OF THEOREM 3.2. For the sake of simplicity, we will restrict ourselves to the case $u_0=0$, $f\ge 0$ and $r=q$, so that the solution u is nonnegative thanks to the maximum principle, and

$$q>\frac{N}{2}+1=\frac{N+2}{2}.$$

For $k>0$, define

$$A_k=\{(x,t)\in Q: u(x,s)>k\}, \tag{3.17}$$

and take $G_k(u)$ as a test function in (2.1), where $G_k(s) = (s-k)^+$. Integrating in $(0,t] \times \Omega$, where $t \le T$, and using assumption (2.3) we get

$$\frac{1}{2}\int_\Omega G_k(u)^2(t)dx + \alpha \int_0^t\!\!\int_\Omega |\nabla G_k(u)|^2 dxdt \le \int_{Q_t} fG_k(u)dxdt.$$

Taking the supremum for $t \in (0,T]$ on the right, we have (since both f and $G_k(u)$ are nonnegative)

$$\frac{1}{2}\int_\Omega G_k(u)^2(t)dx + \alpha \int_0^t\!\!\int_\Omega |\nabla G_k(u)|^2 dxdt \le \int_{Q} fG_k(u)dxdt,$$

which implies

$$\frac{1}{2}\int_\Omega G_k(u)^2(t)dx \le \int_{Q} fG_k(u)dxdt,$$

and

$$\alpha \int_{Q_t} |\nabla G_k(u)|^2 dxdt \le \int_{Q} fG_k(u)dxdt.$$

Taking the supremum on t in $(0,T]$ on the left, and summing up, we therefore get

$$C_0\left[\|G_k(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla G_k(u)\|^2_{L^2(Q)}\right] \le \int_Q fG_k(u)\,dx\,dt. \tag{3.18}$$

By Corollary 1.10, since

$$\int_Q G_k(u)^{2\frac{N+2}{N}}\,dx\,dt \le \|G_k(u)\|^{\frac{4}{N}}_{L^\infty(0,T;L^2(\Omega))}\,\|\nabla G_k(u)\|^2_{L^2(Q)},$$

we have

$$\int_Q G_k(u)^{2\frac{N+2}{N}}\,dx\,dt \le C\left(\int_Q fG_k(u)\right)^{\frac{N+2}{N}}$$

Using Hölder inequality with exponents $2\frac{N+2}{N}$ and $\frac{2N+4}{N+4}$, and observing that both integrals are on A_k, we have

$$\int_{A_k} G_k(u)^{2\frac{N+2}{N}}\,dx\,dt \le C\left(\int_{A_k} f^{\frac{2N+4}{N+4}}\,dx\,dt\right)^{\frac{N+4}{2N}}\left(\int_{A_k} G_k(u)^{2\frac{N+2}{N}}\,dx\,dt\right)^{\frac{1}{2}},$$

from which it follows

$$\int_{A_k} G_k(u)^{2\frac{N+2}{N}}\,dx\,dt \le C\left(\int_{A_k} f^{\frac{2N+4}{N+4}}\,dx\,dt\right)^{\frac{N+4}{N}}.$$

Since $q > \frac{N+2}{2} > \frac{2N+4}{N+4}$, a further use of the Hölder inequality yields

$$\int_{A_k} G_k(u)^{2\frac{N+2}{N}}\,dx\,dt \le C\,\|f\|^{\frac{2(N+2)}{Nq}}_{L^q(Q)}\ \mathrm{meas}(A_k)^{\frac{N+4}{N}-\frac{2(N+2)}{Nq}}.$$

If $h > k > 0$, we then deduce, since $G_k(u) \ge h-k$ on $A_h \subseteq A_k$,

$$\mathrm{meas}(A_h)\,(h-k)^{2\frac{N+2}{N}} \le \int_{A_k} G_k(u)^{2\frac{N+2}{N}}\,dx\,dt \le C\,\mathrm{meas}(A_k)^{\frac{N+4}{N}-\frac{2(N+2)}{Nq}}.$$

Setting $\varphi(h) = \text{meas}(A_h)$ we then have

$$\varphi(h) \le \frac{C}{(h-k)^{2\frac{N+2}{N}}} \varphi(k)^{\frac{N+4}{N} - \frac{2(N+2)}{Nq}}.$$

If $q > \frac{N+2}{2}$ we can apply Lemma 3.3, to conclude that there exists a constant d, depending only on q, $\|f\|_{L^q(Q)}$ and α, such that $\varphi(d) = 0$, that is

$$\|u\|_{L^\infty(Q)} \le d.$$

□

What happens if r and q do not satisfy (3.15) but satisfy

$$2 < \frac{2}{r} + \frac{N}{q} \le \min\left\{2 + \frac{N}{r}, 2 + \frac{N}{2}\right\}, \qquad r \ge 1, \tag{3.19}$$

that is, zone 2 and 3 in Figure 1? Ladyženskaja, Solonnikov and Ural'ceva (see Theorem 9.1, cap. 3 in [**LSU**]) proved that any weak solution of (2.1) belonging to $V_2(Q)$ satisfies also

$$|u(x,t)|^\gamma \in V_2(Q), \tag{3.20}$$

where γ is a constant greater than one that is given by an explicit formula in terms of N, r and q. This value of γ and Theorem 1.10 will then imply that

$$u \in L^s(Q), \qquad s = \frac{(N+2)qr}{Nr + 2q - 2qr} \tag{3.21}$$

A natural question that arises is whether there exists at least a solution of (2.1) belonging to $V_2(Q)$ if the summability exponents (r,q) of f satisfy (3.19).

