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PARTIAL LINE SPACES AND ALGEBRAIC VARIETIES

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1. Introduction

The present paper is meant to be a survey paper on the recent results the Italian school achieved in the matter of characterizing special algebraic varieties as partial line spaces.

A partial line space is a pair $(S, \mathscr{L})$, where S is a non-empty set whose elements are called points and $\mathscr{L}$ is a subset of the power set of S, whose elements are called lines, such that on any line at least two points lie, $\mathscr{L}$ is a covering of S, through any two distinct points at most one line passes. Such a definition is extremely general; for instance, any (simple) graph is a partial line space. Therefore, the addition is allowed of further axioms to those defining a partial line space which turns out to be a fruitful idea.

Any ruled algebraic manifold, in either a projective or an affine space, when attention is focused on its lines, is a partial line space for which suitable incidence properties hold. Thus, the question arises how can these algebraic varieties be characterized as partial line spaces satisfying a proper choice of axioms; this means that, up to isomorphism, any partial line space for which the given axioms are true is the variety the axioms started from.

A thorough look at the essential geometric properties of some special algebraic varieties suggested the simple and nice axioms which can provide the required characterizations. From this standpoint the investigations forming the subject of this paper were carried out.

As a matter of fact, this approach leads to characterize Grassmann varieties (sect.s 3 and 4), C. Segre's product varieties (sect. 5), Veronese varieties

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(sect. 6), and Schubert varieties (sect. 7).

Furthermore, since a graph is a partial line space, the above mentioned characterizations provide graph theoretical characterization theorems too.

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2. Partial line spaces: definitions and notation

A *partial line space*, for short *PLS*, is defined as a pair $(S, \mathscr{L})$, where S is a non-empty set and $\mathscr{L}$ is a proper family of subsets of S, such that:

(2.1) for any $\ell \in \mathscr{L}, |\ell| \geqslant 2$;

(2.2) $\mathscr{L}$ is a covering of S;

(2.3) $\ell, \ell' \in \mathscr{L},\ \ell \neq \ell'$ imply $|\ell \cap \ell'| \leqslant 1$.

The elements of S will be called points, those of $\mathscr{L}$ lines. By (2.3), through any two distinct points x and y at most one line passes; x and y are said to be *collinear*, $x \sim y$, if the line xy through them exists; otherwise, x and y are *non-collinear*, $x \nsim y$. If any two points in $(S, \mathscr{L})$ are collinear, then $(S, \mathscr{L})$ is a *line space;* if this is not the case, $(S, \mathscr{L})$ is a *proper* partial line space, *PPLS*. A partial line space $(S, \mathscr{L})$ is said to be *irreducible* if any line contains at least three points.

An isomorphism between two *PLS*'s $(S, \mathscr{L})$ and $(S', \mathscr{L}')$ is a bijection $f: S \to S'$ which maps lines onto lines and the same is true for its inverse mapping.

PLS's are investigated up to isomorphism, i.e. any two isomorphic spaces are identified. Aut $(S, \mathscr{L})$ will denote the automorphism group of $(S, \mathscr{L})$.

Let $(S, \mathscr{L})$ be a *PLS*. A subset T of S is called a *subspace* of $(S, \mathscr{L})$ if

(2.4) $x, y \in T, x \neq y \Rightarrow x \sim y$ and $xy \subset T$.

If $|T| \geqslant 2$, denote by $\mathscr{L}_T$ the set of lines in $\mathscr{L}$ which are contained in T; thus, $(T, \mathscr{L}_T)$ is a line space. Let $\mathscr{C}_T$ be the family of subspaces of $(S, \mathscr{L})$ contained in T, i.e. the family of subspaces of $(T, \mathscr{L}_T)$. Then the following hold:

(2.5) $\begin{cases} \phi \in \mathscr{C}_T;\ \forall x \in T, \{x\} \in \mathscr{C}_T;\ \mathscr{L}_T \subset \mathscr{C}_T; \\ \mathscr{C}_T \text{ is a closure system for } T. \end{cases}$

Therefore, $\mathscr{C}_T$ defines a closure operator on T. Consequently, independent sets, generators and bases are defined. Namely, X is an independent subset of T if, for any x in X, $x \notin \overline{X \setminus \{x\}}$; X is a generator if $\overline{X} = T$; X is a basis for T if it is an independent generator.

$(S, \mathscr{L})$ will be said to be of *finite rank r* if for any basis B of whichever of its subspaces $|B| \leqslant r$ and a subspace T exists having a basis with r elements.

If for any subspace T of $(S, \mathscr{L})$ the following exchange axiom holds

(2.6) $X \subseteq S, x, y \in S: y \notin \overline{X}, y \in \overline{X \cup \{x\}} \Rightarrow x \in \overline{X \cup \{y\}}$,

then $(S, \mathscr{L})$ will be called a *combinatorial PLS*. Thus, if $(S, \mathscr{L})$ is a finite rank combinatorial *PLS*, then all bases of any subspace T have the same cardinality, rank T. Furthermore, if T_1 and T_2 are subspace of T, then

(2.7) $\operatorname{rank} T_1 + \operatorname{rank} T_2 \geqslant \operatorname{rank} (T_1 \cap T_2) + \operatorname{rank} (\overline{T_1 \cup T_2})$.

