CHARACTERIZATION OF REMARKABLE HYPERVECTOR SPACES

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Abstract

We characterize the hypervector spaces such that all their factor spaces are not classical and provide examples showing that such spaces do exist.

We define hypervector space over a field k a quadruplet $(V, +, \circ, k)$ such that $(V, +)$ is an abelian group and

$$\circ : k \times V \to P'(V)$$

is a mapping of $k \times V$ into the power set of V (deprived of the empty set), such that the following properties holds:

$$\forall\, a, b \in k, \forall\, x \in V, \qquad (a+b) \circ x \subseteq (a \circ x) + (b \circ x)\,, \tag{1}$$

$$\forall\, a \in k, \forall\, x, y \in V, \qquad a \circ (x+y) \subseteq (a \circ x) + (a \circ y)\,, \tag{2}$$

$$\forall\, a, b \in k, \forall\, x \in V, \qquad a \circ (b \circ x) = (ab) \circ x\,, \tag{3}$$

$$\forall\, x \in V, \qquad x \in 1 \circ x\,. \tag{4}$$

If in (1) the equality holds, the hypervector space is called *strongly left distributive*. If in (2) the equality holds, the hypervector space is called *strongly right distributive*.

Such spaces have been completely characterized in a previous work [1]. A deep study of the distributive hypervector spaces is contained in [2], where we provide characterizations of several classes of such spaces.

Moreover we study in further papers the matroidal hypervector spaces, that is such that the closure system of their subspaces satisfies the exchange axiom [4].

Here our aim is to claim that the structure of hypervector space (which is a multivalued algebraic structure) is a real generalization of the notion of classical vector space, in the sense that even factor spaces of hypervector spaces are not classical.

More precisely we characterize the hypervector spaces not containing subspaces such that their factor spaces are classical and provide examples and prove that such hypervector spaces exist. To do this we need to define some notions as follows.

Let V be a hypervector space over the field k; $H \subseteq V$ is a *subspace* of V, if

$$H \neq \phi,\ H - H \subseteq H,\ \forall\, a \in k,\ a \circ H \subseteq H \tag{5}$$

If H is a subspace, of V we consider the factor group $W = (V, +)/(H, +)$. An element of W is the coset $[x] = x + H$.

Set:

$$\forall\, [x] \in W,\ \forall\, a \in k,\ a * [x] = [a \circ x] = \{[y] \in W : y \in a \circ x\}\,. \tag{6}$$

We prove that the set $a * [x]$ does not depend on the element chosen in the class $[x]$.

We prove that $(W, +, *, k)$ is a hypervector space. In a natural way then we call $(W, +, *, k)$ *factor hypervector space* of V with respect to H and write $W = V/H$. Obviously if V is strongly distributive either left or right, the same happens to $W = V/H$.

Let $(V, +, \circ, k)$ and $(V', +', \circ', k)$ be two hypervector spaces over the same field k. We call *homomorphism* between V and V' a mapping

$$f : V \to V'$$

such that:

$$\forall\, x, y \in V,\ f(x + y) = f(x) +' f(y)\,, \tag{7}$$

$$\forall\, a \in k,\ \forall\, x \in V, f(a \circ' x) \subseteq a \circ' f(x)\,. \tag{8}$$

Obviously this notion in the case of classical vector spaces coincides with the known one.

We call *strong homomorphism* an homomorphism such that in (8) the equality holds. In the following we consider only strong homomorphisms, which we call simply homomorphisms. The following Theorems hold [10]:

Theorem 1 *Let V and V' be hypervector spaces. If H is a subspace of V, then $f(H)$ is a subspace of V'. In particular* $\operatorname{Im} f = f(V)$ *is a subspace of V'. Conversely if H' is a subspace of V', then $f^{-1}(H')$ is a subspace of V.*

Theorem 2 $\ker f = f^{-1}(\mathbf{0}')$ *is a subspace of V, if and only if*

$$\forall\, a \in k,\ a \circ' \mathbf{0}' = \{\mathbf{0}'\}\,.$$

A homomorphism $f : V \to V'$ is called *good* if and only if $\ker f$ is a subspace of V, that is:

$$f : V \to V' \text{ good } \Leftrightarrow \forall\, a \in k,\ a \circ' \mathbf{0}' = \{\mathbf{0}'\}$$

The following Theorems hold:

Theorem 3 *Let f be a good homomorphism*

$$f : V \to V'.$$

Then $V/\ker f$ is isomorphic to $\operatorname{Im} f$.

Theorem 4 *Let H be a subspace of V. The mapping (that is the canonical epimorphism)*

$$p : x \in V \to [x] \in V/H$$

is a good homomorphism.

By the previous results it follows that the hypervector spaces with respect to the general homomorphisms are a cathegory, since the product of homomorphisms is an homomorphism and the identity is a homomorphism. A subcathegory of this is the cathegory of hypervector spaces with respect to the strong homomorphisms. A hypervector space is called *good* if

$$\forall\, a \in k, \quad a \circ \mathbf{0} = \{\mathbf{0}\} \tag{9}$$

If V is good, the identity is good.

Moreover, the product of good homomorphisms is good. It follows that the good hypervector spaces with respect to good homomorphisms are a cathegory. Also the not good hypervector spaces form a cathegory with respect to their homomorphisms (which are necessarily not good).

Obviously *every classical vector space is good*. However wide classes of not classical good hypervector spaces exist.