If (3.19) holds with $r \ge 2$ (zone 3 in Figure 1) then the function f belongs also to the space $L^2(0,T;H^{-1}(\Omega))$, so that it is very easy to deduce the existence of at least such a weak solution.

If otherwise (3.19) holds with $r < 2$ (zone 2 in Figure 1) then f does not belong to $L^2(0,T;L^{(2^*)'}(\Omega))$, but again there exists at least a solution of (2.1) belonging to $V_2(Q)$ as proved in [**LSU**] for linear operators (see [**BDGO**] for more general nonlinear operators).

Indeed in [**LSU**] (Theorem 4.1 cap. 3) it is proved the previous existence result when the summability exponent of f satisfies

$$\frac{1}{r} + \frac{N}{2q} = 1 + \frac{N}{4} \quad q \in \left[\frac{2N}{N+2}, 2\right], \quad r \in [1,2],$$

but this implies that the result is true for every choice of exponents (r,q) satisfying

$$\frac{1}{r} + \frac{N}{2q} \le 1 + \frac{N}{4}, \quad q \ge \frac{2N}{N+2}, \tag{3.22}$$

see zones 2, 3 and 4 in Figure 2.

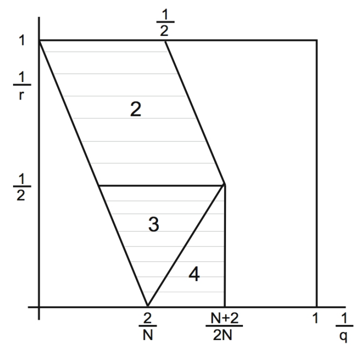


FIGURE 2. Existence results, zone 2, 3 and 4.

Notice that the previous zone includes strictly zone 2 and 3 of Figure 1. The theorem is thus the following.

THEOREM 3.4. *Let u be a solution of (2.1), and suppose that f belongs to $L^q(Q)$ with r and q satisfying (3.19). Then u belongs to $L^s(Q)$, with s given by (3.21).*

PROOF. For the sake of simplicity, we take $u_0 = 0$ and $f \geq 0$, so that $u \geq 0$. We will also prove the result in the case $r = q$, so that q satisfies the inequalities

$$\frac{2(N+2)}{N+4} \leq q < \frac{N+2}{2}, \tag{3.23}$$

and the value of s becomes $s = \frac{(N+2)q}{N+2-2q}$.

We fix $\gamma \geq 0$ and choose $T_k(u)^{2\gamma+1}$ as test function; here $T_k(s) = \min(s,k)$. Integrating on $(0,t])$, with $0 < t \leq T$ we have, denoting with $\Theta_k(s)$ the primitive function of $T_k(s)^{2\gamma+1}$ which is zero in zero,

$$\int_\Omega \Theta_k(u)(t)dx + \alpha(2\gamma+1)\int_{Q_t} |\nabla T_k(u)|^2\, T_k(u)^{2\gamma} dxdt \leq \int_{Q_t} f\, T_k(u)^{2\gamma+1} dxdt.$$

Reasoning as in the proof of Theorem 3.2, we first take the supremum on t in $(0,T]$ on the right, and then on the left. We obtain

$$\|\Theta_k(u)\|_{L^\infty(0,T;L^1(\Omega))} + \int_Q |\nabla T_k(u)|^2\, T_k(u)^{2\gamma} dxdt \leq C \int_Q f\, T_k(u)^{2\gamma+1} dxdt.$$

Since $\Theta_k(u) \geq C\, T_k(u)^{2\gamma+2}$, and since $|\nabla T_k(u)|^2\, T_k(u)^{2\gamma} = C\, |\nabla (T_k(u)^{\gamma+1}|^2$ we then have

$$\|T_k(u)^{\gamma+1}\|^2_{L^\infty(0,T;L^2(Q))} + \|\nabla T_k(u)^{\gamma+1}\|^2_{L^2(Q)} \leq C \int_Q f\, T_k(u)^{2\gamma+1} dxdt.$$

Setting $v = T_k(u)^{\gamma+1}$ and recalling Corollary 1.10, we have

$$\int_Q v^{2\frac{N+2}{N}}\, dxdt \leq \|v\|^{\frac{4}{N}}_{L^\infty(0,T;L^2(\Omega))}\, \|\nabla v\|^2_{L^2(Q)},$$

from which it follows

$$\int_Q v^{2\frac{N+2}{N}}\,dxdt \le C\left(\int_Q f\,v^{\frac{2\gamma+1}{\gamma+1}}\,dxdt\right)^{\frac{N+2}{N}}.$$

Recalling that f belongs to $L^q(Q)$ and using Hölder inequality, we obtain

$$\int_Q v^{2\frac{N+2}{N}}\,dxdt \le C\,\|f\|_{L^q(Q)}^{\frac{N+2}{N}}\left(\int_Q v^{q'\frac{2\gamma+1}{\gamma+1}}\,dxdt\right)^{\frac{N+2}{Nq'}}.$$

We now choose γ such that

$$q'\frac{2\gamma+1}{\gamma+1} = 2\frac{N+2}{N},$$

that is

$$\gamma+1 = \frac{Nq}{2(N+2-2q)}.$$

The condition $\gamma \ge 0$ is satisfied if and only if

$$\frac{2(N+2)}{N+4} \le q < \frac{N+2}{2},$$

which is true by (3.23). Since $q < \frac{N+2}{2}$, the exponent $\frac{N+2}{Nq'}$ is smaller than 1, so that we obtain

$$\left(\int_Q v^{2\frac{N+2}{N}}\,dxdt\right)^{1-\frac{N+2}{Nq'}} \le C\,\|f\|_{L^q(Q)}^{\frac{N+2}{N}}.$$