Let $(S, \mathscr{L})$ be a proper *PLS;* a subspace M is a *maximal* subspace if no subspace exists in which M is properly contained. With the help of Zorn's Lemma, any subspace can be proved to be contained in at least one maximal subspace; therefore, the maximal subspaces form a covering of S. If any maximal subspace is a projective space, which occurs iff Veblen axiom holds (i.e. for any two skew lines ℓ and ℓ' and any point p not on them at most one line exists through p meeting both ℓ and ℓ'), then $(S, \mathscr{L})$ is said to be a *projective PLS*. If this is the case, then equality holds in (2.7).

A *polygonal path* in $(S, \mathscr{L})$ will mean an m-tuple of distinct lines $(\ell_1, \ell_2, \ldots, \ell_m)$ such that $\ell_i \cap \ell_{i+1} \neq \phi$, $i = 1, 2, \ldots, m-1$. The polygonal path will be either *closed* or *open* according to either $\ell_1 \cap \ell_m \neq \phi$ or $\ell_1 \cap \ell_m = \phi$. $(S, \mathscr{L})$ will be said to be *connected* if for any $x, y \in S$ a polygonal path exists through x and y. If $(S, \mathscr{L})$ is not connected then the relation ρ, defined by

(2.8) $x, y \in S, x \rho y \Longleftrightarrow$ a polygonal path exists through x and y,

is an equivalence relation on S. For any equivalence class $S_i, i \in I$, take the set $\mathscr{L}_i$ of lines in $\mathscr{L}$ which are contained in S_i; since $\mathscr{L} = \cup(\mathscr{L}_i : i \in I)$ and $\mathscr{L}_i \cap \mathscr{L}_j = \phi$, $i \neq j$, $(S_i, \mathscr{L}_i)$ is a connected *PLS*, $i \in I$, which will be called a *connected component* of $(S, \mathscr{L})$. Thus, any $(S, \mathscr{L})$ either is connected or can be partitioned into its connected components. Therefore, the investigation of *PLS*'s can be carried out for the connected ones only.

In the sequel the following notations will be used for a partial line space $(S, \mathscr{L})$. For $p \in S$ and $\ell \in \mathscr{L}$,

(2.9) $p \sim \ell$ (i.e. p collinear with ℓ) iff any point on ℓ is collinear with p;

(2.10) $p \nsim \ell$ (i.e. p non-collinear with ℓ) iff no point exists on ℓ which is collinear with p;

(2.11) $p \dashv \ell$ iff precisely one point q exists on ℓ such that $p \sim q$;

(2.12) $p, q, t \in S$, $q \not\approx_p t$ iff either $q \sim p \sim t$ or $q \not\sim p \not\sim t$.

3. A characterization of the Grassmann variety representing the lines in a projective space

It is well known that the lines in $PG(n, F)$, the n-dimensional projective space over the field F, can be represented as points in $PG(N, F)$, the N-dimensional projective space over F, where $N = n(n+1)/2 - 1$. Furthermore, such points form an algebraic manifold $\mathcal{G}$, which is called the *Grassmann variety* representing the lines in $PG(n, F)$. Let S denote the set of points on $\mathcal{G}$ and $\mathcal{L}$ the set of lines in $PG(N, F)$ belonging to $\mathcal{G}$; the pair $G = (S, \mathcal{L})$ is a *PLS* and a one-to-one and onto correspondence exists between the lines in $\mathcal{L}$ and the pencils of lines in $PG(n, F)$, a pencil of lines being the set of all lines through a point on a plane. Moreover, the subspaces of G are projective spaces and the maximal subspaces can be partitioned into two families $\mathcal{S}$ and $\mathcal{T}$. The elements of $\mathcal{S}$ are $(n-1)$-dimensional spaces and a bijection exists between them and the stars of lines in $PG(n, F)$, a star of lines being the set of all lines through a point. On the other hand, the elements of $\mathcal{T}$ are projective planes which correspond to ruled planes, i.e. the planes considered as the sets of their lines, in $PG(n, F)$, the correspondence being one-to-one and onto. It is easy to check that $\mathcal{S}$ and $\mathcal{T}$ satisfy the following:

(α) Any three pairwise collinear points belong to a subspace.

(i) $T, T' \in \mathcal{S}$, $T \neq T'$ imply $|T \cap T'| = 1$.

(ii) $T \in \mathcal{S}$, $\pi \in \mathcal{T}$ imply either $T \cap \pi = \phi$ or $T \cap \pi \in \mathcal{L}$.

(iii) For any $\ell \in \mathcal{L}$, a unique $T \in \mathcal{S}$ and a unique $\pi \in \mathcal{T}$ exist such that $\ell \subseteq T \cap \pi$.

REMARK Let $\mathbb{P}$ be a projective space of dimensions greater than two and assume $\mathbb{P}$ cannot be coordinatized by a field. Again, with $\mathbb{P}$ a *PLS* $G(\mathbb{P}) = (S(\mathbb{P}), \mathcal{L}(\mathbb{P}))$ can be associated whose points are the lines in $\mathbb{P}$ and whose lines are the pencils of lines in $\mathbb{P}$. It is straightforward to prove that the maximal subspaces of $G(\mathbb{P})$ are again partitioned into two families $\mathcal{S}(\mathbb{P})$ and $\mathcal{T}(\mathbb{P})$ satisfying (α), (i), (ii), and (iii). Obviously, when $\mathbb{P}$ can be coordinatized by a field, this construction recovers the incidence structure of points and lines provided by the Grassmann variety representing the lines in $\mathbb{P}$.