For instance, if V is a hypervector space containing a subspace H, different from V, then V/H is a good hypervector space. In particular if the set

$$U(\mathbf{0}) = \bigcup_{a \in k} a \circ \mathbf{0}$$

is a subspace and $U(\mathbf{0}) \neq V$ if V contains at least 2 elements, then $V/U(\mathbf{0})$ is good.

We remark that there are hypervector spaces which do not contain any proper subspace. In a natural way we call such spaces *simple hypervector spaces*. Moreover, if V is simple, the only good homomorphism $f : V \to V'$ is the trivial one (that is such that $V \to \mathbf{0}'$, where $\mathbf{0}' \in V'$, V' any good hypervector space).

An example of simple hypervector space is the following:

I. Let $(V,+)$ be any abelian group, $|H| \geq 3$. Set

$$\begin{array}{rrcl} \forall\, a \in k,\ \forall\, x \in V - \{\mathbf{0}\}, & a \circ x & = & V - \{\mathbf{0}\}, \\ & 1 \circ \mathbf{0} & = & V\,, \\ & 0 \circ \mathbf{0} & = & V - \{\mathbf{0}\}\,. \end{array}$$

It is easy to prove that such a V is simple. The automorphisms of V form a group which consists of good automorphisms if and only if V is good.

We can define the *dual* of a hypervector space, by giving the notion of *linear form* over V, that is a homomorphism $f : V \to k$, where k is considered as the classical vector space over itself. Obviously every linear form is a good homomorphism. The set V^* of all linear forms over V is a classical vector space (the product times a scalar is obviously the following: $\forall\, a \in k$, $(af)(x) = af(x)$, $\forall\, x \in V$). So the dual of a hypervector space is a classical vector space. We define the mapping

$$\bar{u} : f \in V^* \to f(u) \in k, \quad \forall\, u \in V\,.$$

$\bar{u}$ is a linear mapping of V^* into k, that is an element of V^{**}. Then we can define the mapping

$$\chi : u \in V \to \bar{u} \in V^{**}\,.$$

χ is a homomorphism of the hypervector space V into the classical space V^{**}, therefore it is a good homomorphism. It follows that $\ker \chi$ is a subspace of V.

The homomorphism χ is called *canonical homomorphism* and $\ker \chi$ is called *canonical subspace.*

We can now construct a contravariant functor between the cathegory of the hypervector spaces with respect to the strong homomorphisms and the classical cathegory of vector spaces by associating with every hypervector space V, its dual V^* and with every homomorphism $h : V \to V'$ the dual homomorphism

$$h^* : V'^* \to V^*\,.$$

This functor is called duality functor.

The final aim which we announced at the beginning can now be explained.

Set

$$C = \{x \in V : |a \circ x| = 1,\ \forall\, a \in k\}\,. \tag{10}$$

Then the following Theorems hold:

Theorem 5 *If C is not empty, C is a classical subspace of V, which contains every classical subspace of V. If C is empty, in V classical subspaces do not exist.*

Theorem 6 *In V there are classical subspaces, if and only if V is good.*

Let $V = (V, +, \circ, k)$ be a hypervector space over the field k. Our aim is to determine the subspaces H of V such that $W = V/H$ is classical.

If V/H is classical

$$\forall\, a \in k,\ \forall\, [x] \in W = V/H, \quad |a * [x]| = 1\,.$$

(where $\forall\, a \in k,\ a * [x] = [a \circ x] = \{[y] \in W : y \in a \circ x\}$).

Then the following Theorem holds.

Theorem 7 *$W = V/H$ is classical if and only if*

$$\forall\, a \in k,\ \forall\, x \in V,\ (a \circ x) - (a \circ x) \subseteq H\,.$$

Set

$$M = \bigcup_{a \in k} \bigcup_{x \in V} \{(a \circ x) - (a \circ x)\}\,.$$

Let

$$L = \overline{M} \tag{11}$$

be the closure of M (that is the intersection of all the subspaces containing M). By Theorem (7), if a subspace H is such that $W = V/H$ is classical, then H must contain M and therefore L. Conversely if H is a subspace containing L, it contains M and then by Theorem (7), $W = V/H$ is classical. So the following Theorem holds (which characterizes the hypervector spaces such that V/H is classical):

Theorem 8 *A subspace H of V is such that V/H is classical if and only if it contains L. In particular V/L is classical and $L \subseteq \ker \chi$.*

By the previous arguments it follows that we can associate with every hypervector space V the subspaces L (see (11)), $\ker \chi$ and, if V is good, the subspace C (see (10)).

Finally we characterize the hypervector spaces such that V/H is not classical:

Theorem 9 *V does not contain proper subspaces such that V/H is classical if and only if $L = V$.*

We remark that there are hypervector spaces V such that $L = V$. An example is provided by the simple hypervector space already explained. A further example is the following.

II. Let $(\mathbb{R}^n, +)$ be the classical additive group over $\mathbb{R}^n$ and $\forall\, a \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^n$, let $a \circ x$ be the subset of $\mathbb{R}^n$ consisting of the vectors of the closed segment from the origin to the end point of the vector ax. The stucture $(\mathbb{R}^n, +, \circ, \mathbb{R})$ is a hypervector space over $\mathbb{R}$.

Such a space is not strongly distributive either right, or left. Moreover it is a good hypervector space (since $a \circ \mathbf{0} = \mathbf{0}$). But it is such that $C = (\mathbf{0})$ and $L = \mathbb{R}^n$. Therefore it does not contain either proper subspaces whose factor space is classical, or classical subspaces different from $(\mathbf{0})$, that is it is a simple hypervector space.

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