Recalling the definition of v, and the choice of γ, we then have

$$\left(\int_Q T_k(u)^{\frac{(N+2)q}{N+2-2q}}\,dxdt\right)^{1-\frac{N+2}{Nq'}} \le C\,\|f\|_{L^q(Q)}^{\frac{N+2}{N}}.$$

Letting k tend to infinity we then have, by Fatou lemma,

$$\|u\|_{L^s(Q)}^{\frac{N+2}{N}} \le C\,\|f\|_{L^q(Q)}^{\frac{N+2}{N}},$$

with $s = \frac{(N+2)q}{N+2-2q}$, as desired. □

What happens in the remaining zone (i.e., zone 4 in Figure 2) where, as just said, there exists at least a weak solution belonging to $V_2(Q)$?
Moreover, what happens outside of these zones? Are there other zones where there exist $V_2(Q)$ solutions?
In addition, where it is not reasonable to expect solutions in $V_2(Q)$, as for example when r and q are not too large (that is just for $q < (2^*)'$), and also when this regularity occurs, which is the starting regularity which ensures more summability properties of the solutions (of all the solutions) and which is the possibly optimal Lebesgue summability exponent of the solutions?

Recall that outside zone 1 in Figure 1 it is possible to show examples of unbounded solutions: does the same happen with the previous regularity results?

Surprisingly there are no exhaustive answers to these questions in literature. We just mention a regularity result concerning data f belonging to $L^2(0,T;H^{-1}(\Omega)) \cap L^r(0,T;L^q(\Omega))$ and solutions in the energy space $L^2(0,T;H_0^1(\Omega))$ (see [**GM**]). The remaning open questions has been recently faced in [**BPP**].

Remark 3.5. The theorem proved for elliptic equations stated the following: if f belongs to $L^q(\Omega)$ and $q > \frac{N}{2}$, then u belongs to $L^\infty(\Omega)$; if $\frac{2N}{N+2} \leq q < \frac{N}{2}$, then u belongs to $L^r(\Omega)$, with $r = \frac{Nq}{N-2q}$. Observe that the bounds on the exponents, as well as the summability result on the solutions, in the parabolic case can be derived from those in the elliptic case by making the substitution $N \mapsto N+2$ (i.e., taking into account the elliptic exponents for two more space dimensions). The fact that "adding one dimension" yields "add 2 to the exponents" is due to the fact that (heuristically) the only time derivative counts "one half" than the double space derivatives.

3.2. $L^1(Q)$ data

If f belongs to $L^1(Q)$, then none of the results of the preceding section can be applied, and in general a solution in the "energy space" $V_2(Q)$ does not exist. As for elliptic equations, one can then reason by approximation, choosing a sequence of regular (say, bounded) data converging to f, and proving *a priori* estimates in order to prove existence of solutions in the sense of distributions. Since in this case the solution u is not an admissible test function, the energy methods used in order to prove uniqueness are no longer available; furthermore, the nonzero elliptic solution z given by Serrin counterexample is a nonzero parabolic solution with z itself as initial datum, and is different from the solution obtained by approximation.

As Stampacchia did in the elliptic framework, we are now going to introduce a method to select the *right solution* for the parabolic problem.

This notion starts from the clever idea to test the problem with smooth solutions of the *dual* problem. The argument is so powerful that allow us to prove existence of solutions (in this *duality sense*) even with very irregular data, namely measures.

Let us straightforwardly extend this definition to the parabolic case.

Definition 3.6. Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$ A function $u \in L^1(Q)$ is a *duality solution* of problem

$$
\begin{cases} u_t - \operatorname{div}(A(x,t)\nabla u) = f & \text{in } \Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{3.24}
$$

if

$$
-\int_\Omega u_0 w(0)\, dx + \int_Q u\, g\, dxdt = \int_Q f\, w\, dx, \tag{3.25}
$$

for every $g \in L^\infty(Q)$, where w is the solution of the *backward problem*

$$
(3.26) \qquad \begin{cases} -w_t - \operatorname{div}(A^*(x,t)\nabla w) = g & \text{in } (0,T)\times\Omega,\\ w(x,T) = 0 & \text{in } \Omega,\\ w(x,t) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}
$$

where $A^*(x,t)$ is the transposed matrix of $A(x,t)$.

Remark 3.7. Notice that all terms in (3.25) are well defined thanks to Theorem 3.2. Moreover, it is quite easy to check that any duality solution of problem (3.24) actually turns out to be a distributional solution of the same problem. Finally recall that any duality solution turns out to coincide with the renormalized solution of the same problem (see [**Pe**]); this notion introduced in [**DMOP**] for the elliptic case, and then adapted to the parabolic case in [**Pe**], is the right one to ensure uniqueness also in the nonlinear framework. Finally notice that solutions of a *forward* parabolic problem and its associated *backward* problem are the same through the change of variable $t \mapsto -t$.