In [22] G. Tallini proved that, up to some properties to avoid degenerate cases, (α), (i), (ii), and (iii) characterize the *PLS*'s of type $G(\mathbb{P})$. Namely, he proved the following theorem.

THEOREM. Let $(S, \mathcal{L})$ be a *PLS* for which the following hold:

P1. Any three pairwise collinear points belong to a subspace.

P2. No line is a maximal subspace and the collection of maximal subspaces of $(S, \mathcal{L})$ can be partitioned into two families $\mathcal{S}$ and $\mathcal{T}$ such that

(i) $T, T' \in \mathcal{S}$, $T \neq T'$ imply $|T \cap T'| = 1$;

(ii) $T \in \mathcal{S}$, $\pi \in \mathcal{T}$ imply either $T \cap \pi = \phi$ or $T \cap \pi \in \mathcal{L}$;

(iii) for any $\ell \in \mathcal{L}$, both a unique $T \in \mathcal{S}$ and a unique $\pi \in \mathcal{T}$ exist such that $\ell \subseteq T \cap \pi$.

Under these assumptions, a projective space $\mathbb{P}$ exists of dimensions at least three such that $(S, \mathcal{L})$ is isomorphic to $G(\mathbb{P}) = (S(\mathbb{P}), \mathcal{L}(\mathbb{P}))$.

In particular, if $(S, \mathcal{L})$ is a finite, irreducible *PLS*, then it is isomorphic to the Grassmann variety representing the lines in a Galois space $PG(n, q)$. If $(S, \mathcal{L})$ is a graph, in which case P1 always holds, then the *PLS* is isomorphic to the line graph of a complete graph.

The above stated theorem was the starting point of several investigations aiming to study the incidence structures of Grassmann varieties of any index, to generalize the results to the affine spaces and to look at other remarkable algebraic varieties from this point of view. In the following sections the main results obtained will be summed up.

4. Grassmann spaces

Let Σ be a finite or infinite, affine or projective space of dimension at least three; for any integer h, $1 \leqslant h < \dim \Sigma - 1$, let $G^h(\Sigma)$ denote the family of all h-dimensional subspace in Σ.

The *h-th Grassmann space associated with* Σ, or the Grassmann space of the h-dimensional subspaces of Σ, is defined as the partial line space $\Gamma^h(\Sigma) = (G^h(\Sigma), F^h(\Sigma))$ where $F^h(\Sigma)$ denotes the family of all h-pencils of Σ; an h-pencil is the set of all h-dimensional subspaces passing through an $(h-1)$-dimensional space and belonging to an $(h+1)$-dimensional subspace. It is straightforward that $\Gamma^h(\Sigma)$ is a proper and connected *PLS*.

Notice that when $\Sigma = PG(n, F)$ is the n-dimensional projective space over the field F, $\Gamma^h(\Sigma)$ is isomorphic to the Grassmann variety representing the h-dimensional subspaces of Σ.

Denote by $\mathcal{S}(\Sigma)$ and $\mathcal{T}(\Sigma)$ the families of maximal subspaces in $\Gamma^h(\Sigma)$ defined as follows:

$S \in \mathcal{S}(\Sigma) \iff$ S is a complete h-star of Σ, i.e. the set of all h-dimensional subspaces of Σ through an $(h-1)$-dimensional subspace, the centre of S, possibly at infinity when Σ is an affine space.

$T \in \mathcal{T}(\Sigma) \iff$ T is the family of all h-dimensional subspaces of belonging to an $(h+1)$-dimensional subspace.

It is easy to check that $\mathcal{S}(\Sigma)$ and $\mathcal{T}(\Sigma)$ partition the set $\mathfrak{M}(\Sigma)$ of all maximal subspaces of $\Gamma^h(\Sigma)$. Of course, when Σ is an affine space a partition of $\mathcal{S}(\Sigma)$ can be considered into $\mathcal{S}_2(\Sigma)$ and $\mathcal{S}_1(\Sigma)$ according to the centre of the star being or not at infinity.

The space $\Gamma^h(\Sigma)$ satisfies the following incidence properties:

A1. Any three pairwise collinear points belong to a subspace.

A2. No line is a maximal subspace. The family $\mathfrak{M}(\Sigma)$ of maximal subspaces can be partitioned into three classes $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{T}$ such that, with $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, the following hold:

(i) $S, S' \in \mathcal{S}_2$, $S \neq S'$ imply $S \cap S' = \phi$;

(ii) $S \in \mathcal{S}$, $T \in \mathcal{T}$ imply either $S \cap T = \phi$ or $S \cap T$ is a line;

(iii) for any f in $F^h(\Sigma)$, both a unique $S \in \mathcal{S}$ and a unique $T \in \mathcal{T}$ exist such that $f \subseteq S \cap T$;

(iv) (Veblen axiom on S) given S_1, $S_2 \in \mathcal{S}$, either meeting at a point or both belonging to $\mathcal{S}_2$, if S_1', $S_2' \in \mathcal{S}$ are distinct and both meet S_1 and S_2 at distinct points, then either S_1' and S_2' meet or they both belong to $\mathcal{S}_2$.