A unique duality solution for problem (3.24) exists, in fact we have the following

Theorem 3.8. *Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, then there exists a unique duality solution of problem (3.24).*

Proof. Let us fix $r, q \in \mathbb{R}$ such that

$$
r,\, q > 1, \qquad \frac{N}{q} + \frac{2}{r} < 2\,,
$$

and let us consider $g \in L^r(0,T;L^q(\Omega))$. Let w be the weak solution of problem (3.26); we know that w is bounded (Theorem 3.2) and continuous with values in $L^2(\Omega)$ (Theorem 2.5). We actually have

$$
\|w\|_{L^\infty(Q)} \le C\|g\|_{L^r(0,T;L^q(\Omega))};
$$

therefore, the linear functional

$$
\Lambda : L^r(0,T;L^q(\Omega)) \mapsto \mathbb{R},
$$

defined by

$$
\Lambda(g) = \int_Q f\, w\; dx + \int_\Omega u_0 w(0)\,,
$$

is well-defined and continuous, since

$$
|\Lambda(g)| \le (\|f\|_{L^1(Q)} + \|u_0\|_{L^\infty(\Omega)})\|w\|_{L^\infty(Q)} \le C\|g\|_{L^r(0,T;L^q(\Omega))}.
$$

So, by *Riesz's representation theorem* there exists a unique u belonging to $L^{r'}(0,T;L^{q'}(\Omega))$ such that

$$
\Lambda(g) = \int_Q u\, g\; dxdt,
$$

for any $g \in L^r(0,T;L^q(\Omega))$. So we have that, if $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, then there exists a (unique by construction) duality solution of problem (3.24). Note that since u belongs to $L^{r'}(0,T;L^{q'}(\Omega))$, the bounds on r and q imply that u belongs to $L^\sigma(0,T;L^\tau(\Omega))$ with σ and τ such that

$$\frac{2}{\sigma} + \frac{N}{\tau} > N.$$

If $r = q$, then $\sigma = \tau$ satisfies $\tau < \frac{N+2}{N}$ (once again, the elliptic exponent $\frac{N}{N-2}$ becomes the parabolic one with the substitution $N \mapsto N+2$). $\square$

CHAPTER 4

Asymptotic behavior of the solutions

4.1. Naïve idea and main assumptions

Let us give a naïve idea of what happens to a solution for large times. Let $u(x,t)$ be the solution of the 1-D heat equation

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0<t<\infty)\times(0<x<1) \\ u(0,x) = u_0(x) & \text{on } 0\le x\le 1, \\ u(t,0) = u(t,1) = 0 & \text{in } 0\le t<\infty, \end{cases}$$

with smooth u_0 ($u_0(0) = u_0(1) = 0$). Since we can write the initial datum u_0 as the uniform convergent series

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

then the solution $u(x,t)$ is the explicitly given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \sin(n\pi x),$$

and so, $u(x,t)$ tends to zero (with exponential rate!) as $t\to\infty$. Let us just emphasize (actually, my mom too should be able to easily check it!) that $z(x)\equiv 0$ solves the associated elliptic Laplace equation

$$\begin{cases} -z_{xx} = 0 & \text{in } (0<x<1) \\ z(0) = z(1) = 0. \end{cases}$$

That is, the solution u tends to something constant (in time), and so its derivative with repect to t converges, in some sense, to zero.

A large number of papers has been devoted to the study of asymptotic behavior for solutions of parabolic problems under various assumptions and in different contexts: for a review on classical results see [**F**] and the references therein. More recently in [**Pe1**] and [**LP**] the case of nonlinear monotone operators, and quasilinear problems with nonlinear absorbing terms having natural growth, have been considered; in particular, in [**Pe1**], we dealt with nonnegative measures μ absolutely continuous with respect to the parabolic p-capacity (the so called *soft measures*). Here we analyze the case of linear operators with nonnegative data regular enough, following the outlines of [**Pe1**].

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, $T > 0$; as usual, we denote by Q the cylinder $(0,T) \times \Omega$. We are interested in the study of the asymptotic behavior with respect to the time variable t of the solution of the linear parabolic problem

$$\begin{cases} u_t + L(u) = f & \text{in } (0,T) \times \Omega, \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{4.27}$$

with $f \in L^2(Q)$, $u_0 \in L^2(\Omega)$, and

$$L(u) = -\mathrm{div}(A(x)\nabla u),$$

where A is a matrix with bounded, measurable entries, and satisfying the ellipticity assumption

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \tag{4.28}$$

for any $\xi \in \mathbb{R}^N$, with $\alpha > 0$.

First observe that by Theorem 2.5 a unique solution is well defined for all $t > 0$.

Let us state our main result:

THEOREM 4.1. *Let $0 \leq f \in L^2(Q)$ be independent on the variable t. Let $u(x,t)$ be the solution of problem (4.27) with $u_0 \in L^2(\Omega)$, and let $v(x)$ be the solution of the corresponding elliptic problem*

$$\begin{cases} L(v) = f & \textit{in } \Omega, \\ v = 0 & \textit{on } \partial\Omega. \end{cases} \tag{4.29}$$

$$\lim_{t \to +\infty} u(x,t) = v(x),$$

strongly in $L^2(\Omega)$.

We first observe that, since $f \in L^2(Q)$ is independent on time, then the solution v of the elliptic problem (4.29) is also the unique solution of the parabolic problem (4.27), with $u_0 = v$, for any fixed $T > 0$.

We then have the following comparison principle for solutions of parabolic problems.

LEMMA 4.2. *Let $0 \leq f \in L^2(Q)$ be independent on time, and let u_0 and u_1 in $L^2(\Omega)$ be such that $0 \leq u_0 \leq u_1$. If w and z are solutions of (4.27) with $w(0) = u_0$ and $z(0) = u_1$, then $0 \leq w(x,t) \leq z(x,t)$ for every $t > 0$.*

PROOF. Let $u = z - w$. Then u solves

$$\begin{cases} u_t + L(u) = 0 & \text{in } (0,T) \times \Omega, \\ u(0) = u_1 - u_0, & \text{in } \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

Since $u_1 - u_0 \geq 0$, then $u \geq 0$ by the maximum principle, and so $z \geq w$. The fact that $w \geq 0$ follows again from the maximum principle since both f and u_0 are nonnegative. $\square$

PROOF OF THEOREM 4.1. We split the proof in few steps.

Step 1. We suppose that $u_0 = 0$.

Observe that $u(x,t) \geq 0$ for every $t > 0$ by Lemma 4.2. Therefore, if we consider $s > 0$, since we have that both $u(x,t)$ and $u(x,t+s)$ are solutions of problem (4.27) with, respectively, 0 and $u(x,s) \geq 0$ as initial datum, then again by Lemma 4.2 we deduce that $u(x,t+s) \geq u(x,t)$ for $t, s > 0$. Therefore u is a monotone nondecreasing function in t and so it converges to a function $\tilde{v}(x)$ almost everywhere as t tends to infinity. Since v is a solution of (4.27) with $v \geq 0$ itself as initial datum, Lemma 4.2 yields that $u(x,t) \leq v(x)$ for every t. Therefore, since $0 \leq u(x,t) \leq v(x)$, we have that $u(x,\cdot)$ converges to $\tilde{v}$ strongly in $L^2(\Omega)$ (actually, in the same space that v belongs to).

Now, let us consider $u^n(x,t)$ as the solution of

$$
\begin{cases}
u^n_t - \operatorname{div}(A(x)\nabla u^n) = f & \text{in } (0,1)\times\Omega,\\
u^n(0,x) = u(n,x) & \text{in } \Omega\\
u^n = 0 & \text{on } (0,1)\times\partial\Omega.
\end{cases}
\tag{4.30}
$$

We clearly have $u(x,t) = u^n(x,t-n)$ for $t \in [n, n+1]$ (since both $u(x,t)$ and $u^n(x,t-n)$ solve the same problem with the same data), and so

$$
u(x,n) = u^n(x,0) \leq u^n(x,t) \leq u^n(x,1) = u(x,n+1),
$$

for every t in $[0,1]$. Since both $u(x,n)$ and $u(x,n+1)$ converge to $\tilde{v}(x)$, then

$$
\lim_{n\to+\infty} u^n(x,t) = \tilde{v}(x), \quad \forall t \in [0,1],
$$

and the convergence is strong in $L^2(\Omega)$. Choosing u^n as test function, we then have

$$
\int_\Omega (u^n(1))^2 dx - \int_\Omega (u^n(0))^2 dx + \alpha \int_0^1\!\!\int_\Omega |\nabla u_n|^2 dxdt \leq \int_0^1\!\!\int_\Omega f\, u^n dxdt.
$$

The uniform boundedness of $u^n(x,\cdot)$ in $L^2(\Omega)$ (which implies the boundedness of u^n in $L^2(\Omega\times(0,1))$) then easily implies that u^n is bounded in $L^2(0,1;H^1_0(\Omega))$. Therefore u^n converges weakly in the same space to $\tilde{v}$ (there is no need of extracting subsequences since the limit is independent on the chosen subsequence). We now fix a function φ in $H^1_0(\Omega)$, and choose it as test function in the equation satisfied by u^n. We have, since $\varphi_t \equiv 0$,

$$
\begin{aligned}
&\int_\Omega u^n(1)\,\varphi(x)dx - \int_\Omega u^n(0)\,\varphi(x)dx\\
&\qquad + \int_0^1\!\!\int_\Omega A(x)\nabla u^n\cdot\nabla\varphi\, dxdt = \int_0^1\!\!\int_\Omega f\,\varphi dxdt.
\end{aligned}
$$

Since both $u^n(1)$ and $u^n(0)$ converge to $\tilde{v}$ in $L^2(\Omega)$, we have

$$
\lim_{n\to+\infty} \int_\Omega u^n(1)\,\varphi(x)dx - \int_\Omega u^n(0)\,\varphi(x)dx = 0,
$$

while we clearly have

$$\lim_{n\to+\infty}\int_0^1\!\!\int_\Omega f\,\varphi dxdt = \lim_{n\to+\infty}\int_\Omega f\,\varphi dxdt = \int_\Omega f\,\varphi dx.$$

Finally, the weak convergence of u^n to $\tilde{v}$ implies

$$\lim_{n\to+\infty}\int_0^1\!\!\int_\Omega A(x)\nabla u^n\cdot\nabla\varphi\,dxdt = \int_\Omega A(x)\nabla\tilde{v}\cdot\nabla\varphi\,dx.$$

Therefore,

$$\int_\Omega A(x)\,\nabla\tilde{v}\cdot\nabla\varphi\,dx = \int_\Omega f\,\varphi\,dx, \quad \forall\varphi\in H_0^1(\Omega),$$

so that $\tilde{v}$ is a solution of (4.29); by uniqueness, $\tilde{v}=v$, as desired.

Step 2. We suppose $u_0(x)=\lambda\,v(x)$ for some $\lambda\geq 1$.

Since λv is a solution of

$$\begin{cases} z_t + L(z) = \lambda f & \text{in } (0,T)\times\Omega,\\ z(0)=\lambda v, & \text{in } \Omega,\\ z=0 & \text{on } (0,T)\times\partial\Omega,\end{cases}$$

and since $\lambda f\geq f$ (being f nonnegative), then we have $0\leq u(x,t)\leq\lambda v(x)$ for every $t>0$. Therefore, since for every $s>0$ the function $u(x,t+s)$ is a solution of (4.27) with the same datum f, and initial datum $u(x,t)\leq\lambda v = u(x,0)$, we have (by Lemma 4.2)

$$0\leq u(x,t+s)\leq u(x,t),$$

for every $t>0$ and $s>0$. Thus, the function $t\mapsto u(x,t)$ is decreasing, and so $u(x,t)$ converges (strongly in $L^2(\Omega)$) to some function $\tilde{v}(x)$ as t tends to infinity. The same argument as in Step 1 implies that $\tilde{v}=v$.