A3. If $\mathcal{S}_2 \neq \phi$; then any point belongs to at least one maximal subspace in $\mathcal{S}_2$.

Obviously, either $\mathcal{S}_2 \neq \phi$ or $\mathcal{S}_2 = \phi$ according to Σ being either an affine or a projective space. Furthermore, when $\mathcal{S}_2 = \phi$ both (i) and A3 are empty.

It was proved that A1, A2, A3 characterize the Grassmann spaces associated with affine and projective spaces in the sense that will be made precise after the next definitions.

A proper and connected $PLS, \Gamma = (G, F)$, is called an *affine (projective) Grassmann space* if A1, A2, A3 hold and $\mathcal{S}_2 \neq \phi$ ($\mathcal{S}_2 = \phi$). An affine or projective Grassmann space is said to be of *finite index* h if it contains a length $h+1$ saturated chain C of subspaces whose least element is a line and whose greatest element is a subspace in $\mathcal{T}$. This notion is well defined since if such a chain C exists, then any other chain of the same kind can be proved to have length $h+1$.

The next theorem characterizes Grassmann spaces.

THEOREM 1. If Γ is a projective (affine) Grassmann space of finite index h, then a projective (affine) space $\mathbb{P}$ ($\mathbb{A}$) exists such that Γ and $\Gamma^h(\mathbb{P})$ ($\Gamma^h(\mathbb{A})$) are isomorphic. When $h = 1$, (iv), i.e. Veblen axiom on S, is a consequence of the other axioms defining a Grassmann space.

Theorem 1 was first proved for $h = 1$ by G. Tallini (see sect. 3) in the projective case and by A. Bichara and F. Mazzocca in the affine case [1, 4]. The proof of the general result is based on an induction argument which requires the above mentioned papers and was carried out by A. Bichara and G. Tallini in the projective case [10], [11] and by A. Bichara and F. Mazzocca in the affine case [3], [6].

The nature and the number of axioms defining a Grassmann space suggested to investigate their possible dependence. The answer in the negative was given by A. Bichara and F. Mazzocca who proved the following results [2], [5].

THEOREM 2. The axioms defining both affine and projective Grassmann spaces are independent.

Theorem 1 characterizes spaces $\Gamma^h(\Sigma)$ with the help of incidence properties of the two families $\mathcal{S}(\Sigma)$ and $\mathcal{T}(\Sigma)$ of maximal subspaces. Thus the obvious question arises whether just one of these families, namely $\mathcal{S}(\Sigma)$, is enough. Actually, the incidence properties of the family $\mathcal{T}(\Sigma)$ alone are too general to provide such a characterization.

This problem is far from being completely solved. Anyhow, under the assumptions Σ is a projective space and $h = 1$, N. Melone and D. Olanda proved the next theorem [19], [21].

THEOREM 3. Let $\Gamma = (G, F)$ be a proper PLS none of whose lines is a maximal subspace. Γ is isomorphic to the Grassmann space representing the lines in a projective space iff a covering $\mathcal{S}$ exists of Γ consisting of maximal subspaces such that:

If $S \in \mathcal{S}$ and $p \in G \setminus S$, any S' in $\mathcal{S}$ through p meets S at a unique point and the set of all such points is a line all whose points are precisely the points in S collinear with p.

The proof of this statement is achieved constructing another family $\mathcal{T}$ of maximal subspaces of Γ so that for $\mathcal{S}$ and $\mathcal{T}$ A1 and A2 in Theorem 1 hold when $h = 1$ and $\mathcal{S}_2 = \phi$.

Another possible characterization of the spaces $\Gamma^h(\Sigma)$ uses incidence properties involving lines only, i.e. no maximal subspace property is required. Again, when $h > 1$, no result is known yet. On the other hand, the next theorem provides an answer in case $h = 1$.

THEOREM 4. An irreducible *PLS* (G, F) is isomorphic to the Grassmann space representing the lines of a projective space iff the following hold:

(4.1) $\ell \in F, p \in G \setminus \ell \Rightarrow$ either $p \not\sim \ell$, or $p \dashv \ell$, or $p \sim \ell$.

(4.2) $\ell \in F, p, q, t \in G \setminus \ell : p \sim \ell, q \sim \ell, t \sim \ell, q \underset{p}{\not\approx} t \Rightarrow q \sim t$.

(4.3) $\ell \in F \Rightarrow p$ and q exist in G such that $p \not\sim q$, $p \sim \ell$ and $q \sim \ell$.

(4.4) $\ell \in F, p \in G \setminus \ell, p \not\sim \ell \Rightarrow \hat{\ell}_p = \{x \in G : x \sim p, x \sim \ell\} \in F$.

(4.5) $\ell, \ell' \in F, p \in G : \ell \cap \ell' = \phi, \ell \sim \ell', p \not\sim \ell, p \not\sim \ell' \Rightarrow \hat{\ell}_p = \hat{\ell}'_p$.

This theorem was first proved by P.M. Lo Re and D. Olanda [12], [13] for a finite *PLS*; they showed that the family of maximal subspaces of (G, F) can be partitioned into $\mathscr{S}$ and $\mathscr{T}$ so that A1 and A2 in Th. 1 hold for $h = 1$ and $\mathscr{S}_2 = \phi$. N. Melone and D. Olanda [19, 21] extendend this result to the infinite case, proving that under the assumptions in Theorem 4, (G, F) as a family of maximal subspaces satisfying the conditions in Theorem 3.

For an affine space, a similar result was proved by F. Mazzocca and D. Olanda [14], [15].

THEOREM 5. Let (G, F) be a *PLS* whose lines can be partitioned into two families F_1 and F_2. Then (G, F) is isomorphic to the Grassmann space representing the lines in an affine space iff the following hold:

(5.1) $\ell \in F, p \in G \setminus \ell \Rightarrow$ either $p \not\sim \ell$, or $p \dashv \ell$, or $p \sim \ell$.

(5.2) For any ℓ in F, p and q exist in $G \setminus \ell$, $p \neq q$, such that $p \not\sim q$, $p \sim \ell$ and $q \sim \ell$.

(5.3) For any line $\ell \in F_1$ and any point $p \in G \setminus \ell$, $p \not\sim \ell$, the set of all points q such that $q \sim p$ and $q \sim \ell$ is a line $\ell(p)$ in F_1. Furthermore, for any point $t \in \ell$ and any $q \in \ell(p)$, the line through t and q belongs to F_1.

(5.4) Let p, q, t be any three distinct points. If p, q and q, t belong to lines in F_2, then p and t are collinear and the line through them belongs to F_2.

(5.5) Let ℓ be a line and p, q two points collinear with any point on ℓ. Then p and q are of the same type with respect to ℓ (see [15] pg. 626) iff they are collinear.

Again, the proof is achieved with the help of Theorem 1. Namely, the family of maximal subspaces is proved to be partitionable into three classes meeting the hypotheses in Theorem 1 for $h = 1$.

5. Pseudoproduct spaces and C. Segre's varieties.

Let $\mathbb{P}_n$ and $\mathbb{P}'_m$ be two projective spaces of dimensions n and m, respectively. Set $S = \mathbb{P}_n \times \mathbb{P}'_m$. Denote by $\mathscr{L}$ the family of subsets of S consisting of $\{p\} \times \ell'$ and $\ell \times \{q'\}$, where $p \in \mathbb{P}_n$, $q' \in \mathbb{P}'_m$ and ℓ, ℓ' are lines in $\mathbb{P}_n$ and $\mathbb{P}'_m$, respectively. The pair $(S, \mathscr{L})$ is a *PLS* that will be called the *C. Segre's product space of type* $\{n, m\}$ and denoted by $S_{n,m}$.

REMARK. When $\mathbb{P}_n$ and $\mathbb{P}'_m$ are coordinatized by the same field F, $S_{n,m}$ is isomorphic to C. Segre's variety, the projective model of the product $\mathbb{P}_n \times \mathbb{P}'_m$.

The space $S_{n,m} = (S, \mathscr{L})$ satisfies the following:

S1. $\ell \in \mathscr{L}$, $p \in S \setminus \ell$, $p \not\sim \ell$ imply a unique $t_{p,\ell}$ exists in S such that $p \sim t_{p,\ell}$ and $t_{p,\ell} \sim \ell$;

S2. $p, q \in S$, $p \not\sim q$ imply that precisely two points exist, $\hat{p}$ and $\hat{q}$, such that $p \sim \hat{p} \sim q$ and $p \sim \hat{q} \sim q$.

Thus the definition of a *pseudoproduct space* is suggested as a proper $PLS(S, \mathscr{L})$ satisfying Veblen axiom and S1 and S2.

N. Melone and D. Olanda investigated pseudoproduct spaces, for short *PPS*, [18], [20], and the main results they achieved will be now summed up.

An irreducible *PPS* can be proved to contain *singular* points, i.e. points which are collinear with any point, iff it is a projective space. Therefore, it is enough to investigate *PPS*'s without singular points. Let p and q be any two non-collinear points in such a space $(S, \mathscr{L})$; the quadrangle with respect to (w.r.t.) p and q is defined as the set consisting of the four points $p, q, \hat{p}, \hat{q}$ and the four lines $(p, \hat{p})$, $(p, \hat{q})$, $(q, \hat{p})$, $(q, \hat{q})$ and denoted by $\langle p, q, \hat{p}, \hat{q} \rangle$. If $\mathscr{L}_p$ denotes the set of lines in $(S, \mathscr{L})$ through p, then

(5.1) $$| \mathscr{L}_p | \geq 2.$$

PROP. 5.2. In an irreducible *PPS* $(S, \mathscr{L})$ the following conditions are equivalent:

(i) For any $p \in S$, $| \mathscr{L}_p | = 2$.

(ii) $(S, \mathscr{L})$ is isomorphic to a *C.* Segre's product space $S_{1,1}$.

(iii) S has rank two.

PROP 5.3. In an irreducible *PPS* $(S, \mathcal{L})$ the following conditions are equivalent:

(i) At least one point p exists in S such that $|\mathcal{L}_p| \geqslant 3$.

(ii) A non-collinear point-line pair exists.

(iii) S has rank greater than two.