Step 3. We suppose $0\leq u_0(x)\leq\lambda v(x)$ for some $\lambda\geq 1$.

Let $u(x,t)$ be the solution of (4.27). Then, by Lemma 4.2, we have

$$w(x,t)\leq u(x,t)\leq z(x,t),$$

where $w(x,t)$ is the solution of (4.27) with $w(x,0)=0$, and $z(x,t)$ is the solution of (4.27) with $z(x,0)=\lambda v(x)$. We have proved in Step 1 that $w(x,t)$ converges to $v(x)$ strongly in $L^2(\Omega)$ as t tends to infinity, and we have proved in Step 2 that $z(x,t)$ converges to $v(x)$ strongly in $L^2(\Omega)$ as t tends to infinity. Therefore, $u(x,t)$ converges to $v(x)$ strongly in $L^2(\Omega)$ as t tends to infinity.

Step 4. We suppose $0\leq u_0(x)$, and $f\not\equiv 0$.

We first observe that, since $f\geq 0$ and $f\not\equiv 0$, then the strong maximum principle for elliptic equations implies $v>0$ in Ω. Therefore, $\lambda v(x)$ tends to infinity everywhere in Ω as λ tends to infinity. Define now

$$u_{0,\lambda}(x)=\min(u_0(x),\lambda v(x)),$$

so that, by Lebesgue theorem, $u_{0,\lambda}$ converges to u_0 strongly in $L^2(\Omega)$ as λ tends to infinity. If u^λ is the solution of

$$\begin{cases} u_t^\lambda + L(u^\lambda) = f & \text{in } (0,T)\times\Omega, \\ u^\lambda(0) = u_{0,\lambda}, & \text{in } \Omega, \\ u^\lambda = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

then, for every fixed λ, $u_\lambda(x,t)$ converges to $v(x)$ as t tends to infinity thanks to Step 3. If u is the solution of (4.27) with initial datum u_0, then $u - u^\lambda$ is a solution of (4.27) with $f \equiv 0$ and initial datum $u_0 - u_{0,\lambda}$. Therefore

$$\|u(t) - u^\lambda(t)\|_{L^2(\Omega)} \le \|u_0 - u_{0,\lambda}\|_{L^2(\Omega)}, \quad \forall t > 0. \tag{4.31}$$

Given $\varepsilon > 0$, we first choose λ such that

$$\|u_0 - u_{0,\lambda}\|_{L^2(\Omega)} \le \varepsilon,$$

and then $t_\varepsilon > 0$ such that

$$\|u^\lambda(t) - v\|_{L^2(\Omega)} \le \varepsilon, \quad \forall t \ge t_\varepsilon.$$

Therefore, if $t \ge t_\varepsilon$, from (4.31) we have

$$\|u(t) - v\|_{L^2(\Omega)} \le \|u(t) - u^\lambda(t)\|_{L^2(\Omega)} + \|u^\lambda(t) - v\|_{L^2(\Omega)} \le 2\varepsilon,$$

and this implies the result.

Step 5. We suppose $0 \le u_0(x)$, and $f \equiv 0$.

If $\varepsilon \ge 0$, and u^ε is the solution of

$$\begin{cases} u_t^\varepsilon + L(u^\varepsilon) = \varepsilon & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(0) = u_0, & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

and u is the solution of problem (4.27) with $f \equiv 0$, then Lemma 4.2 implies

$$0 \le u(x,t) \le u^\epsilon(x,t) \le u^1(x,t), \tag{4.32}$$

for every $\varepsilon \le 1$. By Step 5, u^ε tends to v^ε, the solution of

$$\begin{cases} L(v^\varepsilon) = \varepsilon & \text{in } \Omega, \\ v^\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

Therefore,

$$0 \le \limsup_{t\to+\infty} u(x,t) \le \lim_{t\to+\infty} u^\varepsilon(x,t) = v^\varepsilon(x),$$

for every $\varepsilon \le 1$. Since v^ε tends to zero as ε tends to zero, we have that $u(x,t)$ tends to zero as t tends to infinity (and the convergence is strong in $L^2(\Omega)$ by Lebesgue theorem thanks to (4.32)).

Step 6. We suppose u_0 in $L^2(\Omega)$.

If $u_0 \ge v$ we define $z(x,t) = u(x,t) - v(x)$, which solves problem (4.27) with $u_0 - v$ as initial data and $f \equiv 0$. Since $z(x,t)$ tends to zero in $L^2(\Omega)$ as t diverges thanks to Step 5 (being $z(x,0) \ge 0$), we have that $u(x,t)$ tends to $v(x)$. If $u_0 \le v$, then $z(x,t) = v(x) - u(x,t)$ solves problem (4.27) with $f \equiv 0$ and a nonnegative initial datum, so that (again by Step 5), $z(x,t)$ tends to zero and so $u(x,t)$ tends to $v(x)$.

Now, if $u^{\oplus}$ and $u^{\ominus}$ solve problem (4.27) with, respectively, $\max(u_0, v)$ and $\min(u_0, v)$ as initial data, then, by Lemma 4.2, we have

$$u^{\ominus}(x,t) \le u(x,t) \le u^{\oplus}(x,t)$$

for any t, and this concludes the proof since the result holds true for both $u^{\oplus}$ and $u^{\ominus}$.