By prop. 5.2, just the irreducible *PPS*'s of rank greater than two are to be investigated. Under these assumptions, let p, ℓ be a non-collinear point-line pair in $(S, \mathcal{L})$ and for any point x on ℓ, let $\langle p, x, t_{p,\ell}, \hat{x}\rangle$ be the quadrangle w.r.t. p, x. $(S, \mathcal{L})$ will be said to be *regular* if for any point p and any line ℓ non-collinear with p, the point $\hat{x}$ spans a line as x ranges on ℓ.

For any two distinct collinear points p, q in $(S, \mathcal{L})$ set

(5.4) $$V(p, q) = \{x \in S : x \sim p \text{ and } x \sim q\}.$$

PROP. 5.5. Let $(S, \mathcal{L})$ be an irreducible *PPS* of rank greater than two. Then

(i) For any pair (p, q) of distinct collinear points, $V(p, q)$ is a maximal subspace of $(S, \mathcal{L})$.

(ii) Any maximal subspace of $(S, \mathcal{L})$ is a $V(p, q)$.

(iii) Any two maximal subspaces share at most one point.

(iv) Any point p in S belongs to exactly two distinct maximal subspaces and any line through p belongs to either of these spaces.

(v) The set of maximal subspaces in $(S, \mathcal{L})$ is partitioned into two families Σ and $\mathcal{T}$ such that subspaces of the same family are pairwise skew and two subspaces of distinct families meet at precisely one point.

Next, let V, W be two skew maximal subspaces and $\omega_{V,W}: V \to W$ the mapping defined as follows. For any $x \in V$, $\omega_{V,W}(x)$ is the point at which W meets the maximal subspace through x other than V.

PROP. 5.6. An irreducible *PPS* is regular iff for any two skew maximal subspaces V and W the mapping $\omega_{V,W}$ is a collineation.

Taking into account the above argument, C. Segre's product spaces are characterized as follows.

THEOREM. An irreducible *PPS* is isomorphic to a C. Segre's product space iff it is regular.

Next, a sketch of the proof will be given, pointing out that, since Veblen axiom holds, any subspace of $(S, \mathcal{L})$ is a projective space. Denote by n, m the dimensions of the maximal subspaces belonging to Σ and $\mathcal{T}$, respectively, and take $V \in \Sigma$ and $W \in \mathcal{T}$; let $S_{n,m}$ be the C. Segre's product space on $V \times W$. For any $(p, q) \in V \times W$, the maximal subspaces τ_p and σ_q through p and q, respectively, other than V and W meet at a point. The mapping

$$f: (p, q) \in V \times W \to \tau_p \cap \sigma_q \in S$$

turns out to be an isomorphism and the theorem follows.

6. Veronese spaces

Let $\mathbb{P}_n$ be a projective space of finite dimensions $n \geqslant 3$ and $V(\mathbb{P}_n) = (S, \mathcal{L})$ the *PLS* whose points are the unordered pairs of primes in $\mathbb{P}_n$ and whose lines are the subsets of pairs with a fixed component, the other one spanning the primes in a pencil of primes in $\mathbb{P}_n$. The space $V(\mathbb{P}_n)$ will be called the *Veronese space* associated with the projective space $\mathbb{P}_n$.

REMARK. If $\mathbb{P}_n$ is coordinatized by a field F of characteristic other than two, then $V(\mathbb{P}_n)$ is isomorphic to the incidence structure point-line provided by the variety of the spaces tangent to a Veronese manifold of indeces $(n, 2)$.

It is easy to check that $V(\mathbb{P}_n) = (S, \mathcal{L})$ satisfied the following:

V1. For any $\ell, \ell' \in \mathcal{L}$, $\ell \nsim \ell'$, a unique point p exists in S such that $p \sim \ell$ and $p \sim \ell'$.

V2. If $\ell \in \mathcal{L}, p, q \in S, p \sim \ell, q \sim \ell$, then $p \sim q$.

V3. For any $\ell \in \mathcal{L}, p \in S, p \nsim \ell, |\{x \in S : p \sim x \sim \ell\}| \leqslant 2$.

V4. For any $\ell \in \mathcal{L}$, a unique d_ℓ exists in S such that $d_\ell \sim \ell$ and $d_\ell \nsim x$ for every $x \in S$ and $x \nsim \ell$.

These properties suggest to call *Veronese space* any proper *PLS* $(S, \mathcal{L})$ whose lines are not maximal subspaces for which V1, V2, V3, and V4 hold.

Next, the results N. Melone [16, 17] obtained on these spaces are summarized.

A Veronese space $(S, \mathcal{L})$ is said to be projective if Veblen axiom holds. For any line $\ell \in \mathcal{L}$, set

(6.1) $$T(\ell) = \{x \in S : x \sim \ell\}.$$

PROP. 6.2. For a Veronese space $(S, \mathcal{L})$ the following hold:

(i) For any line ℓ, $T(\ell)$ is the unique maximal subspace to which ℓ belongs.

(ii) Any maximal subspace of $(S, \mathscr{L})$ is a $T(\ell)$, $\ell \in \mathscr{L}$; therefore, these subspaces cover S.

(iii) Any two distinct maximal subspaces meet at a unique point.

(iv) Any point belongs to at most two maximal subspaces.

A *double point* will be a point $d \in S$ through which just one maximal subspace passes; V will denote the set of all double points.

PROP 6.3. For any line $\ell \in \mathscr{L}$, d_ℓ is the unique double point in the subspace $T(\ell)$. Consequently, any maximal subspace contains a unique double point.