$\square$

APPENDIX A

Basic tools in integration and measure theory

We set by $\mathbb{R}^N$ the N-Euclidian space (simply $\mathbb{R}$ if $N = 1$) on which the standard Lebesgue measure is defined on the σ-algebra of Lebesgue measurable sets. The scalar product between two vectors a, b in $\mathbb{R}^N$ will be denoted by $a \cdot b$ or simply ab in most cases. Given a bounded open set Ω of $\mathbb{R}^N$, whose boundary will be denoted by $\partial\Omega$, and given a positive T, we shall consider the cylinder $Q_T = (0, T) \times \Omega$ (or simply Q where there is no possibility of confusion), setting by $C_0(Q)$ and $C_0^\infty(Q)$, the space of continuous, respectively C^∞, functions with compact support in Ω, while $C(\overline{\Omega})$ will denote functions that are continuous in the whole closed set $\overline{\Omega}$; moreover we will indicate by $C_0^\infty([0, T] \times \Omega)$ (resp. $C_0^\infty([0, T) \times \Omega)$) the set of all C^∞ functions with compact support on the set $[0, T] \times \Omega)$ (resp. on $[0, T) \times \Omega)$.

For the sake of simplicity here we will denote by D any bounded open subset of $\mathbb{R}^N$. We will deal with the space $M(D)$ of Radon measures μ on D that, by means of *Riesz's representation theorem*, turns out to coincide with the dual space of $C_0(D)$ with the topology of locally uniform convergence; we shall identify the element μ in $M(D)$ with the real valued additive set function associated, which is defined on the σ-algebra of Borel subsets of D and is finite on compact subsets. Thus with μ^+ and μ^- we mean, respectively, the positive and the negative variation of the Hahn decomposition of μ, that is $\mu = \mu^+ - \mu^-$, while the total variation of μ will be denoted by $|\mu| = \mu^+ + \mu^-$. Since we will always deal with the subset of $M(D)$ of the measures with bounded total variation on D, to simplify the notation we will denote also by $M(D)$ this subset. The restriction of a measure μ on a subset E is denoted by $\mu \llcorner E$ and is defined as follows:

$$(\mu \llcorner E)(B) = \mu(E \cap B), \quad \text{for every Borel subset } B \subseteq D.$$

If $\mu = \mu \llcorner E$ we will say that μ is concentrated on E.

For $1 \leq p \leq \infty$, we denote by $L^p(D)$ the space of Lebesgue measurable functions (in fact, equivalence classes, since almost everywhere equal functions are identified) $u : D \to \mathbb{R}$ such that, if $p < \infty$

$$\|u\|_{L^p(D)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} < \infty,$$

or which are essentially bounded (w.r.t. Lebesgue measure) if $p = \infty$. For the definition, the main properties and results on Lebesgue spaces we refer to [**B**]. For a function u in a Lebesgue space we set by $\dfrac{\partial u}{\partial x_i}$ (or simply u_{x_i}) its partial derivative in the direction x_i defined in the sense of distributions, that is

$$\langle u_{x_i}, \varphi \rangle = -\int_D u \varphi_{x_i}\, dx,$$

and we denote by $\nabla u = (u_{x_1}, \ldots, u_{x_N})$ the gradient of u defined this way.

The Sobolev space $W^{1,p}(D)$ with $1 \le p \le \infty$, is the space of functions u in $L^p(D)$ such that $\nabla u \in (L^p(D))^N$, endowed with its natural norm $\|u\|_{W^{1,p}(D)} = \|u\|_{L^p(D)} + \|\nabla u\|_{L^p(D)}$, while $W_0^{1,p}(D)$ will indicate the closure of $C_0^\infty(D)$ with respect to this norm. We still follow [**B**] for basic results on Sobolev spaces. Let us just recall that, for $1 < p < \infty$, the dual space of $L^p(D)$ can be identified with $L^{p'}(D)$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent of p, and that the dual space of $W_0^{1,p}(D)$ is denoted by $W^{-1,p'}(D)$. By a well known result, any element of $T \in W^{-1,p'}(D)$ can be written in the form $T = -\mathrm{div}(G)$ where $G \in (L^{p'}(D))^N$.

For every $0 < p < \infty$, we introduce the Marcinkiewicz space $M^p(D)$ of measurable functions f such that there exists $c > 0$, with

$$\mathrm{meas}\{x : |f(x)| \ge k\} \le \frac{c}{k^p},$$

for every positive k; it turns out to be a Banach space endowed with the norm

$$\|f\|_{M^p(D)} = \inf\left\{c > 0 : \mathrm{meas}\{x : |f(x)| \ge k\} \le \left(\frac{c}{k}\right)^p\right\}.$$

Let us recall that, since D is bounded, then for $p > 1$ we have the following continuous embeddings

$$L^p(D) \hookrightarrow M^p(D) \hookrightarrow L^{p-\varepsilon}(D),$$

for every $\varepsilon \in (0, p-1]$.

We already said that we refer to [**B**] for most basic tools in Lebesgue theory and Sobolev spaces; however, among them, let us recall explicitly some that will play a crucial role in the methods we use.

(1) *Generalized Young's inequality:* for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and any positive ε we have:

$$ab \le \varepsilon^p \frac{a^p}{p} + \frac{1}{\varepsilon^{p'}} \frac{b^{p'}}{p'}, \qquad \forall a, b > 0.$$

(2) *Hölder's inequality:* for $1 < p < \infty$, $p' = \frac{p}{p-1}$, we have, for every $f \in L^p(D)$ and every $g \in L^{p'}(D)$:

$$\int_D |fg|\, dx \le \left(\int_D |f|^p\right)^{\frac{1}{p}} \left(\int_D |g|^{p'}\right)^{\frac{1}{p'}}.$$