Let T, T' be any two distinct maximal subspaces and d_T, $d_{T'}$ their double points; for any $x \in T \setminus \{d_T\}$, denote by T_x the maximal subspace through x other than T. Define the mapping $\omega_{TT'}: T \to T'$ as follows:

$$(6.4) \qquad \omega_{TT'}(x) = \begin{cases} d_{T'} & \text{if } x = T \cap T'; \\ T \cap T' & \text{if } x = d_T; \\ T' \cap T_x & \text{if } x \in T \setminus \{d_T, T \cap T'\}. \end{cases}$$

$(S, \mathscr{L})$ will be called a *regular* Veronese space if $\omega_{TT'}$ is a collineation for any pair of distinct maximal subspaces T, T'.

Next, let ℓ be any line, T the maximal subspace through ℓ; the set

$$(6.5) \qquad a(\ell) = \begin{cases} \{d_{T_x} \in V : x \in \ell\} & \text{if } d_T \notin \ell, \\ \{d_{T_x} \in V : x \in \ell \setminus d_T\} \cup \{d_T\} & \text{if } d_T \in \ell, \end{cases}$$

will be called the *arc* associated with ℓ. Denote by $\mathscr{M}$ the set of all arcs.

PROP. 6.6. The pair $(V, \mathscr{M})$ is a line space, more precisely, a projective space.

The above argument leads to the final result.

THEOREM. Let $(S, \mathscr{L})$ be a Veronese space, V the set of its double points and $\mathscr{M}$ the family of its arcs. If $(S, \mathscr{L})$ is projective, regular and of finite rank, then it is isomorphic to the Veronese space associated with a projective space, namely the dual of $(V, \mathscr{M})$.

7. Schubert spaces

Let $\mathbb{P}$ be a projective space of dimensions $n > 1$ and denote by $\mathscr{P}_h$, $0 \leqslant h \leqslant n-1$, the family of h-dimensional subspaces of $\mathbb{P}$. An n-tuple $(S_0, S_1, \ldots, S_{n-1})$ of subspaces in $\mathbb{P}$ will be called a *flag* if for any $i = 0, 1, \ldots, n-2$, $S_i \subset S_{i+1}$ and $\dim S_j = j$ for all j's in $\{0, 1, \ldots, n-1\}$. Let $\mathscr{S}(\mathbb{P})$ be the family of all flags in $\mathbb{P}$.

Let $h \in \{0, 1, \ldots, n-1\}$; for any chain of subspaces $\overline{S}_0 \subset \overline{S}_1 \subset \ldots \subset \overline{S}_{h-1} \subset \overline{S}_{h+1} \subset \ldots \subset \overline{S}_{n-1}$ such that $\dim \overline{S}_i = i$, denote by $f_h(\overline{S}_0, \ldots, \overline{S}_{h-1}, \overline{S}_{h+1}, \ldots, \overline{S}_{n-1})$ the set of flags $\{(\overline{S}_0, \ldots, \overline{S}_{h-1}, S_h, \overline{S}_{h+1}, \ldots, \overline{S}_{n-1}) : S_h \in \mathscr{P}_h\}$ and by $\mathscr{F}_h = \mathscr{F}_h(\mathbb{P})$ the family of subsets in $\mathscr{S}(\mathbb{P})$ any of which is an $f_h(\overline{S}_0, \ldots, \overline{S}_{h-1}, \overline{S}_{h+1}, \ldots, \overline{S}_{n-1})$. Set

$$\mathscr{F} = \mathscr{F}(\mathbb{P}) = (\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_{n-1}) \text{ and}$$

$$\tilde{\mathscr{F}} = \tilde{\mathscr{F}}(\mathbb{P}) = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \ldots \cup \mathscr{F}_{n-1}.$$

The pair $\Sigma(\mathbb{P}) = (\mathscr{S}(\mathbb{P}), \tilde{\mathscr{F}}(\mathbb{P}))$ is called the *flag space* of $\mathbb{P}$ and it is easy to check that $(\mathscr{S}(\mathbb{P}), \tilde{\mathscr{F}}(\mathbb{P}))$ is a proper connected *PLS*. Furthermore, $\Sigma(\mathbb{P})$ satisfies the following:

B1. Any polygonal path containing exactly one line in $\mathscr{F}_i(\mathbb{P})$, $i = 0, 1, \ldots, n-1$, is open.

B2. A polygonal path of length four is closed iff its lines belong to $\mathscr{F}_i \cup \mathscr{F}_j$, $i < j$, $j \neq i+1$, $i = 0, \ldots, n-3$, $j = 2, \ldots, n-1$.

B3. Any polygonal path of length five whose lines belong to $\mathscr{F}_i \cup \mathscr{F}_{i+1}$, $i = 0, 1, \ldots, n-2$, is contained in a closed polygonal path of length six all whose lines belong to $\mathscr{F}_i \cup \mathscr{F}_{i+1}$.

REMARK. If $\mathbb{P}$ is coordinatized by a field, then the space $(\mathscr{S}(\mathbb{P}), \tilde{\mathscr{F}}(\mathbb{P}))$ is isomorphic to the incidence structure of points and lines provided by the Schubert variety representing the flags in $\mathbb{P}$.

The previous argument suggests the following definition. Let $\mathscr{S}$ be a non-empty set and $\mathscr{F} = (\mathscr{F}_0, \ldots, \mathscr{F}_{n-1})$, $n \geqslant 2$, an n-tuple of partitions of $\mathscr{S}$ which pairwise share no block. Set $\tilde{\mathscr{F}} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \ldots \cup \mathscr{F}_{n-1}$. The pair $\Sigma = (\mathscr{S}, \mathscr{F})$ will be called a *Schubert space* of finite index n if the pair $(\mathscr{S}, \tilde{\mathscr{F}})$ is a proper connected *PLS* satisfying B1, B2 and B3.

Schubert spaces were defined and investigated by A. Bichara and C. Somma who proved the following result [7], [8].

THEOREM. Any Schubert space of finite index n is isomorphic to the flag space of some n-dimensional projective space.

Recently, A. Bichara and C. Somma extendend to affine spaces the results summed up in this section. Namely, they defined and completely characterized the flag space of an affine space [9].

REFERENCES

[1] A. BICHARA & F. MAZZOCCA, *Su una caratterizzazione dello spazio di Grassmann delle rette di uno spazio affine,* Quad. Sem. Geom. Comb. n. 31, 1980, Ist. Mat. «G. Castelnuovo», Univ. Roma.

[2] A. BICHARA & F. MAZZOCCA, *Sull'indipendenza degli assiomi che definiscono gli spazi di Grassmann affini e proiettivi,* Rapp. Int. n. 8, Ist. Mat. Fac. Ing. Napoli, Ottobre 1981.

[3] A. BICHARA & F. MAZZOCCA, *On a characterization of Grassmann spaces associated with an affine space,* Rapp. Int. n. 5, Ist. Mat. Fac.Ing. Napoli, Sept. 1981.

[4] A. BICHARA & F. MAZZOCCA, *On a characterization of Grassmann space representing the lines in an affine space,* Simon Stevin, vol. 56, (1982), 129 - 141.

[5] A. BICHARA & F. MAZZOCCA, *On the independence of the axioms defining the affine and projective Grassmann spaces,* Ann. Discr. Math. 14 (1982), 123 - - 128.

[6] A. BICHARA & F. MAZZOCCA, *On a characterization of the Grassmann spaces associated with an affine space,* Ann. Discr. Math. 18 (1983),95 - 112.

[7] A. BICHARA & C. SOMMA, *A characterization of Schubert manifold associated with a 3-dimensional projective space,* Atti Convegno Geometria combinatoria e di incidenza, La Mendola (Italy), July 1982. Rend. Sem. Mat. Brescia, 7 (1984), 89 - 110.

[8] A. BICHARA & C. SOMMA, *On Schubert manifold associated with a projective space,* to appear in Rend. Mat. Univ. Roma.

[9] A. BICHARA & C. SOMMA, *On the flag space associated with an affine or projective space,* to appear in Rend. Mat. Univ. Roma

[10] A. BICHARA & G. TALLINI, *On a characterization of the Grassmann manifold representing the planes in a projective space,* Ann. Discr. Math., 14 (1982), 129 - 150.

[11] A. BICHARA & G. TALLINI, *On a characterization of the Grassmann space representing the h-dimensional subspaces in a projective space,* Ann. Discr. Math. 18 (1983), 113 - 132.

[12] P.M. LO RE & D. OLANDA, *Spazi di Grassmann,* Ist. Mat. «R. Cacciopoli», Univ. Napoli, 1980.

[13] P.M. LO RE & D. OLANDA, *Grassmann spaces,* J. Geometry, 17 (1981),50 - 60.

[14] F. MAZZOCCA & D. OLANDA, *A graphic characterization of the lines of an affine space,* Rapp. int. n. 6, Ist. Mat. Fac. Ing. Univ. Napoli, Ottobre 1981.

[15] F. MAZZOCCA & D. OLANDA, *A graphic characterization of the lines of an affine space,* Ann. Discr. Math. 18 (1983), 625 - 634.

[16] N. MELONE, *Veronese spaces,* Pubbl. Ist. Mat. Univ. Napoli, n. 42, 1982.

[17] N. MELONE, *Veronese spaces,* J. Geometry, 20 (1983), 169 - 180.

[18] N. MELONE & D. OLANDA, *Su certe strutture di incidenza che generalizzano le varietà di C. Segre,* Quad. Sem. Geom. Comb. n. 22, Febbraio 1980, Ist. Mat. «G. Castelnuovo», Univ. Roma.

[19] N. MELONE & D. OLANDA, *Una proprietà caratteristica della Grassmanniana delle rette di uno spazio proiettivo,* Publ. Ist. Mat. Univ. Napoli, n. 2, 1981.

[20] N. MELONE & D. OLANDA, *Spazi pseudoprodotti e varietà di C. Segre,* Rend. Mat. pura e appl., VII, I, 1981, 381 - 397.

[21] N. MELONE & D. OLANDA, *A characteristic property of the Grassmann manifold representing the lines of a projective space,* to appear in European J. Combin.

[22] G. TALLINI, *On a characterization of the Grassmann manifold representing the lines in a projective space,* in «Finite Geometries and Designs», London Math. Soc. Lecture Note Series 49, Cambridge U.P., (1981), 354 - 358.