(3) Let $1 < p < \infty$, $p' = \frac{p}{p-1}$, $\{f_n\} \subset L^p(D)$, $\{g_n\} \subset L^{p'}(D)$ be such that f_n strongly converges to f in $L^p(D)$ and g_n weakly converges to g in $L^{p'}(D)$. Then

$$\lim_{n\to\infty} \int_D f_n\, g_n \; dx = \int_D fg \; dx\,.$$

The same conclusion holds true if $p = 1$, $p' = \infty$ and the weak convergence of g_n is replaced by the $*$-weak convergence in $L^\infty(D)$. Moreover, if f_n strongly converges to zero in $L^p(D)$, and g_n is bounded in $L^{p'}(D)$, we also have

$$\lim_{n\to\infty} \int_D f_n\, g_n \; dx = 0\,.$$

(4) Let f_n converge to f in measure and suppose that:

$$\exists\, C > 0, \quad q > 1: \quad \|f_n\|_{L^q(D)} \le C, \quad \forall\, n.$$

Then

$$f_n \longrightarrow f \quad \text{strongly in } L^s(D), \text{ for every } 1 \le s < q.$$

(5) *Fatou's lemma:* Let $\{f_n\} \subset L^1(D)$ be a sequence such that $f_n \to f$ a.e. in D and $f_n \ge h(x)$ with $h(x) \in L^1(D)$, then

$$\int_D f \; dx \le \liminf_{n\to\infty} \int_D f_n \; dx.$$

(6) *Generalized Lebesgue theorem:* Let $1 \le p < \infty$, and let $\{f_n\} \subset L^p(D)$ be a sequence such that $f_n \to f$ a.e. in D and $|f_n| \le g_n$ with g_n strongly convergent in $L^p(D)$, then $f \in L^p(D)$ and f_n strongly converges to f in $L^p(D)$.

(7) Let $\{f_n\} \subset L^1(D)$ and $f \in L^1(D)$ be such that, $f_n \ge 0$, $f_n \to f$ a.e. in D, and

$$\lim_{n\to\infty} \int_D f_n \; dx = \int_D f \; dx,$$

then f_n strongly converges to f in $L^1(D)$.

(8) *Vitali's theorem:* Let $1 \le p < \infty$, and let $\{f_n\} \subset L^p(D)$ be a sequence such that $f_n \to f$ a.e. in D and

$$\lim_{\text{meas}(E)\to 0} \sup_n \int_E |f_n|^p \; dx = 0. \tag{A.33}$$

Then $f \in L^p(D)$ and f_n strongly converges to f in $L^p(D)$.

(9) Let $\{f_n\} \subset L^1(D)$ and $\{g_n\} \subset L^\infty(D)$ be two sequences such that

$$f_n \longrightarrow f \quad \text{weakly in } L^1(D),$$

$$g_n \longrightarrow g \quad \text{a.e. in } D \text{ and } *\text{-weakly in } L^\infty(D).$$

Then

$$\lim_{n\to\infty} \int_D f_n\, g_n \; dx = \int_D fg \; dx\,.$$

Remark A.1. Property (A.33) is the so called equi-integrability property of the sequence $\{|f_n|^p\}$. We recall that *Dunford-Pettis theorem* ensures that a sequence in $L^1(D)$ is weakly convergent in $L^1(D)$ if and only if it is equi-integrable. Moreover, results (4), (6) and (7) can be proven as an easy consequences of Vitali's theorem and so we will refer to them as Vitali's theorem as well. For the same reason we will refer to result (9) as *Egorov theorem*.

For functions in the Sobolev space $W_0^{1,p}(D)$ we will often use *Sobolev's theorem* stating that, if $p < N$, $W_0^{1,p}(D)$ continuously injects into $L^{p^*}(D)$ with $p^* = \frac{Np}{N-p}$; if $p = N$, $W_0^{1,p}(D)$ continuously injects into $L^q(D)$ for every $q < \infty$, while, if $p > N$, $W_0^{1,p}(D)$ continuously injects into $C(\overline{D})$. Let us also recall *Rellich's theorem* stating that, if $p < N$, the injection of $W_0^{1,p}(D)$ into $L^q(D)$ is compact for every $1 \le q < p^*$, and *Poincaré's inequality*, that is, there exists $C > 0$ such that

$$\|u\|_{L^p(D)} \le C\|\nabla u\|_{(L^p(D))^N} ,$$

for every $u \in W_0^{1,p}(D)$, so that $\|\nabla u\|_{(L^p(D))^N}$ can be used as equivalent norm on $W_0^{1,p}(D)$.

We will often use the following result due to G. Stampacchia.

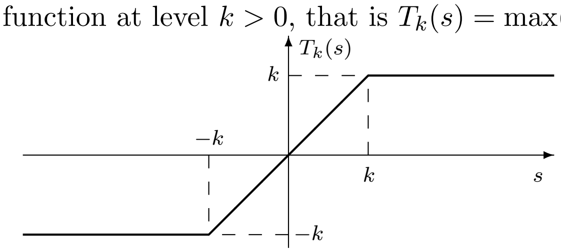
Theorem A.2. *Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then for every $u \in W_0^{1,p}(D)$ we have $G(u) \in W_0^{1,p}(D)$ and $\nabla G(u) = G'(u)\nabla u$ almost everywhere in D.*

Proof. See [**S**]. □

Theorem A.2 has an important consequence, that is

$$\nabla u = 0 \quad \text{a.e. in } F_c = \{x : u(x) = c\},$$

for every $c > 0$. Hence, we are able to consider the composition of function in $W_0^{1,p}(D)$ with some useful auxiliary function. One of the most used will be the truncation function at level $k > 0$, that is $T_k(s) = \max(-k, \min(k, s))$;



thus, if $u \in W_0^{1,p}(D)$, we have that $T_k(u) \in W_0^{1,p}(D)$ and $\nabla T_k(u) = \nabla u \chi_{\{u<k\}}$ a.e. on D, for every $k > 0$.

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